Running example

$x = ACGTA$ and $y = ATCTG$

Running example

$x = ACGTA$ and $y = ATCTG$

	A	T	C	T	G
A					
C					
G					
T					
A					

Running example

$x = ACGTA$ and $y = ATCTG$

		A	T	C	T	G
	0	1	2	3	4	5
A	1	0	1	2	3	4
C	2	1	1	1	2	3
G	3	2	2	2	2	2
T	4	3	2	3	2	3
A	5	4	3	3	3	3

A T C T G
A C G T A

cost:3

A T C - T G
A - C G T A

cost=3

Pseudocode

def EDIT_DISTANCE(x, y):

Pseudocode

def $\textsc{Edit\_Distance}(x, y)$**:**

  for $i = 0, \ldots, m$**:**

Pseudocode

```
def EDIT_DISTANCE(x, y):
    for i = 0, ..., m:
        E(i, 0) = i;
```

Pseudocode

def $\textsc{Edit\_Distance}(x, y)$**:**

```
for i = 0, ..., m:
    E(i, 0) = i;
for j = 0, ..., n:
```

def $\textsc{Edit\_Distance}(x, y)$**:**

    for $i = 0, \ldots, m$**:**

        $E(i, 0) = i$;

    for $j = 0, \ldots, n$**:**

        $E(0, j) = j$;

Pseudocode

def $\textsc{Edit\_Distance}(x, y)$**:**

```
for i = 0, ..., m:
    E(i, 0) = i;
for j = 0, ..., n:
    E(0, j) = j;
for i = 1, ..., m:
```

Pseudocode

```
def EDIT_DISTANCE(x, y):
    for i = 0, ..., m:
        E(i, 0) = i;
    for j = 0, ..., n:
        E(0, j) = j;
    for i = 1, ..., m:
        for j = 1, ..., n:
```

Pseudocode

```
def EDIT_DISTANCE(x, y):
    for i = 0, ..., m:
        E(i, 0) = i;
    for j = 0, ..., n:
        E(0, j) = j;
    for i = 1, ..., m:
        for j = 1, ..., n:
            E(i, j) =
              min{1 + E(i−1, j), 1 + E(i, j−1), diff(i, j) + E(i−1, j−1)};
```

Pseudocode

```
def EDIT_DISTANCE(x, y):
    for i = 0, ..., m:
        E(i, 0) = i;
    for j = 0, ..., n:
        E(0, j) = j;
    for i = 1, ..., m:
        for j = 1, ..., n:
            E(i, j) =
             min{1 + E(i−1, j), 1 + E(i, j−1), diff(i, j) + E(i−1, j−1)};
    return E(m, n);
```

Pseudocode

```
def EDIT_DISTANCE(x, y):
    for i = 0, ..., m:
        E(i, 0) = i;
    for j = 0, ..., n:
        E(0, j) = j;
    for i = 1, ..., m:
        for j = 1, ..., n:
            E(i, j) =
             min{1 + E(i-1, j), 1 + E(i, j-1), diff(i, j) + E(i-1, j-1)};
    return E(m, n);
```

Running time: $O(mn)$

Finding the alignment

We use an extra table prev to record where each entry of $E(i, j)$ was coming from:

Finding the alignment

We use an extra table prev to record where each entry of $E(i, j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i-1,j-1) \end{cases}$$

Finding the alignment

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def PRING_ALIGNMENT(x, y, prev)**:**

Finding the alignment

We use an extra table prev to record where each entry of $E(i, j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i-1,j-1) \end{cases}$$

def $\textsc{Pring\_Alignment}(x, y, \mathrm{prev})$**:**

Set $i = m, j = n$;

Finding the alignment

We use an extra table prev to record where each entry of $E(i,j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i-1,j-1) \end{cases}$$

def $\textsc{Pring\_Alignment}(x, y, \mathrm{prev})$**:**

  Set $i = m, j = n$;

  if $\mathrm{prev}(i,j) = (i-1, j-1)$**:**

Finding the alignment

We use an extra table prev to record where each entry of $E(i,j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i-1,j-1) \end{cases}$$

def PRING_ALIGNMENT(x, y, prev)**:**

Set $i = m, j = n$;

if $\mathrm{prev}(i,j) = (i-1, j-1)$**:**

print_back$\binom{y_i}{x_i}$;

Finding the alignment

We use an extra table prev to record where each entry of $E(i,j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i-1,j-1) \end{cases}$$

def $\textsc{Pring\_Alignment}(x, y, \mathrm{prev})$**:**

   Set $i = m, j = n$;

   if $\mathrm{prev}(i,j) = (i-1,j-1)$**:**

      $\mathrm{print\_back}\binom{y_i}{x_i}$;

   if $\mathrm{prev}(i,j) = (i-1,j)$**:**

Finding the alignment

We use an extra table prev to record where each entry of $E(i, j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i-1,j-1) \end{cases}$$

def $\textsc{Pring\_Alignment}$*(x, y, prev)***:**

Set $i = m, j = n$;

if $\mathrm{prev}(i,j) = (i-1, j-1)$**:**

$\quad \mathrm{print\_back}\binom{y_i}{x_i}$;

if $\mathrm{prev}(i,j) = (i-1, j)$**:**

$\quad \mathrm{print\_back}\binom{\text{-}}{x_i}$;

Finding the alignment

We use an extra table prev to record where each entry of $E(i, j)$ was coming from:

$$\mathrm{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \mathrm{diff}(i,j) + E(i-1,j-1) \end{cases}$$

def $\textsc{Pring\_Alignment}(x, y, \mathrm{prev})$**:**

Set $i = m, j = n$;

if $\mathrm{prev}(i,j) = (i-1,j-1)$**:**

$\mathrm{print\_back}\binom{y_i}{x_i}$;

if $\mathrm{prev}(i,j) = (i-1,j)$**:**

$\mathrm{print\_back}\binom{\text{-}}{x_i}$;

if $\mathrm{prev}(i,j) = (i, j-)$**:**

Finding the alignment

We use an extra table prev to record where each entry of $E(i,j)$ was coming from:

$$\text{prev}(i,j) = \begin{cases} (i-1,j) & \text{if } E(i,j) = 1 + E(i-1,j) \\ (i,j-1) & \text{if } E(i,j) = 1 + E(i,j-1) \\ (i-1,j-1) & \text{if } E(i,j) = \text{diff}(i,j) + E(i-1,j-1) \end{cases}$$

Print_back

000

def $\textsc{Pring\_Alignment}(x, y, \text{prev})$**:**

- Set $i = m, j = n$;
- **if** $\text{prev}(i,j) = (i-1, j-1)$**:**
  - $\text{print\_back}\binom{y_i}{x_i}$;
- **if** $\text{prev}(i,j) = (i-1, j)$**:**
  - $\text{print\_back}\binom{\text{-}}{x_i}$;
- **if** $\text{prev}(i,j) = (i, j-)$**:**
  - $\text{print\_back}\binom{y_j}{\text{-}}$;

Dynamic Programming

0-1 Knapsack (Textbook Section 6.4)

0-1 Knapsack

W $1, 2, \cdots, n$

0-1 Knapsack Problem

A Thief has a backpack with certain capacity. There is a set of items with certain weight and value. **Goal:** pack the backpack with the largest value

w_j v_j

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- But it has the optimal substructure property:

0-1 Knapsack

0-1 Knapsack Problem

A Thief has a backpack with certain capacity. There is a set of items with certain weight and value. **Goal:** pack the backpack with the largest value

- Doesn't have the greedy choice property
- But it has the optimal substructure property:
  Suppose the optimal packing has weight $\leq W$. If we remove item j from it, the remaining packing must be the optimal packing for capacity $W - w_j$ with items excluding j

Subproblem

- **Subproblem**: $K(w, j)$ — the maximum value achievable using a backpack of capacity w and items $\underline{1, \ldots, j}$ $1, \ldots, n$
  W

Subproblem

- **Subproblem**: $K(w, j)$ — the maximum value achievable using a backpack of capacity w and items $1, \ldots, j$
- **Optimal solution:** $K(W, n)$

consider $K(w, j) \leftarrow$

focus on j

case ①: j is picked $K(w,j) = K(\cancel{w}, j-1) + v_j$ ($w - w_j$)

case ②: j is not picked. $K(w,j) = K(w, j-1)$

Subproblem

- **Subproblem**: $K(w, j)$ — the maximum value achievable using a backpack of capacity w and items $1, \ldots, j$
- **Optimal solution:** $K(W, n)$
- **Recurrence:**

$$K(w, j) = \max\{K(w - w_j, j - 1) + v_j, K(w, j - 1)\}$$

Base case:

$K(0, j) = 0 \leftarrow$

$K(w, 0) = 0 \leftarrow$

Subproblem

- **Subproblem**: $K(w, j)$ — the maximum value achievable using a backpack of capacity w and items $1, \ldots, j$
- **Optimal solution:** $K(W, n)$
- **Recurrence:**

$$K(w, j) = \max\{K(w - w_j, j - 1) + v_j, K(w, j - 1)\}$$

- **Base case:** $K(0, j) = 0$ for all j and $K(w, 0) = 0$ for all w

Pseudocode

def $\textsc{Knapsack}(W, w, v)$:

Pseudocode

def $\textsc{Knapsack}(W, w, v)$**:**

Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

Pseudocode

def $\textsc{Knapsack}(W, w, v)$:

  Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

  for $j = 1, \ldots, n$:

Pseudocode

```
def KNAPSACK(W, w, v):
    Set K(0, j) = 0, K(w, 0) = 0 for all j, w;
    for j = 1, ..., n:
        for w = 1, ..., W:
```

Pseudocode

def KNAPSACK(W, w, v)**:**

   Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

   for $j = 1, \ldots, n$**:**

      for $w = 1, \ldots, W$**:**

         if $w_j > w$**:**

Pseudocode

def $\textsc{Knapsack}(W, w, v)$**:**

Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

for $j = 1, \ldots, n$**:**

  for $w = 1, \ldots, W$**:**

    if $w_j > w$**:**

      $K(w, j) = K(w, j - 1)$;

Pseudocode

```
def KNAPSACK(W, w, v):
    Set K(0, j) = 0, K(w, 0) = 0 for all j, w;
    for j = 1, ..., n:
        for w = 1, ..., W:
            if w_j > w:
                K(w, j) = K(w, j − 1);
            else:
```

Pseudocode

```
def KNAPSACK(W, w, v):
    Set K(0, j) = 0, K(w, 0) = 0 for all j, w;
    for j = 1, ..., n:
        for w = 1, ..., W:
            if w_j > w:
                K(w, j) = K(w, j − 1);
            else:
                K(w, j) = max{K(w − w_j, j − 1) + v_j, K(w, j − 1)};
```

Pseudocode

capacity

$w_1, w_2, \cdots, w_n$

def KNAPSACK(W, w, v)**:** $\rightarrow v_1, v_2, \cdots, v_n$

    Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;
    for $j = 1, \ldots, n$**:**
        for $w = 1, \ldots, W$**:**
            if $w_j > w$**:**
                $K(w, j) = K(w, j-1)$;
            else:
                $K(w, j) = \max\{K(w - w_j, j-1) + v_j, K(w, j-1)\}$;
    return $K(W, n)$;

1. How many entries $O(Wn)$
2. cost for computing each entry : $O(1)$

Pseudocode

```
def KNAPSACK(W, w, v):
    Set K(0, j) = 0, K(w, 0) = 0 for all j, w;
    for j = 1, ..., n:
        for w = 1, ..., W:
            if w_j > w:
                K(w, j) = K(w, j − 1);
            else:
                K(w, j) = max{K(w − w_j, j − 1) + v_j, K(w, j − 1)};
    return K(W, n);
```

Running time: $O(nW)$

Pseudocode

def KNAPSACK(W, w, v)**:**

Set $K(0, j) = 0, K(w, 0) = 0$ for all j, w;

for $j = 1, \ldots, n$**:**

for $w = 1, \ldots, W$**:**

if $w_j > w$**:**

$K(w, j) = K(w, j-1)$;

else:

$K(w, j) = \max\{K(w - w_j, j-1) + v_j, K(w, j-1)\}$;

return $K(W, n)$;

represent W: $\underbrace{O(\log W)}_{\text{input size}}$ bits

Running time: $O(nW)$

Question: is this a polynomial-time algorithm? No

Pseudocode

```
def KNAPSACK(W, w, v):
    Set K(0, j) = 0, K(w, 0) = 0 for all j, w;
    for j = 1, ..., n:
        for w = 1, ..., W:
            if w_j > w:
                K(w, j) = K(w, j − 1);
            else:
                K(w, j) = max{K(w − w_j, j − 1) + v_j, K(w, j − 1)};
    return K(W, n);
```

Running time: $O(nW)$

Question: is this a polynomial-time algorithm? No!

Running example

Example: $W = 10$

item	1	2	3	4
w_j	6	3	4	2
v_j	30	14	16	9

Running example

Example: $W = 10$

item	1	2	3	4
w_j	$\boxed{6}$	3	4	2
v_j	30	14	16	9

The K table:

$w \backslash j$	0	1	2	3	4
0	0	0	0	0	0
1	0	0			
2	0	0			
3	0	0			
4	0	0			
5	0	0			
6	0	30			
7	0				
8	0				
9	0				
10	0				

+30

if $w_j > w$:

$K(w,j) = K(w, j-1)$

else:

$$\boxed{K(w,j) = \max\left\{ K(w-w_j, j-1)+v_j,\ K(w, j-1) \right\}}$$

$K(\underset{w}{1}, \underset{j}{1})$

$6 > 1$

$K(1,1) = K(1,0)$

$K(\underset{w}{6}, \underset{j}{1})$

$w_j = 6 \quad w = 6$

$K(\underbrace{w-w_j}_{0}, 0) + \underset{30}{v_1}$

$K(w, j-1) = K(6,0) = 0$

Running example

Example: $W = 10$

item	1	2	3	4
w_j	6	3	4	2
v_j	30	14	16	9

The K table:

$w \backslash j$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	9
3	0	0	14	14	14
4	0	0	14	16	16
5	0	0	14	16	23
6	0	30	30	30	30
7	0	30	30	30	30
8	0	30	30	30	39
9	0	30	44	44	44
10	0	30	44	46	46

Running example

Example: $W = 10$

item	1	2	3	4
w_j	6	3	4	2
v_j	30	14	16	9

The K table:

$w \backslash j$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	9
3	0	0	14	14	14
4	0	0	14	16	16
5	0	0	14	16	23
6	0	30	30	30	30
7	0	30	30	30	30
8	0	30	30	30	39
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Running example

Example: $W = 10$

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w_j	6	3	4	2
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The K table:

$w \backslash j$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	9
3	0	0	14	14	14
4	0	0	14	16	16
5	0	0	14	16	23
6	0	30	30	30	30
7	0	30	30	30	30
8	0	30	30	30	39
9	0	30	44	44	44
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Running example

Example: $W = 10$

item	1	2	3	4
w_j	6	3	4	2
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The K table:

$w \backslash j$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	9
3	0	0	14	14	14
4	0	0	14	16	16
5	0	0	14	16	23
6	0	30	30	30	30
7	0	30	30	30	30
8	0	30	30	30	39
9	0	30	44	44	44
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Running example

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The K table:

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0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	9
3	0	0	14	14	14
4	0	0	14	16	16
5	0	0	14	16	23
6	0	30	30	30	30
7	0	30	30	30	30
8	0	30	30	30	39
9	0	30	44	44	44
10	0	30	44	46	46

Running example

Example: $W = 10$

item	1	2	3	4
w_j	6	3	4	2
v_j	30	14	$\boxed{16}$	9

The K table:

$w \backslash j$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	+30 0	0	0	9
3	0	0	14	14	14
4	0	0	14	16	16
5	0	0	14	16	23
6	0	30	30	30	30
7	0	30	30	16 30	30
8	0	30	30	30	39
9	0	30	44	44	44
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Dynamic Programming

Chain matrix multiplication (Textbook Section 6.5)

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$
Need to compute

$$M_1 \cdot M_2 \cdots M_n$$

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$

Need to compute

$$M_1 \cdot M_2 \cdots M_n$$

The dimensions of these matrices are:

$$M_1 \in \boxed{\mathbb{R}^{m_0 \times m_1}}, M_2 \in \mathbb{R}^{m_1 \times m_2}, \ldots, M_n \in \mathbb{R}^{m_{n-1} \times m_n}$$

$\mathbb{R}^{m \times n}$: the class of all matrices of dim. $m \times n$ where each entry is a real number. $\mathbb{R}$: all the real number

$\mathbb{C}$: all the complex numbers

$\mathbb{C}^{m \times n}$

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$
Need to compute

$$M_1 \cdot M_2 \cdots M_n$$

The dimensions of these matrices are:

$$M_1 \in \mathbb{R}^{m_0 \times m_1}, M_2 \in \mathbb{R}^{m_1 \times m_2}, \ldots, M_n \in \mathbb{R}^{m_{n-1} \times m_n}$$

Recall if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then the cost for computing $A \cdot B$ is $m \cdot n \cdot p$

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$

Need to compute

$$M_1 \cdot M_2 \cdots M_n$$

The dimensions of these matrices are:

$$M_1 \in \mathbb{R}^{m_0 \times m_1}, M_2 \in \mathbb{R}^{m_1 \times m_2}, \ldots, M_n \in \mathbb{R}^{m_{n-1} \times m_n}$$

Recall if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then the cost for computing $A \cdot B$ is $m \cdot n \cdot p$

Also, matrix multiplication is associative:
$A \cdot B \cdot C = (A \cdot B) \cdot C = A \cdot (B \cdot C)$

Chain matrix multiplication

We have n matrices $M_1, M_2, \ldots, M_n$

Need to compute

$$M_1 \cdot M_2 \cdots M_n$$

The dimensions of these matrices are:

$$M_1 \in \mathbb{R}^{m_0 \times m_1}, M_2 \in \mathbb{R}^{m_1 \times m_2}, \ldots, M_n \in \mathbb{R}^{m_{n-1} \times m_n}$$

Recall if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then the cost for computing $A \cdot B$ is $m \cdot n \cdot p$

Also, matrix multiplication is associative:
$A \cdot B \cdot C = (A \cdot B) \cdot C = A \cdot (B \cdot C)$

Question: what's the best way for computing $M_1 \cdot M_2 \cdots M_n$? i.e., where to put the parentheses?

Example of chain matrix multiplication

Consider $M_1 \in \mathbb{R}^{50\times 20}, M_2 \in \mathbb{R}^{20\times 1}, M_3 \in \mathbb{R}^{1\times 10}, M_4 = \mathbb{R}^{10\times 100}$

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There are many ways to do multiplication

- $\overset{50\times 20}{M_1} \cdot ((\underbrace{\boxed{\overset{20\times 10}{M_2 \cdot M_3}}) \cdot \overset{10\times 100}{M_4}}_{20\times 100})$

Cost: $20\times 1\times 10 + 20\times 10\times 100 + 50\times 20 \times 100$

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- $M_1 \cdot ((M_2 \cdot M_3) \cdot M_4)$
  Cost: $20 \cdot 1 \cdot 10 + 20 \cdot 10 \cdot 100 + 50 \cdot 20 \cdot 100 = 10200$

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  Cost: $20 \cdot 1 \cdot 10 + 50 \cdot 20 \cdot 10 + 50 \cdot 10 \cdot 100 = 60200$
- $(\overbrace{M_1 \cdot M_2}^{50\times 1}) \cdot (\overbrace{M_3 \cdot M_4}^{1\times 100})$

Cost: $50\times 20\times 1 + 1\times 10\times 100 + 50\times 1\times 100$

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- $(M_1 \cdot M_2) \cdot (M_3 \cdot M_4)$
  Cost: $50 \cdot 20 \cdot 1 \cdot 10 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100 = 7000$

Example of chain matrix multiplication

$M_1 , \underbrace{M_2 \cdots}\, M_n$

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- $(M_1 \cdot M_2) \cdot (M_3 \cdot M_4)$
  Cost: $50 \cdot 20 \cdot 1 \cdot 10 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100 = 7000$

Goal: find a way to do multiplication with the minimum cost

Optimal solution: $C(1, n)$

- **Subproblem:**
  $C(i, j)$ — the minimum cost for multiplying $M_i, M_{i+1}, \ldots, M_j$

Dynamic programming

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$$\underbrace{(M_i M_{i+1} \cdots M_k)}_{m_{i-1} \times m_k} \bullet \underbrace{(M_{k+1} M_{k+2} \cdots M_j)}_{m_k \times m_j}$$

$m_{i-1} \times m_i$ $\quad$ $m_{k-1} \times m_k$

$C(i,j) = C(i,k) + C(k+1,j) + m_{i-1} \cdot m_k \cdot m_j$

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- **Subproblem:**
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$$\underbrace{(\overset{m_{i-1} \times m_i}{M_i} M_{i+1} \cdots \overset{m_{k-1} \times m_k}{M_k})}_{m_{i-1} \times m_k} (M_{k+1} M_{k+2} \cdots M_j)$$

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So, $$C(i,j) = \min_{i \le k < j} \{C(i,k) + C(k+1,j) + m_{i-1}\cdot m_k \cdot m_j\}$$

- Base case? $C(i,i) = 0$

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- **Optimal solution:** $C(1,n)$

Pseudocode

def CHAIN_MATRIX(*m*):

Pseudocode

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  for $i = 1 \ldots n$:

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def $\textsc{Chain\_Matrix}(m)$**:**

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Pseudocode

def $\text{CHAIN\_MATRIX}(m)$**:**

  for $i = 1 \ldots n$**:**

    $C(i,i) = 0;$

  for $s = 1 \ldots n-1$**:**

Pseudocode

```
def CHAIN_MATRIX(m):
    for i = 1 ... n:
        C(i, i) = 0;
    for s = 1 ... n − 1:
        for i = 1 ... n − s:
```

Pseudocode

```
def CHAIN_MATRIX(m):
    for i = 1 ... n:
        C(i, i) = 0;
    for s = 1 ... n − 1:
        for i = 1 ... n − s:
            j = i + s;
```

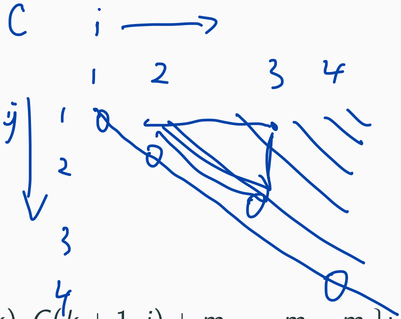
Pseudocode

```
def CHAIN_MATRIX(m):
    for i = 1 ... n:
        C(i, i) = 0;
    for s = 1 ... n − 1:
        for i = 1 ... n − s:
            j = i + s;
            C(i, j) = min_{i≤k<j}{C(i, k), C(k + 1, j) + m_{i−1} · m_k · m_j};
```

Pseudocode

def CHAIN_MATRIX(m):

- **for** $i = 1 \dots n$:
  - $C(i, i) = 0$;
- **for** $s = 1 \dots n - 1$:
  - **for** $i = 1 \dots n - s$:
    - $j = i + s$;
    - $C(i, j) = \min_{i \le k < j}\{C(i, k), C(k + 1, j) + m_{i-1} \cdot m_k \cdot m_j\}$;
- **return** $C(1, n)$;



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Running time:

Pseudocode

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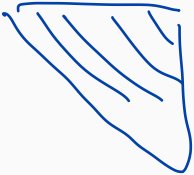
```

Running time:

$O(n^2)$ entries to fill; $O(n)$ operations to fill in each entry

Pseudocode



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```
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$$C(i,j) = \min_{i \le k < j}\{C(i,k), C(k+1,j) + m_{i-1} \cdot m_k \cdot m_j\};$$

Running time:

$O(n^2)$ entries to fill; $O(n)$ operations to fill in each entry

Total running time: $O(n^3)$