Running example

$x = ACGTA$ and $y = ATCTG$

Running example

$x = ACGTA$ and $y = ATCTG$

	A	T	C	T	G
A					
C					
G					
T					
A					

Running example

$x = ACGTA$ and $y = ATCTG$

		A	T	C	T	G
	0	1	2	3	4	5
A	1	0	1	2	3	4
C	2	1	1	1	2	3
G	3	2	2	2	2	2
T	4	3	2	3	2	3
A	5	4	3	3	3	3

A T C T G
A C G T A

Cost: 3

A T C - T G
A - C G T A

Cost: 3

Pseudocode

def EDIT_DISTANCE(x, y)**:**

Pseudocode

def $\textsc{Edit\_Distance}(x, y)$**:**

  for $i = 0, \ldots, m$**:**

Pseudocode

def $\textsc{Edit\_Distance}(x, y)$**:**

  for $i = 0, \ldots, m$**:**

    $E(i, 0) = i$;

Pseudocode

def EDIT_DISTANCE(x, y)**:**

```
for i = 0, ..., m:
    E(i, 0) = i;
for j = 0, ..., n:
```

Pseudocode

```
def EDIT_DISTANCE(x, y):
    for i = 0, ..., m:
        E(i, 0) = i;
    for j = 0, ..., n:
        E(0, j) = j;
```

Pseudocode

```
def EDIT_DISTANCE(x, y):
    for i = 0, ..., m:
        E(i, 0) = i;
    for j = 0, ..., n:
        E(0, j) = j;
    for i = 1, ..., m:
```

Pseudocode

def $\textsc{Edit\_Distance}(x, y)$**:**

```
for i = 0, ..., m:
    E(i, 0) = i;
for j = 0, ..., n:
    E(0, j) = j;
for i = 1, ..., m:
    for j = 1, ..., n:
```

Pseudocode

def $\text{EDIT\_DISTANCE}(x, y)$:

```
for i = 0, ..., m:
    E(i, 0) = i;
for j = 0, ..., n:
    E(0, j) = j;
for i = 1, ..., m:
    for j = 1, ..., n:
        E(i, j) =
          min{1 + E(i-1, j), 1 + E(i, j-1), diff(i, j) + E(i-1, j-1)};
```

Pseudocode

```
def EDIT_DISTANCE(x, y):
    for i = 0, ..., m:
        E(i, 0) = i;
    for j = 0, ..., n:
        E(0, j) = j;
    for i = 1, ..., m:
        for j = 1, ..., n:
            E(i, j) =
             min{1 + E(i-1, j), 1 + E(i, j-1), diff(i, j) + E(i-1, j-1)};
    return E(m, n);
```

Pseudocode

```
def EDIT_DISTANCE(x, y):
    for i = 0, ..., m:
        E(i, 0) = i;
    for j = 0, ..., n:
        E(0, j) = j;
    for i = 1, ..., m:
        for j = 1, ..., n:
            E(i, j) =
              min{1 + E(i − 1, j), 1 + E(i, j − 1), diff(i, j) + E(i − 1, j − 1)};
    return E(m, n);
```

Running time: $O(mn)$

Finding the alignment

We use an extra table prev to record where each entry of $E(i, j)$ was coming from:

Finding the alignment

We use an extra table prev to record where each entry of $E(i,j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + \cancel{E(i,j)} \end{cases}$$

$E(i-1, j-1)$

Finding the alignment

We use an extra table prev to record where each entry of $E(i, j)$ was coming from:

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def $\textsc{Pring\_Alignment}(x, y, \mathrm{prev})$**:**

Finding the alignment

We use an extra table prev to record where each entry of $E(i,j)$ was coming from:

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def $\textsc{Pring\_Alignment}(x, y, \mathrm{prev})$**:**

Set $i = m, j = n$;

Finding the alignment

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Finding the alignment

We use an extra table prev to record where each entry of $E(i, j)$ was coming from:

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def $\textsc{Pring\_Alignment}(x, y, \mathrm{prev})$**:**

  Set $i = m, j = n$;

  if $\mathrm{prev}(i,j) = (i-1, j-1)$**:**

    $\mathrm{print\_back}\binom{y_i}{x_i}$;

Finding the alignment

We use an extra table prev to record where each entry of $E(i,j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i,j) \end{cases}$$

def PRING_ALIGNMENT(x, y, prev)**:**

```
    Set i = m, j = n;
    if prev(i, j) = (i − 1, j − 1):
        print_back(y_i over x_i);
    if prev(i, j) = (i − 1, j):
```

Finding the alignment

We use an extra table prev to record where each entry of $E(i,j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i,j) \end{cases}$$

def $\textsc{Pring\_Alignment}(x, y, \mathrm{prev})$**:**

Set $i = m, j = n$;

if $\mathrm{prev}(i,j) = (i-1,j-1)$**:**

print_back$\binom{y_i}{x_i}$;

if $\mathrm{prev}(i,j) = (i-1,j)$**:**

print_back$\binom{\text{-}}{x_i}$;

Finding the alignment

We use an extra table prev to record where each entry of $E(i,j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i,j) \end{cases}$$

def PRING_ALIGNMENT(x, y, prev)**:**

  Set $i = m, j = n$;

  if $\mathrm{prev}(i,j) = (i-1,j-1)$**:**

    $\mathrm{print\_back}\binom{y_i}{x_i}$;

  if $\mathrm{prev}(i,j) = (i-1,j)$**:**

    $\mathrm{print\_back}\binom{-}{x_i}$;

  if $\mathrm{prev}(i,j) = (i,j-)$**:**

Finding the alignment

We use an extra table prev to record where each entry of $E(i, j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1, j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i, j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1, j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i,j) \end{cases}$$

def $\textsc{Pring\_Alignment}(x, y, \mathrm{prev})$**:**

- Set $i = m, j = n$;
- **if** $\mathrm{prev}(i,j) = (i-1, j-1)$**:**
  - $\mathrm{print\_back}\binom{y_i}{x_i}$;
- **if** $\mathrm{prev}(i,j) = (i-1, j)$**:**
  - $\mathrm{print\_back}\binom{\text{-}}{x_i}$;
- **if** $\mathrm{prev}(i,j) = (i, j-)$**:**
  - $\mathrm{print\_back}\binom{y_j}{\text{-}}$;

Print_back

Dynamic Programming

0-1 Knapsack (Textbook Section 6.4)

0-1 Knapsack

0-1 Knapsack Problem

A Thief has a backpack with certain capacity. There is a set of items with certain weight and value. **Goal:** pack the backpack with the largest value

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- Doesn't have the greedy choice property
- But it has the optimal substructure property:

0-1 Knapsack

0-1 Knapsack Problem

W

A Thief has a backpack with certain capacity. There is a set of items with certain weight and value. **Goal:** pack the backpack with the largest value

- Doesn't have the greedy choice property
- But it has the optimal substructure property:
  Suppose the optimal packing has weight $\leq W$. If we remove item j from it, the remaining packing must be the optimal packing for capacity $W - w_j$ with items excluding j

Subproblem

- **Subproblem**: $K(w, j)$ — the maximum value achievable using a backpack of capacity w and items $1, \ldots, j$

case 1: j is used in the optimal packing for $K(w,j)$

$$K(w,j) = K(\underbrace{\boxed{w}}_{w-w_j}, j) + \boxed{val(j)}$$

case 2: j is not used

$$K(w,j) = K(w, j-1)$$

Subproblem

- **Subproblem**: $K(w, j)$ — the maximum value achievable using a backpack of capacity w and items $1, \ldots, j$
- **Optimal solution:** $K(W, n)$

Subproblem

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- **Optimal solution:** $K(W, n)$
- **Recurrence:**

$$\boxed{K(w,j) = \max\{K(w - w_j, j - 1) + v_j, K(w, j - 1)\}}$$

Base case:

$K(0, j) = 0 \qquad K(w, 0) = 0$

Subproblem

- **Subproblem**: $K(w, j)$ — the maximum value achievable using a backpack of capacity w and items $1, \ldots, j$
- **Optimal solution:** $K(W, n)$
- **Recurrence:**

Value of item j

$$K(w, j) = \max\{K(w - w_j, j - 1) + v_j, K(w, j - 1)\}$$

- **Base case:** $K(0, j) = 0$ for all j and $K(w, 0) = 0$ for all w

Pseudocode

def KNAPSACK(W, w, v):

Pseudocode

def $\textsc{Knapsack}(W, w, v)$**:**

Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

Pseudocode

def $\textsc{Knapsack}(W, w, v)$**:**

Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

for $j = 1, \ldots, n$**:**

Pseudocode

def $\textsc{Knapsack}(W, w, v)$**:**

Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

for $j = 1, \ldots, n$**:**

for $w = 1, \ldots, W$**:**

Pseudocode

def KNAPSACK(W, w, v)**:**

  Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

  for $j = 1, \ldots, n$**:**

    for $w = 1, \ldots, W$**:**

      if $w_j > w$**:**

Pseudocode

```
def KNAPSACK(W, w, v):
    Set K(0, j) = 0, K(w, 0) = 0 for all j, w;
    for j = 1, ..., n:
        for w = 1, ..., W:
            if w_j > w:
                K(w, j) = K(w, j − 1);
```

Pseudocode

def $\textsc{Knapsack}(W, w, v)$**:**

    Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

    for $j = 1, \ldots, n$**:**

        for $w = 1, \ldots, W$**:**

            if $w_j > w$**:**

                $K(w, j) = K(w, j-1)$;

            else:

Pseudocode

```
def KNAPSACK(W, w, v):
    Set K(0, j) = 0, K(w, 0) = 0 for all j, w;
    for j = 1, ..., n:
        for w = 1, ..., W:
            if w_j > w:
                K(w, j) = K(w, j − 1);
            else:
                K(w, j) = max{K(w − w_j, j − 1) + v_j, K(w, j − 1)};
```

Pseudocode

capacity $\quad w_1, w_2, \cdots, w_n$

def KNAPSACK(W, w, v): $\longrightarrow v_1, v_2, \cdots v_n$

- Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;
- **for** $j = 1, \ldots, n$**:**
  - **for** $w = 1, \ldots, W$**:**
    - **if** $w_j > w$**:**
      - $K(w, j) = K(w, j - 1)$;
    - **else:**
      - $K(w, j) = \max\{K(w - w_j, j - 1) + v_j, K(w, j - 1)\}$;
- **return** $K(W, n)$; $\quad n \cdot W$.

Pseudocode

def $\textsc{Knapsack}(W, w, v)$**:**

Set $K(0,j) = 0, K(w,0) = 0$ for all j, w;

for $j = 1, \ldots, n$**:**

for $w = 1, \ldots, W$**:**

if $w_j > w$**:**

$K(w,j) = K(w, j-1)$;

else:

$K(w,j) = \max\{K(w - w_j, j-1) + v_j, K(w, j-1)\}$;

return $K(W, n)$;

Running time: $O(nW)$

Pseudocode

def $\text{KNAPSACK}(W, w, v)$**:**

  Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

  for $j = 1, \ldots, n$**:**

    for $w = 1, \ldots, W$**:**

      if $w_j > w$**:**

        $K(w, j) = K(w, j - 1)$;

      else:

        $K(w, j) = \max\{K(w - w_j, j - 1) + v_j, K(w, j - 1)\}$;

  return $K(W, n)$;

$O(\log W)$ bits for W

input size

Running time: $O(nW)$

Question: is this a polynomial-time algorithm?

Pseudocode

$\boxed{j = 1}$, $w = 6$

$w_j = w_1 = 6$

def $\textsc{Knapsack}(W, w, v)$**:**

    Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;
    for $j = 1, \ldots, n$**:**
        for $w = 1, \ldots, W$**:**
            if $w_j > w$**:**
                $K(w, j) = K(w, j-1)$;
            else:
                $K(w, j) = \max\{K(w - w_j, j-1) + v_j, K(w, j-1)\}$;
    return $K(W, n)$;

$j = 1$, $w = 1$

$K(1,1) = K(1,0)$

$\underset{6}{w_1} > \underset{1}{w}$

$\underbrace{K(0,0) + V_1}_{30}$, $\underbrace{K(6,0)}_{0}$

Running time: $O(nW)$

Question: is this a polynomial-time algorithm? No!

Running example

Example: $W = 10$

item	1	2	3	4
w_j	6	3	4	2
v_j	30	14	16	9

Running example

Example: $W = 10$

item	1	2	3	4
w_j	6	3	4	2
v_j	30	14	16	9

$K(w - w_j, j-1) + v_j$

$\boxed{K(w, j-1)}$

The K table:

$w \backslash j$	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0	0			
3	0	0			
4	0	0			
5	0	0			
6	0	30			
7	0				
8	0				
9	0				
10	0				

$K(1,1) =$

$K(6,1) =$

Running example

Example: $W = 10$

item	1	2	3	4
w_j	6	3	4	2
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The K table:

$w \backslash j$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	9
3	0	0	14	14	14
4	0	0	14	16	16
5	0	0	14	16	23
6	0	30	30	30	30
7	0	30	30	30	30
8	0	30	30	30	39
9	0	30	44	44	44
10	0	30	44	46	46

Running example

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4	0	0	14	16	16
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Dynamic Programming

Chain matrix multiplication (Textbook Section 6.5)

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$
Need to compute

$$M_1 \cdot M_2 \cdots M_n$$

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$

Need to compute

$$M_1 \cdot M_2 \cdots M_n$$

$\mathbb{R}^{m \times n}$ the class of all the matrices of dim. $m \times n$, where each entry is a real number $\mathbb{R}$

The dimensions of these matrices are:

$$M_1 \in \mathbb{R}^{m_0 \times m_1}, M_2 \in \mathbb{R}^{m_1 \times m_2}, \ldots, M_n \in \boxed{\mathbb{R}^{m_{n-1} \times m_n}}$$

$\boxed{\mathbb{C}^{m \times n}}$

complex number

Chain matrix multiplication

We have n matrices $M_1, M_2, \dots, M_n$
Need to compute

$$M_1 \cdot M_2 \cdots M_n$$

The dimensions of these matrices are:

$$M_1 \in \mathbb{R}^{m_0 \times m_1}, M_2 \in \mathbb{R}^{m_1 \times m_2}, \dots, M_n \in \mathbb{R}^{m_{n-1} \times m_n}$$

Recall if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then the cost for computing $A \cdot B$ is $m \cdot n \cdot p$

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$
Need to compute

$$M_1 \cdot M_2 \cdots M_n$$

The dimensions of these matrices are:

$$M_1 \in \mathbb{R}^{m_0 \times m_1}, M_2 \in \mathbb{R}^{m_1 \times m_2}, \ldots, M_n \in \mathbb{R}^{m_{n-1} \times m_n}$$

Recall if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then the cost for computing $A \cdot B$ is $m \cdot n \cdot p$

Also, matrix multiplication is associative:
$A \cdot B \cdot C = (A \cdot B) \cdot C = A \cdot (B \cdot C)$

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$

Need to compute

$$M_1 \cdot M_2 \cdots M_n$$

The dimensions of these matrices are:

$$M_1 \in \mathbb{R}^{m_0 \times m_1}, M_2 \in \mathbb{R}^{m_1 \times m_2}, \ldots, M_n \in \mathbb{R}^{m_{n-1} \times m_n}$$

Recall if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then the cost for computing $A \cdot B$ is $m \cdot n \cdot p$

Also, matrix multiplication is associative:
$A \cdot B \cdot C = (A \cdot B) \cdot C = A \cdot (B \cdot C)$

Question: what's the best way for computing $M_1 \cdot M_2 \cdots M_n$? i.e., where to put the parentheses?

Example of chain matrix multiplication

Consider $M_1 \in \mathbb{R}^{50\times 20}, M_2 \in \mathbb{R}^{20\times 1}, M_3 \in \mathbb{R}^{1\times 10}, M_4 = \mathbb{R}^{10\times 100}$

Example of chain matrix multiplication

Consider $M_1 \in \mathbb{R}^{50\times 20}, M_2 \in \mathbb{R}^{20\times 1}, M_3 \in \mathbb{R}^{1\times 10}, M_4 = \mathbb{R}^{10\times 100}$

There are many ways to do multiplication

Example of chain matrix multiplication

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There are many ways to do multiplication

- $M_1 \cdot \boxed{(\overbrace{(M_2 \cdot M_3)}^{20\times 10} \cdot M_4)}$ 20×100

$20\times 1\times 10 + 20\times 10\times 100 + 50\times 20\times 100$

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- $\underbrace{(M_1 \cdot (\overbrace{M_2 \cdot M_3}^{20\times 10}))}_{50\times 10} \cdot M_4$

  $20\times 1\times 10 + 50\times 20\times 10 + 50\times 10\times 100$

Example of chain matrix multiplication

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- $(M_1 \cdot M_2) \cdot (M_3 \cdot M_4)$

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- $(M_1 \cdot M_2) \cdot (M_3 \cdot M_4)$
  Cost: $50 \cdot 20 \cdot 1 \cdot 10 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100 = 7000$

Example of chain matrix multiplication

$M_1 M_2 \cdots M_n$

Consider $M_1 \in \mathbb{R}^{50\times 20}$, $M_2 \in \mathbb{R}^{20\times 1}$, $M_3 \in \mathbb{R}^{1\times 10}$, $M_4 = \mathbb{R}^{10\times 100}$

There are many ways to do multiplication

- $M_1 \cdot (\underbrace{(M_2 \cdot M_3)}_{20\times 10} \cdot M_4)$ 20×100
  Cost: $20 \cdot 1 \cdot 10 + 20 \cdot 10 \cdot 100 + 50 \cdot 20 \cdot 100 = 10200$
- $(M_1 \cdot ((M_2 \cdot M_3)) \cdot M_4$
  Cost: $20 \cdot 1 \cdot 10 + 50 \cdot 20 \cdot 10 + 50 \cdot 10 \cdot 100 = 60200$
- $(M_1 \cdot M_2) \cdot (M_3 \cdot M_4)$
  Cost: $50 \cdot 20 \cdot 1 \cdot 10 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100 = 7000$

Goal: find a way to do multiplication with the minimum cost

Dynamic programming

optimal solution: $C(1,n)$

- **Subproblem:**
  $C(i,j)$ — the minimum cost for multiplying $M_i, M_{i+1}, \ldots, M_j$

Dynamic programming

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$m_{i-1} \times m_i$ $\quad$ $m_{k-1} \times m_k$

$$\boxed{C(i,j) = C(i,k-1) + C(k+1,j) + \underline{m_{i-1} \cdot m_k \cdot m_j}}$$

Dynamic programming

- **Subproblem:**
  $C(i, j)$ — the minimum cost for multiplying $M_i, M_{i+1}, \ldots, M_j$
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$$\Big(\underbrace{\overset{m_{i-1}\times m_i}{M_i} M_{i+1}\cdots \overset{m_{k-1}\times m_k}{M_k}}_{m_{i-1}\times m_k}\Big)\Big(M_{k+1}M_{k+2}\cdots M_j\Big)$$

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So, $\quad C(i,j) = \min\limits_{i \le k < j} \{C(i,k) + C(k+1,j) + m_{i-1} \cdot m_k \cdot m_j\}$

Base case? $\quad C(i,i) = 0$

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- **Base case:** $C(i, i) = 0$
- **Optimal solution:** $C(1, n)$

Pseudocode

def CHAIN_MATRIX(*m*):

Pseudocode

def CHAIN_MATRIX(m):

  for $i = 1 \dots n$:

Pseudocode

def $\textsc{Chain\_Matrix}(m)$:

  for $i = 1 \ldots n$:

    $C(i, i) = 0$;

Pseudocode

```
def CHAIN_MATRIX(m):
    for i = 1 ... n:
        C(i, i) = 0;
    for s = 1 ... n − 1:
```

Pseudocode

def $\textsc{Chain\_Matrix}(m)$:

```
for i = 1 ... n:
    C(i, i) = 0;
for s = 1 ... n - 1:
    for i = 1 ... n - s:
```

Pseudocode

```
def Chain_Matrix(m):
    for i = 1 ... n:
        C(i, i) = 0;
    for s = 1 ... n − 1:
        for i = 1 ... n − s:
            j = i + s;
```

Pseudocode

def $\textsc{Chain\_Matrix}(m)$**:**

```
for i = 1 ... n:
    C(i, i) = 0;
for s = 1 ... n − 1:
    for i = 1 ... n − s:
        j = i + s;
        C(i, j) = min_{i≤k<j} {C(i, k), C(k + 1, j) + m_{i−1} · m_k · m_j};
```

Pseudocode

$C(i,j)$

$j \geq i$

list of dims

def CHAIN_MATRIX(m):

    for $i = 1 \dots n$:

        $C(i,i) = 0$;

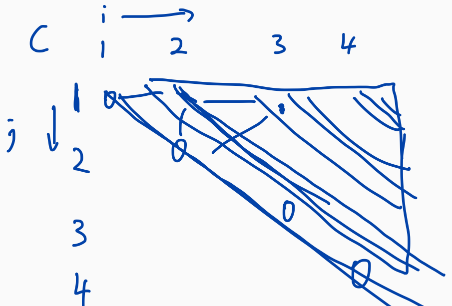
    for $s = 1 \dots n-1$:

        for $i = 1 \dots n-s$:

            $j = i + s$;

            $C(i,j) = \min_{i \leq k < j}\{C(i,k), C(k+1,j) + m_{i-1} \cdot m_k \cdot m_j\}$;

    return $C(1,n)$;



$O(n^2)$ entries.

cost for each entry : $O(n)$

Pseudocode

def CHAIN_MATRIX(m)**:**

```
for i = 1 ... n:
    C(i, i) = 0;
for s = 1 ... n − 1:
    for i = 1 ... n − s:
        j = i + s;
        C(i, j) = min_{i ≤ k < j} {C(i, k), C(k + 1, j) + m_{i−1} · m_k · m_j};
return C(1, n);
```

Running time:

Pseudocode

```
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Running time:

$O(n^2)$ entries to fill; $O(n)$ operations to fill in each entry

Pseudocode

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def CHAIN_MATRIX(m):
    for i = 1 ... n:
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        for i = 1 ... n − s:
            j = i + s;
            C(i, j) = min_{i≤k<j}{C(i, k), C(k + 1, j) + m_{i−1} · m_k · m_j};
    return C(1, n);
```

Running time:

$O(n^2)$ entries to fill; $O(n)$ operations to fill in each entry

Total running time: $O(n^3)$