Dynamic Programming

All-pair shortest path (Textbook Section 6.6)

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$|V| \cdot O(|V||E|)$

If for all u, v, APSP takes $\boxed{O(|V|^2|E|)}$ time

$|E| \leq |V|^2$

$|E| = O(|V|^2)$

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Rethink this problem using DP.

Subproblem

$|V| = n$

WLOG, index the vertices as $V = \{1, 2, \ldots, n\}$

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To find out the recurrence relation, we need to relate $\mathrm{sp}(u, v, k)$ to smaller subproblems $\mathrm{sp}(u, v, k - 1)$

Recurrence

Suppose $\mathrm{sp}(\mathrm{u}, \overset{v}{\cancel{\mathrm{k}}}, \mathrm{k}) = P$

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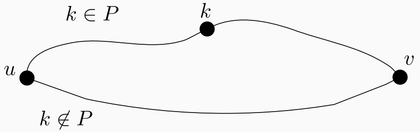

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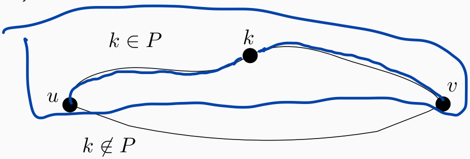

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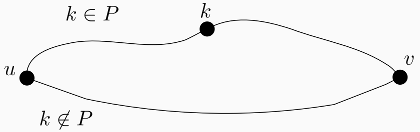

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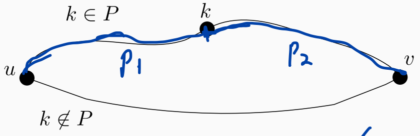
P

uses $\{1, 2, \ldots k\}$



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$$P: \quad u \xrightarrow{P_1} k \xrightarrow{P_2} v$$

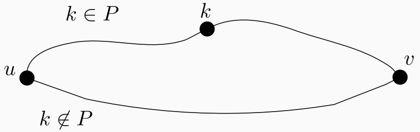
$P_1 = sp(u, k, k-1)$ $\boxed{P_2 = sp(k, v, k-1)}$

$sp(u, v, k) = ?$

$\boxed{|P_1| + |P_2|}$

compare

Suppose $\mathrm{sp(u, n, k)} = P$



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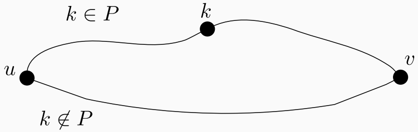

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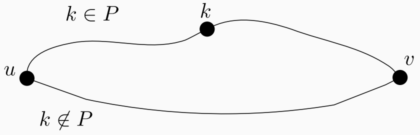

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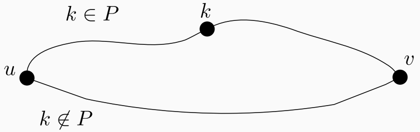

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$$\text{Hence,} \quad P_1 = \mathrm{sp}(u, k, k-1), P_2 = \mathrm{sp}(k, v, k-1)$$

  Using k is better if

$$|\mathrm{sp}(i, k, k-1)| + |\mathrm{sp}(k, v, k-1)| \leq |\mathrm{sp}(i, v, k-1)|$$

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- **Optimal solution:** $\mathrm{dist}(\cdot, \cdot, n)$ $\quad$ $\mathrm{dist}(u, v, n)$

- Base case:

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    for u = 1 ... n:
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where $\mathrm{dist}(u,v,0) = \begin{cases} w_{u,v} & \text{if } (u,v) \in E \\ \infty & \text{otherwise} \end{cases}$ and $\mathrm{dist}(u,v,k) = \min\{\mathrm{dist}(u,v,k-1), \mathrm{dist}(u,k,k-1) + \mathrm{dist}(k,v,k-1)\}$.

Running time: $O(n^3) = O(|V|^3)$