MATH 455: HOMEWORK 11

Problem 1. Find the first two iterations of both the Jacobi and the Gauss-Seidel methods for the following linear systems, using $\mathbf{X}^0 = \mathbf{0}$.

a.
$$\begin{cases} 3x_1 - x_2 + 2x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{cases}$$

b.
$$\begin{cases} 10x_1 - x_2 &= 9 \\ -x_1 + 10x_2 - 2x_3 &= 7 \\ -2x_2 + 10x_3 &= 6 \end{cases}$$

Problem 2. Find the first two iterations of the SOR method with $\omega = 1.1$, $\omega = 1.2$ and $\omega = 1.3$ for the following linear systems, using $\mathbf{X}^0 = \mathbf{0}$.

a.
$$\begin{cases} 3x_1 - x_2 + 2x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{cases}$$

b.
$$\begin{cases} 4x_1 + x_2 - x_3 &= 5 \\ -x_1 + 3x_2 + x_3 &= -4 \\ 2x_1 + 2x_2 + 5x_3 &= 1 \end{cases}$$

Problem 3. Consider the following linear system

$$\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

a. Does the Jacobi method converge for solving this linear system? If so, prove $\rho(T_J) < 1$.

b. Does the Gauss-Seidel method converge for solving this linear system? If so, prove $\rho(T_{GS}) < 1$.

Problem 4. Consider the following linear system

$$\begin{pmatrix} 1+\delta & -1 & 0 \\ -1 & 2+\delta & -1 \\ 0 & -1 & 1+\delta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

a. For any $\delta > 0$, does the Jacobi method converge for solving this linear system? If so, prove $\rho(T_J) < 1$.

b. For any $\delta > 0$, does the Gauss-Seidel method converge for solving this linear system? If so, prove $\rho(T_{GS}) < 1$.

Problem 5. Consider the following linear system

$$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

- Does the Gauss-Seidel method converge for solving this linear system? If so, prove $\rho(T_{GS}) < 1$.

Problem 1. Find the first two iterations of both the Jacobi and the Gauss-Seidel methods for the following linear systems, using $\mathbf{X}^0 = \mathbf{0}$.

a.

$$\begin{cases} 3x_1 - x_2 + 2x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{cases}$$

b.

$$\begin{cases} 10x_1 - x_2 &= 9 \\ -x_1 + 10x_2 - 2x_3 &= 7 \\ -2x_2 + 10x_3 &= 6 \end{cases}$$

Gauss-Seidel

$$\begin{bmatrix} 3 & -1 & 2 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$X^1_{n+1} = \frac{1}{3}\left(1 + X^2_n - 2X^3_n\right) \qquad x^1_1 = \frac{1}{3}(1+0-0) = \frac{1}{3}$$

$$X^2_{n+1} = \frac{1}{6}\left(0 - 3X^1_{n+1} - 2X^3_n\right) \qquad x^2_1 = \frac{1}{6}\left(0 - 3\cdot\frac{1}{3} - 2\cdot 0\right) = \frac{1}{6}\cdot -1 = -\frac{1}{6}$$

$$x^3_{n+1} = \frac{1}{7}\left(4 - 3x^1_{n+1} - 3x^2_{n+1}\right) \qquad x^3_1 = \frac{1}{7}\left(4 - 3\cdot\frac{1}{3} - 3\cdot\frac{1}{6}\right) = \frac{1}{7}\left(4 - 1 + \frac{1}{2}\right)$$

$$= \frac{1}{7}\left(3 + \frac{1}{2}\right) = \frac{7}{14} = \frac{1}{2}$$

$$X^1_2 = \frac{1}{3}\left(1 + -\frac{1}{6} - 2\cdot\frac{1}{2}\right) = \frac{1}{3}\left(-\frac{1}{6}\right) = -\frac{1}{18}$$

$$X^2_2 = \frac{1}{6}\left(0 + 3 \cdot -\frac{1}{18} - 2\cdot\frac{1}{2}\right) = \frac{1}{6}\left(-\frac{7}{6}\right)$$

$$= \frac{7}{36}$$

$$X^3_2 = \frac{1}{7}\left(4 - 3\cdot -\frac{1}{18} - 3\cdot\frac{7}{36}\right) = \frac{1}{7}\left(4 + \frac{1}{6} - \frac{7}{12}\right) = \frac{1}{7}\left(\frac{48+2-7}{12}\right) = \frac{1}{7}\cdot\frac{43}{12}$$

$$= \frac{43}{82}$$

Jacobi

$x_0 = 0$

$$x^1_{n+1} = \frac{1}{3}\left(1 + x^2_n - 2x^3_n\right)$$

$$x^2_{n+1} = \frac{1}{6}\left(0 - 3x^1_n - 2x^3_n\right)$$

$$x^3_{n+1} = \frac{1}{7}\left(4 - 3x^1_n - 3x^2_n\right)$$

$$x^1_1 = \frac{1}{3}\left(1 + 0 - 0\right) = \frac{1}{3}$$

$$x^2_1 = \frac{1}{6}\left(0 - 0 - 0\right) = 0$$

$$x^3_1 = \frac{1}{7}\left(4 - 0 - 0\right) = \frac{4}{7}$$

$$x^1_2 = \frac{1}{3}\left(1 + 0 - 2 \cdot \frac{4}{7}\right) = \frac{1}{3} \cdot \frac{(7+8)}{7} = \frac{15}{21}$$

$$x^2_2 = \frac{1}{6}\left(0 - 3 \cdot \frac{1}{3} - 2 \cdot \frac{4}{7}\right) = \frac{1}{6}\left(-\frac{7}{7} - \frac{8}{7}\right) = \frac{-15}{42}$$

$$x^3_2 = \frac{1}{7}\left(4 - 3 \cdot \frac{1}{3} - 3 \cdot 0\right) = \frac{1}{7}\left(4 - 1 - 0\right) = \frac{3}{7}$$

b.

$$\begin{cases} 10x_1 - x_2 = 9 \\ -x_1 + 10x_2 - 2x_3 = 7 \\ -2x_2 + 10x_3 = 6 \end{cases} \rightarrow \begin{bmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix}$$

Jacobi

$$x_1^{n+1} = \frac{1}{10}(9 + x_2^n)$$

$$x_2^{n+1} = \frac{1}{10}(7 + x_1^n + 2x_3^n)$$

$$x_3^{n+1} = \frac{1}{10}(6 + 2x_2^n)$$

$$x^0 = 0$$

$$x_1^1 = \frac{1}{10}(9) = \frac{9}{10}$$

$$x_2^1 = \frac{1}{10}(7 + 0 + 0) = \frac{7}{10}$$

$$x_3^1 = \frac{6}{10}$$

$$x_1^2 = \frac{1}{10}\left(9 + \frac{7}{10}\right) = \frac{97}{100}$$

$$x_2^2 = \frac{1}{10}\left(7 + \frac{9}{10} + 2\frac{6}{10}\right) = \frac{91}{100}$$

$$x_3^2 = \frac{1}{10}\left(6 + 2\cdot\frac{7}{10}\right) = \frac{74}{100}$$

b. G-S
$$\begin{cases} 10x_1 - x_2 &= 9 \\ -x_1 + 10x_2 - 2x_3 &= 7 \\ -2x_2 + 10x_3 &= 6 \end{cases} \rightarrow \begin{bmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix}$$

$$x_1^{n+1} = \frac{1}{10}\left(9 + x_2^{n}\right)$$
$$x_2^{n+1} = \frac{1}{10}\left(7 + x_1^{n+1} + 2x_3^{n}\right)$$
$$x_3^{n+1} = \frac{1}{10}\left(6 + 2x_2^{n+1}\right)$$

$$x_1^1 = \frac{9}{10}$$
$$x_2^1 = \frac{1}{10}\left(7 + \frac{9}{10} + 0\right) = \frac{79}{100}$$
$$x_3^1 = \frac{1}{10}\left(6 + 2\cdot\frac{79}{100}\right) = \frac{758}{1000}$$

$$x_1^2 = \frac{1}{10}\left(9 + \frac{79}{100}\right) = \frac{979}{1000}$$

$$x_2^2 = \frac{1}{10}\left(7 + \frac{979}{1000} + 2\cdot\frac{758}{1000}\right) = \frac{7000 + 979 + 1516}{10000} = \frac{9495}{10000}$$

$$x_3^2 = \frac{1}{10}\left(6 + 2\cdot\frac{9495}{10000}\right) = \frac{60000 + 18990}{100000} = \frac{78990}{100000}$$

Problem 2. Find the first two iterations of the SOR method with $\omega = 1.1$, $\omega = 1.2$ and $\omega = 1.3$ for the following linear systems, using $\mathbf{X}^0 = \mathbf{0}$.

a.
$$\begin{cases} 3x_1 - x_2 + 2x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{cases} \rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$x_1^{n+1} = (1-\omega)x_1^n + \frac{\omega}{3}(1 + x_2^n - 2x_3^n)$$
$$x_2^{n+1} = (1-\omega)x_1^n + \frac{\omega}{6}(0 - 3x_1^{n+1} - 2x_3^n) \qquad X^0 = 0$$
$$x_3^{n+1} = (1-\omega)x_1^n + \frac{\omega}{7}(4 - 3x_1^{n+1} - 3x_2^{n+1})$$

$\omega = 1.1$

$$x_1^1 = (1-1.1)\,0 + \frac{1.1}{3}(1+0-0)$$
$$= \frac{1.1}{3}$$
$$x_2^1 = (1-1.1)\,0 + \frac{1.1}{6}\left(0 - 3\,\frac{1.1}{3} \cdot 2.0\right)$$
$$= \frac{1.1}{6} \cdot 1.1$$
$$= \frac{-1.21}{6}$$
$$x_3^1 = (1-1.1)\,0 + \frac{1.1}{7}\left(4 - 3\,\frac{1.1}{3} - 3 \cdot \frac{-1.21}{6}\right)$$
$$= \frac{1.1}{7}\left(4 - 1.1 + \frac{1.21}{2}\right)$$
$$= \frac{1.1}{7}\left(\frac{8 - 2.2 + 1.21}{2}\right)$$
$$= \frac{1.1(7.01)}{14}$$
$$= \frac{7.711}{14}$$

$x_1^2 = -0.1479$

$x_2^2 = 0.1004$

$x_3^2 = 0.6905$

$\omega = 1.2$

$x_1^1 = \frac{1.2}{3}$

$x_2^1 = -0.4$

$x_3^1 = 0.6034$

$x_1^2 = -0.2587$

$x_2^2 = -0.0381$

$x_3^2 = 0.7137$

$\omega = 1.3$

$x_1^1 = 0.4333$

$x_2^1 = -0.2816$

$x_3^1 = 0.6583$

$x_1^2 = -0.1292$

$x_2^2 = -0.1168$

$x_3^2 = 0.6824$

b.

$$\begin{cases} 4x_1 + x_2 - x_3 &= 5 \\ -x_1 + 3x_2 + x_3 &= -4 \\ 2x_1 + 2x_2 + 5x_3 &= 1 \end{cases}$$

$$\begin{bmatrix} 4 & 1 & -1 \\ -1 & 3 & 1 \\ 2 & 2 & 5 \end{bmatrix} \quad \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \qquad X^0 = 0$$

$\omega = 1.1$

$x_1^1 = 1.375$

$x_2^1 = -0.9625$

$x_3^1 = 0.0385$

$x_1^2 = 1.5128$

$x_2^2 = -0.8299$

$x_3^2 = -0.843$

$\omega = 1.2$

$x_1^1 = \frac{3}{2}$

$x_2^1 = -1$

$x_3^1 = 0$

$x_1^2 = \frac{3}{2}$

$x_2^2 = -0.8$

$x_3^2 = -0.046$

$\omega = 1.3$

$x_1^1 = 1.625$

$x_2^1 = -1.029$

$x_3^1 = -0.049$

$x_1^2 = 1.456$

$x_2^2 = -0.772$

$x_3^2 = -0.0805$

Problem 3. Consider the following linear system

$$\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$\rightarrow \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a. Does the Jacobi method converge for solving this linear system? If so, prove $\rho(T_J) < 1$.

b. Does the Gauss-Seidel method converge for solving this linear system? If so, prove $\rho(T_{GS}) < 1$.

a. Jacobi

$T_J = -D^{-1}(L+U)$

$L = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad D^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

$T_J = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -\frac{2}{3} \\ -\frac{2}{3} & 0 \end{bmatrix}$

$|T - \lambda I| = 0$

$\begin{bmatrix} -\lambda & -\frac{2}{3} \\ -\frac{2}{3} & -\lambda \end{bmatrix} = 0$

$\lambda^2 - \frac{4}{9} = 0$

$\lambda = \sqrt{\frac{4}{9}}$

$\lambda = \pm \frac{2}{3}$

$\max \lambda = \frac{2}{3} < 1$

So Jacobi method will converge

b) G-S method

$$-(D+L)^{-1}U$$

$$(D+L)^{-1} = \begin{bmatrix} 3 & 0 \\ -2 & 3 \end{bmatrix}^{-1}$$

$$= \frac{1}{9}\begin{bmatrix} 3 & 0 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{2}{9} & \frac{1}{3} \end{bmatrix}$$

$$-(D+L)^{-1}U = -\begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{2}{9} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{2}{3} \\ 0 & \frac{4}{9} \end{bmatrix}$$

$|T_{GS} - \lambda|$ $\quad \lambda_1 = 0 \quad \lambda_2 = \frac{4}{9}$

$$\max(|\lambda_1|, |\lambda_2|) = \frac{4}{9} < 1$$

Converge

Problem 4. Consider the following linear system

$$\begin{pmatrix} 1+\delta & -1 & 0 \\ -1 & 2+\delta & -1 \\ 0 & -1 & 1+\delta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

a. For any $\delta > 0$, does the Jacobi method converge for solving this linear system? If so, prove $\rho(T_J) < 1$.

b. For any $\delta > 0$, does the Gauss-Seidel method converge for solving this linear system? If so, prove $\rho(T_{GS}) < 1$.

a)

$$|1+\delta| > |-1|$$

$$|2+\delta| > |-1| + |-1|$$

$$|1+\delta| > |-1|$$

$$T = D + R$$

$$D = \begin{pmatrix} 1+\delta & 0 & 0 \\ 0 & 2+\delta & 0 \\ 0 & 0 & 1+\delta \end{pmatrix} \qquad R = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$T_J = -D^{-1}R = \begin{pmatrix} \frac{1}{1+\delta} & 0 & 0 \\ 0 & \frac{1}{2+\delta} & 0 \\ 0 & 0 & \frac{1}{1+\delta} \end{pmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{1+\delta} & 0 \\ \frac{1}{2+\delta} & 0 & -\frac{1}{2+\delta} \\ 0 & -\frac{1}{1+\delta} & 0 \end{bmatrix}$$

$\because 2+\delta > 2$

$$\rho(T_J) = \frac{2}{2+\delta} < 1$$

b)

$$A_{GS} = -(D+L)^{-1} U$$

$$(D+L) = \begin{pmatrix} 1+\delta & 0 & 0 \\ 0 & 2+\delta & 0 \\ 0 & 0 & 1+\delta \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{GS} = \begin{bmatrix} 0 & \frac{1}{1+\delta} & 0 \\ 0 & \frac{1}{(1+\delta)(2+\delta)} & \frac{1}{(2+\delta)} \\ 0 & \frac{1}{(1+\delta)^2(2+\delta)} & \frac{1}{(1+\delta)(2+\delta)} \end{bmatrix}$$

$$\rho(A_{GS}) = \max\left(\frac{1}{1+\delta}, \frac{1}{2+\delta}, \frac{1}{(1+\delta)^2(2+\delta)}\right)$$

$$= \frac{1}{1+\delta} < 1$$

Problem 5. Consider the following linear system

$$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

- Does the Gauss-Seidel method converge for solving this linear system? If so, prove $\rho(T_{GS}) < 1$.

$$T_{(GS)} = -(D+L)^{-1} U$$

$$= -\begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{6} & \frac{2}{3} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$|\lambda - T_{GS}|$$

$$\rho(T_{GS}) = \max\{\lambda_1, \lambda_2, \lambda_3\}$$

$$= 0.7723 < 1$$