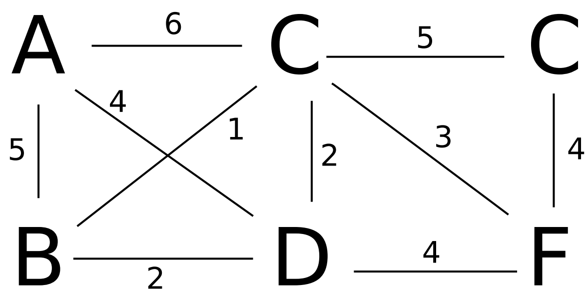
CMPSC 465 Data Structures & Algorithms
Spring 2022 Mingfu Shao and Chunhao Wang

Worksheet 8

Mar 16, 2022

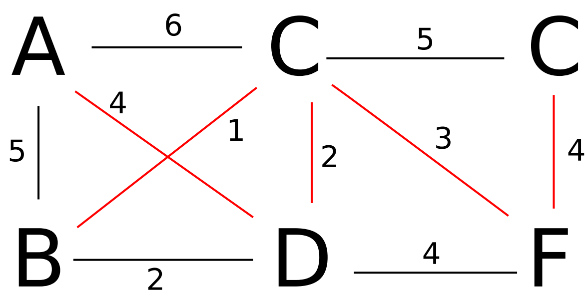
1. (pts.) Kruskal's algorithm

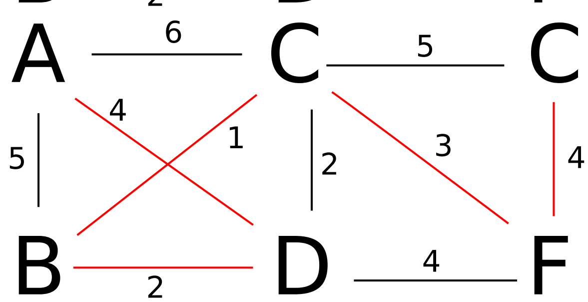
Given a graph as follows, use Kruskal's algorithm to find a minimum spanning tree. What is the minimum weight?



There are two possible solutions. Try to find all of them.

Solution





The weight is 14 for both solutions.

2. (pts.) Tree properties

A *tree* is an undirected graph that is connected and acyclic. Prove that any connected, undirected graph $G = (V, E)$ is a tree if and only if $|E| = |V| - 1$.

Solution

(See page 129 in the text book.)

We first prove "only if", i.e., if a graph is a tree, then it holds that $|E| = |V| - 1$. To see this, consider the way of building a tree one edge at a time, starting with an empty graph. Initially each of the $|V|$ vertices is disconnected from the others, in a connected component by itself. When a particular edge $\{u, v\}$ comes up, we can be sure that u and v lie in separate connected components, for otherwise there would already be a path between them and this edge could create a cycle. Adding this edge then merges these two components, thereby reducing the total number of connected components by one. Over the course of this incremental process, the number of components decreases from $|V|$ to one, meaning that $|V| - 1$ edges must have been added along the way.

Now we prove the "if direction", i.e., if $|E| = |V| - 1$ then an undirected and connected G is a tree. For this, we just need to show that G is acyclic. Consider the following procedure. Suppose an undirected and connected G has $|E| = |V| - 1$ but G is not a tree: it contains a cycle. We just remove an edge on this cycle, which does not affect the connectivity of G. This process yields a graph $G' = (V, E')$ with $E' \subseteq E$, which is undirected, connected, and acyclic, thus a tree. By the "only if" direction shown above, $|E'| = |V| - 1$. It turns out that $|E'| = |E| = |V| - 1$, which means no edges were removed. Hence G was acyclic to start with.