CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Dynamic Programming

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Prelude

Dynamic programming vs. Greedy algorithms

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- Difference: greedy choice property

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Sometimes, the greedy choice won't work — we need to check many subproblems to find the optimal solution $\rightarrow$ **Dynamic programming**

General steps for Dynamic Programming

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- Use information from smaller subproblems to solve a larger subproblem

Warm-up: Longest increasing subsequence

Problem (Longest increasing subsequence)

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Example:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
5	2	8	6	3	6	9	7

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$i_1 = 2, i_2 = 5, i_3 = 6, i_4 = 7$

Solving LIS using DAG

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$$\begin{aligned} a_8 &= 7 \\ a_7 &= 9 \\ a_6 &= 6 \\ a_5 &= 3 \\ a_4 &= 6 \\ a_3 &= 8 \\ a_2 &= 2 \\ a_1 &= 5 \end{aligned}$$

Solving LIS using DAG

We can model the longest increasing subsequence problem as a directed acyclic graph

- There is a link $i \to j$ if $a_i < a_j$

$$a_8 = 7$$
$$a_7 = 9$$
$$a_6 = 6$$
$$a_5 = 3$$
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$$a_3 = 8$$
$$a_2 = 2$$
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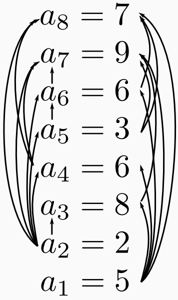

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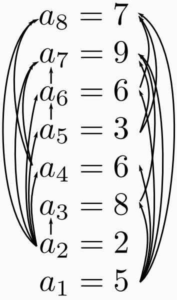

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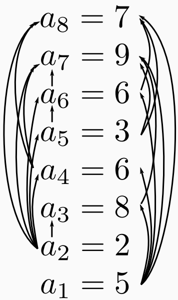
Use $L(j)$ to denote the length of the longest path (longest increasing subsequence) ending with a_j



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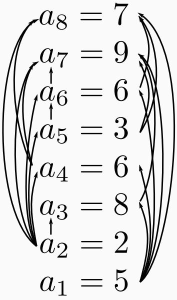

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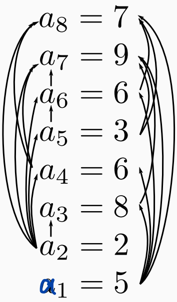

def LIS_DAG(*GAG* $G = (V, E)$ *for* $a_1, \ldots, a_n$):

    for $j = 1, \ldots, n$:

$$L(j) = \begin{cases} 1 + \max\{L(i) : (i,j) \in E\} \\ 1 & \text{if no such edge} \end{cases}$$

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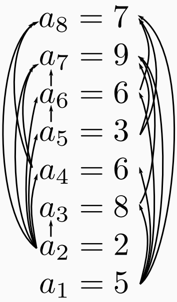

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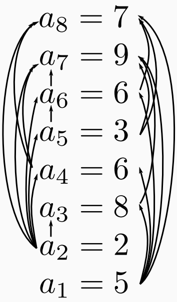

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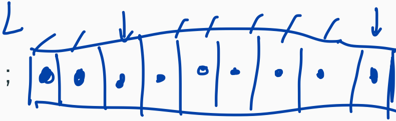

Running example

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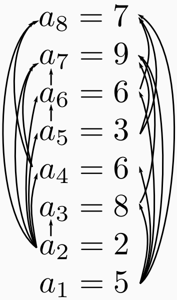

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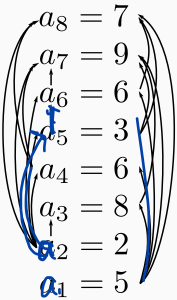

a_i	5	2	8	6	3	6	9	7
i	1	2	3	4	5	6	7	8
L	1	1	2	2	2	3	4	4

1+L(1) 1+L(1) 1+L(2) 1+L(5) 1+L(6) 1+L(6)

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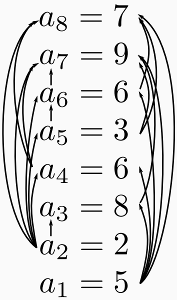

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$\rightarrow$ for $j = 1, \ldots, n$

$L(j) = 1$

for $i = 1, \ldots, j$?

if $a_i < a_j$ and $L(i)+1 > L(j)$

$L(j) = L(i)+1$

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Costs more than greedy: need to check more subproblems

The actual subsequence

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  for $j = 1, \ldots, n$:

    $L(j) = 1$, $\mathrm{prev}(j) = \cdot$;

    for $i = 1, \ldots, j$:

      if $L(i) > L(j)$: $a_i < a_j$ and $L(i)+1 > L(j)$:

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      if $\cancel{L(i) > L(j)}$**:** $a_i < a_j$, and $L(i)+1 > L(j)$

        $L(j) = L(i) + 1$, $\mathrm{prev}(j) = i$;

  return $\max_j L(j)$; $1 + L(1)$

a_i	5	2	8	6	3	6	9	7
i	1	2	3	4	5	6	7	8
L	1	1	2	2	2	3	4	4
prev	·	·	1	1	2	5	6	6

a_2, a_5 a_6 a_7

2 3 6 9

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```
def LIS(GAG G = (V, E) for a_1, ..., a_n):
    for j = 1, ..., n:
        L(j) = 1, prev(j) = ·;
        for i = 1, ..., j:
            if L(i) > L(j):
                L(j) = L(i) + 1, prev(j) = i;
    return max_j L(j);
```

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   e.g. $L(j) = 1 + \max\{L(i) : a_i < a_j\}$
3. Base case