CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Greedy algorithms

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Set Cover (Textbook Section 5.4)

The set cover problem

Problem (Set Cover)

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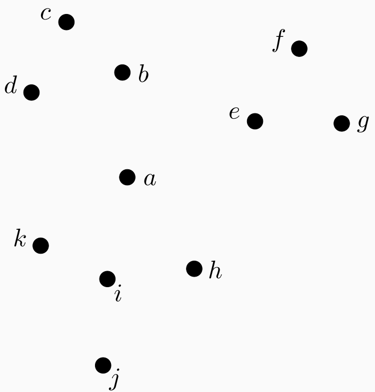
Goal: *minimize the number of selected subsets*

Set cover: example

Example: Each post office can serve 30 miles. Where to build post offices in centre county?



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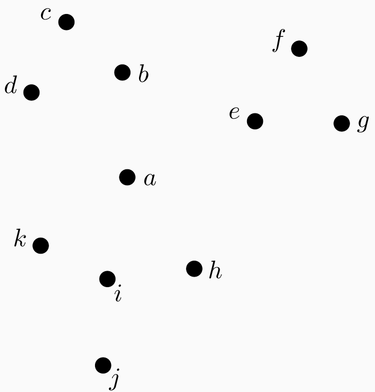

Draw an edge if two towns are within 30 miles

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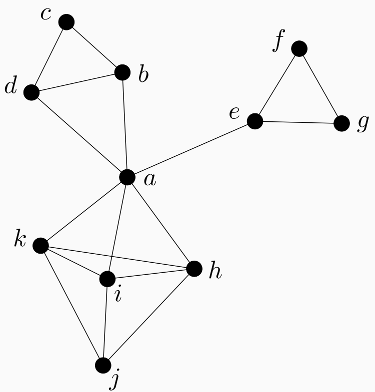

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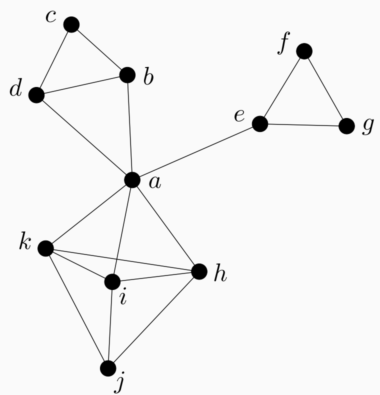

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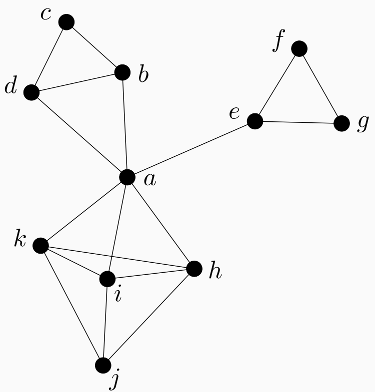

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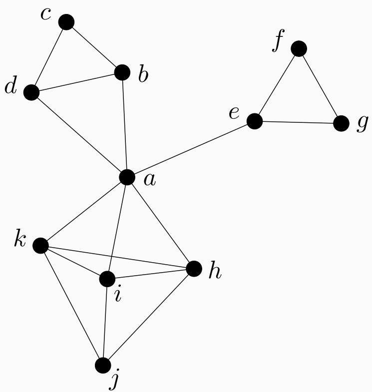

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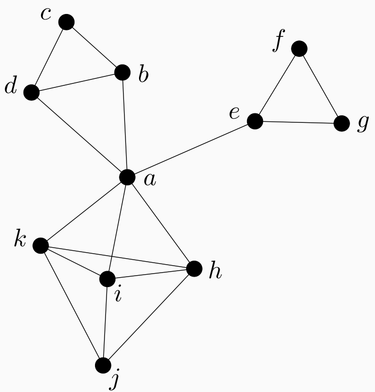

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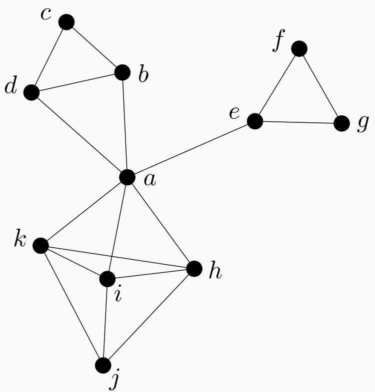

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S_x: the towns within 30 miles of x

Set cover: greedy heuristic

Greedy heuristic: choose the next subset with the most number of uncovered items, until B gets covered

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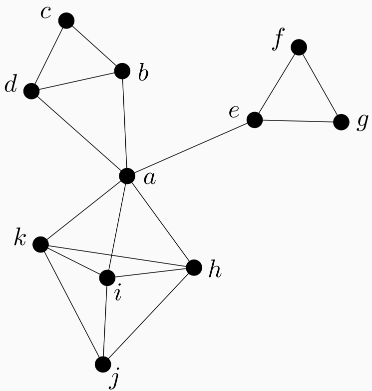

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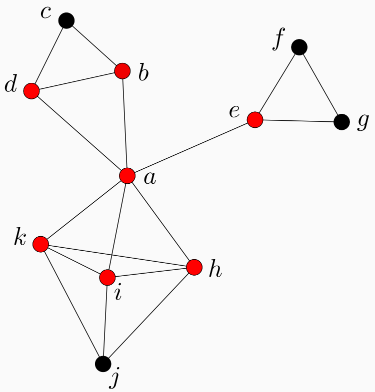

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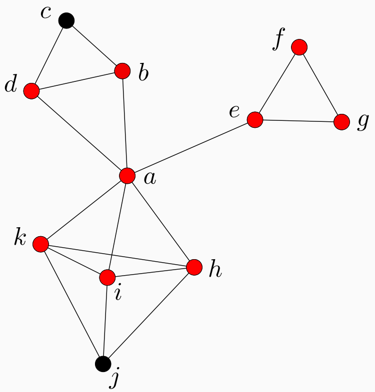

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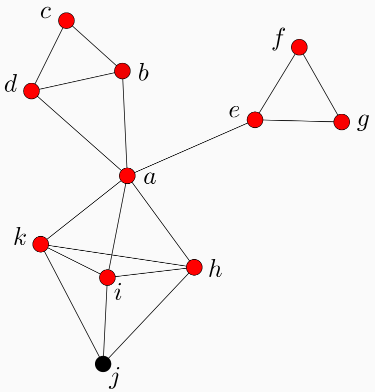

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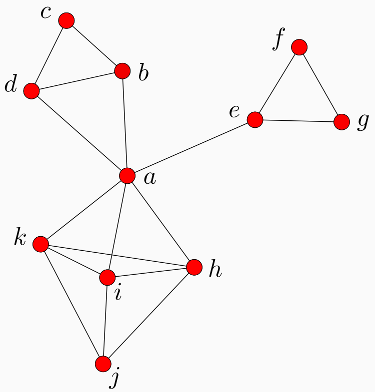

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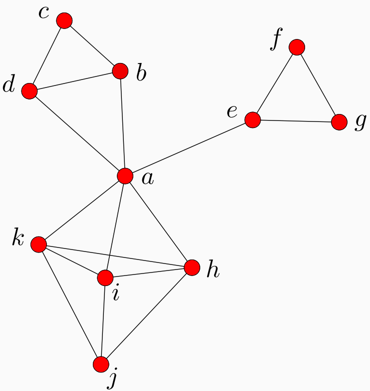

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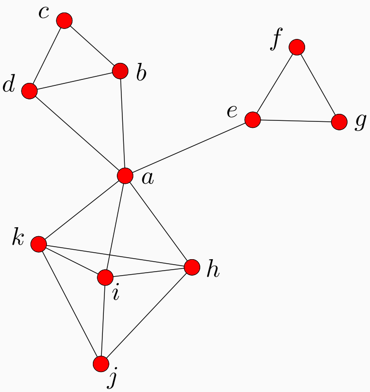

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Optimal solution: S_b, S_e, S_i

Greedy solution is not too bad

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Theorem

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$\ln(n)$: *approximation ratio*

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$\ln(n)$: *approximation ratio*

More about **approximation algorithms**: CSE 565

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Greedy algorithm terminates when $n_t < 1$. Let's find out what t makes $n_t < 1$

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$f'(x) = -e^{-x} + 1$. Critical point at $x = 0$, achieving minimum $\square$