CMPSC 465
Data Structures and Algorithms
Spring 2022

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Dynamic Programming

Dynamic Programming

All-pair shortest path (Textbook Section 6.6)

All-pair shortest path

Consider $G = (V, E)$ weighted, directed graph without negative cycles

How to compute the $\mathrm{shortest\_path}(u, v)$?

Recall Bellman-Ford: $\mathrm{shortest\_path}(u, v)$ for fixed u, all v takes $O(|V| \cdot |E|)$ time

If for all u, v, APSP takes $O(|V|^2|E|)$ time
When $|E| = O(|V|^2)$, its running time becomes $O(|V|^4)$

Rethink this problem using DP.

Subproblem

WLOG, index the vertices as $V = \{1, 2, \ldots, n\}$

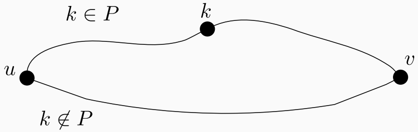
Subproblem: find the shortest path $u \to v$ using intermediate vertices from $\{1, \ldots, k\} \subseteq V$. Denote it by $\mathrm{sp}(u, v, k)$

Optimal solution: the entries $\mathrm{sp}(u, v, n)$ for all u, v

To find out the recurrence relation, we need to relate $\mathrm{sp}(u, v, k)$ to smaller subproblems $\mathrm{sp}(u, v, k - 1)$

Recurrence

Suppose $\mathrm{sp}(\mathrm{u}, \mathrm{v}, \mathrm{k}) = P$



- if $k \notin P$, then $\mathrm{sp}(u, v, k) = \mathrm{sp}(u, v, k-1)$
- if $k \in P$, then consider

$$P: \quad u \xrightarrow{P_1} k \xrightarrow{P_2} v$$

P_1, P_2 are paths whose intermediate vertices are from $\{1, \dots, k-1\}$. Because there's no negative cycles, there's no repeated vertices in shortest path

$$\text{Hence,} \quad P_1 = \mathrm{sp}(u, k, k-1), P_2 = \mathrm{sp}(k, v, k-1)$$

Using k is better if

$$|\mathrm{sp}(i, k, k-1)| + |\mathrm{sp}(k, v, k-1)| \leq |\mathrm{sp}(i, v, k-1)|$$

Dynamic programming

Let $\mathrm{dist}(u, v, k) = |\mathrm{sp}(u, v, k)|$

- **Recurrence:**

$$\mathrm{dist}(u, v, k) = \min\{\mathrm{dist}(u, v, k-1), \mathrm{dist}(u, k, k-1) + \mathrm{dist}(k, v, k-1)\}$$

- **Optimal solution:** $\mathrm{dist}(\cdot, \cdot, n)$
- **Base case:**

$$\mathrm{dist}(u, v, 0) = \begin{cases} w_{u,v} & \text{if } (u, v) \in E \\ \infty & \text{otherwise} \end{cases}$$

Pseudocode

The Floyd-Warshall algorithm:

```
def FLOYD_WARSHALL(G, w):
    for u = 1 ... n:
        for v = 1 ... n:
            dist(u, v, 0) = { w_{u,v}   if (u, v) ∈ E
                            { ∞         otherwise        ;
    for k = 1 ... n:
        for u = 1 ... n:
            for v = 1 ... n:
                dist(u, v, k) =
                  min{dist(u, v, k − 1), dist(u, k, k − 1) + dist(k, v, k − 1)};
    return dist(·, ·, n);
```

Running time: $O(n^3) = O(|V|^3)$