Greedy algorithms

Set Cover (Textbook Section 5.4)

The set cover problem

Problem (Set Cover)

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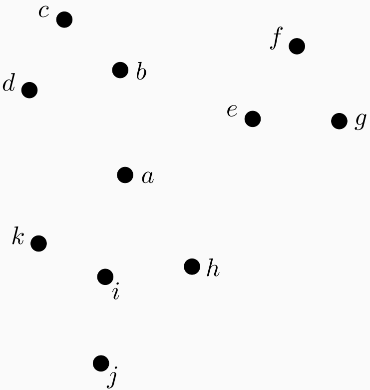
Goal: *minimize the number of selected subsets*

Set cover: example

Example: Each post office can serve 30 miles. Where to build post offices in centre county?



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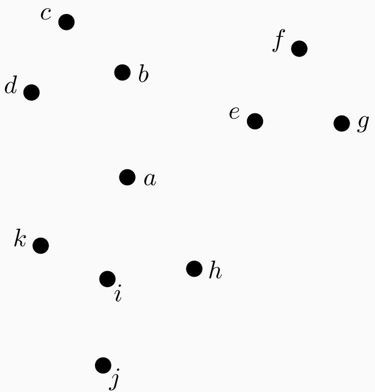

Draw an edge if two towns are within 30 miles

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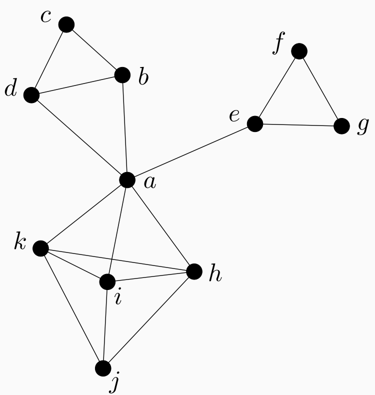

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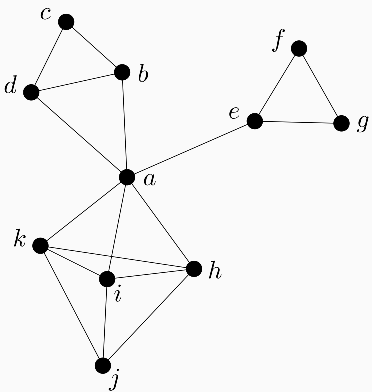

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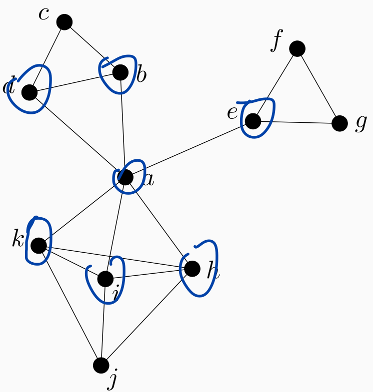

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$$S_a = \{a, b, d, e, h, i, k\}$$

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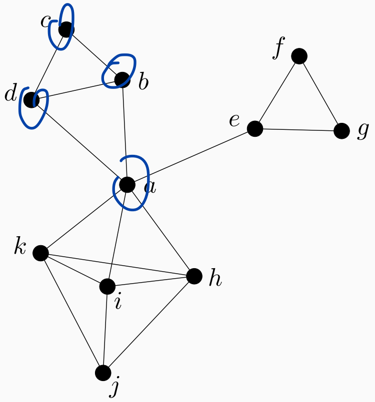

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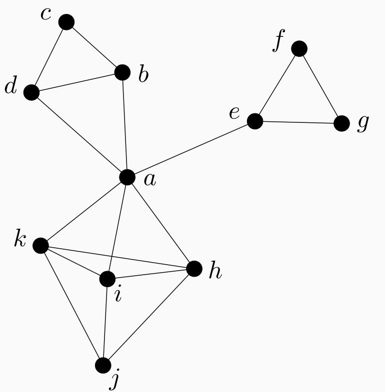

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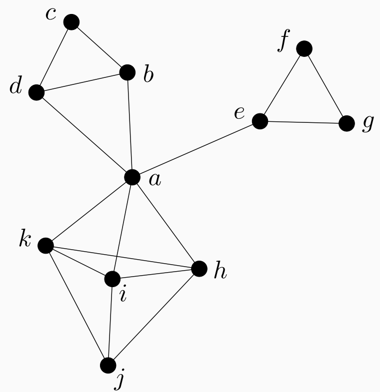

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S_x: the towns within 30 miles of x

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Greedy heuristic: choose the next subset with the most number of uncovered items, until B gets covered

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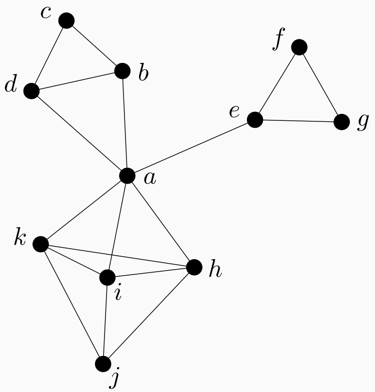

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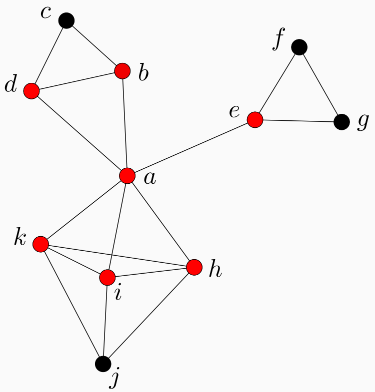

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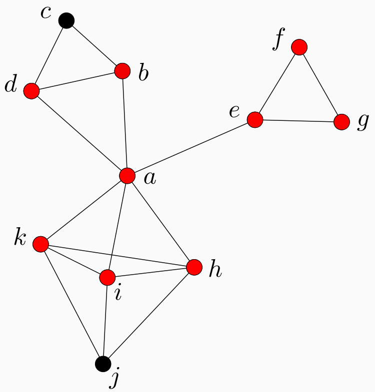

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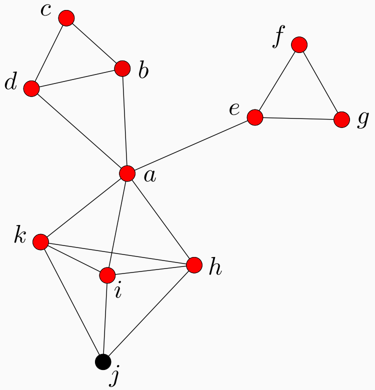

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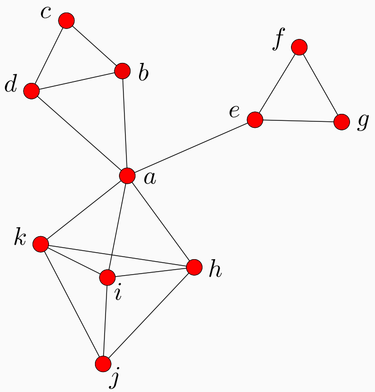

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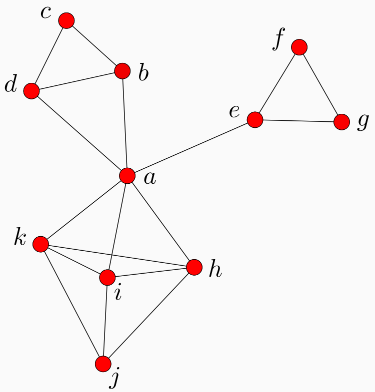

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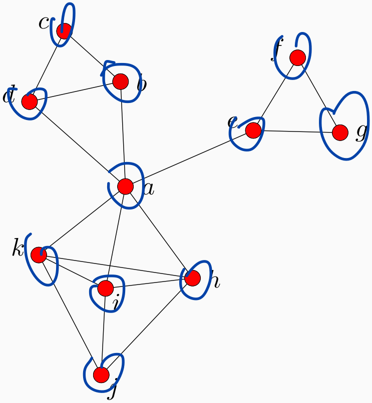

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Optimal solution: S_b, S_e, S_i

Greedy solution is not too bad

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Theorem

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$\ln(n)$: *approximation ratio*

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$\ln = \log_e$

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More about **approximation algorithms**: CSE 565

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Suppose not. all subsets have $< \frac{n_t}{k}$ of the uncovered elements

total number of elements covered by these k subsets $< \frac{n_t}{k} \cdot k = n_t$

Proof: Let $\boxed{n_t}$ be the number of elements not covered by the greedy algorithm after t iterations. These remaining n_t elements are covered by the optimal k subsets. So some subsets has $\geq \frac{n_t}{k}$ of these uncovered elements, and the greedy algorithm will pick a set of size at least $\boxed{\frac{n_t}{k}}$.

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So, $n_{t+1} \leq n_t - \frac{n_t}{k} = n_t \left(1 - \frac{1}{k}\right)$

$$n_t \leq n_{t-1}\left(1-\frac{1}{k}\right) \leq n_{t-2}\left(1-\frac{1}{k}\right)^2 \leq \cdots \leq \underset{?}{n_0}\left(1-\frac{1}{k}\right)^t = n\left(1-\frac{1}{k}\right)^t$$

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Using the fact: $1 - x \leq e^{-x}$ (equality when $x = 0$)

$$\left(1-\frac{1}{k}\right) \leq e^{-\frac{1}{k}}$$

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Greedy algorithm terminates when $n_t < 1$. Let's find out what t makes $n_t < 1$

$\Rightarrow n_t < 1$

Since $n_t < ne^{-t/k}$, it suffices to make $\boxed{ne^{-t/k} \leq 1}$ what t makes it happen?

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Solving $ne^{-t/k} \leq 1$ $\iff e^{-t/k} \leq \frac{1}{n}$

$$\iff -\frac{t}{k} \leq \ln\left(\frac{1}{n}\right)$$

$$\iff t \geq -k \ln\left(\frac{1}{n}\right) = k \ln n$$

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if $t \geq k\ln(n)$ then $n_t < 1$, the greedy alg finishes

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At $\underline{t = k\ln(n)}$, $n_t < 1$. Everything is covered $\square$

t: # of iterations, also # of subsets

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Proof of the fact $1 - x \le e^{-x}$ (equality when $x = 0$):

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Consider $f(x) = e^{-x} - (1 - x) \geq 0$

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Solving $ne^{-t/k} \le 1$

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Consider $f(x) = e^{-x} - (1 - x) \ge 0$

$f'(x) = -e^{-x} + 1$. Critical point at $x = 0$, achieving minimum $\square$