# CMPSC 465
# Data Structures and Algorithms
# Spring 2022

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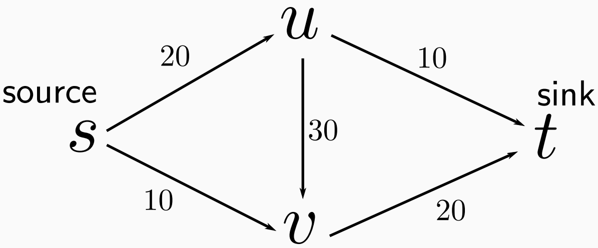
# Flow network
# (Textbook, Section 7.2
# Kleinberg & Tardos Section 7.1)

# Motivation

We use graphs to model "transportation networks"
i.e., networks whose edges allow some sort of traffic, and vertices act as switches

In this setting, edges have **capacities**, and we assume there is a **source vertex** generating traffic, a **sink vertex**, which absorbs traffic.

Example:



## Definition

A **flow network** is a directed graph $G = (V, E)$ s.t.

1. there is a capacity $c_e \geq 0$ for all $e \in E$
2. there is a single source $s \in V$
3. there is a single sink $t \in V$

# $s$-$t$ flow

**Definition**

An **s-t flow** is a function $f : E \to \mathbb{R}^{\geq 0}$ satisfying

- **Capacity constraint:** $\forall e \in E$, $0 \leq f(e) \leq c_e$
- **Conservation condition:** $\forall v \in V - \{s, t\}$

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

Intuition: $f(e)$ is the amount of flow/traffic carried out by $e$

**Definition**

The **value** of a flow $f$ is

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

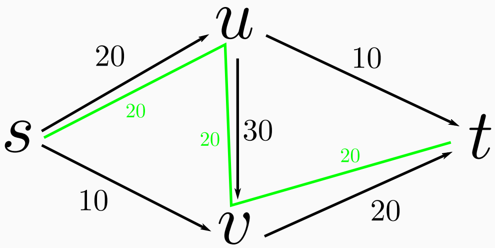
Intuition: $v(f)$ shows how much traffic can be accommodated

**Problem (The Max-Flow problem)**

*Given a flow network, find a flow of the maximum possible value*

Natural idea: Find an $s$-$t$ path and push a flow of value $\min\{c_e : e \text{ along the path}\}$. Then keep doing this

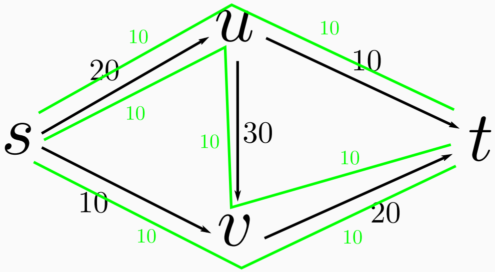
Example:




$f(s,u) = 20$
$f(u,v) = 20$
$f(v,t) = 20$
$f(s,v) = 0$
$f(u,t) = 0$

$v(f) = 20$. Can we do better?

# A better flow




$$f(s,u) = 20$$
$$f(u,v) = 10$$
$$f(v,t) = 20$$
$$f(s,v) = 10$$
$$f(u,t) = 10$$

$v(f) = 30$.

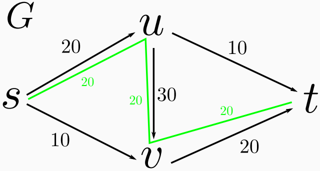
# Residual graph

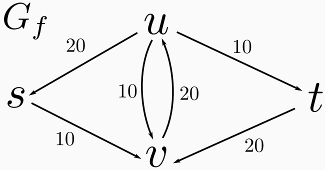
We use a residual graph to keep track of the "leftover" capacity

Given a flow network $G$, flow $f$, construct the **residual graph** $G_f$ as follows

- the vertex set of $G_f$ is the same as $G$
- for each edge $e$ of $G$ with $f(e) < c_e$ (leftover), include $e$ in $G_f$ with capacity $c_e - f(e)$ (**forward edge**)
- for each edge $e = (u, v)$ in $G$ with $f(e) > 0$ (how much we can undo), include $e' = (v, u)$ in $G_f$ with capacity $f(e)$ (**backward edge**)

Capacity of an edge in the residual graph is called **residual capacity**



## Better idea

We introduce **augmenting paths**

Let $P$ be an $s$-$t$ path in $G_f$, we can improve our flow by augmenting along $P$ as follows

Define $\mathrm{bottleneck}(P, f)$ as the minimum residual capacity of edges of $P$

**def** AUGMENT*(f, P)*:
- Set $b = \mathrm{bottleneck}(P, f)$;
- **for** *each* $e = (u, v) \in P$:
  - **if** *e is a forward edge*:
    - Increase $f(e)$ by $b$;
  - **else:**
    - Decrease $f(e)$ by $b$;
- **return** $f$;

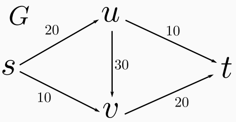
Any $s$-$t$ path in $G_f$ is called an **augmenting path**

# The Ford-Fulkerson algorithm

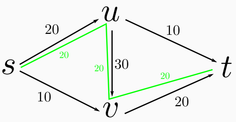
**def** $\textsc{Ford-Fulkerson}$*(G, capacities c)*:

- Set $f(e) = 0$ for all $e \in G$;
- **while** $\exists$ *s-t path in the residual graph* $G_f$:
  - Let $P$ be a simple $s$-$t$ path in $G_f$;
  - $f' = \textsc{Augment}(f, P)$;
  - $G_f = G_{f'}$;
  - $f = f'$;
- **return** $f$;

# Running example



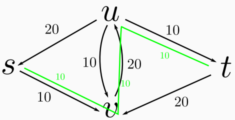

$\xrightarrow{G_f}$




$P : s \to u \to v \to t$

$b : 20$

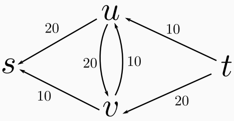
$\xrightarrow{G_f}$




$P : s \to v \to u \to t$

$b : 10$

$G_f \downarrow$




No more $s$-$t$ path

| $e$ | $f_1(e)$ | $f_2(e)$ | $f_3(e)$ |
|---|---|---|---|
| $s \to u$ | 0 | 20 | 20 |
| $s \to v$ | 0 | 0 | 10 |
| $u \to v$ | 0 | 20 | 10 |
| $u \to t$ | 0 | 0 | 10 |
| $v \to t$ | 0 | 20 | 20 |