# Packet 2: Functions of Random Variables

## Chap 5.1 Function of One Random Variable

STAT 414 has introduced distributions that are meaningful and mathematically continent.

*Motivation*:

- What if the distribution of a random variable is uncommon but some transformation leads to a common distribution?

- Sometimes, we can also apply simple tools to the transformed variable.
  E.g. the time series of world population is as follows:

| **Year** | **Pop Size (million)** | **log(Size) (base 10)** |
|---|---|---|
| 1 | 170 | 8.23 |
| 400 | 190 | 8.28 |
| 800 | 220 | 8.34 |
| 1200 | 360 | 8.56 |
| 1600 | 545 | 8.74 |
| 1800 | 900 | 8.95 |
| 1850 | 1200 | 9.08 |
| 1900 | 1625 | 9.20 |
| 1950 | 2500 | 9.40 |
| 1975 | 3900 | 9.59 |
| 2000 | 6080 | 9.78 |

- Change the support of the random variable, e.g. logit transformation on a proportion variable.

Question: After the distribution of transformed variable is estimated, how about the original one? Let us assume the distribution of a continuous random variable $X$ follows a common distribution, e.g. log income follows a normal distribution. What is the distribution of the income at the original scale, $Y = \exp(X)$?

So we need a theory to figure out the distribution of $Y = u(X)$ for any transformation $u(.)$ that is a 1-1 mapping.

Example 5.1-1 (p.g. 163) Let $X$ have a gamma distribution with p.d.f.

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta},$$

where $0 < x < \infty$, $\alpha > 0$ and $\theta > 0$.

Question: what is the distribution of $Y = e^X$?

*Shortcut for the change of variable (univariate case):*

Let X be a continuous random variable with p.d.f. f($x$). When $u(.)$ is a 1-1 mapping:

- $Y = u(X)$ is an strictly increasing function of X with inverse function $X = v(Y)$.

- $Y = u(X)$ is an strictly decreasing function of X with inverse function $X = v(Y)$.

When $u(.)$ is not a 1-1 mapping:

Example: X has a Cauchy distribution with p.d.f.

$$f(x) = \frac{1}{x(1+x^2)},$$

where $-\infty < x < \infty$.

Question, what is the distribution of $Y = X^2$?

More examples 5.1-3, 5.1-5.

*A special case. Theorem 5.1-1:* Let $F(x) = P(X < x)$ have the properties of a c.d.f.: strictly increasing on $a < x < b$, $F(a) = 0$ and $F(b) = 1$.

If $Y \sim U(0,1)$ and $Y = F(X)$ , then the random variable $X = F^{-1}(Y)$ will have c.d.f. $F(x)$.

## Chap 5.2 Function of Two Random Variables

Question: What if two random variables are involved in the transformation?

Answer: The rule is the same as that in the univariate case, with derivative being replaced by the Jacobian. (p.g. 171)

Example 5.2-1: Let $X_1, X, 2$ have the joint p.d.f. $f(x_1, x_2) = 2$ for $0 < x_1 < x_2 < 1$.

Question: What is the joint distribution of $Y_1 = X_1/X_2$ and $Y_2 = X_2$?

More examples: 5.2-2, 5.2-3.

Example 5.2-4: Let two independent variables $U \sim \chi^2(r_1)$ $V \sim \chi^2(r_2)$.

Question: What is the distribution of $F = \frac{U/r_1}{V/r_2}$?