CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Linear Programming
(Textbook, Section 7.1)

Advertisement

Please consider taking

CMPSC 497 — Quantum Computation in Fall 2022

if you are interested in learning **Quantum Computing**

Optimization: we want to maximize some function $f(\mathbf{x})$ for $\boxed{\mathbf{x} \in \mathbb{R}^n,}$ subject to constraints

$$C(\mathbf{x}) \leq \mathbf{b}, \text{ for } \mathbf{b} \in \mathbb{R}^n$$

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$\vdots$

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How to allocate your time?

LP formulation

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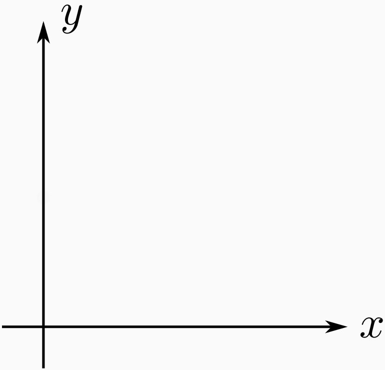

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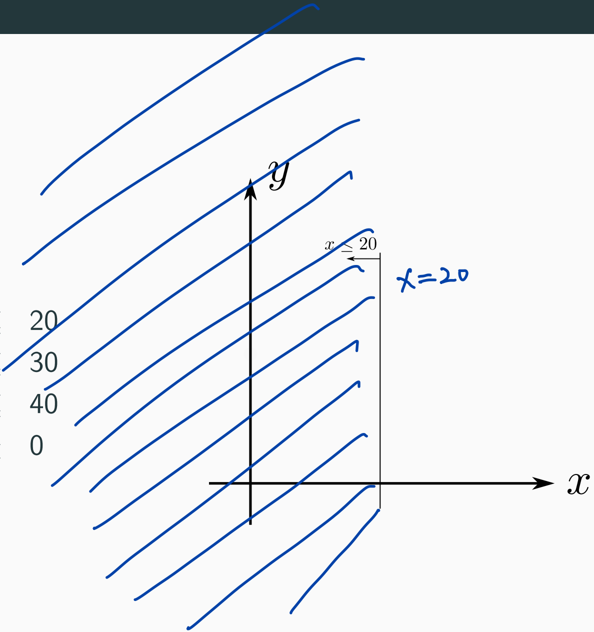

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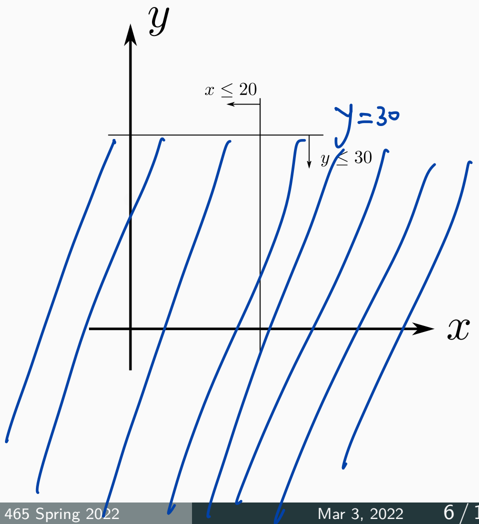

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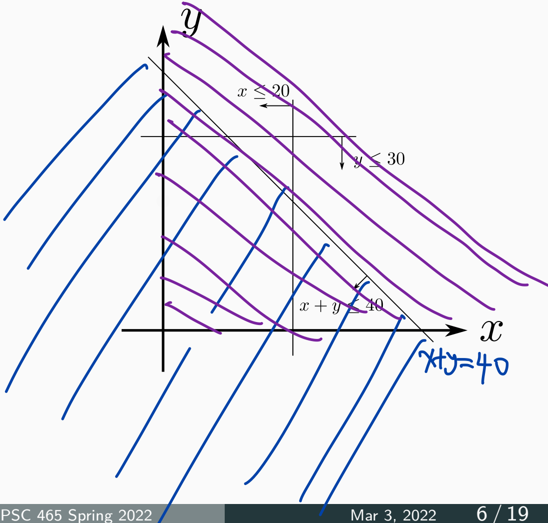

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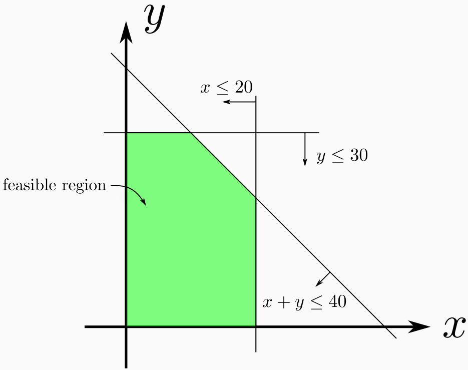

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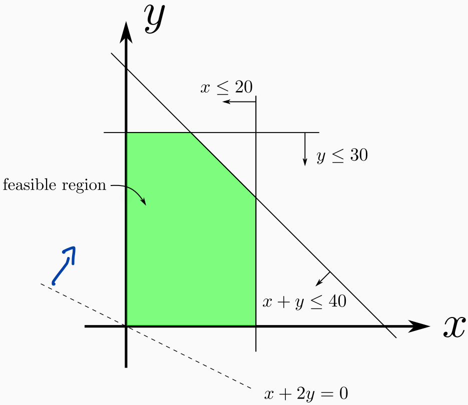

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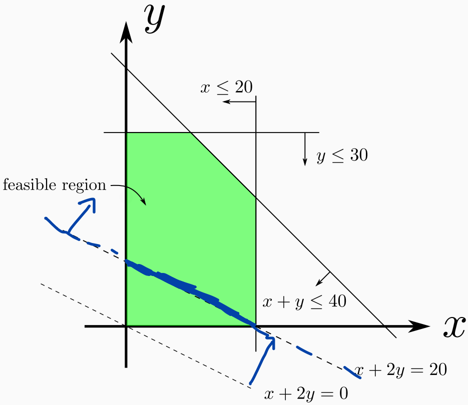

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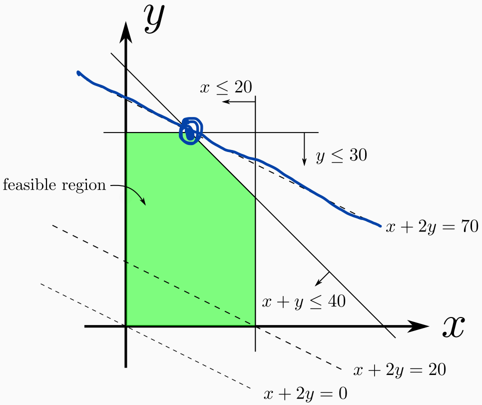

$$\begin{cases} x+2y=70 \\ y=30 \end{cases}$$
$$\Downarrow$$
$$(10,30)$$

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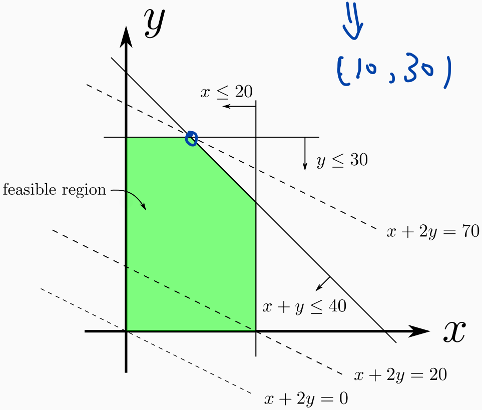

Optimal solution: $x+2y=70$ at $x=10$, $y=30$

Algorithm for solving LP

Observation: (search for an optimal solution)

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Theorem

For an LP with bounded, nonempty feasible region, the maximum value will be attained at some vertex of the feasible region

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The hill climbing approach (the simplex method)

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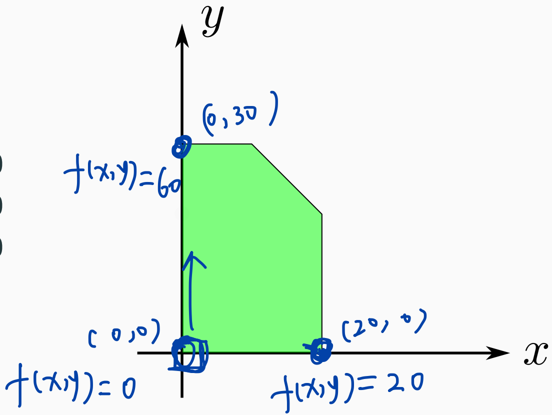

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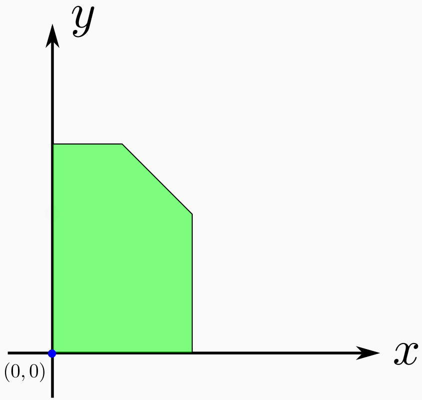

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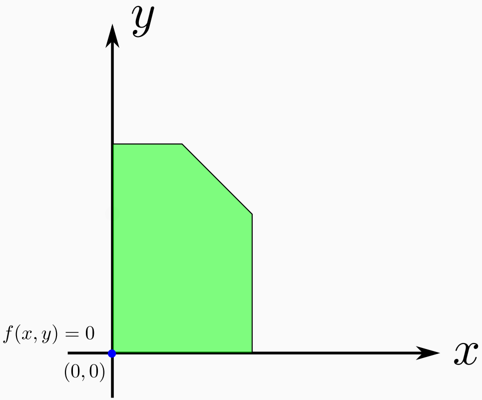

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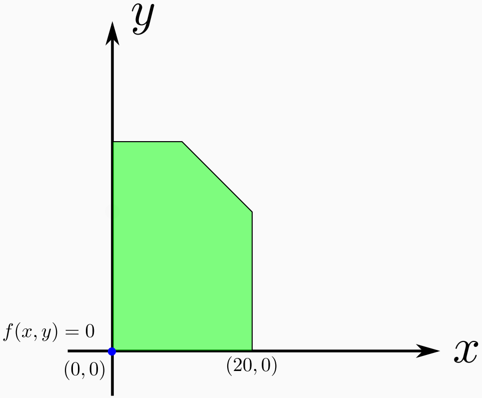

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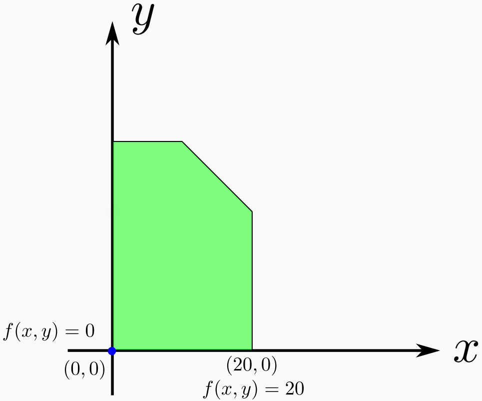

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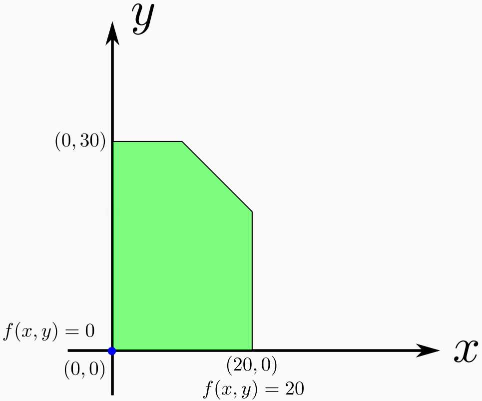

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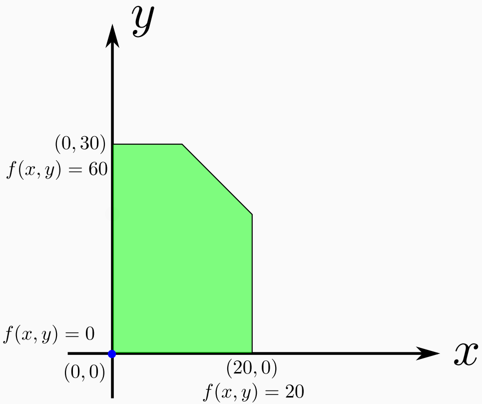

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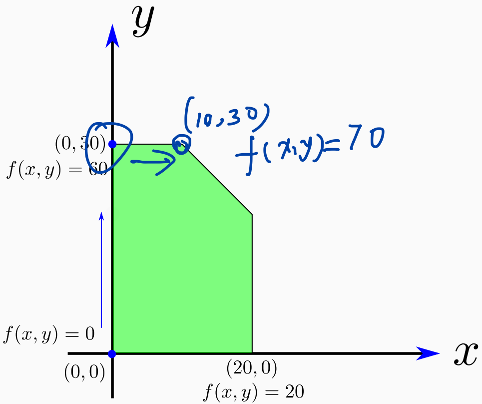

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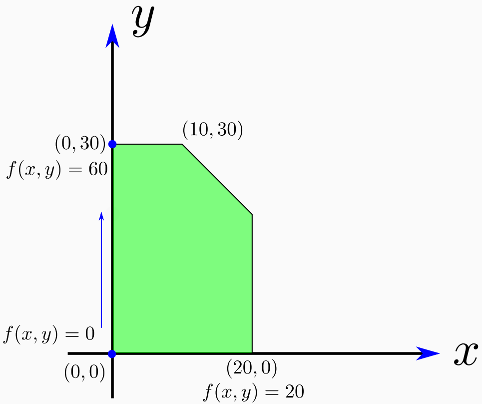

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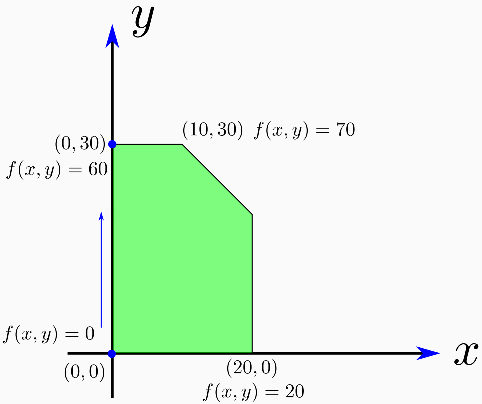

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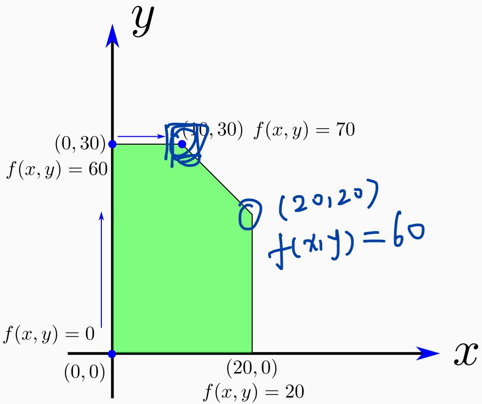

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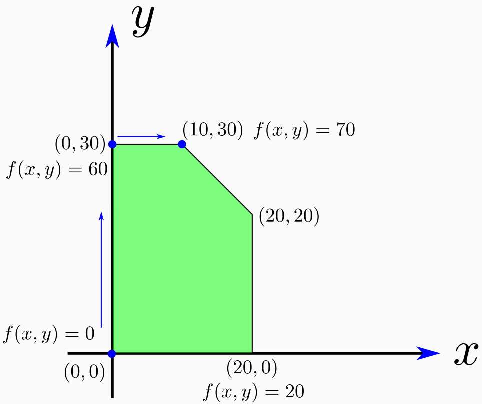

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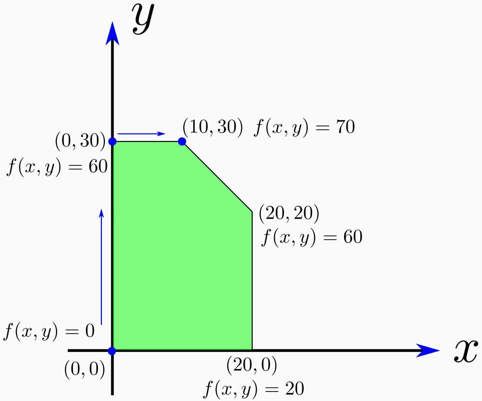

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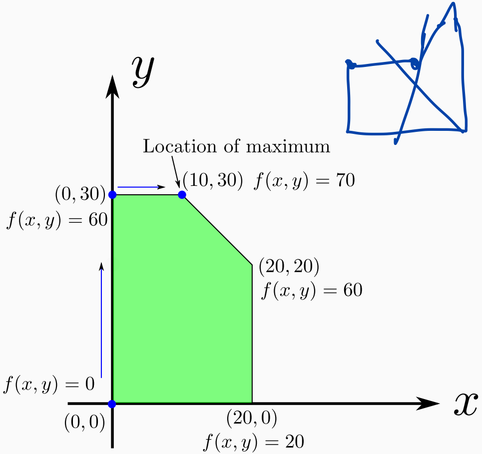

Standard forms

LP solvers, such as MOSEK, Gurobi, CVX, and COIN are implementations of the simplex method. They require the LP to be in certain standard form

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Standard form 1

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \boxed{A\mathbf{x} \leq \mathbf{b}} \\ & \mathbf{x} \geq 0 \\ & \mathbf{x}, \mathbf{c}, \mathbf{b} \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n} \end{aligned}$$

$(Ax)_1 \leq b_1$

$(Ax)_2 \leq b_2$

$\vdots$

$(Ax)_n \leq b_n$

Standard forms

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Standard form 1

$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

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$c = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad c^T \begin{pmatrix} x \\ y \end{pmatrix} = (1, 2)\begin{pmatrix} x \\ y \end{pmatrix}$

$= x+2y$

Example:

$$\begin{aligned} \text{maximize} \quad & x + 2y \\ \text{subject to} \quad & x \leq 20 \\ & y \leq 30 \\ & x + y \leq 40 \\ & x, y \geq 0 \end{aligned}$$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix}$

$\leq \begin{pmatrix} 20 \\ 30 \\ 40 \end{pmatrix}$

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\equiv
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\text{maximize} & (1,2)\begin{pmatrix} x \\ y \end{pmatrix} & & \\
\text{subject to} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} & \leq & \begin{pmatrix} 20 \\ 30 \\ 40 \end{pmatrix} \\
 & x, y & \geq & 0
\end{array}
$$

Convert to the standard form

- Minimization to maximization

$$\begin{array}{llll} \min & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array} \equiv$$

max

Convert to the standard form

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$$
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\text{s. t.} & A\mathbf{x} & \leq & \mathbf{b} \\
& \mathbf{x} & \geq & 0
\end{array}
\quad \equiv \quad
\begin{array}{lrcl}
\max & -\mathbf{c}^T\mathbf{x} & & \\
\text{s. t.} & A\mathbf{x} & \leq & \mathbf{b} \\
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\end{array}
$$

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- Equality to inequality

$$\begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & x_1 + x_2 & = & 7 \end{array} \quad \equiv$$

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- Wrong inequality direction

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- Missing nonnegative constraints

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s. t.} \quad & x_1 \leq 20 \\ & x_1 + x_2 \leq 40 \\ & x_1 \geq 0 \end{aligned} \quad \equiv$$

missing $x_2 \geq 0$

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\equiv
\begin{array}{lrcl}
 & \multicolumn{3}{c}{\text{rewrite } x_2 = x_2^+ - x_2^-} \\
\max & x_1 + 2(x_2^+ - x_2^-) & & \\
\text{s. t.} & x_1 & \leq & 20 \\
 & x_1 + (x_2^+ - x_2^-) & \leq & 40 \\
 & x_1 & \geq & 0 \\
 & x_2^+ & \geq & 0 \\
 & x_2^- & \geq & 0
\end{array}
$$

Another Standard form

Standard form 2

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20 is bigger than x_1 by some positive amount, call it s

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The new variable s is call the *slack variable*

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We just **reduced** $\mathrm{shortest\_path}$ to LP

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$f(e)$

$f_{s,u} = \boxed{f_e}$ $e = (s, u)$

LP formulation

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\text{s.t.} & f_e & \leq & c_e & \forall e \in E \\
& \sum_{(u,v)\in E} f_{u,v} - \sum_{(v,w)\in E} f_{v,w} & = & 0 & \\
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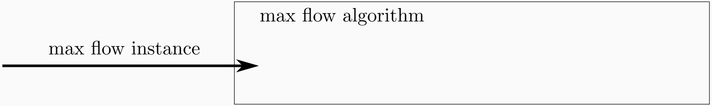

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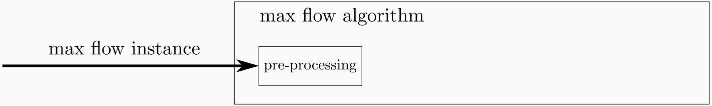

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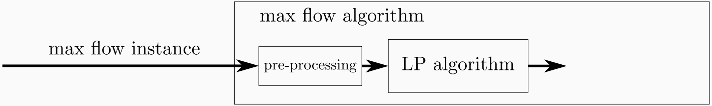

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