CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

NP and Computational Hardness

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P, NP, and NP-completeness (Kleinberg-Tardos, Section 8.3, 8.4)

The Cook-Levin Theorem

Theorem (Cook-Levin)

circuit-SAT is **NP***-complete*

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Proof sketch.

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To decide if $s \in$ X, we check if there exists a string t of length $p(|s|)$ s.t. $B(s, t) =$ yes

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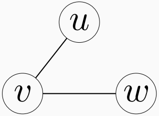
Ask if C_s is satisfiable. If yes, there exists such t so $s \in$ X.
If no, there's such t that $B(s, t) =$ yes. So $s \notin$ X $\square$

Example of such C_s

Decide if there's an IS of size 2



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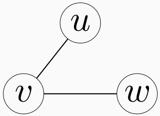

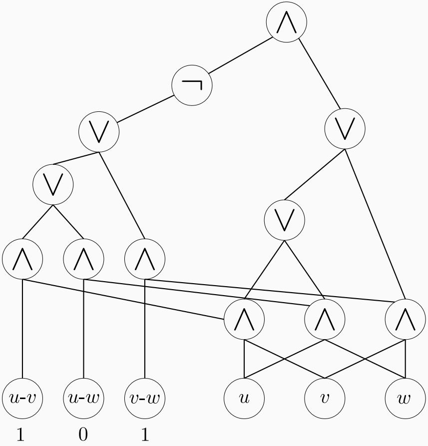

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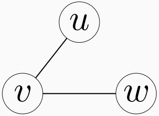

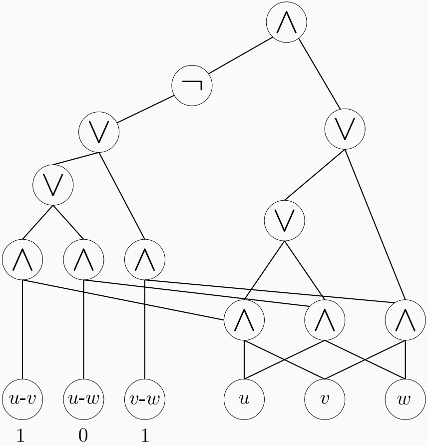

$\binom{n}{2}$ hardwired inputs (graph description)

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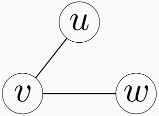

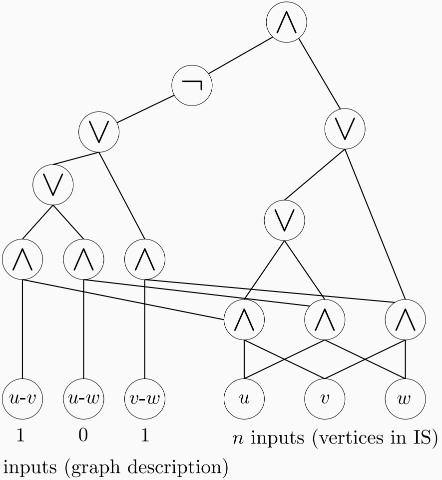

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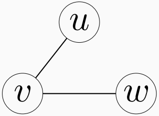

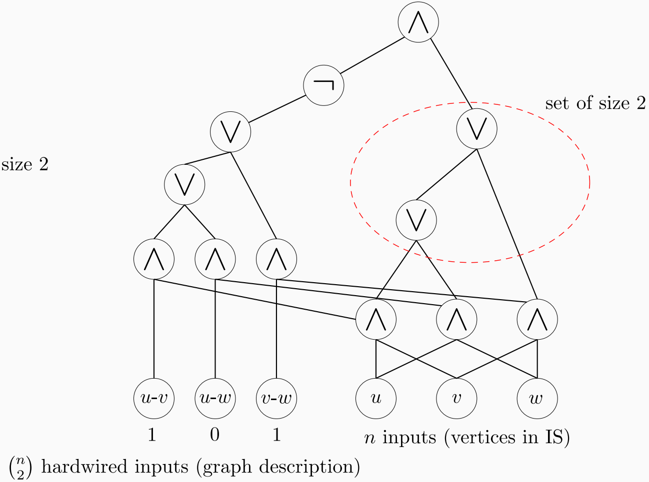

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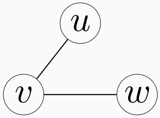

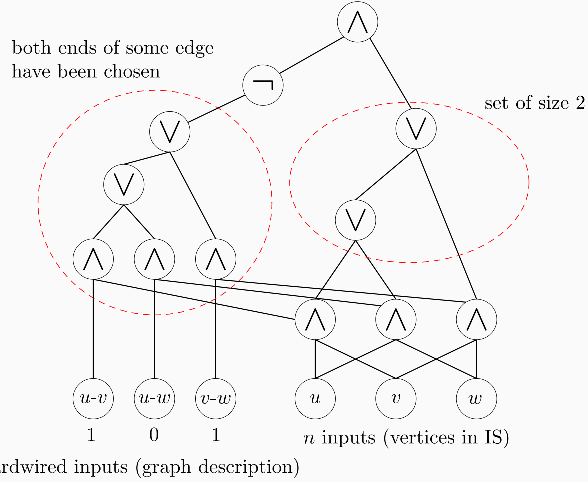

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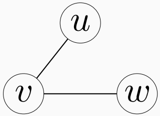

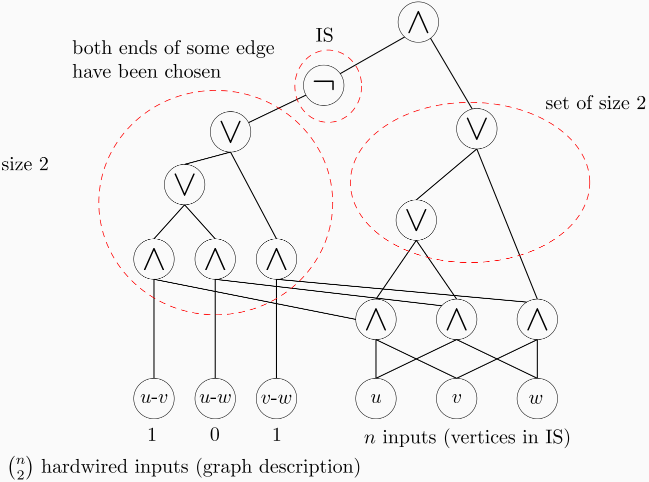

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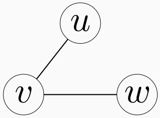

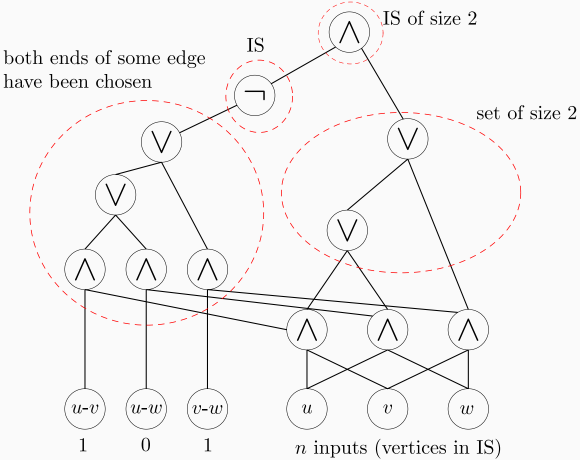

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Proving NP-completeness

Recipe for proving Y is **NP**-complete

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Step 1: Prove $Y \in \mathbf{NP}$

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Observation

If X *is* **NP***-complete,* $\mathrm{Y} \in \mathbf{NP}$*, and* $\mathrm{X} \leq_P \mathrm{Y}$*, then* Y *is* **NP***-complete*

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Observation

If X *is* **NP***-complete,* Y $\in$ **NP***, and* X $\leq_P$ Y*, then* Y *is* **NP***-complete*

Proof.

Let W be any problem in **NP**. Then W $\leq_P$ X $\leq_P$ Y implies that W $\leq_P$ Y. Therefore, Y is **NP**-complete $\square$

3-SAT is NP-complete

Theorem

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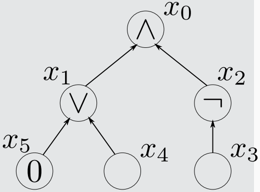

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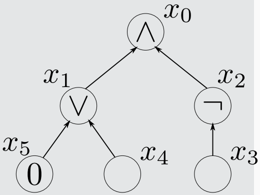

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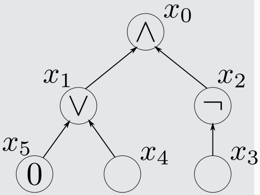

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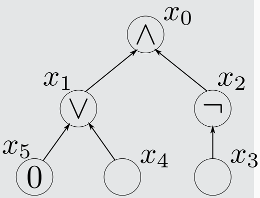

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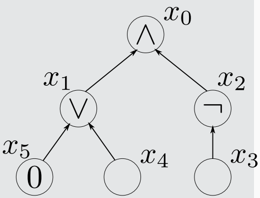

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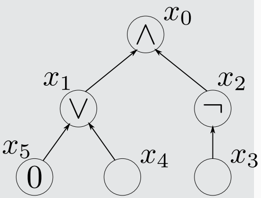

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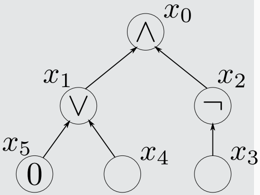

Turn clauses of length < 3 into clauses of length exactly 3 $\square$

Other NP-complete problems

From last lecture:

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Want to learn more about this topic? Take CMPSC 464