Bellman-Ford Algorithm

Bellman-Ford algorithm can be used to solve the (single-source) shortest path problem with negative edge length, and its extension can also be used to detect if a graph contains negative cycle (reachable from the given source).

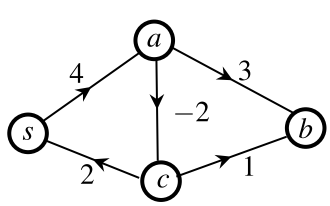
Bellman-Ford algorithm is quite simple. It only maintain an array, *dist* of size $|V|$, as its data structure. And it just does a bunch of "update" operations. An "update" function takes an edge $e = (u,v)$ as input, and updates $dist[v]$ as $dist[u] + l(u,v)$ if the former is larger than the latter.

procedure update(edge $(u,v) \in E$)
    if ($dist[v] > dist[u] + l(u,v)$)
        $dist[v] = dist[u] + l(u,v)$;
    end if;
end procedure;

Bellman-Ford algorithm iterates $(|V| - 1)$ rounds, and in each round, updates *all* edges, in an arbitrary order. If the given G does contain negative cycle reachable from given s, when the algorithm terminates, we will have that $dist[v] = distance(s,v)$ for every $v \in V$.

Algorithm Bellman-Ford ($G = (V,E)$, $l(e)$ for any $e \in E$, $s \in V$)
    init an array *dist* of size $|V|$;
    $dist[s] = 0$; $dist[v] = \infty$ for any $v \neq s$;
    for $k = 1 \to |V| - 1$
        for each edge $(u,v) \in E$
            $update(u,v)$;
        end for;
    end for;
end algorithm;

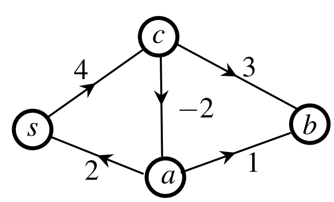
Since *update* function takes constant time, clearly, Bellman-Ford algorithm runs in $\Theta(|V| \cdot |E|)$ time. See examples below.



graph

	s	a	b	c
init	0	∞	∞	∞
$R1$	0	4	∞	∞
$R2$	0	4	3	2
$R3$	0	4	3	2

dist array



graph

	s	a	b	c
init	0	∞	∞	∞
$R1$	0	∞	∞	4
$R2$	0	2	7	4
$R3$	0	2	3	4

dist array

Figure 1: The dist array (after each round) running Bellman-Ford algorithm on each example. In each example, in each round, we choose to update all edges in lexicographic order, i.e., $(a,b),(a,c),(c,b),(c,s),(s,a)$.

Now let's see why this algorithm is correct. We first show an invariant about the data structure *dist* array:

Fact 1. Throughout the algorithm, if $dist[v] \neq \infty$ then $dist[v]$ represents the length of some path from s to v. In other words, $dist[v] \geq distance(s,v)$ throughout the algorithm, as $dist[v]$ represents the length of *some* path from s to v, while $distance(s,v)$ represents the length of the *shortest* path from s to v.

Clearly, in the initialization step which sets $dist[s] = 0$ and $dist[v] = \infty$ for all $v \neq s$, above claim holds, as $dist[s]$ stores a path from s to s without any edge and therefore its length is 0. Now to show above fact is correct throughout the algorithm, we just need to show that the "update" operation keeps this invariant (as this algorithm does nothing else but "update" operations).

Fact 2. The update operation keeps the invariant that $dist[v]$ represents the length of some path from s to v when $dist[v] \neq \infty$, i.e., $dist[v] \geq distance(s,v)$, for every $v \in V$.

Proof. We prove this by induction w.r.t. the sequence of update operations. Assume that up to the n-th update opertion above claim holds, i.e., $dist[v]$ stores the length of some path from s to v when $dist[v] \neq \infty$. Now consider the $(n+1)$-th update operation on edge $e = (u,v)$. Assume that $dist[v] > dist[u] + l(u,v)$, as otherwise this operation does not change $dist$ and the claim continues to be true. Now $dist[v]$ is updated as $dist[u] + l(u,v)$. Since, according to the inductive assumption, $dist[u]$ stores the length of some path from s to u, we have that $dist[v]$ stores the length of the path that consists of the aforementioned path from s to u followed by edge (u,v). □

In Bellman-Ford algorithm, $dist[v]$ starts from a trivial upper bound (i.e., infinity) of $distance(s,v)$, and will get closer and closer to $distance(s,v)$ through the "update" procedures, and eventually reach $distance(s,v)$. We now state the conditions for this to happen.

Fact 3. If edge (u,v) is the last edge on one shortest path from s to v and $dist[u] = distance(s,u)$, then after $update(u,v)$ we will have $dist[v] = distance(s,v)$.

Since edge (u,v) is the last edge on one shortest path from s to v, according to Property 3 of Lecture A14, we know that $distance(s,v) = distance(s,u) + l(u,v) = dist[u] + l(u,v)$. Through $update(u,v)$, $dist[v]$ will be compared with $dist[u] + l(u,v) = distance(s,v)$. The first case will be that $dist[v] \leq dist[u] + l(u,v) = distance(s,v)$. Notice that in this case we must have $dist[v] = distance(s,v)$ according to above fact, i.e., $dist[v]$ already stores the distance. The second case will be that $dist[v] > dist[u] + l(u,v) = distance(s,v)$, and in this case the "update" function will set $dist[v] = dist[u] + l(u,v) = distance(s,v)$. Hence, in either case, we will have $dist[v] = distance(s,v)$ after updating edge (u,v). □

Suppose that $s \to v_1 \to v_2 \to \cdots \to v$ is one shortest path from s to v. In the initialization step we have $dist[s] = distance(s,s) = 0$. If at a later time, $update(s,v_1)$ is executed, then following above Fact 3, we know that $dist[v_1] = distance(s,v_1)$ after this update (reasons: $dist[s] = distance(s,s)$, and (s,v_1) is the last edge on one shortest path from s to v_1 according to the optimal substrcture property). Once $dist[v_1]$ becomes $distance(s,v_1)$, $dist[v_1]$ will stay as $distance(s,v_1)$ according to Fact 1. If at a later time $update(v_1,v_2)$ happens then following Fact 3, we know that $dist[v_2] = distance(s,v_2)$. Note that it doesn't matter if additional updates happen between $update(s,v_1)$ and $update(v_1,v_2)$. We can continue this argument; a general form is summarized below.

Fact 4. If there exists a sequence of udpate procedures (not necessarily consecutive) that update all the edges following one shortest path from s to v, then after that we will have $dist[v] = distance(s,v)$. Again, there can be other "update"(s) between any two "update"s in this sequence.

But we don't know the the shortest path in advance. That's fine. As the number of edges in the shortest path will not exceed $(|V|-1)$, the Bellman-Ford algorithm simply update *all* edges in each round, and do this $(|V|-1)$ times. This therefore guarantees that the i-th edge on the shortest path can be updated during the *i*-th round. Consequently, this guarantees the existence of a sequence of update procedures that update all edges following the shortest path. This analysis leads to the following conclusion, which actually proves the correctness of Bellman-Ford algorithm. See an illustration in Figure 2.

Fact 5. If G does not contain negative cycle, then we have $dist[v] = distance(s,v)$ for $v \in V$ after $|V|-1$ rounds. In particular, let p be one shortest path from s to v with k edge. Then after k rounds of the Bellman-Ford algorithm, $dist[v] = distance(s,v)$.

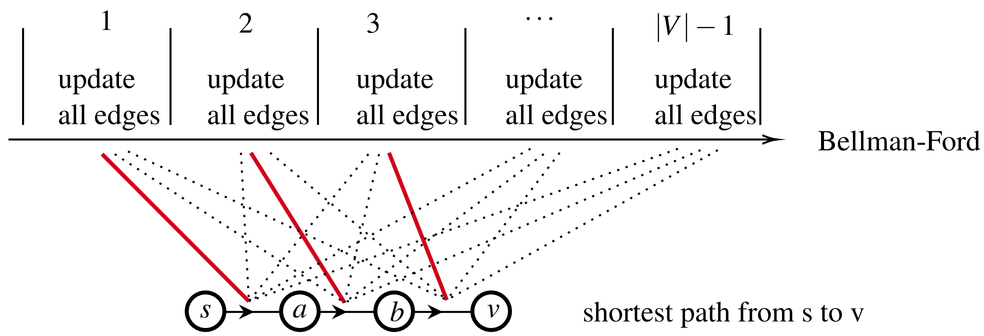


Figure 2: Illustration of the correctness of the Bellman-Ford algorithm. Dotted lines represent additional updates on the corresponding edge.

Detecting Negative Cycles

We can slightly modify Bellman-Ford algorithm to detect if a given graph contains negative cycle that is reachable from s. The algorithm does one more round of updates, in which it determines if some *dist* value can be further reduced.

Algorithm Bellman-Ford-Detect-Negative-Cycle ($G = (V,E)$, $l(e)$ for any $e \in E$, $s \in V$)
  init an array *dist* of size $|V|$;
  $dist[s] = 0$; $dist[v] = \infty$ for any $v \neq s$;
  for $k = 1 \to |V|-1$
    for each edge $(u,v) \in E$
      $update(u,v)$;
    end for;
  end for;
  for each edge $(u,v) \in E$
    if ($dist[v] > dist[u] + l(u,v)$): report G contains negative cycle and exit
  end for;
  report that G does not contain negative cycle
end algorithm;

Let's show that above algorithm is correct. We first prove that, if G does not contain negative cycle (reachable from s), then in above additional round $dist[v] > dist[u] + l(u,v)$ will never happen, i.e., we will get the report that "G does not contain negative cycle". As per Fact 5 and the assumption that G does not contain negative cycle, we know that $dist[v] = distance(s,v)$ after $|V|-1$ rounds. Also, according to Fact 2, update function will never make $dist[v]$ smaller than $distance(s,v)$ when G does not contain negative cycle. Hence, during the $|V|$-th round in above algorithm, none of the *dist* value can be further reduced.

We then prove that, if G contains negaive cycle (reachable from s), then in above additional round, there must exist an edge (u,v) such that $dist[v] > dist[u] + l(u,v)$. Suppose conversely that, in above additional round, all edges satisfy $dist[v] \leq dist[u] + l(u,v)$. Let $C = v_1 \to v_2 \to \cdots \to v_{k-1} \to v_k \to v_1$ be one negative reachable from s. We have $\sum_{e \in C} l(e) < 0$ as C is a nagative cycle. Applying $dist[v] \leq dist[u] + l(u,v)$ to all edges in C gives:

$$\begin{aligned}
dist[v_2] &\leq dist[v_1] + l(v_1, v_2) \\
dist[v_3] &\leq dist[v_2] + l(v_2, v_3) \\
&\cdots \\
dist[v_k] &\leq dist[v_{k-1}] + l(v_{k-1}, v_k) \\
dist[v_1] &\leq dist[v_k] + l(v_k, v_1)
\end{aligned}$$

Summing up both sides of all above inequalities gives $\sum_{e \in C} l(e) \geq 0$, a contradiction.