ECE 476: Power System Analysis — Fall 2020

Lecture 6: September 10

Alejandro D. Domínguez-García

- TRANSMISSION LINE MODELING (CONTINUED)

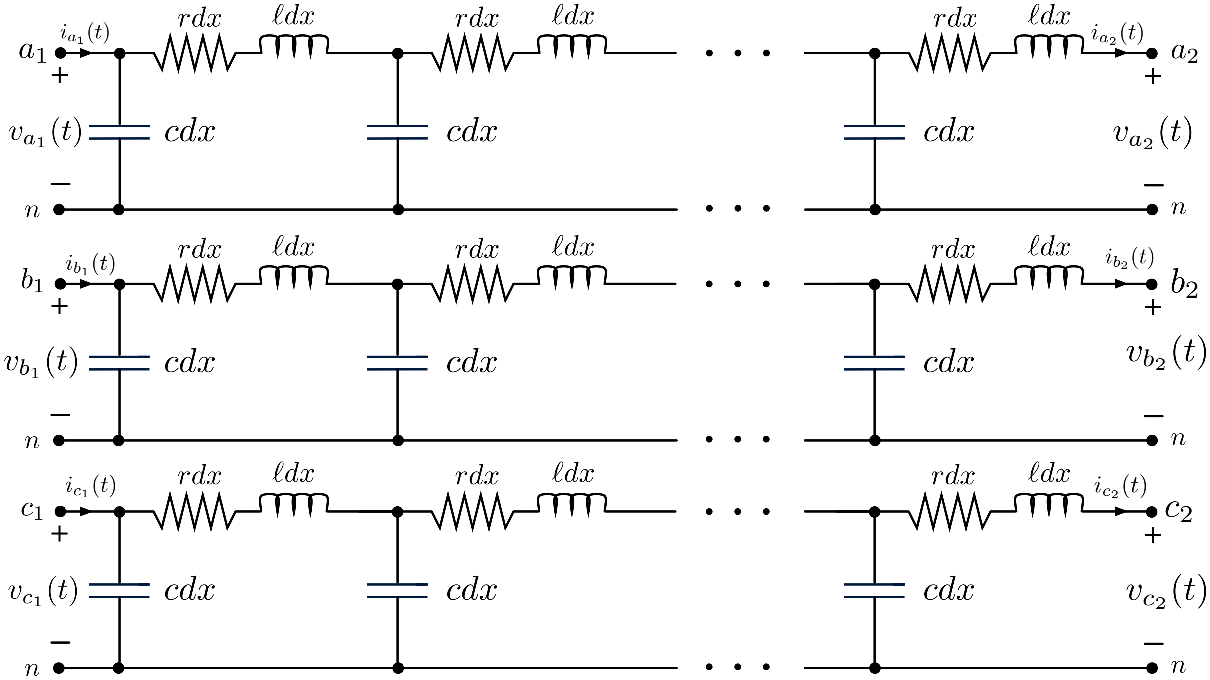

Figure 6.1: Per-phase distributed equivalent circuit models for a balanced three-phase transmission with no coupling between phases.

Transmission Line Modeling (CONTINUED)

A transmission line can be hundreds of miles in length. Thus, in order to capture its electrical behavior at any point of its length, we need to use a distributed parameter circuit model (as opposed to standard lumped parameter circuit models). To this end, consider the per-phase equivalent circuit models in Fig. 6.1 where r [Ω/m] denotes the per-unit length resistance of the wire, ℓ [H/m] denotes the per-unit length inductance, and c [F/m] denotes the per-unit length capacitance, and dx denotes the differential length element. The value of r depends on the conductivity of the wire material and the number of wires per phase (there could be more than one in what is called conductor bundling). The value of ℓ depends on the permeability of the material of the wire, the number of wires per phase, the separation between phases, and the spatial arrangement of the phases. The value of c depends on the number of wires per phase and the separation between phases, and the spatial arrangement of the phases. Note that in the model in Fig. 6.1, we are making the assumption that phases are decoupled (this is consistent with the earlier discussion), i.e., there is no inductive coupling between phases and we are also not considering the capacitance between the different phases but the capacitance between the wire and the neutral terminal. In practice, there is both inductive and capacitive coupling between phases. However, by changing the physical position of the wires along the length of a transmission line, the average per-unit length inductance and capacitance of all the phases ends up being the same; such procedure is referred to as line transposition. Thus, by assuming a line is transposed along its length, we can effectively decouple the phases and describe its electrical behavior by the per-phase equivalent circuit models in Fig. 6.1. Since all three are identical, in the remainder, we just focus on analyzing one of them. Thus, in order to simplify notation, we will relabel all variables as shown in Fig. 6.2, i.e., instead of using $v_{a_1}(t)$, $i_{a_1}(t)$, etc., we use $v_1(t)$, $i_1(t)$, etc., to denote the corresponding generic phase variables. We will often refer to the port formed by terminals 1 and n on the left as the transmission line sending end, whereas the port formed by terminals 2 and n is referred to as the receiving end of the line.

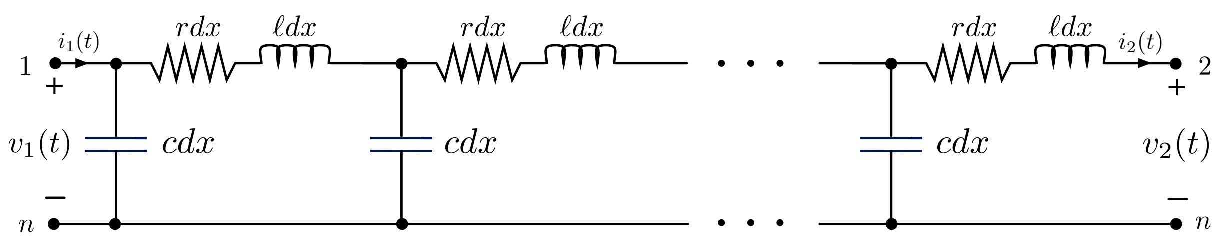

Figure 6.2: Generic per-phase distributed equivalent circuit model for a balanced three-phase transmission with no coupling between phases.

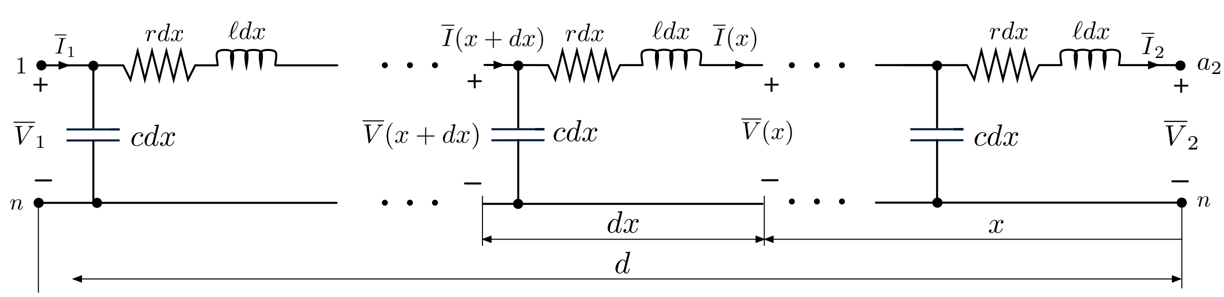

Figure 6.3: Transmission line section of differential length located at x m from the receiving end of the transmission line.

Derivation of Voltage-Current Terminal Relations

We will assume sinusoidal regime, i.e., $v_1(t)$, $v_2(t)$, $i_1(t)$, and $i_2(t)$ are sinuous with frequency ω, and will denote their associated phasors by $\overline{V}_1$, $\overline{V}_2$, $\overline{I}_1$, and $\overline{I}_2$, respectively. Thus, the voltage and current at any point of the transmission line located at a distance x of the line receiving end are also sinusoids with frequency ω, and will denote their corresponding phasors by $\overline{V}(x)$ and $\overline{I}(x)$, respectively. Now, consider a section of the transmission line of differential length, dx, as shown in Fig. 6.3 (the right side of the section is located at a distance x of the line receiving end). Then, by using KCL, it follows that

$$\begin{aligned}\overline{I}(x+dx) &= \overline{I}(x) + d\overline{I}(x)\\ &= \overline{I}(x) + j\omega c\, dx\big(\overline{V}(x) + d\overline{V}(x)\big)\\ &= \overline{I}(x) + j\omega c\, dx\overline{V}(x) + j\omega c\, \underbrace{dx d\overline{V}(x)}_{<<dx\,\overline{V}(x)}\\ &\approx \overline{I}(x) + j\omega c\, dx\overline{V}(x),\end{aligned} \tag{6.1}$$

which yields the following first-order ordinary differential equation (ODE)

$$\frac{d\overline{I}(x)}{dx} = \overline{y}\,\overline{V}(x), \tag{6.2}$$

where $\overline{y} = j\omega c$ is referred to as the per-unit length admittance of the transmission line. Similarly, by using KVL, it follows that

$$\begin{aligned}\overline{V}(x+dx) &= \overline{V}(x) + d\overline{V}(x)\\ &= \overline{V}(x) + (r + j\omega\ell)\, dx\overline{I}(x),\end{aligned} \tag{6.3}$$

which yields the following first-order ODE

$$\frac{d\overline{V}(x)}{dx} = \overline{z}\overline{I}(x), \tag{6.4}$$

where $\overline{z} = r + j\omega\ell$ is referred to as the per-unit length impedance of the transmission line.

One can now check that the solution of (6.2) and (6.4) is of the form

$$\begin{aligned}\overline{V}(x) &= c_1 e^{\gamma x} + c_2 e^{-\gamma x},\\ \overline{I}(x) &= c_3 e^{\gamma x} + c_4 e^{-\gamma x},\end{aligned} \tag{6.5}$$

where $\gamma = \sqrt{\overline{z}\,\overline{y}}$ is referred to as the *propagation constant*, and c_1, c_2, c_3, c_4 are constants. To determine the value of c_1, c_2, c_3, c_4, we use the fact that $\overline{V}(0) = \overline{V}_2$ and and $\overline{I}(0) = \overline{I}_2$; then, by using (6.5), we have that

$$\begin{aligned}\overline{V}_2 &= c_1 + c_2,\\ \overline{I}_2 &= c_3 + c_4.\end{aligned} \tag{6.6}$$

Also note that

$$\frac{d\overline{I}(0)}{dx} = \overline{y}\,\overline{V}_2,$$
$$\frac{d\overline{V}(0)}{dx} = \overline{z}\overline{I}_2; \tag{6.7}$$

then, by using (6.5), we obtain that

$$\gamma(c_3 - c_4) = \overline{y}(c_1 + c_2),$$
$$\gamma(c_1 - c_2) = \overline{z}(c_3 + c_4). \tag{6.8}$$

Then, by using (6.6), (6.8), and the fact that $\gamma = \sqrt{\overline{z}\,\overline{y}}$, we can solve for c_1, c_2, c_3, and c_4, which yields

$$c_1 = \frac{1}{2}\left(\overline{V}_2 + \sqrt{\frac{\overline{z}}{\overline{y}}}\overline{I}_2\right),$$
$$c_2 = \frac{1}{2}\left(\overline{V}_2 - \sqrt{\frac{\overline{z}}{\overline{y}}}\overline{I}_2\right),$$
$$c_3 = \frac{1}{2}\left(\sqrt{\frac{\overline{y}}{\overline{z}}}\overline{V}_2 + \overline{I}_2\right),$$
$$c_4 = \frac{1}{2}\left(-\sqrt{\frac{\overline{y}}{\overline{z}}}\overline{V}_2 + \overline{I}_2\right), \tag{6.9}$$

and by plugging into (6.5), we obtain that

$$\overline{V}(x) = \left(\frac{e^{\gamma x} + e^{-\gamma x}}{2}\right)\overline{V}_2 + \sqrt{\frac{\overline{z}}{\overline{y}}}\left(\frac{e^{\gamma x} - e^{-\gamma x}}{2}\right)\overline{I}_2,$$
$$\overline{I}(x) = \sqrt{\frac{\overline{y}}{\overline{z}}}\left(\frac{e^{\gamma x} - e^{-\gamma x}}{2}\right)\overline{V}_2 + \left(\frac{e^{\gamma x} + e^{-\gamma x}}{2}\right)\overline{I}_2. \tag{6.10}$$

Now define

$$\overline{Z}_c = \sqrt{\frac{\overline{z}}{\overline{y}}}, \tag{6.11}$$

which we refer to as the transmission line *characteristic impedance*. Then, by noting that

$$\cosh(\gamma x) = \frac{e^{\gamma x} + e^{-\gamma x}}{2},$$
$$\sinh(\gamma x) = \frac{e^{\gamma x} - e^{-\gamma x}}{2}, \tag{6.12}$$

we can rewrite (6.10) more compactly as follows:

$$\begin{aligned} \overline{V}(x) &= \cosh(\gamma x)\overline{V}_2 + \overline{Z}_c \sinh(\gamma x)\overline{I}_2, \\ \overline{I}(x) &= \frac{1}{\overline{Z}_c}\sinh(\gamma x)\overline{V}_2 + \cosh(\gamma x)\overline{I}_2. \end{aligned} \tag{6.13}$$

Finally, by noting that $\overline{V}(d) = \overline{V}_1$ and $\overline{I}(d) = \overline{I}_1$, where d denotes the length of the transmission line, we can use (6.13) to obtain the relation between the line end voltages and currents, $\overline{V}_1$, $\overline{I}_1$, $\overline{V}_2$, $\overline{I}_2$, which yields

$$\begin{aligned} \overline{V}_1 &= \cosh(\gamma d)\overline{V}_2 + \overline{Z}_c \sinh(\gamma d)\overline{I}_2, \\ \overline{I}_1 &= \frac{1}{\overline{Z}_c}\sinh(\gamma d)\overline{V}_2 + \cosh(\gamma d)\overline{I}_2. \end{aligned} \tag{6.14}$$

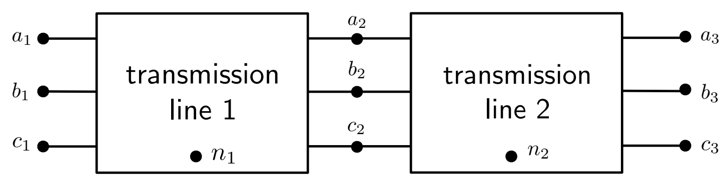

Figure 6.4: Two three-phase transmission lines connected in series.

Transmission Matrix

The voltage-current terminal voltage relations in (6.14) can be rewritten in matrix form as follows:

$$\begin{bmatrix} \overline{V}_1 \\ \overline{I}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{=:T} \begin{bmatrix} \overline{V}_2 \\ \overline{I}_2 \end{bmatrix} \tag{6.15}$$

where

$$\begin{aligned} A &= \cosh(\gamma d), \\ B &= \overline{Z}_c \sinh(\gamma d), \\ C &= \frac{1}{\overline{Z}_c}\sinh(\gamma d), \\ D &= \cosh(\gamma d), \end{aligned} \tag{6.16}$$

are referred to as the transmission parameters and the matrix T is referred to as transmission matrix. The matrix description above is useful for describing the terminal behavior of two or more transmission lines connected in parallel. For example, consider the setting in Fig. 6.4 showing two-three phase transmission lines, labelled as transmission line 1 and transmission line 2, connected in series with known transmission matrices, denoted by T_1 and T_2, respectively. Then, we have that

$$\begin{bmatrix}\overline{V}_1\\ \overline{I}_1\end{bmatrix} = T_1 \begin{bmatrix}\overline{V}_2\\ \overline{I}_2\end{bmatrix},$$
$$\begin{bmatrix}\overline{V}_2\\ \overline{I}_2\end{bmatrix} = T_2 \begin{bmatrix}\overline{V}_3\\ \overline{I}_3\end{bmatrix}. \tag{6.17}$$

Then, if we want to find the relations between the two ends of the series arrangement, i.e., the relations between the vectors $\left[\overline{V}_1, \overline{I}_1\right]^\top$, and $\left[\overline{V}_3 \overline{I}_3\right]^\top$, we just need to multiply the transmission matrices as follows:

$$\begin{bmatrix}\overline{V}_1\\ \overline{I}_1\end{bmatrix} = T_1 T_2 \begin{bmatrix}\overline{V}_3\\ \overline{I}_3\end{bmatrix}. \tag{6.18}$$

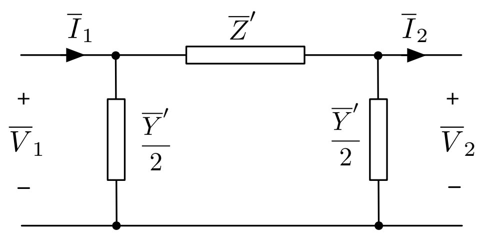

Figure 6.5: Transmission line π-equivalent circuit model.

Lumped-Parameter Circuit Model

Consider the voltage-current terminal relations of a transmission line given in (6.14), i.e.,

$$\overline{V}_1 = \cosh(\gamma d)\overline{V}_2 + \overline{Z}_c \sinh(\gamma d)\overline{I}_2, \tag{6.19}$$
$$\overline{I}_1 = \frac{1}{\overline{Z}_c}\sinh(\gamma d)\overline{V}_2 + \cosh(\gamma d)\overline{I}_2. \tag{6.20}$$

Here, we would like to find a lumped-parameter circuit model whose electrical behavior matches that of (6.19) – (6.20). Such circuit model is useful, because as we will see later, it allows one to describe the network of a power system as a linear circuit and use well-known circuit techniques such as nodal analysis. While there are several choices for constructing a circuit whose electrical behavior matches that prescribed by (6.19) – (6.20), a convenient choice is the one shown in Fig. 6.5, referred to as the π-equivalent circuit model. The objective now is to find the values of $\overline{Z}'$ and $\overline{Y}'$ so that the voltage-current terminal relations of the π-equivalent circuit model in Fig. 6.5 match the expressions in (6.19) – (6.20). First, by using KVL, we obtain that

$$\begin{aligned}\overline{V}_1 &= \overline{V}_2 + \overline{Z}'\left(\overline{I}_2 + \frac{\overline{Y}'}{2}\overline{V}_2\right) \\ &= \left(1 + \overline{Z}'\frac{\overline{Y}'}{2}\right)\overline{V}_2 + \overline{Z}'\overline{I}. \end{aligned} \tag{6.21}$$

Now by matching the second term on the right-hand side of (6.19) to the second term on the right-hand side of (6.21), it follows that

$$\overline{Z}' = \overline{Z}_c \sinh(\gamma d), \tag{6.22}$$

$$\tag{6.23}$$

and by multiplying and diving the right-hand side by γd and taking into account that $\overline{Z}_c = \sqrt{\overline{z}/\overline{y}}$ and $\gamma = \sqrt{\overline{z}\,\overline{y}}$, we obtain that

$$\begin{aligned}\overline{Z}' &= \sqrt{\frac{\overline{z}}{\overline{y}}}\frac{\gamma d}{\gamma d}\sinh(\gamma d) \\ &= \overline{z}\,d\frac{\sinh(\gamma d)}{\gamma d};\end{aligned} \tag{6.24}$$

thus,

$$\overline{Z}' = \overline{Z}\,\frac{\sinh(\gamma d)}{\gamma d}, \tag{6.25}$$

where

$$\overline{Z} = \overline{z}\,d. \tag{6.26}$$

In words, the value of the series impedances in the lumped-parameter circuit model, $\overline{Z}'$, is the product of two terms, $\overline{Z}$ and $\frac{\sinh(\gamma d)}{\gamma d}$. The first one, $\overline{Z}$, is equal to the transmission line per-unit length impedance, $\overline{z}$ [Ω/m] multiplied by the length of the line, d [m]; thus, one can think of $\overline{Z}$ as an impedance. The second term, $\frac{\sinh(\gamma d)}{\gamma d}$, is dimensionless and one can think of it as a correction factor when computing $\overline{Z}'$ from $\overline{Z}$ that takes into account the length of the transmission line if one were to think of $\overline{Z}'$ as the "series" impedance of the transmission line.

Next, by matching the first term on the right-hand side of (6.19) to the first term on the right-hand side of (6.21) and taking into account the expression in (6.22), it follows that

$$\begin{aligned}\frac{\overline{Y}'}{2} &= \frac{1}{\overline{Z}'}\big(\cosh(\gamma d) - 1\big)\\ &= \frac{1}{\overline{Z}_c}\frac{\cosh(\gamma d) - 1}{\sinh(\gamma d)}\\ &= \frac{1}{\overline{Z}_c}\tanh\left(\frac{\gamma d}{2}\right),\end{aligned} \tag{6.27}$$

and by multiplying and dividing the right hand side by $\frac{\gamma d}{2}$ and taking into account that $\overline{Z}_c = \sqrt{\overline{z}/\overline{y}}$ and $\gamma = \sqrt{\overline{z}\,\overline{y}}$, we obtain that

$$\begin{aligned}\frac{\overline{Y}'}{2} &= \frac{\frac{\gamma d}{2}}{\overline{Z}_c}\,\frac{\tanh\left(\frac{\gamma d}{2}\right)}{\frac{\gamma d}{2}}\\ &= \frac{\sqrt{\overline{z}\,\overline{y}}\frac{d}{2}}{\sqrt{\frac{\overline{z}}{\overline{y}}}}\,\frac{\tanh\left(\gamma\frac{d}{2}\right)}{\frac{\gamma d}{2}}\\ &= \frac{\overline{y}\,d}{2}\,\frac{\tanh\left(\frac{\gamma d}{2}\right)}{\frac{\gamma d}{2}};\end{aligned} \tag{6.28}$$

thus,

$$\frac{\overline{Y}'}{2} = \frac{\overline{Y}}{2}\,\frac{\tanh\left(\frac{\gamma d}{2}\right)}{\frac{\gamma d}{2}}, \tag{6.29}$$

where

$$\frac{\overline{Y}}{2} = \frac{\overline{y}\,d}{2}. \tag{6.30}$$

In words, the value of the shunt admittance in the lumped-parameter circuit model, $\overline{Y}'/2$, is the product of two terms, $\overline{Y}/2$ and $\frac{\tanh(\gamma d/2)}{\gamma\frac{d}{2}}$. The first one, $\overline{Y}/2$, is equal to the transmission line per-unit length shunt admittance, $\overline{y}$ [Ω^{-1}/m] multiplied by half of the length of the transmission, $d/2$ [m]; thus, one can think of it as an admittance. The second term, $\frac{\tanh(\gamma d/2)}{\frac{\gamma d}{2}}$, is dimensionless and one can think of it as a correction factor when computing $\overline{Y}'/2$ from $\overline{Y}/2$ that takes into account the length of the transmission line if one were to think of $\overline{Y}/2$ as half of the "shunt" admittance of the transmission line computed the obvious way.

Simplified Models

Recall that the series impedance and shunt admittance of the π-equivalent circuit model, denoted by $\overline{Z}'$ and $\overline{Y}'/2$, respectively, are given by

$$\begin{aligned}\overline{Z}' &= \overline{Z}\,\frac{\sinh(\gamma d)}{\gamma d},\\ \frac{\overline{Y}'}{2} &= \frac{\overline{Y}}{2}\,\frac{\tanh\left(\frac{\gamma d}{2}\right)}{\frac{\gamma d}{2}}.\end{aligned} \tag{6.31}$$

These expressions are exact for any d but they can be simplified as follows. First note that

$$\begin{aligned}\sinh(\gamma d) &= \frac{e^{\gamma d} - e^{-\gamma d}}{2},\\ \tanh\left(\frac{\gamma d}{2}\right) &= \frac{e^{\gamma d/2} - e^{-\gamma d/2}}{e^{\gamma d/2} + e^{-\gamma d/2}};\end{aligned} \tag{6.32}$$

then, for γd small enough, we have that $e^{\gamma d} = 1 + \gamma d$, $e^{-\gamma d} = 1 - \gamma d$, $e^{\gamma d} = 1 + \gamma d/2$, and $e^{-\gamma d} = 1 - \gamma d/2$; thus,

$$\begin{aligned}\frac{\sinh(\gamma d)}{\gamma d} &\approx 1,\\ \frac{\tanh\left(\frac{\gamma d}{2}\right)}{\frac{\gamma d}{2}} &\approx 1,\end{aligned} \tag{6.33}$$

from where it follows that

$$\begin{aligned}\overline{Z}' &\approx \overline{Z}\\ &= \overline{z}d,\\ \frac{\overline{Y}'}{2} &\approx \frac{\overline{Y}}{2};\\ &= \frac{\overline{y}\,d}{2}.\end{aligned} \tag{6.34}$$

Thus, the series impedance and shunt admittance elements in the π-model can be approximated the obvious way, i.e., the series impedance element can be approximated as the product of the transmission line per-unit length series impedance and the transmission line length, whereas the shunt admittance elements can be approximated as the product of the transmission line per-unit length shunt admittance and the transmission line half lenght. For typical values of $\overline{z}$ and $\overline{y}$, such approximation is valid when $d < 200$ miles. Furthremore, when $d < 50$ miles, the shunt admittance element can be completely neglected. The results above are summarized in Table 6.1.

Table 6.1: Parameters of the π-equivalent circuit model for different transmission line lengths.

length d	$\overline{Z}'$	$\frac{\overline{Y}'}{2}$
$d > 200$ miles	$\overline{Z}\frac{\sinh(\gamma d)}{\gamma d}$	$\frac{\overline{Y}}{2}\frac{\tanh(\frac{\gamma d}{2})}{\frac{\gamma d}{2}}$
50 miles$<d<200$ miles	$\overline{Z}$	$\frac{\overline{Y}}{2}$
$d < 50$ miles	$\overline{Z}$	0

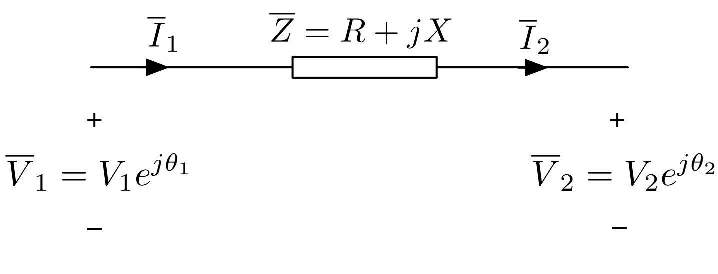

Figure 6.6:

Power Transfer Through a Short Transmission Line

Now we will study the amount of active and reactive power that can be transferred from the sending end to the receiving end of a short transmission line, i.e., $d < 50$ miles; see Fig. 6.6 for the resulting π-equivalent circuit model. To this end, first, write the impedance of the transmission line, $\overline{Z} = R + jX$, in polar coordinates as follows:

$$\overline{Z} = Ze^{j\phi}, \tag{6.35}$$

where $Z = \sqrt{R^2 + X^2}$ and $\phi = \arctan(X/R)$. Let $\overline{S}_1$ denote the complex power that enters the transmission line via its sending end, then, we have that:

$$\begin{aligned}
\overline{S}_1 &= \overline{V}_1 \overline{I}_1^* \\
&= \overline{V}_1 \frac{\overline{V}_1^* - \overline{V}_2^*}{\overline{Z}^*} \\
&= V_1 e^{j\theta_1} \frac{V_1 e^{-j\theta_1} - V_2 e^{-j\theta_2}}{Z^{-j\phi}} \\
&= \frac{V_1^2}{Z} e^{j\phi} - \frac{V_1 V_2}{Z} e^{j(\theta_1 - \theta_2 + \phi)}.
\end{aligned} \tag{6.36}$$

Thus, the active and reactive power entering the transmission line via its sending end, denoted by P_1 and Q_1, respectively, are given by

$$\begin{aligned} P_1 &= \frac{V_1^2}{Z}\cos(\phi) - \frac{V_1 V_2}{Z}\cos(\theta_1 - \theta_2 + \phi), \\ Q_1 &= \frac{V_1^2}{Z}\sin(\phi) - \frac{V_1 V_2}{Z}\sin(\theta_1 - \theta_2 + \phi). \end{aligned} \tag{6.37}$$

Let $\overline{S}_2$ denote the complex power that enters the the transmission line via its receiving end; this can be computed as follows:

$$\begin{aligned} \overline{S}_2 &= -\overline{V}_2 \overline{I}_2^* \\ &= -\overline{V}_2 \frac{\overline{V}_1^* - \overline{V}_2^*}{\overline{Z}^*} \\ &= -V_2 e^{j\theta_2} \frac{V_1 e^{-j\theta_1} - V_2 e^{-j\theta_2}}{Z^{-j\phi}} \\ &= \frac{V_2^2}{Z} e^{j\phi} - \frac{V_1 V_2}{Z} e^{j(\theta_2 - \theta_1 + \phi)}. \end{aligned} \tag{6.38}$$

Thus, the active and reactive power entering the transmission line via its receiving end, denoted by P_2 and Q_2, respectively, are given by

$$\begin{aligned} P_2 &= \frac{V_2^2}{Z}\cos(\phi) - \frac{V_1 V_2}{Z}\cos(\theta_2 - \theta_1 + \phi), \\ Q_2 &= \frac{V_2^2}{Z}\sin(\phi) - \frac{V_1 V_2}{Z}\sin(\theta_2 - \theta_1 + \phi). \end{aligned} \tag{6.39}$$

Let $P_{\overline{Z}}$ and $Q_{\overline{Z}}$ respectively denote the active and reactive power consumed by the impedance $\overline{Z}$; then, since from the conservation of complex power principle studied earlier, we have that

$$\begin{aligned} P_{\overline{Z}} &= P_1 + P_2 \\ &= \frac{V_1^2 + V_2^2}{Z}\cos(\phi) - \frac{V_1 V_2}{Z}\big(\cos(\theta_2 - \theta_1 + \phi) + \cos(\theta_1 - \theta_2 + \phi)\big) \\ Q_{\overline{Z}} &= Q_1 + Q_2 \\ &= \frac{V_1^2 + V_2^2}{Z}\sin(\phi) - \frac{V_1 V_2}{Z}\big(\sin(\theta_2 - \theta_1 + \phi) + \sin(\theta_1 - \theta_2 + \phi)\big). \end{aligned} \tag{6.40}$$

If the line is lossless, i.e., $R = 0$, we have that $\overline{Z} = jX$; thus, $\phi = \pi/2$, therefore, the expressions in (6.37) simplify to

$$\begin{aligned}
P_1 &= -\frac{V_1V_2}{X}\cos(\theta_1-\theta_2+\pi/2)\\
&= \frac{V_1V_2}{X}\sin(\theta_1-\theta_2),\\
Q_1 &= \frac{V_1^2}{X}-\frac{V_1V_2}{X}\sin(\theta_1-\theta_2+\pi/2)\\
&= \frac{V_1^2}{X}-\frac{V_1V_2}{X}\cos(\theta_1-\theta_2). \qquad (6.41)
\end{aligned}$$

Similarly, the expressions in (6.39) simplify to

$$\begin{aligned}
P_2 &= \frac{V_1V_2}{X}\sin(\theta_2-\theta_1),\\
Q_2 &= \frac{V_2^2}{X}-\frac{V_1V_2}{X}\cos(\theta_2-\theta_1). \qquad (6.42)
\end{aligned}$$

Finally, the expressions in (6.40) simplify to

$$\begin{aligned}
P_{\overline{Z}} &= 0,\\
Q_{\overline{Z}} &= \frac{V_1^2+V_2^2}{X}-2\frac{V_1V_2}{X}\cos(\theta_1-\theta_2). \qquad (6.43)
\end{aligned}$$

The fact that the calculation yields $P_{\overline{Z}}=0$ is reassuring because we assumed the line to be lossless; therefore, it cannot dissipate any active power. The expression for $Q_{\overline{Z}}$ can also be easily obtained by noting that the reactive power consumed by $\overline{Z}=jX$ is given by

$$Q_{\overline{Z}} = \frac{V_{12}^2}{X}, \qquad (6.44)$$

where

$$\begin{aligned}
V_{12}^2 &= (\overline{V}_1-\overline{V}_2)(\overline{V}_1^*-\overline{V}_2^*)\\
&= V_1^2+V_2^2-V_1V_2e^{j(\theta_1-\theta_2)}-V_1V_2e^{j(\theta_2-\theta_1)}\\
&= V_1^2+V_2^2-2V_1V_2\cos(\theta_1-\theta_2). \qquad (6.45)
\end{aligned}$$

Lossless Transmission Lines and Surge Impedance Loading

When the ohmic losses are zero, we have that $\overline{z}=j\omega\ell$ and $\overline{y}=j\omega c$; then, the propagation constant is given by

$$\begin{aligned}
\gamma &= \sqrt{\overline{z}\,\overline{y}}\\
&= j\omega\sqrt{\ell c}, \qquad (6.46)
\end{aligned}$$

i.e., it is a purely imaginary number, and the characteristic impedance is given by

$$\begin{aligned}\overline{Z}_c &= \sqrt{\frac{\overline{z}}{\overline{y}}} \\ &= \sqrt{\frac{j\omega\ell}{j\omega c}} \\ &= \sqrt{\frac{\ell}{c}},\end{aligned} \tag{6.47}$$

i.e., it is a real number, and it is typically refer to as the *surge impedance*. Let $\alpha = \sqrt{\frac{\ell}{c}}$ and $\beta = \omega\sqrt{\ell c}$; then, the voltage-current terminal relations in (6.14) reduce to

$$\overline{V}_1 = \cos(\beta d)\overline{V}_2 + j\alpha\sin(\beta d)\overline{I}_2, \tag{6.48}$$

$$\overline{I}_1 = j\frac{1}{\alpha}\sin(\beta d)\overline{V}_2 + \cos(\beta d)\overline{I}_2. \tag{6.49}$$

Let us study now what happens when we load the receiving end of a transmission line with an impedance $\overline{Z}_l$ whose value is equal to that of the surge impedance, i.e., $\overline{Z}_l = \alpha$. Thus, in this case, the relation between $\overline{V}_2$ and $\overline{I}_2$ is given by

$$\begin{aligned}\overline{I}_2 &= \frac{\overline{V}_2}{\overline{Z}_l} \\ &= \frac{\overline{V}_2}{\alpha},\end{aligned} \tag{6.50}$$

which by plugging into (6.48) yields

$$\begin{aligned}\overline{V}_1 &= \cos(\beta d)\overline{V}_2 + j\sin(\beta d)\overline{V}_2 \\ &= \big(\cos(\beta d) + j\sin(\beta d)\big)\overline{V}_2,\end{aligned} \tag{6.51}$$

from where it follows that

$$\overline{V}_2 = e^{-j\beta d}\overline{V}_1; \tag{6.52}$$

therefore,

$$V_2 = V_1. \tag{6.53}$$

In words, if we load a lossless transmission line loaded with its surge impedance, then the magnitude of the voltage at the receiving end of the line, V_2, is equal to the magnitude of the voltage at the sending end of the transmission line, V_1. Furthermore because the load at the receiving end of the transmission line is purely resistive (recall that the surge impedance is a real number), the active and reactive power consumed by it, P_l and Q_l, respectively, are

$$\begin{aligned} P_l &= \frac{V_2^2}{\alpha} \\ &= \frac{V_1^2}{\alpha}, \\ Q_l &= 0. \end{aligned} \tag{6.54}$$

The relation between the current in the sending and receiving ends can be obtained by using (6.49) and (6.50) as follows:

$$\begin{aligned} \overline{I}_1 &= j\frac{1}{\alpha}\sin(\beta d)\overline{V}_2 + \cos(\beta d)\overline{I}_2 \\ &= j\sin(\beta d)\overline{I}_2 + \cos(\beta d)\overline{I}_2, \end{aligned} \tag{6.55}$$

from where it follows that

$$\overline{I}_2 = e^{-j\beta d}\overline{I}_1; \tag{6.56}$$

therefore,

$$I_2 = I_1. \tag{6.57}$$

Thus, the magnitudes of the current at both sending and receiving ends are identical.

Let $\overline{S}_l$ denote the complex power delivered to the load at the receiving end of the transmission line; then, we have that

$$\begin{aligned} \overline{S}_l &= \overline{V}_2\overline{I}_2^* \\ &= \overline{V}_1\overline{I}_1^*, \end{aligned} \tag{6.58}$$

where the last line follows from (6.52) and (6.56). Now let $\overline{S}_1$ denote the complex power that enters the transmission line via the sending end; thus,

$$\overline{S}_1 = \overline{V}_1\overline{I}_1^*. \tag{6.59}$$

Therefore, we have that

$$\overline{S}_1 = \overline{S}_l. \tag{6.60}$$

In words, the complex power consumed by the load is equal to the complex power entering the transmission line via its sending end. Furthermore, recall that the reactive power consumed by $\overline{Z}_l = \alpha$ is zero; therefore, $Q_1 = Q_l = 0$.