Munkres Topology Answers

§30

1. a)

Suppose X is a first-countable T_1 space. Let $x \in X$. Let $\{B_n\}$ be a countable basis at x. Then $x \in \bigcap B_n$. For all $y \in X$ such that $y \neq x$, X - $\{y\}$ is an open set (one element sets are closed in a T_1 space) so there is a $B_n \subset X$ - $\{y\}$. Thus $y \notin \bigcap B_n$. So $\bigcap B_n = x$ which means X is a G_δ set.

b)?

2. Suppose X has a countable basis $\{B_k\}$. Let $\mathcal{C}$ be a basis of X. For each n, m for which it is possible, choose $C_{n,m} \in \mathcal{C}$ such that $B_n \subset C_{n,m} \subset B_m$. Since $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable, $\{C_{n,m}\}$ is countable. Let U be an open subset of X and $x \in X$. Since $\{B_n\}$ is a basis, $x \in B_m \subset U$ for some m $\in \mathbb{Z}_+$. Since $\mathcal{C}$ is a basis, there exists $C \in \mathcal{C}$ such that $x \in C \subset B_m$. Similarly, since $\{B_k\}$ is a basis, there exists n $\in \mathbb{Z}_+$ such that $x \in B_n \subset C \subset B_m$. $x \in C_{n,m} \subset U$ so $\{C_{n,m}\}$ is a basis for X by Lemma 13.2.

3. Suppose X is has a countable basis and A is an uncountable subset of X. Let $\{B_n\}$ be a countable basis of X. Suppose $x \in A$ - A'. Then $\exists$ B_n such that $B_n \cap A = \{\text{x}\}$. Let $C_n = B_n \cap A$ $\forall$ $n \in \mathbb{Z}_n$ so $\{C_n\}$ is a countable basis for A. $S = \{a \in A \mid \exists\ C_n = \{a\}\}$ is countable; if it were not, then $\{C_n\}$ would not be countable since each a has an associated C_n. So for A to be uncountable, A - S must be uncountable and A - $S = A \cap A'$.

4. Suppose X is a compact metrizable space. Let d be a metric that induces the topology on X. For each $n \in \mathbb{Z}_+$, the open cover $\{B_d(x, \frac{1}{n})\}_{x \in X}$ has a finite subcover $\mathcal{A}_n$. Let $\mathcal{B} = \bigcup_{n \in \mathbb{Z}_+} \mathcal{A}_n$. $\mathcal{B}$ is countable since it is the countable union of countable sets. Let U be an open set of X and $x \in U$. $\exists$ $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$ and $\exists$ $N \in \mathbb{Z}_+$ such that $\frac{1}{N} < \epsilon$. Then $x \in B_d(x, \frac{1}{N}) \subset B_d(x, \epsilon) \subset$ U so $\mathcal{B}$ is a countable basis for X.