Shi Qin

MATH 455: HOMEWORK 6

Problem 1. 5. Consider the equation $x^4 = x^3 + 10$

(a) Find an interval $[a, b]$ of length one inside which the equation has a solution.

(b) Starting with $[a, b]$, how many steps of the Bisection Method are required to calculate the solution within 10^{-10} ? Answer with an integer.

Problem 2. Consider the bisection method to find a root for $f(x) = 0$ where f is a continuous function. We take the $[0, 1]$ as the initial interval provided that $f(0)f(1) < 0$. We take the following practical stopping criteria:

$$|b_n - a_n| \le \epsilon, \qquad \epsilon = 1 \times 10^{-8}.$$

How many steps of the bisection method are needed to obtain an approximation to the root?

Problem 3. Given a function $f(x) = \mathrm{e}^{-x} - \cos(x)$.

(1) Show that there is a root inside the interval $[1.1, 1.6]$.

(2) Using the fixed point iteration

$$x_{n+1} = g_1(x_n), \qquad \text{where} \qquad g_1(x) = f(x) + x,$$

with the starting point $x_0 = 1.6$. Perform 4 iterations to compute the values x_1, x_2, x_3, x_4. Does this scheme converge? Why?

(3) Using another fixed point iteration

$$x_{n+1} = g_2(x_n), \qquad \text{where} \qquad g_2(x) = x - f(x),$$

with the starting point $x_0 = 1.6$. Perform 4 iterations to compute the values x_1, x_2, x_3, x_4. Does this scheme converge? Why?

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$$x^4 = x^3 + 10$$

$$x^4 - x^3 - 10$$

$x = 1 \quad 1 - 1 - 10 = -10 < 0$

$x = 2 \quad 16 - 8 - 10 = -2 < 0$

$x = 3 \quad 81 - 27 - 10 = 44 > 0$

$$[2, 3]$$

$$|x_n - x'| < \frac{b-a}{2^{n+1}}$$

$[2, 3]$

$$b - a = \frac{1}{2^{n+1}} \leq 10^{-10}$$

$$n \geq 32.219$$

$$n \approx 33$$

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$$[0, 1]$$

$$n = \frac{\log(b-a) - \log \epsilon}{\log(2)}$$

$$= \frac{\log(1-0) - \log(10^{-8})}{\log(2)}$$

$$\approx 26.58$$

$$= \boxed{27}$$

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$f(x) = e^{-x} - \cos(x)$ $\quad [1.1, 1.6]$

$f(1.1) = e^{-1.1} - \cos(1.1) < 0$

$f(1.6) = e^{-1.6} - \cos(1.6) > 0$

According to IVT, there at least exist a root between 1.1 & 1.6

$f(x) = e^{x} - \cos(x) = 0$

$x = x + e^{-x} - \cos(x)$

$g(x) = x + e^{-x} - \cos(x)$

$x_0 = 1.6$

$g'(x) = 1 - e^{-x} + \sin(x)$

$|g'(1.6)| = |1 - e^{-1.6} + \sin(1.6)|$

$= 1.798 > 1$

do not converge

$x_1 = g'(x_0)$

$= 1.6 + e^{-1.6} - \cos(1.6)$

$= 1.8309$

$x_2 = g'(x_1)$

$= 1.8304 + e^{-1.8304} - \cos(1.8304)$

$= 2.2482$

$x_3 = g'(x_2)$

$= 2.2482 + e^{-2.2482} - \cos(2.2482)$

$= 2.9804$

$x_4 = g'(x_3)$

$= 2.9804 + e^{-2.9804} - \cos(2.9804)$

$= 4.0181$

$\{x_0, x_1, x_2, x_3, x_4\}$

$= \{1.6, 1.83, 2.25, 2.98, 4.02\}$

3 $g(x) = x - f(x)$

$g(x) = x - e^{-x} + \cos(x)$

$x_1 = g(x_0) = 1.6 - e^{-1.6} + \cos(1.6)$

$= 1.369$

$x_2 = g(x_1) = 1.369 - e^{-1.369} + \cos(1.369)$

$= 1.315$

$x_3 = g(x_2) = 1.315 - e^{-1.315} + \cos(1.315)$

$$= 1.299$$

$$x_4 = g(x_3) = 1.299 - e^{-1.299} + \cos(1.299$$

$$= 1.294$$

$$g'(x) = 1 + e^{-x} - \sin(x)$$

$$g'(1.6) = 1 + e^{-1.6} - \sin(1.6)$$

$$= 0.202 < 1$$

will converge