Lecture 5: September 8

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- PER-PHASE ANALYSIS (CONTINUED)
- TRANSMISSION LINE MODELING

Per-Phase Analysis (Continued)

Consider the system in Fig. 5.1 (left), where $\overline{V}_{aa'} = Ve^{j0}$, $\overline{V}_{bb'} = Ve^{-j2\pi/3}$, $\overline{V}_{cc'} = Ve^{j2\pi/3}$. As we saw earlier, the analysis of such system can be performed by independently analyzing the circuits in Fig. 5.1 (right) because the neutral terminals, n and n', are at the same potential. Furthermore, we also saw that since $\overline{V}_{aa'}$, $\overline{V}_{bb'}$, and $\overline{V}_{cc'}$, so are $\overline{I}_a$, $\overline{I}_b$, and $\overline{I}_c$; thus, in practice, it is sufficient with analyzing one of the single phase circuits, e.g., the top one on the right.

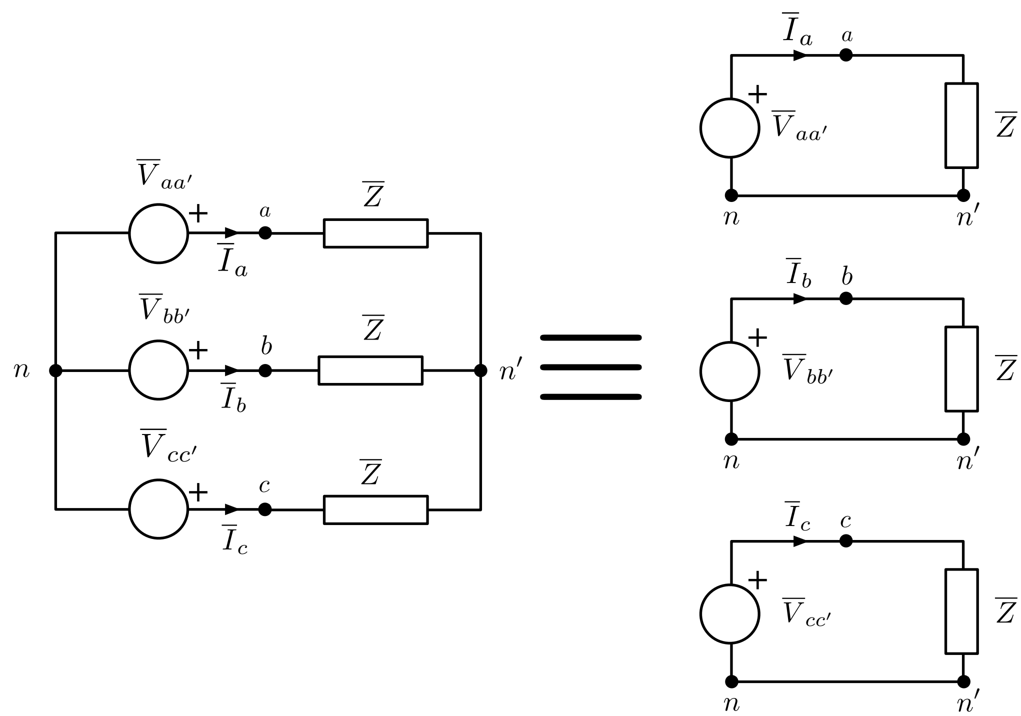

Figure 5.1: Wye-wye three-phase circuit and corresponding single-phase equivalent circuits.

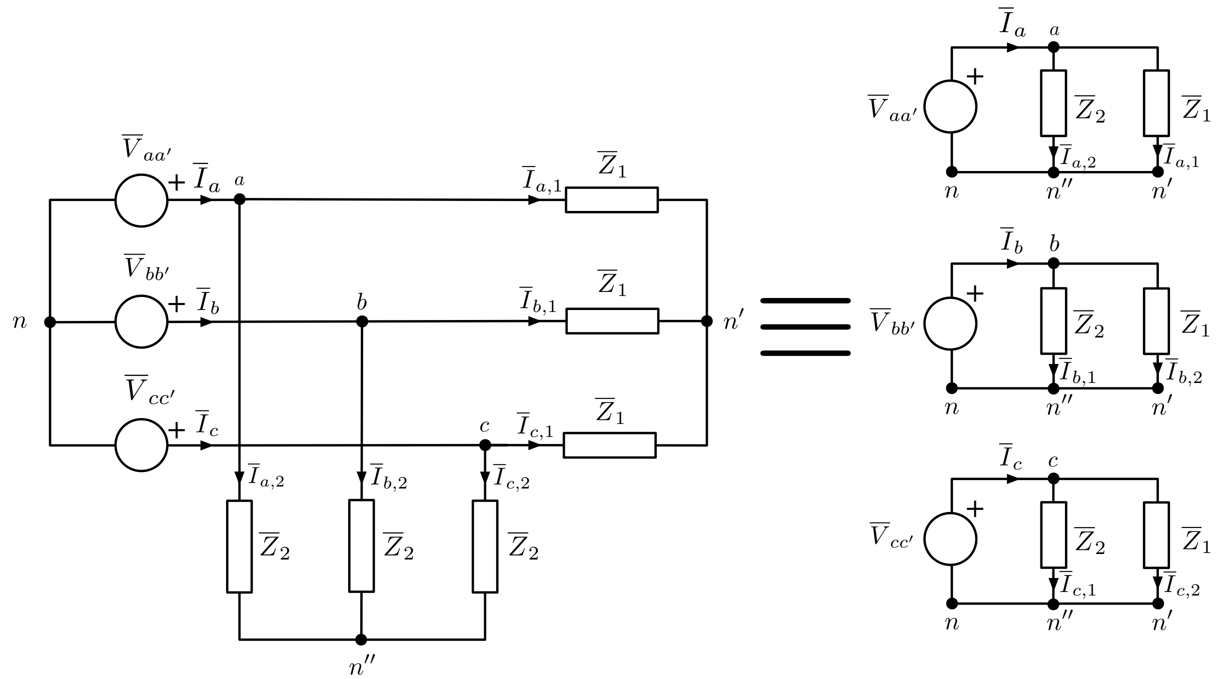

Figure 5.2: Three-phase circuit with one wye-connected source and two wye-connected passive loads.

Now consider the slightly more complicated system in Fig. 5.2 (left), where $\overline{V}_{aa'} = Ve^{j0}$, $\overline{V}_{bb'} = Ve^{-j2\pi/3}$, $\overline{V}_{cc'} = Ve^{j2\pi/3}$. One can conduct a similar analysis to that for the system in Fig. 5.1 (left) and show that neutral terminals, n, n', and n'' are all at the same potential; therefore, the phases are completely decoupled and it is possible to analyze the circuit by analyzing the circuit in Fig. 5.2 (right). Furthermore, because $\overline{V}_{aa'}$, $\overline{V}_{bb'}$ and $\overline{V}_{cc'}$ are balanced, so are triplets $\left\{\overline{I}_{a,1}, \overline{I}_{b,1}, \overline{I}_{c,1}\right\}$ and $\left\{\overline{I}_{a,2}, \overline{I}_{b,2}, \overline{I}_{c,2}\right\}$, it is sufficient to, e.g., use the top circuit on the right of Fig. 5.2 to compute $\overline{I}_a$, $\overline{I}_{a,1}$, and $\overline{I}_{a,2}$, and then use the fact that $\overline{I}_b$, $\overline{I}_{b,1}$, and $\overline{I}_{b,2}$ lag $\overline{I}_a$, $\overline{I}_{a,1}$, and $\overline{I}_{a,2}$, respectively, by $2\pi/3$, and that $\overline{I}_c$, $\overline{I}_{c,1}$, and $\overline{I}_{c,2}$ lead $\overline{I}_a$, $\overline{I}_{a,1}$, and $\overline{I}_{a,2}$, respectively, by $2\pi/3$.

More generally, assume that we are given a balanced three-phase system where all voltage and current sources are connected in wye and have either positive or negative sequence, and all the passive loads are connected in wye. Also, assume there are no mutual inductances between phases. Then, all the neutral terminals are at the same potential; therefore the phases are decoupled. Furthermore, all voltage and current triplets are balanced and have the same phase sequence as that of the sources. Then, the system can be analyzed by constructing three independent circuits, each of which describing the behavior of a particular phase. The topology of these circuits is identical and they only defer in the values taken by the voltage and current sources.

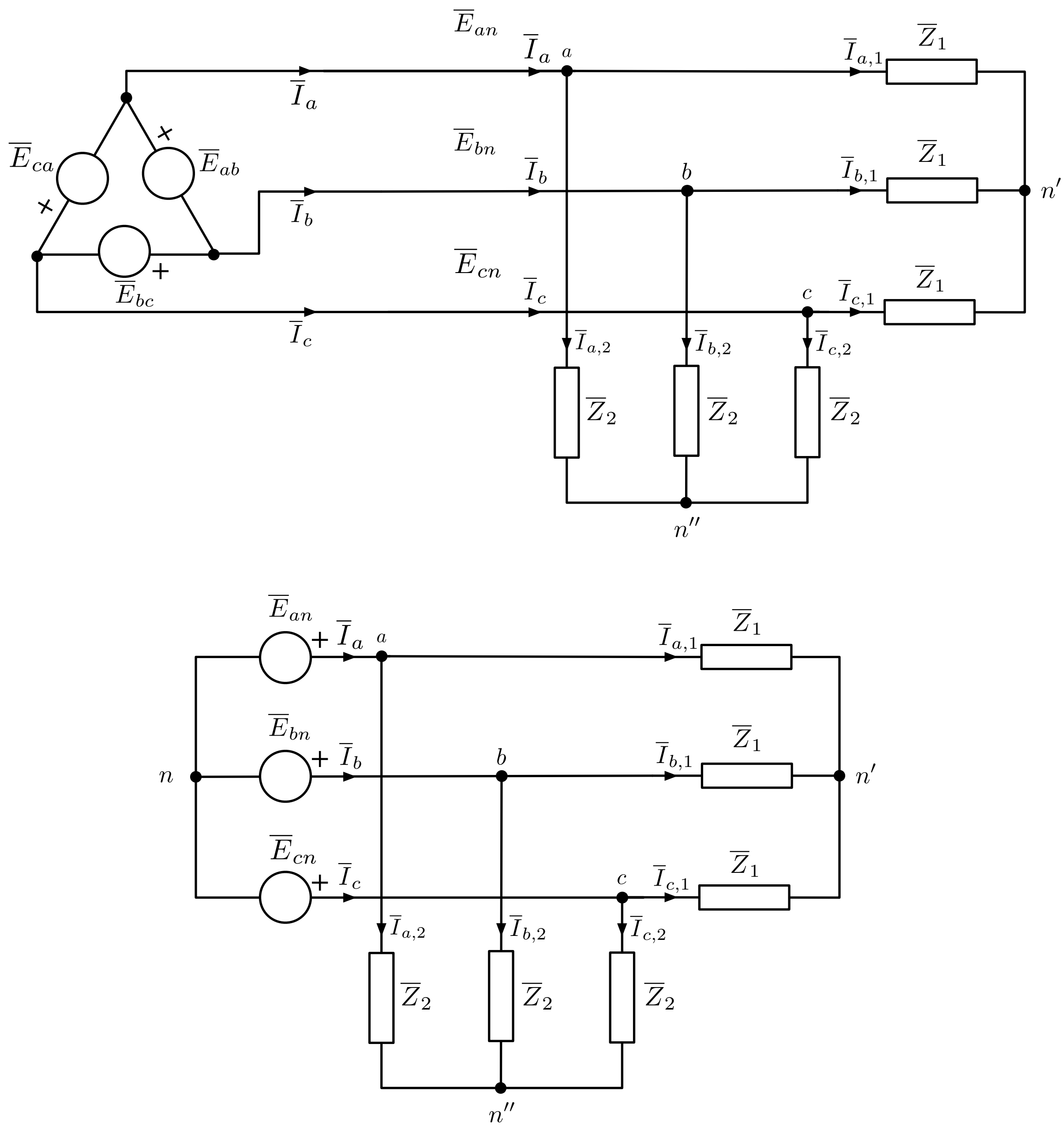

Figure 5.3: Per-Phase Equivalents - 3

Analysis of More General Systems

So far, we have developed a method to analyze balance three-phase systems whose sources and loads are all connected in wye. The question that arises now is how do we analyze balanced three-phase systems with both wye-connected and delta-connected devices. For example, consider the system in Fig. 5.3 (top) with a delta-connected voltage source such that $\overline{E}_{ab}$, $\overline{E}_{bc}$, and $\overline{E}_{ca}$ are balanced, and two balanced wye-connected passive loads. If we were able to transform the delta-connected source into a wye-connected source with equivalent terminal behavior, then we would be able to use the method described earlier for analyzing systems whose elements are all connected in wye. To this end, consider the system in Fig. 5.3 (bottom). Then, if we choose $\overline{E}_{an}$, $\overline{E}_{bn}$, and $\overline{E}_{cn}$ as follows:

$$\begin{aligned}
\overline{E}_{an} &= \frac{1}{\sqrt{3}}\overline{E}_{ab}e^{-j\pi/6}, \\
\overline{E}_{bn} &= \frac{1}{\sqrt{3}}\overline{E}_{bc}e^{-j\pi/6}, \\
\overline{E}_{cn} &= \frac{1}{\sqrt{3}}\overline{E}_{ca}e^{-j\pi/6},
\end{aligned} \tag{5.1}$$

the line voltages of both circuits, are identical. To see this, note that in the circuit on the left, the line voltages are $\overline{V}_{ab} = \overline{E}_{ab}$, $\overline{V}_{bc} = \overline{E}_{bc}$, and $\overline{V}_{ca} = \overline{E}_{ca}$. Now, for the circuit on the right, we have that

$$\begin{aligned}
\overline{V}_{ab} &= \overline{E}_{an} - \overline{E}_{bn} \\
&= \frac{1}{\sqrt{3}}\overline{E}_{ab}e^{-j\pi/6} - \frac{1}{\sqrt{3}}\overline{E}_{bc}e^{-j\pi/6} \\
&= \frac{1}{\sqrt{3}}\overline{E}_{ab}\left(e^{-j\pi/6} - e^{-j5\pi/6}\right) \\
&= \frac{1}{\sqrt{3}}\overline{E}_{ab}\big(\underbrace{\cos(-\pi/6)}_{=\sqrt{3}/2} + j\underbrace{\sin(-\pi/6)}_{=-1/2} - \underbrace{\cos(-5\pi/6)}_{=-\sqrt{3}/2} - j\underbrace{\sin(-5\pi/6)}_{=-1/2}\big) \\
&= \overline{E}_{ab}.
\end{aligned} \tag{5.2}$$

A similar calculation yields,

$$\begin{aligned}
\overline{V}_{bc} &= \overline{E}_{bc}, \\
\overline{V}_{bc} &= \overline{E}_{bc}.
\end{aligned} \tag{5.3}$$

Now, we can use the method described earlier to analyze the system in Fig. 5.3 (top), with the resulting values of the line currents for the source and the passive loads being identical to those of the corresponding elements in the system in Fig. 5.3 (bottom).

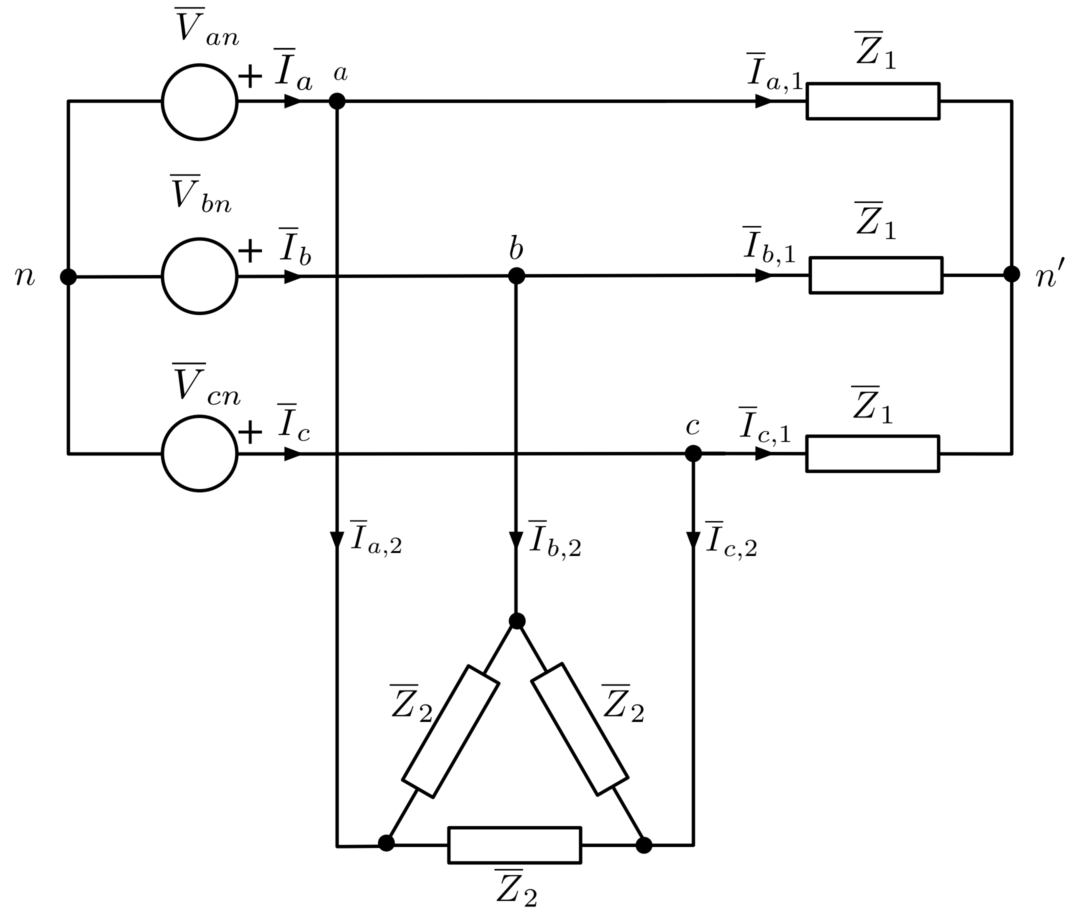

Figure 5.4: Three-phase circuit with one wye-connected source, one wye-connected passive load, and one delta-connected passive load.

Next, consider the system in Fig. 5.4, where the voltage source and one of the passive loads is connected in wye while the other passive load is connected in delta. Again, if we were able to transform the delta-connected passive load into a wye-connected passive load with identical terminal behavior, we would be able to utilize the technique developed earlier to analyze the resulting system with all three-phase elements connected in wye. More specifically, given the delta-connected and wye-connected passive loads in Fig. 5.5, both of which are excited with the same balanced three-phase set of line voltages, $\overline{V}_{ab}$, $\overline{V}_{bc}$, and $\overline{V}_{ca}$, we would like to determine the relation between $\overline{Z}_{\mathrm{Y}}$ and $\overline{Z}_{\Delta}$ so that the lines currents are identical, i.e.,

$\overline{I}_a = \overline{I}'_a$, $\overline{I}_b = \overline{I}'_b$, and $\overline{I}_c = \overline{I}'_c$. On the one hand, for the delta-connected load, we have that

$$\begin{aligned}\overline{I}_a &= \overline{I}_{ab} - \overline{I}_{ca} \\ &= \frac{\overline{V}_{ab}}{\overline{Z}_\Delta} - \frac{\overline{V}_{ca}}{\overline{Z}_\Delta} \\ &= \frac{\overline{V}_{ab} - \overline{V}_{ca}}{\overline{Z}_\Delta}. \end{aligned} \tag{5.4}$$

On the other hand, for the wye-connected load, we have that

$$\overline{V}_{ab} = \overline{Z}_\mathrm{Y}\overline{I}'_a - \overline{Z}_\mathrm{Y}\overline{I}'_b, \tag{5.5}$$

$$\overline{V}_{ca} = \overline{Z}_\mathrm{Y}\overline{I}'_c - \overline{Z}_\mathrm{Y}\overline{I}'_a, \tag{5.6}$$

and by subtracting the second equation from the first one, we obtain

$$\begin{aligned}\overline{V}_{ab} - \overline{V}_{ca} &= 2\overline{Z}_\mathrm{Y}\overline{I}'_a - \overline{Z}_\mathrm{Y}\underbrace{\left(\overline{I}'_b + \overline{I}'_c\right)}_{=-\overline{I}'_a} \\ &= 3\overline{Z}_\mathrm{Y}\overline{I}'_a; \end{aligned} \tag{5.7}$$

thus,

$$\overline{I}'_a = \frac{\overline{V}_{ab} - \overline{V}_{ca}}{3\overline{Z}_\mathrm{Y}}. \tag{5.8}$$

Then, by using (5.4) and (5.8), one has that $\overline{I}_a = \overline{I}'_a$ if and only if

$$\overline{Z}_\mathrm{Y} = \frac{1}{3}\overline{Z}_\Delta. \tag{5.9}$$

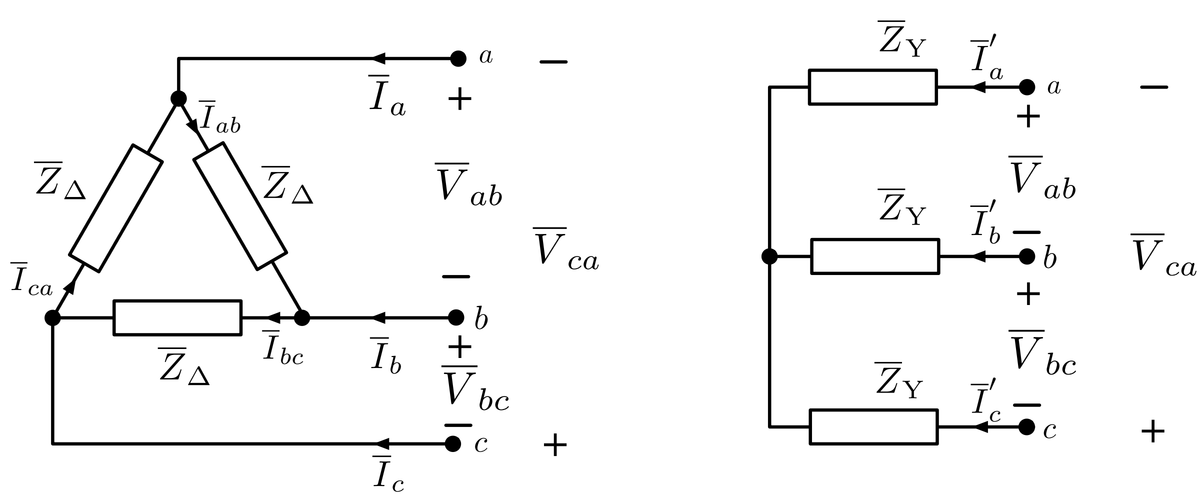

Figure 5.5: Equivalence between and wye-connected and delta-connected passive loads.

Conservation of Power Solution Approach

As we showed earlier, the complex power conservation approach is very powerful when analyzing AC circuits and in many instances, it is not necessary to use phasor diagrams. Here, we showcase how to use such approach for soling balanced three-phase systems. Consider a three-phased balanced system with line voltage $V_L = 480$ V and three loads as follows:

- Load # 1. Wye-connected load: 100 KVA (3ϕ), at 0.9 PF lagging.
- Load # 2. Wye-connected load: 60 KW (3ϕ), 0.7 PF leading.
- Load # 3. Delta-connected load: 75 A (phase current), 0.9 PF lagging.

a) Total Complex Power (3ϕ)

- **Load # 1:**
  $\overline{S}_1 = 100\big(0.9 + j\sin(\cos^{-1}(0.9)\big) = 90 + j43.6$
- **Load # 2:**
  $\overline{S}_2 = 60 - j61.21$
- **Load # 3:**
  $I_\phi = 75$ A, $I_L = \sqrt{3} \cdot 75 = 129.9$ A,
  $S_3 = \sqrt{3} \cdot V_L \cdot I_L = \sqrt{3} \cdot 480 \cdot 129.9 = 108$ KVA,
  $\overline{S}_3 = 108\big(0.9 + j\sin(\cos^{-1}(0.9)\big) = 97.2 + j47.06$
- **Total Complex Power:**
  $\overline{S}_T = \overline{S}_1 + \overline{S}_2 + \overline{S}_3 = (90 + 60 + 97.2) + j(43.6 - 61.21 + 47.06)$
  $= 247.2 + j29.45$
  $\boxed{S_T = 248.94\ KVA}$

b) Line Current

- $I_L = \frac{S_T}{\sqrt{3}V_L} = \frac{248.94 \cdot 10^3}{\sqrt{3} \cdot 480} = 299.43A \approx 300$ A

c) Per-Phase Reactive Power Needed for Unity PF

- $Q_T = +29.45 \rightarrow$ need to add $Q_C = -29.45 \rightarrow \frac{Q_C}{3} = -9.82$ KVAr to be added per phase.

d) New Line Current

- $I'_L = \frac{S'_T}{\sqrt{3}V_L} = \frac{247.2 \cdot 10^3}{\sqrt{3} \cdot 480} = 297.33$ A

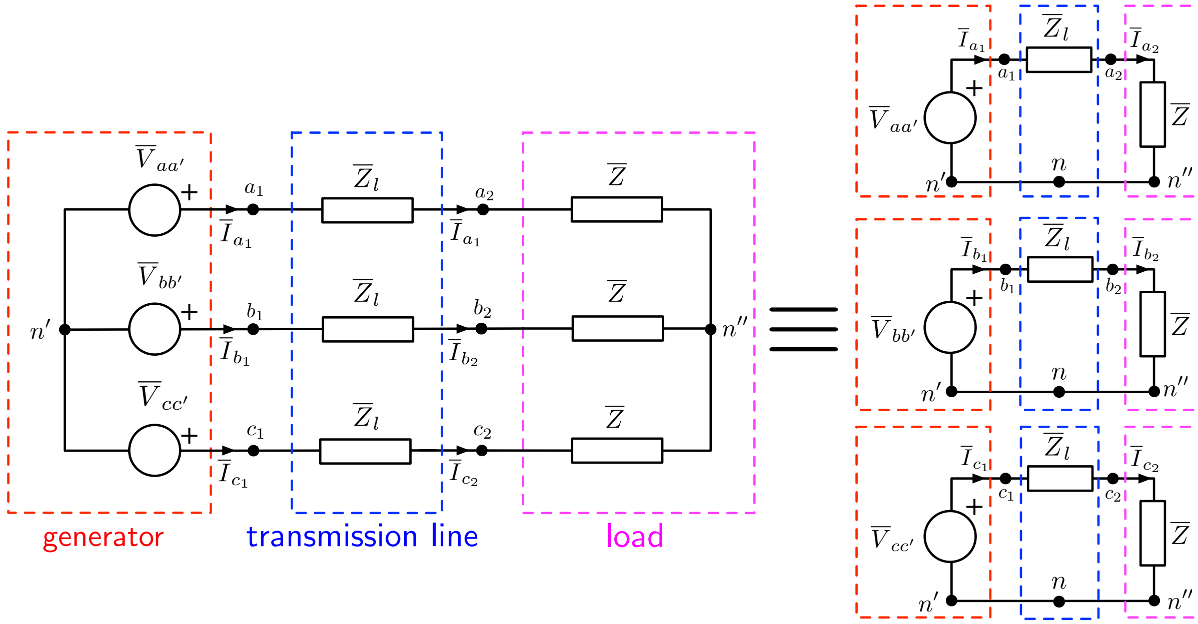

Figure 5.6: Three-phase power system model comprising a generator modeled as three voltage sources connected in wye, a transmission line modeled as three impedances (one per phase), and a load modeled as three impedances connected in wye.

Transmission Line Modeling

To motivate the problem, consider the three-phase linear circuit in Fig. 5.6 (left), where $\overline{V}_{aa'} = Ve^{j0}$, $\overline{V}_{bb'} = Ve^{-j2\pi/3}$, $\overline{V}_{cc'} = Ve^{j2\pi/3}$, i.e., the circuit is balanced. One can think of such circuit as a simple three-phase power system, where the voltage sources on the left connected in wye describe the electrical behavior of a generator, the three impedances in the middle describe the electrical behavior of a transmission line, and the three impedances on the right connected in wye describe the electrical behavior of a load. As discussed earlier, because the system is balanced, we can equivalently describe this three-phase system model by the three "per-phase" equivalent circuits in Fig. 5.6 (right). By inspecting these equivalent circuits, one notices that each of them is comprised by individual "per-phase" equivalent elements each of which corresponds to a three-phase element of the original circuit. For example, the voltage sources of the per-phase equivalent circuits correspond to the voltages sources comprising the three-phase voltage source in the original three-phase circuit. Similarly, the impedances on the right of the equivalent circuits corresponds to the impedances comprising the passive load in the original three-phase circuit. Finally, the impedance in the middle of the equivalent circuits correspond to the series impedances of the transmission line model in the original three-phase circuit. Thus, the transmission line per-phase equivalent models are described by the three-terminal networks in Fig. 5.7.

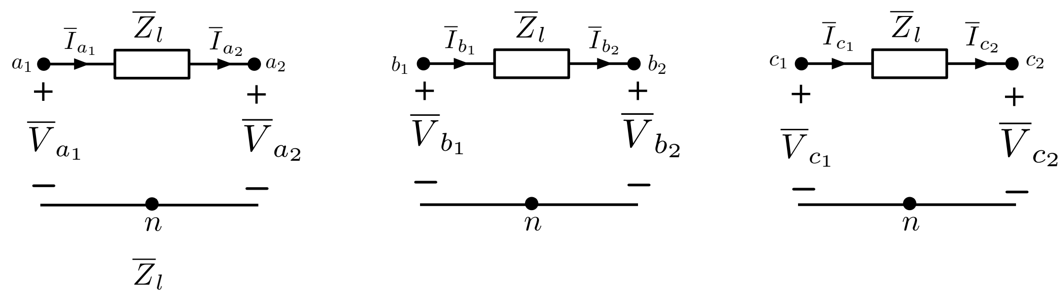

Figure 5.7: Three-terminal circuit models equivalent to the three-phase transmission line model in Fig. 5.6 (left).

While in some instances the circuit model in Fig. 5.6 (left) might accurately capture the behavior of an actual three-phase power system, in general, the models describing the elements comprising are more sophisticated that those shown in Fig. 5.6. However, we can still model a power system model as a collection of interconnected multi-terminal elements, each of which corresponds to an actual component of the system; see Fig. 5.8 (left). For example, a three-phase generator can be described by a four-terminal element. Similarly, a three-phase load, can be described as a four-terminal element. Finally, a three-phase transmission line can be described by a seven-terminal element. Now we assume that: i) the system is balanced, ii) loads and generators are connected in wye (otherwise, we can always find an equivalent wye-connected representation), and iii) there is no coupling between the phases. Then, we can equivalently describe the three-phase system in Fig. 5.8 (left) by the three (decoupled) per-phase equivalent models in Fig. 5.8 (right). The goal here is to fill in the details of what is inside the boxes in Fig. 5.8 (right). We start with those labelled as transmission line per-phase equivalent model.

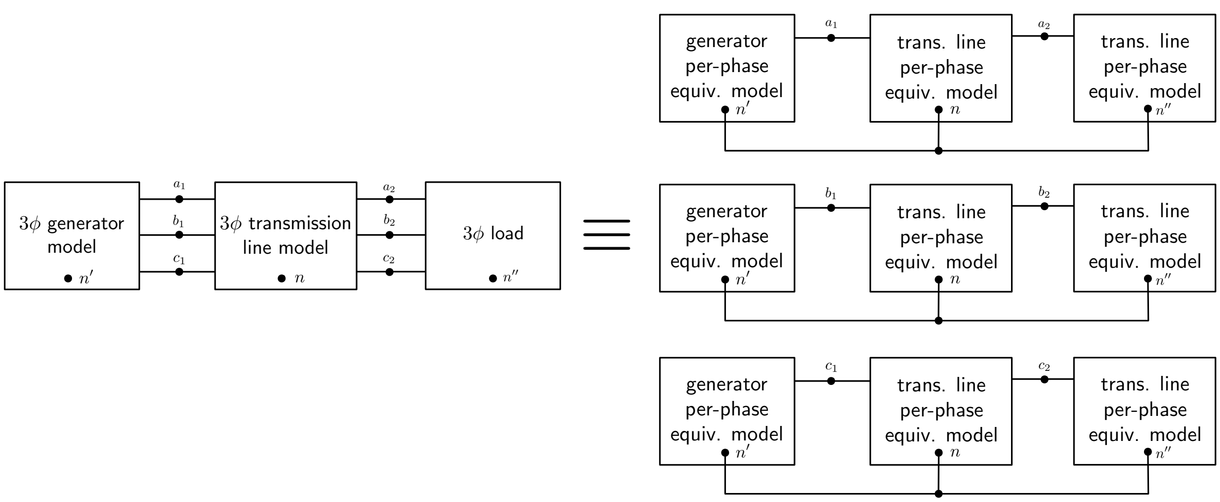

Figure 5.8: Three-phase power system model comprising a generator, a transmission line, and a load described by generic four-terminal circuit models.

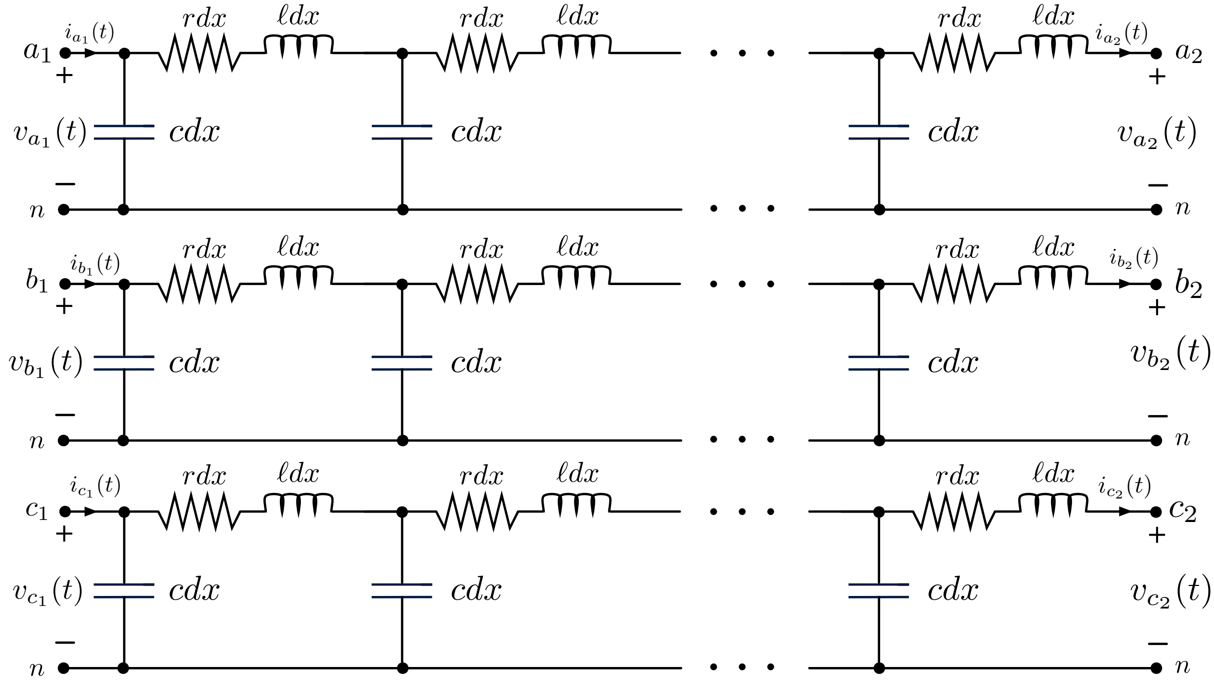

Figure 5.9: Per-phase distributed equivalent circuit models for a balanced three-phase transmission with no coupling between phases.

A transmission line can be hundreds of miles in length. Thus, in order to capture its electrical behavior at any point of its length, we need to use a distributed parameter circuit model (as opposed to standard lumped parameter circuit models). To this end, consider the per-phase equivalent circuit models in Fig. 5.9 where r [Ω/m] denotes the per-unit length resistance of the wire, ℓ [H/m] denotes the per-unit length inductance, and c [F/m] denotes the per-unit length capacitance, and dx denotes the differential length element. The value of r depends on the conductivity of the wire material and the number of wires per phase (there could be more than one in what is called conductor bundling). The value of ℓ depends on the permeability of the material of the wire, the number of wires per phase, the separation between phases, and the spatial arrangement of the phases. The value of c depends on the number of wires per phase and the separation between phases, and the spatial arrangement of the phases. Note that in the model in Fig. 5.9, we are making the assumption that phases are decoupled (this is consistent with the earlier discussion), i.e., there is no inductive coupling between phases and we are also not considering the capacitance between the different phases but the capacitance between the wire and the neutral terminal. In practice, there is both inductive and capacitive coupling between phases. However, by changing the physical position of the wires along the length of a transmission line, the average per-unit length inductance and capacitance of all the phases ends up being the same; such procedure is referred to as line transposition. Thus, by assuming a line is transposed along its length, we can effectively decouple the phases and describe its electrical behavior by the per-phase equivalent circuit models in Fig. 5.9. Since all three are identical, in the remainder, we just focus on analyzing one of them. Thus, in order to simplify notation, we will relabel all variables as shown in Fig. 5.10, i.e., instead of using $v_{a_1}(t)$, $i_{a_1}(t)$, etc, we use $v_1(t)$, $i_1(t)$, etc., to denote the corresponding generic phase variables. We will often refer to the port formed by terminals 1 and n on the left as the transmission line sending end, whereas the port formed by terminals 2 and n is referred to as the receiving end of the line.

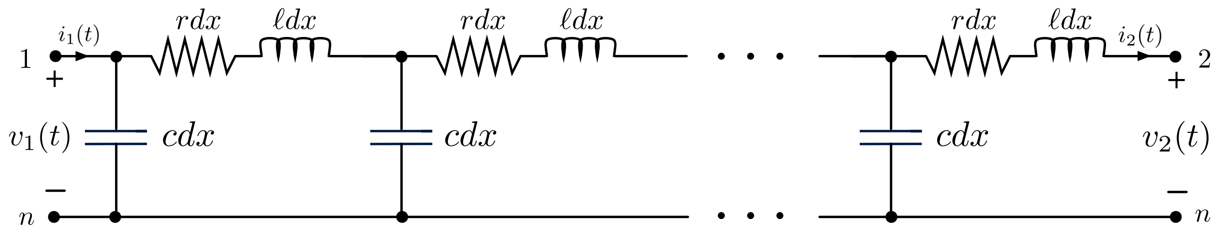

Figure 5.10: Generic per-phase distributed equivalent circuit model for a balanced three-phase transmission with no coupling between phases.

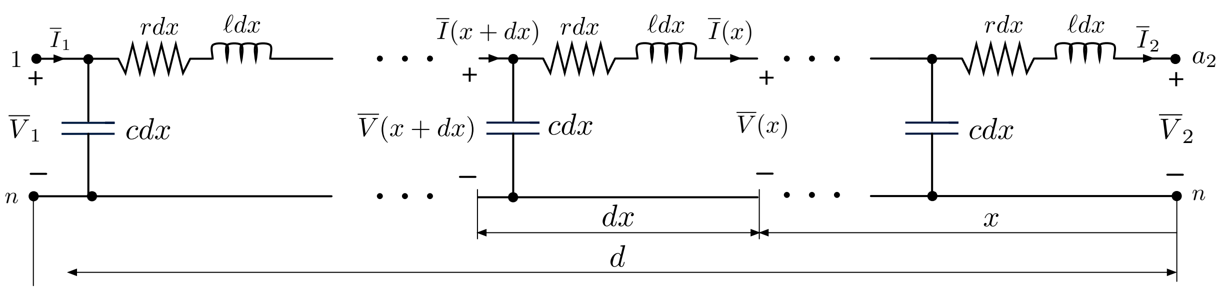

Figure 5.11: Transmission line section of differential length located at x m from the receiving end of the transmission line.

Derivation of Voltage-Current Relations

We will assume sinusoidal regime, i.e., $v_1(t)$, $v_2(t)$, $i_1(t)$, and $i_2(t)$ are sinuous with frequency ω, and will denote their associated phasors by $\overline{V}_1$, $\overline{V}_2$, $\overline{I}_1$, and $\overline{I}_2$, respectively. Thus, the voltage and current at any point of the transmission line located at a distance x of the line receiving end are also sinusoids with frequency ω, and will denote their corresponding phasors by $\overline{V}(x)$ and $\overline{I}(x)$, respectively. Now, consider a section of the transmission line of differential length, dx, as shown in Fig. 5.11 (the right side of the section is located at a distance x of the line receiving end). Then, by using KCL, it follows that

$$\begin{aligned}\overline{I}(x+dx) &= \overline{I}(x) + d\overline{I}(x)\\ &= \overline{I}(x) + j\omega c\, dx\big(\overline{V}(x) + d\overline{V}(x)\big)\\ &= \overline{I}(x) + j\omega c\, dx\overline{V}(x) + j\omega c\, \underbrace{dx d\overline{V}(x)}_{<<dx\,\overline{V}(x)}\\ &\approx \overline{I}(x) + j\omega c\, dx\overline{V}(x), \end{aligned} \tag{5.10}$$

which yields the following first-order ordinary differential equation (ODE)

$$\frac{d\overline{I}(x)}{dx} = \overline{y}\,\overline{V}(x), \tag{5.11}$$

where $\overline{y} = j\omega c$ is referred to as the per-unit length admittance of the transmission line. Similarly, by using KVL, it follows that

$$\begin{aligned}\overline{V}(x+dx) &= \overline{V}(x) + d\overline{V}(x)\\ &= \overline{V}(x) + (r + j\omega\ell)\, dx\overline{I}(x), \end{aligned} \tag{5.12}$$

which yields the following first-order ODE

$$\frac{d\overline{V}(x)}{dx} = \overline{z}\overline{I}(x), \tag{5.13}$$

where $\overline{z} = r + j\omega\ell$ is referred to as the per-unit length impedance of the transmission line.

One can now check that the solution of (5.11) and (5.13) is of the form

$$\begin{aligned}\overline{V}(x) &= c_1 e^{\gamma x} + c_2 e^{-\gamma x},\\ \overline{I}(x) &= c_3 e^{\gamma x} + c_4 e^{-\gamma x}, \end{aligned} \tag{5.14}$$

where $\gamma = \sqrt{\overline{z}\,\overline{y}}$ is referred to as the *propagation constant*, and c_1, c_2, c_3, c_4 are constants. To determine the value of c_1, c_2, c_3, c_4, we use the fact that $\overline{V}(0) = \overline{V}_2$ and and $\overline{I}(0) = \overline{I}_2$; then, by using (5.14), we have that

$$\begin{aligned}\overline{V}_2 &= c_1 + c_2,\\ \overline{I}_2 &= c_3 + c_4. \end{aligned} \tag{5.15}$$

Also note that

$$\begin{aligned}\frac{d\overline{I}(0)}{dx} &= \overline{y}\,\overline{V}_2,\\ \frac{d\overline{V}(0)}{dx} &= \overline{z}\overline{I}_2; \end{aligned} \tag{5.16}$$

then, by using (5.14), we obtain that

$$\begin{aligned}\gamma(c_3 - c_4) &= \overline{y}(c_1 + c_2),\\ \gamma(c_1 - c_2) &= \overline{z}(c_3 + c_4).\end{aligned} \tag{5.17}$$

Then, by using (5.15) and (5.17), we can solve for c_1, c_2, c_3, and c_4, and by plugging the result into (5.14), we obtain that

$$\begin{aligned}\overline{V}(x) &= \left(\frac{e^{\gamma x} + e^{-\gamma x}}{2}\right)\overline{V}_2 + \sqrt{\frac{\overline{z}}{\overline{y}}}\left(\frac{e^{\gamma x} - e^{-\gamma x}}{2}\right)\overline{I}_2,\\ \overline{I}(x) &= \sqrt{\frac{\overline{y}}{\overline{z}}}\left(\frac{e^{\gamma x} - e^{-\gamma x}}{2}\right)\overline{V}_2 + \left(\frac{e^{\gamma x} + e^{-\gamma x}}{2}\right)\overline{I}_2.\end{aligned} \tag{5.18}$$

Now define

$$\overline{Z}_c = \sqrt{\frac{\overline{z}}{\overline{y}}}, \tag{5.19}$$

which we refer to as the transmission line *characteristic impedance*. Then, by noting that

$$\begin{aligned}\cosh(\gamma x) &= \frac{e^{\gamma x} + e^{-\gamma x}}{2},\\ \sinh(\gamma x) &= \frac{e^{\gamma x} - e^{-\gamma x}}{2},\end{aligned} \tag{5.20}$$

we can rewrite (5.18) more compactly as follows:

$$\begin{aligned}\overline{V}(x) &= \cosh(\gamma x)\overline{V}_2 + \overline{Z}_c \sinh(\gamma x)\overline{I}_2,\\ \overline{I}(x) &= \frac{1}{\overline{Z}_c}\sinh(\gamma x)\overline{V}_2 + \cosh(\gamma x)\overline{I}_2.\end{aligned} \tag{5.21}$$

Finally, by noting that $\overline{V}(d) = \overline{V}_1$ and $\overline{I}(d) = \overline{I}_1$, where d denotes the length of the transmission line, we can use (5.21) to obtain the relation between the line end voltages and currents, $\overline{V}_1$, $\overline{I}_1$, $\overline{V}_2$, $\overline{I}_2$, which yields

$$\begin{aligned}\overline{V}_1 &= \cosh(\gamma d)\overline{V}_2 + \overline{Z}_c \sinh(\gamma d)\overline{I}_2,\\ \overline{I}_1 &= \frac{1}{\overline{Z}_c}\sinh(\gamma d)\overline{V}_2 + \cosh(\gamma d)\overline{I}_2.\end{aligned} \tag{5.22}$$