

CMPSC 465

Data Structures and Algorithms

Spring 2022

Instructor: Chunhao Wang

Dynamic Programming

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Prelude

Key steps to design DP algorithms

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1. Identify subproblems
2. Recurrence
e.g. $L(j) = 1 + \max\{L(i) : a_i < a_j\}$
3. Base case

Dynamic Programming

Edit Distance (Textbook Section 6.3)

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cost : 5

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x:	A	C	-	-	G	T	A
		↕	↕	↕		↕	↕
y:	A	T	C	T	G	-	-

cost : 5

So $d(x, y) = 3$

Edit distance — subproblem

Consider two strings

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Define

$$E(i, j) = d(x_1 \cdots x_i, y_1 \cdots y_j)$$

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How to use the solution to the subproblems to solve $E(i, j)$?

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Contributes 1 to the cost plus the cost of alignment

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$$E(i, j) = 1 + E(i, j - 1)$$

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$$\text{Case 3} \quad \begin{array}{cccc} x_1 & \cdots & x_{i-1} & x_i \\ y_1 & \cdots & y_{j-1} & y_j \end{array}$$

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$$E(i, j) = \begin{cases} E(i - 1, j - 1) & \text{if } x_i = y_j \\ 1 + E(i - 1, j - 1) & \text{otherwise} \end{cases}$$

Recurrence (II)

The recurrence:

$$E(i, j) = \min\{1 + E(i - 1, j), 1 + E(i, j - 1), \text{diff}(i, j) + E(i - 1, j - 1)\},$$

where

$$\text{diff}(i, j) = \begin{cases} 1 & \text{if } x_i \neq y_j \\ 0 & \text{otherwise} \end{cases}$$

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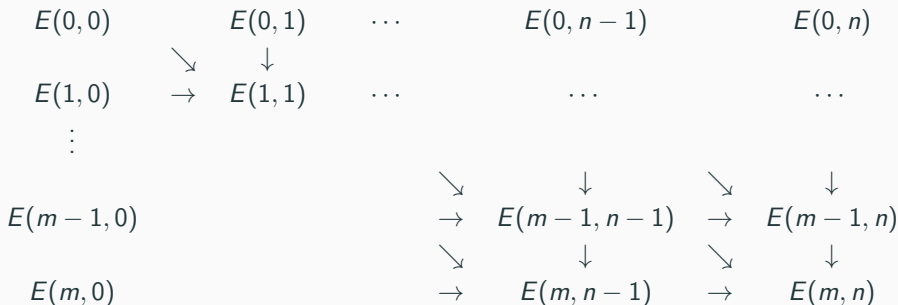
Base case: $E(0, 0) = 0$, $E(i, 0) = i$, $E(0, j) = j$

Filling the table

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A T C T G

A

C

G

T

A

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		A	T	C	T	G
	0	1	2	3	4	5
A	1	0	1	2	3	4
C	2	1	1	1	2	3
G	3	2	2	2	2	2
T	4	3	2	3	2	3
A	5	4	3	3	3	3