Greedy algorithms

Matroid, Task Scheduling (Cormen et al. 16.4, 16.5)

Warning

Very abstract!

"Computer Science is a science of **abstraction** — creating the right model for a problem and devising the appropriate mechanizable techniques to solve it."

— Alfred Aho

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For a matroid $M = (S, \mathcal{I})$, each $A \in \mathcal{I}$ is called an **independent subset**

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Why M_G is a matroid?

$B \in \mathcal{I}$,

• Hereditary property: $A \subseteq B$, $A \in \mathcal{I}$?

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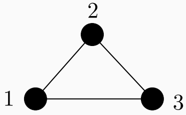
$S = \{(1,2), (2,3), (1,3)\}$

$\mathcal{I} = \{$



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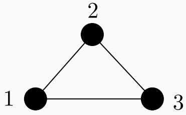

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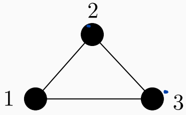

$S = \{(1,2),(2,3),(3,1)\}$

$\mathcal{I} = \{\emptyset, \{(1,2)\}, \{(2,3)\}, \{(1,3)\}, \{(1,2),(2,3)\}, \{(1,3),(1,2)\}, \{(1,3),(2,3)\}\}$

$B = \{(1,2),(2,3)\}$ $\quad A = \{(1,2)\}$

$\exists x \in B - A$ s.t. $A \cup \{x\} \in \mathcal{I}$ $\quad x = (2,3)$

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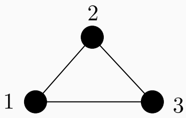

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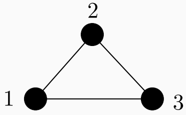

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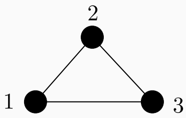

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Say $A = \{(2,3)\}, B = \{(1,3), (1,2)\}$
$x \in B - A$, for example, $x = (1,3)$
then $A \cup \{x\} = \{(2,3), (1,3)\} \subseteq \mathcal{I}$

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For connected undirected G, every maximal independent subset of M_G must be a tree with $|V| - 1$ edges. Hence it is a spanning tree

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Note: for graphic matroids, weight of M_G is corresponding to edge weights

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Hence a max-weighted indep. subset of M_G corresponds to an MST of G

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Running time: let $n = |S|$

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Proof of correctness: Cormen et al. 16.4

Running time: let $n = |S|$

Assume checking if $A \cup \{x\} \in \mathcal{I}$ takes $O(f(n))$.

Pseudocode for finding max-weighted independent subset

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1 def GREEDY(M = (S, 𝓘), weights w):
2     Set A := { };
3     Sort S in decreasing order of w ;          // O(n log n)
4     for x ∈ S:
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6             A := A ∪ {x};
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Application: task scheduling

Problem (Task scheduling)

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task	a	b	c	d
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1	2	3	4	
b	d	a	c	penalty : 5
✓	✓	×	✓	
a	d	c	b	
✓	✓	✓	×	penalty : 10

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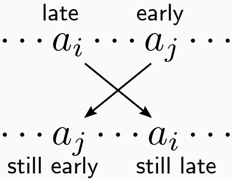

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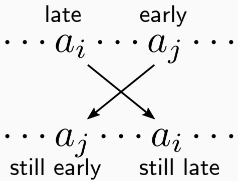

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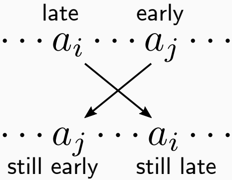

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Finding an optimal schedule $\equiv$ finding max-weighted indep. subset of M