CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Greedy algorithms

Greedy algorithms

Matroid, Task Scheduling
(Cormen et al. 16.4, 16.5)

Warning

Very abstract!

"Computer Science is a science of **abstraction** — creating the right model for a problem and devising the appropriate mechanizable techniques to solve it."

— Alfred Aho

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For a matroid $M = (S, \mathcal{I})$, each $A \in \mathcal{I}$ is called an **independent subset**

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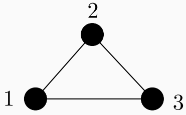

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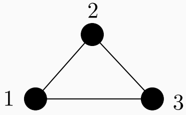

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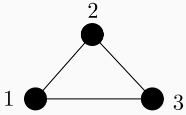

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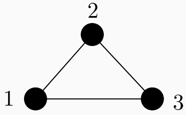

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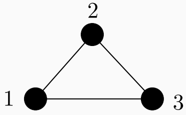

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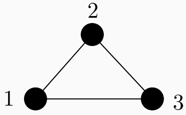

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Say $A = \{(2,3)\}, B = \{(1,3),(1,2)\}$
$x \in B - A$, for example, $x = (1,3)$
then $A \cup \{x\} = \{(2,3),(1,3)\} \subseteq \mathcal{I}$

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For connected undirected G, every maximal independent subset of M_G must be a tree with $|V| - 1$ edges. Hence it is a spanning tree

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Note: for graphic matroids, weight of M_G is corresponding to edge weights

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- For M_G, $w'(e)$ are positive
- For max-weighted independent subset A $w'(A) = (|V| - 1)c - w(A)$, so $w(A)$ is minimized

Hence a max-weighted indep. subset of M_G corresponds to an MST of G

Pseudocode for finding max-weighted independent subset

def GREEDY($M = (S, \mathcal{I})$, *weights* w)**:**

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    Set A := { };
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    Sort S in decreasing order of w ;   // $O(n \log n)$

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Proof of correctness: Cormen et al. 16.4

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Assume checking if $A \cup \{x\} \in \mathcal{I}$ takes $O(f(n))$.

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1 def GREEDY(M = (S, I), weights w):
2     Set A := { };
3     Sort S in decreasing order of w ;        // O(n log n)
4     for x ∈ S:
5         if A ∪ {x} ∈ I:
6             A := A ∪ {x};
7     return A;
```

Proof of correctness: Cormen et al. 16.4

Running time: let $n = |S|$
Assume checking if $A \cup \{x\} \in \mathcal{I}$ takes $O(f(n))$. Lines 5-6 takes $O(n \cdot f(n))$

Pseudocode for finding max-weighted independent subset

1 **def** $\textsc{Greedy}(M = (S, \mathcal{I}),\ \textit{weights}\ w)$**:**
2   Set $A := \{\,\}$;
3   Sort S in decreasing order of w ;   // $O(n \log n)$
4   **for** $x \in S$**:**
5   **if** $A \cup \{x\} \in \mathcal{I}$**:**
6             $A := A \cup \{x\}$;
7   **return** A;

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Running time: let $n = |S|$
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Total running time: $O(n \log n + n \cdot f(n))$

Application: task scheduling

Problem (Task scheduling)

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- *n unit-time tasks $a_1, \dots, a_n$*

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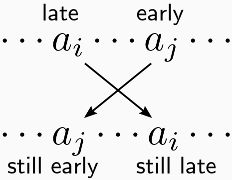

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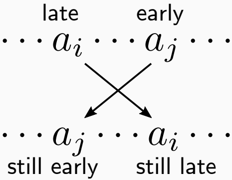

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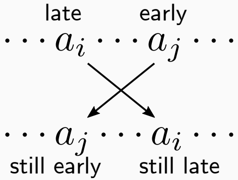

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Finding optimal schedule using matroid

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Finding an optimal schedule $\equiv$ finding max-weighted indep. subset of M