$\cos^2 x \quad \lambda^2 - \lambda\cos(x) - \frac{\cos^2(x)}{4}$

$(x - \frac{\cos x}{2})^2$

MATH 455: HOMEWORK 8

Problem 1 (3pts). Solve

$$f(x) = x^2 - x\cos(x) + \frac{1}{4} - \frac{\sin^2(x)}{4} = 0, \text{ with } x_0 = \frac{\pi}{2}.$$

(1) Does Newton's method converge quadratically to the root $r = r_1 \in [0, 1]$? If not, explain why?

(2) Find the multiplicity of the root $r = r_1$ of $f(x)$.

(3) Write out the Modified Newton's Method such that we have quadratical convergence.

Problem 2 (2 pts). Apply two steps of the Secant Method with $x_0 = 1$ and $x_1 = 2$ to solve $f(x) = x^3 - 2x - 2 = 0$.

Problem 3 (Page 84 of the book, Exercises 2) (4 pts). Find the LU factorization of the given matrices. Check by matrix multiplication.

(a) $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{bmatrix}$

Problem 4 (Page 85 of the book, Exercises 4) (2 pts). Solve the system by finding the LU factorization and then carrying out the two-step back substitution.

(a) $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

1)

$$x^2 - x\cos(x) + \frac{1}{4} - \frac{\sin^2(x)}{4} = 0$$

$x^2 - x\cos(x) + \frac{1}{4}(1 - \sin^2 x) = 0$

$x^2 - x\cos(x) + \frac{1}{4}\cos^2 x = 0$

$(x - \frac{1}{2}\cos x)^2 = 0$

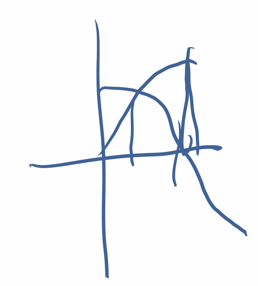
$f'(x) = 2(x - \frac{\cos(x)}{2})(1 + \frac{\sin x}{2})$

Fail to converge

$f''(x) = 2(x - \frac{\cos x}{2})(\frac{\cos x}{2}) + 2(1 + \frac{\sin x}{2})^2$

$\neq 0$

$m = 2$



$\frac{\pi}{2} - \frac{1}{2}$

$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$

$= \frac{\pi}{2} - \frac{2(\frac{\pi}{2})^2}{4(\frac{\pi}{2})(1+\frac{3}{2})}$

$\sin 2x$

$= \frac{\pi}{2} - \frac{2\frac{\pi^2}{4}}{\frac{6\pi}{2}}$

$= \frac{3\pi - 2\pi}{6}$

$= \frac{\pi}{6}$

2) $x_{n+1} = x_n - \dfrac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$

$x_2 = 2 - \dfrac{f(2)}{f(2) - f(1)}$

$= 2 - \dfrac{2}{2 - -3}$

$= 2 - \frac{2}{5}$

$= \dfrac{8}{5}$

$x_3 = \frac{8}{5} - (-0.142)$

$= 1.742$

Problem 3 (Page 84 of the book, Exercises 2) (4 pts). Find the LU factorization of the given matrices. Check by matrix multiplication.

(a) $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{bmatrix}$

$$\begin{matrix} ① \\ ② \\ ③ \end{matrix}\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \xrightarrow{②-2\times①} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 3 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{③-1\times①}$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{bmatrix} \xrightarrow{3 - 1} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & 2 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{3 - 2 \times 2} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & -1 \end{bmatrix} \xrightarrow{4 - 2} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} U$$

$$L \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{bmatrix}$$

Problem 4 (Page 85 of the book, Exercises 4) (2 pts). Solve the system by finding the LU factorization and then carrying out the two-step back substitution.

(a) $\underbrace{\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}}_{b}$

$LU = A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 8 \end{bmatrix}$ $\quad L \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$② - 2 \times ①$

$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 3 & 1 & 5 \end{bmatrix} \xrightarrow{③ - ①} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$Lc = b \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$c_1 = 0$

$c_2 = 1$

$c_3 = 3$

$$U_x = c$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{cases} x_3 = 1 \\ x_2 = 1 \\ x_1 = -1 \end{cases}$$