Dynamic Programming

All-pair shortest path (Textbook Section 6.6)

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Rethink this problem using DP.

Subproblem

$|V| = n$

WLOG, index the vertices as $V = \{1, 2, \ldots, n\}$

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To find out the recurrence relation, we need to relate $\mathrm{sp}(u, v, k)$ to smaller subproblems $\mathrm{sp}(u, v, k - 1)$

Recurrence

Suppose $\mathrm{sp(u, v, k)} = P$

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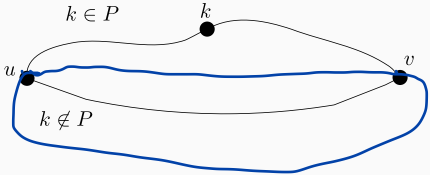

• $k \notin P$ $\quad sp(u, v, k) \quad\quad sp(u, v, k-1)$

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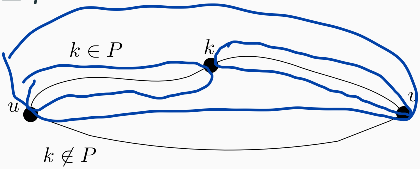

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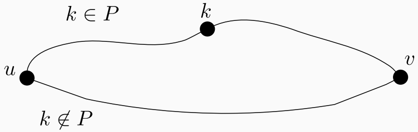

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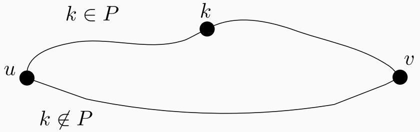

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$$P: \quad u \xrightarrow{P_1} k \xrightarrow{P_2} v$$

$P_1 =$ $P_2 =$

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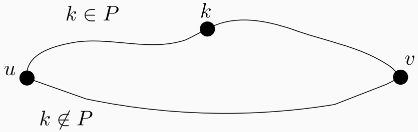

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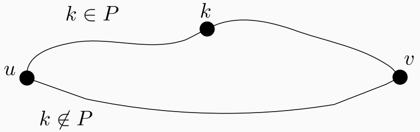

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Because there's no negative cycles, there's no repeated vertices in shortest path

$P_1 = (u, k, k-1)$ $\quad$ $P_2 = (k, v, k-1)$

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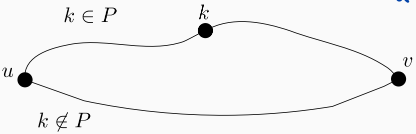

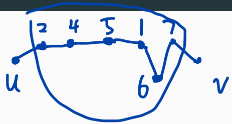

$\{1, 2, \ldots 7\}$

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$$\text{Hence,} \quad P_1 = \mathrm{sp}(u, k, k-1), P_2 = \mathrm{sp}(k, v, k-1)$$

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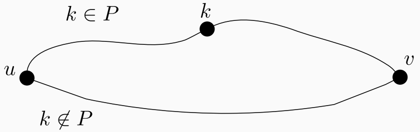

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$$\text{Hence,} \quad P_1 = \mathrm{sp}(u, k, k - 1), P_2 = \mathrm{sp}(k, v, k - 1)$$

Using k is better if

$$|\mathrm{sp}(i, k, k - 1)| + |\mathrm{sp}(k, v, k - 1)| \leq |\mathrm{sp}(i, v, k - 1)|$$

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- **Optimal solution:** $\mathrm{dist}(\overset{u}{\cdot}, \overset{v}{\cdot}, n)$

- Base case:

$$\mathrm{dist}(u, v, 0) =$$

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- **Optimal solution:** $\mathrm{dist}(\cdot, \cdot, n)$
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$$\mathrm{dist}(u, v, 0) = \begin{cases} w_{u,v} & \text{if } (u, v) \in E \\ \infty & \text{otherwise} \end{cases}$$

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    for $u = 1 \ldots n$**:**

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def FLOYD_WARSHALL(G, w):
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    return $\text{dist}(\cdot, \cdot, n)$;

How many entries : $O(n^3)$

Running time :

time for each entry : $O(1)$

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Running time: $O(n^3) = O(|V|^3)$