5. FLOW NETWORKS

Aleks Ignjatović, ignjat@cse.unsw.edu.au
office: K17 504
Course Admin: Song Fang, cs3121@cse.unsw.edu.au

School of Computer Science and Engineering
UNSW Sydney

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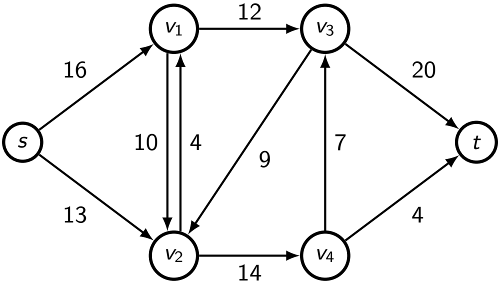

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2. *Flow conservation:* for all vertices $v \in V \setminus \{s, t\}$ we require

$$\sum_{(u,v) \in E} f(u, v) = \sum_{(v,w) \in E} f(v, w),$$

   i.e. the flow into any vertex (other than the source and the sink) equals the flow out of that vertex.

Definition

The *value* of a flow is defined as

$$|f| = \sum_{v:(s,v)\in E} f(s,v) = \sum_{v:(v,t)\in E} f(v,t),$$

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Given a flow network, our goal is to find a flow of maximum value.

Integrality Property

If all capacities are integers (as we assumed earlier), then there is a flow of maximum value such that $f(u, v)$ is an integer for each edge $(u, v) \in E$.

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Note

This means that there is always at least one solution entirely in integers. We will only consider integer solutions hereafter.

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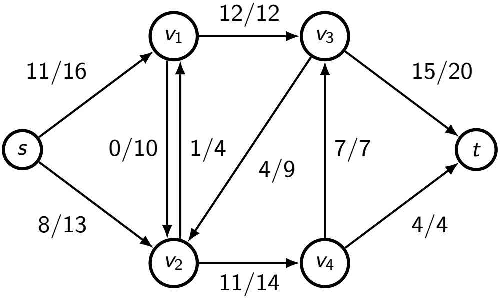

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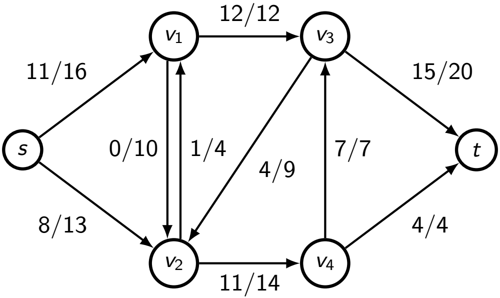

The pictured flow has a value of 19 units, and it does not appear possible to send another unit of flow. But we can do better!

Here is a flow of value 23 units in the same flow network.

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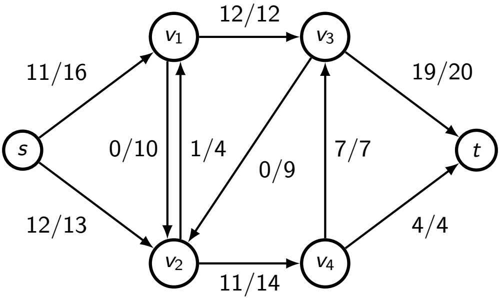

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- It would have been better to send those four units of flow to t directly, but this may not have been obvious at the time this decision was made.
- We need a way to correct mistakes! We would like to send flow from v_2 back to v_3 so as to "cancel out" the earlier allocation.

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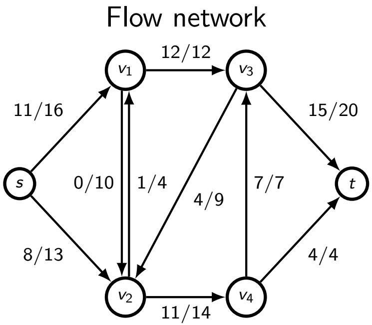

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Residual Flow Network

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  1. an edge from v to w with capacity $c - f$, and
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- Edges of capacity zero (when $f = 0$ or $f = c$) need not be included.

Augmenting Paths

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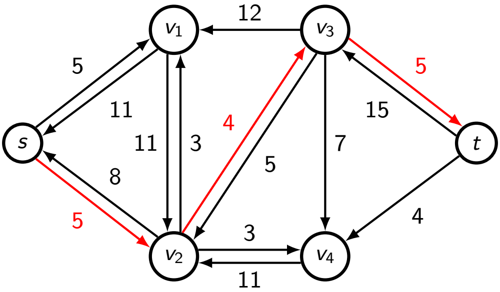

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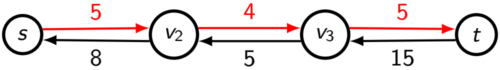
- Suppose we have an augmenting path of capacity f, including an edge from v to w. We should:
  - cancel up to f units of flow being sent from w to v,
  - add the remainder of these f units to the flow being sent from v to w,
  - increase the residual capacity from w to v by f, and
  - reduce the residual capacity from v to w by f.

Recall that the augmenting path was as follows.



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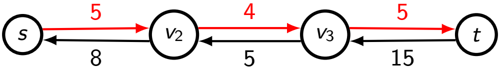

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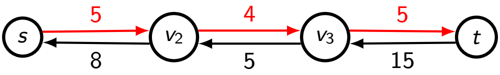

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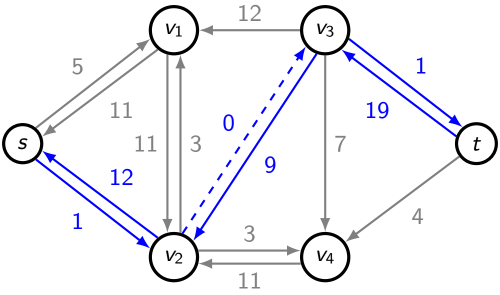

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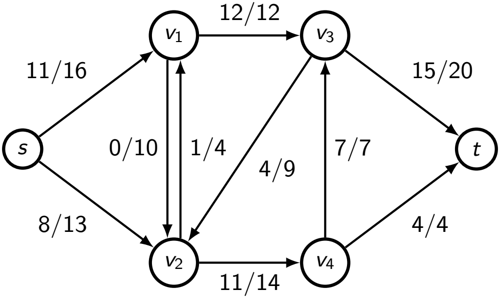
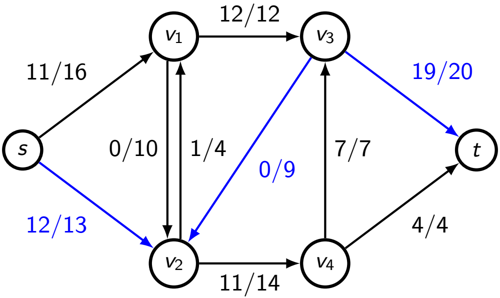


Pictured below is the new flow, after sending four units of flow along the path $s \to v_2 \to v_3 \to t$.



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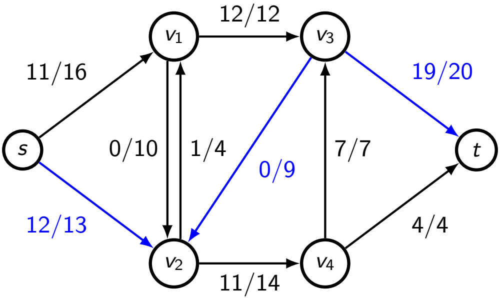

Note that the four units of flow previously sent from v_3 to v_2 have been cancelled out.

- Keep adding flow through new augmenting paths for as long as it is possible.

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- When there are no more augmenting paths, you have achieved the largest possible flow in the network.

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- We conclude that the process must terminate eventually.

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- Maybe we have created bottlenecks by choosing bad augmenting paths; maybe better choices of augmenting paths could produce a larger total flow through the network?
- This is not at all obvious, and to show that this is not the case we need a mathematical proof!

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1. $S \cup T = V$
2. $S \cap T = \emptyset$
3. $s \in S$ and $t \in T$.

Definition

The *capacity* $c(S, T)$ of a cut (S, T) is the sum of capacities of all edges leaving S and entering T, i.e.

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Note that the capacities of edges going in the opposite direction, i.e. from T to S, do not count.

Definition

Given a flow f, the *flow* $f(S,T)$ through a cut (S,T) is the total flow through edges from S to T minus the total flow through edges from T to S, i.e.

$$
\begin{aligned}
f(S,T) = & \sum_{(u,v)\in E} \{f(u,v) : u \in S, v \in T\} \\
& - \sum_{(u,v)\in E} \{f(u,v) : u \in T, v \in S\}.
\end{aligned}
$$

Fact

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This is a consequence of *flow conservation*.

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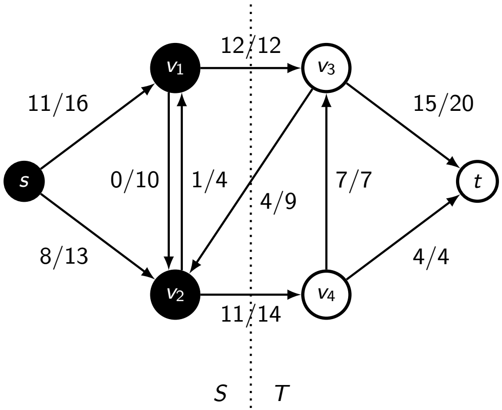
- Therefore $f(S, T) \leq c(S, T)$.
- It follows that $|f| \leq c(S, T)$, so the value of any flow is at most the capacity of any cut.

Cuts in a Flow Network: Example



- In the above example the net flow across the cut is given by

$$f(S,T) = f(v_1, v_3) + f(v_2, v_4) - f(v_2, v_3) = 12 + 11 - 4 = 19.$$

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- As we have mentioned, we add only the capacities of edges from S to T and not of edges in the opposite direction.

Theorem

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- Thus, if we find a flow f which equals the capacity of some cut (S, T), then such flow must be maximal and the capacity of such a cut must be minimal.

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- Thus, if we find a flow f which equals the capacity of some cut (S, T), then such flow must be maximal and the capacity of such a cut must be minimal.
- We now show that when the Ford-Fulkerson algorithm terminates, it produces a flow equal to the capacity of an appropriately defined cut.

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- Define T to be the set of all vertices for which there is no such path.

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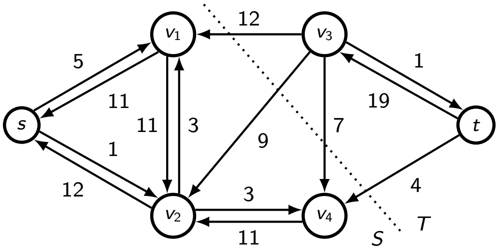
- Define S to be the source s and all vertices u such that there is a path in the residual flow network from the source s to that vertex u.
- Define T to be the set of all vertices for which there is no such path.
- Since there are no more augmenting paths from s to t, clearly the sink t belongs to T.



12
v_1
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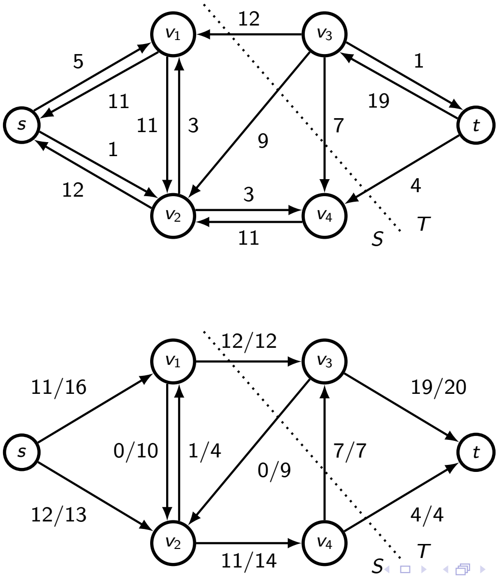

v1
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v2
3
v4
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T
S
v1
12/12
v3
11/16
19/20
s
0/10
1/4
7/7
t
0/9
12/13
4/4
v2
v4
T
11/14
S

Claim

All the edges from S to T are fully occupied with flow, and all the edges from T to S are empty.

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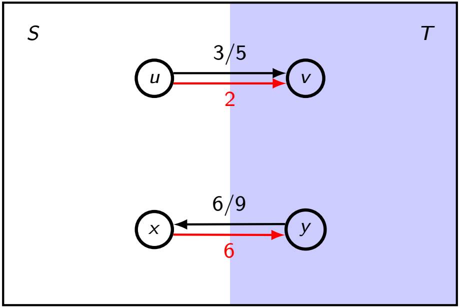

Proof

- Suppose an edge (u, v) from S to T has any additional capacity left. Then in the residual flow network, the path from s to u could be extended to a path from s to v. This contradicts our assumption that $v \in T$.

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- Suppose an edge (y, x) from T to S has any flow in it. Then in the residual flow network, the path from s to x could be extended to a path from s to y. This contradicts our assumption that $y \in T$.

- Since all edges from S to T are occupied with flows to their full capacity, and also there is no flow from T to S, the flow across the cut (S, T) is precisely equal to the capacity of this cut, i.e., $f(S, T) = c(S, T)$.

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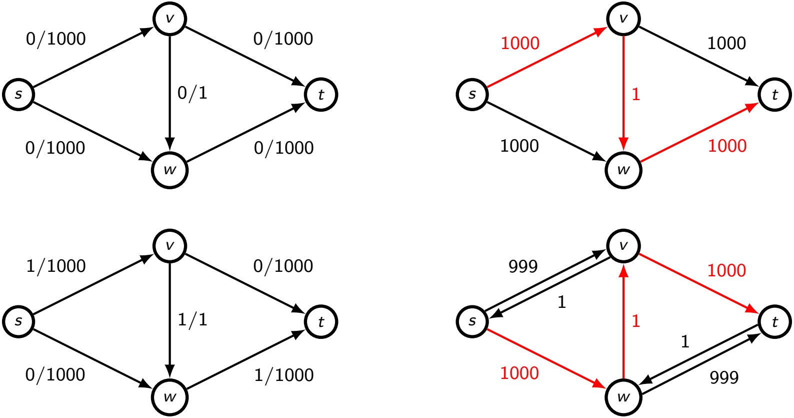
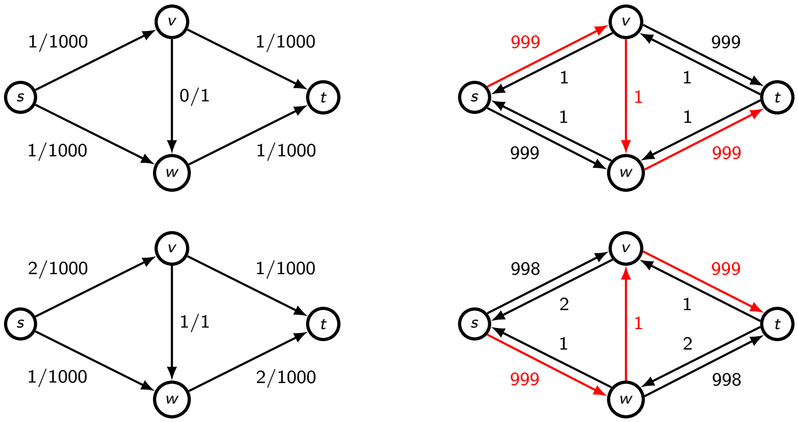
- Thus, such a flow is maximal and the corresponding cut is a minimal cut, regardless of the particular way in which the augmenting paths were chosen.

How efficient is the Ford-Fulkerson algorithm?





v
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2/1000
1/1000
1/1
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2/1000
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Ford-Fulkerson Algorithm: Time Complexity

- The number of augmenting paths can be up to the value of the max flow, denoted $|f|$.

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- Each augmenting path is found in $O(V + E)$, e.g. by DFS. In any sensible flow network, $V \leq E + 1$, so we can write this as simply $O(E)$.

Ford-Fulkerson Algorithm: Time Complexity

- The number of augmenting paths can be up to the value of the max flow, denoted $|f|$.
- Each augmenting path is found in $O(V + E)$, e.g. by DFS. In any sensible flow network, $V \leq E + 1$, so we can write this as simply $O(E)$.
- Therefore the time complexity of the Ford-Fulkerson algorithm is $O(E|f|)$.

- Recall that the input specifies V vertices and E edges, as well as a capacity for each edge.

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- If the capacities are all $\leq C$, then the length of the input (i.e. the number of bits required to encode it) is $O(V + E \log C)$.
- However, the value of the maximum flow $|f|$ can be as large as VC in general.
- Therefore, the time complexity $O(E|f|)$ can be exponential in the size of the input, which is unsatisfactory.

- The Edmonds-Karp algorithm improves the Ford-Fulkerson algorithm in a simple way: always choose the augmenting path which consists of the *fewest edges*.

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- At each step, we find the next augmenting path using *breadth-first search* in $O(V+E) = O(E)$ time.

Edmonds-Karp Algorithm

- The Edmonds-Karp algorithm improves the Ford-Fulkerson algorithm in a simple way: always choose the augmenting path which consists of the *fewest edges*.

- At each step, we find the next augmenting path using *breadth-first search* in $O(V + E) = O(E)$ time.

- Note that this choice is somewhat counter-intuitive: augmenting paths are chosen based only on length, so we may end up flowing edges with small capacities before edges with large capacities.

Edmonds-Karp Algorithm

Question

What is the time complexity of the Edmonds-Karp algorithm?

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Answer

It can be proven (see CLRS pp.727–730) that the number of augmenting paths is $O(VE)$, and since each takes $O(E)$ to find, the time complexity is $O(VE^2)$.

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It can be proven (see CLRS pp.727–730) that the number of augmenting paths is $O(VE)$, and since each takes $O(E)$ to find, the time complexity is $O(VE^2)$.

Note also that Edmonds-Karp is a specialisation of Ford-Fulkerson, so the original $O(E|f|)$ bound also applies.

Faster max flow algorithms

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- Max flow algorithms based on augmenting paths tend to perform better in practice than their worst case complexity might suggest, but we can't rely on this especially in this course.
- In March 2022, Chen et al. developed an "almost linear" time algorithm for max flow.

Table of Contents

Networks with multiple sources and sinks

- Flow networks with multiple sources and sinks are reducible to networks with a single source and single sink by adding a "super-source" and "super-sink" and connecting them to all sources and sinks, respectively, by edges of infinite capacity.

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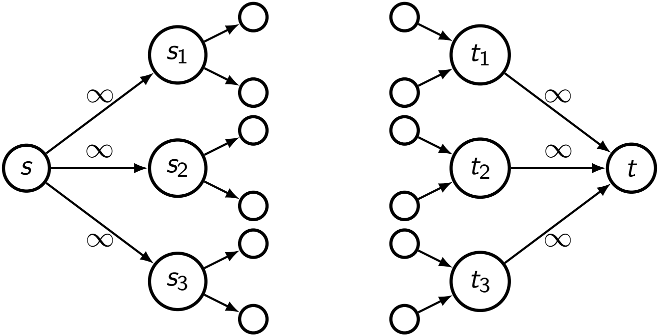

- Sometimes not only the edges but also the vertices v_i of the flow graph might have capacities $C(v_i)$, which limit the total throughput of the flow coming to the vertex (and, consequently, also leaving the vertex):

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- We can handle this by reducing it to a situation with only edge capacities!

- Suppose vertex v has capacity $C(v)$.

Networks with vertex capacities

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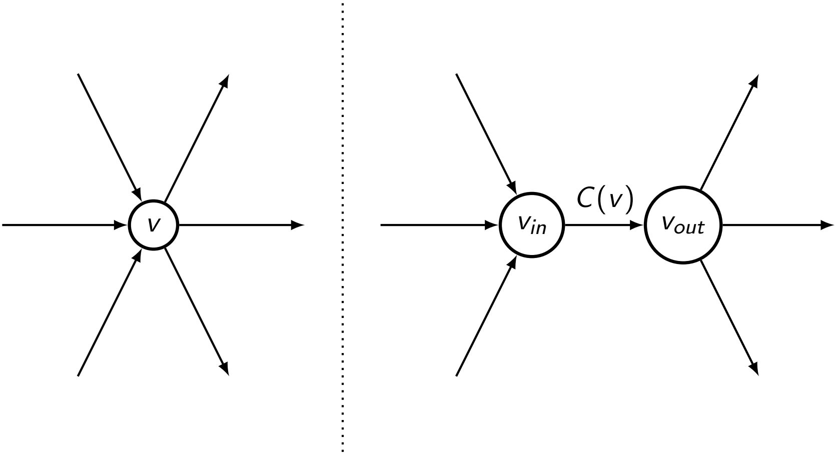
- Connect v_{in} and v_{out} with an edge $e^* = (v_{in}, v_{out})$ of capacity $C(v)$.



v
v_{in}
$C(v)$
v_{out}

Problem

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Task: Design an algorithm which runs in time polynomial in n and k and dispatches the largest possible number of movies.

Construct a flow network with:

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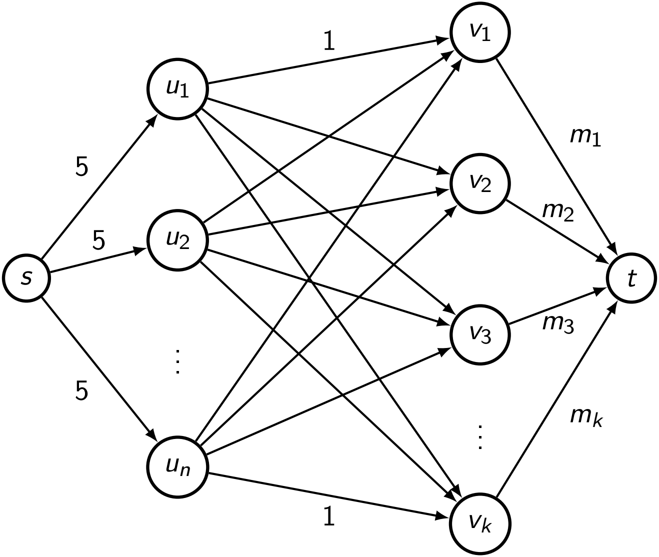

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- Since the value of any flow is constrained by the total capacity from s, which in this case is $5n$, we can achieve a tighter bound of $O(E|f|) = O(n(nk + n + k)) = O(n^2k)$.

Problem

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You are given the capacity of every cell; the cell in row i and column j has capacity $C(i, j)$. To ensure the stability of the ship, the total weight in each row i must not exceed $C_r(i)$ and the total weight in each column j must not exceed $C_c(j)$.

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Task: Design an algorithm which runs in time polynomial in n and m and allocates the maximum possible weight of cargo without exceeding any of the cell, row or column capacities.

	col 1	col 2	col 3	col 4	row cap
row 1	$C(1,1)$	$C(1,2)$	$C(1,3)$	$C(1,4)$	$C_r(1)$
row 2	$C(2,1)$	$C(2,2)$	$C(2,3)$	$C(2,4)$	$C_r(2)$
row 3	$C(3,1)$	$C(3,2)$	0	$C(3,4)$	$C_r(3)$
row 4	$C(4,1)$	0	$C(4,3)$	0	$C_r(4)$
row 5	$C(5,1)$	$C(5,2)$	0	$C(5,4)$	$C_r(5)$
col cap	$C_c(1)$	$C_c(2)$	$C_c(3)$	$C_c(4)$	

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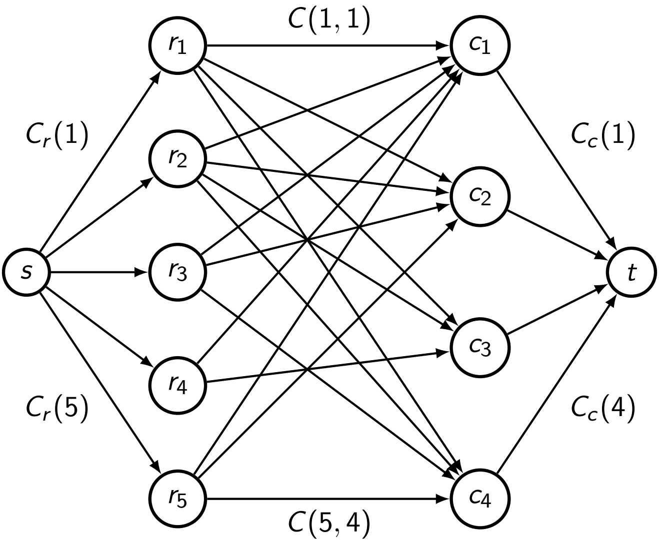

$C(1,1)$
r_1
c_1
$C_r(1)$
r_2
$C_c(1)$
c_2
s
r_3
t
c_3
r_4
$C_r(5)$
$C_c(4)$
r_5
c_4
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- To maximise the cargo allocated, we find the maximum flow using the Edmonds-Karp algorithm.
- There are $n+m+2$ vertices and up to $nm+n+m$ edges, so the time complexity is

$$\begin{aligned} &O((n+m+2)(nm+n+m)^2) \\ &= O((n+m)(nm)^2), \end{aligned}$$

which is polynomial in n and m as required.

Problem

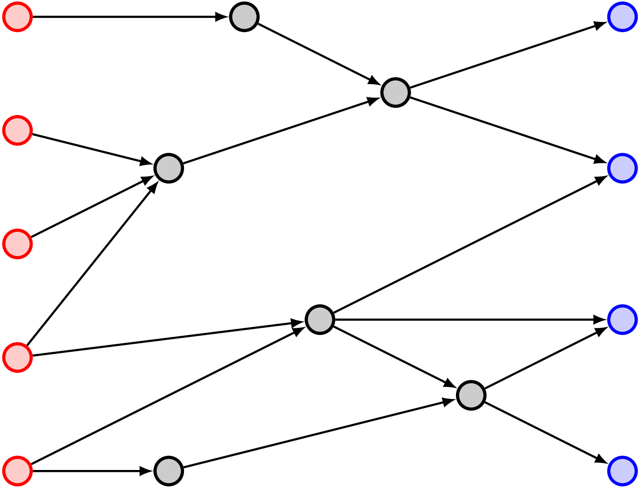
Instance: You are given a directed graph G with n vertices and m edges. Of these vertices, r are painted red, b are painted blue, and the remaining $n - r - b$ are black. Red vertices have only outgoing edges and blue vertices have only incoming edges.

Example problem: Vertex-Disjoint Paths

Problem

Instance: You are given a directed graph G with n vertices and m edges. Of these vertices, r are painted red, b are painted blue, and the remaining $n - r - b$ are black. Red vertices have only outgoing edges and blue vertices have only incoming edges.

Task: Design an algorithm which runs in time polynomial in n and m and determines the largest possible number of vertex-disjoint (i.e. non-intersecting) paths in this graph, each of which starts at a red vertex and finishes at a blue vertex.



- The red vertices function as sources, and the blue vertices as sinks.

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- The red vertices function as sources, and the blue vertices as sinks.
- To use our usual max flow algorithms, we need to introduce a super-source and super-sink.
- To ensure that no black vertex is used twice, we should impose a vertex capacity of 1 for each of them.

Construct a flow network with:

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- super-source s joined to each red vertex,

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- super-source s joined to each red vertex,
- each blue vertex joined to super-sink t,
- for each black vertex, two vertices v_{in} and v_{out}, joined by an edge,
- each edge of the original graph, with edges from a black vertex drawn from the corresponding out-vertex, and edges to a black vertex drawn to the corresponding in-vertex.

Construct a flow network with:

- super-source s joined to each red vertex,
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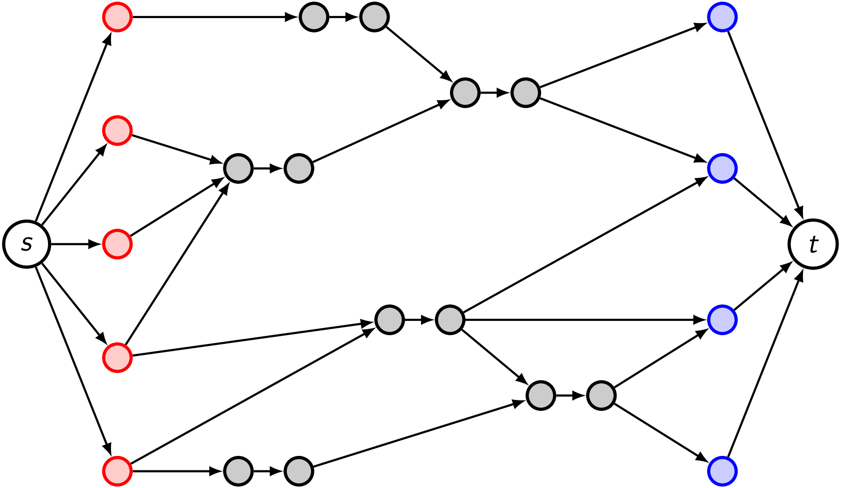
All edges have capacity 1.


s
t

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- *Flow conservation* ensures that each unit of flow travels from s to a red vertex, then to zero or more black vertices, before arriving at a blue vertex and terminating at t.

- By the *capacity constraint*, each red vertex receives at most one unit of flow from s, so *flow conservation* ensures that at most one edge from that vertex is flowed, i.e. we use this vertex in at most one path.

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- Similarly, each blue vertex is used in at most one path also.
- Likewise, each in-vertex and out-vertex pair contributes to at most one path, so the paths are indeed vertex-disjoint.

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Definition

A graph $G = (V, E)$ is said to be *bipartite* if its vertices can be divided into two disjoint sets A and B such that every edge $e \in E$ has one end in the set A and the other in the set B.

Definition

A *matching* in a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each vertex of the graph belongs to at most one edge in M.

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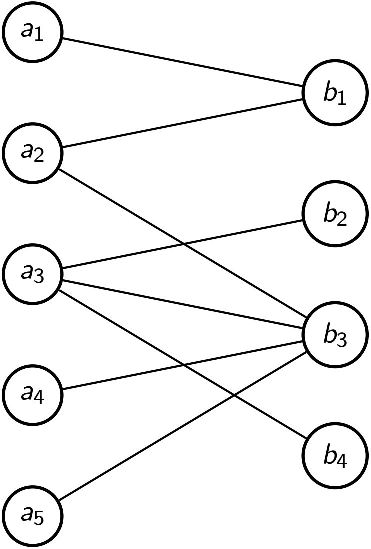
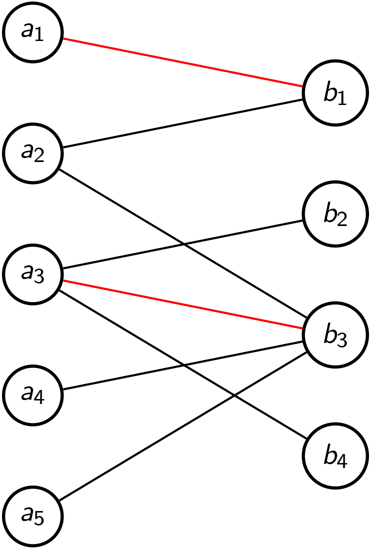
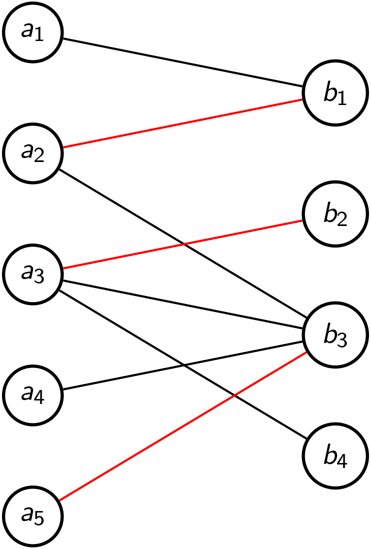
A *matching* in a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each vertex of the graph belongs to at most one edge in M.

A *maximum matching* in G is a matching containing the largest possible number of edges.


a_1
a_2
a_3
a_4
a_5
b_1
b_2
b_3
b_4


a_1
a_2
a_3
a_4
a_5
b_1
b_2
b_3
b_4

Matchings in a Bipartite Graph



Problem

Given a bipartite graph G, find the size (i.e. the number of pairs matched) in a maximum matching.

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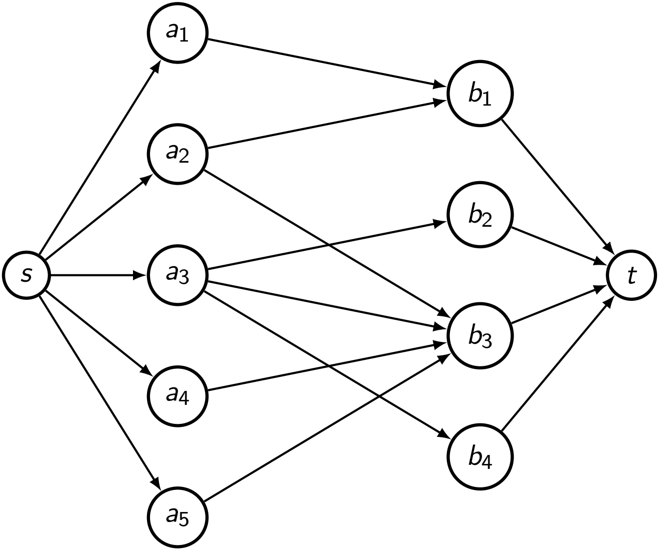
Question

How can we turn a Maximum Bipartite Matching problem into a Maximum Flow problem?

Answer

Create two new vertices, s and t (the source and sink). Construct an edge from s to each vertex in A, and from each vertex in B to t. Orient the existing edges from A to B. Assign capacity 1 to all edges.


a_1
a_2
a_3
a_4
a_5
b_1
b_2
b_3
b_4
s
t

Property

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Since all capacities in the flow network are 1, we need only denote the direction of the edge in the residual graph!

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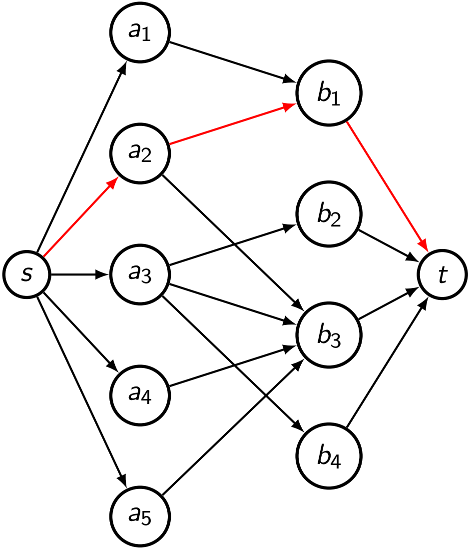
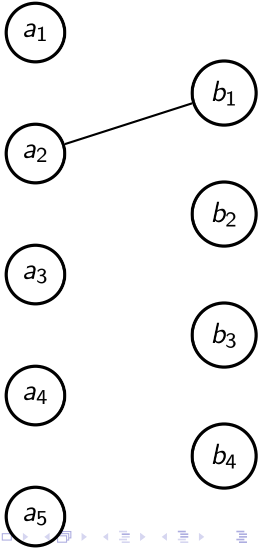
As always, the residual graph allows us to correct mistakes and increase the total flow.



a_1
a_2
a_3
a_4
a_5
b_1
b_2
b_3
b_4
s
t



a_1
a_2
a_3
a_4
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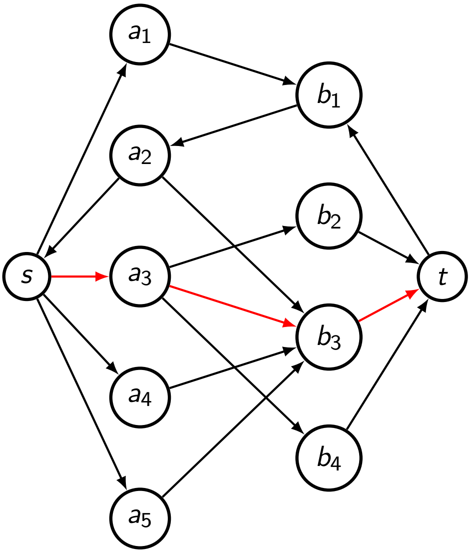
b_3
b_4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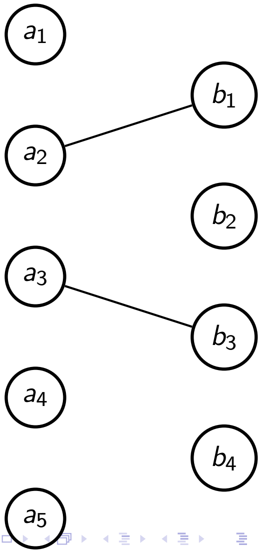
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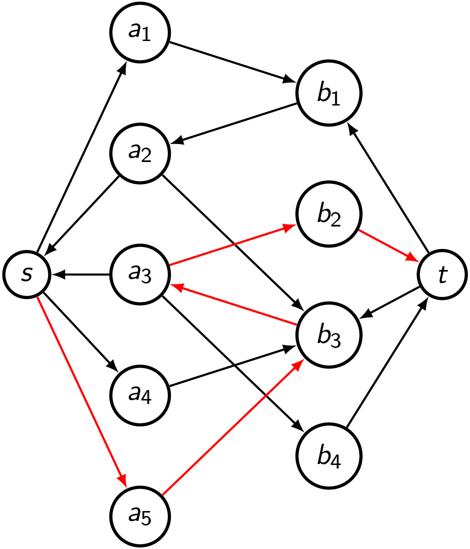
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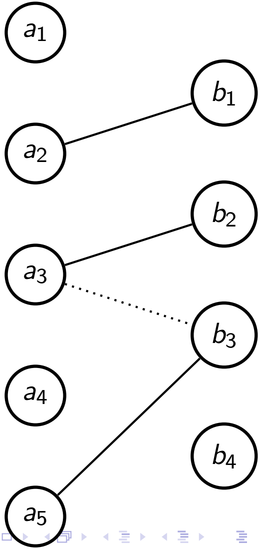
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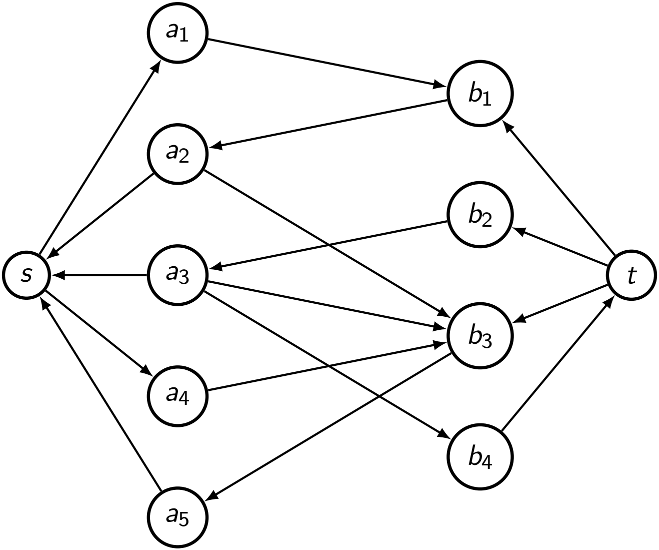
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Problem

Instance: You are running a job centre. In your country, there are k recognised qualifications. There are n unemployed people, each holding a subset of the available qualifications. There are also m job openings, each requiring a subset of the qualifications.

Example problem: Job Centre

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Instance: You are running a job centre. In your country, there are k recognised qualifications. There are n unemployed people, each holding a subset of the available qualifications. There are also m job openings, each requiring a subset of the qualifications.

Task: Design an algorithm which runs in time polynomial in n, m and k, and places as many people as possible into jobs for which they are qualified. No worker can take more than one job, and no job can employ more than one worker.

- Create an unweighted, undirected graph with vertices $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$, representing the workers and jobs respectively.

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- Once again, the tightest bound on the runtime of Edmonds-Karp is of the form $O(E|f|)$, where $E \leq nm + n + m$ and $|f| \leq \min(n, m)$.
- Therefore the total time complexity is $O(nm(k + \min(m, n)))$, which is polynomial in n, m and k as required.

Problem

You are taking two kinds of medicines, A and B, which each come in identical bottles of 30 pills. Pills of medicine A are completely indistinguishable from pills of medicine B. You take one pill of each medicine every day. The pharmacy will only refill your pill bottles every 28 days.

One day, you are down to the last two pills in each bottle when you drop both bottles on the floor, spilling all four pills. Since you cannot tell which are of type A and which are of type B, how can you continue to take one pill of each medicine for the remaining two days?

That's All, Folks!!