CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Flow network
(Textbook, Section 7.2
Kleinberg & Tardos Section 7.1)

Motivation

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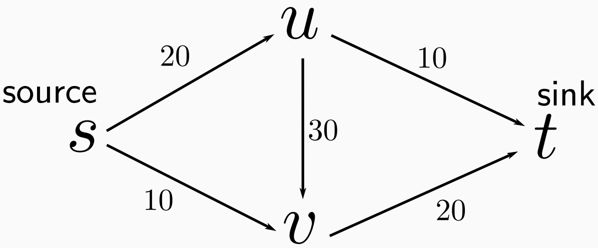
Example:



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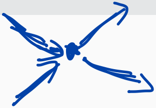

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Intuition: $v(f)$ shows how much traffic can be accommodated

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Problem (The Max-Flow problem)

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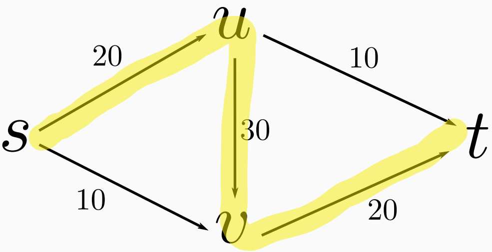

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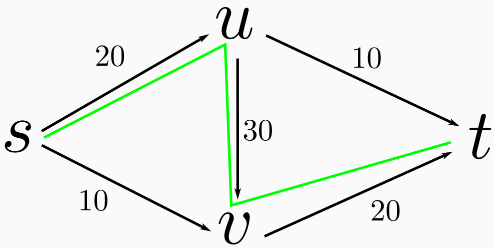

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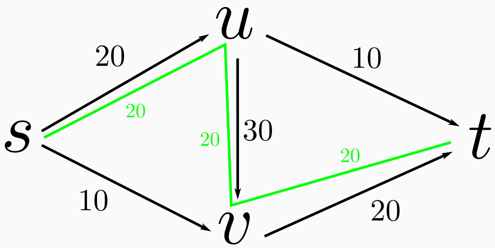

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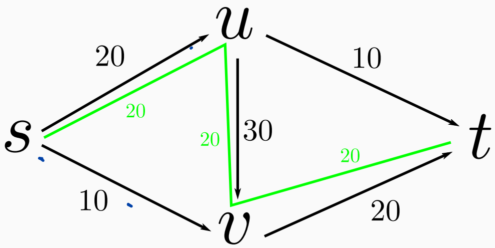

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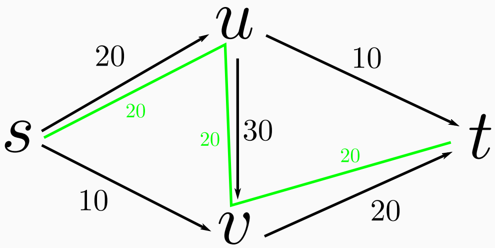

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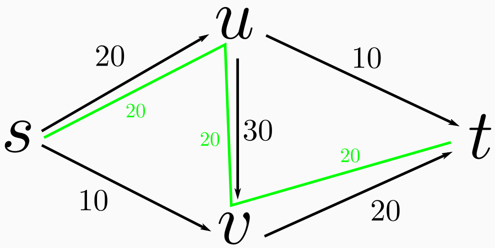

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$v(f)=20$. Can we do better?

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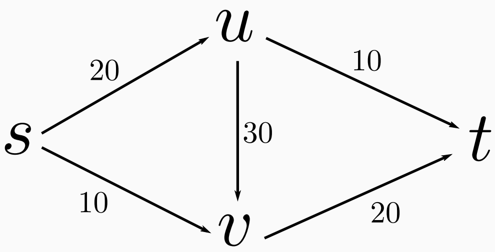

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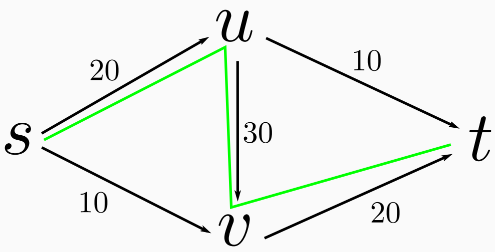

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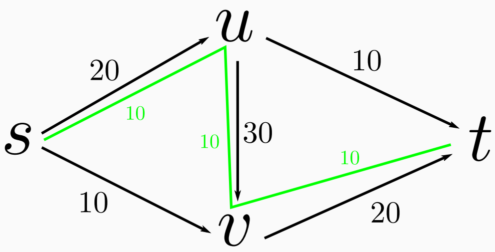

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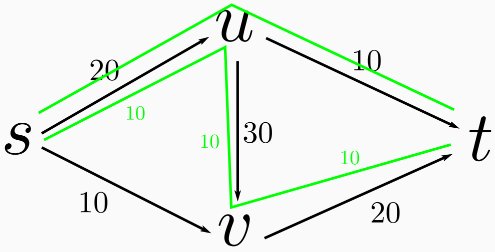

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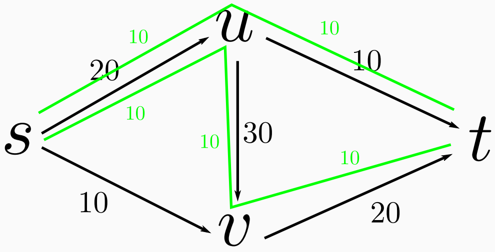

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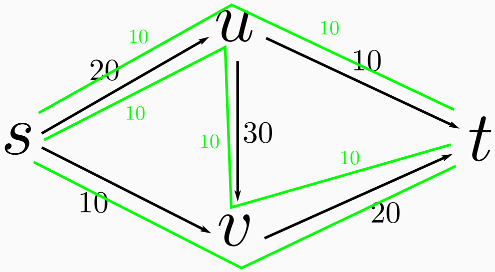

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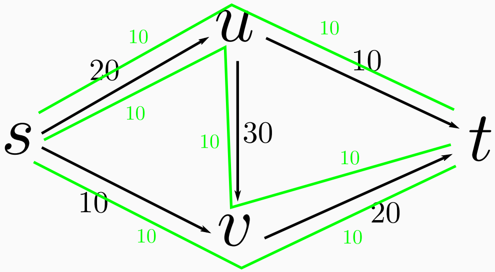

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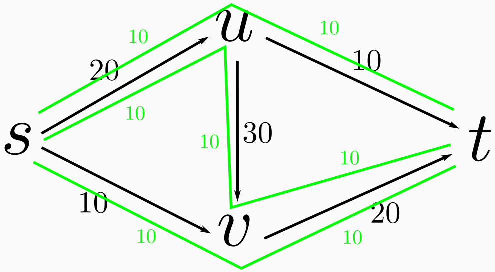

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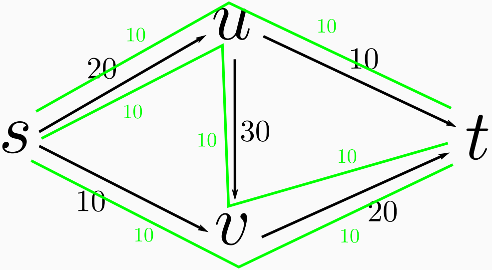

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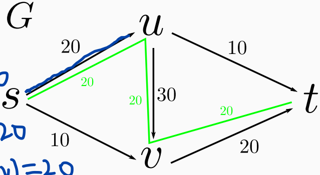

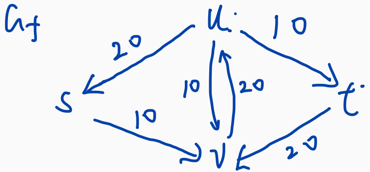

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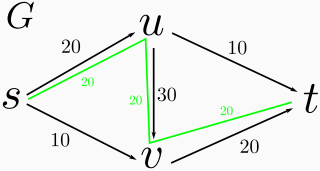

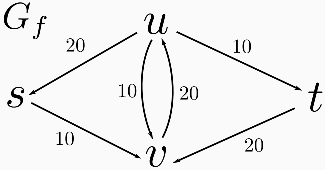

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Any s-t path in G_f is called an **augmenting path**

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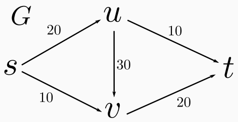

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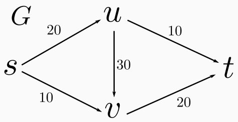

G_f

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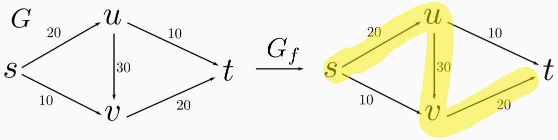

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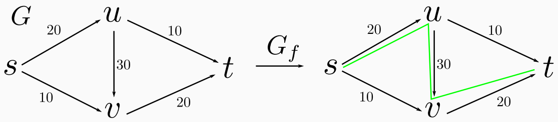
$u \to v$	0		
$u \to t$	0		
$v \to t$	0		

Running example

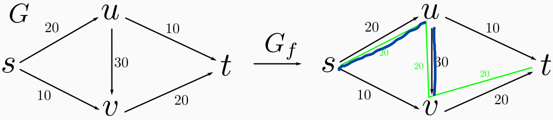


$P : s \to u \to v \to t$

b:

e	$f_1(e)$		
$s \to u$	0		
$s \to v$	0		
$u \to v$	0		
$u \to t$	0		
$v \to t$	0		

Running example

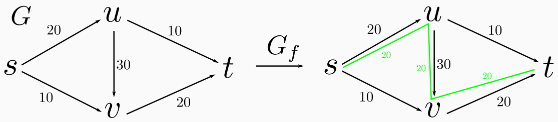


$P : s \to u \to v \to t$

$b : 20$

e	$f_1(e)$	$f(e)$	
$s \to u$	0	20	
$s \to v$	0	0	
$u \to v$	0	20	
$u \to t$	0	0	
$v \to t$	0	20	

Running example

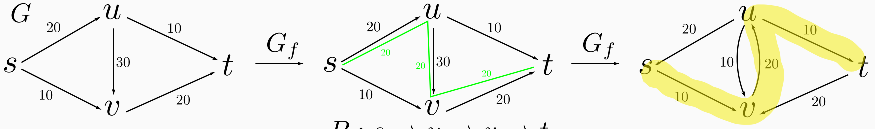


$P : s \to u \to v \to t$

$b : 20$

e	$f_1(e)$	$f_2(e)$
$s \to u$	0	20
$s \to v$	0	0
$u \to v$	0	20
$u \to t$	0	0
$v \to t$	0	20

Running example

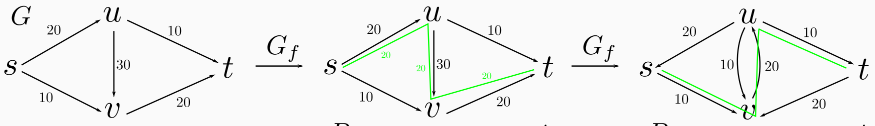


$P : s \to u \to v \to t$

$b : 20$

e	$f_1(e)$	$f_2(e)$	
$s \to u$	0	20	
$s \to v$	0	0	
$u \to v$	0	20	
$u \to t$	0	0	
$v \to t$	0	20	

Running example



$P : s \to u \to v \to t$

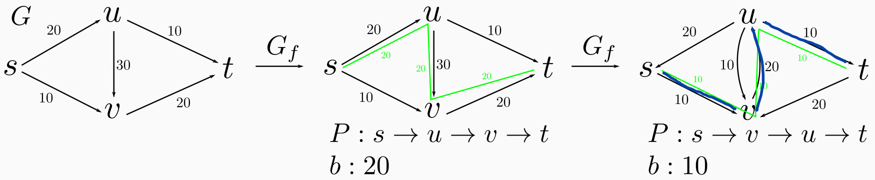
$b : 20$

$P : s \to v \to u \to t$

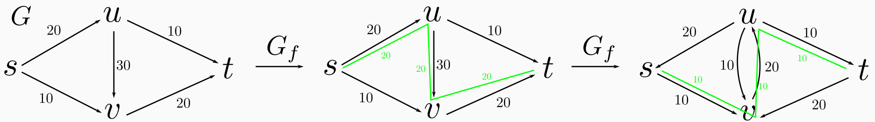
b: 10

e	$f_1(e)$	$f_2(e)$	
$s \to u$	0	20	
$s \to v$	0	0	
$u \to v$	0	20	
$u \to t$	0	0	
$v \to t$	0	20	

Running example



e	$f_1(e)$	$f_2(e)$	$f(e)$
$s \to u$	0	20	20
$s \to v$	0	0	10
$u \to v$	0	20	10
$u \to t$	0	0	10
$v \to t$	0	20	20

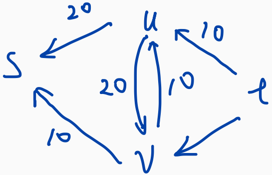


$P : s \to u \to v \to t$
$b : 20$

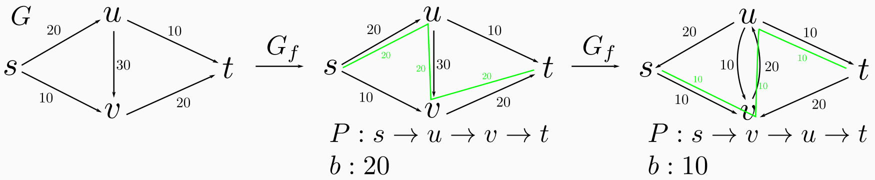
$P : s \to v \to u \to t$
$b : 10$

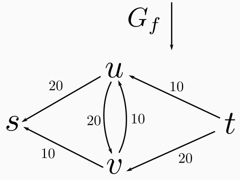
v

e	$f_1(e)$	$f_2(e)$	$f_3(e)$
$s \to u$	0	20	20
$s \to v$	0	0	10
$u \to v$	0	20	10
$u \to t$	0	0	10
$v \to t$	0	20	20



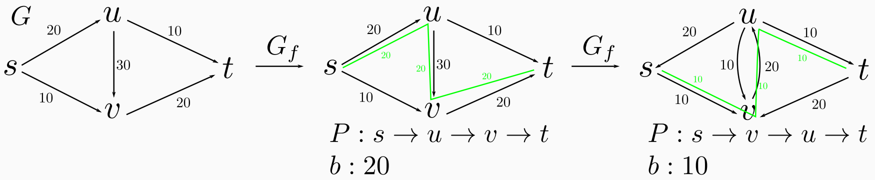
Running example



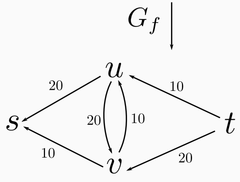


e	$f_1(e)$	$f_2(e)$	$f_3(e)$
$s \to u$	0	20	20
$s \to v$	0	0	10
$u \to v$	0	20	10
$u \to t$	0	0	10
$v \to t$	0	20	20

Running example



e	$f_1(e)$	$f_2(e)$	$f_3(e)$
$s \to u$	0	20	20
$s \to v$	0	0	10
$u \to v$	0	20	10
$u \to t$	0	0	10
$v \to t$	0	20	20



No more s-t path