CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Linear Programming
(Textbook, Section 7.1)

Duality of LP (I)

Consider

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \leq 20 \\ & x_2 \leq 30 \\ & x_1 + x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Duality of LP (I)

Consider

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Can we show the optimal solution is at least 60?

Duality of LP (I)

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Can we show the optimal solution is at least 60? Check (0, 30)

Duality of LP (I)

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Can we show the optimal solution is at least 60? Check (0, 30)

Can we show that optimal solution is at most 90?

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Consider

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Can we show the optimal solution is at least 60? Check (0, 30)

Can we show that optimal solution is at most 90? Use linear combinations constraints

Duality of LP (II)

Define a variable for each constraint

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \le 20 \\ & x_2 \le 30 \\ & x_1 + x_2 \le 40 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Duality of LP (II)

Define a variable for each constraint

$$
\begin{array}{rll}
\text{maximize} & x_1 + 2x_2 & \\
\text{subject to} & x_1 \le 20 & y_1 \\
& x_2 \le 30 & y_2 \\
& x_1 + x_2 \le 40 & y_3 \\
& x_1, x_2 \ge 0 &
\end{array}
$$

Duality of LP (II)

Define a variable for each constraint

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 & \\
\text{subject to} \quad & x_1 \le 20 & \color{red}{y_1} \\
& x_2 \le 30 & \color{red}{y_2} \\
& x_1 + x_2 \le 40 & \color{red}{y_3} \\
& x_1, x_2 \ge 0 &
\end{aligned}
$$

Adding them together:

Duality of LP (II)

Define a variable for each constraint

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 & \\ \text{subject to} \quad & x_1 \leq 20 & \color{red}{y_1} \\ & x_2 \leq 30 & \color{red}{y_2} \\ & x_1 + x_2 \leq 40 & \color{red}{y_3} \\ & x_1, x_2 \geq 0 & \end{aligned}$$

Adding them together:

$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 20y_1 + 30y_2 + 40y_3$

Duality of LP (II)

Define a variable for each constraint

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\text{maximize} \quad & x_1 + 2x_2 & \\
\text{subject to} \quad & x_1 \leq 20 & \qquad {\color{red} y_1} \\
& x_2 \leq 30 & \qquad {\color{red} y_2} \\
& x_1 + x_2 \leq 40 & \qquad {\color{red} y_3} \\
& x_1, x_2 \geq 0 &
\end{aligned}
$$

Adding them together:
$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 20y_1 + 30y_2 + 40y_3$
We let $y_1 + y_3 \geq 1$ and $y_2 + y_3 \geq 2$ to get an upper bound on $x_1 + 2x_2$:

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$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \le 20y_1 + 30y_2 + 40y_3$

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$x_1 + 2x_2 \le (y_1 + y_3)x_1 + (y_2 + y_3)x_2 \le 20y_1 + 30y_2 + 40y_3$

Duality of LP (III)

Primal LP

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \leq 20 \\ & x_2 \leq 30 \\ & x_1 + x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Duality of LP (III)

Primal LP

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& x_2 \le 30 \\
& x_1 + x_2 \le 40 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
\text{minimize} \quad & 20y_1 + 30y_2 + 40y_3 \\
\text{subject to} \quad & y_1 + y_3 \ge 1 \\
& y_2 + y_3 \ge 2 \\
& y_1, y_2, y_3 \ge 0
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$$

Duality of LP (III)

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Optimal solution: $(x_1, x_2) = (10, 30) \implies x_1 + 2x_2 = 70$

Dual LP

$$\begin{aligned} \text{minimize} \quad & 20y_1 + 30y_2 + 40y_3 \\ \text{subject to} \quad & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Duality of LP (III)

Primal LP

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \le 20 \\ & x_2 \le 30 \\ & x_1 + x_2 \le 40 \\ & x_1, x_2 \ge 0 \end{aligned}$$

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Dual LP

$$\begin{aligned} \text{minimize} \quad & 20y_1 + 30y_2 + 40y_3 \\ \text{subject to} \quad & y_1 + y_3 \ge 1 \\ & y_2 + y_3 \ge 2 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

Optimal solution: $(y_1, y_2, y_3) = (0, 1, 1) \implies 20y_1 + 30y_2 + 40y_3 = 70$

Duality of LP(IV)

More generally

Primal LP

$$
\begin{aligned}
\max \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
& \quad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

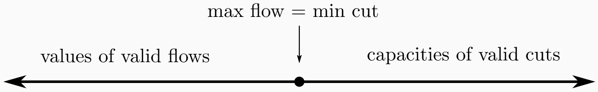
Duality of LP(IV)

More generally

Primal LP	Dual LP
$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\ & x_1, x_2, \ldots, x_n \ge 0 \end{aligned}$$	$$\begin{aligned} \min \quad & b_1y_1 + b_2y_2 + \cdots + b_my_m \\ \text{s.t.} \quad & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1 \\ & a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2 \\ & \vdots \\ & a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n \\ & y_1, y_2, \ldots, y_m \ge 0 \end{aligned}$$

Duality of LP (V)

Duality of flow and cut



Duality of LP (V)

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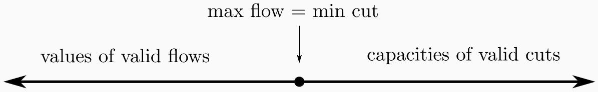


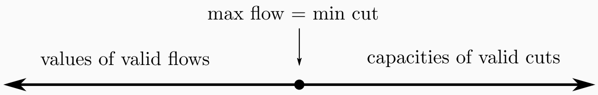
For LP we have:

Theorem (Weak Duality)

A feasible solution to the dual LP is an upper bound on any feasible solution to the primal LP

Duality of LP (V)

Duality of flow and cut



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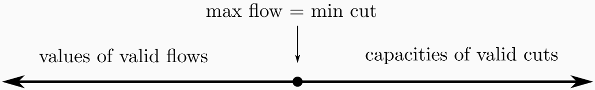
A feasible solution to the dual LP is an upper bound on any feasible solution to the primal LP

Theorem (Strong Duality)

The optimal solution to the dual LP is equal to the optimal solution to the primal LP

Duality of LP (V)

Duality of flow and cut



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