§26 Compact Spaces - Solutions to Exercises

1. (a) Suppose X is compact under τ'. Let $\mathcal{A} \subset \tau'$ be an open cover of X. Since $\tau \subset \tau'$, $\mathcal{A} \subset \tau'$. Then by the definition of compactness, there exists a finite subcollection of $\mathcal{A}$ covering X. So X is compact under τ as well.

   X compact under τ does not say anything about the compactness of X under τ'. For example, suppose X is infinite and compact under τ. If τ' is $\mathcal{P}(X)$ (the set of all sets of X), then the open cover $\mathcal{A} = \{ \{x\} \,|\, x \in X \} \subset \tau'$ does not admit a finite subcover in τ'.

   (b) Suppose X is compact and Hausdorff under both τ and τ'. Suppose τ and τ' are comparable. Without loss of generality, assume $\tau \subset \tau'$. Let $U \in \tau'$. Then $X - U$ is closed in (X, τ'). Since X compact, $X - U$ is compact as a subspace of (X, τ') by Theorem 26.2. By the result of part (a) above, $X - U$ is compact in (X, τ). Since (X, τ) is Hausdorff, $X - U$ is closed in (X, τ) by Theorem 26.3. By definition of closed set, $U \in \tau$. Thus, $\tau' \subset \tau$, which implies $\tau' = \tau$.

2. (a) Let U be a subset of $\mathbb{R}$. If $U = \emptyset$ it is vacuously true that U is compact. If $U \neq \emptyset$, let $\{A_\alpha\}$ be an open cover of U. Pick a nonempty member of the open cover, say A_β. $\mathbb{R} - A_\beta$ is finite so $U \cap (\mathbb{R} - A_\beta)$ is also finite. If $U \cap (\mathbb{R} - A_\beta) = \emptyset$, then $U = A_\beta$ which would mean U is compact. If $U \cap (\mathbb{R} - A_\beta) \neq \emptyset$, then for each $p \in U \cap (\mathbb{R} - A_\beta)$, there exists $A_{\alpha_p} \in \{A_\alpha\}$ that contains p. Then the set $\{ A_{\alpha_p} \,|\, p \in U \cap (\mathbb{R} - A_\beta) \} \cup \{A_\beta\}$ is a finite open cover of U.

   (b) Let $K_m = \{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \text{ and } n \geq m \}$. Then $\{\mathbb{R} - K_n\}_{n \in \mathbb{Z}_+}$ is an open cover of $[0, 1]$. For every finite subcollection of these sets, there is a maximum N such that $\mathbb{R} - K_N$ is in the subcollection. $\mathbb{R} - K_N$ does not contain $\frac{1}{n}$ $\forall\, n \geq N$ nor does any $\mathbb{R} - K_m$ $\forall\, m \leq N$. So $[0, 1]$ cannot be compact.

3. Let $U_1, \ldots, Un$ be a collection of compact subspaces of X. Let $\mathcal{A}_0$ be an open covering of $\bigcup_{i=1}^{n} U_i$. Since $\mathcal{A}_0$ is an open cover of each U_i, there is a finite subcollection $\mathcal{A}_i$ that covers each U_i. Since $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_n$ is finite and covers $U_1 \cup \ldots \cup U_n$. Thus, $\bigcup_{i=1}^{n} U_i$ is compact.

4. Let X be a metric space (with metric d) and A be a compact subspace of X. Every metric space is Hausdorff so A is closed by Theorem 26.3. the set of balls $\{B_d(x, 1)\}_{x \in A}$ is an open covering of A. So there exists a finite subcover $B_d(x_1, 1), \ldots, B_d(x_n, 1)$. Let $M = \max_{1 \leq i,j \leq n} d(x_i, x_j) + 2$. Let $a_1, a_2 \in A$. Then $a_1 \in B_d(x_{m_1}, 1)$ and $a_2 \in B_d(x_{m_2}, 1)$ for some $m_1, m_2 \in \{1, \ldots, n\}$. By the triangle inequality,

   $$d(a_1, a_2) \leq d(a_1, x_{m_1}) + d(x_{m_1}, x_{m_2}) + d(x_{m_2}, a_2) < 1 + \max_{1 \leq i,j \leq n} d(x_i, x_j) + 1 = M$$

   So A is bounded.

   $\mathbb{R}$ equipped with the standard bounded metric is closed and bounded but not compact.