CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Linear Programming
(Textbook, Section 7.1)

Advertisement

Please consider taking

CMPSC 497 — Quantum Computation in Fall 2022

if you are interested in learning **Quantum Computing**

Background

Optimization: we want to maximize some function $f(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$, subject to constraints

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Example

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How to allocate your time?

LP formulation

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How to solve an LP

Consider a simpler LP:

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\begin{aligned}
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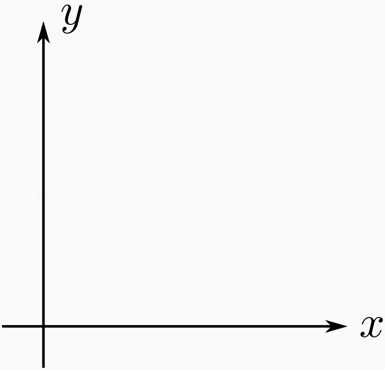

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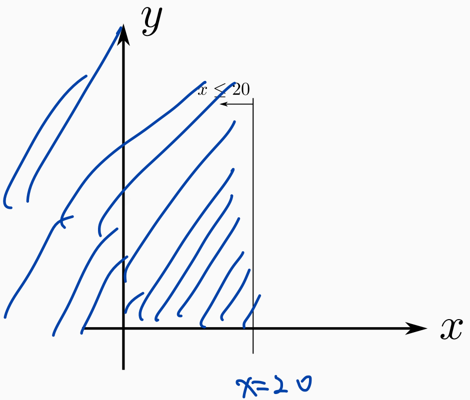

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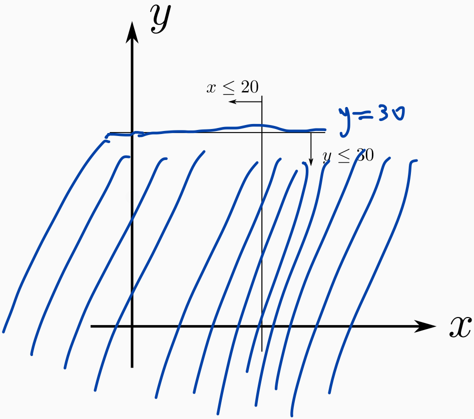

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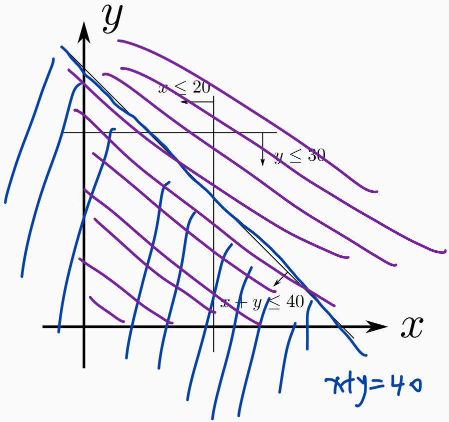

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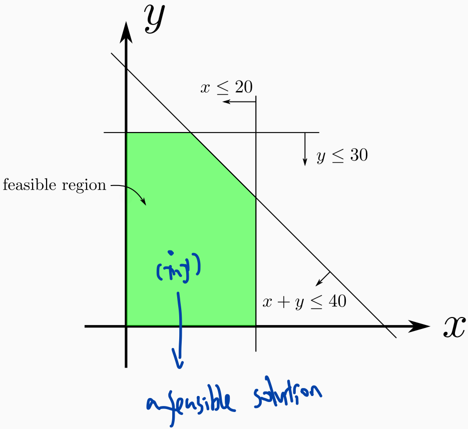

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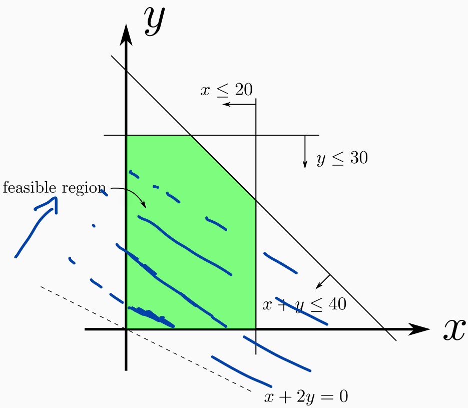

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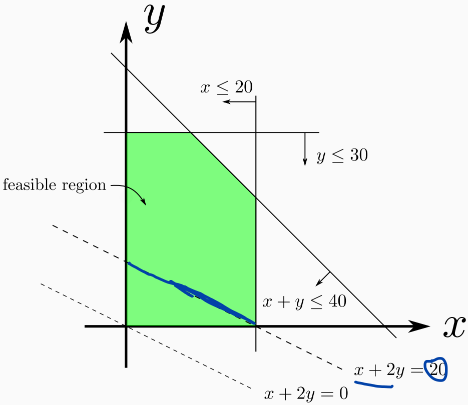

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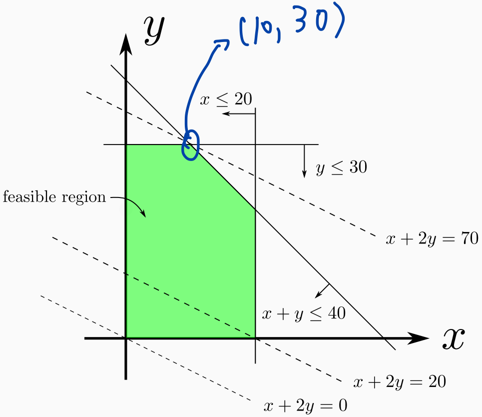

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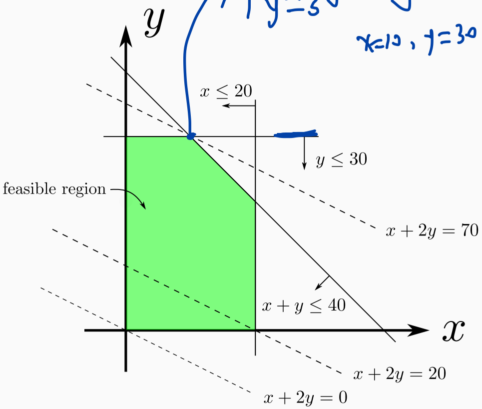

$\left\{ \begin{array}{l} x+2y=70 \\ y=30 \end{array} \right. \Rightarrow x=10,\ y=30$

Optimal solution: $x + 2y = 70$

$x=10,\ y=30$

Algorithm for solving LP

Observation: (search for an optimal solution)

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Theorem

For an LP with bounded, nonempty feasible region, the maximum value will be attained at some vertex of the feasible region

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The hill climbing approach (the simplex method)

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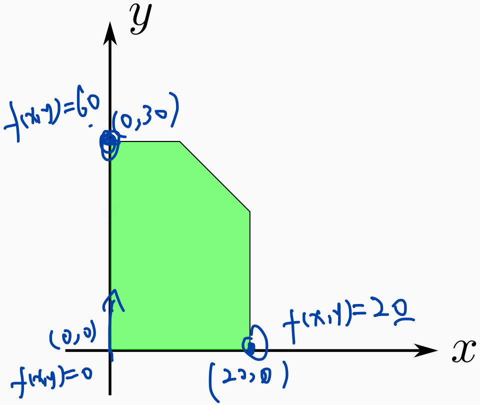

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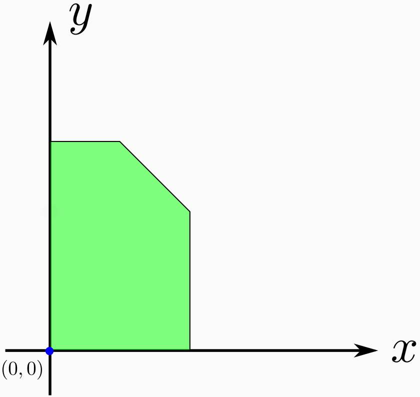

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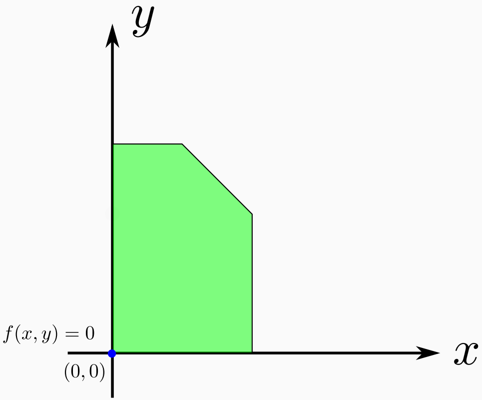

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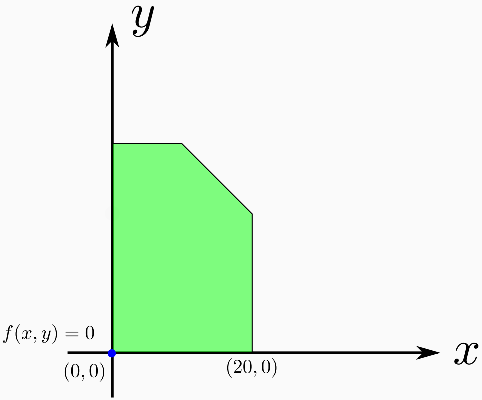

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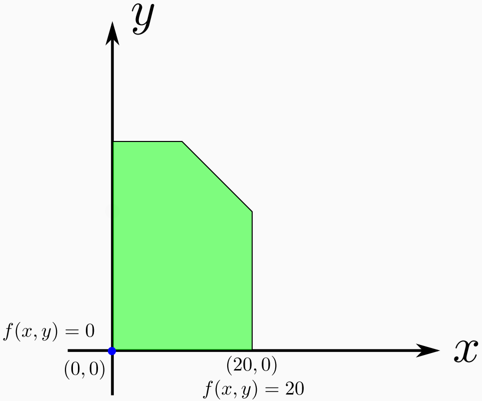

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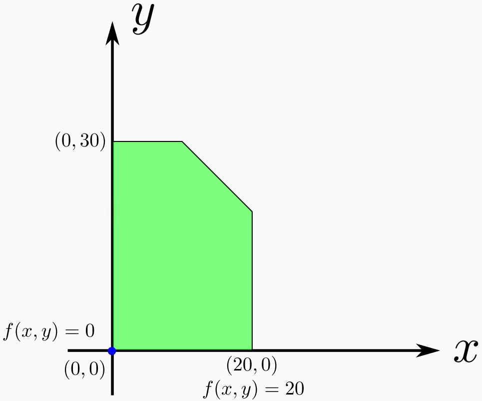

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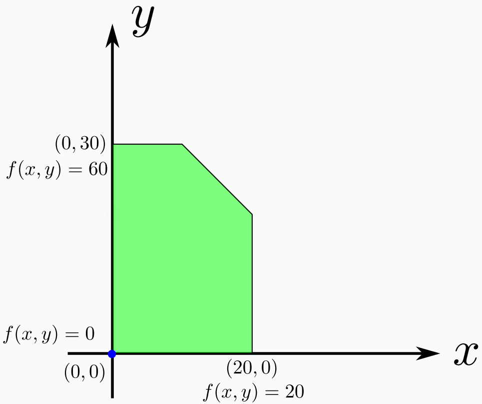

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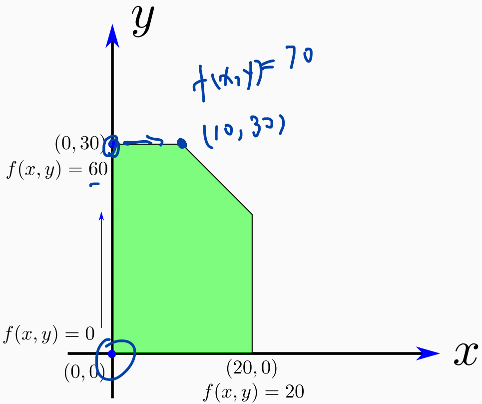

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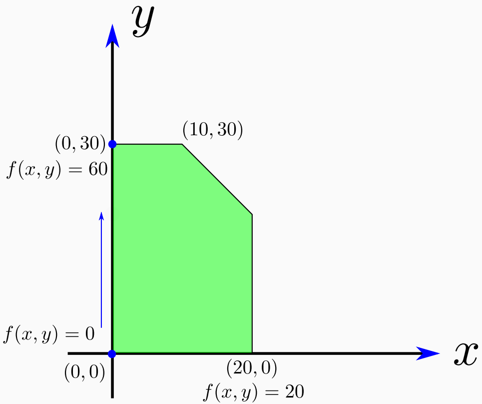

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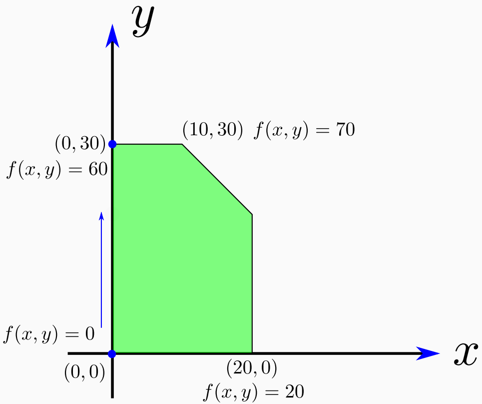

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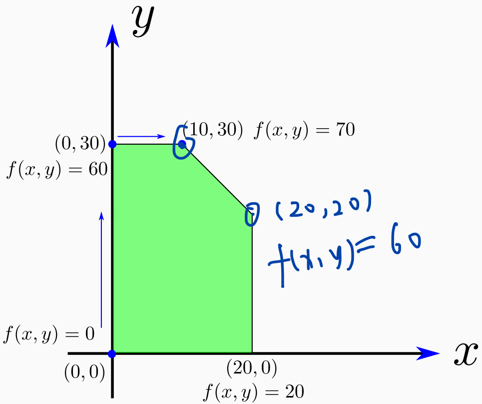

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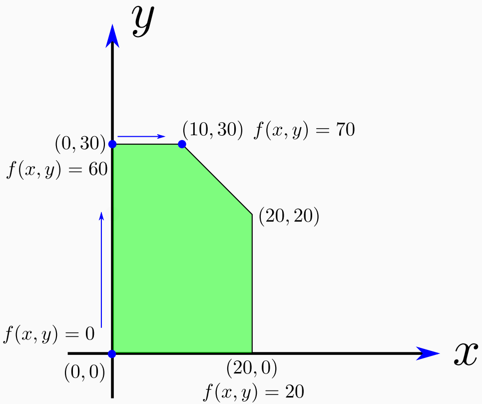

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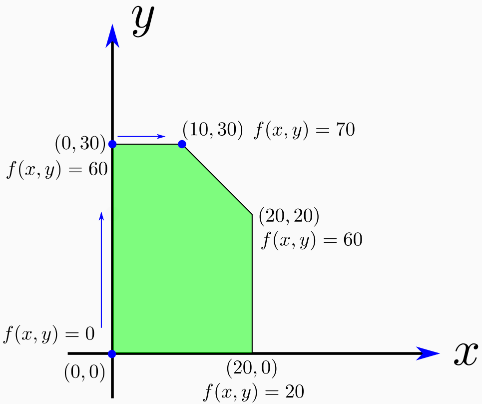

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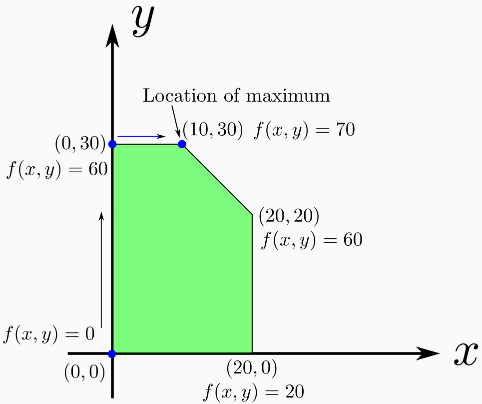

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Standard form 1

$$\begin{array}{lrcl} \text{maximize} & \mathbf{c}^T\mathbf{x} & & \\ \text{subject to} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \\ & \mathbf{x}, \mathbf{c}, \mathbf{b} & \in & \mathbb{R}^n, A \in \mathbb{R}^{m \times n} \end{array}$$

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$$

$c = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$c^T x$

Example:

$$
\begin{array}{lrcl}
\text{maximize} & x + 2y & & \\
\text{subject to} & x & \leq & 20 \\
 & y & \leq & 30 \\
 & x + y & \leq & 40 \\
 & x, y & \geq & 0
\end{array}
$$

$\max \quad (1, 2)\begin{pmatrix} x \\ y \end{pmatrix} = x + 2y$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix}$

$b = \begin{pmatrix} 20 \\ 30 \\ 40 \end{pmatrix}$

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$x_1 \geq 0$ $x_2 \geq 0$ ... $x_n \geq 0$

Example:

$$\begin{array}{lrcl} \text{maximize} & x + 2y & & \\ \text{subject to} & x & \leq & 20 \\ & y & \leq & 30 \\ & x+y & \leq & 40 \\ & x, y & \geq & 0 \end{array} \equiv \begin{array}{lrcl} \text{maximize} & (1,2)\begin{pmatrix} x \\ y \end{pmatrix} & & \\ \text{subject to} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} & \leq & \begin{pmatrix} 20 \\ 30 \\ 40 \end{pmatrix} \\ & x, y & \geq & 0 \end{array}$$

Convert to the standard form

- Minimization to maximization

$$
\begin{array}{llll}
\min & \mathbf{c}^T\mathbf{x} & & \\
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- Equality to inequality

$$\begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & x_1 + x_2 & = & 7 \end{array} \quad \equiv$$

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- Missing nonnegative constraints

$$\begin{array}{llcl} \max & x_1 + 2x_2 & & \\ \text{s. t.} & x_1 & \leq & 20 \\ & x_1 + x_2 & \leq & 40 \\ & x_1 & \geq & 0 \end{array} \quad \equiv$$

missing $x_2 \geq 0$

$\boxed{-x_2^-}$

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Another Standard form

Standard form 2

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20 is bigger than x_1 by some positive amount, call it s

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The new variable s is call the *slack variable*

Applications of LP — shortest path

We are given $G = (V, E), w : E \to \mathbb{R}$

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minimize

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\text{subject to} & d_v & \le & d_u + w(u, v) & \forall (u, v) \in E \\
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We just **reduced** $\mathrm{shortest\_path}$ to LP

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& f_e & \geq & 0 &
\end{array}
$$

$f_e \quad e = (s, u)$

conservation

capacity

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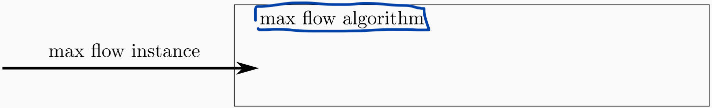

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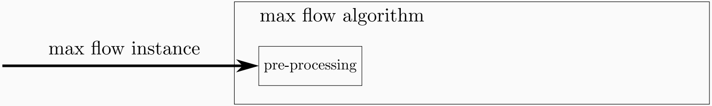

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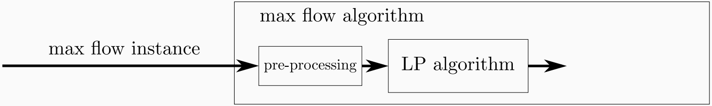

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