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- FORD-FULKERSON outputs the max flow

Running time analysis (I)

For simplicity, we assume the capacities are all integers

$\frac{1}{3}$ $\frac{1}{4}$

$\downarrow$ $\downarrow$

4 3

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The first edge of P leaves s, and P doesn't revisit s again. Moreover, it's a forward edge. So

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$$v(f') = v(f) + \mathrm{bottleneck}(P, f)$$

s $\xrightarrow{e}$

$f'(e) = f(e) + b$

□

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So total running time is $O(C \cdot |E|)$

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  Let $e = (u, v) \in P$. $\mathrm{bottleneck}(P, f)$ is at most the residual capacity of e
  - if e is a forward edge, then residual capacity: $c_e - f(e)$

$$0 \le f(e) \le f'(e) = f(e) + \mathrm{bottleneck}(P, f) \le f(e) + c_e - f(e) = c_e$$

So, $0 \le f'(e) \le c_e$

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residual capacity: $f(e) \geq \mathrm{bottleneck}(P,f)$

$$c_e \geq f(e) \geq f'(e) = f(e) - \mathrm{bottleneck}(P,f) \geq f(e) - f(e) = 0$$

$$\Rightarrow 0 \leq f'(e) \leq c_e$$

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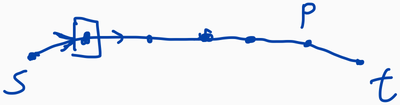
- **Conservation condition.** It suffices to observe that for every vertex, additional amount of flow, 0, or $\mathrm{bottleneck}(P, f)$ entering this vertex equals the additional amount of flow, 0, or $\mathrm{bottleneck}(P, f)$ leaving it



□

Correctness of Ford-Fulkerson (I)

Flow and Cut

Definition

An **s-t cut** is a partition of V, (A, B) where $s \in A$ and $t \in B$

$(\{s\}, V-\{s\})$

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$$\boxed{c(A, B)} = \sum_{e \text{ out of } A} c_e$$

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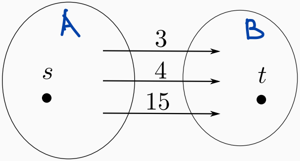
How does a cut help? $c(A,B) = 22$



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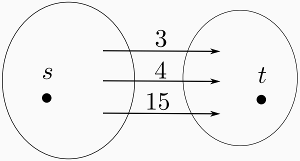

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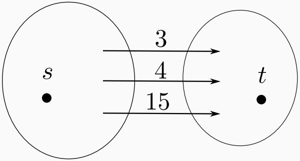

The flow must have a value ≤ 22
Capacity of a cut put a bound on the flow value

Correctness of Ford-Fulkerson (II)

Lemma

Let f be an s-t flow, (A, B) be an s-t cut. Then $v(f) \leq c(A, B)$

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$$v(f) = \sum_{e \text{ out } s} f(e) = f^{\text{out}}(s) = f^{\text{out}}(s) - \underbrace{f^{\text{in}}(s)}_{=0}$$

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for all $v \in A - \{s, t\}$ $\quad f^{out}(v) = f^{in}(v) \Rightarrow f^{out}(v) - f^{in}(v) = 0$

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$\implies \boxed{f^{\text{out}}(v) - f^{\text{in}}(v)} = 0$ for all $v \neq s, t$

$v(f) = f^{out}(s) - f^{in}(s)$ $+0 +0 +0 +0+ \cdots +0$

$= \sum_{v \in A} \left(f^{out}(v) - f^{in}(v) \right)$

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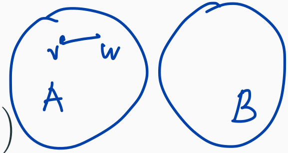

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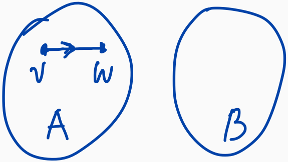

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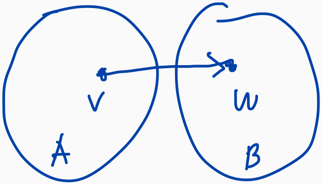
- if $v \in A, w \notin A$, this edge contributes $f(e)$



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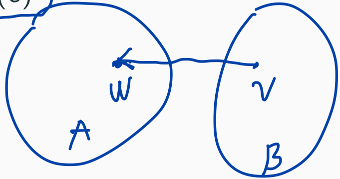

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$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \leq \sum_{e \text{ out of } A} f(e) = c(A, B)$$

$\square$

$v(f) \leq c(A, B)$)

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Lemma

Let f be a flow s.t. there's no s-t path in G_f. Then there exists an s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$

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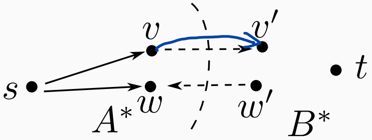
- for all edge $e = (v, v') \in E$ with $v \in A^*, v' \in B^*$, we have $f(e) = c_e$ Otherwise, (v, v') is an edge in G_f with capacity $c_e - f(e) \neq 0$.



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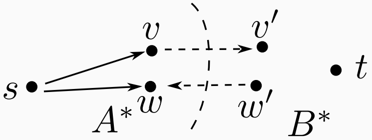
Lemma

Let f be a flow s.t. there's no s-t path in G_f. Then there exists an s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$

Proof. Let A^* be the set of vertices reachable from s in G_f. Let B^* be $V - A^*$. We have the follow facts

- (A^*, B^*) is an s-t cut: $s \in A^*$, $t \in B^*$ (no s-t path in G_f)
- for all edge $e = (v, v') \in E$ with $v \in A^*, v' \in B^*$, we have $f(e) = c_e$ Otherwise, (v, v') is an edge in G_f with capacity $c_e - f(e) \neq 0$. Forward edge. So v' is reachable from s in G_f (contradiction)



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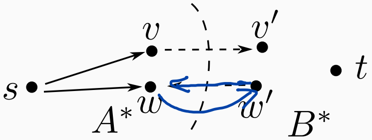

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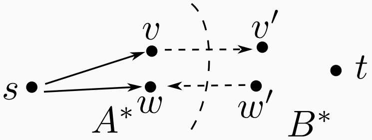

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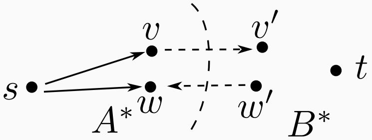

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$$v(f) = \underbrace{\sum_{e \text{ out of } A^*} f(e)}_{c_e} - \underbrace{\sum_{e \text{ into } A^*} f(e)}_{0} \quad \text{(from the proof of } v(f) \leq c(A, B)\text{)}$$

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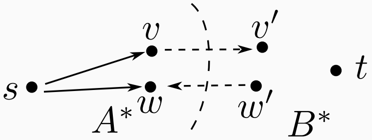

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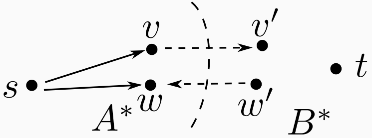

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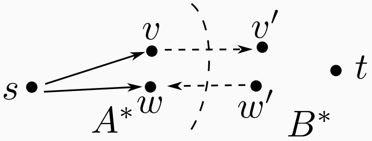

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$\square$

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- If all capacities of a flow network are integers, then there is a max flow f s.t. $f(e)$ is an integer for all e