## Homework 16

**Problem 1.** Apply Householder reflectors to find the full QR factorization of the following matrices:

(a) $\begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & -4 \\ -2 & 7 \\ 4 & -5 \end{bmatrix}$

**Problem 2.** Use the two-point forward-difference formula and the three-point centered-difference formula to approximate $f'(1)$, and find the approximation error, where $f(x) = \ln x$, for (a) $h = 0.1$ (b) $h = 0.01$.

**Problem 3.** Use the three-point centered-difference formula for the second derivative to approximate $f''(1)$, where $f(x) = x^{-1}$, for (a) $h = 0.1$ (b) $h = 0.01$. Find the approximation error.

**Problem 4.** Given function $f(x) = e^{-x}$, we study different numerical approximations to the integral

$$\int_{0.0}^{0.8} f(x)\,dx.$$

We will use the values of $f(x)$ at the points $0.0, 0.2, 0.4, 0.6, 0.8$. Generate the data set before you start the numerical integration. Use 6-digits accuracy.

a). Write out the trapezoid rule and compute the numerical integration with 6 digits.

b). Write out the Simpson's rule and compute the numerical integration with 6 digits.

c). What is the exact value of the integral? What is the absolute error by using trapezoid and Simpson's rule? Which method is better?

d). The error formula for the trapezoid rule with n+1 points yields

$$E_T(f;h) = -\frac{b-a}{12}h^2 f''(\xi_0),\ \ h = \frac{b-a}{n},$$

for some $\xi_0 \in (a, b)$. The error for Simpson's rule with $(2n+1)$ points yields

$$E_S(f;h) = -\frac{b-a}{180} h^4 f^{(4)}(\xi_1), \;\; h = \frac{b-a}{2n},$$

for some $\xi_1 \in (a, b)$. If we wish the absolute value of the error to be smaller than $10^{-4}$, how many points would be needed for each method?

**Problem 1.** Apply Householder reflectors to find the full QR factorization of the following matrices:

$$\text{(a)} \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} -4 & -4 \\ -2 & 7 \\ 4 & -5 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\|a_1\| = \sqrt{2^2 + (-2)^2 + 1^2}$$

$$= 3$$

$$V_1 = 3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$H_1 = I - 2\frac{VV^T}{V^TV} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$H_1 A = H_1 \cdot \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 0 & 0 \\ 0 & -3 \end{bmatrix} \qquad \hat{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\|\hat{x}\| = 3$$

$$\hat{H}_2 = I - 2\frac{vv^T}{v^Tv} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 9 & 9 \\ 9 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$R = H_2 \cdot H_1 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$$Q = H_1 \cdot H_2 = \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{15} & \frac{1}{15} \\ -\frac{1}{3} & -\frac{2}{15} & \frac{14}{15} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & -\frac{3}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$Q \cdot R = A$$

b) $\begin{bmatrix} -4 & -4 \\ -2 & 7 \\ 4 & -5 \end{bmatrix}$

$\downarrow$

$b_1 = \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$ $\qquad \|b_1\| = \sqrt{4^2 + 2^2 + 4^2} = 6$

$w = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ $\quad v = w - x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ -4 \end{bmatrix}$

$H_1 = I - 2\dfrac{vv^T}{v^Tv} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{14}{15} & \frac{2}{15} \\ \frac{2}{3} & \frac{2}{15} & \frac{11}{15} \end{bmatrix}$ $\qquad H_1A = \begin{bmatrix} 6 & -3 \\ 0 & \frac{36}{5} \\ 0 & -\frac{27}{5} \end{bmatrix}$ $\qquad \hat{x} = \begin{bmatrix} \frac{36}{5} \\ -\frac{27}{5} \end{bmatrix}$ $\|\hat{x}\| = 9$

$\hat{w} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$ $\quad \hat{v} = \begin{bmatrix} \frac{9}{5} \\ \frac{27}{5} \end{bmatrix}$

$\hat{H}_2 = I - 2\dfrac{vv^T}{v^Tv} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$

$R = H_2 H_1 A = \begin{bmatrix} 6 & -3 \\ 0 & -9 \\ 0 & 0 \end{bmatrix}$

$Q = H_1 H_2 = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

**Problem 2.** Use the two-point forward-difference formula and the three-point centered-difference formula to approximate $f'(1)$, and find the approximation error, where $f(x) = \ln x$, for (a) $h = 0.1$ (b) $h = 0.01$.

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

2.

$$h = 0.1 = \frac{\ln 1.1 - \ln 1}{0.1} = 0.9531$$

$err = 0.9531 - f'(1)$

$= 0.0469$

$$h = 0.01 = \frac{\ln 1.01 - \ln 1}{0.01} = 0.995$$

$err = 0.995 - 1$

$= 0.00497$

3. $h = 0.1$

$$\rightarrow \frac{f(x+0.1) - f(x-0.1)}{2 \cdot 0.1} = \frac{\ln 1.1 - \ln 0.9}{0.2} = 1.0035$$

$err = 1 - 1.0035$

$= 0.0035$

$h = 0.01$

$$\rightarrow \frac{\ln 1.01 - \ln 0.99}{0.02} = 1.000033$$

$err = 1 - 1.000033$

$= 0.000033$

**Problem 3.** Use the three-point centered-difference formula for the second derivative to approximate $f''(1)$, where $f(x) = x^{-1}$, for (a) $h = 0.1$ (b) $h = 0.01$. Find the approximation error.

$f(x) = \frac{1}{x}$ $\quad f'(x) = -\frac{1}{x^2}$ $\quad f''(x) = \frac{2}{x^3}$

and $f''(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

a) $h = 0.1$

$f(1) = \frac{1}{1} = 1$

$$= \frac{\frac{1}{1.1} - 2 \cdot 1 + \frac{1}{0.9}}{0.1^2}$$

$$= \frac{\frac{1}{1.1} - 2 + \frac{1}{0.9}}{0.01} = 2.0202$$

b) $h = 0.01$

$$\frac{\frac{1}{1.01} - 2 + \frac{1}{0.99}}{(0.01)^2} = 2.0002$$

**Problem 4.** Given function $f(x) = e^{-x}$, we study different numerical approximations to the integral

$$\int_{0.0}^{0.8} f(x)\,dx.$$

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$$\frac{0.8-0}{4} = 0.2$$

a) $x = 0.0$
$x = 0.2$
$x = 0.4$
$x = 0.6$
$x = 0.8$

$$\int_0^{0.8} f(x)\,dx = \frac{0.2}{2}\left(f(0.0) + 2f(0.2) + 2\cdot f(0.4) + 2\cdot f(0.6) + f(0.8)\right)$$

$$= \frac{1}{10}(5.52505)$$

$$= 0.552505$$

b) $$\int_0^{0.8} f(x)\,dx = \frac{0.2}{3}\left(f(0.0) + 4f(0.2) + 2\cdot f(0.4) + 4\cdot f(0.6) + f(0.8)\right)$$

$$= \frac{1}{15}(8.26014)$$

$$= 0.550676$$

c) $\int_0^{0.8} f(x)\,dx = -e^{-x}\Big|_0^{0.8}$

$= -e^{-0.8} + e^{0}$

$= 1 - e^{-0.8}$

$= 0.55067$

$|0.55067 - 0.552505| = 0.001834$

$|0.55067 - 0.550676| = 4.9641 \times 10^{-6}$

d) $E_T = -\frac{b-a}{12} h^2 \quad x \in [a,b] \, |f''(x)| \leq 10^{-4}$

trapezoidal $= -\frac{0.8-0}{12} \left(\frac{0.8}{n}\right)^2 (e^{-0.4}) \leq 10^{-4}$

$= \frac{0.8 \cdot 0.64}{12 \cdot h^2}(e^{-0.4}) \leq 10^{-4}$

$n > 16.955$

$n \approx 17$

Simp $= -\frac{b-a}{180} h^4 \max_{x \in [a,b]} |f^4(x)| \leq 10^{-4}$

$= -\frac{0.8-0}{180} \left(\frac{0.8}{n}\right)^4 (e^{0.4}) \leq 10^{-4}$

$n > 1.22203$

$n \simeq 2$