§31 Exercises

1.

Suppose X is regular and $x,\ y \in X$ such that $x \neq y$. Then since X is Hausdorff, there exists disjoint neighbourhoods U and V of x and y respectively. Since X is T_1, $\{x\}$ and $\{y\}$ is closed and so there exists open sets $A,\ B$ such that $x \subset \bar{A} \subset U$ and $y \subset \bar{B} \subset V$ by Lemma 3.31. A and B are disjoint because U and V are disjoint.

2.

Similar to the proof above.

3.

If X has less than two elements, it is clear that it is regular. Suppose X has at least two elements and is an ordered set with the order topology. Let $x \in X$ and let U be a neighbourhood of x. If x is not the smallest or largest element, then U contains a set of the form $(a,\ b)$ that contains x. Let c be the immediate predecessor if it exists, otherwise choose $c \in (a,\ x)$. Let d be the immediate successor if it exists, otherwise choose $d \in (x,\ b)$. The closure of $(c,\ d)$ $= [c,\ d]$ and $[c,\ d] \subset (a,\ b) \subset U$. If x is the smallest, then U contains a set of the form $[x,\ a)$. Let b be the immediate successor if it exists, otherwise choose $b \in [x,\ a)$. Then if b is the immediate successor, $[x,\ b) = \{x\}$ which is both open and closed and contained in U. If x has no immediate successor, then the closure of $[x,\ b)$ is $[x,\ b]$ which is a subset of $[x,\ a)$ so it is contained in U. A similar argument for the case when x is the largest element applies. By Lemma 31.1, X is regular.

4.

If X is Hausdorff, then X' is Hausdorff. If X' is Hausdorff, this does not imply X is Hausdorff. e.g. Let $X' = \{1,\ 2\}$ with the discrete topology and let $X = \{1,\ 2\}$ with the trivial topology. X regular does not imply X' regular because $\mathbb{R}$ is regular but $\mathbb{R}_K$ is not. Since normal implies regular, X normal does not imply X' normal because $\mathbb{R}$ is normal but $\mathbb{R}_K$ is not. ??If X' is regular or normal I don't think X is necessarily regular or normal too.

5.

Let $S = \{x \mid f(x) = g(x)\}$. Let $x \in X$ - S. Then $f(x) \neq g(x)$ so there exists disjoint neighbourhoods U and V of $f(x)$ and $g(x)$ respectively since Y is Hausdorff. $f^{-1}(U)$ and $g^{-1}(V)$ are both neighbourhoods of x. The intersection $f^{-1}(U) \cap g^{-1}(V)$ is a neighbourhood of x and if $y \in f^{-1}(U) \cap g^{-1}(V)$, $f(y) \in U$ and $g(y) \in V$ so $f(y) \neq g(y)$. So X - S is open.

6.

Suppose $p : X \longrightarrow Y$ is a closed, continuous surjective map and X is normal.

Lemma: Let $y \in Y$. If U is an open set containing $p^{-1}(\{y\})$, then $X - U$ is closed. p is a closed map so $p(X - U)$ is closed and $W = Y - p(X - U)$ is open. Suppose $a \in p^{-1}(W)$. Then $p(a) \in W$. Which means $p(a) \notin p(X$ - $U)$ so $a \notin X - U \implies a \in U$. So $p^{-1}(W) \subset U$.

Suppose $A,\ B$ are disjoint closed sets of Y. Then their preimages are disjoint and closed in X. Because X is normal, there are disjoint open sets $U,\ V$ containing $p^{-1}(A)$ and $p^{-1}(B)$ respectively. For each $y \in A$, there is an neighbourhood W_y of y such that $p^{-1}(W_y) \subset U$. For each $y \in B$, there is an neighbourhood W_y of y such that $p^{-1}(W_y) \subset V$. If $x \in A$ and $y \in B$, then W_x and W_y are disjoint; if there was a point in the intersection, then their preimages would intersect and that would mean $U,\ V$ intersect. So the sets $\bigcup\limits_{x \in A} W_x and \bigcup\limits_{x \in B} W_x$ are disjoint open sets containing A and B respectively. Thus Y is normal.

7.

a) Suppose X is Hausdorff. Let x, y be distinct points of Y. $p^{-1}(\{x\})$ and $p^{-1}(\{y\})$ are compact. For each $a \in p^{-1}(\{x\})$, we can find disjoint open sets U_a and V_a containing a and $p^{-1}(\{y\})$ respectively by Lemma 26.4. $\{V_a\}_{a \in p^{-1}(\{x\})}$ is an open cover of $p^{-1}(\{x\})$ and since it is compact, there are points $\{a_1, ..., a_n\}$ such that $\{U_{a_1}, ..., U_{a_n}\}$ covers $p^{-1}(\{x\})$. Then $U = \bigcup\limits_{i=1}^{n} U_{a_i}$ and $V = \bigcap\limits_{i=1}^{n} V_{a_i}$ are disjoint open sets that contain $p^{-1}(\{x\})$ and $p^{-1}(\{y\})$. Using the Lemma from Exercise 6, we obtain neighbourhoods W_x and W_y of x and y respectively such that $p^{-1}(W_x) \subset U$ and $p^{-1}(W_y) \subset V$. W_x and W_y are disjoint for if there was a point of intersection, then that would imply U and V intersect. Therefore Y is Hausdorff.

b)