CS 461

Programming Language Concepts

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Lambda Calculus

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Readings

- Ch11.7 of the supplemental materials of the textbook
  - See the schedule page of the course website

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History

- History
  - Introduced by Alonzo Church
  - Greek letter lambda, which is used to introduce functions
  - No significance to the letter lambda
  - Calculus means there is a way to
    - calculate the result of applying functions to arguments
- Most PLs are rooted in lambda calculus
  - It provides a basic mechanism for function abstraction and application
  - Functional PLs: Lisp, ML, Haskell, other languages
  - Java, C++, and C# all support lambda functions
- Important part of CS history and foundations
- Warning:
  - We'll study formalism

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Syntax

- <term> ::= <var> | λ<var>.<term> | <term> <term>
- t ::= x | λx. t | t1 t2
  - where x may be any variable
  - Function abstraction (function definition): λx. t
    - Define a new function whose parameter is x and whose body is t
    - Racket: (lambda (x) t)
  - Function application (function call): t1 t2
    - t1 should eval to a function; t2 is the argument to the function
    - Racket: (t1 t2)
    - Math: t1(t2)

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Examples

- Function abstraction
  - λx. x
    - there is no need to write explicit returns; x is the returning result
  - λx. (x+3)
    - assume + is a built-in function
  - λf. λx. f (f x)
    - multi-parameter function, in curried notation
    - Only curried functions are supported in lambda calculus
- Function application
  - (λx. x) 3 -> 3
  - (λx. (x+y)) 3 -> 3 + y
  - (λx. λy. (x+y)) 3 4 -> 3 + 4
  - (λz. (x + 2*y + z)) 5 -> x + 2*y + 5

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Parsing convention

- The lambda-calculus grammar is ambiguous
  - E.g., t1 t2 t3 can be parsed in different ways
  - We'll use parentheses and associativity to disambiguate
- Convention
  - function abstraction: the scope of functions extends as far to the right as possible (unless encountering parentheses)
    - – λf. f x = λf.(f x), not (λf. f) x
  - function application is left associative
    - – t 2 3 = ((t 2) 3), not t (2 3), suppose f = λx. λy. x + y

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Reduction (Informally)

- (λx. x) 3 = 3
  - using 3 to replace x
- (λy. (y+1)) 3
- (λx. x) (λz. z)
- (λx. x) (λx. x)
- (λf. λx. f (f x)) (λy. y+1)

= λx. (λy. y+1) ((λy. y+1) x)

= λx. (λy. y+1) (x+1)

= λx. (x+1)+1

- (λf. λx. f (f x)) (λy. y*y)

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Free and Bound Variables

- "λx. t" binds a new var x and its scope is t
  - Occurrences of x in t are said to be bound
    - Variable x is bound in λx. (x+y)
  - A bound variable has a scope: In "λx. t", the scope of x is t
  - A bound variable is a "placeholder" and can be renamed
    - Function λx. (x+y) is the same function as λz. (z+y)
- Names of free (=unbound) variables matter
  - Variable y is free in λx. (x+y)
  - Function λx. (x+y) is *not* the same as λx. (x+z)
- Example: λx. ((λy. y+2) x) + y
  - y in "y+2" is bound, while the second occurrence of y is free

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Formal def. of free variables

Goal: define FV(t), the set of free variables of t

$FV(x) = \{x\}$
$FV(t_1\ t_2) = FV(t_1) \cup FV(t_2)$
$FV(\lambda x.\ t) = FV(t) - \{x\}$

- FV(λx. x) = FV(x) – {x} = {}
- FV(λf. λx. f (g x)) = FV(λx. f (g x)) – {f}
  = FV(f (g x)) – {f,x} = {f,g,x} – {f,x} = {g}
- Exercise
  - FV((λx. x) (λy. y))
  - FV(λx. ((λy. y+2) x) + y)

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Alpha renaming (rename bound variables)

λx. t = λy. [y/x] t (α)
when y is not free in t

- λx. x = λy. y
- λx. ((λy. y+2) x) + y, rename the first y to z
  - Becomes λx. ((λz. z+2) x) + y
- λx. λy. x - y = λy. λx. y – x, rename x to y and y to x

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Capture-Avoiding Substitution

- Notation: [t/x] t' means using t to replace all **free** occurrences of x in t'
  - Note: bound occurrences of x should not be affected
- Definition of [t/x] t'

[t/x] x = t,
[t/x] y = y, where y is a variable different from x
[t/x] (t1 t2) = ([t/x] t1) ([t/x] t2)
[t/x] (λx. t1) = λx. t1
[t/x] (λy. t1) = λy. ([t/x] t1), where y is not free in t

- [λx. x / x] x = λx. x
- [3/y] (λx. x + y) = λx. x + 3
- [3/x] (λx. x + y) = λx. x + y
- [y/x] (λy. x+y) = [y/x] (λz. x+z) = λz. y+z

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Reduction (Formal Semantics)

- Basic computation rule is β-reduction

  (λx. t') t -> [t/x] t'

  where substitution involves renaming as needed
- Reduction sequence:
  - Apply the β-reduction rule to any subterm
  - Repeat until no β-reduction is possible
- Normal form: a lambda-calculus term that cannot be further reduced
- Example:
  - (λf. λx. f (f x)) (λy. y+1) 3

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Reduction Maybe Nonderterministic

- An example of two beta-reduction sequences
  - (λy. y) ((λy. y) 2) -> (λy. y) 2 -> 2
  - (λy. y) ((λy. y) 2) -> ((λy. y) 2) -> 2
- Confluence (Church-Rosser theorem):
  - Final result (if there is one) is uniquely determined

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Reduction May Not Terminate

Ω Combinator: $\lambda x.(x\ x)$

Evaluate: Ω $(\lambda v.v)$ -> $(\lambda x.(x\ x))$ $(\lambda v.v)$
-> $(\lambda v.v)$ $(\lambda v.v)$ -> $(\lambda v.v)$

Evaluate: Ω Ω -> $(\lambda x.(x\ x))$ $(\lambda x.(x\ x))$
-> $(\lambda x.(x\ x))$ $(\lambda x.(x\ x))$ -> …
Infinite loop!

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Importance of Renaming Bound Variables

- Function application

  $\underbrace{(\lambda f.\ \lambda x.\ f\ (f\ x))}_{\text{apply twice}}\ \underbrace{(\lambda y.\ y+x)}_{\text{add x to argument}}$
- Substitute "blindly" and wrong result

  Wrong step

  [(λy. y+x) / f] ~~(λx. f (f x))~~

  = λx. [(λy. y+x) ((λy. y+x) x)] = λx. x+x+x
- Rename bound variables

  (λf. λz. f (f z)) (λy. y+x)

  = λz. ((λy. y+x) ((λy. y+x) z))) = λz. z+x+x

Easy rule: always rename bound variables to be distinct

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Programming in Lambda Calculus

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Declarations as "Syntactic Sugar"

- Informal Examples
  - let x = 3 in x + 4
  - let x = 3 let y = 4 in x + y + y
  - let f = λ x. x+1 in f(3)
  - let g = λ f. λ x. f(f (x)) in
    let h = λ x. x+1
    g h 2
- Encoding of let
  - let x = N in M same as (λx. M) N
- Syntactic sugar: the let is sweeter to write, but we can think of it as a syntactic magic

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Declarations as "Syntactic Sugar"

```
function f(x)
    return x+2
end;
f(5);
```

- same as let f =λx. x+2 in (f 5)

$\underbrace{(\lambda f.\ f(5))}_{\text{block body}}\ \underbrace{(\lambda x.\ x+2)}_{\text{declared function}}$

Extra reading: Tennent, *Language Design Methods Based on Semantics Principles.* Acta Informatica, 8:97-112, 197

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Encoding: Boolean

Booleans

$\text{TRUE} \triangleq \lambda x.\lambda y.x \qquad \text{FALSE} \triangleq \lambda x.\lambda y.y$

Encoding "if" so that

$$\text{Spec: IF } b\ t1\ t2 = \begin{cases} t1 \text{ when } b \text{ is TRUE} \\ t2 \text{ when } b \text{ is FALSE} \end{cases}$$

Definition: $\text{IF} \triangleq \lambda b.\lambda t1.\lambda t2.(b\ t1\ t2)$

Check IF TRUE t1 t2 = t1 and IF FALSE t1 t2 = t2

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Encoding: Boolean

Booleans

$\text{TRUE} \triangleq \lambda x.\lambda y.x \qquad \text{FALSE} \triangleq \lambda x.\lambda y.y$

Encoding of "and"

$$\text{Spec: AND } b_1\ b_2 = \begin{cases} \text{TRUE when } b_1, b_2 \text{ are both TRUE} \\ \text{FALSE otherwise} \end{cases}$$

Definition: $\text{AND} \triangleq \lambda b_1.\lambda b_2.\ (b_1\ (b_2 \text{ TRUE FALSE}) \text{ FALSE})$

Check AND TRUE TRUE = TRUE and
AND FALSE TRUE = FALSE

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Encoding: Boolean

Booleans

$\text{TRUE} \triangleq \lambda x.\lambda y.x \qquad \text{FALSE} \triangleq \lambda x.\lambda y.y$

Encoding of "or"

$$\text{Spec: OR } b_1\ b_2 = \begin{cases} \text{TRUE when } \mathit{either}\ b_1\ \mathit{or}\ b_2 \text{ is TRUE} \\ \text{FALSE otherwise} \end{cases}$$

Definition: $\text{OR} \triangleq \lambda b_1.\lambda b_2.\ (b_1 \text{ TRUE } (b_2 \text{ TRUE FALSE}))$

Check OR TRUE TRUE = TRUE and
OR FALSE FALSE = FALSE

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Church Encoding of Numbers

Natural numbers

Church numerals: $\text{n} \triangleq \lambda f.\lambda z.\underbrace{f\ (f\ \ldots (f}_{\text{n invocations of f}}\ z) \ldots)$

$0 \triangleq \lambda f.\lambda z.z$
$1 \triangleq \lambda f.\lambda z.(f\ z)$
$2 \triangleq \lambda f.\lambda z.(f\ (f\ z))$
…

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Church Numerals

Encoding of "+1":
$\text{SUCC} \triangleq \lambda n.\lambda f.\lambda z.\ \ (f\ (n\ f\ z))$

Check "SUCC 2"= 3

Encoding of PLUS
$\text{PLUS} \triangleq \lambda n_1.\lambda n_2.\ \ (n_1 \text{ SUCC } n_2)$

Check "PLUS 1 2" = 3

Multiplication and exponentiation can also be encoded.

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Pure vs. Applied λ-Calculus

- Pure λ-Calculus: the calculus discussed so far

- Applied λ-Calculus:
  - Built-in values and data structures
    - (e.g., 1, 2, 3, true, false, (1 2 3))
  - Built-in functions
    - (e.g., +, *, /, and, or)
  - Named functions
  - Recursion

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