COMP3121/9101
Algorithm Design

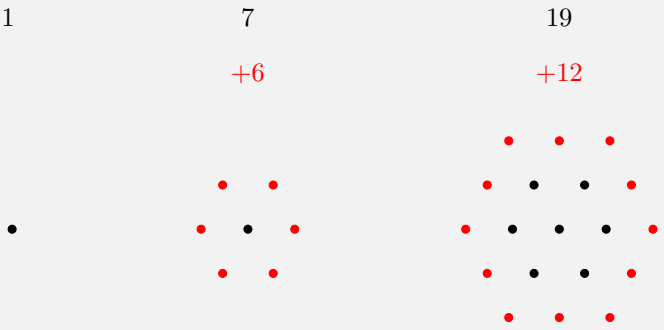
Practice Problem Set 5 – Dynamic Programming

[**K**] – key questions [**H**] – harder questions [**E**] – extended questions [**X**] – beyond the scope of this course

Contents

§ SECTION ONE: MEMOISATION

[K] Exercise 1. The *nth centred hexagonal number* is the number of vertices required to fill a hexagon of radius n. The first few hexagonal numbers are given with an illustration.



Design an $O(n)$ algorithm that returns the nth centred hexagonal number. Can you find a closed form for the nth centred hexagonal number, and thus, design an $O(1)$ algorithm for the nth centred hexagonal number?

Solution. We define each subproblem $P(i)$: *let* OPT(i) *be the ith centred hexagonal number.*

Let $H(n)$ be the nth centred hexagonal number. To get the nth layer of the hexagon, we need to add n vertices to each edge. Therefore, there are $6n$ vertices we need to add. However, there are exactly 6 vertices that belong on two edges (these are the corner vertices of the hexagon). Therefore, we need to subtract the double counting of the corner vertices. In other words, our recursion becomes

$$H(n) = H(n-1) + 6n - 6 = H(n-1) + 6(n-1),$$

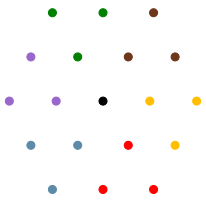
with $H(1) = 1$. This gives us a nice procedure for our recursive algorithm.

If $n = 1$, then our algorithm returns $H(1) = 1$. Otherwise, for each i from $2, \ldots, n$, we return

$$H(i) = H(i-1) + 6(i-1).$$

We cache each intermediary subproblem into an array of length n so that the lookup time is $O(1)$.

We can actually partition the vertices (except for the centre) of the hexagon into six triangles as follows.



These form the $(n-1)$ triangular numbers which have the closed form

$$1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}.$$

Therefore, the nth centred hexagonal number can be thought of six $(n-1)$ triangular numbers with the centre vertex

which gives the closed form

$$\begin{aligned} H(n) &= 6\left(\frac{n(n-1)}{2}\right) + 1 \\ &= 3n(n-1) + 1 \\ &= 3n^2 - 3n + 1. \end{aligned}$$

In other words, we can output the nth centred hexagonal number in constant time by returning $H(n) = 3n^2 - 3n + 1$. □

[K] Exercise 2. You are given a $2 \times n$ rectangular board and identical 1×2 tiles. Each tile can be laid either horizontally or vertically.

(a) Design an $O(n)$ algorithm to count the number of ways to completely cover the board with the 1×2 tiles. Can you find a closed form?

(b) We now build up a way to generate the number of ways to completely tile a $3 \times n$ rectangular board. Let $f(n)$ be the number of ways to tile a $3 \times n$ board with 1×2 tiles.

(i) Explain why $f(2k+1) = 0$ for each $k \in \mathbb{N}$. For the remainder of this problem, we will let $f(n)$ be the number of ways to tile a $3 \times 2n$ board with 1×2 tiles.

(ii) Show that $f(1) = 3$ and $f(2) = 11$.

(iii) Generate a suitable recursion based on the tilings found in $f(1)$.

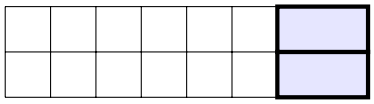
> To ensure that your recursion is correct, check that $f(2) = 11$.

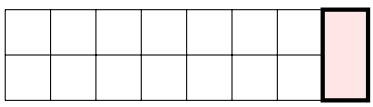
(iv) Hence, design an $O(n)$ algorithm that counts the number of ways to completely cover the $3 \times 2n$ board with the tiles.

As you found, constructing a general recurrence for an $m \times n$ board is quite difficult.

Solution.

(a)
- **Subproblem**: Let $\text{NUM}(i)$ be the number of ways to tile a $2 \times i$ rectangular board using 1×2 tiles.
- **Recurrence**: We observe that, to tile a $2 \times i$ tile, we have the two ending possibilities as follows:





If we place one of the blue tiles, the other 1×2 blue tile must appear to complete the grid. Therefore, in this case, it suffices to count the number of ways to tile the $1 \times (i-2)$ board. If we decide to place a red tile, then we see that it suffices to count the number of ways to tile the $1 \times (i-1)$ board. Therefore, we obtain the natural recurrence

$$\text{NUM}(i) = \text{NUM}(i-1) + \text{NUM}(i-2).$$

- **Base cases**: Since the recurrence relies on $\text{NUM}(i-1)$ and $\text{NUM}(i-2)$, we require the two base cases:

$$\text{NUM}(1) = 1, \quad \text{NUM}(2) = 2.$$

- **Order of Computation and Final Solution**: By our recurrence, we observe that each subproblem relies on computing smaller sized boards. Therefore, the natural ordering is to solve the subproblems in increasing order of board size; that is, we compute $\text{NUM}(1), \text{NUM}(2), \text{NUM}(3), \text{NUM}(4), \ldots, \text{NUM}(n)$, with the final solution being $\text{NUM}(n)$.
- **Time Complexity**: There are n subproblems, each of which can be computed in $O(1)$ time. Therefore, the time complexity is $O(n)$.

If we define $\text{NUM}(0) = 1$, then we observe that we obtain the Fibonacci sequence by our recurrence; specifically,

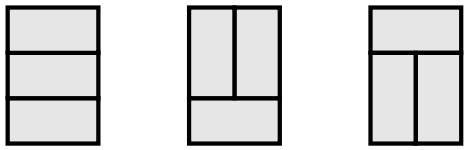
$$\begin{aligned}
\text{NUM}(0) &= 1 = F_1, \\
\text{NUM}(1) &= 1 = F_2, \\
\text{NUM}(2) &= 2 = F_3, \\
\text{NUM}(3) &= 3 = F_4, \\
&\cdots \\
\text{NUM}(n) &= F_{n-1}.
\end{aligned}$$

Therefore, we can simply return $\text{NUM}(n) = F_{n-1}$, which has a well-known closed form; that is,

$$\text{NUM}(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}.$$

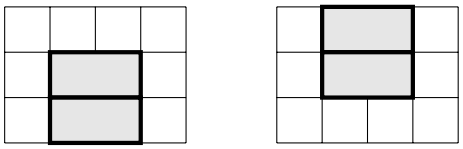
(b) (i) We see that every 1×2 tile must cover exactly two distinct squares of the board. Therefore, to cover a board with 1×2 tiles, the number of squares in the board must be even. However, any $3 \times (2k+1)$ board will contain an odd number of squares. Therefore, any tiling will have at least one square remaining if no tiles can intersect.

(ii) We first show that $f(1) = 3$. This is easy to list out. We have

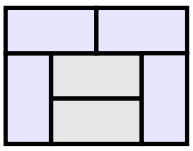
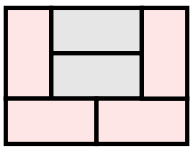


Therefore, we have $f(1) = 3$. To derive an expression for $f(2)$, we can break this into two 3×2 boards. Each board has exactly three configurations as shown above. Therefore, there are $3 \times 3 = 9$ configurations if tiles do not overlap between the two 3×2 boards. We now look at all configurations where tiles can overlap over the two boards.

Consider the following configurations.



This forces the remaining tiles to be placed in the following positions.

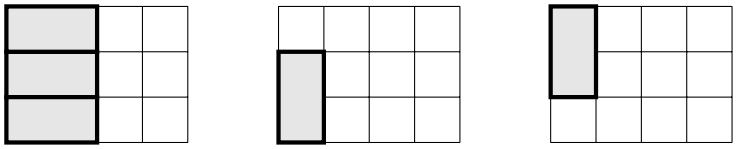
Note that placing three 1×2 tiles in the middle of the board reduces the problem to a 3×1 board which has no solution. Therefore, we note that these are the only additional solutions that aren't accounted for in the previous disjoint case. Thus, we obtain $f(2) = 9 + 2 = 11$ number of ways to tile a 3×4 board.

(iii) Let $f(n)$ denote the number of ways to tile a $3 \times 2n$ board and $g(n)$ denote the number of ways to tile a $3 \times (2n + 1)$ board with its first square missing.

To tile a $3 \times 2n$ board, we can either:

- tile it so that the three left-most squares are all covered by the horizontal 1×2 tiles, leaving us with a $3 \times (2n - 2)$ board,
- or tile two of the left-most squares leaving one square vacant.

The following are illustrated below.



This gives $f(n) = f(n - 1) + 2g(n - 1)$.

In a similar light, to tile a $3 \times (2n+1)$ board with one tile missing, the other two squares have to be covered by either:

- a single tile, leaving a $3 \times 2n$ board,
- or two tiles leaving a missing square in the next column.

This gives $g(n) = f(n) + g(n - 1)$.

Therefore the recurrence becomes

$$\begin{aligned} f(n) &= f(n-1) + 2g(n-1) \\ &= f(n-1) + 2\left(f(n-1) + g(n-2)\right) \\ &= f(n-1) + 2\left(f(n-1) + \frac{f(n-1) - f(n-2)}{2}\right) \\ &= f(n-1) + 2f(n-1) + f(n-1) - f(n-2) \\ &= 4f(n-1) - f(n-2). \end{aligned}$$

(c) We therefore devise a dynamic programming solution based on the recurrence above.

- **Subproblem**: Let $f(n)$ denote the number of ways to tile a $3 \times 2n$ board using 1×2 dominoes.
- **Recurrence**: The recurrence is given by

$$f(n) = 4f(n-1) - f(n-2).$$

- **Base case**: $f(1) = 3$ and $f(2) = 11$.
- **Order of Computation and Final Solution**: We solve each subproblem in increasing order of i with the final solution being $f(n)$.

- **Time Complexity**: There are n subproblems and each subproblem can be computed in cosntant time giving an $O(n)$ algorithm.

□

[K] Exercise 3. In this problem, we introduce the *Tower of Hanoi* game. For students who might not be familiar with the game, we have three pegs and n disks. Each disk has a unique radius. In other words, no two distinct disks share the same radius. The rules of the game are as follows:

- The game state initially starts out with all disks on the first peg. The disks are stacked in increasing order of radius with the smallest disk on the top of the stack.
- In each move, you can only move one disk from one peg to another. You can only move one disk from one peg to another peg if the top of the stack of the peg has a bigger radius. In other words, you cannot stack a disk on top of a smaller disk.
- The game ends when all disks are placed in the third peg, in increasing order of radius with the smallest disk on the top of the stack of the third peg.

To familiarise yourself with the game, feel free to click here to play a game! In this problem, we will design an algorithm that finds the minimal number of moves to win the game, given n disks.

Let $f(n)$ be the minimum number of moves to win the *Tower of Hanoi* game with n disks.

(a) Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

(b) For $n \geq 2$, explain why $f(n) = 2f(n-1) + 1$.

(c) Hence, design an $O(n)$ algorithm that computes the minimum number of moves to win the game with n disks.

Solution.

(a) Denote $a \to b$ as moving the topmost disk of peg a to peg b. Then:

- for $n = 1$, the solution is $1 \to 3$,
- for $n = 2$, the solution is $1 \to 2$, $1 \to 3$, $2 \to 3$,
- for $n = 3$, the solution is $1 \to 3$, $1 \to 2$, $3 \to 2$, $1 \to 3$, $2 \to 1$, $2 \to 3$, $1 \to 3$, and
- for $n = 4$, the solution is $1 \to 2$, $1 \to 3$, $2 \to 3$, $1 \to 2$, $3 \to 1$, $3 \to 2$, $1 \to 2$, $1 \to 3$, $2 \to 3$, $2 \to 1$, $3 \to 1$, $2 \to 3$, $1 \to 2$, $1 \to 3$, $2 \to 3$.

Therefore $f(1) = 1$, $f(2) = 3$, $f(3) = 7$ and $f(4) = 15$.

(b) In order to move the largest disk to the target position, we must first move all $n-1$ disks above it to peg 2. This takes $f(n-1)$ moves. Once we move the largest disk, we must then move all $n-1$ smaller disks from peg 2 to peg 3, again taking $f(n-1)$ moves. Therefore the task both requires at least $2f(n-1)+1$ moves and can be accomplished in exactly this many moves, completing the proof.

(c) **Subproblems:** for $i \geq 1$, let $f(i)$ be the minimum number of moves to win the *Tower of Hanoi* game with i disks.

Recurrence: for $i > 1$, $f(i) = 2f(i-1) + 1$.

Base case: $f(1) = 1$.

Order of computation: increasing from $i = 1$ to $i = n$.

Final answer: $f(n)$.

Time complexity: each of n subproblems is solved in constant time, for a total time complexity of $O(n)$.

Note: we can also prove (by induction) that $f(n) = 2^n - 1$, and compute this directly for a faster algorithm.

□

[H] Exercise 4. Strings are formed from the alphabet $\Sigma = \{0, 1, 2, 3\}$. We want to count the number of strings of length n that do not contain 12 or 20 as substrings. Let $f(n)$ be the number of such strings of length n.

(a) Find $f(1)$ and $f(2)$.

(b) Show that $f(3) = 49$.

Hint: First, how many strings are there without any restriction? Then find the number of length 3 strings which contain 12 as a substring. Then find the number of length 3 strings which contain 20 as a substring. Are there any strings which contain *both*, 12 and 20 as substrings?

(c) Find a suitable recursion in terms of $n-1$, $n-2$, $n-3$ that finds the number of length n strings that do not contain 12 or 20 as substrings, and hence, design an $O(n)$ algorithm that finds the number of n length strings that does not contain 12 or 20 as substrings.

> Which subproblems would we need to consider to define our recursion?

Solution.

(a) $f(1) = 4$, as all single-character strings are valid.
$f(2) = 16 - 2 = 14$, as all two-character strings except "12" and "20" are valid.

(b) There are four strings of each of the following forms: "12x", "x12", "20x" and "x20". This suggests deducting sixteen from the set of sixty-four three character strings.

However "120" is double-counted, as it counts towards both the forms above. Therefore only fifteen three-character strings are invalid, giving the answer $f(3) = 49$.

(c) To obtain a suitable recurrence, let $f(n-1)$ denote the number of such strings of length $n-1$. We now look at what happens when we append each of the characters. We will define four new recurrences: $a(n)$, $b(n)$, $c(n)$, $d(n)$ for such strings ending in 0, 1, 2, 3 respectively such that $f(n) = a(n) + b(n) + c(n) + d(n)$.

- The nth character is a 0. Then we cannot have the $(n-1)$th character be a 2. Therefore, the $(n-1)$th character is either a 0, 1 or 3. This gives the recurrence $a(n) = a(n-1) + b(n-1) + d(n-1)$.
- The nth character is a 1. Then every string of length $n-1$ forms a valid string. Therefore, the number of such strings is $b(n) = f(n-1)$.
- The nth character is a 2. Then we cannot have the $(n-1)$th character be a 1. Therefore, the $(n-1)$th character is either a 0, 2 or 3. This gives the recurrence $c(n) = a(n-1) + c(n-1) + d(n-1)$.
- The nth character is a 3. Then every string of length $n-1$ forms a valid string. Therefore, the number of such strings is $d(n) = f(n-1)$.

We now simplify this expression entirely in terms of $f(n)$. Note that $b(n)$ and $d(n)$ are recurrences purely in terms of $f(n)$. We also see that

$$f(n-1) - b(n-1) = a(n-1) + c(n-1) + d(n-1) = c(n).$$

Therefore, we have that

$$\begin{aligned} c(n) &= f(n-1) - b(n-1) \\ &= f(n-1) - b(n-1) \\ &= f(n-1) - f(n-2). \end{aligned}$$

Similarly, we see that $a(n) = f(n-1) - c(n-1)$ and

$$a(n) = f(n-1) - f(n-2) - f(n-3).$$

Thus, we deduce that the recurrence is

$$\begin{aligned} f(n) &= a(n) + b(n) + c(n) + d(n) \\ &= (f(n-1) - f(n-2) - f(n-3)) + f(n-1) + (f(n-1) - f(n-2)) + f(n-1) \\ &= 4f(n-1) - 2f(n-2) - f(n-3). \end{aligned}$$

We now design an $O(n)$ algorithm based on the above.

- **Subproblem**: Let $f(n)$ denote the number of strings of length n that do not contain 12 or 20 as substrings.
- **Recurrence**: $f(n) = 4f(n-1) - 2f(n-2) - f(n-3)$ for $n > 3$.
- **Base case**: $f(1) = 4$, $f(2) = 14$, $f(3) = 49$.
- **Order of Computation and Final Solution**: We solve the subproblems in increasing order with the final solution being $f(n)$.
- **Time Complexity**: There are n subproblems and each subproblem can be computed in $O(1)$ time; therefore, the running time is $O(n)$.

□

§ SECTION TWO: OPTIMISATION

[K] Exercise 5. (*Making Change*) You are given n types of denominations of values with $v(1) < v(2) < \cdots < v(n)$ and $v(1) = 1$. You are additionally given a positive integer C. Design an $O(nC)$ algorithm which makes change of value C using as few coins as possible. You may assume that you have an infinite supply of such coins.

Provide a suitable subproblem that allows you to define a simple recursion.

Solution. To provide a formal dynamic programming solution, we need to propose a suitable subproblem for which our recursion falls out naturally.

Let $P(i)$ be the subproblem: *what is the smallest number of coins required to make change of value i?*. Let $\text{OPT}(i)$ be the solution to the subproblem $P(i)$. Note that, to arrive at subproblem $P(i)$, we need to consider what happens if we take out an arbitrary coin. In particular, each coin will only add one to our tally. Therefore, to obtain the minimal number of coins, each previous subproblem needs to also maintain minimality. Therefore, we minimise each of the previous subproblems by considering all of the coin denominations; this gives us the following recursion

$$\text{OPT}(i) = \min\{\text{OPT}(i - v(k)) : 1 \leq k \leq n\} + 1,$$

for $1 \leq i \leq C$. To obtain the actual change sequence, suppose that the optimal k that was chosen was m. In other words, $\text{OPT}(i) = \text{OPT}(i - v(m)) + 1$. We can construct a table where index i stores the value m. Then backtracking to obtain the sequence is a matter of figuring out the right lookup based on the kth denomination for each subproblem.

We can define the base case to be $\text{OPT}(0) = 0$ since no coins are required to obtain a value of 0. The final solution is $\text{OPT}(C)$ and the natural order of computing the subproblems is in increasing order of i.

We finally justify the time complexity. There are C subproblems (one for each i) and each subproblem requires a single loop through all of the n denominations. Therefore, each subproblem is solved in $O(n)$ time, which implies that our algorithm runs in $C \cdot O(n) = O(nC)$ time. It is important to note that C is not a constant, but rather part of our input. Therefore, we can't reduce this to $O(n)$. □

[K] Exercise 6. (*Edit Distance*) You are given string A of size n and string B of size m, and you want to transform A to B. You are allowed to insert a character, delete a character, and replace a character with another. However, each operation comes with a cost.

- Inserting a character costs c_I;
- Deleting a character costs c_D;
- Replacing a character costs c_R.

Design an $O(mn)$ algorithm to find the lowest total cost to transform A into B.

How should you relate strings A and B together? Provide a suitable subproblem that allows you to do so.

Solution. The trick behind this exercise is to provide a natural subproblem that allows you to connect strings A and B together. In particular, we want to ensure that the first k letters of both strings are matching, so that we only need to change subsequent characters.

Therefore, we shall let $P(i, j)$ be the subproblem: *what is the lowest total cost to transform the sequence $A[1 \ldots i]$ to $B[1 \ldots j]$?*. Let $\text{OPT}(i, j)$ be the solution to subproblem $P(i, j)$.

To obtain the recursion, we need to find all of the ways to obtain $A[1 \dots i]$ and $B[1 \dots j]$. If the current operation is delete, then we must have transformed $A[1 \dots i-1]$ into $B[1 \dots j]$ and then deleted $A[i]$. If the current operation is insert, then we must have transformed $A[1 \dots i]$ into $B[1 \dots j-1]$ and then appended $B[j]$. Otherwise, we must have transformed $A[1 \dots i-1]$ into $B[1 \dots j-1]$. In this case, if $A[i] = B[j]$, there is nothing to do. Otherwise, if $A[i] \neq B[j]$, then we do a simple replacement. This defines our recursion as follows:

$$\textsc{opt}(i,j) = \min \begin{cases} \textsc{opt}(i-1,j) + c_D, & \\ \textsc{opt}(i,j-1) + c_I, & \\ \textsc{opt}(i-1,j-1) & \text{if } A[i] = B[j], \\ \textsc{opt}(i-1,j-1) + c_R & \text{if } A[i] \neq B[j], \end{cases}$$

with $1 \leq i \leq n$ and $1 \leq j \leq m$.

The base case is $\textsc{opt}(i,0) = i \cdot c_D$ and $\textsc{opt}(0,j) = j \cdot c_I$. The overall solution is $\textsc{opt}(n,m)$. Note that there are mn subproblems (one for each character of A and one for each character of B). However, updating each of the subproblem takes constant time. Therefore, the overall running time of the algorithm is $O(mn)$. □

[K] Exercise 7. A *palindrome* is a word that can be read the same both forwards and backwards. For example, the word "kayak" is a palindrome as reading it forwards is the same as reading it backwards. Similarly, "kaayak" is not a palindrome because reading it backwards reads "kayaak" which is not the same as "kaayak". Given a string of length n, design an $O(n^2)$ algorithm that finds the minimum number of characters to delete such that the resulting string after deletion is a palindrome.

For example, "kaayak" only requires one deletion to form a palindrome, while the string "abccab" requires two deletions.

Solution. Let $s[1, \dots, n]$ be our input string.

- **Subproblem**: Let $\textsc{opt}(i,j)$ denote the minimum number of characters to delete such that the string $s[i, i+1, \dots, j]$ is a palindrome.
- **Recurrence**: Now, to derive a recursion, we first observe that $\textsc{opt}(i+1, j-1)$ gives us the smallest number of deletions required such that $s[i+1, \dots, j-1]$ is a palindrome.

$$\underbrace{s[i+1]\, s[i+2] \dots s[j-1]}_{\textsc{opt}(i+1,j-1) \implies \text{ palindrome}}$$

  To ensure that $s[i, \dots, j]$ is a palindrome, we now determine whether $s[i] = s[j]$. If so, then there are no extra deletions required to form a palindrome from index i to j. In other words, the number of deletions required is just the number of deletions that forms the palindrome $s[i+1, \dots, j-1]$; in other words, we have that $\textsc{opt}(i,j) = \textsc{opt}(i+1, j-1)$ in the case where $s[i] = s[j]$.

  We now jump to the case where $s[i] \neq s[j]$. In this case, we need to delete one of $s[i]$ or $s[j]$. To determine which character to delete, we compute both subproblems and take the minimum number of deletions among the two. Therefore, under the case where $s[i] \neq s[j]$, we have

$$\textsc{opt}(i,j) = \min\{\textsc{opt}(i+1,j), \textsc{opt}(i,j-1)\} + 1.$$

  In other words, the recurrence becomes

$$\textsc{opt}(i,j) = \begin{cases} \textsc{opt}(i+1,j-1) & \text{if } s[i] = s[j], \\ \min\{\textsc{opt}(i+1,j), \textsc{opt}(i,j-1)\} + 1 & \text{if } s[i] \neq s[j]. \end{cases}$$

  For the edge case where $j - 1 \leq i$, set $\textsc{opt}(i,j) = 0$ since we already have a palindrome.

- **Base case**: For the base case, note that $\text{OPT}(i, i) = 0$ for each $1 \leq i \leq n$ since a single character is a palindrome.
- **Order of Computation and Final Solution**: The final solution is $\text{OPT}(1, n)$. To obtain the order of evaluation, observe that each subproblem relies on computing "smaller" length strings. Therefore, we should be solving the subproblems in increasing order of string size (i.e. in increasing order of $j - i$).
- **Time Complexity**: To determine the time complexity, observe that there are n^2 many subproblems. Each subproblem takes $O(1)$ to compute; therefore, the running time of the algorithm is $O(n^2)$.

□

[K] Exercise 8. You are given a set of n types of rectangular boxes, where the i^{th} box has height h_i, width w_i and depth d_i. You want to create a stack of boxes which is as tall as possible, but you can only stack a box on top of another box if the dimensions of the 2-D base of the lower box are each strictly larger than those of the 2-D base of the higher box. Of course, you can rotate a box so that any side functions as its base. It is also allowable to use multiple instances of the same type of box. Design an $O(n^2)$ algorithm that returns the maximum height of the stack of boxes.

How can we reduce this to a problem without rotations?

Solution. To simplify the problem, we distinguish among all of the rotations. For each box, there are six rotations since for each face, we can exchange its width and height and there are three faces. Thus, consider the problem of having $6n$ types of rectangular boxes, which allows us to assume that there are no rotations involved. We now solve the original problem.

Using merge sort, we can order the boxes in decreasing order based on the surface area of their base (remember that each box is fixed and there are no rotations). In this way, if B_1 can be stacked on top of B_0, then B_0 must appear before B_1 when we ordered the boxes.

- **Subproblem**: Let $\text{OPT}(i)$ denote the maximum height possible for a stack if the top box is box number i.
- **Recurrence**: Observe that we look at *all* boxes whose base is strictly larger than box B_i and look at what the maximum height can be produced from placing box B_i at the top. That is, for each $1 \leq i \leq 6n$:

$$\text{OPT}(i) = \max\{\text{OPT}(j) + h_i : \text{ over all } j \text{ such that } w_j > w_i,\ d_j > d_i\}.$$

- **Base case**: $\text{OPT}(i) = h_i$, for each i if we cannot place a box below B_i.
- **Order of Computation and Final Solution**: By focusing on the recurrence, we observe that each subproblem relies on smaller widths and smaller depth. Therefore, we compute the subproblems in increasing order of w_i and d_i (i.e. in increasing order of $w_i + d_i$). The final solution is $\max_{1 \leq i \leq 6n} \text{OPT}(i)$.
- **Time Complexity**: Ordering the boxes using merge sort takes $O(n \log n)$ time. For the dynamic programming component, we have $6n$ subproblems and for each subproblem, we perform a $O(n)$ search to find boxes whose base is large enough to stack the current box. Thus, we have an $O(n \log n) + O(6n^2) = O(n^2)$ algorithm.

□

[K] Exercise 9. Given a sequence of n positive or negative integers $A_1, A_2, \ldots, A_n$, determine a contiguous subsequence A_i to A_j for which the sum of elements in the subsequence is maximised.

Solution.

- **Subproblem**: Let $\text{OPT}(i)$ denote the maximum sum of elements ending with integer i.

- **Recurrence**: Suppose that we have chosen the sequence of elements that end with $i-1$. If we now consider the ith element of A, we can either choose to include it into our subsequence or A_i starts the new block.
  - If $A_i + \text{OPT}(i-1)$ is negative, then A_i should start a new chain.
  - If $A_i + \text{OPT}(i-1)$ is non-negative, then it is safe to keep A_i as part of the current chain.

  Therefore, we obtain the recurrence

$$\text{OPT}(i) = A_i + \max\{\text{OPT}(i-1), 0\}.$$

  We can determine the contiguous subsequence by keeping track of the starting element of the chain. To $\text{OPT}(i-1) \leq 0$, then the ith element forms the start of the chain. Therefore, define $\text{start}(i)$ to denote the start of the chain that includes the A_i. Then, for all $1 \leq i \leq n$, we have the recurrence

$$\text{start}(i) = \begin{cases} \text{start}(i-1) & \text{if } \text{OPT}(i-1) > 0, \\ i & \text{if } \text{OPT}(i-1) \leq 0. \end{cases}$$

- **Base case**: If there are no elements chosen, then the maximum sum of the elements is 0. Therefore, $\text{OPT}(0) = 0$.
- **Order of Computation and Final Solution**: We observe that the recurrence lies on smaller values of i. Therefore, the natural order of evaluation is in increasing order of i with the final solution being $\max_{1 \leq i \leq n} \text{OPT}(i)$. We can return the contiguous subproblem by returning the interval $[\text{start}(i), i]$ for the given index i.
- **Time Complexity**: There are n subproblems and each subproblem takes $O(1)$ to compute; therefore, the running time of the algorithm is $O(n)$.

□

[K] Exercise 10. Due to the recent droughts, n proposals have been made to dam the Murray river. The ith proposal asks to place a dam x_i meters from the head of the river (i.e., from the source of the river) and requires that there is not another dam within r_i metres (upstream or downstream). Design an algorithm that returns the maximal number of dams that can be built, you may assume that $x_i < x_{i+1}$ for all $i = 1, \ldots, n-1$.

Solution.

- **Subproblem**: Let $\text{OPT}(i)$ denote the maximum number of dams that can be built among proposals $1, \ldots, i$ such that the ith dam is built.
- **Recurrence**: We first observe that, if we build the ith dam, then we cannot place any dam where $|x_i - x_j| \leq r_i$ *and* $|x_i - x_j| \leq r_j$. Therefore, we need to look at all dams such that $|x_i - x_j| > \max\{r_i, r_j\}$. By our subproblem formulation, we look at all proposals from $1, \ldots, i$. Therefore, we look at all possible dams $j < i$ that satisfy the above constraint that maximises the number of dams that we can build. This is equivalent to building dam i and then building dams using proposals $1, \ldots, j$, giving us the recurrence

$$\text{OPT}(i) = 1 + \max_{j<i}\{\text{OPT}(j) : x_i - x_j > \max(r_i, r_j)\}.$$

- **Base case**: If there is only one dam, then we build the dam. Therefore, $\text{OPT}(1) = 1$.
- **Order of Computation**: To obtain the appropriate recurrence, we need to look at all proposals from $1, \ldots, j, \ldots, i$. Therefore, the natural order of evaluation is in increasing order of i.
- **Overall answer**: Considering all possible choices for the last dam to be built, the answer is

$$\max_{1 \leq i \leq n} opt(i).$$

- **Time Complexity**: There are n subproblems. However, each subproblem requires us to look at $j < i$ subproblems. This gives $1 + 2 + \cdots + n = O(n^2)$ many subproblems to compute altogether. Therefore, the running time of the algorithm is $O(n^2)$.

□

[K] Exercise 11. You are travelling by canoe down a river and there are n trading posts along the way. Before starting your journey, you are given for each $1 \leq i < j \leq n$ the fee $F(i, j)$ for renting a canoe from post i to post j. These fees are arbitrary, for example it is possible that $F(1, 3) = 10$ and $F(1, 4) = 5$. You begin at trading post 1 and must end at trading post n (using rented canoes). Your goal is to design an efficient algorithm that produces the sequence of trading posts where you change your canoe which minimizes the total rental cost.

Solution.

- **Subproblem**: Let $\text{OPT}(i)$ denote the minimum cost it would take to reach post i.
- **Recurrence**: We note that, to arrive at post i, we must come from some post j. We look at all possible posts that we can come from and choose the post that minimises the overall cost. This gives us

$$\text{OPT}(i) = \min_{1 \leq j \leq i} \{\text{OPT}(j) + F(j, i)\}.$$

- **Base case**: We begin at trading post 1; therefore, the base case is $\text{OPT}(1) = 0$.
- **Order of Computation and Final Solution**: Since our subproblem recurrence relies on previous values of i, the natural ordering is in increasing order of i with the final solution being $\text{OPT}(n)$.

  We then reconstruct the sequence of trading posts the canoe had to have visited. For $i > 1$ we define the following function:

$$\text{from}(i) = \operatorname*{argmin}_{1 \leq j < i} \{\text{OPT}(j) + F(j, i)\},$$

  where argmin returns the value j that minimises $\text{OPT}(j) + F(j, i)$. To obtain the sequence, we backtrack from n, giving the sequence:

$$\{n, \text{from}(n), \text{from}(\text{from}(n)), \ldots, 1\}$$

  and reversing this sequence gives the boat's journey.
- **Time Complexity**: There are n subproblems and each subproblem takes $O(n)$ time to find the previous trading post. Therefore, the overall running time is $O(n^2)$.

□

[K] Exercise 12. You have an amount of money M and you are in a candy store. There are n kinds of candies and for each candy, you know how much pleasure you get by eating it, which is a number between 1 and K, as well as the price of each candy. Your task is to choose which candies you are going to buy to maximise the total pleasure you will get by gobbling them all.

Solution. This is a knapsack problem with duplicated values. The pleasure score is the value of the item, the cost of a particular type of candy is its weight, and the money M is the capacity of the knapsack. The complexity is $O(Mn)$.
□

[K] Exercise 13. Your shipping company has just received N shipping requests (jobs). For each request i, you know it will require t_i trucks to complete, paying you d_i dollars. You have T trucks in total. Out of these N jobs you can take as many as you would like, as long as no more than T trucks are used total. Devise an efficient algorithm to select jobs that will bring you the largest possible amount of money.

Solution. We can recognise that this is nothing but the standard knapsack problem with t_i being the size of the ith item, d_i its value and with T as the capacity of the knapsack. Since each transportation job can be executed only once, it is a knapsack problem with no duplicated items allowed. The complexity follows the knapsack problem which results to be $O(NT)$. □

[K] Exercise 14. Again your shipping company has just received N shipping requests (jobs). This time, for each request i, you know it will require e_i employees and t_i trucks to complete, paying you d_i dollars. You have E employees and T trucks in total. Out of these N jobs you can take as many of them as you would like, as long as no more than E employees and T trucks are used in total. Devise an efficient algorithm to select jobs which will bring you the largest possible amount of money.

Solution. This is a slight modification of the knapsack problem with two constraints on the total size of all jobs; think of a knapsack which can hold items of total weight not exceeding E units of weight and total volume not exceeding T units of volume, with item i having a weight of e_i integer units of weight and t_i integer units of volume.

- **Subproblem**: For each triplet $e \leq E, t \leq T, i \leq N$, let $\textsc{opt}(i, e, t)$ denote the subcollection of items $1, \ldots, i$ that fits in a knapsack of capacity e and t units of volume which is of largest possible value.
- **Recurrence**: The recurrence is then as follows:

$$\textsc{opt}(i, e, t) = \max\{\textsc{opt}(i-1, e, t),\ \textsc{opt}(i-1, e-e_i, t-t_i) + d_i\},$$

  and the rest follows as with the knapsack problem. The time complexity is $O(NET)$.

□

[H] Exercise 15. There are m levels in a video game, and in each level, there are n doors you can choose from. Each door has a treasure chest with value $a_{i,j}$, where $a_{i,j}$ represents the treasure value of door j in level i. In each level, you can only choose one door and once you choose that particular door, you can only choose from doors directly adjacent to the chosen door (if they exist) in the next level. In other words, once you choose door i in level j, then you can only choose between doors $i-1$, i or $i+1$ on level $j+1$. On level 1, you can choose any door.

Design an $O(mn)$ algorithm to maximise the total value accrued by choosing one door in each of the m levels.

Solution.

- **Subproblem**: Let possible(i, k) denote whether the player can reach level i after playing exactly k levels.
- **Recurrence**: To reach level i using exactly k levels, we can either arrive to level i directly from level $i-1$ using $k-1$ levels previously, or arrive from a door at some other level $j < i-1$ using exactly $k-1$ levels. To make our recurrence easier to compute, we define the following predicate link(j, i) to denote whether level j has a secret exit to level i. That is, link(j, i) is true if and only if level j has a secret exit to level i. The recurrence then becomes

$$\text{possible}(i, k) = \text{possible}(i-1, k-1) \vee \left(\bigvee_{1 \leq j < i-1} \text{possible}(j, k-1) \wedge \text{link}(j, i) \right).$$

Note that the right hand side of the recurrence only suggests that there must be *some* level that can be reached using only $k-1$ levels and has a secret exit to level i.

- **Base case**: Since reaching 0 levels is trivially true, we have that possible$(0,0) = \mathsf{true}$.
- **Order of Computation and Final Solution**: Since we need to compute all levels for each value of k, we solve each subproblem in increasing order of i and then increasing order of k, with the final solution being possible(K, n).
- **Time Complexity**: There are $O(nK)$ subproblems in total, and it appears at first that we need to iterate through all $j < i$ for each one. However, we can do better. Since only n links exist, we can precompute for each level i which earlier levels j link to it, and iterate through only this list. In this way, all subproblems $O(\cdot, k)$ take a total of $O(n)$ time, and therefore the total time complexity is $O(nK)$.

□

[H] Exercise 16. You have n_1 items of size s_1 and n_2 items of size s_2. You would like to pack all of these items into bins, each of capacity $C > s_1, s_2$, using as few bins as possible. Design an algorithm that returns the minimal number of bins required.

Solution.

- **Subproblem**: Let $\textsc{opt}(i, j)$ denote the minimum number of bins required to pack i many items of size s_1 and j many items of size s_2.
- **Recurrence**: Let K denote the ratio C/s_1. This denotes the maximum number of items we can pack into one bin using items of size s_1. We find the value k such that k items of size s_1 can be placed in one bin. We also find the value k such that we can fill in as many items of size s_2 into the same bin. This gives

$$\textsc{opt}(i,j) = 1 + \min_{1 \le k \le K} \left\{ \textsc{opt}\left(i-k, j - \left\lfloor \frac{C - ks_1}{s_2} \right\rfloor \right) \right\}.$$

- **Base case**: We can compute the base case of $\textsc{opt}(1,1)$ by considering the values of s_1, s_2, C.
- **Order of Computation and Final Solution**: We compute each of the subproblems in increasing order of j and i.
- **Time Complexity**: There are K many placements for each (i, j) pair. Therefore, the running time is $O(K \cdot n_1 n_2) = O(C \cdot n_1 n_2)$.

□

[H] Exercise 17. We are given a checkerboard which has 4 rows and n columns, and has an integer written in each square. We are also given a set of $2n$ pebbles, and we want to place some or all of these on the checkerboard (each pebble can be placed on exactly one square) so as to maximise the sum of the integers in the squares that are covered by pebbles. There is one constraint: for a placement of pebbles to be legal, no two of them can be on horizontally or vertically adjacent squares (diagonal adjacency is fine).

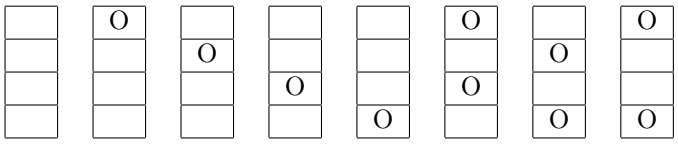
(a) Determine the number of legal *patterns* that can occur in any column (in isolation, ignoring the pebbles in adjacent columns) and describe these patterns.

Call two patterns *compatible* if they can be placed on adjacent columns to form a legal placement. Let us consider sub-problems consisting of the first k columns $1 \le k \le n$. Each sub-problem can be assigned a type, which is the pattern occurring in the last column.

(b) Using the notions of compatibility and type, give an $O(n)$-time algorithm for computing an optimal placement.

Solution.

(a) There are 8 patterns, listed below.



(b) Let t denote the type of pattern so that $1 \leq t \leq 8$.

- **Subproblem**: Let $\text{OPT}(k, t)$ denote the maximum score that can be achieved by only placing pebbles in the first k columns such that the kth column contains pattern t.
- **Recurrence**: We first define $\text{score}(k, t)$ to be the score obtained by using pattern t on column k. Then the recurrence becomes

$$\text{OPT}(k, t) = \text{score}(k, t) + \max\left\{\text{OPT}(k-1, s) : s \text{ is compatible with } t\right\}.$$

- **Base case**: If there is no column, then any pattern on the column gives a score of 0. Therefore, $\text{OPT}(0, t) = 0$ for each $1 \leq t \leq 8$.
- **Order of Computation and Final Solution**: For each column, we need to solve for every pattern. Therefore, we solve in increasing order of t and then increasing order of k, with the final solution being $\max_{1 \leq t \leq 8} \text{OPT}(n, t)$.
- **Time Complexity**: Since the number of patterns is fixed, there are only $O(n)$ many subproblems and each subproblem only needs at most eight many subproblems to check. Therefore, the running time is $O(n)$.

□

[H] Exercise 18. A company is organising a party for its employees. The organisers of the party want it to be a fun party, and so have assigned a fun rating to every employee. The employees are organised into a strict hierarchy, i.e. a tree rooted at the president. There is one restriction, though, on the guest list to the party: an employee and their immediate supervisor (parent in the tree) cannot both attend the party (because that would be no fun at all). Give an algorithm that makes a guest list for the party that maximises the sum of the fun ratings of the guests.

Solution. Suppose that $T(i)$ defines the subtree of the tree T of all employees that is rooted by employee i. Then, for each subtree, we will look at the maximum sum of the fun ratings in two ways: one which includes employee i and one that does not include employee i.

For each subtree, we define the following quantities:

- $I(i)$ is the maximal sum of fun factors $\text{fun}(i)$ that satisfies all of the constraints but *includes* root i,
- $E(i)$ is the maximal sum of fun factors $\text{fun}(i)$ that satisfies all of the constraints but *excludes* root i.

With these quantities defined, we now compute the dynamic programming solution.

- **Subproblem**: Let $\text{OPT}(i)$ denote the maximum sum of the fun ratings of $T(i)$.
- **Recurrence**: For the recurrence, we need to compute two quantities, $I(i)$ and $E(i)$. For each non-leaf node, we need to consider excluding the subordinates (because i is the immediate supervisor of these workers) if we choose to include employee i, or if we exclude i, then we can either choose to exclude the children or include the children of i, whichever value is greater. Thus, we compute $I(i)$ by

$$I(i) = \text{fun}(i) + \sum_{1 \leq k \leq m} E(j_k),$$

where $j_1, \ldots, j_m$ are the subordinates of i.

We compute $E(i)$ by

$$E(i) = \sum_{1 \le k \le m} \max\left(I(j_k), E(j_k)\right),$$

where $j_1, \ldots, j_m$ are defined as above. In this way, *for each* child of i, we can either include them or exclude them.

Then, we have that

$$\text{OPT}(i) = \max(I(i), E(i)).$$

- **Base case**: For each i that is a leaf node, $\text{OPT}(i) = \text{fun}(i)$.
- **Order of Computation and Final Solution**: We solve the subproblems from the leaves of the tree to the root of the leaves, where the final solution is simply $\text{OPT}(n) = \max(I(n), E(n))$ where n is the root of the tree.
- **Time Complexity**: The time complexity is linear in the number of employees because we only need to scan through each of the employees once. Each employee only has one parent. In the worst case, we need to ask every employee exactly once.

□

[H] Exercise 19. You have to cut a wood stick into several pieces. The most affordable company, Analog Cutting Machinery (ACM), charges money according to the length of the stick being cut. Their cutting saw allows them to make only one cut at a time. It is easy to see that different cutting orders can lead to different prices. For example, consider a stick of length 10 m that has to be cut at 2, 4, and 7 m from one end. There are several choices. One can cut first at 2, then at 4, then at 7. This leads to a price of $10 + 8 + 6 = 24$ because the first stick was of 10 m, the resulting stick of 8 m, and the last one of 6 m. Another choice could cut at 4, then at 2, then at 7. This would lead to a price of $10 + 4 + 6 = 20$, which is better for us. Your boss demands that you design an $O(n^2)$ algorithm to find the minimum possible cutting cost for any given stick.

Solution. Define $x(i)$ to be the distance along the stick where the ith cut occurs. We, then, approach with dynamic programming.

- **Subproblem**: Let $\text{OPT}(i, j)$ denote the minimum cost of cutting up the stick from the cutting point i to cutting point j.
- **Recurrence**: Choosing to cut at points i and j will decrease the distance to $x(j) - x(i)$. In this interval, we then need to find the smallest possible cut(s) in terms of cost; therefore, we need to compute $\text{OPT}(i, k) + \text{OPT}(k, j)$ for each $i < k < j$. Thus, we have

$$\text{OPT}(i, j) = x(j) - x(i) + \min\left\{\text{OPT}(i, k) + \text{OPT}(k, j) : 1 < k < j\right\}.$$

- **Base case**: Fixing i, the base case occurs when $j = i + 1$ in which case, $\text{OPT}(i, i + 1) = 0$ for each i.
- **Order of Computation and Final Solution**: We solve each subproblem in increasing order of j and then increasing order of i with the final solution being $\text{OPT}(1, n)$.
- **Time Complexity**: For each i, we need to solve $n - 1$ subproblems. Thus, there are $O(n^2)$ many subproblems, each of which take constant time. Thus, the time complexity is $O(n^2)$.

□

[H] Exercise 20. You have been handed responsibility for a business in Texas for the next n days. Initially, you have K illegal workers. At the beginning of each day, you may hire an illegal worker, keep the number of illegal workers the same or fire an illegal worker. At the end of each day, there will be an inspection. The inspector on the i^{th} day will check that you have between l_i and r_i illegal workers (inclusive). If you do not, you will fail the inspection. Design an $O(n^2)$ algorithm that determines the fewest number of inspections you will fail if you hire and fire illegal employees optimally.

Solution. Since we can either hire or fire an illegal worker (or do nothing), then we observe that the maximum on the evening of day i can be obtained by continuously hiring an illegal worker whereas the minimum on day i can be obtained by continuously firing an illegal worker. Therefore, the maximum number of illegal workers is given by $K+i$ while the minimum number of illegal workers is given by $\max(K-i, 0)$. We will use this as part of our recurrence.

- **Subproblem**: Let $\text{OPT}(i, j)$ denote the minimum number of inspections to fail such that there are j many illegal workers on day i.
- **Recurrence**: We first require that $\max(0, K-i) \leq j \leq K+i$. To help us with the recurrence, denote $\text{failed}(i, j)$ to indicate whether $l_i \leq j \leq r_i$. That is,

$$\text{failed}(i,j) = \begin{cases} 0 & \text{if } l_i \leq j \leq r_i, \\ 1 & \text{otherwise.} \end{cases}$$

  Then we either need to consider hiring an illegal worker, firing an illegal worker or do nothing. We look at each of the cases separately to give the recurrence

$$\text{OPT}(i,j) = \text{failed}(i,j) + \min \begin{cases} \text{OPT}(i-1, j-1) & \text{if } j-1 \leq \max(0, K-(i-1)), \\ \text{OPT}(i-1, j), & \\ \text{OPT}(i-1, j+1) & \text{if } j+1 \leq K+(i-1). \end{cases}$$

- **Base case**: On the zeroth day, we have no failed inspections. Therefore, we obtain $\text{OPT}(0, K) = 0$.
- **Order of Computation and Final Solution**: Observe that our recurrence relies on all possible values of j, for each day i. Therefore, we solve our subproblems in increasing order of j and then in increasing order of i, with the final solution being

$$\min\{\text{OPT}(n, j) : \max(K-n, 0) \leq j \leq K+n\}.$$

- **Time Complexity**: There are at most n days and $2n+1$ possibility of illegal workers since there needs to be a subproblem for all workers between $K-n \leq j \leq K+n$. Therefore, there are $n(2n+1) = O(n^2)$ subproblems, each of which is computed in $O(1)$ time. Thus, the overall running time of the algorithm is $O(n^2)$.

□

§ SECTION THREE: COUNTING

[K] Exercise 21. Given an array of n positive integers, find the number of ways of splitting the array up into contiguous blocks of sum at most k in $O(n^2)$ time.

Solution.

- **Subproblem**: Let $\text{NUM}(i)$ denote the number of ways to split the first i elements into contiguous blocks of sum at most k.
- **Recurrence**: We look at all of the possible ways to place a divider such that all elements in the same group of the divider sum to at most k. Denote $\text{sum}(j, i)$ to be the sum of all elements between (and including) j and i. Then we have that
$$\text{NUM}(i) = \sum_{\substack{1 \le j \le i, \\ \text{sum}(j,i) \le k}} \text{NUM}(j-1).$$
- **Base case**: There is exactly one way to split 0 elements; therefore, $\text{NUM}(0) = 1$.
- **Order of Computation and Final Solution**: We solve the subproblems in increasing order of i. For a particular subproblem $1(i)$, we consider the j values in *decreasing* order, so that we can update $\sum(j, i)$ value in constant time by adding one more term at each step until the early exit. The final solution is $\text{NUM}(n)$.
- **Time Complexity**: There are n subproblems and each subproblem requires an $O(n)$ search. Therefore, the running time of the algorithm is $O(n^2)$.

□

[K] Exercise 22. Let A and B be two strings of lengths n and m respectively. We say that a string A occurs as a subsequence of another string B if we can obtain A by deleting some of the letters of B. Given strings A and B, design an $O(mn)$ algorithm that gives the number of different occurrences of A in B; that is, we want to compute the number of ways one can delete some of the symbols of B to get A.

Solution. We begin by defining some notations. Let A_i denote the ith letter of A and B_j as the jth letter of B.

- **Subproblem**: Let $\text{ways}(i, j)$ denote the number of times the first i letters of A appears as a subsequence in the first j letters of B.
- **Recurrence**: If $A_i \neq B_j$, then this is equivalent to removing the jth character from B and checking if the first i characters appear in the substring $B_1, \ldots, B_{j-1}$. This gives $\text{ways}(i, j-1)$ number of ways this could happen.

  On the other hand, if $A_i = B_j$, then we check to see if $A_{i-1} = B_{j-1}$. This gives $\text{ways}(i-1, j-1)$ number of ways. However, the substring A ending at A_i can also be contained entirely inside the string $B_1, \ldots, B_{j-1}$. Therefore, we also compute $\text{ways}(i, j-1)$. In other words, we obtain the recurrence
$$\text{ways}(i, j) = \begin{cases} \text{ways}(i, j-1) & \text{if } A_i \neq B_j, \\ \text{ways}(i-1, j-1) + \text{ways}(i, j-1) & \text{if } A_i = B_j. \end{cases}$$
- **Base case**: If A is empty, then $\text{ways}(0, j) = 1$ for each $1 \le j \le m$. If B is empty, then $\text{ways}(i, 0) = 0$ for each $1 \le i \le n$.
- **Order of Computation and Final Solution**: We solve each subproblem in increasing order of j and then increasing order of i.

- **Time Complexity**: There are mn many subproblems (one for i and one for j) and each subproblem takes constant time to compute; therefore, the running time of the algorithm is $O(mn)$.

□

[K] Exercise 23. A partition of a number n is a sequence $\langle p_1, p_2, \ldots, p_k \rangle$ such that $1 \leq p_1 \leq p_2 \leq \cdots \leq p_k \leq n$. We call each p_i a *part*. Define $\text{NUMPART}(n, k)$ to be the number of partitions of n such that each part is at most k.

(a) Explain why $\text{NUMPART}(n, 1) = 1$ for each n.

(b) Devise a recurrence to determine the number of partitions of n in which every part is at most k, where n, k are two given integers such that $2 \leq k \leq n$.

- To obtain the right recursion, consider two different cases depending on the largest possible value of each part.

(c) Hence, find the total number of partitions of n.

(d) How many subproblems are there, and for each subproblem, what is the time complexity to compute? Hence, analyse the time complexity of the algorithm.

Solution.

(a) Note that $\text{NUMPART}(n, 1)$ counts the number of ways of expressing n such that every part is at most 1. This can be done precisely when every part is 1.

(b) We split up the partitions by the value of their largest part p_t. More precisely, we consider the cases where $p_t = k$ and $p_t < k$ separately.

For the first case, we have the equation

$$p_1 + \cdots + p_{t-1} + k = n,$$

and subtracting k from both sides gives

$$p_1 + \ldots + p_{t-1} = n - k.$$

We observe that the remaining sequence $\langle p_1, \ldots, p_{t-1} \rangle$ is a partition of $n - k$ in which no part exceeds k, so there are $\text{NUMPART}(k, n - k)$ such sequences.

For the second case, if $p_t < k$ then every other term of the sequence is also less than k. Therefore $\langle p_1, \ldots, p_t \rangle$ is a partition of n in which no part exceeds $k - 1$, so there are $\text{NUMPART}(n, k - 1)$ such sequences.

Note that these cases will not overlap (i.e. you can't have a case where the greatest part is k and not k simultaneously). Therefore, the number of partitions is simply

$$\text{NUMPART}(n, k) = \text{NUMPART}(n - k, k) + \text{NUMPART}(n, k - 1).$$

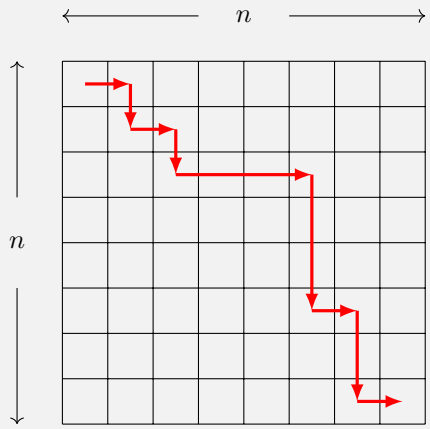
We can solve these subproblems in lexicographic order, i.e. increasing order of k, breaking ties in increasing order of n.

(c) If we allow p_t to take any value up to n, then any partition of n is allowed. Therefore the total number of (unrestricted) partitions of n is simply $\text{NUMPART}(n, n)$.

(d) Note that there is one subproblem for each number of partitions and one subproblem for the maximum size of each partition. Therefore, there are n^2 many subproblems. For each subproblem, it takes $O(1)$ time to compute. Thus, the overall running time is $O(n^2)$.

□

[H] Exercise 24. You are given an $n \times n$ chessboard with an integer in each of its n^2 squares. You start from the top left corner of the board; at each move you can go either to the square immediately below or to the square immediately to the right of the square you are at the moment; you can never move diagonally. The goal is to reach the right bottom corner so that the sum of integers at all squares visited is minimal.



(a) Describe a greedy algorithm which attempts to find such a minimal sum path and show by an example that such a greedy algorithm might fail to find such a minimal sum path.

(b) Describe an algorithm which always correctly finds a minimal sum path and runs in time n^2.

(c) Describe an algorithm which computes the number of such minimal paths.

Solution.

(a) Consider the following greedy solution. Starting at the top left square, we move to the square below or to the right that contains the smallest integer.

Using this solution, consider the following counterexample.

0	1	1
5	1000	1000
5	5	0

The algorithm would choose the path 1-1-1000-0 for a score of 1002, while the correct path would be 5-5-5-0 for a score of 15.

(b) We define the quantity BOARD(i, j) to be the integer inside the cell located on row i and column j. We then begin with the dynamic programming approach.

- **Subproblem**: Let OPT(i, j) denote the best score that can be achieved if we arrive at cell (i, j).
- **Recurrence**: If we land at cell (i, j), observe that the previous move must have been either $(i-1, j)$ (i.e. we move to the right to land at the cell) or $(i, j-1)$ (i.e. we move downwards to land at the cell). Thus, we need to check which way yielded us with the minimum value *so far*. Another way to see this is that, if we land at the final square (n, n), then the path that we would need to trace needs to be the path where it is minimal ending at either cell $(n-1, n)$ or $(n, n-1)$, and we repeat this process until we land back at the original square.

Thus, the recursion is

$$\textsc{opt}(i,j) = \textsc{board}(i,j) + \min\left\{\textsc{opt}(i-1,j), \textsc{opt}(i,j-1)\right\}.$$

- **Base case**: The base case is $\textsc{opt}(1,1) = \textsc{board}(1,1)$. To ensure that we never consider illegal moves (i.e. moves that land outside of the board), we also have that $\textsc{opt}(i,j) = \infty$ if cell (i,j) does not exist (or is off the board).
- **Order of Computation and Final Solution**: Observe that, in order to solve $\textsc{opt}(i,j)$, we need to know $\textsc{opt}(i-1,j)$ and $\textsc{opt}(i,j-1)$. Thus, the order in which we solve the problem is that we solve the subproblems in increasing order of i, and then increasing order of j to break ties. The final solution is simply solving $\textsc{opt}(n,n)$. To return the actual path, we can make a note of the choice that the recursion makes by choosing either $\textsc{opt}(i-1,j)$ or $\textsc{opt}(i,j-1)$ by backtracking through the subproblems that are chosen. This gives us a path from (n,n) to $(1,1)$ which yields the path taken.
- **Time Complexity**: To analyse the time complexity, we require two pieces of information: the number of subproblems and the time taken to compute each subproblem. There are n^2 subproblems for each $1 \leq i,j \leq n$. Now, for each subproblem $P(i,j)$, we would have computed $\textsc{opt}(i-1,j)$ and $\textsc{opt}(i,j-1)$ previously. Hence, we have $O(1)$ lookup time and the computation for $\textsc{opt}(i,j)$ is simply a comparison and an arithmetic operation, all of which takes $O(1)$ time. Thus, the overall time complexity is $n^2 \cdot O(1) = O(n^2)$.

(c) We approach this similar to part (b); in fact, we will use $\textsc{opt}(i,j)$ as defined above to our advantage.

- **Subproblem**: Let $\textsc{ways}(i,j)$ denote the minimum number of ways to reach cell (i,j) with score $\textsc{opt}(i,j)$.
- **Recurrence**: In much the same way, if we land at cell (i,j), the previous move must have been either from $(i-1,j)$ or $(i,j-1)$. We, therefore, need to check the minimum path from both of these directions. There are three cases to consider:
  - If the minimum path happens to come from $(i-1,j)$, then we ignore all of the paths from $(i,j-1)$ because those paths will not be minimal.
  - If the minimum path happens to come from $(i,j-1)$, then we ignore all of the paths from $(i-1,j)$.
  - However, if they are equal, then we consider *all* possible paths to land at (i,j).

  With this in mind, it is easy to see that the recurrence becomes

$$\textsc{ways}(i,j) = \begin{cases} \textsc{ways}(i-1,j) & \text{if } \textsc{opt}(i-1,j) < \textsc{opt}(i,j-1), \\ \textsc{ways}(i,j-1) & \text{if } \textsc{opt}(i-1,j) > \textsc{opt}(i,j-1), \\ \textsc{ways}(i-1,j) + \textsc{ways}(i,j-1) & \text{if } \textsc{opt}(i-1,j) = \textsc{opt}(i,j-1). \end{cases}$$

- **Base case**: The base case is $\textsc{ways}(1,1) = 1$ since there is only one way to get to cell $(1,1)$. We also define $\textsc{ways}(i,j) = 0$ for all cells that fall outside of the grid since there are no ways to get to those cells.
- **Order of Computation and Final Solution**: We solve the problems in increasing order of i, and then increasing order of j to break ties for the same reason as the previous part. The final solution is $\textsc{ways}(n,n)$.
- **Time Complexity**: The time complexity is $O(n^2)$ since there are still n^2 subproblems and each subproblem takes $O(1)$ time to compute.

□

§ SECTION FOUR: GAMES AND GRAPHS

[K] Exercise 25. Consider two deck of n and m cards respectively. A new deck of cards is formed out of the $n+m$ cards. We say that the new deck is an *interleaving* if the ordering of the two original decks are preserved in the new deck. Given the two deck of cards and the new deck, say X, Y, Z, respectively, design an $O(nm)$ algorithm that determines if Z is an interleaving of X and Y.

Solution.

- **Subproblem**: Let $\text{OPT}(i, j)$ denote whether the first i cards of X and the first j cards of Y form an interleaving of the first $i+j$ cards of Z.
- **Recurrence**: First, denote X_i, Y_i, Z_i to be the ith card from decks X, Y, and Z respectively. We note that, if $\text{OPT}(i, j)$ forms an interleaving, then we need to check which deck of cards come first.
  - If X comes first, then the ith card of X must form the $(i+j)$th card of Z *and* the first $i-1$ cards from X forms an interleaving with the first j cards from Y. This gives the recurrence
  $$\text{OPT}(i, j) = \text{OPT}(i-1, j) \wedge (Z_{i+j} = X_i).$$
  - If Y comes first, then the jth card of Y must form the $(i+j)$th card of Z *and* the first $j-1$ cards from Y forms an interleaving with the first i cards from X. This gives the recurrence
  $$\text{OPT}(i, j) = \text{OPT}(i, j-1) \wedge (Z_{i+j} = Y_j).$$

  This gives the recurrence
  $$\text{OPT}(i, j) = (\text{OPT}(i-1, j) \wedge (Z_{i+j} = X_i)) \vee (\text{OPT}(i, j-1) \wedge (Z_{i+j} = Y_j)).$$
- **Base case**: Our base case occurs when we reach the empty string, giving us $\text{OPT}(0, 0) = \mathsf{true}$. For any $i < 0$ or $j < 0$, we also set $\text{OPT}(i, j) = \mathsf{false}$.
- **Order of Computation and Final Solution**: We compute the subproblems in increasing order of j and then i, with the final solution being $\text{OPT}(m, n)$.
- **Time Complexity**: We have mn many subproblems and each takes constant time to compute; therefore, our algorithm takes $O(mn)$ running time.

□

[K] Exercise 26. Some people think that the bigger an elephant is, the smarter it is. To disprove this, you want to analyse a collection of elephants and place as large a subset of elephants as possible into a sequence whose weights are increasing but their IQ's are decreasing. Design an $O(n \log n)$ algorithm which, given the weights and IQ's of n elephants, will find a longest sequence of elephants such that their weights are increasing but IQ's are decreasing.

Solution. Using merge sort (or equivalent), sort the elephants in decreasing order of IQ. This problem now reduces to the longest increasing subsequence problem in terms of the weight of the elephants. Sorting takes $O(n \log n)$ and the longest increasing subsequence problem also takes $O(n \log n)$. Hence, the overall time complexity is $O(n \log n)$. □

[K] Exercise 27. Let $G = (V, E)$ be a directed and weighted graph on n vertices. In other words, $|V| = n$. For two vertices $u, v \in V$, we define the *distance* $\delta(u, v)$ from u to v to be the weight of the shortest path from u to v. However, we alter the distances from u to v for two special cases:

- If there is no path from u to v, define $\delta(u, v) = \infty$.
- If there is a path from u to v that contains a negative-weight cycle, define $\delta(u, v) = -\infty$.

Given a graph G, design an $O(n^3)$ algorithm that computes $\delta(u, v)$ for all pairs $u, v \in V$.

Note that Floyd-Warshall's algorithm assumes that G contains no negative-weight cycles. How would you modify the algorithm?

Solution. Label the vertices $1, \ldots, n$.

- **Subproblem**: Let $\delta_k(i, j)$ denote the distance from i to j through paths using the intermediary vertices $\{1, \ldots, k\}$.
- **Recurrence**: To obtain a solution for $\delta_k(i, j)$, we either include the kth vertex or we exclude it from the shortest path.
  - If we exclude the kth vertex, then we use the first $(k-1)$ vertices and the recurrence becomes $\delta_k(i, j) = \delta_{k-1}(i, j)$.
  - If we include the kth vertex, then we find the distance of the shortest path from i to k via the intermediary vertices $\{1, \ldots, k-1\}$ and then find the distance of the shortest path from k to j. Now, it could be the case that $\delta_{k-1}(k, k) < 0$; therefore, we will define the following notation $(\delta_{k-1}(k, k))^*$ to be $-\infty$ if $\delta_{k-1}(k, k) < 0$. In this case, we obtain the recurrence

$$\delta_k(i, j) = \delta_{k-1}(i, k) + (\delta_{k-1}(k, k))^* + \delta_{k-1}(k, j).$$

  Thus, the recurrence becomes

$$\delta_k(i, j) = \min\left\{\delta_{k-1}(i, j), \delta_{k-1}(i, k) + (\delta_{k-1}(k, k))^* + \delta_{k-1}(k, j)\right\}.$$

- **Base case**: If (i, j) is not an edge, then set $w(i, j) = \infty$, where $w(i, j)$ denotes the weight of the edge from i to j. For each vertex $i \in V$, we note that $\delta_0(i, i) = \min\{0, w(i, i)\}$ and $\delta_0(i, j) = w(i, j)$ if $i \neq j$.
- **Order of Computation and Final Solution**: We order the subproblems in increasing order of k, with the final solution being a grid of all $O(n^2)$ shortest paths that satisfy the two constraints in the problem.
- **Time Complexity**: For each $k = 0, \ldots, n$, we solve $O(n^2)$ many subproblems. Therefore, the overall time complexity is $O(n^3)$.

□

[H] Exercise 28. Let $G = (V, E)$ be a connected and undirected graph, where $|V| = n$ and $|E| = m$. Given two vertices $s, t \in V$, design an $O(n^2)$ dynamic programming algorithm to count the number of shortest paths (in the sense of number of edges required to connect s and t) from s to t.

Solution. We first preprocess a `dist` array which keeps track of the shortest path from s to every vertex $u \in V$. In other words, `dist[u]` is the length of the shortest path from s to u, which can be done using breadth-first search. We now begin the dynamic programming.

- **Subproblem**: Let NUM(u) denote the number of shortest paths from s to u.
- **Recurrence**: To get to vertex u, we need to find the number of shortest paths that end at the neighbours of u whose distance is one less than u. Therefore, we only count along the paths that end with vertex v which satisfy the two properties:

- $(u, v) \in E$ is an edge in the graph G (i.e. v is a neighbouring vertex to u).
- `dist[v]` $=$ `dist[u]` $- 1$.

Once we find v, we then count the number of shortest paths from s to v and this determines the number of shortest paths to u. Hence, our recurrence is

$$\text{NUM}(u) = \sum_{\substack{v:(v,u)\in E,\\ \texttt{dist[v]}=\texttt{dist[u]}-1}} \text{NUM}(v).$$

- **Base case**: We hit the base case precisely when we arrive at the starting vertex s. There is exactly one shortest path from s to s; therefore, $\text{NUM}(s) = 1$.
- **Order of Computation and Final Solution**: The final solution is $\text{NUM}(t)$. Note that each subproblem relies on information about neighbouring vertices which have a smaller "breadth-first search" distance. Therefore, we solve each subproblem in increasing order of distance computed by the precomputation of breadth-first search.
- **Time complexity**: Even though each subproblem requires at most $O(n)$ work, observe that we only need to worry about computing the subproblems for each *neighbouring* vertex. Therefore, each subproblem requires $O(\deg(u))$ work. Hence, the total work required is equivalent to $\sum_{u\in V} \deg(u)$ amount of work. But this is the same as $2m$, by the Handshaking Lemma; therefore, the total running time is $O(m) = O(n^2)$.

□

[H] Exercise 29. There are $2n$ students in a class and these students are sitting around a circular table. We form n pairs of students. When two students are paired up (say student i and student j), we connect a line from student i to student j. Design an $O(n^2)$ algorithm to find the number of ways we can form n pairs such that no two lines intersect.

Look at small cases of n. Once we connect two students, can we arbitrarily connect two other students?

Solution. The key insight is that, once we draw a line that connect two points on the circle, then this divides our circle into two smaller sets, say S_1 and S_2. If we connect an edge connecting a point in S_1 and a point in S_2, then this must intersect the chord that we drew. Therefore, this is equivalent to solving the same problem on the smaller sets, S_1 and S_2.

- **Subproblem**: Let $\text{NUM}(i)$ denote the number of ways to pair up $2i$ points to form i pairs with non-intersecting lines.
- **Recurrence**: Note that, once we fix a pair, we subdivide into two regions with $2j$ and $2(i-j-1)$ points, with $j = 0, \ldots, i-1$. It, therefore, suffices to find the number of pairs that we can form on both of these separate regions. In the first region with $2j$ points, we can form $\text{NUM}(j)$ many pairs while the second region with $2(i-j-1)$ points can form $\text{NUM}(i-j-1)$ pairs. We take all possible subdivisions, giving

$$\text{NUM}(i) = \sum_{j=0}^{i-1} \text{NUM}(j) \cdot \text{NUM}(i-j-1).$$

- **Base case**: If there are only two points, then we can only form one pair; therefore, $\text{NUM}(1) = 1$ and $\text{NUM}(0) = 0$.
- **Order of Computation and Final Solution**: We solve this in increasing order of i with the final solution being $\text{NUM}(n)$.
- **Time Complexity**: There are n subproblems and each subproblem requires at most $O(n)$ work, giving us an $O(n^2)$ algorithm.

□

[E] Exercise 30. Let $G = (V, E)$ be a directed graph, where $|V| = n$ and $|E| = m$. Given two vertices $s, t \in V$, design an algorithm that determines whether s and t are connected. Aim to do this by only using $O(\log^2 n)$ amount of space. Your algorithm would have exponential running time.

This problem is the basis for an important breakthrough result in computational complexity theory, Savitch's theorem.

Solution. We combine dynamic programming with divide and conquer.

- **Subproblem**: Let $\text{CONNECTED}(u, v, k)$ denote whether $u, v \in V$ are connected in k steps.
- **Recurrence**: To do this properly, we will invoke non-determinism. Guess a vertex w. Then notice that $\text{CONNECTED}(u, v, k)$ is true if and only if $\text{CONNECTED}(u, w, k/2)$ and $\text{CONNECTED}(w, v, k/2)$ are both simultaneously true since we have the path

$$\underbrace{u \to \cdots \to w}_{k/2 \text{ steps}} \underbrace{\to \cdots \to v}_{k/2 \text{ steps}}.$$

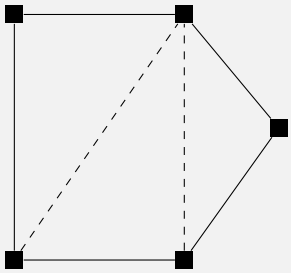
  Thus, we have the recurrence

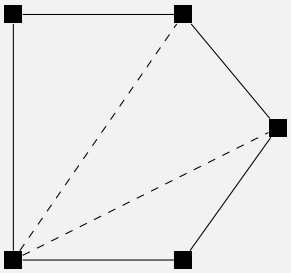
$$\text{CONNECTED}(u, v, k) = \bigvee_{w \in V} (\text{CONNECTED}(u, w, k/2) \wedge \text{CONNECTED}(w, v, k/2))$$

- **Base case**: If $u = v$, then $\text{CONNECTED}(u, v, k) = \mathsf{true}$. If $k = 1$, then we just need to check if there is an edge from u to v.
- **Order of Computation and Final Solution**: We solve in increasing order of k, only storing the vertex w such that $\text{CONNECTED}(u, w, k/2)$ and $\text{CONNECTED}(w, v, k/2)$ are both true. The final solution is $\text{CONNECTED}(s, t, n)$.
- **Space Complexity**: At each iteration, we only store the solution vertex w. Since there are $\log n$ many iterations, we only require $O(\log n)$ storage non-deterministically. Savitch's theorem ensures that there is a deterministic solution only requiring $O(\log^2 n)$ space.

□

[E] Exercise 31. A convex polygon is a closed shape where none of its edges bend inwards. Another way to think of this is, if you connect any two points inside the polygon, the line segment remains completely inside the polygon. A *triangulation* of a convex polygon is a deconstruction of the convex polygon into triangles such that the diagonals do not intersect. The following diagram describes possible triangulations of the same convex polygon.





We define the weight of a triangle to be the perimeter of the triangle, and we define the cost of a triangulation to be the sum of the weights of its component triangles. Given n points and some corresponding graph data structure to define the convex polygon, design an $O(n^3)$ algorithm to find the minimum cost triangulation of the convex polygon.

Solution. Label the vertices $1, \ldots, n$.

- **Subproblem**: Let $\text{OPT}(i, j)$ denote the minimum cost polygon triangulation using vertices $i, \ldots, j$.
- **Recurrence**: Define $\text{COST}(i, j, k)$ to be the sum of the perimeter of the triangle with vertices i, j, k. To obtain a suitable recurrence, we look at all possible line segments. Note that each line segment divides our convex polygon into two smaller convex polygons. Therefore, we look at *all* possible ways to divide our polygon into two smaller polygons and form a triangle with the vertices i, k, j where $i + 1 \leq k \leq j - 1$. This gives us a cost of $\text{COST}(i, k, j)$ and we obtain the polygons $\text{OPT}(i, k)$ and $\text{OPT}(k, j)$. Thus, we obtain

$$\text{OPT}(i, j) = \min_{i<k<j} \{\text{OPT}(i, k) + \text{OPT}(k, j) + \text{COST}(i, k, j)\}.$$

- **Base case**: If $i = j$, then we have no cost. Therefore, $\text{OPT}(i, i) = 0$.
- **Order of Computation and Final Solution**: We solve these subproblems in increasing order of the size of the polygons; therefore, we solve in increasing order of $j - i$. The final solution is $\text{OPT}(1, n)$.
- **Time Complexity**: There are n^2 many subproblems and we look at all possible triangulations. This gives us $O(n)$ work for each pair of indices, giving us an $O(n^3)$ algorithm.

□