It is interesting to observe that the usual two-sample problem is actually a linear model. Let $\beta_1 = \mu_1$ and $\beta_2 = \mu_2$, and consider n pairs of (x_1, x_2) that equal $(1,0)$ and m pairs that equal $(0,1)$. This would require each of the first n variables $Y_1, Y_2, \ldots, Y_n$ to have the mean

$$\beta_1 \cdot 1 + \beta_2 \cdot 0 = \beta_1 = \mu_1$$

and the next m variables $Y_{n+1}, Y_{n+2}, \ldots, Y_{n+m}$ to have the mean

$$\beta_1 \cdot 0 + \beta_2 \cdot 1 = \beta_2 = \mu_2.$$

This is the background of the two-sample problem, but with the usual $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ replaced by $Y_1, Y_2, \ldots, Y_n$ and $Y_{n+1}, Y_{n+2}, \ldots, Y_{n+m}$, respectively.

Exercises

7.6-1. The mean of Y when $x = 0$ in the simple linear regression model is $\alpha - \beta\overline{x} = \alpha_1$. The least squares estimator of α_1 is $\widehat{\alpha} - \widehat{\beta}\,\overline{x} = \widehat{\alpha}_1$.

(a) Find the distribution of $\widehat{\alpha}_1$ under the usual model assumptions.

(b) Obtain an expression for a $100(1-\gamma)\%$ two-sided confidence interval for α_1.

7.6-2. Obtain a two-sided $100(1-\gamma)\%$ prediction interval for the average of m future independent observations taken at the same X-value, x^*.

7.6-3. For the data given in Exercise 6.5-3, with the usual assumptions,

(a) Find a 95% confidence interval for $\mu(x)$ when $x = 68, 75$, and 82.

(b) Find a 95% prediction interval for Y when $x = 68, 75$, and 82.

7.6-4. For the data given in Exercise 6.5-4, with the usual assumptions,

(a) Find a 95% confidence interval for $\mu(x)$ when $x = 2, 3$, and 4.

(b) Find a 95% prediction interval for Y when $x = 2, 3$, and 4.

7.6-5. For the cigarette data in Exercise 6.5-7, with the usual assumptions,

(a) Find a 95% confidence interval for $\mu(x)$ when $x = 5, 10$, and 15.

(b) Determine a 95% prediction interval for Y when $x = 5, 10$, and 15.

7.6-6. A computer center recorded the number of programs it maintained during each of 10 consecutive years.

(a) Calculate the least squares regression line for the data shown.

(b) Plot the points and the line on the same graph.

(c) Find a 95% prediction interval for the number of programs in year 11 under the usual assumptions.

Year	Number of Programs
1	430
2	480
3	565
4	790
5	885
6	960
7	1200
8	1380
9	1530
10	1591

7.6-7. For the ACT scores in Exercise 6.5-6, with the usual assumptions,

(a) Find a 95% confidence interval for $\mu(x)$ when $x = 17, 20, 23, 26$, and 29.

(b) Determine a 90% prediction interval for Y when $x = 17, 20, 23, 26$, and 29.

7.6-8. By the method of least squares, fit the regression plane $y = \beta_1 + \beta_2 x_1 + \beta_3 x_2$ to the following 12 observations of (x_1, x_2, y): (1, 1, 6), (0, 2, 3), (3, 0, 10), If $p_1 - p_2 = 0$ is not in this interval, we reject H_0: $p_1 - p_2 = 0$ at the $\alpha = 0.05$ significance level. This is equivalent to saying that we reject H_0: $p_1 - p_2 = 0$ if

$$\frac{\left|\dfrac{y_1}{n_1} - \dfrac{y_2}{n_2}\right|}{\sqrt{\dfrac{(y_1/n_1)(1 - y_1/n_1)}{n_1} + \dfrac{(y_2/n_2)(1 - y_2/n_2)}{n_2}}} \ge 1.96.$$

In general, if the estimator $\widehat{\theta}$ (often, the maximum likelihood estimator) of θ has an approximate (sometimes exact) normal distribution $N(\theta, \sigma^2_{\widehat{\theta}})$, then H_0: $\theta = \theta_0$ is rejected in favor of H_1: $\theta \neq \theta_0$ at the approximate (sometimes exact) α significance level if

$$\theta_0 \notin (\widehat{\theta} - z_{\alpha/2}\,\sigma_{\widehat{\theta}}, \widehat{\theta} + z_{\alpha/2}\,\sigma_{\widehat{\theta}})$$

or, equivalently,

$$\frac{|\widehat{\theta} - \theta_0|}{\sigma_{\widehat{\theta}}} \ge z_{\alpha/2}.$$

Note that $\sigma_{\widehat{\theta}}$ often depends upon some unknown parameter that must be estimated and substituted in $\sigma_{\widehat{\theta}}$ to obtain $\widehat{\sigma_{\widehat{\theta}}}$. Sometimes $\sigma_{\widehat{\theta}}$ or its estimate is called the **standard error** of $\widehat{\theta}$. This was the case in our last illustration when, with $\theta = p_1 - p_2$ and $\widehat{\theta} = \widehat{p_1} - \widehat{p_2}$, we substituted y_1/n_1 for p_1 and y_2/n_2 for p_2 in

$$\sqrt{\frac{p_1(1 - p_1)}{n_1} + \frac{p_2(1 - p_2)}{n_2}}$$

to obtain the standard error of $\widehat{p_1} - \widehat{p_2} = \widehat{\theta}$. ■

Exercises

8.3-1. Let Y be $b(100, p)$. To test H_0: $p = 0.08$ against H_1: $p < 0.08$, we reject H_0 and accept H_1 if and only if $Y \le 6$.

(a) Determine the significance level α of the test.

(b) Find the probability of the Type II error if, in fact, $p = 0.04$.

8.3-2. A bowl contains two red balls, two white balls, and a fifth ball that is either red or white. Let p denote the probability of drawing a red ball from the bowl. We shall test the simple null hypothesis H_0: $p = 3/5$ against the simple alternative hypothesis H_1: $p = 2/5$. Draw four balls at random from the bowl, one at a time and with replacement. Let X equal the number of red balls drawn.

(a) Define a critical region C for this test in terms of X.

(b) For the critical region C defined in part (a), find the values of α and β.

8.3-3. Let Y be $b(192, p)$. We reject H_0: $p = 0.75$ and accept H_1: $p > 0.75$ if and only if $Y \ge 152$. Use the normal approximation to determine

(a) $\alpha = P(Y \ge 152; p = 0.75)$.

(b) $\beta = P(Y < 152)$ when $p = 0.80$.

8.3-4. Let p denote the probability that, for a particular tennis player, the first serve is good. Since $p = 0.40$, this player decided to take lessons in order to increase p. When the lessons are completed, the hypothesis H_0: $p = 0.40$ will be tested against H_1: $p > 0.40$ on the basis of $n = 25$ trials. Let y equal the number of first serves that are good, and let the critical region be defined by $C = \{y : y \ge 13\}$.

(a) Determine $\alpha = P(Y \ge 13; p = 0.40)$. Use Table II in the appendix.

(b) Find $\beta = P(Y < 13)$ when $p = 0.60$; that is, $\beta = P(Y \le 12; p = 0.60)$. Use Table II.

8.3-5. If a newborn baby has a birth weight that is less than 2500 grams (5.5 pounds), we say that the baby has a low birth weight. The proportion of babies with a low birth weight is an indicator of lack of nutrition for the mothers. For the United States, approximately 7% of babies have a low birth weight. Let p equal the proportion of babies born in the Sudan who weigh less than 2500 grams. We shall test the null hypothesis $H_0\colon p = 0.07$ against the alternative hypothesis $H_1\colon p > 0.07$. In a random sample of $n = 209$ babies, $y = 23$ weighed less than 2500 grams.

(a) What is your conclusion at a significance level of $\alpha = 0.05$?

(b) What is your conclusion at a significance level of $\alpha = 0.01$?

(c) Find the p-value for this test.

8.3-6. It was claimed that 75% of all dentists recommend a certain brand of gum for their gum-chewing patients. A consumer group doubted this claim and decided to test $H_0\colon p = 0.75$ against the alternative hypothesis $H_1\colon p < 0.75$, where p is the proportion of dentists who recommend that brand of gum. A survey of 390 dentists found that 273 recommended the given brand of gum.

(a) Which hypothesis would you accept if the significance level is $\alpha = 0.05$?

(b) Which hypothesis would you accept if the significance level is $\alpha = 0.01$?

(c) Find the p-value for this test.

8.3-7. The management of the Tigers baseball team decided to sell only low-alcohol beer in their ballpark to help combat rowdy fan conduct. They claimed that more than 40% of the fans would approve of this decision. Let p equal the proportion of Tiger fans on opening day who approved of the decision. We shall test the null hypothesis $H_0\colon p = 0.40$ against the alternative hypothesis $H_1\colon p > 0.40$.

(a) Define a critical region that has an $\alpha = 0.05$ significance level.

(b) If, out of a random sample of $n = 1278$ fans, $y = 550$ said that they approved of the new policy, what is your conclusion?

8.3-8. Let p equal the proportion of drivers who use a seat belt in a state that does not have a mandatory seat belt law. It was claimed that $p = 0.14$. An advertising campaign was conducted to increase this proportion. Two months after the campaign, $y = 104$ out of a random sample of $n = 590$ drivers were wearing their seat belts. Was the campaign successful?

(a) Define the null and alternative hypotheses.

(b) Define a critical region with an $\alpha = 0.01$ significance level.

(c) What is your conclusion?

8.3-9. According to a population census in 1986, the percentage of males who are 18 or 19 years old and are married was 3.7%. We shall test whether this percentage increased from 1986 to 1988.

(a) Define the null and alternative hypotheses.

(b) Define a critical region that has an approximate significance level of $\alpha = 0.01$. Sketch a standard normal pdf to illustrate this critical region.

(c) If $y = 20$ out of a random sample of $n = 300$ males, each 18 or 19 years old, were married (*U.S. Bureau of the Census, Statistical Abstract of the United States: 1988*), what is your conclusion? Show the calculated value of the test statistic on your figure in part (b).

8.3-10. Because of tourism in the state, it was proposed that public schools in Michigan begin after Labor Day. To determine whether support for this change was greater than 65%, a public poll was taken. Let p equal the proportion of Michigan adults who favor a post–Labor Day start. We shall test $H_0\colon p = 0.65$ against $H_1\colon p > 0.65$.

(a) Define a test statistic and an $\alpha = 0.025$ critical region.

(b) Given that 414 out of a sample of 600 favor a post–Labor Day start, calculate the value of the test statistic.

(c) Find the p-value and state your conclusion.

(d) Find a 95% one-sided confidence interval that gives a lower bound for p.

8.3-11. A machine shop that manufactures toggle levers has both a day and a night shift. A toggle lever is defective if a standard nut cannot be screwed onto the threads. Let p_1 and p_2 be the proportion of defective levers among those manufactured by the day and night shifts, respectively. We shall test the null hypothesis, $H_0\colon p_1 = p_2$, against a two-sided alternative hypothesis based on two random samples, each of 1000 levers taken from the production of the respective shifts.

(a) Define the test statistic and a critical region that has an $\alpha = 0.05$ significance level. Sketch a standard normal pdf illustrating this critical region.

(b) If $y_1 = 37$ and $y_2 = 53$ defectives were observed for the day and night shifts, respectively, calculate the value of the test statistic. Locate the calculated test statistic on your figure in part (a) and state your conclusion.

8.3-12. Let p equal the proportion of yellow candies in a package of mixed colors. It is claimed that $p = 0.20$.

(a) Define a test statistic and critical region with a significance level of $\alpha = 0.05$ for testing $H_0\colon p = 0.20$ against a two-sided alternative hypothesis.