CSE 461: Programming Language Concepts – Homework 5
Gang Tan, Spring 2022

1. (a) FV((λx. λy. y x) (λz. y))
   = FV(λx. λy. y x) ∪ FV(λz. y)
   = (FV(λy. y x)-{x}) ∪ FV(λz. y)
   = ((FV(y x)−{y})−{x}) ∪ FV(λz. y)
   = (((FV(y) ∪ FV(x))−{y})−{x}) ∪ FV(λz. y)
   = ((({y} ∪ {x})−{y})−{x}) ∪ FV(λz. y)
   = (({x, y}−{y})−{x}) ∪ FV(λz. y)
   = ({x}−{x}) ∪ FV(λz. y)
   = {} ∪ FV(λz. y)
   = FV(λz. y)
   = FV(y)-{z}
   = {y}-{z}
   = {y}
   Therefore, the final y in the original expression is free.
   (b) (λx. λy. y x) (λz. y)
   $\Rightarrow$ [(λz. y)/x] (λy. y x) = [(λz. y)/x] (λb. b x) = λb. b (λz. y)

(c) If you fail to rename the bound variables, the result would instead be (λy. y (λz. y)) which appears as if the free y that was substituted in is instead bound by λy. Any application to this expression will result in two substitutions instead of only one.

2. (a) $(\lambda f.\ \lambda x.\ f\ (f\ x))\ (\lambda y.\ 2 + y)\ 1$

$\Rightarrow ([(\lambda y.\ 2 + y)/f]\ \lambda x.\ f\ (f\ x))\ \ 1$

$= (\lambda x.\ (\lambda y.\ 2 + y)\ ((\lambda y.\ 2 + y)\ x))\ 1$

$\Rightarrow [1/x]\ ((\lambda y.\ 2 + y)\ ((\lambda y.\ 2 + y)\ x))$

$= (\lambda y.\ 2 + y)\ ((\lambda y.\ 2 + y)\ 1)$

$\Rightarrow (\lambda y.\ 2 + y)\ ([1/y]\ (2 + y))$

$= (\lambda y.\ 2 + y)\ (2 + 1)$

$\Rightarrow (\lambda y.\ 2 + y)\ 3$

$\Rightarrow [3/y]\ (2 + y)$

$= 2 + 3$

$\Rightarrow 5$

3. $(\lambda z.\ z)\ (\lambda y.\ y\ y)\ (\lambda x.\ x\ a)$

$\Rightarrow ([(\lambda y.\ y\ y)/z]\ z)\ (\lambda x.\ x\ a)$

$= (\lambda y.\ y\ y)\ (\lambda x.\ x\ a)$

$\Rightarrow [(\lambda x.\ x\ a)/y]\ (y\ y)$

$= (\lambda x.\ x\ a)\ (\lambda x.\ x\ a)$

$\Rightarrow [(\lambda x.\ x\ a)/x]\ (x\ a)$

$= (\lambda x.\ x\ a)\ a$

$\Rightarrow [a/x]\ (x\ a)$

$= a\ a$

3. ADD 2 1

$= (\lambda n1.\ \lambda n2.\ n1\ SUCC\ n2)\ 2\ 1$

$\Rightarrow\ ([2/n1]\ (\lambda n2.\ n1\ SUCC\ n2)\ )\ 1$

$= (\lambda n2.\ 2\ SUCC\ n2)\ 1$

$\Rightarrow\ [1/n2]\ (2\ SUCC\ n2)$

$= 2\ SUCC\ 1$

$= (\lambda f.\ \lambda x.\ f\ (f\ x))\ SUCC\ 1$

$\Rightarrow\ ([SUCC/f]\ (\lambda x.\ f\ (f\ x)))\ 1$

$= (\lambda x.\ SUCC\ (SUCC\ x))\ 1$

$\Rightarrow\ [1/x]\ (SUCC\ (SUCC\ x)\ )$

$= SUCC\ (SUCC\ 1)$

$= SUCC\ ((\lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x))\ 1)$

$\Rightarrow\ SUCC\ ([1/n]\ (\lambda f.\ \lambda x.\ f\ (n\ f\ x)))$

$= SUCC\ (\lambda f.\ \lambda x.\ f\ (1\ f\ x))$

$= SUCC\ (\lambda f.\ \lambda x.\ f\ ((\lambda a.\ \lambda x.\ a\ x)\ f\ x))$

$\Rightarrow\ SUCC\ (\lambda f.\ \lambda x.\ f\ (([f/a]\ (\lambda x.\ a\ x))\ x))$

$= SUCC\ (\lambda f.\ \lambda x.\ f\ ((\lambda x.\ f\ x)\ x))$

$\Rightarrow\ SUCC\ (\lambda f.\ \lambda x.\ f\ ([x/x](f\ x)\ ))$

$= SUCC\ (\lambda f.\ \lambda x.\ f\ (f\ x))$

$= (\lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x))\ (\lambda f.\ \lambda x.\ f\ (f\ x))$

$\Rightarrow\ [(\lambda f.\ \lambda x.\ f\ (f\ x))/n]\ (\lambda f.\ \lambda x.\ f\ (n\ f\ x))$

$= \lambda f.\ \lambda x.\ f\ ((\lambda f.\ \lambda x.\ f\ (f\ x))\ f\ x)$

$= \lambda f.\ \lambda x.\ f\ ((\lambda a.\ \lambda b.\ a\ (a\ b))\ f\ x)$

$\Rightarrow\ \lambda f.\ \lambda x.\ f\ ([f/a](\lambda b.\ a\ (a\ b))\ \ x)$

$= \lambda f.\ \lambda x.\ f\ ((\lambda b.\ f\ (f\ b))\ x)$

$\Rightarrow\ \lambda f.\ \lambda x.\ f\ ([x/b](f\ (f\ b))\ )$

$= \lambda f.\ \lambda x.\ f\ (f\ (f\ x))$

$= 3$