RUSS DEFOREST

MATH 486 - GAME THEORY

LECTURE NOTES

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Part I

Non-cooperative Games

Math 486 Week 1 Lecture Notes

1.1 Introduction

Game Theory is a wide ranging subject with applications in fields ranging from evolutionary biology to computer science. This course will provide an introduction to non-cooperative and evolutionary games.

A sample of some results and applications:

- We will see that chess is theoretically **solvable**: one of the following three possibilities is true:
  - Player 1 (white) can always force a win
  - Player 2 (black) can always force a win
  - the players can always force a stalemate (tie).

  No one knows which of these three possibilities is true and it is unlikely to ever be known.
- What can we learn from a mathematical model of decision making about the actions of U.S. and Soviet leaders during the Cuban Missile Crisis?
- Under what conditions do the incentives of self-interested individuals act as obstacles to achieving outcomes that would leave all individuals better off, as in global action on climate change?
- Auctions: How are ad spots on a google search allocated and what does this have to do with game theory?
- Why does bluffing play such an important role in the game of poker?

- Why do some animal species settle territorial or mating disputes through ritualized conflict? Under what conditions are such behaviors evolutionarily favored?

We begin the course with non-cooperative games, formalizing strategic interactions between rational, self-interested agents. We will develop **solution concepts** for these strategic interactions, beginning with **extensive form games** that can be solved through **backward induction** and **dominance solvable normal form games**, working toward more general solution concepts that will apply to a wider range of games. Along the way we will also raise questions about the limitations of our mathematical models as we examine real applications.

Later in the course we study evolutionary games. In this setting we leave behind assumptions of self-interested rational agents and instead assume inferior strategies or behaviors within a population are displaced by more successful strategies over time. This topic will require some use of modeling with differential equations. While some background in differential equations is helpful, it is not a prerequisite for the course and we will develop the necessary background.

1.2 *Extensive Form Games with Perfect Information*

1.2.1 *An Introductory Example*

Consider the following interaction between two **players**, **A** and **B**:

- **A** has caused a minor accident resulting in minor cosmetic damage to **B**'s truck.
- **B** gets an estimate of $300 for repairing the damage. However, **B**'s truck is not new; it has many pre-existing dings and dents.
- **A** assumes that if they pay $300 to **B** for the repair then **B** will pocket the money and not pay to have the repair done.
- We will assume that **A** has two choices: pay **B** $300 for the repair, or demand a receipt and agree to reimburse **B** for the cost of the repair after it is done.

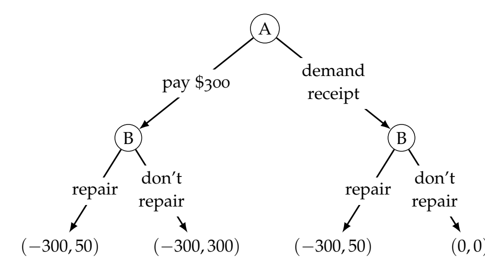

Figure 1.1: A game tree for the car accident scenario.

A game tree for this scenario is depicted in Fig. 1.1.

If **A** demands a receipt we expect **B** to have the repair done (why?); the payoff for **A** will then be -300 regardless of their action. But

Useful Questions

We assumed a payoff of 50 for **B** if **A** demands a receipt and **B** has the repair done. How can you interpret this payoff?

perhaps this first model is incomplete and does not capture all of **A**'s choices.

Consider a second model where **A** has a third option: pay half now and demand a receipt before paying the rest. The game tree is shown in Fig. 1.2. If **A** pays \$150 we expect **B** to keep the money and not get the repair. The payoffs will then be $(-150, 150)$.

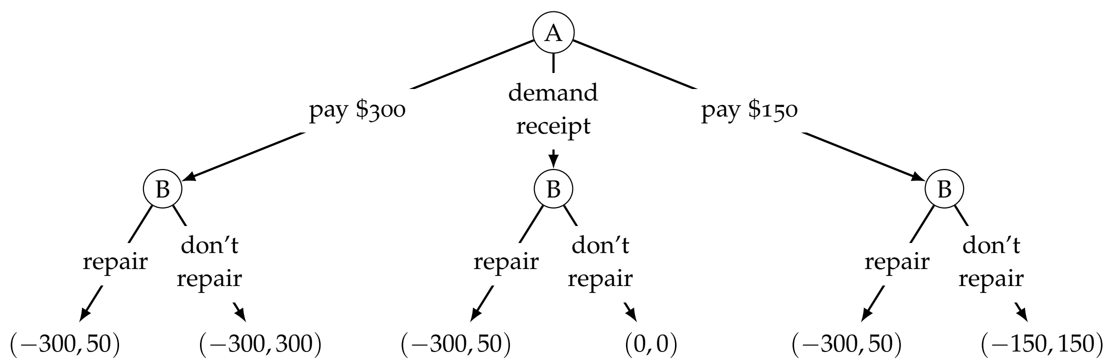

Figure 1.2: A game tree for the car accident scenario where **A** has three options.

1.2.2 *Terminology*

The models in Figures 1.1 and 1.2 are each an example of an **extensive form game with perfect information**. We take a closer look at the structure of a **game tree** in Figure 1.3.

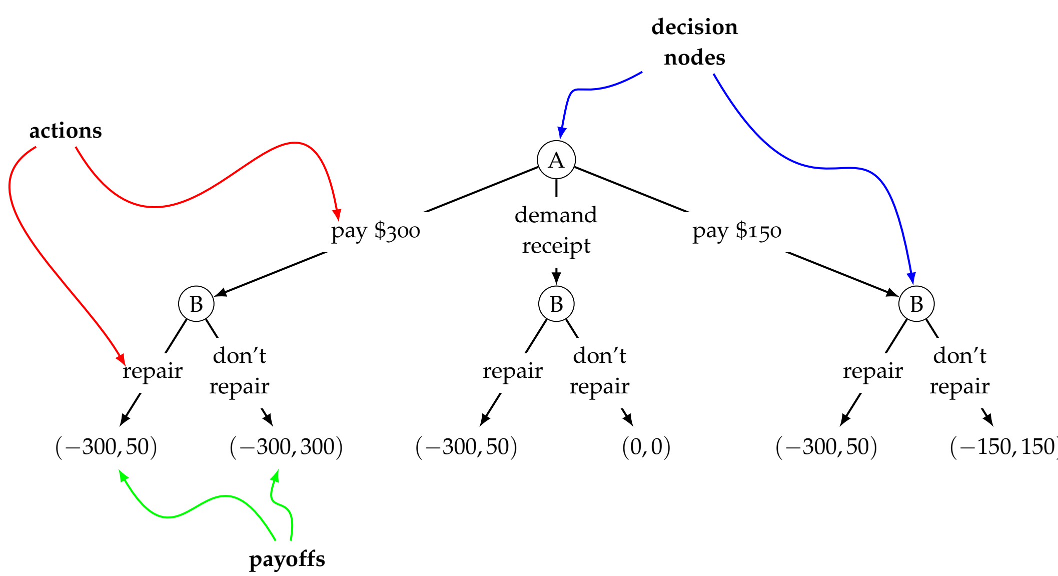

Figure 1.3: the caption

- Player **A** has a single **decision node** which is also the **root** of the game tree. We view **A** as "player 1" because they take the first move.
- Each action from the root node leads to a decision node for **B**. Each action for **B** leads to a **terminal node** with **payoffs** specified for both players: (<payoff to **A**>,<payoff to **B**>).
- In an n-player game we will list the payoffs as a tuple:

  (<payoff to player 1>, <payoff to player 2>, . . . , <payoff to player n>)
- Informally, **perfect information** means that each player knows where they are at in the game tree when they take an action. In games with imperfect information, some information or actions of other players may be hidden when a player takes an action. We will include additional structure in the game tree (information sets) to represent games with imperfect information.

Here is the formal definition of an extensive form game:

Definition: Extensive Form Game with Perfect Information

A finite extensive form game with perfect information is a **directed rooted tree** along with the following:

- A set of players, $P = \{1, 2, \ldots, n\}$.
- A set of actions or moves, M.
- A set of terminal nodes, T.
- A set of non-terminal **decision nodes**, H.
- A labeling function $\ell : H \to P$ that assigns each decision node to a player $i \in P$.
- An **action function** $\alpha : H \to 2^M$. $\alpha(h)$ is a subset of M. It is the set of actions or moves available to player $i = \ell(h)$ at node h.
- A collection of **successor functions** $\nu_h : \alpha(h) \to H \cup T$. For each h and for each action $a \in \alpha(h)$, $\nu_h(a)$ is a **successor** of node h in the game tree.
- A collection of **payoff functions** $\pi_i : T \to \mathbb{R}$. The terminal nodes in T determine the outcomes of the game. The payoff function π_i determines the value or **payoff** of each outcome for player i.
- The sets P, M, T, and H are all finite.

A **tree** in the graph theory sense is a graph wherein any two nodes are connected by exactly one path. For more on graphs, directed graphs, and trees, see the appendix.

For any node, h, in the game tree there is **exactly one path** from the root node to node h. This path corresponds to a **history** or **path of play**: a sequence of moves in the game leading from the root node to node **h**. A **complete history** is a path from the root node to a terminal node. In the extensive form game definition we took the set of terminal nodes T as the domain of each payoff function π_i. We could instead take the set of complete histories as the domain of π_i. In a **finite horizon** game these are equivalent. We will later consider infinitely repeated games. In these games some complete histories do not have terminal nodes and we will need to use the set of complete histories as the domain for each payoff function.

Information in Games: The information players have about the structure of the game and where they are in the game tree when taking actions plays a central in game theory. For now we are considering games with **complete** and **perfect** information.

- **Complete Information:** In an extensive form game with **complete information** the players in the game have complete knowledge of the structure of the game tree. Each player knows the actions available to all players along with the payoffs to each player for all outcomes.
- **Perfect Information:** In an extensive form game with **perfect information** each player has complete knowledge of the history or path of play at each node. At each decision node the players know where they are in the game tree and how they arrived there.
- Unless otherwise stated we will assume players have complete information. When we later consider games with incomplete information we will introduce **chance nodes** and reframe these as games with complete but imperfect information.

1.2.3 *Strategies in an Extensive Form Game with Perfect Information*

Informally, a strategy in an extensive form game is a **complete contingency plan** on how to play the game. For a game with perfect information, a strategy for player i indicates one action at each of player i's decision nodes.

Definition: Strategies in an Extensive Form Game with Perfect Information

Let G be an extensive form game with perfect information and let $\left\{h_i^1, h_i^2, \ldots, h_i^{k_i}\right\}$ be the set of player i's decision nodes. Recall that $\alpha(h_i^k)$ is the set of actions available to player i at node h_i^k.

Player i's strategy set, S_i, is the Cartesian product of their action sets:

$$S_i = \alpha(h_i^1) \times \alpha(h_i^2) \times \cdots \times \alpha(h_i^{k_i}),$$

and each of player i's strategies indicates one action for each decision node: If $s_i \in S_i$ then

$$s_i = a_1\ a_2\ \cdots a_{k_i},$$

where each a_k is an action in $\alpha(h_i^k)$.

1.2.4 *Extensive Form Games with Imperfect Information*

Consider the two player game, *Matching Pennies*:

- The players **simultaneously** choose either H or T (heads or tails).
- If the choices match, then player 1 wins and the payoffs are $(1, -1)$.
- If the choices do not match, then player 2 wins and the payoffs are $(-1, 1)$.

This game is shown in extensive form in Fig. 1.4. The key difference between this game and games with **perfect information** is that the players choose their actions simultaneously. Player 2 must select H or T before knowing Player 1's choice. We connect player 2's nodes to indicate these nodes are in a single **information set.** Player 2 cannot distinguish between these two nodes.

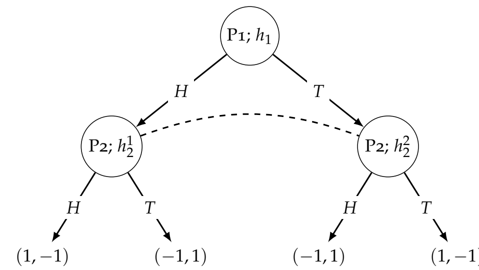

Figure 1.4: Matching pennies as an extensive form game with **imperfect information.** We've labeled each player node for reference. Player 2 has an **information set** with the two nodes h_2^1 and h_2^2. This is used to represent the idea that player 2 cannot distinguish between these two nodes and does not know where they are in the game tree when it is time to choose an action.

Because the players move simultaneously, we could also represent the game with Player 2 in the root position. Then player 1 would have two nodes in a single information set.

Definition: Extensive Form Game with Imperfect Information

A finite extensive form game with imperfect information consists of a finite extensive form game with perfect information with the following additional structure:

- For each player j, let I_j be a collection of **information sets**,

$$I_j = \left\{ I_j^1, I_j^2, \ldots, I_j^{k_j} \right\}$$

If two nodes h and h' belong to the same information set I_j^k then both nodes must be assigned to player j:

$$\ell(h) = \ell(h') = j$$

and player j has the same set of actions available at both nodes:

$$\alpha(h) = \alpha(h')$$

An information set can be a **singleton set** containing one node. In an extensive form game with perfect information, each information set is a singleton set. However, in games with perfect information we won't typically use information sets since they don't provide any additional structure.

When player j has just one information set, we will drop the superscript notation and label this information set as I_j.

Example 1.1

In the matching pennies game shown in Fig. 1.4, each player has one information set. Player 1's information set is the singleton set consisting of the root node: $I_1 = \{h_1\}$. Player 2's information set, $I_2 = \{h_2^1, h_2^2\}$.

In games of imperfect information we need to make a modification to the definition of a strategy. A player's strategy indicates an action for each **information set**.

Definition: Strategies in an Extensive Form Game with Imperfect Information

Let G be an extensive form game with imperfect information and let $\left\{I_j^1, I_j^2, \ldots, h_j^{k_j}\right\}$ be the collection of player j's information sets. We can use $\alpha(I_j^k)$ to denote the set of actions available to player j at the information set I_j^k. Player j's strategy set, S_j, is then the Cartesian product of their information sets:

$$S_j = \alpha(I_j^1) \times \alpha(I_j^2) \times \cdots \times \alpha(I_j^{k_j}),$$

and each of player j's strategies indicates one action for each information set: If $s_j \in S_j$ then

$$s_j = a_1\ a_2\ \cdots a_{k_j},$$

where each a_k is an action in $\alpha(I_j^k)$.

Since a decision node in an extensive form game with perfect information can be viewed as a singleton information set, this strategy definition includes the definition for extensive form games with perfect information as a degenerate case.

1.3 Normal Form Games

Every extensive form game can be represented in **normal form** (also called **strategic form**).

In a normal form game we specify the set of players, the strategies available to each player, and the payoffs for each outcome.

Definition: Normal Form Game

A normal form game G consists of:

- A set of players, $P = \{1, 2, \ldots, N\}$
- For each player i, a strategy set S_i.
  The Cartesian product of each strategy set is the set of **strategy profiles**:

  $$S = S_1 \times S_2 \times \cdots \times S_N$$

  A strategy profile $s \in S$ indicates one strategy for each player in the game and determines one outcome of the game.
- For each player i, a payoff function $\pi_i : S \to \mathbb{R}$.

Example 1.2

In the matching pennies game the strategy sets are $S_1 = \{H, T\}$ and $S_2 = \{H, T\}$. A two player game with small strategy sets can be represented in normal form as a **matrix game**.

	Player 2		
		H	T
Player 1	H	$(1, -1)$	$(-1, 1)$
	T	$(-1, 1)$	$(1, -1)$

By convention, player 1 is the *row player*, player 2 is the *column player*, and the payoffs are listed in the order (player 1, player 2)

In this example there are four **strategy profiles:** and

$$S = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$$

The payoffs depend on the strategy profile. For example,

$$\pi_1(H, T) = -1, \qquad \pi_2(H, T) = 1$$

1.4 *Rationality in an Extensive Form Game with Perfect Information*

We are assuming that our games model strategic interactions between **self-interested** players. We need a notion of **rational play**. Here is our first notion of rationality.

Consider an extensive form game with perfect information and consider a particular decision node h_j for player j. Suppose that a and a' are two different actions available to player j at node h_j: $a, a' \in \alpha(h_j)$.

Further suppose that playing a will "lead to" a strictly higher payoff for player j than playing a' at this node. We will need to clarify later what we mean by an action leading to a higher payoff. It is only straightforward for decision nodes whose successors are all terminal nodes. In general this will depend on the actions of other players.

If player j is **rational**, then player j will not play action a' at node h_j.