CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Flow network
(Textbook, Section 7.2
Kleinberg & Tardos Section 7.1)

Analysis

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So total running time is $O(C \cdot |E|)$

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- **Conservation condition.** It suffices to observe that for every vertex, additional amount of flow, 0, or $\text{bottleneck}(P, f)$ entering this vertex equals the additional amount of flow, 0, or $\text{bottleneck}(P, f)$ leaving it

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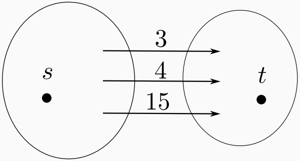

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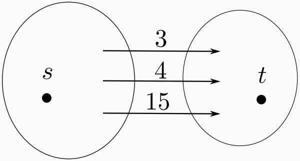

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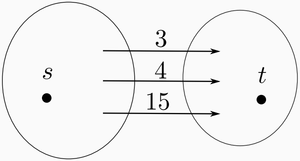

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Capacity of a cut put a bound on the flow value

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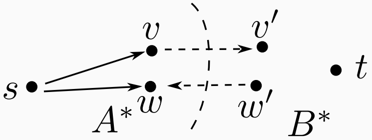

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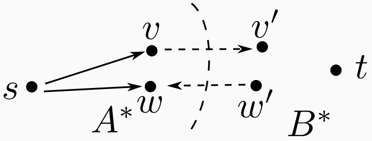
  Forward edge. So v' is reachable from s in G_f (contradiction)



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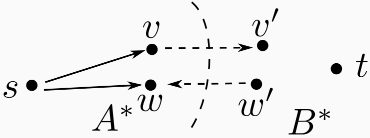

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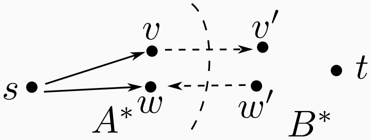

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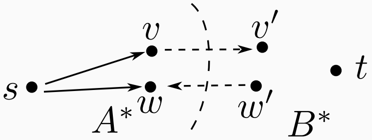

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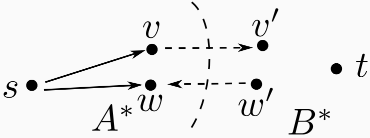

Then

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v(f) &= \sum_{e \text{ out of } A^*} f(e) - \sum_{e \text{ into } A^*} f(e) \quad \text{(from the proof of } v(f) \leq c(A, B)) \\
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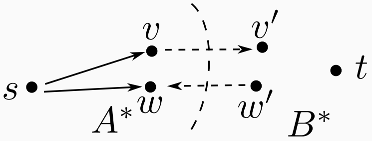

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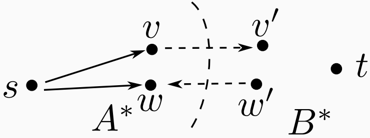

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□

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- If all capacities of a flow network are integers, then there is a max flow f s.t. $f(e)$ is an integer for all e