CMPSC 465
Data Structures and Algorithms
Spring 2022

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Linear Programming
(Textbook, Section 7.1)

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Please consider taking

CMPSC 497 — Quantum Computation in Fall 2022

if you are interested in learning **Quantum Computing**

Background

Optimization: we want to maximize some function $f(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$, subject to constraints

$$C(\mathbf{x}) \leq \mathbf{b}, \text{ for } \mathbf{b} \in \mathbb{R}^n$$

- If no structures of f or C are known: general purpose constraint optimization
- Given some restrictions on f or C, e.g., f is convex and C is a convex region: convex optimization
- Lots of stuff in between: quadratic programming, 2nd-order cone programming (SOCP), semidefinite programming (SDP)
- Simplest non-trivial (but still powerful) case: f and C are linear functions, e.g., $f(\mathbf{x}) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$
  — **Linear Programming**

Example

Resource allocation: 168 hours in a week

S: study time; P: fun/party time; E: everything else

- to survive: $E \geq 56$
- to to pass classes: $S \geq 60$
- to stay sane: $P + E \geq 70$
- $2S + E - 3P \geq 150$: need more study time if had too much fun or not enough sleep
- happiness: $2P + E$ **objective function**

  i.e., $f(S, P, E) = 2P + E$

How to allocate your time?

LP formulation

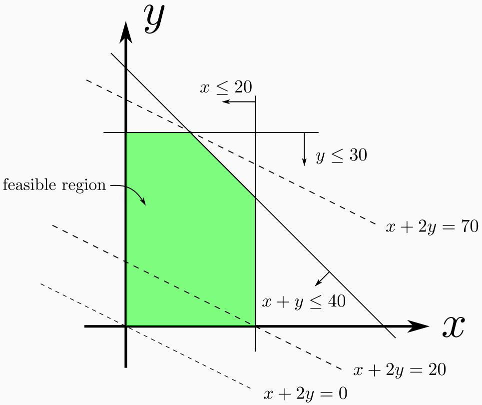
Maximize happiness: LP formulation:

$$
\begin{array}{lrcl}
\text{maximize} & 2P + E & & \\
\text{subject to} & E & \geq & 56 \\
& S & \geq & 60 \\
& 2S + E - 3P & \geq & 150 \\
& S, P, E & \geq & 0 \\
& S + P + E & \leq & 168
\end{array}
$$

How to solve an LP

Consider a simpler LP:

$$\begin{array}{llrcl} \text{maximize} & x+2y & & & \\ \text{subject to} & & x & \leq & 20 \\ & & y & \leq & 30 \\ & & x+y & \leq & 40 \\ & & x,y & \geq & 0 \end{array}$$



Optimal solution: $x + 2y = 70$

Algorithm for solving LP

Observation: (search for an optimal solution)

Objective function is linear, and feasible region is convex. So a unique direction of maximal increase of objective function exists. Follow it and you will run into the boundary. At the boundary, moving in any direction will

(a) Decrease objective function $\rightarrow$ don't go this way

(b) Increase objective function $\rightarrow$ follow to a vertex

(c) Objective function stays constant $\rightarrow$ follow to a vertex

Theorem

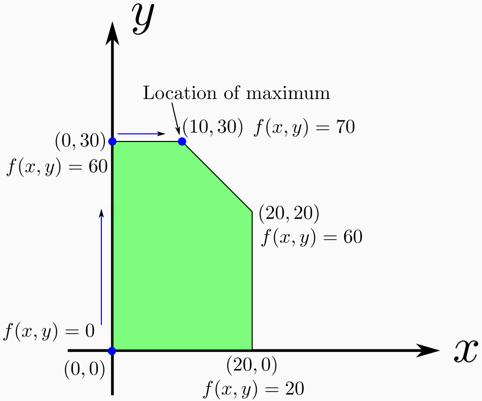
For an LP with bounded, nonempty feasible region, the maximum value will be attained at some vertex of the feasible region

Algorithm idea

The hill climbing approach (the simplex method)

Start at a vertex, look at adjacent vertices, move in the direction of largest increase to the objective function

$$
\begin{array}{lrcl}
\text{maximize} & x + 2y & & \\
\text{subject to} & x & \leq & 20 \\
 & y & \leq & 30 \\
 & x + y & \leq & 40 \\
 & x, y & \geq & 0
\end{array}
$$



Standard forms

LP solvers, such as MOSEK, Gurobi, CVX, and COIN are implementations of the simplex method. They require the LP to be in certain standard form

Standard form 1

$$
\begin{array}{lrcl}
\text{maximize} & \mathbf{c}^T\mathbf{x} & & \\
\text{subject to} & A\mathbf{x} & \leq & \mathbf{b} \\
 & \mathbf{x} & \geq & 0 \\
 & \mathbf{x}, \mathbf{c}, \mathbf{b} & \in & \mathbb{R}^n, A \in \mathbb{R}^{m\times n}
\end{array}
$$

Example:

$$
\begin{array}{lrcl}
\text{maximize} & x+2y & & \\
\text{subject to} & x & \leq & 20 \\
 & y & \leq & 30 \\
 & x+y & \leq & 40 \\
 & x,y & \geq & 0
\end{array}
\quad \equiv \quad
\begin{array}{lrcl}
\text{maximize} & (1,2)\begin{pmatrix} x \\ y \end{pmatrix} & & \\
\text{subject to} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} & \leq & \begin{pmatrix} 20 \\ 30 \\ 40 \end{pmatrix} \\
 & x,y & \geq & 0
\end{array}
$$

Convert to the standard form

- Minimization to maximization

$$\begin{array}{llll} \min & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array} \quad \equiv \quad \begin{array}{llll} \max & -\mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

- Equality to inequality

$$\begin{array}{llll} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & x_1 + x_2 & = & 7 \end{array} \quad \equiv \quad \begin{array}{llll} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & x_1 + x_2 & \leq & 7 \\ & x_1 + x_2 & \geq & 7 \end{array}$$

- Wrong inequality direction

$$\begin{array}{llll} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & x_1 + x_2 & \geq & 7 \end{array} \quad \equiv \quad \begin{array}{llll} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & -x_1 - x_2 & \leq & -7 \end{array}$$

- Missing nonnegative constraints

$$
\begin{array}{lrcl}
\max & x_1 + 2x_2 & & \\
\text{s. t.} & x_1 & \leq & 20 \\
 & x_1 + x_2 & \leq & 40 \\
 & x_1 & \geq & 0
\end{array}
\quad \equiv \quad
\begin{array}{lrcl}
 & \text{rewrite } x_2 = x_2^+ - x_2^- & & \\
\max & x_1 + 2(x_2^+ - x_2^-) & & \\
\text{s. t.} & x_1 & \leq & 20 \\
 & x_1 + (x_2^+ - x_2^-) & \leq & 40 \\
 & x_1 & \geq & 0 \\
 & x_2^+ & \geq & 0 \\
 & x_2^- & \geq & 0
\end{array}
$$

Another Standard form

Standard form 2

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \\ & \mathbf{x}, \mathbf{c}, \mathbf{b} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n} \end{aligned}$$

- Inequality to equality: use **slack variables**

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & x_1 \leq 20 \\ & x_1 \geq 0 \end{aligned} \quad \equiv \quad \begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} + 0 \cdot s \\ \text{subject to} \quad & x_1 + s = 20 \\ & x_1 \geq 0 \\ & s \geq 0 \end{aligned}$$

20 is bigger than x_1 by some positive amount, call it s

The new variable s is call the *slack variable*

Applications of LP — shortest path

We are given $G = (V, E), w : E \to \mathbb{R}$
Want to compute $\text{shortest\_path}(s, t)$ for given $s, t \in V$
Can we model this as an LP?

Recall Bellman-Ford: we calculate d_v for all $v \in V$, s.t. $d_v \le d_u + w(u, v)$
So we had the greatest lower bound: $d_v = \min_{u \text{ s.t. } (u,v)\in E}\{d_u + w(u, v)\}$
i.e., d_v is the largest value s.t. $d_v \le d_u + w(u, v)$ for all $(u, v) \in E$

So we have

$$\begin{array}{lrcll} \text{minimize} & d_t & & & \\ \text{subject to} & d_v & \le & d_u + w(u,v) & \forall (u,v) \in E \\ & d_s & = & 0 & \end{array}$$

There are $|V|$ variables, $|E|$ constraints

We just **reduced** shortest_path to LP

Application of LP — network flow

We are given $G = (V, E)$, $s, t \in V$, capacity c_e for all $e \in E$ Find a flow $f : E \to \mathrm{R}^{\geq 0}$ s.t.

- $0 \leq f(e) \leq c_e$
- $\sum_{(u,v)\in E} f(u, v) = \sum_{(v,w)\in E} f(v, w)$

LP formulation

$$
\begin{array}{lrcll}
\text{max} & \sum_{(s,u)\in E} f_{s,u} & & & \\
\text{s.t.} & f_e & \leq & c_e & \forall e \in E \\
 & \sum_{(u,v)\in E} f_{u,v} - \sum_{(v,w)\in E} f_{v,w} & = & 0 & \\
 & f_e & \geq & 0 &
\end{array}
$$

We just **reduced** $\mathrm{max\_flow}$ to LP