STAT/MATH 415 HW#3

September 22, 2017

EXERCISES

6.4.3 A random sample $X_1, X_2, ..., X_n$ of size n is taken from a Poisson distribution with a mean of λ, $0 < \lambda < \infty$.

a) Show that the maximum likelihood estimator for λ is $\hat{\lambda} = \bar{X}$.

Answer: Since $X_i \sim Pois(\lambda)$, the pdf of X is

$$f(x_i; \lambda) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$L(\lambda) = \prod_{a_i}^{n} f(x_i; \lambda) = \frac{\lambda^{x_1+x_2+...+x_n} e^{-\lambda n}}{x_1!\ x_2!\ ...x_n!} = \frac{\lambda^{n\bar{x}} e^{-\lambda n}}{x_1!\ x_2!\ ...x_n!} \qquad \bar{x} = \frac{x_1 + x_2 + ... + x_n}{n}$$

Since $L(\lambda)$ is a continuous function of λ, thus $\hat{\lambda}$ can be found at either derivative = 0 or the boundary of parameter space.

$$\frac{\partial L}{\partial \lambda} = \frac{1}{x_1!\ x_2!\ ...x_n!}[(n\bar{x})\lambda^{n\bar{x}-1}e^{-\lambda n} + \lambda^{n\bar{x}}e^{-\lambda n}(-n)]$$

$$= \frac{1}{x_1!\ x_2!\ ...x_n!}(n\bar{x} - n\lambda)\lambda^{n\bar{x}-1}e^{-\lambda n} = 0 \qquad \Rightarrow \qquad \bar{x} = \lambda$$

$$\lim_{\lambda \to 0} L(\lambda) = 0 \qquad \text{(check the boundary)}$$

Thus, the maximum likelihood estimator is $\hat{\lambda} = \bar{X}$

b) Let X equal the number of flaws per 100 feet of a used computer tape. Assume that X has a Poisson distribution with a mean of λ. If 40 observations of X yielded 5 zeros, 7 ones, 12 twos, 9 threes, 5 fours, 1 five and 1 six. Find the maximum likelihood estimate of λ.

Answer: Since $X \sim Pois(\lambda)$, $f(x; \lambda) = \dfrac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, 2...$

$$L(\lambda) = (f(0, \lambda))^5 \times (f(1, \lambda))^7 \times (f(2, \lambda))^{12} \times (f(3, \lambda))^9 \times (f(4, \lambda))^5 \times (f(5, \lambda))^1 \times (f(6, \lambda))^1$$

$$= e^{-5\lambda} \times (\lambda^7 e^{-7\lambda}) \times (\frac{\lambda^{24}e^{-12\lambda}}{2^{12}}) \times (\frac{\lambda^{27}e^{-9\lambda}}{6^9}) \times (\frac{\lambda^{20}e^{-5\lambda}}{24^5}) \times (\frac{\lambda^5 e^{-\lambda}}{120}) \times (\frac{\lambda^6 e^{-\lambda}}{720})$$

$$= \frac{\lambda^{89}e^{-40\lambda}}{2^{12} \times 6^9 \times 24^5 \times 120 \times 720}$$

$$\log L(\lambda) = 89 \log \lambda - 40\lambda - \log(2^{12} \times 6^9 \times 24^5 \times 120 \times 720)$$

$$\frac{\partial L}{\partial \lambda} = \frac{89}{\lambda} - 40 = 0 \qquad \Rightarrow \qquad \hat{\lambda} = 2.225$$

$$\lim_{\lambda \to 0} \log L(\lambda) = -\infty \qquad \text{(check the boundary)}$$

Thus, the maximum likelihood estimator is $\hat{\lambda} = 2.225$

6.4.9 Let $X_1, X_2, ..., X_n$ be a random sample of size n from the exponential distribution whose pdf is $f(x;\theta) = (1/\theta)e^{-x/\theta},\ 0 < x < \infty,\ 0 < \theta < \infty$

a) Show that $\bar{X}$ is an unbiased estimator of θ

Answer: We can assume that $X_1, ..., X_n$ are independent

$$E(X) = \int_0^{\infty} xf(x)dx = \frac{1}{\theta}\int_0^{\infty} xe^{-x/\theta}dx = \frac{1}{\theta}\theta^2 = \theta$$

$$E(\bar{X}) = E(\frac{X_1 + X_2 + ... + X_n}{n}) = \frac{1}{n}E(X_1 + X_2 + ... + X_n) = \frac{E(X_1) + E(X_2) + ... + E(X_n)}{n} = \theta$$

Thus $\bar{X}$ is an unbiased estimator of θ

b) Show that the variance of $\bar{X}$ is θ^2/n

Answer: Similarly, we can have

$$Var(\bar{X}) = Var(\frac{X_1 + X_2 + ... + X_n}{n}) = \frac{1}{n^2}Var(X_1 + X_2 + ... + X_n) = \frac{Var(X_1) + ... + Var(X_n)}{n^2}$$

$$Var(X) = E(X^2) - E(X)^2 = \int_0^{\infty} x^2 f(x)dx - \theta^2 = 2\theta^2 - \theta^2 = \theta^2$$

$$Var(\bar{X}) = \frac{n\theta^2}{n^2} = \frac{\theta^2}{n}$$

c) What is a good estimator of θ if a random sample of size 5 yielded the sample values 3.5, 8.1, 0.9, 4.4 and 0.5?

Answer: From a and b we can know that $\bar{X}$ is an unbiased estimate of θ and its variance $\to 0$ as n $\to \infty$, so $\bar{X}$ is a good estimator. In this case,

$$\bar{X} = \frac{3.5 + 8.1 + 0.9 + 4.4 + 0.5}{5} = 3.48$$

6.4.13 Let $X_1, X_2, ..., X_n$ be a random sample from a uniform distribution on the interval $(\theta - 1, \theta + 1)$

a) Find the method-of-moments estimator of θ

Answer: Since $X_i \sim U(\theta - 1, \theta + 1)$, the pdf is $f(x) = \frac{1}{2},\ \theta - 1 < x < \theta + 1$

$$E(X) = \int_{\theta-1}^{\theta+1} xf(x)dx = \frac{1}{4}[(\theta + 1)^2 - (\theta - 1)^2] = \theta$$

$$\bar{X} = \frac{X_1 + X_2 + ... + X_n}{n} = E(X) \qquad \Rightarrow \qquad \hat{\theta} = \frac{X_1 + X_2 + ... + X_n}{n}$$

b) Is your estimator in part a an unbiased estimator of θ?

Answer: Assume that $X_i s$ are independent. Here shows $\bar{X}$ is an unbiased estimator of θ

$$E(\hat{\theta}) = E(\frac{X_1 + X_2 + ... + X_n}{n}) = \frac{1}{n}[E(X_1) + E(X_2) + ... + E(X_n)] = \frac{n\theta}{n} = \theta$$

c) Given the following $n = 5$ observations of X, give a point estimate of θ: 6.61 7.70 6.98 8.36 7.26

Answer: Based on a and b, a point estimate of θ is $\bar{X}$

$$\hat{\theta} = \bar{X} = \frac{6.61 + 7.70 + 6.98 + 8.36 + 7.26}{5} = 7.382$$

d) The method-of-moments estimator actually has greater variance than the maximum likelihood estimator of θ, namely $[\min(X_i)+\max(X_i)]/2$. Compute the value of the latter estimator for the $n = 5$ observations in c.

Answer: The minimum is 6.61 and the maximum is 8.36, thus the maximum likelihood estimator is

$$\hat{\theta}_{MLE} = \frac{6.61 + 8.36}{2} = 7.485$$

7.1.3 To determine the effect of 100% nitrate on the growth of pea plants, several specimens were planted and then watered with 100% nitrate every day. At the end of two weeks, the plants were measured. Here are data on seven of them: 17.5 14.5 15.2 14.0 17.3 18.0 13.8 Assume that these data are a random sample from normal distribution $N(\mu, \sigma^2)$

a) Find the value of a point estimate of μ

Answer: Based on maximum likelihood estimate, we know that $\bar{X}$ is a good estimator for μ

$$\hat{\mu} = \bar{X} = \frac{17.5 + 14.5 + 15.2 + 14.0 + 17.3 + 18.0 + 13.8}{7} = 15.757$$

b) Find the value of a point estimate of σ

Answer: Similarly, the maximum likelihood estimate of $\hat{\sigma} = \sqrt{2.751} = 1.659$

$$\hat{\sigma^2} = \frac{\sum_{i=1}^{7}(x_i - \bar{x})^2}{7} = 2.751$$

c) Give the endpoints for a 90% confidence interval for μ

Answer: Here $\alpha = 0.1,\ t_{0.05(6)} = 1.943$

$$s^2 = \frac{\sum_{i=1}^{7}(x_i - \bar{x})^2}{7-1} = 3.210 \qquad \Rightarrow \qquad s = 1.792$$

$$\bar{X} \pm t_{0.05(6)}\frac{s}{\sqrt{n}} = 15.757 \pm 1.943\frac{1.792}{\sqrt{7}} = 15.757 \pm 1.316$$

Thus the 90% confidence interval for μ is $(14.441, 17.073)$

7.1.7 Thirteen tons of cheese, including "22-pound" wheels is stored in some gypsum mines. A random sample of $n = 9$ of these wheels (label weight) yielded the following weights in pounds: 21.50 18.95 18.55 19.40 19.15 22.35 22.90 22.20 23.10 Assuming that the distribution of the weights of the wheels of cheese is $N(\mu, \sigma^2)$, find a 95% confidence interval for μ

Answer: Here $\alpha = 0.05,\ t_{0.025(8)} = 2.306$

$$\bar{X} = \frac{\sum x_i}{9} = 20.9$$

$$s^2 = \frac{\sum(x_i - \bar{x})^2}{9-1} = 3.454$$

$$\bar{X} \pm t_{0.025(8)}\frac{s}{\sqrt{n}} = 20.9 \pm 2.306\frac{1.858}{3} = 20.9 \pm 1.429$$

Thus the 95% confidence interval for μ is $(19.471, 22.329)$

7.1.11 Students took $n = 35$ samples of water from the east basin of Lake Macatawa and measured the amount of sodium in parts per million. For their data, they calculated $\bar{x} = 24.11$ and $s^2 = 24.44$. Find an approximate 90% confidence interval for μ, the mean of the amount of sodium in parts per million.

Answer: Here $\alpha = 0.1,\ t_{0.05(34)} \approx 1.645$

$$\bar{X} \pm t_{0.05(34)} \frac{s}{\sqrt{n}} = 24.11 \pm 1.645 \frac{4.944}{5.916} = 24.11 \pm 1.37 = [22.74, 25.48]$$

Thus the 90% confidence interval for μ is $(22.74, 25.48)$