CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Dynamic Programming

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Prelude

Dynamic programming vs. Greedy algorithms

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Sometimes, the greedy choice won't work — we need to check many subproblems to find the optimal solution → **Dynamic programming**

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- Use information from smaller subproblems to solve a larger subproblem

Warm-up: Longest increasing subsequence

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Example:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
5	2	8	6	3	6	9	7

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$i_1 = 2, i_2 = 5, i_3 = 6, i_4 = 7$

Solving LIS using DAG

We can model the longest increasing subsequence problem as a directed acyclic graph

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$$
\begin{aligned}
a_8 &= 7 \\
a_7 &= 9 \\
a_6 &= 6 \\
a_5 &= 3 \\
a_4 &= 6 \\
a_3 &= 8 \\
a_2 &= 2 \\
a_1 &= 5
\end{aligned}
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- There is a linke $i \to j$ if $a_i < a_j$

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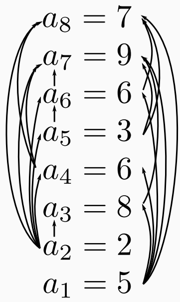

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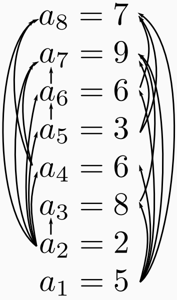

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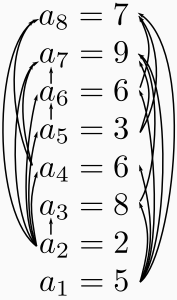

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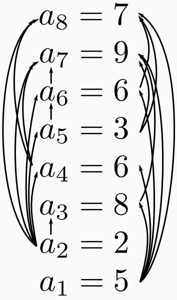

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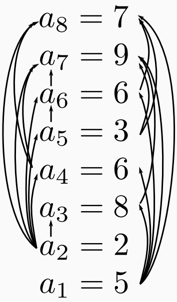

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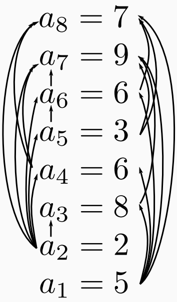

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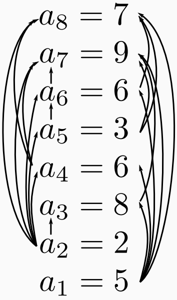

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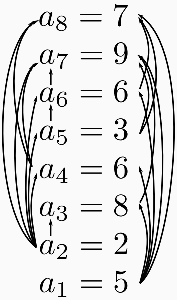

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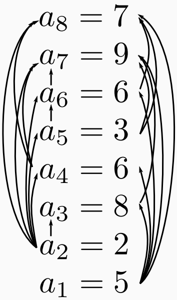

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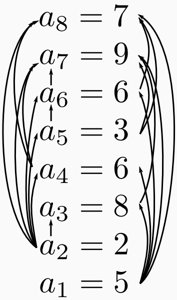

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Costs more than greedy: need to check more subproblems

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- **for** $j = 1, \ldots, n$**:**
  - $L(j) = 1$, $\mathrm{prev}(j) = \cdot$;
  - **for** $i = 1, \ldots, j$**:**
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L	1	1	2	2	2	3	4	4
prev	$\cdot$	$\cdot$	1	1	2	5	6	6

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3. Base case