CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Greedy algorithms

Greedy algorithms

Huffman Encoding (Textbook Section 5.2)

Huffman Encoding

An encoding scheme used in, e.g., MP3 encoding

Data: a string S of symbols over an alphabet Γ
Goal: find a binary encoding e of Γ resulting in minimum encoded length of S

Denote the encoded string by S_e

Example: ASCII encoding
$$\begin{array}{ll} a & 01100001 \\ b & 01100010 \\ \vdots & \vdots \end{array}$$

Different encodings

Consider $\Gamma = \{a, b, c\}$

Stats on S: a appears 45 times, b 16 times, and c twice

- Fixed-length encoding

$$e_1 : \begin{array}{l} a \to 00 \\ b \to 01 \\ c \to 10 \end{array} \qquad |S_{e_1}| = 45 \times 2 + 16 \times 2 + 2 \times 2 = 126$$

- Variable-length encoding

$$e_2 : \begin{array}{l} a \to 0 \\ b \to 10 \\ c \to 11 \end{array} \qquad |S_{e_2}| = 45 \times 1 + 16 \times 2 + 2 \times 2 = 81$$

- Be careful! $e_2 : \begin{array}{l} a \to 0 \\ b \to 1 \\ c \to 01 \end{array}$ Decoding will lead to ambiguity

Prefix-free encoding

Consider the bad encoding e_2 : $\begin{array}{l} a \to 0 \\ b \to 1 \\ c \to 01 \end{array}$ How to decode 010110?

ababba?, *ccba*?, *abcba*?, or ...?

To avoid ambiguity, we need the encoding to be prefix-free

Definition

An encoding is **prefix-free** if no codeword is a prefix of any other codewords

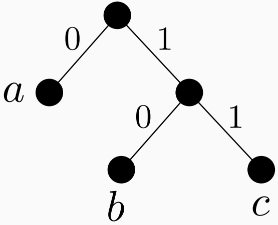
Tree representation of a prefix-free encoding (I)

Definition

A **full binary tree** is a binary tree where each node is either a leaf or it has two children

We use a full binary tree to represent a prefix-free encoding

- leaves are corresponding to symbols in Γ
- label edge to the left child with 0
- label edge to the right child with 1



To obtain the encoding, read edge labels from root to a symbol

$a \to 0$, $b \to 10$, $c \to 11$,

Depth of a leaf $\equiv$ length of its codeword

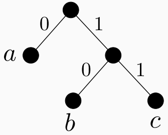
It guarantees to be prefix-free

Tree representation of a prefix-free encoding (II)

Let e be an encoding represented by a tree

For string S, let f_v be the symbol count in S for each $v \in \Gamma$

$$|S_e| = \sum_{v \in \Gamma} f_v \cdot \mathrm{depth}(v)$$



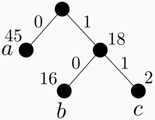
$$|S_e| = 45 \times 1 + 16 \times 2 + 2 \times 2 = 81$$

A useful re-write: label internal nodes with counts of descendants

For all non-root node v, define

$\mathrm{cost}(v) :=$ sum of leaf node counts descending from v

$$|S_e| = \sum_{v \in T - \{\mathrm{root}\}} \mathrm{cost}(v)$$



$$|S_e| = 45 + 16 + 2 + 18 = 81$$

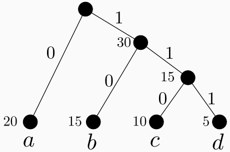
Constructing the prefix-free encoding tree

Idea: put more frequent symbols at smaller depth

Greedy approach: continually merge least frequent symbols/nodes until you have a full binary tree encoding all symbols

Constructing the prefix-free encoding tree – examples

- $a : 20, b : 15, c : 10, d : 5$



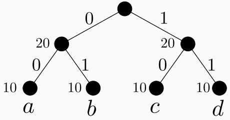
$a \to 0$

$b \to 10$

$c \to 110$

$d \to 111$

- $a : 10, b : 10, c : 10, d : 10$



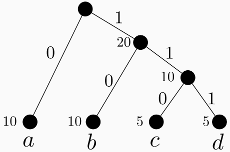
$a \to 00$

$b \to 01$

$c \to 10$

$d \to 11$

- $a : 10, b : 10, c : 5, d : 5$



$a \to 0$

$b \to 10$

$c \to 110$

$d \to 111$

Proof of optimality (I)

Proof sketch

- Greedy choice property:

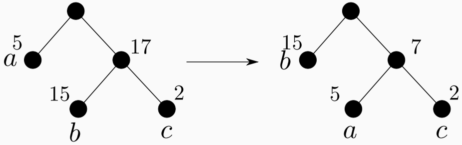
Claim

Every optimal solution has two lowest frequent symbols as leaves connected to an internal node of greatest depth

Proof. (exchange argument).

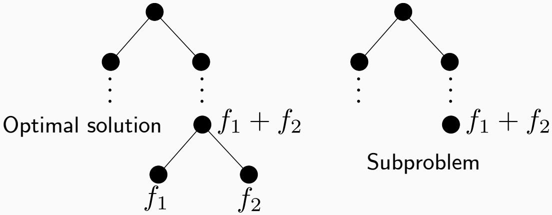
Suppose we have a tree T with two lowest frequent symbols not as deep as possible. Then at least one has a smaller depth. Switch it with one of the deepest nodes that is more frequent.

This improves the encoding length. Thus T is not optimal □



Proof of optimality (II)

- Optimal substructure
  Consider a global optimal solution for $(f_1, f_2, \ldots, f_n)$ with f_1, f_2 the least frequencies.
  Our subproblem is: find an optimal solution for $(f_1 + f_2, f_3, \ldots, f_n)$



Thus, the greedy solution will lead to the global optimal solution

Pseudocode

```
1  def Huffman(f): // f : f[1],...,f[n]; Γ has n symbols
2      T: empty tree;
3      H: priority queue ordered by f;
4      for i := 1 in n:
5          insert(H, i);
6      for k := n+1 in 2n−1:
7          i := extract_min(H);
8          j := extract_min(H);
9          Creat a node k in T with children i and j;
10         f[k] := f[i] + f[j];
11         insert(H, k);
```

Binary heap: insert $O(\log n)$, extract_min: $O(\log n)$

Lines 4-5: $O(n \log n)$; Lines 6-11: $O(n \log n)$

Total cost: $O(n \log n)$

More about the pseudocode

Question: why $2n - 1$ in line 6?

Answer: if a full binary tree has n leaves, then it has $2n - 1$ total nodes

More examples

$a : 5, b : 5, c : 7, d : 10, e : 15, f : 17$