UNSW SYDNEY

1. INTRODUCTION

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School of Computer Science and Engineering
UNSW Sydney

Term 2, 2023

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1. Admin

2. Solving problems using algorithms

3. Proofs

4. An example of the role of proofs

5. Asymptotic Notation

6. Puzzles

- Only prerequisite: COMP2521/9024

Prerequisite courses

- Only prerequisite: COMP2521/9024
- Highly desirable (but not officially required)
  - For undergrads (COMP3121): MATH1081 Discrete Mathematics (proofs, graphs)
  - For postgrads (COMP9101): COMP9020 Foundations of Computer Science

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- Use the index to locate these notions in the textbook and/or search the Wikipedia to find articles for reviewing these notions

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Required knowledge and skills

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  - No programming involved, emphasis on problem solving, proving correctness of algorithms and estimating efficiency of algorithms

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  - Smarthinking for writing help

- Tuesday 9:00 - 11:00 and Friday 11:00 - 13:00, weeks 1–10
- Face to face at Keith Burrows Theatre (K-J14-G5)
- Slides and recordings are available on Moodle

Consultation

- Friday 15:00 - 17:00, weeks 1–10
- Join live on MS Teams
- Exam consultation TBA

- Weeks 1–5, 7–10 (no tutorials in flex week)
- Most face to face, some online
- Will demonstrate problem solving and reasoning
- Prepare by reading the tutorial sheet before attending

Practice Problems

- Six sets of practice problems will be released on Moodle, with written solutions (approx weeks 1, 2, 3, 5, 7, 9).
- Discuss these with your friends and on the Ed forum.
- You can also bring these to tutorials.

- Take-home quiz
  - Released week 1, attempt before your second tutorial
  - Due before Friday at Noon in week 4
  - Incorporate tutor's feedback until task complete
  - Weighted 5% of course mark

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- Assignments
  - 3 assignments
  - Each consists of 3 questions
  - Each assignment weighted 15% of course mark

- Final Exam
  - Section I: 5 multiple choice questions
  - Section II: 3 algorithm design problems
  - INSPERA (ON campus): more info here
  - Weighted 50% of course mark

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- Forum participation
  - Up to 5 bonus marks

Recommended textbook

Kleinberg and Tardos: *Algorithm Design*
paperback edition available at UNSW Bookshop

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An alternative textbook

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4th edition now available at UNSW Bookshop, 3rd edition also useful

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- excellent: to be used later as a reference manual;
- not so good: somewhat formalistic and written in a rather dry style.

Feedback

- Changes from last year:
  - weekly tutorials
  - fewer assignments and fewer questions per assignment
  - FAQ on Moodle and Ed
  - earlier feedback via take-home quiz

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- Feedback is always welcome, e.g.
  - myExperience survey
  - feedback post on Ed (can post anonymously)
  - email

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What is an algorithm?

- An algorithm is a collection of precisely defined steps that can be executed *mechanically*, i.e. without intelligent decision-making.
- Designing a recipe involves creativity, executing it does not; the same is true of algorithms.
- The word "algorithm" comes by corruption of the name of Muhammad ibn Musa al-Khwarizmi, a Persian scientist 780–850 AD, who wrote an important book on algebra, *"Al-kitab al-mukhtasar fi hisab al-gabr wal-muqabala"*.

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- the action of each step is not randomised, so the algorithm always gives the same result for the same input.

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Course content:

- a survey of algorithm **design techniques**
- particular algorithms will be mostly used to illustrate design techniques
- emphasis on development of your algorithm design **skills**

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- Interview questions at good companies often involve asking you to solve a tricky problem.
- Some of the practice problems in this course are actually Google and Microsoft interview questions.

Problem

Alice and Bob have robbed a warehouse and have to split a pile of items. There is no objective valuation of the items (e.g. price tags); instead, Alice and Bob each have a valuation in mind, and they might not agree.
Design an algorithm to split the pile so that each thief values their own pile as at least half the loot.

Example: Two Thieves

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Solution

Alice splits the pile in two parts, so that she believes that both parts are of equal value. Bob then chooses the part that he believes is no worse than the other.

Note

We are assuming that it's always possible to split up the loot into whatever fraction we like. With discrete items this is more complicated than it might appear!

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Question

If there are n items, and Alice values the ith at v_i dollars, can Alice efficiently split the loot into two equal piles?

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Question

If there are n items, and Alice values the ith at v_i dollars, can Alice efficiently split the loot into two equal piles?

Answer

There is no known algorithm that is significantly more efficient than the brute force (try all choices, of which there are 2^n).

Example: Three Thieves

Problem

Alice, Bob and Carol have robbed a warehouse and have to split a pile of items. Each thief has their own valuation of the items. Design an algorithm to split the pile so that each thief values their own pile as at least one third of the loot.

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Example: Three Thieves

- The problem is much harder with 3 thieves!
- Let us try do the same trick as in the case of two thieves. Say Alice splits the loot into three piles which she thinks are of equal value; then Bob and Carol each choose which pile they want to take.
- If they choose different piles, they can each take the piles they have chosen and Alice gets the remaining pile; in this case clearly each thief thinks that they got at least one third of the loot.

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- One might think that in this case, Alice can pick either of the other two piles, after which the remaining two piles are put together for Bob and Carol to split them as in the earlier problem with only two thieves.
- Unfortunately this does not work!

- Suppose that Alice splits the loot into three piles X, Y, Z, and that Bob thinks that

$$X = 50\%, Y = 40\%, Z = 10\%$$

of the total value, while Carol thinks that

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- However, if Alice picks pile Y, then Bob will object that (in his eyes) only 60% of the loot remains, so he is guaranteed to get only 30% of the loot i.e. he is not guaranteed to get at least one-third of the total.

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- However, if Alice picks pile Y, then Bob will object that (in his eyes) only 60% of the loot remains, so he is guaranteed to get only 30% of the loot i.e. he is not guaranteed to get at least one-third of the total.
- If instead Alice picks pile Z, then Carol will object for the same reason.

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- Now we ask Bob to reduce the pile X until he believes it to be $1/3$ of the total. Alice values the new pile as $< 1/3$, so she is happy to split the remainder of the loot (worth $> 2/3$) with one other thief.

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- Bob asks Carol whether she thinks that $X \leq 1/3$.
- If Carol says NO, then Carol takes X, and Alice and Bob split the rest.
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Exercise

Try generalising this to n thieves.

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Hint

There is a *nested recursion* happening even with 3 thieves!

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Question

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Answer

When this is not obvious by inspecting the algorithm using common sense!
Mathematical proofs are **NOT** academic embellishments; we use them to justify things which are not obvious to common sense!

Algorithm

We recursively apply the following algorithm to the subarray $A[\ell..r]$.

If $\ell = r$, i.e. the subarray has only one element, we simply exit. This is the base case of our recursion.

Otherwise:

- first define $m = \left\lfloor \frac{\ell+r}{2} \right\rfloor$, the midpoint of the subarray,
- then apply the algorithm recursively to $A[\ell..m]$ and $A[m+1..r]$, and
- finally merge the subarrays $A[\ell..m]$ and $A[m+1..r]$.

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Example: Merge-Sort

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- Thus, MERGE-SORT always terminates, and in fact it terminates in $O(n \log_2 n)$ steps.
- Merging two sorted arrays always produces a sorted array, thus, the output of MERGE-SORT will be a sorted array.
- The above is essentially a proof by induction, but we will never bother formalising proofs of (essentially) obvious facts.

- However, sometimes it is **NOT** clear from a description of an algorithm that such an algorithm will not enter an infinite loop and fail to terminate.

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- Sometimes it is **NOT** clear that an algorithm will not run in exponentially many steps (in the size of the input), which is usually almost as bad as never terminating.
- Sometimes it is **NOT** clear from a description of an algorithm why such an algorithm, after it terminates, produces a desired solution.

- Proofs are needed for such circumstances; in a lot of cases they are **the only way** to know that the algorithm does the job.

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- For that reason we will **NEVER** prove the obvious (the CLRS textbook sometimes does just that, by sometimes formulating and proving trivial little lemmas, being too pedantic!). We will prove only what is genuinely nontrivial.

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- For that reason we will **NEVER** prove the obvious (the CLRS textbook sometimes does just that, by sometimes formulating and proving trivial little lemmas, being too pedantic!). We will prove only what is genuinely nontrivial.
- **However, BE VERY CAREFUL what you call trivial!!**

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- Suppose there are n hospitals in the state, and n new doctors have graduated from university. Each hospital wants to employ exactly one new doctor. (This is to keep the problem simpler; one can design an algorithm which works for more general cases where there are more doctors than hospitals and each hospital employs more than one doctor.)

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- Every hospital submits a list of preferences, which ranks all the doctors, **and** every doctor submits a list of preferences, which ranks all the hospitals.

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- Every hospital submits a list of preferences, which ranks all the doctors, **and** every doctor submits a list of preferences, which ranks all the hospitals.
- We'd like to assign all n doctors to different hospitals in order to "make everyone happy" in some sense.

Definition

A *matching* is an assignment of doctors to hospitals so that no doctor has more than one job and no hospital has more than one new employee.

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Definition

A *perfect matching* is a matching involving *all* doctors and *all* hospitals.

Our task is to design a perfect matching that somehow satisfies the hospitals and doctors, with regards to their preferences.

Question

Can we assign every doctor and every hospital their first preference?

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Answer

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Answer

Not necessarily!

- We'll have to lower our expectations. Let's aim for an allocation that doesn't make anyone too unhappy.
- If a hospital and doctor were both very unhappy with their allocations, they might leave our system and arrange hiring directly.

Definition

A *stable matching* is a perfect matching in which there are no two pairs (h, d) and (h', d') such that:

- hospital h prefers doctor d' to doctor d, **and**
- doctor d' prefers hospital h to hospital h'.

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Stable matchings are self-enforcing; no hospital h and no doctor d' will leave the system to do better outside of it.

We will design an algorithm which produces a stable matching.

$h_1 : d_1, d_2$

$h_2 : d_1, d_2$

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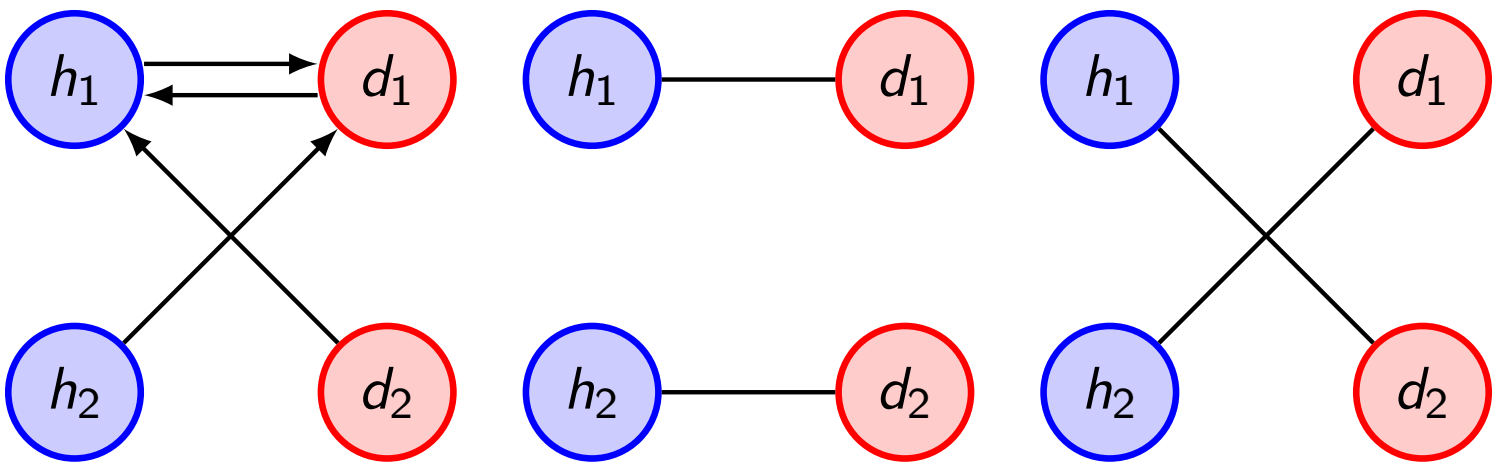
$d_2 : h_1, h_2$

$h_1 : d_1, d_2$ $\quad$ $d_1 : h_1, h_2$

$h_2 : d_1, d_2$ $\quad$ $d_2 : h_1, h_2$

Preferences Stable Not stable



Stable Matching Problem: Example 2

$h_1 : d_1, d_2$

$h_2 : d_2, d_1$

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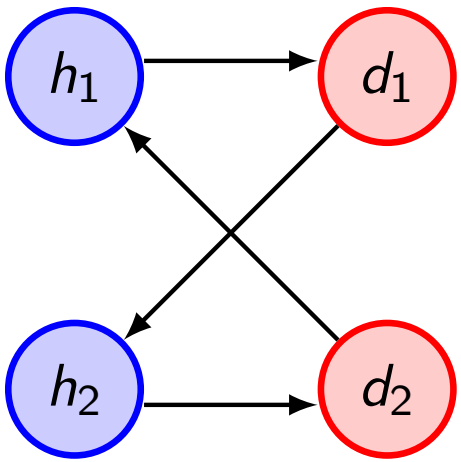
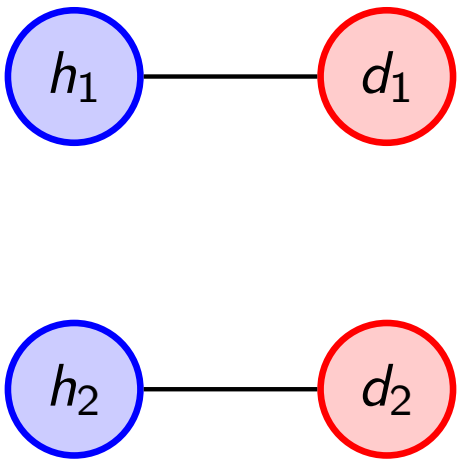
$h_1 : d_1, d_2$ $\quad$ $d_1 : h_2, h_1$

$h_2 : d_2, d_1$ $\quad$ $d_2 : h_1, h_2$

Preferences



Stable



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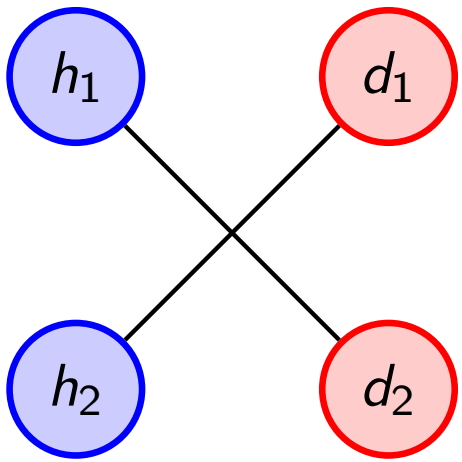

Question

Given n hospitals and n doctors, how many ways are there to match them, without regard for preferences?

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Answer

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    - otherwise, since d prefers her current hospital h', she rejects the offer from h and stays at h'.

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- Thus there can be no more than n^2 many rounds.

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- **"Big Oh" notation:** $f(n) = O(g(n))$ is an abbreviation for:

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- Clearly, multiplying constants c of interest will be larger than 1, thus "enlarging" $g(n)$.

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- **"Theta" notation:** $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$; thus, $f(n)$ and $g(n)$ have the same asymptotic growth rate.

- Note: $f(n) = O(g(n))$ is NOT a linear ordering; there are functions $f(n)$ and $g(n)$ such that it is neither $f(n) = O(g(n))$ nor $g(n) = O(f(n))$.

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- Example: $f(n) = n^2 + n$ and $g(n) = n^2$. Note that $g(n) \leq f(n)$; thus $g(n) \leq c\, f(n)$ with $c = 1$, and so $g(n) = O(f(n))$.

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  However $f(n) \leq 2\, g(n)$; thus $f(n) \leq c\, g(n)$ with $c = 2$. So also $f(n) = O(g(n))$. Thus $f(n) = \Theta(g(n))$.

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- Also, for small inputs the efficiency might depend on operating system overhead.
- Asymptotic estimates of efficiency make these facts irrelevant.

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Problem

Tom and his wife Mary went to a party where nine more couples were present.

- Not every one knew everyone else, so people who did not know each other introduced themselves and shook hands.
- People who knew each other from before did not shake hands.
- Later that evening Tom got bored, so he walked around and asked all other guests (including his wife) how many hands they had shaken that evening, and got 19 different answers.
- How many hands did Mary shake?
- How many hands did Tom shake?

That's All, Folks!!