CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Dynamic Programming

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Prelude

Dynamic programming vs. Greedy algorithms

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Sometimes, the greedy choice won't work — we need to check many subproblems to find the optimal solution $\rightarrow$ **Dynamic programming**

General steps for Dynamic Programming

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- Use information from smaller subproblems to solve a larger subproblem

Warm-up: Longest increasing subsequence

Problem (Longest increasing subsequence)

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Example:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
5	2	8	6	3	6	9	7

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$i_1 = 2, i_2 = 5, i_3 = 6, i_4 = 7$

Solving LIS using DAG

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$$
\begin{aligned}
a_8 &= 7\\
a_7 &= 9\\
a_6 &= 6\\
a_5 &= 3\\
a_4 &= 6\\
a_3 &= 8\\
a_2 &= 2\\
a_1 &= 5
\end{aligned}
$$

Solving LIS using DAG

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- There is a link $i \to j$ if $a_i < a_j$

$a_8 = 7$
$a_7 = 9$
$a_6 = 6$
$a_5 = 3$
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$a_3 = 8$
$a_2 = 2$
$a_1 = 5$

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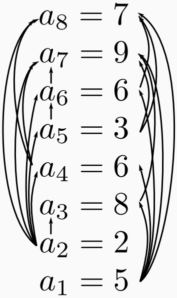

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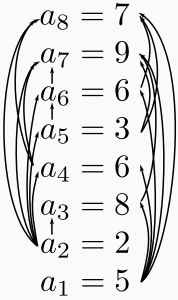

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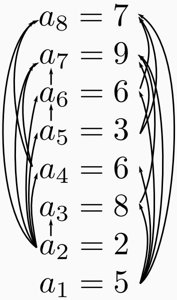
Use $L(j)$ to denote the length of the longest path (longest increasing subsequence) ending with a_j



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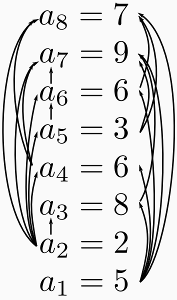

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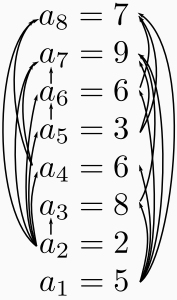

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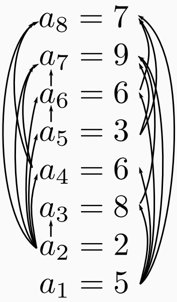

def $\mathrm{LIS\_DAG}$(*GAG* $G = (V, E)$ *for* $a_1, \ldots, a_n$):

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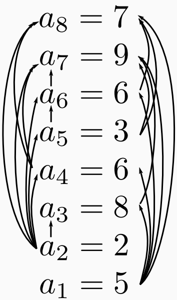

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Running example

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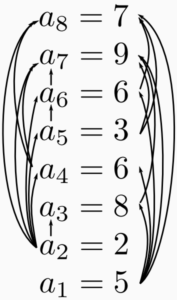

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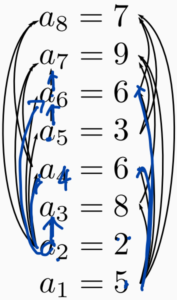

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$1 + L(6)$

$1+L(2)$ $1+L(1)$ $1+L(2)$ $1+L(3)$ $1+L(6)$

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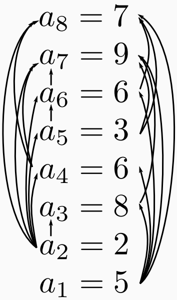

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$\Theta(n^2)$

Running time: $O(n^2)$

Costs more than greedy: need to check more subproblems

The actual subsequence

The above dynamic programming algorithm only computes the length of the longest increasing subsequence, but how to find the subsequence?

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  for $j = 1, \ldots, n$**:**

    $L(j) = 1$, $\boxed{\mathrm{prev}(j)} = \cdot$;

    for $i = 1, \ldots, j$**:**

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        $\boxed{L(j) = L(i) + 1}$, $\mathrm{prev}(j) = i$;

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$L(1)+1$ $L(1)+1$ $L(2)+1$ $L(5)+1$

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prev	•	•	1	1	2	5	6	6

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```
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    for i = 1, ..., j:
        if L(i) > L(j):
            L(j) = L(i) + 1, prev(j) = i;
return max_j L(j);
```

$j = 3$

$L(3) = \cancel{1}$

$L(3) = 1 + 1 = 2$

$pre(3) = 1$

$L(2)+1$

a_i	5	2	8	6	3	6	9	7
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   e.g. $L(j) = 1 + \max\{L(i) : a_i < a_j\}$
3. Base case