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~~5.1~~-5. The pdf of X is $f(x) = \theta x^{\theta-1}$, $0 < x < 1, 0 < \theta < \infty$. Let $Y = -2\theta \ln X$. How is Y distributed?

$$f(x) = \theta x^{\theta-1}$$

$$Y = -2\theta \ln x$$

$$X = e^{-\frac{Y}{2\theta}} \qquad Y \in (0, \infty)$$

$$\frac{dy}{dx} = \frac{-2\theta}{x}$$

$$f_Y(y) = \theta x^{\theta-1} \cdot \frac{x}{2\theta} \qquad Y \in (0, \infty)$$

$$= \frac{1}{2} (x)^{\theta}$$

$$= \frac{1}{2} \left(e^{\frac{-Y}{2\theta}}\right)^{\theta}$$

$$= \frac{1}{2} e^{\frac{-Y}{2}}$$

$$Y \sim \exp\left(\frac{1}{2}\right)$$

~~5.1~~-15. Let $Y = X^2$.

(a) Find the pdf of Y when the distribution of X is $N(0,1)$.

(b) Find the pdf of Y when the pdf of X is $f(x) = (3/2)x^2, \ -1 < x < 1$.

a) $X \sim N(0,1)$

$Y = X^2$

$X = \pm\sqrt{Y}$

for $X = 0$

$Y = 0$

$$f(y) = \frac{\frac{1}{2}^{\frac{1}{2}}}{\sqrt{\frac{1}{2}}} y^{\frac{1}{2}-1} e^{-\frac{1}{2}y}$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-\frac{x}{2}} \qquad 0 < y < \infty$$

b $f(x) \frac{3}{2} x^2 \qquad -1 < x < 1$

$F(x) = \int_{1}^{x} \frac{3}{2} t^2 dt$

$= \frac{1}{2}(x^3 + 1)$

$y = x^2$

$x = \pm\sqrt{y}$

$$F(y) = \frac{y^{\frac{3}{2}} + 1}{2} - \frac{-y^{\frac{3}{2}} - 1}{2}$$

$$= y^{\frac{3}{2}}$$

$$f(y) = \frac{d}{dy} F(y) = \frac{d}{dy} y^{\frac{3}{2}} = \frac{3}{2}\sqrt{y} \qquad 0 < y < 1$$

~~5.2~~-1. Let X_1, X_2 denote two independent random variables, each with a $\chi^2(2)$ distribution. Find the joint pdf of $Y_1 = X_1$ and $Y_2 = X_2 + X_1$. Note that the support of Y_1, Y_2 is $0 < y_1 < y_2 < \infty$. Also, find the marginal pdf of each of Y_1 and Y_2. Are Y_1 and Y_2 independent?

$Y_1 = x_1$ $\qquad$ $x_1 = Y_1$

$Y_2 = X_2 + X_1$ $\qquad$ $X_2 = Y_2 - X_1 = Y_2 - Y_1$

$$f(x_1, x_2) = \frac{1}{2} e^{-\frac{x_1}{2}} \frac{1}{2} e^{-\frac{x_2}{2}}$$

$$= \frac{1}{4} e^{-\frac{x_1 + x_2}{2}}$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$f(y_1, y_2) = f(x_1, x_2)\,|J|$$

$$= \frac{1}{4} e^{-\frac{y_2}{2}}$$

$$f(y_1) = \int_{y_1}^{\infty} \frac{1}{4} e^{\frac{y_1}{2}}\, dy_1$$

$$= \frac{1}{4} e^{\frac{y_1}{2}}$$

$$f(y_2) = \frac{1}{4} e^{\frac{y_2}{2}}$$

$$f(y_1, y_2) \neq f(y_1) \cdot f(y_2)$$

So they are not independent.

5.~~2~~-5. Let the distribution of W be $F(8,4)$. Find the following:

(a) $F_{0.01}(8,4)$.

(b) $F_{0.99}(8,4)$.

(c) $P(0.198 \le W \le 8.98)$.

a $F_{001}(8,4) = 14.8$

b $F_{0.99}(8,4) = \frac{1}{F_{0.01}(8,4)}$

$= \frac{1}{7.01}$

$= 0.142653$

c. $P(0.198 \le W \le 8.98)$

$P(W \le 8.98) = 0.975$

$P(W \le 0.198) = 0.025$

$P(0.198 \le W \le 8.98) = 0.975 - 0.025 = 0.95$

~~5.2~~-6. Let X_1 and X_2 have independent gamma distributions with parameters α, θ and β, θ, respectively. Let $W = X_1/(X_1 + X_2)$. Use a method similar to that given in the derivation of the F distribution (Example 5.2-4) to show that the pdf of W is

$$g(w) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, w^{\alpha-1}(1-w)^{\beta-1}, \qquad 0 < w < 1.$$

We say that W has a beta distribution with parameters α and β. (See Example 5.2-3.)

$$W = \frac{X_1}{X_1+X_2}$$

$$g(w) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1}(1-w)^{\beta-1} \qquad 0<w<1$$

X_1 & X_2 are independent gamma dist

with α,θ & β,θ

~~5.3~~-4. Let X_1 and X_2 be a random sample of size $n = 2$ from the exponential distribution with pdf $f(x) = 2e^{-2x}$, $0 < x < \infty$. Find

(a) $P(0.5 < X_1 < 1.0, 0.7 < X_2 < 1.2)$.

(b) $E[X_1(X_2 - 0.5)^2]$.

a) $f(x_1, x_2) = 2e^{-2x_1} \cdot 2e^{-2x_2} = 4e^{-2x_1-2x_2}$

$P(0.5 < X_1 < 1.0, 0.7 < X_2 < 1.2)$

$$\int_{0.5}^{1.0}\int_{0.7}^{1.2} 4e^{-2x_1-2x_2}\, dx_2\, dx_1$$

$$= 4\int_{0.5}^{1} e^{-2x_1} \left.\frac{e^{-2x_2}}{-2}\right|_{0.7}^{1.2} dx_1$$

$$= 0.31176 \left.\frac{e^{-2x_1}}{-2}\right|_{0.5}^{1}$$

$$= 0.03625$$

$$E(X_1(X_2-0.5)^2) = E(X_1(X_2^2+0.25-X_2))$$

$$= E(X_1X_2^2+0.25X_1-X_1X_2)$$

$$= E(X_1X_2^2)+E(0.25X_1)-E(X_1X_2)$$

$$= E(X_1)E(X_2^2)+0.25E(X_1)-E(X_1)\cdot E(X_2)$$

$$= \frac{1}{2}\cdot\frac{2}{4}+\frac{1}{4}\cdot\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2}$$

$$= \frac{1}{4}+\frac{1}{8}-\frac{1}{4}$$

$$= \frac{1}{8}$$

5.3-11. Let X_1, X_2, X_3 be three independent random variables with binomial distributions $b(4, 1/2)$, $b(6, 1/3)$, and $b(12, 1/6)$, respectively. Find

(a) $P(X_1 = 2, X_2 = 2, X_3 = 5)$.

(b) $E(X_1 X_2 X_3)$.

(c) The mean and the variance of $Y = X_1 + X_2 + X_3$.

$$X_1 \sim b(4, \tfrac{1}{2})$$
$$X_2 \sim b(6, \tfrac{1}{3})$$
$$X_3 \sim b(12, \tfrac{1}{6})$$

a) $P(X_1 = 2, X_2 = 2, X_3 = 5) = P(X_1 = 2)P(X_2 = 2)P(X_3 = 5)$

$$= \binom{4}{2}\left(\tfrac{1}{2}\right)^2\left(1-\tfrac{1}{2}\right)^{4-2} \cdot \binom{6}{2}\left(\tfrac{1}{3}\right)^2\left(1-\tfrac{1}{3}\right)^{6-2}$$
$$\cdot \binom{12}{5}\left(\tfrac{1}{6}\right)^5\left(1-\tfrac{1}{6}\right)^{12-5}$$

$$= 0.375 \cdot 0.329 \cdot 0.0284$$
$$= 0.00351$$

b) $E(X_1 X_2 X_3) = E(X_1) \cdot E(X_2) \cdot E(X_3)$

$$= 4 \cdot \tfrac{1}{2} \cdot 6 \cdot \tfrac{1}{3} \cdot 12 \cdot \tfrac{1}{6}$$
$$= 2^3$$
$$= 8$$

c. mean & var of $Y = X_1 + X_2 + X_3$

$$E(Y) = E(X_1 + X_2 + X_3)$$

$$= E(X_1) + E(X_2) + E(X_3)$$

$$= 2 + 2 + 2$$

$$= 6$$

$$Var(Y) = Var(X_1) + Var(X_2) + Var(X_3)$$

$$= 4 \cdot \frac{1}{2} \cdot \frac{1}{2} + 6 \cdot \frac{1}{3} \cdot \frac{2}{3} + 12 \cdot \frac{1}{6} \cdot \frac{5}{6}$$

$$= 1 + \frac{12}{9} + \frac{60}{36}$$

$$= 4$$