Packet 2: Functions of Random Variables

Chap 5.3 Several Independent Random Variables

Change of variable formula can be applied to multiple random variables, but the derivation becomes challenging when the dimension gets high.

Knowing the expectation and variance could be much easier than knowing the exact density function.

Expectation is a measure of the average behavior of an experiment.

Variance is a measure of the uncertainty.

Independence: $X_1, X_2, ..., X_n$ are independent if

Basic Properties:

1. If a and b are constants,

   $E(aX + b) = aE(X) + b$

   $Var(aX + b) = a^2 Var(X)$

2. For any transformation $u(X)$,

   $Var(u(X)) = E[u(X)^2] - E[u(X)]^2$

3. For transformations $u_1(X_1)$, $u_2(X_2)$, ... $u_n(X_n)$,

   $E[u_1(X_1) + u_2(X_2) + ... + u_n(X_n)] = E[u_1(X_1)] + E[u_2(X_2)] + ... + E[u_n(X_n)]$

4. If X_1, X_2, ... X_n are independent,

   $E[u_1(X_1) \times u_2(X_2) \times ... \times u_n(X_n)] = E[u_1(X_1)] \times E[u_2(X_2)] \times ... \times E[u_n(X_n)]$

5. If X_1, X_2, ... X_n are independent,

   $Var[u_1(X_1) + u_2(X_2) + ... + u_n(X_n)] = Var[u_1(X_1)] + Var[u_2(X_2)] + ... + Var[u_n(X_n)]$

Example 1: X_1, X_2 are independent, $X_1 \sim N(1,1)$, $X_2 \sim N(-1,1)$.

1. $E(X_1 + X_2) =$

2. $E(X_1 X_2) =$

3. $Var(X_1 + X_2) =$

4. $P(X_1 < 0.5, X_2 > -0.5) =$

Example 2 (Theorem 5.3-2): X_1, X_2, ... X_n are independent random variables with respective means, $\mu_1, \mu_2, ..., \mu_n$, and variances, $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$. What is the mean and the variance of $Y = \sum_{i=1}^{n} a_i X_i$?

Chap 5.4 The Moment Generating Function

The moment-generating function of a random variable is an alternative specification of a probability distribution.

It is particularly useful in finding the exact distribution of $Y = \sum_{i=1}^{n} a_i X_i$ if X_1, X_2, ... X_n are independent.

$$M_Y(t) = E(e^{tY}).$$

Example 1 (5.4-1): X_1 and X_2 are independent discrete random variables, both follow Uniform distributions on $\{1, 2, 3, 4\}$. What is the distribution of $Y = X_1 + X_2$?

Interpretation of the Moment Generating Function:

Example 2 (Theorem 5.4-1): If X_1, X_2, ... X_n are independent random variables with m.g.f. $M_{X_i}(t) = E(e^{X_i t})$, then $Y = \sum_{i=1}^n a_i X_i$ has m.g.f. $M_Y(t) = \prod_{i=1}^n E(e^{a_i X_i t})$. *Proof:*

Example 3 (Theorem 5.4-2): If X_1, X_2, ... X_n are independent χ^2 random variables with r_1, r_2, ..., r_n degrees of freedom. X_i has p.d.f.

$$f_i(x) = \frac{1}{\Gamma(r_i/2)2^{r_i/2}} x^{r_i/2-1} e^{-x/2}, x > 0,$$

then

$$Y = \sum_{i=1}^n X_i \sim \chi^2(\sum_{i=1}^n r_i).$$

Proof: