# CMPSC 465
# Data Structures and Algorithms
# Spring 2022

Instructor: Chunhao Wang

# Flow network
# (Textbook, Section 7.2
# Kleinberg & Tardos Section 7.1)

## Motivation

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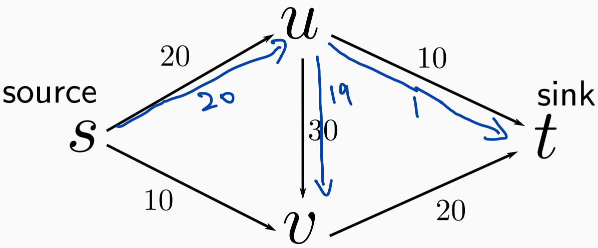

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Intuition: $v(f)$ shows how much traffic can be accommodated

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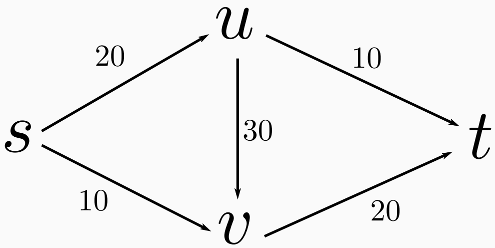

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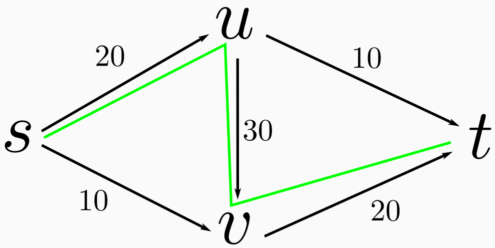

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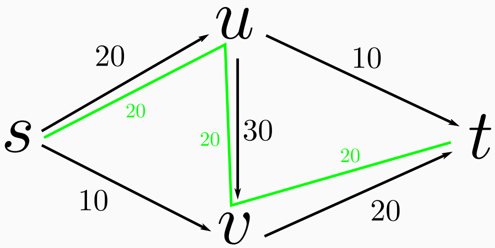

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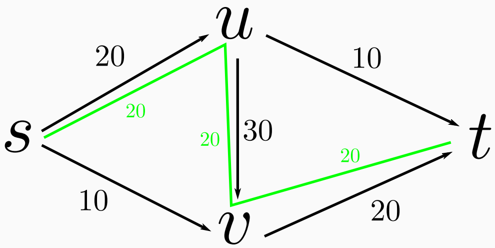


$V(f)=20$

$$f(s,u) = 20$$
$$f(u,v) = 20$$
$$f(v,t) = 20$$
$$f(s,v) = 0$$
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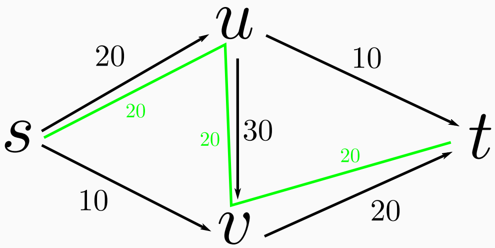


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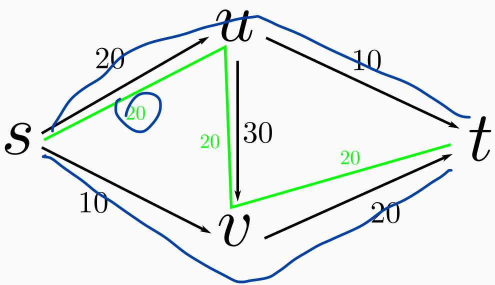


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$v(f) = 20$. Can we do better?

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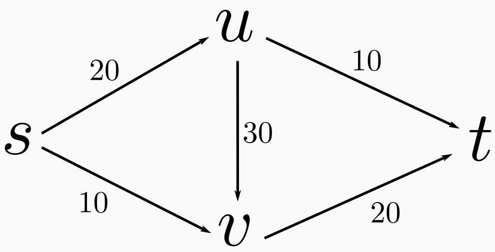

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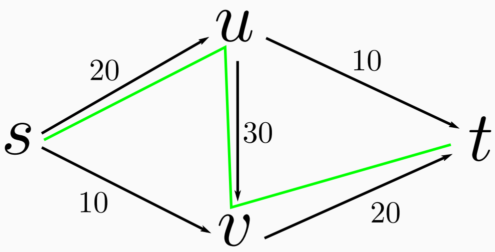

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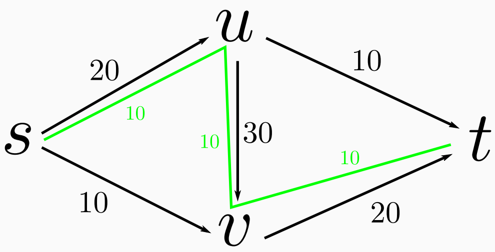

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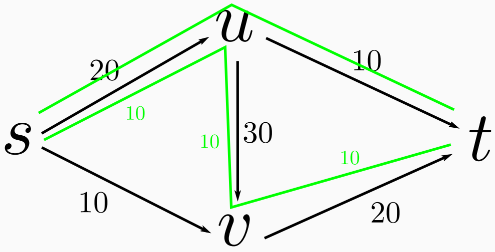

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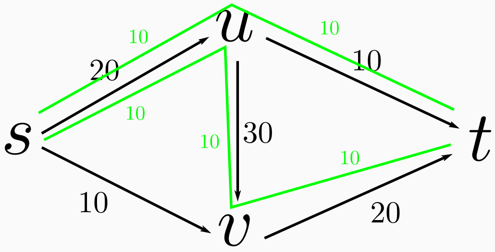

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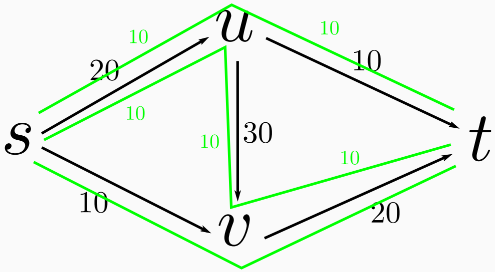

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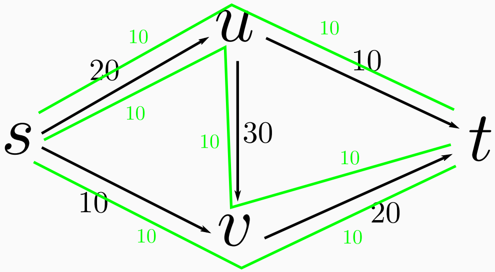

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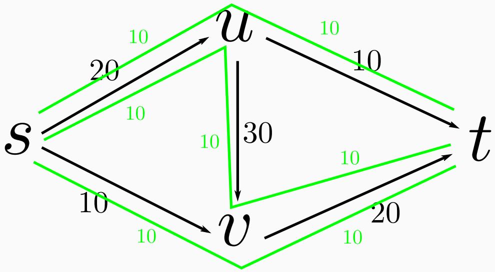


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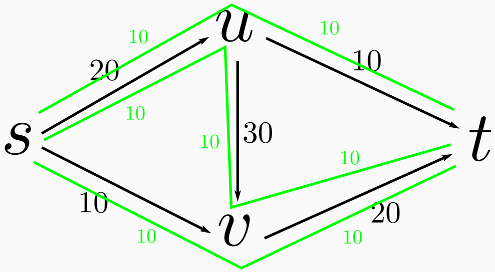


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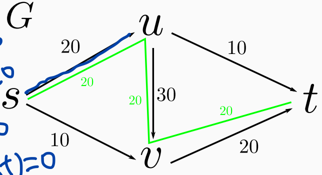


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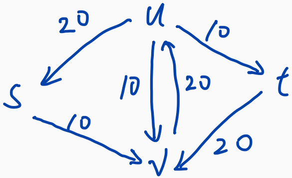

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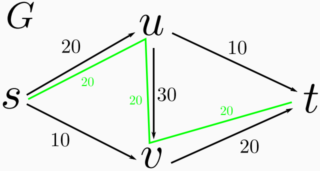


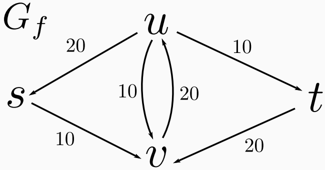

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Any $s$-$t$ path in $G_f$ is called an **augmenting path**

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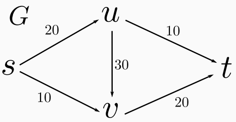
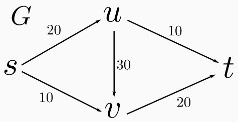
  - $f = f'$;
- **return** $f$;

# Running example




| $e$ | f(e) | | |
|---|---|---|---|
| $s \to u$ | 0 | | |
| $s \to v$ | 0 | | |
| $u \to v$ | 0 | | |
| $u \to t$ | 0 | | |
| $v \to t$ | 0 | | |

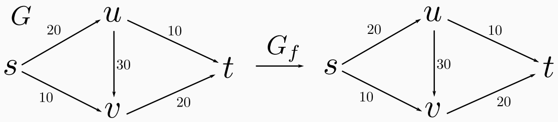
# Running example




$G_f$:

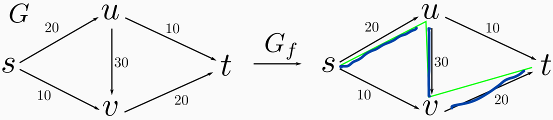
| $e$ | $f_1(e)$ | | |
|---|---|---|---|
| $s \to u$ | 0 | | |
| $s \to v$ | 0 | | |
| $u \to v$ | 0 | | |
| $u \to t$ | 0 | | |
| $v \to t$ | 0 | | |

# Running example




| $e$ | $f_1(e)$ | | |
|---|---|---|---|
| $s \to u$ | 0 | | |
| $s \to v$ | 0 | | |
| $u \to v$ | 0 | | |
| $u \to t$ | 0 | | |
| $v \to t$ | 0 | | |

# Running example

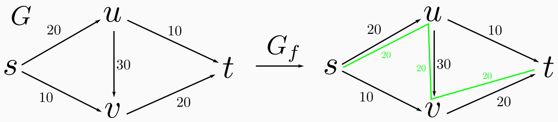



$P : s \to u \to v \to t$

b: 20

| $e$ | $f_1(e)$ | | |
|---|---|---|---|
| $s \to u$ | 0 | 20 | |
| $s \to v$ | 0 | 0 | |
| $u \to v$ | 0 | 20 | |
| $u \to t$ | 0 | 0 | |
| $v \to t$ | 0 | 20 | |

# Running example

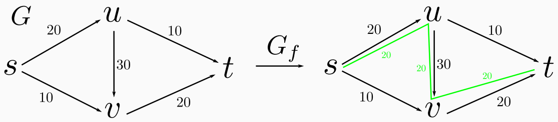



$P: s \to u \to v \to t$

$b: 20$

| $e$ | $f_1(e)$ | | |
|---|---|---|---|
| $s \to u$ | 0 | | |
| $s \to v$ | 0 | | |
| $u \to v$ | 0 | | |
| $u \to t$ | 0 | | |
| $v \to t$ | 0 | | |

# Running example

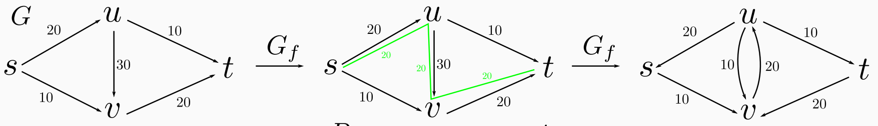



$P : s \to u \to v \to t$

$b : 20$

| $e$ | $f_1(e)$ | $f_2(e)$ | |
|---|---|---|---|
| $s \to u$ | 0 | 20 | |
| $s \to v$ | 0 | 0 | |
| $u \to v$ | 0 | 20 | |
| $u \to t$ | 0 | 0 | |
| $v \to t$ | 0 | 20 | |

# Running example

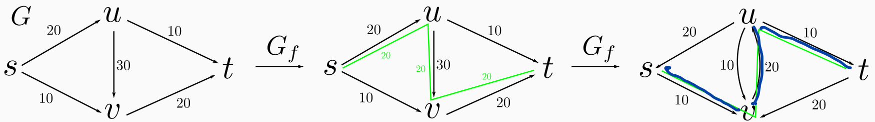



$P : s \to u \to v \to t$

$b : 20$

| $e$ | $f_1(e)$ | $f_2(e)$ | |
|---|---|---|---|
| $s \to u$ | 0 | 20 | |
| $s \to v$ | 0 | 0 | |
| $u \to v$ | 0 | 20 | |
| $u \to t$ | 0 | 0 | |
| $v \to t$ | 0 | 20 | |

# Running example




$P : s \to u \to v \to t$

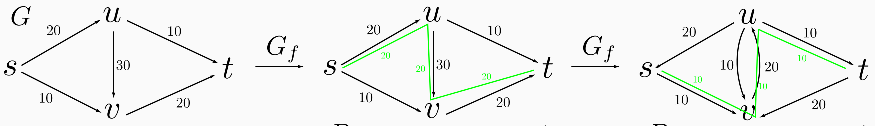
$b : 20$

$P : s \to v \to u \to t$

$b : 10$

| $e$ | $f_1(e)$ | $f_2(e)$ | |
|---|---|---|---|
| $s \to u$ | 0 | 20 | 20 |
| $s \to v$ | 0 | 0 | 10 |
| $u \to v$ | 0 | 20 | 10 |
| $u \to t$ | 0 | 0 | 10 |
| $v \to t$ | 0 | 20 | 20 |

# Running example

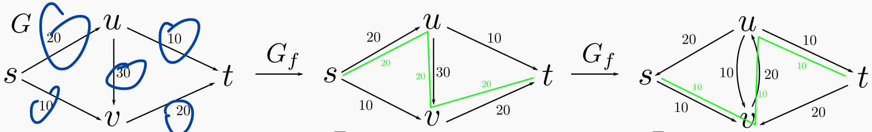



$P : s \to u \to v \to t$
$b : 20$

$P : s \to v \to u \to t$
$b : 10$

| $e$ | $f_1(e)$ | $f_2(e)$ | |
|---|---|---|---|
| $s \to u$ | 0 | 20 | |
| $s \to v$ | 0 | 0 | |
| $u \to v$ | 0 | 20 | |
| $u \to t$ | 0 | 0 | |
| $v \to t$ | 0 | 20 | |

# Running example

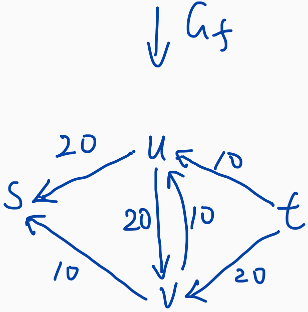



$P: s \to u \to v \to t$

$b: 20$

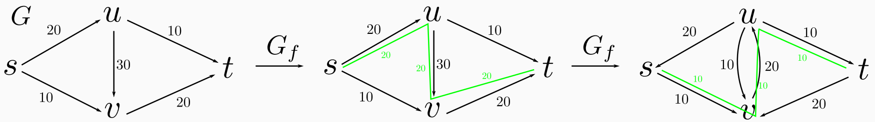
$P: s \to v \to u \to t$

$b: 10$




| $e$ | $f_1(e)$ | $f_2(e)$ | $f_3(e)$ |
|---|---|---|---|
| $s \to u$ | 0 | 20 | 20 |
| $s \to v$ | 0 | 0 | 10 |
| $u \to v$ | 0 | 20 | 10 |
| $u \to t$ | 0 | 0 | 10 |
| $v \to t$ | 0 | 20 | 20 |

# Running example

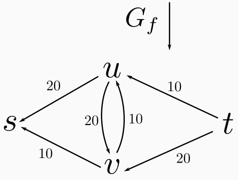



$P: s \to u \to v \to t$

$b: 20$

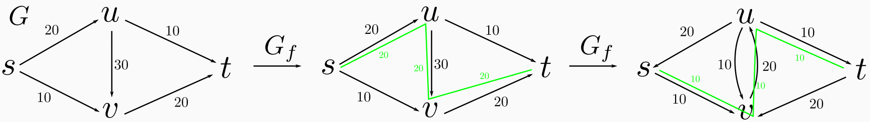
$P: s \to v \to u \to t$

$b: 10$




| $e$ | $f_1(e)$ | $f_2(e)$ | $f_3(e)$ |
|---|---|---|---|
| $s \to u$ | 0 | 20 | 20 |
| $s \to v$ | 0 | 0 | 10 |
| $u \to v$ | 0 | 20 | 10 |
| $u \to t$ | 0 | 0 | 10 |
| $v \to t$ | 0 | 20 | 20 |

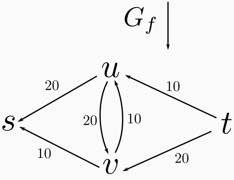
# Running example




$P: s \to u \to v \to t$
$b: 20$

$P: s \to v \to u \to t$
$b: 10$

| $e$ | $f_1(e)$ | $f_2(e)$ | $f_3(e)$ |
|---|---|---|---|
| $s \to u$ | 0 | 20 | 20 |
| $s \to v$ | 0 | 0 | 10 |
| $u \to v$ | 0 | 20 | 10 |
| $u \to t$ | 0 | 0 | 10 |
| $v \to t$ | 0 | 20 | 20 |




No more $s$-$t$ path