CMPSC 465
Data Structures and Algorithms
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Dynamic Programming

Dynamic Programming

Prelude

Dynamic programming vs. Greedy algorithms

- Similarity: optimal substructure
- Difference: greedy choice property

A greedy algorithm makes the greedy choice and it leaves a subproblem to solve

Sometimes, the greedy choice won't work — we need to check many subproblems to find the optimal solution $\rightarrow$ **Dynamic programming**

General steps for Dynamic Programming

- Break problem into smaller subproblems
- Solve smaller subproblems first (**bottom-up**)
- Use information from smaller subproblems to solve a larger subproblem

Warm-up: Longest increasing subsequence

Problem (Longest increasing subsequence)

Given $a_1, \ldots, a_n \in \mathbb{R}$, find the longest subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ s.t. $i_1 < i_2 < \cdots < i_k$ and $a_{i_1} < a_{i_2} < \cdots a_{i_k}$

Example:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
5	2	8	6	3	6	9	7

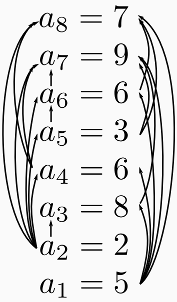
$i_1 = 2, i_2 = 5, i_3 = 6, i_4 = 7$

Solving LIS using DAG

We can model the longest increasing subsequence problem as a directed acyclic graph

- There is a linke $i \to j$ if $a_i < a_j$
- Find the longest path in the DAG:

  Use $L(j)$ to denote the length of the longest path (longest increasing subsequence) ending with a_j



def $\mathrm{LIS\_DAG}$*(DAG $G = (V, E)$ for $a_1, \ldots, a_n$)***:**

  for $j = 1, \ldots, n$**:**

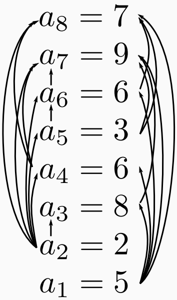
    $$L(j) = \begin{cases} 1 + \max\{L(i) : (i, j) \in E\} \\ 1 \text{ if no such edge} \end{cases};$$

  return $\max_j L(j)$;

Running example

def $\mathrm{LIS\_DAG}$*(GAG $G = (V, E)$ for $a_1, \ldots, a_n$)*:

- **for** $j = 1, \ldots, n$:
  - $L(j) = \begin{cases} 1 + \max\{L(i) : (i, j) \in E\} \\ 1 \text{ if no such edge} \end{cases}$;
- **return** $\max_j L(j)$;



a_j	5	2	8	6	3	6	9	7
j	1	2	3	4	5	6	7	8
L	1	1	2	2	2	3	4	4

A more direct approach

Do we really need to work on a DAG?

def $\mathrm{LIS\_DAG}$*(GAG $G = (V, E)$ for $a_1, \ldots, a_n$)*:

  for $j = 1, \ldots, n$:

    $L(j) = \begin{cases} 1 + \max\{L(i) : (i, j) \in E\} \\ 1 \text{ if no such edge} \end{cases}$;

  return $\max_j L(j)$;

A more direct approach:

def LIS*($a_1, \ldots, a_n$)*:

  for $j = 1, \ldots, n$:

    $L(j) = \begin{cases} 1 + \max\{L(i) : a_i < a_j\} \\ 1 \text{ if no such } i \end{cases}$;

  return $\max_j L(j)$;

Running time: $O(n^2)$

Costs more than greedy: need to check more subproblems

The actual subsequence

The above dynamic programming algorithm only computes the length of the longest increasing subsequence, but how to find the subsequence?

We use an additional table to keep track of the subsequence

```
def LIS(a_1, ..., a_n):
    for j = 1, ..., n:
        L(j) = 1, prev(j) = ·;
        for i = 1, ..., j:
            if a_i < a_j and L(i) + 1 > L(j):
                L(j) = L(i) + 1, prev(j) = i;
    return max_j L(j);
```

a_j	5	2	8	6	3	6	9	7
j	1	2	3	4	5	6	7	8
L	1	1	2	2	2	3	4	4
prev	·	·	1	1	2	5	6	6

Key steps to design DP algorithms

1. Identify subproblems
2. Recurrence
   e.g. $L(j) = 1 + \max\{L(i) : a_i < a_j\}$
3. Base case