CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

NP and Computational Hardness

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Polynomial-time reduction
(Kleinberg-Tardos, Section 8.1, 8.2)

Motivation

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Is Problem B really hard? How do we prove hardness?

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Tool: **polynomial-time** reduction

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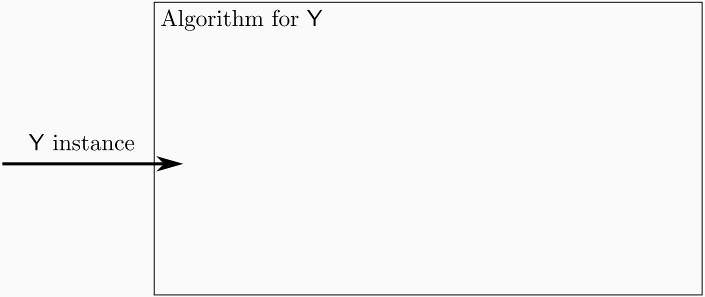

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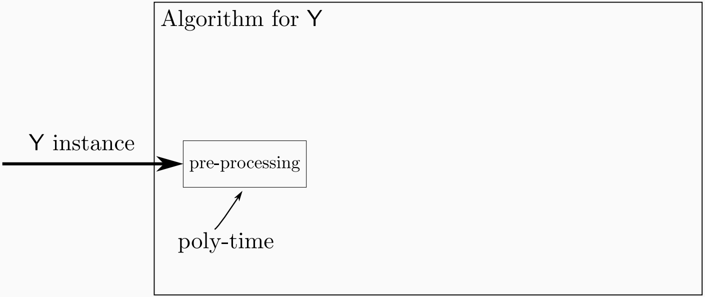

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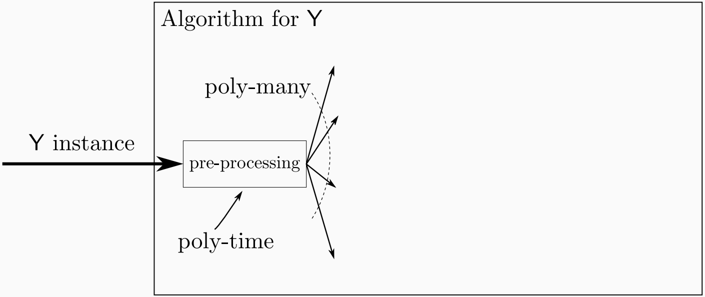

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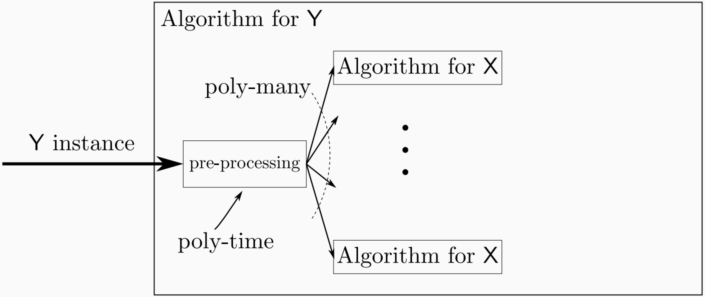

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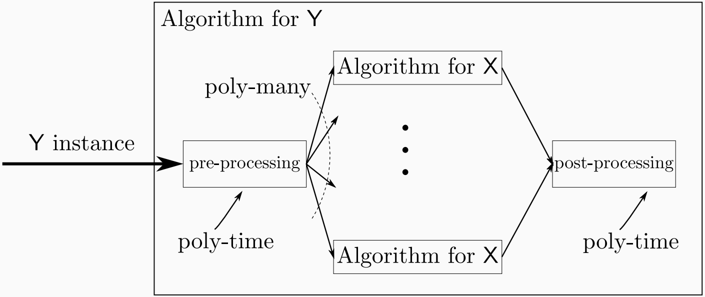

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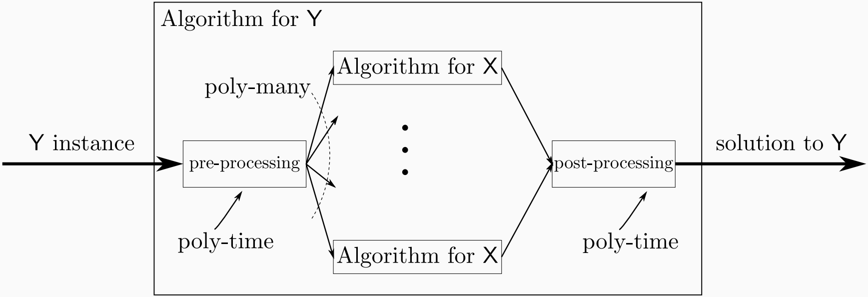

Consequences of polynomial time reductions

Lemma

Suppose $\mathsf{Y} \leq_P \mathsf{X}$. *If* X *can be solved in polynomial time, then*

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Suppose $Y \leq_P X$. *If* Y *cannot be solved in polynomial time, then* X *cannot be solved in polynomial time*

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A set of vertices is said to be **independent**, if no two of them are connected by an edge

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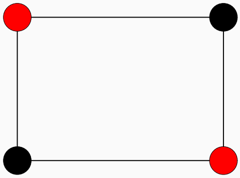

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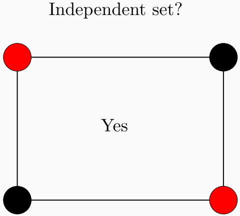

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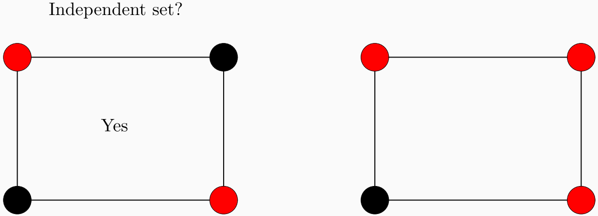

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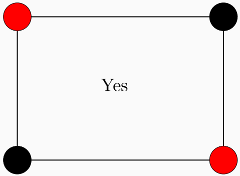

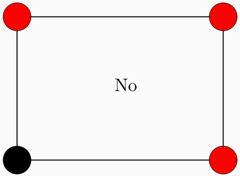

The Maximum Independent Set Problem

The Maximum Independent Set Problem (Decision version)

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Instance: a graph G, a number k

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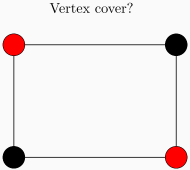

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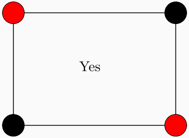

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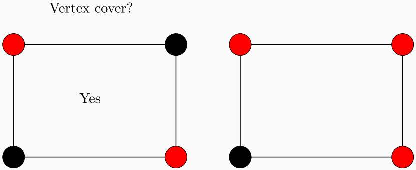

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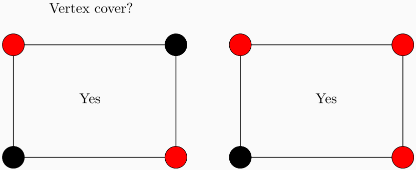

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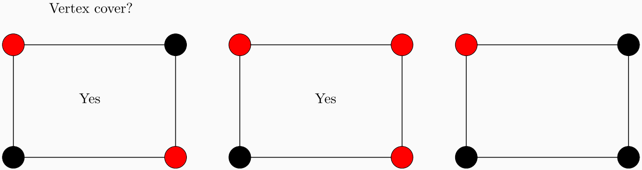

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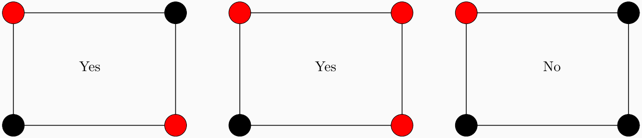

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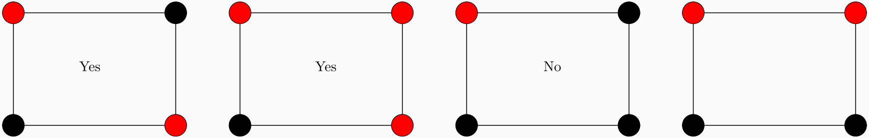

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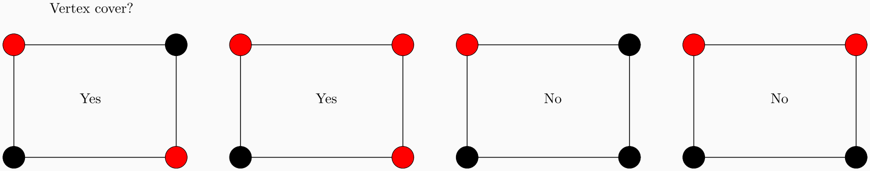

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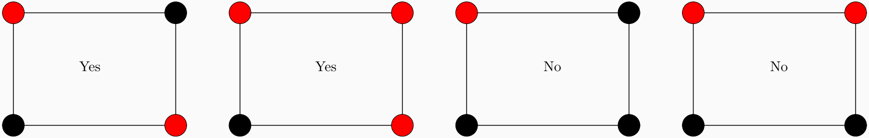

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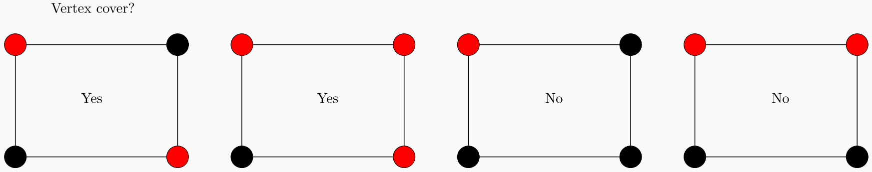

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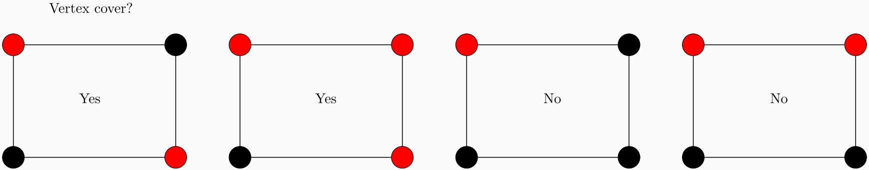

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Let $G = (V, E)$ be a graph. Then S is an independent set if and only if its complement $V - S$ is a vertex cover

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Proof. $\Rightarrow$

- "only if" need to show: S is an IS $\Rightarrow$ $V-S$ is VC

$\Leftarrow$ need to show:

- "if" $V-S$ is a VC $\Rightarrow$ S is an IS

if and only if

iff $\Longleftrightarrow$

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Independent Set vs Vertex Cover (II)

Theorem

- *Independent Set* $\leq_P$ *Vertex Cover*

Independent Set vs Vertex Cover (II)

Theorem

- *Independent Set* $\leq_P$ *Vertex Cover*
- *Vertex Cover* $\leq_P$ *Independent Set*