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Math 455, Sample Exam II
November 1, 2021

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 50 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 10 pages of the test.

Please do NOT write in this box.

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Total	________

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Partial Credit

You must show your work on the partial credit problems to receive credit!

1.(10 pts.)

(a) Find the multiplicity of the root $r = 0$ of $f(x) = x^2 \sin(x)$

$f(0) = 0^2 \sin(0)$
$= 0$

$f'(x) = 2x \sin x + x^2 \cos(x)$

$f'(0) = 0 + 0$

$f''(x) = 2\sin x + 2x\cos x + 2x\cos x - x^2 \sin(x)$
$= 0 + 0 + 0 + 0$

$f''(0) = 0$

$f'''(x) = 2\cos x + 2\cos x - 2x\sin x$

$\neq 0$

$m = 3$

(b) Find the forward error and backward error of the approximation root $c = 0.666$ for the function $f(x) = (3x - 2)^3$.

root $= 3x = 2$
$r = \frac{2}{3}$

For $|r - c| = 0.000666\cdots = 6.6 \times 10^{-4}$

back $= |f(c)| = |(3 \cdot 0.666 - 2)^3|$
$= (0.002)^3$
$= 8 \times 10^{-3}$ ^{3}
$= 8 \times 10^{-9}$

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2.(20 pts.) Solve

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = x^2 - \frac{2}{3}x\cos(x) + \frac{1}{9} - \frac{\sin^2(x)}{9} = 0, \text{ with } x_0 = \frac{\pi}{2}.$$

(a) Write an algorithm to solve $f(x) = 0$ by using Newton's method.

$$f(x) = x^2 - \frac{2}{3}x\cos x + \frac{1-\sin^2 x}{9} = 0$$

$$x^2 - \frac{2}{3}x\cos x + \frac{\cos^2 x}{9} = 0$$

$$(x - \frac{1}{3}\cos x)^2 = 0$$

$$x_0 = \frac{\pi}{2}$$

$$f'(x) = 2(x - \frac{1}{3}\cos x)(1 + \frac{1}{3}\sin x)$$

$$\begin{cases} x_{n+1} = x_n - \dfrac{(x_n - \frac{1}{3}\cos(x_n))^2}{2(x_n - \frac{1}{3}\cos x_n)(1+\frac{1}{3}\sin x_n)} \\ x_0 = \dfrac{\pi}{2} \end{cases}$$

(b) Does Newton's Method converge quadratically to the root $r = r_1 \in [0, \frac{\pi}{2}]$? If not, explain why?

if $f(r_1) = 0$

$$r_1 - \frac{1}{3}\cos r_1 = 0$$

$$f'(r_1) = 0$$

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(c) Find the multiplicity of the root $r = r_1$ of $f(x)$.

$f(x) = (x - \frac{1}{3}\cos x)^2 = 0$

$f'(x) = 2(x - \frac{1}{3}\cos x)(1 + \frac{1}{3}\sin x) = 0$

$f''(x) = 2(x - \frac{1}{3}\cos x)\frac{1}{3}\cos x) + 2(1 + \frac{1}{3}\sin x)^2$

$= 0 + 2(1+\frac{1}{3}\sin x)^2 > 0$

$m = 2$

(d) Write out the Modified Newton's Method such that we have quadratical convergence.

$$x_{n+1} = x_n + \frac{m f(x_n)}{f'(x_n)}$$

$$= x_n + \frac{2 \cdot (x - \frac{1}{3}\cos x)^2}{2(x - \frac{1}{3}\cos x)(1 + \frac{1}{3}\sin x)}$$

$$\begin{cases} x_{n+1} = x_n + \dfrac{x - \frac{1}{3}\cos x}{1 + \frac{1}{3}\sin x} \\ x_0 = \dfrac{\pi}{2} \end{cases}$$

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3.(15 pts.) Consider the linear system

$$\begin{pmatrix} -1 & 2 \\ -2 & 4.01 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6.01 \end{pmatrix}.$$

(a) Find the (l^1-norm) of the relative forward and backward errors and the error magnification factor for the approximate solution $\begin{pmatrix} 599 \\ 301 \end{pmatrix}$.

602

$(1+2)$

$-599+1201.2$

$x_1 = -1 \quad x_2 = 1$

$$\frac{\left\| \begin{pmatrix} 599 \\ 301 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\|_1}{\left\| \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\|_1} = \frac{\left\| \begin{pmatrix} 600 \\ 300 \end{pmatrix} \right\|_1}{\left\| \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\|_1} = \frac{900}{2} = 450$$

$$\begin{pmatrix} -1 & 2 \\ -2 & 4.01 \end{pmatrix} \cdot \begin{pmatrix} 599 \\ 301 \end{pmatrix} = \begin{pmatrix} 3 \\ 9.01 \end{pmatrix}$$

$-1198 + 1207.01$

$$\frac{\| \vec{b} - A\vec{x} \|_1}{\| \vec{b} \|}$$

$$= \frac{\left\| \begin{pmatrix} 3 \\ 6.01 \end{pmatrix} - \begin{pmatrix} 3 \\ 9.01 \end{pmatrix} \right\|_1}{9.01}$$

$$= \frac{\left\| \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right\|}{9.01}$$

$$= \frac{3}{9.01}$$

$$\frac{450}{\frac{3}{9.01}} = 150 \cdot 9.01 = 1351.5$$

$=$

(b) What is the (l^1-norm) condition number of the coefficient matrix?

$\mathrm{Cond}_1(A) = \|A\|_1 \|A^{-1}\|_1$

$$A = \begin{vmatrix} -1 & 2 \\ -2 & 4.01 \end{vmatrix}$$

$$A^{-1} = \frac{1}{-4.01+4} \begin{vmatrix} 4.01 & -2 \\ 2 & -1 \end{vmatrix}$$

$$= \frac{-1}{0.01}$$

$$= \begin{vmatrix} -401 & 200 \\ -200 & 100 \end{vmatrix}$$

$\|A^{-1}\|_1 = 601$

$\mathrm{Cond}_1 A = 6.01 \cdot 601$

$= \quad 6.01$

3606

3612.01

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4.(10pts.) Consider the following linear system

$$\begin{cases} 2x_1 - x_2 + 2x_3 - x_4 &= 2 \\ 2x_1 - 4x_2 + 5x_3 - 3x_4 &= -1 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + x_2 + 3x_4 &= 10 \end{cases}$$

(a) Use Gaussian Elimination with partial pivoting to find its solution.

$$\left[\begin{array}{cccc|c} 2 & -1 & 2 & -1 & 2 \\ 2 & -4 & 5 & -3 & -1 \\ 2 & 2 & 1 & 0 & -1 \\ 2 & 1 & 0 & 3 & 10 \end{array}\right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1}} \left[\begin{array}{cccc|c} 2 & -1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -2 & -3 \\ 0 & 3 & -1 & 1 & -3 \\ & 2 & -2 & 4 & 8 \end{array}\right]$$

$$\xrightarrow{\substack{R_3 + R_2 \\ R_4 + \frac{2}{3}R_2}} \left[\begin{array}{cccc|c} 2 & -1 & 2 & -1 & 2 \\ 0 & -3 & 3 & -2 & -3 \\ 0 & 0 & 2 & -1 & -6 \\ 0 & 0 & 0 & \frac{8}{3} & 6 \end{array}\right] \rightarrow \begin{cases} 2x_1 - x_2 + 2x_3 - x_4 = 2 \\ -3x_2 + 3x_3 - 2x_4 = -3 \\ 2x_3 - x_4 = -6 \\ \frac{8}{3}x_4 = 6 \end{cases}$$

$$x_4 = \frac{18}{8} = \frac{9}{4}$$

$$2x_3 - \frac{9}{4} = -6$$

$$8x_3 - 9 = -24$$

$$8x_3 = -15$$

$$x_3 - \frac{15}{8}$$

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5.(25pts.)

Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 4 & -3 & 5 \\ 2 & -2 & 3 \end{pmatrix}$$

(a)Perform the LU factorization for A and verify your answer.

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & -3 & 5 \\ 2 & -2 & 3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & -1 & 5 \\ 0 & -1 & 3 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 4 & -3 & 5 \\ 2 & -2 & 3 \end{bmatrix}$$

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$-1+\frac{3}{2}\,3$

(b)Perform the PA=LU factorization for A and verify your answer.

$$A=\begin{pmatrix}2&-1&0\\4&-3&5\\2&-2&3\end{pmatrix}\xrightarrow[R_2\to R_1)]{P\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}}\begin{bmatrix}4&-3&5\\2&-1&0\\2&-2&3\end{bmatrix}\xrightarrow[R_2-\frac{1}{2}R_1]{R_3-\frac{1}{2}R_1}\begin{bmatrix}4&-3&5\\\frac{1}{2}&\frac{1}{2}&-\frac{5}{2}\\\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\end{bmatrix}\xrightarrow{R_3+R_2}\begin{bmatrix}4&-3&5\\\frac{1}{2}&\frac{1}{2}&-\frac{5}{2}\\\frac{1}{2}&-1&-2\end{bmatrix}$$

P A L U

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&-1&0\\4&-3&5\\2&-2&3\end{bmatrix}=\begin{bmatrix}1&0&0\\\frac{1}{2}&1&0\\\frac{1}{2}&-1&1\end{bmatrix}\begin{bmatrix}4&-3&5\\0&\frac{1}{2}&-\frac{5}{2}\\0&0&2\end{bmatrix}$$

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6.(10pts.) Consider the following linear system

$$\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

a. Does the Jacobi method converge for solving this linear system? If so, prove $\rho(T_J) < 1$.

b. Does the Gauss-Seidel method converge for solving this linear system? If so, prove $\rho(T_{GS}) < 1$.

$T_J = -D^{-1}(L+U)$

$T_{GS} = -(D+L)^{-1}U$

$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ $L = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ $U = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$

$D^{-1} = \frac{1}{6}\begin{bmatrix} 2 & \\ & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

T_{GS}

$(D+L)^{-1} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{2} \end{bmatrix}$

$T_J = -\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -\frac{2}{3} \\ -\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{2} & 0 \end{bmatrix}$

$T_{GS} = -\begin{bmatrix} & \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$

$\|T_{GS}\|_\infty = \frac{2}{3}$ converge.

$\|T_J\|_\infty = \frac{2}{3} < 1$

$\rho(T_J) < \|T_J\|_\infty = \frac{2}{3} < 1$

$\rho(T_{GS}) < \|T_{GS}\| = \frac{2}{3} < 1$

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7.(10pts.)

Consider the following linear system

$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

.

- Does the Gauss-Seidel method converge for solving this linear system? If so, prove $\rho(T_{GS}) < 1$.

$A^T = A$

$\det(A_{11}) > 0$