CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Dynamic Programming

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Prelude

Key steps to design DP algorithms

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2. Recurrence

   e.g. $L(j) = 1 + \max\{L(i) : a_i < a_j\}$
3. Base case

Dynamic Programming

Edit Distance (Textbook Section 6.3)

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		$\updownarrow$		$\updownarrow$		$\updownarrow$
y:	A	T	C	-	T	G

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Consider the following **alignments**:

x:	A	-	C	G	T	A
y:	A	T	C	-	T	G

cost : 3

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cost : 5

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Consider the following **alignments**:

x:	A	-	C	G	T	A
y:	A	T	C	-	T	G

cost : 3 (optimal)

x:	A	C	-	-	G	T	A
y:	A	T	C	T	G	-	-

cost : 5

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$$\begin{array}{lcccccc} \text{x:} & \mathrm{A} & - & \mathrm{C} & \mathrm{G} & \mathrm{T} & \mathrm{A} \\ & & \updownarrow & & \updownarrow & & \updownarrow \\ \text{y:} & \mathrm{A} & \mathrm{T} & \mathrm{C} & - & \mathrm{T} & \mathrm{G} \end{array}$$

$$\text{cost} : 3 \text{ (optimal)}$$

$$\begin{array}{lccccccc} \text{x:} & \mathrm{A} & \mathrm{C} & - & - & \mathrm{G} & \mathrm{T} & \mathrm{A} \\ & & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow \\ \text{y:} & \mathrm{A} & \mathrm{T} & \mathrm{C} & \mathrm{T} & \mathrm{G} & - & - \end{array}$$

$$\text{cost} : 5$$

So $d(x, y) = 3$

Edit distance — subproblem

Consider two strings

$$x = x_1 x_2 \cdots x_m \quad \text{and} \quad y = y_1 y_2 \cdots y_n$$

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$$E(i,j) = d(x_1 \cdots x_i, y_1 \cdots y_j)$$

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How to use the solution to the subproblems to solve $E(i, j)$?

Recurrence (I)

Look at the rightmost column:

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$$\text{Case 1} \quad \begin{array}{cccc} x_1 & \cdots & x_{i-1} & \color{red}{x_i} \\ y_1 & \cdots & y_j & \color{red}{-} \end{array}$$

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Contributes 1 to the cost plus the cost of alignment $\begin{array}{ccc} x_1 & \cdots & x_{i-1} \\ y_1 & \cdots & y_j \end{array}$

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Case 2
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Look at the rightmost column:

Case 1 $\begin{matrix} x_1 & \cdots & x_{i-1} & {\color{red}x_i} \\ y_1 & \cdots & y_j & {\color{red}\text{-}} \end{matrix}$

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Case 3 $\begin{array}{cccc} x_1 & \cdots & x_{i-1} & x_i \\ y_1 & \cdots & y_{j-1} & y_j \end{array}$

$$E(i,j) = \begin{cases} E(i-1,j-1) & \text{if } x_i = y_j \\ 1 + E(i-1,j-1) & \text{otherwise} \end{cases}$$

Recurrence (II)

The recurrence:

$$E(i,j) = \min\{1 + E(i-1,j), 1 + E(i,j-1), \mathrm{diff}(i,j) + E(i-1,j-1)\},$$

where

$$\mathrm{diff}(i,j) = \begin{cases} 1 & \text{if } x_i \neq y_j \\ 0 & \text{otherwise} \end{cases}$$

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Base case: $E(0,0) = 0$, $E(i,0) = i$, $E(0,j) = j$

Filling the table

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$$\begin{array}{cccccccc}
E(0,0) & & E(0,1) & \cdots & & E(0,n-1) & & E(0,n) \\
& \searrow & \downarrow & & & & & \\
E(1,0) & \rightarrow & E(1,1) & \cdots & & \cdots & & \cdots \\
\vdots & & & & & & & \\
& & & & \searrow & \downarrow & \searrow & \downarrow \\
E(m-1,0) & & & & \rightarrow & E(m-1,n-1) & \rightarrow & E(m-1,n) \\
& & & & \searrow & \downarrow & \searrow & \downarrow \\
E(m,0) & & & & \rightarrow & E(m,n-1) & \rightarrow & E(m,n)
\end{array}$$

Running example

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	A	T	C	T	G
A					
C					
G					
T					
A					

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$x = ACGTA$ and $y = ATCTG$

		A	T	C	T	G
	0	1	2	3	4	5
		$\searrow$				
A	1	0	$\rightarrow$ 1	2	3	4
			$\searrow$	$\searrow$		
C	2	1	1	1	2	3
				$\searrow$ $\downarrow$		
G	3	2	2	2	2	2
					$\searrow$	
T	4	3	2	3	2	3
						$\searrow$
A	5	4	3	3	3	3