Finding optimal schedule using matroid

How to find an optimal schedule?

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Finding optimal schedule using matroid

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   1.1 Find a set A of tasks that are early

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    1.2 Sort the tasks of A in increasing deadlines

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Finding an optimal schedule $\equiv$ finding max-weighted indep. subset of M

Such M for task scheduling is a matroid

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- Exchange property:
  Say $A, B \in \mathcal{I}$ and $|B| > |A|$.

need to show $\exists\, x \in B - A$

s.t. $A \cup \{x\} \in \mathcal{I}$

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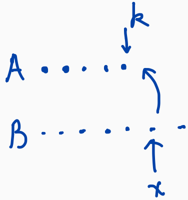

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  Then $A \cup \{x\} \subseteq \mathcal{I}$

Greedy algorithm for finding optimal scheduling

def $\textsc{Greedy}(M = (S, \mathcal{I}), \textit{weights } w)$**:**

Greedy algorithm for finding optimal scheduling

def $\textsc{Greedy}(M = (S, \mathcal{I}),\ \textit{weights } w)$**:**

  Set $A := \{\,\}$;

Greedy algorithm for finding optimal scheduling

```
def GREEDY(M = (S, I), weights w):
    Set A := { };
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    if $A \cup \{x\} \in \mathcal{I}$**:**

      $A := A \cup \{x\}$;

  return A;

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Running time: let $n = |S|$

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6             A := A ∪ {x};
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$O(n\log n + nf(n))$

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Claim: $f(n) = O(n)$ for task scheduling problem (Homework)

Greedy algorithm for finding optimal scheduling

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Claim: $f(n) = O(n)$ for task scheduling problem (Homework)

Total running time: $O(n^2)$

Greedy algorithms

Horn formulas (Textbook Section 5.3)

Horn formulas — Prelude

Consider the following puzzle

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- If Alice has a dog, then Bob has a cat
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Horn formulas — Prelude

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- If Alice has a dog, then Bob has a cat
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Question: what pets do they have?

Basics of boolean formulas

Boolean formulas

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- **Variables:** possibilities
  Knowledge about variables is represented by a special type of boolean formulas

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Boolean formulas

Basics of boolean formulas

- **Variables:** possibilities
  Knowledge about variables is represented by a special type of boolean formulas
  Goal; find a *consistent* explanation of the knowledge when $x=1$
- **Boolean variable:** $x = 1$ (true) or $x = 0$ (false) $\bar{x} = 0$
- **Literal:** x (positive literal), $\bar{x}$ (negative literal) or $\neg x$ when $x=0$ $\bar{x}=1$

Boolean formulas

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- **Variables:** possibilities
  Knowledge about variables is represented by a special type of boolean formulas
  Goal; find a *consistent* explanation of the knowledge
- **Boolean variable:** $x = 1$ (true) or $x = 0$ (false)
- **Literal:** x (positive literal), $\bar{x}$ (negative literal)
- **Clause:** a clause consists of literals connected by
  $\wedge$ (AND), $\vee$ (OR), $\implies$ (implies)

Boolean formulas

x	y	$x \wedge \bar{y}$
0	0	0
0	1	0
1	0	1
1	1	0

Basics of boolean formulas

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  Goal; find a *consistent* explanation of the knowledge
- **Boolean variable:** $x = 1$ (true) or $x = 0$ (false)
- **Literal:** x (positive literal), $\bar{x}$ (negative literal)
- **Clause:** a clause consists of literals connected by
  $\wedge$ (AND), $\vee$ (OR), $\implies$ (implies)
  Examples: $x \wedge \bar{y}$, $\quad (x \wedge y) \implies z$

Horn formulas

In a Horn formula, there are only two types of clauses (**Horn clauses**):

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  - $(x \wedge \bar{y}) \implies z$ ✗
  - $(x \vee y) \implies z$ ✗
  - $\implies z$ $\Leftarrow$ $1 \Rightarrow z$

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  - $(x \vee y) \implies z$ ✗
  - $\implies z$ ✓

$(x \wedge y \wedge z) \implies \bar{w}$ ✗

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    - $(x \vee y) \implies z$ ✗
    - $\implies z$ ✓
- **Pure negative clauses** $\bar{x}_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_n$

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  - $(x \vee y) \implies z$ ✗
  - $\implies z$ ✓
- **Pure negative clauses** $\bar{x}_1 \vee \bar{x}_2 \vee \cdots \vee \bar{x}_n$
  OR of any number of negative literals

Horn formula example

Consider the puzzle:

- If Alice has a dog, then Bob has a cat
- If Charlie and Bob both have pets of the same species, then Alice has a cat
- Charlie and Alice don’t share a pet of the same species

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Define variables:

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Define variables:

- a: Alice has a dog

$a=1 \Rightarrow$ Alice has a dog

$a=0 \Rightarrow$ Alice doesn't have a dog

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Define variables:

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- b: Bob has a dog

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- b: Bob has a dog
- c: Charlie has a dog

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- b: Bob has a dog
- c: Charlie has a dog
- x: Alice has a cat

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Define variables:

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- b: Bob has a dog
- c: Charlie has a dog
- x: Alice has a cat
- y: Bob has a cat

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Consider the puzzle:

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Define variables:

- a: Alice has a dog 0/1
- b: Bob has a dog 0/1
- c: Charlie has a dog 0/1
- x: Alice has a cat 0/1
- y: Bob has a cat 0/1
- z: Charlie has a cat 0/1

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Modelled by a set of Horn clauses:

$a \implies y$

$\cancel{((c \wedge b) \vee (z \wedge y)) \Rightarrow x}$

$\boxed{b \wedge c \Rightarrow x}$

$y \wedge z \Rightarrow x$

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$$a \implies y$$

$$(b \wedge c) \implies x$$

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Modelled by a set of Horn clauses:

$a \implies y$

$(b \wedge c) \implies x$

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$\boxed{\bar{a} \vee \bar{c} \vee \bar{x} \vee \bar{z}}$

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Define variables:

- a: Alice has a dog 0
- b: Bob has a dog 6
- c: Charlie has a dog 0
- x: Alice has a cat 0
- y: Bob has a cat 6
- z: Charlie has a cat 0

Modelled by a set of Horn clauses:

✓ $a \implies y$

✓ $(b \wedge c) \implies x$

✓ $(y \wedge z) \implies x$

✓ $\bar{a} \vee \bar{c}$ $\Longleftrightarrow a \Rightarrow \bar{c}$

✓ $\bar{x} \vee \bar{z}$ $\Longleftrightarrow x \Rightarrow \bar{z}$

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Define variables:

- a: Alice has a dog 0
- b: Bob has a dog 1
- c: Charlie has a dog 1
- x: Alice has a cat 1
- y: Bob has a cat 1
- z: Charlie has a cat 0

Modelled by a set of Horn clauses:

$a \implies y$

$(b \wedge c) \implies x$

$(y \wedge z) \implies x$

$\bar{a} \vee \bar{c}$

$\bar{x} \vee \bar{z}$

Question: satisfying assignment?

Greedy approach for Horn formulas

Problem (Horn Satisfiability)

Given a set of Horn clauses, determine whether or not there is a consistent explanation,

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Given a set of Horn clauses, determine whether or not there is a consistent explanation, i.e., an assignment of 0/1 to variables that satisfy all clauses

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Given a set of Horn clauses, determine whether or not there is a consistent explanation, i.e., an assignment of 0/1 to variables that satisfy all clauses

Example: $\underbrace{(x \wedge y) \implies z}_{\checkmark}$, $\underbrace{\bar{x} \vee \bar{w}}_{\checkmark}$

$\boxed{x=0,\ y=0,\ z=0,\ w=0}$

Greedy approach for Horn formulas

Problem (Horn Satisfiability)

Given a set of Horn clauses, determine whether or not there is a consistent explanation, i.e., an assignment of 0/1 to variables that satisfy all clauses

Example: $(x \wedge y) \implies z$, $\bar{x} \vee \bar{w}$ can be satisfied by $x = 0, y = 0, z = 0, w = 0$

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Example: $(x \wedge y) \implies z, \bar{x} \vee \bar{w}$ can be satisfied by $x = 0, y = 0, z = 0, w = 0$

Greedy heuristic: start with all 0. Only set a variable to 1 if you need to, i.e., when an implication says you need to

$p \Rightarrow q$ $\Rightarrow a$ $1 \Rightarrow a$

"when $p=1$, then q must be 1"

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Given a set of Horn clauses, determine whether or not there is a consistent explanation, i.e., an assignment of 0/1 to variables that satisfy all clauses

Example: $(x \wedge y) \implies z$, $\bar{x} \vee \bar{w}$ can be satisfied by $x = 0, y = 0, z = 0, w = 0$

Greedy heuristic: start with all 0. Only set a variable to 1 if you need to, i.e., when an implication says you need to $\Rightarrow a$

Recall: $p \implies q \iff \bar{p} \vee q$

Pseudocode

def GREEDY_HORN(*set of Horn clauses*):

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Set all variables to 0;

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$\Longrightarrow x$

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$\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \vee \cdots \vee \bar{x}_n$

How does this theorem help?

If all the pure negative clauses cannot be satisfied after the while loop, then there's no such assignment satisfying them

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Running time: Let n be the size of the Horn formula, i.e., the number occurrences of literals.

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