Names shi Qin Section 3 Date 20211018

Math 455 Quiz 2

Total 20 points

- Answer each of the following questions.
- Be sure that your name is on the top of the page.

1. (7 points) Consider a function $f(x) = x^3 + \frac{3}{2}x^2 + 1/8$.

   (a) (3 points) Prove that there exists at least one root of $f(x) = 0$ on $[-2, -1]$.

$f(-2) = (-2)^3 + \frac{3}{2}(-2)^2 + \frac{1}{8}$

$= -8 + \frac{3\cdot\cancel{4}2}{\cancel{2}} + \frac{1}{8}$

$= -8 + 6 + \frac{1}{8}$

$= \frac{-28+1}{8}$

$= \frac{-15}{8} < 0$

$f(-1) = (-1)^3 + \frac{3}{2}(-1)^2 + \frac{1}{8}$

$= -1 + \frac{3}{2} + \frac{1}{8}$

$= \frac{-8+12+1}{8} > 0$

so $f(-1)\cdot f(-2) < 0$

So according to IVT, there exist at least one root.

   (b) (4 points)Consider a fixed point iteration

$$x_{n+1} = g_2(x_n), \qquad \text{where} \qquad g_2(x) = -\frac{\frac{3}{2}x^2 + 1/8}{x^2},$$

with the starting point $x_0 = -1$. Does this scheme converge?

$g_2(x) = -\left(\frac{\frac{3}{2}\cancel{x^2}}{\cancel{x^2}} + \frac{\frac{1}{8}}{x^2}\right)$

$= -\left(\frac{3}{2} + \frac{1}{8x^2}\right)$

$= -\left(\frac{3}{2} + 1\cdot(8x^2)^{-1}\right)$

$g_2'(x) = \frac{1}{4x^3}$

$g_2'(-1) = \frac{1}{4\cdot -1}$

$= \frac{1}{-4}$

$= -\frac{1}{4}$

$|-\frac{1}{4}| = \frac{1}{4} < 1$

converge

2. (a) (3 points) Find the multiplicity of the root $r = 0$ of $f(x) = x\sin^2(x)$

$f(0) = 0 \cdot \sin^2(0) = 0$

$f'(x) = 1 \cdot \sin^2 x + 2x \sin 2x$

$f'(0) = 1 \cdot 0 + 2 \cdot 0 \cdot \sin 2 \cdot 0$

$= 0$

$m = 3$

$f''(x) = 2\sin 2x + 4x\cos 2x + 2\cos x \sin x$

$f''(0) = 2 \cdot 0 + 4 \cdot 0 \cos 0 + 2 \cdot 0 \sin 0$

$= 0$

$f'''(x) = 12\cos(2x) - 8x\sin(2x)$

$= 12 - 8 \cdot 0 \cdot 0 = 12 \neq 0$

(b) (2 points) Find the forward error and backward error of the approximation root $c = 0.66$ for the function $f(x) = (3x - 2)^3$.

$f(x) = (3x - 2)^3$

$(3x-2)^3 = 0$

root $= \frac{2}{3}$

forward $= |x_a - r|$

$= 0.0066\ldots$

$= 6.6 \times 10^{-3}$

backward $= f(0.66) = |(3 \cdot (0.66) - 2)^3|$

$= |(1.98 - 2)^3|$

$= 0.02^3$

3. (8 points) Solve

$$f(x) = x^2 - x\cos(x) + \frac{\cos^2(x)}{4} = 0, \text{ with } x_0 = \frac{\pi}{2}.$$

(a) (4 points) Does Newton's Method converge quadratically? Why?

$f(x) = \left(x - \frac{\cos x}{2}\right)^2 \quad x_0 = \frac{\pi}{2}$

$f'(r_1) = 0$

$f'(x) = \frac{(2x - \cos(x))(\sin(x) + 2)}{2}$

if $f(r_1) = 0$

So it did not converge quadratically.

$\left(r - \frac{\cos r}{2}\right)^2 = 0$

$\left(r - \frac{\cos r}{2}\right) = 0$

$\frac{(2r - \cos r)}{2} = 0$

$(2r - \cos r) = 0$

(b)(4 points) Write out the Modified Newton's Method such that we have quadratical convergence.

$$f''(x) = 2\left(x - \frac{\cos x}{2}\right)\left(\frac{\cos x}{2}\right) + 2\left(1 + \frac{\sin x}{2}\right)^2$$

$$f''(0) \neq 0$$

$$m = 2$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{2\left(x_n - \frac{\cos x_n}{2}\right)^2}{2\left(x_n - \frac{\cos x_n}{2}\right)\left(1 + \frac{\sin x_n}{2}\right)}$$

$$= x_n - \frac{x_n - \frac{\cos x_n}{2}}{1 + \frac{\sin x_n}{2}}$$

$$\text{for } x_1 = \frac{\pi}{2} - \frac{2 \cdot \left(\frac{\pi}{2}\right)^2}{2\left(\frac{\pi}{2}\right)\left(1 + \frac{1}{2}\right)}$$

$$= \frac{\pi}{6}$$