HW4 revision

score: 12/20

: 3 Quesetions.

E1 5/5

E2 0/5

P1 1/5

P2 1/5

Math 486

Lesson 4 Homework

Due Tues, June 14 at 11:59 on Gradescope

Instructions

Please refer to the solution guidelines posted on Canvas under Course Essentials.

Exercise 1.

Suppose we have a game with N players ($N \geq 3$) and suppose that the strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a Nash equilibrium.

(a) Describe in words what it means for the strategy profile s^* to be a Nash equilibrium. This description can be informal—it should reflect the way that you think about the Nash equilibrium concept—but it also must be conceptually correct.

(b) State, mathematically, what it means for s^* to be a Nash equilibrium. You need to use mathematical notation involving payoffs in an N-player game here.

Remark: The purpose here is not memorization of a mathematical definition, but to connect the mathematical definition with the conceptual idea expressed in words in part (a). The overall goal is to strengthen your understanding of what a Nash Equilibrium is, so that you can approach problems requiring you to show that a particular strategy profile is or is not a Nash equilibrium.

a) make S^* to be a Nash equilibrium, it means for the player, there is no better streategy while others choose this strategy, vise versa, and at this time, it is a Nash equilibrium.

b) $\pi_i(s_i^*, s_{-i}^*) - \pi_i(s_i, s_{-i}^*) \geq 0$ for all $s_i \in S_i$

Exercise 2.

Consider the following two player game. Each player has 1 (divisible) unit of resource.

Player 1 chooses some amount of the resource $s_1 \in [0, 1]$. Player 2 simultaneously chooses some amount $s_2 \in [0, 1]$.

The player that chooses the higher amount wins the total of what remains: $(1 - s_1) + (1 - s_2)$. The player that chooses the lower amount gets a payoff of zero (they forfeit the remaining amount to the other player). In the case of a tie, they split what remains.

The payoff for each player can be written as follows:

$$\pi_1(s_1, s_2) = \begin{cases} (1 - s_1) + (1 - s_2), & \text{if } s_1 > s_2, \\ 1 - s_1, & \text{if } s_1 = s_2, \\ 0, & \text{if } s_1 < s_2 \end{cases}, \quad \pi_2(s_1, s_2) = \begin{cases} (1 - s_1) + (1 - s_2), & \text{if } s_2 > s_1, \\ 1 - s_2, & \text{if } s_1 = s_2, \\ 0, & \text{if } s_2 < s_1 \end{cases}$$

(a) Show that when $s_1 = s_2 = 1$, this is a Nash equilibrium of the game.

(b) Show that there are no Nash equilibria where $s_2 < s_1 < 1$.

(c) Show that there are no Nash equilibria where $s_2 < s_1$ with $s_1 = 1$.

(d) Show that there are no Nash equilibria where $s_1 = s_2 < 1$.

$S_1 = S_2 = [0,1]$

$$\pi_1(s_1, s_2) \begin{cases} (1-s_1)+(1-s_2) & \text{if } s_1 > s_2 \\ 1-s_1 & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 < s_2 \end{cases}$$

$$\pi_2(s_1, s_2) \begin{cases} (1-s_1)+(1-s_2) & \text{if } s_1 < s_2 \\ 1-s_1 & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 > s_2 \end{cases}$$

a) if $s_1 = s_2 = 1$, Nash

$\pi_1(1,1) = \pi_2(1,1) = 0$

if $s_1 < 1$ then $\pi_2(s_1, 1) = 0$

So $\pi_1(s_1, 1) \leq \pi_1(1,1)$ for all $s_1 \in [0,1]$

b) No Nash for $s_2 < s_1 < 1$

$\pi_1(s_1, s_2) = (1-s_1)+(1-s_2)$ $\qquad$ $\pi_2(s_1, s_2) = 0$

if $s_2' > s_2$ $\qquad$ then $\pi_2(s_1, s_2') = 1 - s_1 + 1 - s_2' > 0$

c) No Nash for $s_2 < s_1 \geq 1$

$\pi_1(1, s_2) = 1 - s_2$ $\pi_2(1, s_2) = 0$ (No change avilable)

if $s_2 < s_1' < 1$

$\pi_1(s_1', s_2) = 1 - s_1' + 1 - s_2 > 1 - s_2$ so No Nash

d) No Nash $s_1 = s_2 < 1$

$\pi_1(s_1, s_2) = 1 - s_1$ $\pi_2(s_1, s_2) = 1 - s_2$

if $s_1' > s_2$ then $\pi_1(s_1', s_2) = 1 - s_1' + 1 - s_2$

Problem 1.

Recall the Pick a Number Interactive game. Players simultaneously choose an integer in the set $\{1, 2, \ldots, 100\}$. The player whose choice is nearest to 3/4 of the average choice gets a payoff of 4. Other players get a payoff of zero. In the event of a tie, any players who are equally close to 3/4 of the average receive a payoff of 4.

Assume that the number of players is large enough that the effect on the average by any single player is negligible. Denote the payoff to player i as $\pi_i(s_i, \bar{s})$ where $\bar{s}$ is the average. Under our assumption $\bar{s}$ does not change if player i changes s_i (with the other players' strategies fixed).

(a) Show that for any $k \in \{1, \ldots, 100\}$, if $s_i = k$ for all i (if all players choose the same amount), then the resulting strategy profile is a Nash Equilibrium.

(b) For which values of k is the Nash equilibrium from part (a) a **strict Nash equilibrium**. Show that the Nash equilibrium is strict.

a) if $s_i = k$ for all i, it is a Nash

Average $= k$

everyone are same distance to $\frac{3}{4}k$

everyone have same pay off

e.g. if $k = 40$, $\pi_1(40$, all other goes $40) = 4$

$\pi(30$, all other goes $40) = 4$

$\pi(10)$ all other goes $40) = 0$

No change

b) when $k = 1$, it is a strict Nash equilibrium

since $\pi_1(1$, all other goes $1) = 4$

and for any num bigger then 1, for example $\pi(2$, all other goes $1) = 0$ pay off will be 0

Problem 2.

In the Lesson 3 lectures we considered a second-price auction. We showed that the strategy of bidding your value weakly dominates any other strategy. Here we consider a first-price auction. You will show that one particular strategy profile is a Nash equilibrium and that bidding above your value is weakly dominated.

Suppose there are N players participating in a sealed-bid first-price auction ($N \geq 3$). Each player i has a value, $v_i > 0$ for the object being auctioned. Player i makes a sealed-bid $b_i \in [0, \infty)$. You can assume the bids are made simultaneously or that players only know their own bid, so that player i's strategy set is $S_i = [0, \infty)$. The player with the highest bid will win the object. The winner pays the amount of their bid, and so has the payoff $v_i - b_i$ (the value gained minus the cost). The other players pay nothing and have a payoff of zero.

We assume that the values are ranked in the following way (the players know this):

$$v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_N > 0$$

In the case of a tie, where two players have the same highest bid, the winner is the player with the lower index. For example if player 2 and player 4 both have the same highest bid, player 2 will win the object and player 4 will get a payoff of zero.

For player i, let h_i be the maximum bid of the other players, $h_i = \max_{j \neq i} b_j$. Then the payoff function for player i can be written as

$$\pi_i(b_i, b_{-i}) = \begin{cases} v_i - b_i, & \text{if } b_i > h_i \\ v_i - b_i, & \text{if } b_i = h_i, \text{ and } i \text{ is the lowest index with bid equal to } h_i, \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that if $b_1 = v_2$ and $b_i = v_i$ for all $i \geq 2$, then this is a Nash equilibrium of the game.

(b) Show that any strategy $b_i > v_i$ is weakly dominated by the strategy $b_i' = v_i$

(c) Show that the strategy $b_i = v_i$ is weakly dominated by some $b_i' < v_i$.

Note: In part (b), you must demonstrate or explain why both conditions in the definition of a weakly dominated strategy are satisfied.

a) $V_1 \geqslant V_2 \cdots \geqslant V_N > 0$

$b_1 = V_2$ for $i \geqslant 2$, $b_i = V_i$

$\pi_1(b_1, b_{-1}) = V_1 - V_2 \geqslant 0$ $\pi_i(b_i, b_i) = 0$ for $i \geqslant 2$

if $b_1 < V_2$, $\pi_1(b_1, b_{-1}) = 0$

if $b_1 > V_2$, $\pi_1(b_1, b_{-1}) = V_1 - b_1 < V_1 - V_2$ change will not resule a better payoff if $b_i > v_2$ then $\pi_i(b_i, b_{-i}) = v_i - b_i < 0$ negative payoff

b) $\pi_i(v_i, b_{-i}) = 0$ for all b_i

if $b_i > v_i$ and player i wins auction,

$\pi_i(b_i, b_{-i}) = v_i - b_i < 0$

c) $\pi_i(v_i, b_{-i}) = 0$ for all b_i

if $b_i = v_i$ and player i wins auction,

$\pi_i(b_i, b_{-i}) = v_i - b_i = 0$