CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Greedy algorithms

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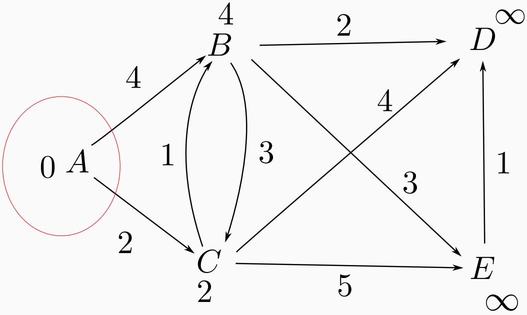
Warm-up

Greedy algorithms

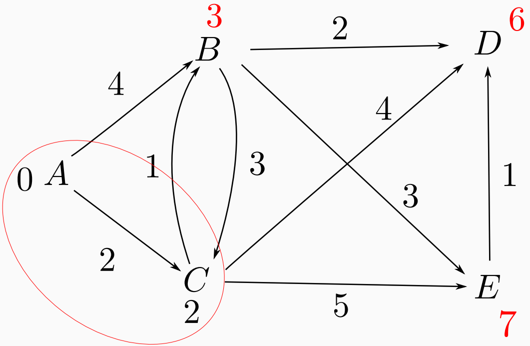
> "Greedy ... is good. Greedy is right. Greedy works."
>
> — Wall Street

Warm-up example: Dijkstra's algorithm (I)



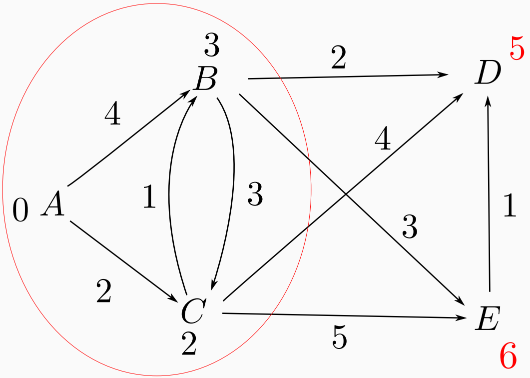
A	
B	4
C	2
D	
E	

Warm-up example: Dijkstra's algorithm (II)



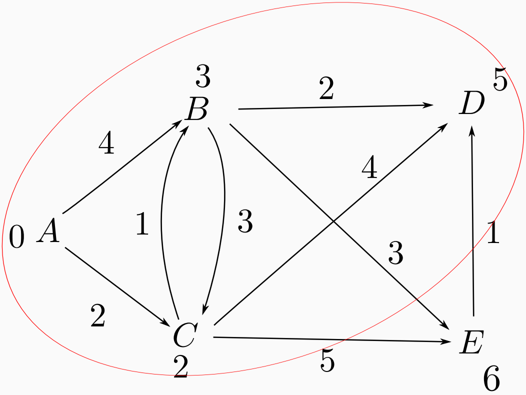
A C	
B	3
D	6
E	7

Warm-up example: Dijkstra's algorithm (III)



A C B	
D	5
E	6

Warm-up example: Dijkstra's algorithm (IV)



A C B D	
E	6

How to design a greedy algorithm?

Key step (the *greedy heuristic*)

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Question: do greedy algorithms always work?

i.e. do they always produce the **optimal solution**?

A failure example (I)

0-1 Knapsack Problem

A Thief has a backpack with certain capacity. There is a set of items with certain weight and value.

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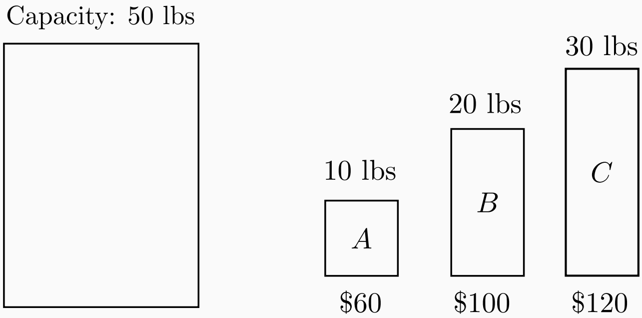

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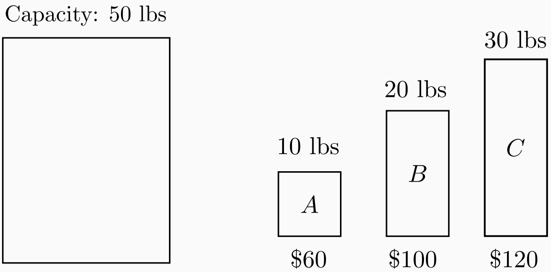

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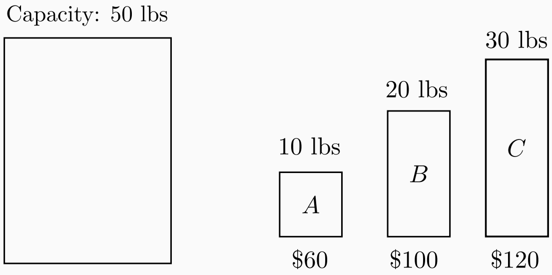

Greedy heuristic:

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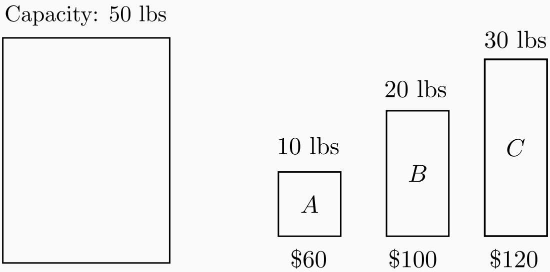

Greedy heuristic: always pick the item with the highest value/weight

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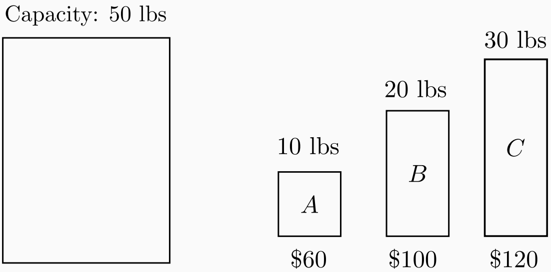

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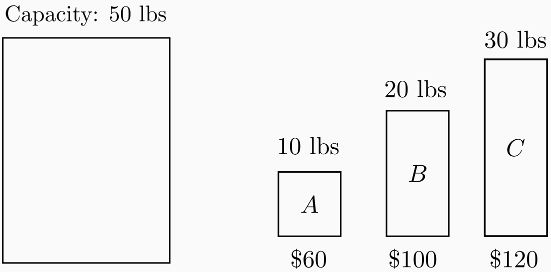

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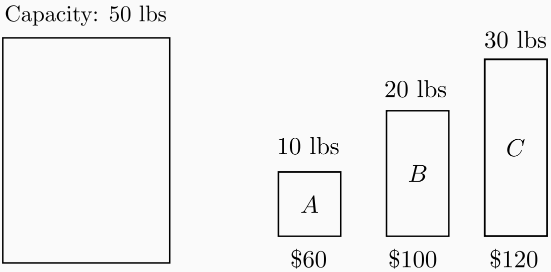

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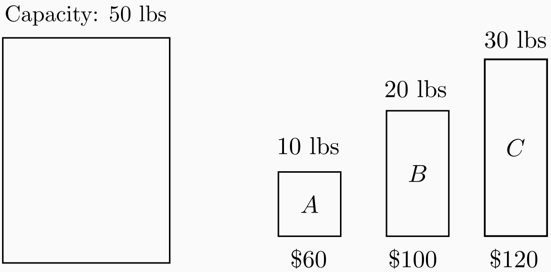

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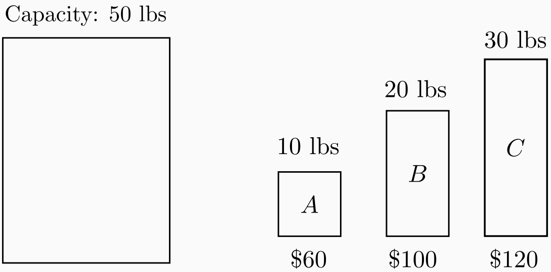

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There exists an optimal solution that makes the greedy choice

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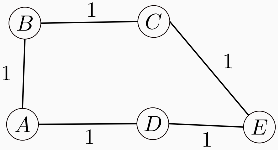

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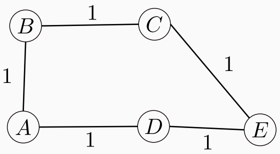

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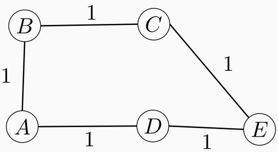

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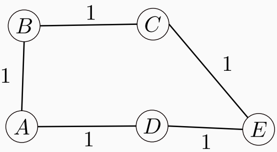

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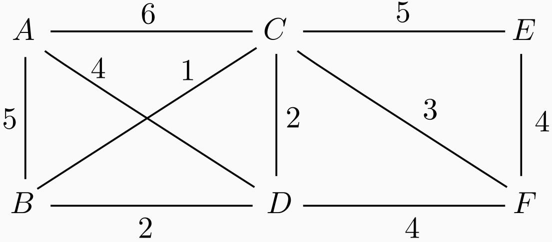
Greedy algorithms

Minimum Spanning Tree

Minimum Spanning Tree (MST)



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Input: undirected graph $G = (V, E)$ with edge weights $w_e \in \mathbb{R}, \forall e \in E$.

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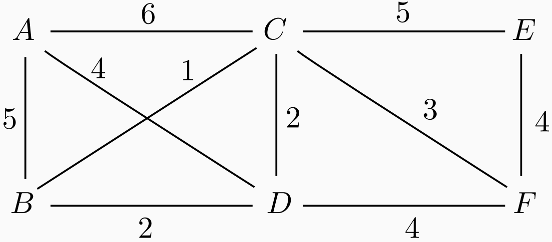

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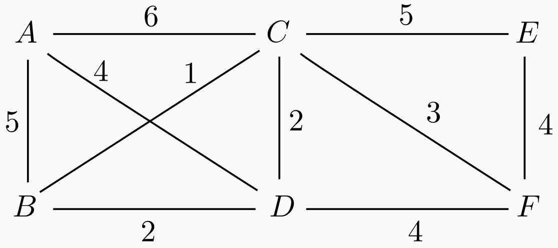

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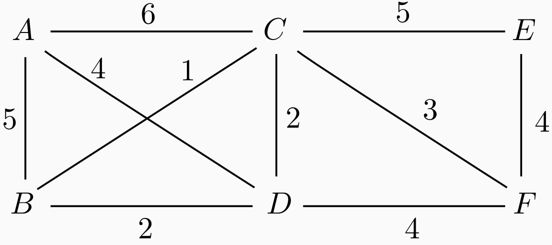

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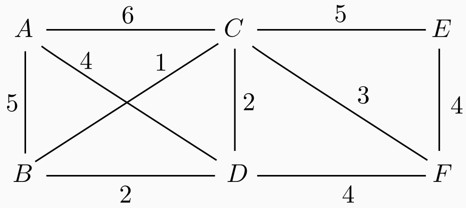

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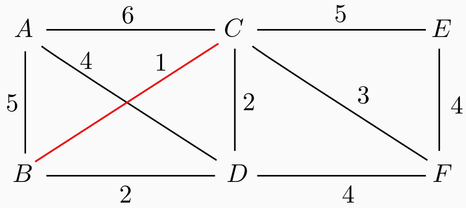

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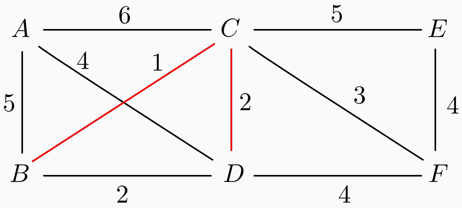

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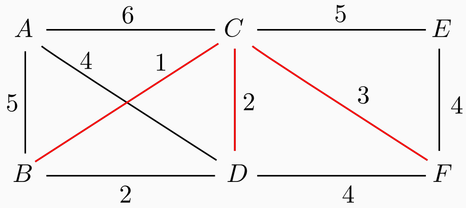

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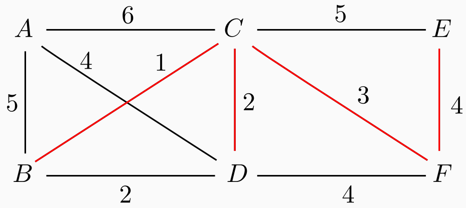

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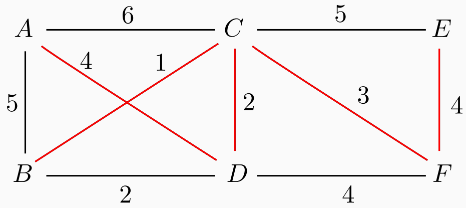

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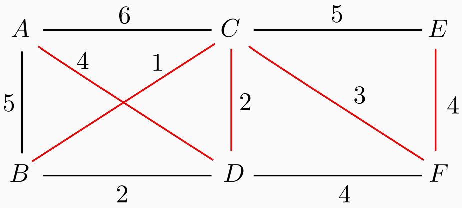

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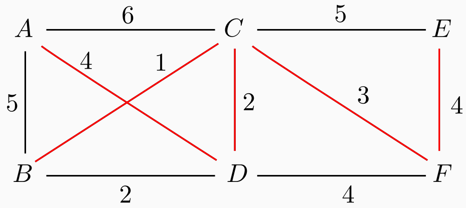

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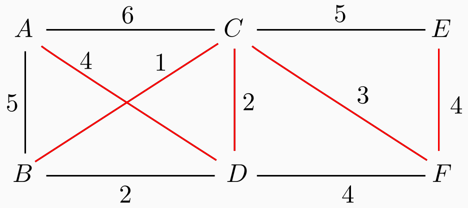

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Optimality?

We need to show greedy choice and optimal substructure

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Some technical preliminaries

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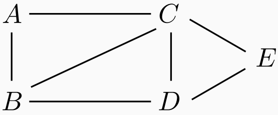

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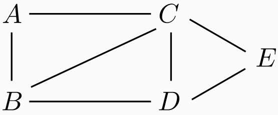

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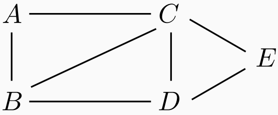

$(\{A, B, C\}, \{D, E\})$ is a cut

$(\{A, B\}, \{D, E\})$ is a not a cut

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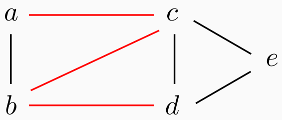

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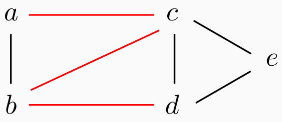

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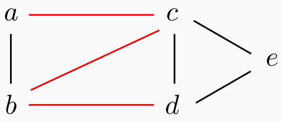

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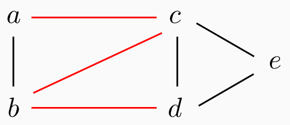

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How does this help?

- It shows the greedy choice property:
  there exists an optimal solution that makes the greedy choice
- It also demonstrates the optimal substructure property
  by applying this theorem multiple times until you can't