Math 486

Lesson 7 Homework

Due Tues, July 12 at 11:59 on Gradescope

Instructions

Please refer to the solution guidelines posted on Canvas under Course Essentials.

Exercise 1.

Suppose that player 1 and player 2 participate in an infinitely repeated game, where each stage of the game is the Prisoner's Dilemma game shown below. As described in the lectures and textbook, the payoff for each player is a discounted sum of the payoffs for each stage, with discount factor $0 < \delta < 1$: if p_n denotes a player's payoff in stage n, then that player's payoff for the repeated game is

$$\sum_{n=0}^{\infty} \delta^n p_n = p_0 + \delta p_1 + \delta^2 p_2 + \cdots$$

		Player 2	
		C	D
Player 1	C	$5, 5$	$-8, 8$
	D	$8, -8$	$0, 0$

(a) Let σ denote the **trigger strategy**. Suppose that player 1 plays σ in the repeated game. Explain in words what this means.

(b) Let τ denote the **tit-for-tat strategy**: Begin by playing C. Thereafter, play whatever the other player played in the previous round. Find the payoffs when player 1 plays σ and player 2 plays τ:

$$\pi_1(\sigma, \tau) \quad \text{and} \quad \pi_2(\sigma, \tau)$$

It should be clear in your work which move each player plays in each stage (you can explain this, or you can demonstrate an established pattern as part of your calculation).

(c) Let ν denote the following **modified tit-for-tat strategy**: Begin by playing D. Thereafter, play whatever the other player played in the previous round. Find the payoffs when player 1 plays τ and player 2 plays ν :

$$\pi_1(\tau, \nu) \quad \text{and} \quad \pi_2(\tau, \nu)$$

It should be clear in your work which move each player plays in each stage (you can explain this, or you can demonstrate an established pattern as part of your calculation).

(d) Find $\delta_0 > 0$ such that the strategy profile (σ, σ) is a Nash equilibrium of the repeated game whenever the discount rate $\delta \geq \delta_0$.

a) player 1 will choose to Coorepate First, and as player 2 Exceeds the threshold, player 1 will punish player2 by choosing D on later rounds.

b) (5,5) since player 1 will chose C initially and player 2 will start with C and follow player 1's move. player 1 did not observed player 2 choose D, so player 1 will continuesly choose C and they will repete C till end.

So $\pi_1(\sigma, \tau) = \frac{5}{1-\delta} = \pi_2(\sigma, \tau)$

c)

	player 1	player 2
initial step,	C	D
2nd step	D	C
3rd step	C	D
4th step	D	C
⋮	C	D
	⋮	⋮

player 1 start with C and player 2 start with D.
and player 1 follows last player 2 and choose D, while player 2 follows last player 1 and choose C and repete but reversed till end.

$\pi_1(\tau, v) = -\frac{8}{1+\delta}$ $\quad\quad$ $\pi_2(\tau, v) = \frac{8}{1+\delta}$

d) payoff for C D

p1 5

Case 1 player plays C in each stage.

$\pi_1(\sigma, \sigma) = 5 + 5\delta + 5\delta^2 \cdots = \frac{5}{1-\delta}$

$\pi_1(\sigma', \sigma) = \pi_1(\sigma, \sigma) = \frac{5}{1-\delta}$

$\pi_1(\sigma', \sigma) = 5 + 5\delta + 5\delta^2 + \cdots 5\delta^{k-1} + 8\delta^k + 0$

$$8\cancel{\delta^k} \leq \frac{\cancel{\delta^k}}{1-\delta}$$

$$8 \leq \frac{5}{1-\delta}$$

$$8 - 8\delta \leq 5$$

$$8\delta \geq 3$$

$$\delta \geq \frac{3}{8}$$

Exercise 2.

Suppose that player 1 and player 2 participate in an infinitely repeated game, where the stage game is shown below. As in Exercise 1, the payoff for each player is a discounted sum of the payoffs for each stage, with discount factor $0 < \delta < 1$: if p_n denotes a player's payoff in stage n, then that player's payoff for the repeated game is

$$\sum_{n=0}^{\infty} \delta^n p_n = p_0 + \delta p_1 + \delta^2 p_2 + \cdots$$

		Player 2	
		C	D
Player 1	C	$4, 6$	$-2, 10$
	D	$6, -1$	$1, 2$

(a) Is the stage game a Prisoner's Dilemma type game? Explain why or why not? (this game is not symmetric, but that is not essential to a prisoner's dilemma type game).

(b) Let σ denote the trigger strategy. Find $\delta_0 > 0$ such that the strategy profile (σ, σ) is a Nash equilibrium of the repeated game whenever the discount rate $\delta \geq \delta_0$.

a) it is a prisoner's Dilemma type game.
since for player1, if player 2 choose C, he will have higher payoff to choose D and if player 2 choose D, he will also have higher payoff then choose C but if they both choose D, their payoff is smaller then they both choose C. vise versa.

for player 1& 2, C is strictly dominated by D, but (C, C) is better than (D, D) b)

start with C

$$\pi_1(\delta',\delta) = 4 + 4\delta + 4\delta^2 \cdots 4\delta^{k-1} + 6\delta^k + 1\delta^{k+1}$$

$$\pi_1(\delta,\delta) = 4 + 4\delta + 4\delta^2 \cdots$$

$$6\delta^k + \delta^{k+1} + \delta^{k+2} \cdots \leq 4\delta^k + 4\delta^{k+1} + 4\delta^{k+2} \cdots$$

$$6 + \frac{1}{1-\delta} \leq \frac{4}{1-\delta}$$

$$\delta \geq \frac{2}{5}$$

$$\pi_2(\delta',\delta) = 6 + 6\delta + 6\delta^2 \cdots + 10\delta^k + 2\delta^{k+1}$$

$$\pi_1(\delta,\delta) = 6 + 6\delta + 6\delta^2 \cdots$$

$$10\delta^k + 2\delta^{k+1} \cdots \leq 6\delta^k + 6\delta^{k+1} \cdots$$

$$10 + \frac{2\delta}{1-\delta} \leq \frac{6}{1-\delta}$$

$$\delta \geq \frac{1}{2}$$

Problem 1.

Consider again the repeated game in Exercise 1, where the stage game is the prisoner's dilemma game shown below and the repeated strategies σ, τ, and ν are as defined in Exercise 1.

		Player 2	
		C	D
Player 1	C	$5, 5$	$-8, 8$
	D	$8, -8$	$0, 0$

(a) Explain how the payoffs for the strategy profile (σ, σ) compare with the payoffs for the strategy profile (τ, τ).

(b) Find $\delta_0 > 0$ such that

$$\pi_1(\nu, \tau) \leq \pi_1(\tau, \tau)$$

whenever the discount factor $\delta \geq \delta_0$. That is, we want to show that for a large enough discount factor, player 1 cannot improve their own payoff by switching from τ to ν when player 2 plays τ.

(c) Let β denote the strategy "Always play D". Find $\delta_0 > 0$ such that

$$\pi_1(\beta, \tau) \leq \pi_1(\tau, \tau)$$

whenever the discount factor $\delta \geq \delta_0$. That is, we want to show that for a large enough discount factor, player 1 cannot improve their own payoff by switching from τ to β when player 2 plays τ.

(d) Note that in parts (b) and (c), we showed two possible strategy changes for player 1, neither of which could improve player 1's payoff over playing τ. It turns out that, for sufficiently large δ, the strategy profile (τ, τ) is a Nash equilibrium of the repeated game. Describe, briefly, what we would need to show to prove this result.

(e) Let ζ denote the following **generous tit-for-tat** strategy: Begin by playing C in stage zero. Also play C in stage one. After this, play whatever the other played in the previous stage. This strategy is called generous or forgiving because it does not give up on cooperation if the other player begins with D. Find δ_0 such that whenever $\delta > \delta_0$,

$$\pi_1(\zeta, \nu) > \pi_1(\tau, \nu)$$

This result can be used to show that, while (τ, τ) is a Nash equilibrium of the repeated game, it is not subgame perfect.

a) payoff for (σ, σ) is $5,5$ and (τ,τ) is $5,5$

so $(\sigma,\sigma) = (\tau,\tau)$ $\quad 5 = 5$

b) $\pi_1(V,T) =$

p1	p2	
D	C	
C	D	$= 8$
D	C	-8
C	D	8
$\vdots$	$\vdots$	-8
		$\vdots$
		$= 0$

$\pi_1(T,T) =$

	p1	p2
	C	C
$= 5$	C	C
5	C	C
5		
5		
$\vdots$		
$= 5$		

$$\pi_1(V,T) = \frac{8}{1+\delta} \qquad \pi_1(T,T) = \frac{5}{1-\delta}$$

$$\frac{8}{1+\delta} \leq \frac{5}{1-\delta}$$

$$8 - 8\delta \leq 5 + 5\delta$$

$$\delta \geq \frac{3}{13} \qquad \delta = \frac{3}{13}$$

c) $\pi_1(B,T) \leq \pi_1(T,T)$

$\pi_1(B,T)$

P_1:	D	D	D	D
P_2:	C	D	D	D

payoff will be 8

$\pi_1(T,T)$

P_1	C	C	C	C
P_2	C	C	C	C

payoff $= \frac{5}{1-\delta}$

when discount factor is large enough,

$$8 < \frac{5}{1-\delta} \qquad 8 - 8\delta < 5$$

$$\delta > \frac{3}{8}$$

d) We can show $\pi_1(T, T)$ is a Nash Equ.

by proving $\pi_1(T, T) \geq \pi_1(\text{any}, T)$

e) $\pi_1(S, V) > \pi_1(T, V)$

P_1	C	C	C	C	...	P_1	C	D	C	D	...
P_2	D	C	C	C	...	P_2	D	C	D	C	...

$$\cancel{-8} + 5\delta + 5\delta^2 + 5\delta^3 \cdots > \cancel{-8} + 8\delta - 8\delta^2$$

$$\frac{5}{1-\delta} > \frac{8}{1+\delta}$$

$$\delta > \frac{3}{13}$$

Problem 2.

Suppose we select 4 students from the class to play the following game:

- I give each student one (infinitely divisible) dollar. Each student chooses (simultaneously) how much of their dollar to put into a common fund.
- I multiply whatever is in the common fund by 3, and distribute the total equally among the 4 students.
- Let x_i denote player i's strategy in this game, so that $0 \leq x_i \leq 1$.

(a) Find the payoff function for player i, $\pi_i(x_i, x_{-i})$. Remember that each player gains \$1 before choosing the amount to put into the common fund.

(b) Show that for each player i, choosing $x_i > 0$ is strictly dominated by choosing $x_i = 0$.

(c) Suppose that the game described is a stage game for an infinitely repeated game with a discount factor δ. The game is repeated using the same 4 students, and each player receives \$1 at the start of every stage. No matter how much a player currently has, they can only choose how much of the dollar they just received to put into the common fund.

We define a "trigger-type" strategy σ_y as follows:

- Begin by contributing $y > 0$ in stage 0 ($0 < y \leq 1$).
- Continue contributing the same amount y in subsequent stages as long as every other student has contributed at least y in the previous stage.
- If one or more players contributes an amount less than y in the previous stage, contribute 0 in the current stage and continue contributing zero every stage thereafter. Find a value δ_0 such that the strategy profile $(\sigma_y, \sigma_y, \sigma_y, \sigma_y)$ is a Nash equilibrium whenever $\delta \geq \delta_0$.

Hints:

- What is the payoff when all players "always contribute y"?
- Just as with the prisoner's dilemma you can frame the problem by thinking about "the first stage where player i contributes zero instead of y. It is sufficient to consider the case where player i "defects" by contributing zero right away in stage 0. What is the payoff for player i in stage zero (with everyone else playing y)? What happens in subsequent rounds? Don't forget that the players keep receiving \$1 in each stage, even if they contribute zero to the fund.

a) $\pi_i(x_i, x_{-i}) = 1 - x_i + \frac{3}{4} x_i$ $\quad$ x_{-i} = total investment.

b) if $x_i = 0$ $\quad$ $x_i \neq 0$

$\pi_1(0, x_i) = 1 - 0 + \frac{3}{4} x_i > \pi_i(x_i, x_{-i}) = 1 - x_i + \frac{3}{4} x_i$

$= 1 + \frac{3}{4} x_i$

So $x_1 > 0$ is strictly dominated by $x_i = 0$

c) if all $\pi_i(y, 4y) = 1 - y + \frac{3}{4} 4y$ $\quad$ $\pi_i(0, 3y) = 1 - 0 + \frac{3}{4} 3y$

$= 1 + \frac{9}{4} y$

$= 1 - y + \frac{12}{4} y$

$= 1 + 2y$

and $\pi_1(y, 4y) = 1 + 2y + (1+2y)\delta + (1+2y)\delta^2 \cdots\cdots$

$= \frac{1+2y}{1-\delta}$

if one choose to contribute 0,

$\pi(y, 4y) = \frac{1+2y}{1-\delta}$

$\pi(y', 4y) = (1+2y) + (1+2y)\delta + (1+2y)\delta^2 - \cdots (1+2y)\delta^{k-1} + (1+\frac{9}{4}y)\delta^k + \delta^{k+1}$

$(1 + \frac{9}{4} y)\delta^k + \delta^{k+1} \cdots \geq \frac{1+2y}{1-\delta}$

$\delta \leq \frac{1}{9}$ $\quad$ so $\delta_0 = \frac{1}{9}$