Half-Plane Intersection

Definitions

Definition 1 (half-planes)**.** A line l on 2D plane with function $y = ax - b$ defines two *half-planes*: the *upper half-plane*: $y \geq ax - b$ and the *lower half-plane*: $y \leq ax - b$.

Definition 2 (upper- and lower-envelop)**.** Let $L = \{y = a_i x - b_i \mid 1 \leq i \leq n\}$ be a set of lines on 2D plane. We define the *upper-envelop* of L, denoted as $UE(L)$, as the intersection of the corresponding n upper half-planes $\{y \geq a_i x - b \mid 1 \leq i \leq n\}$. We define the *lower-envelop* of L, denoted as $LE(L)$, as the intersection of the corresponding n lower half-planes $\{y \leq a_i x - b \mid 1 \leq i \leq n\}$.

Either upper-envolop or lower-envelop of a set of lines can be represented as the list of lines that define its boundary from left to right. In the example below, we can write $UE(L) = (l_1, l_2, l_4, l_7)$ and $LE(L) = (l_7, l_5, l_1)$.

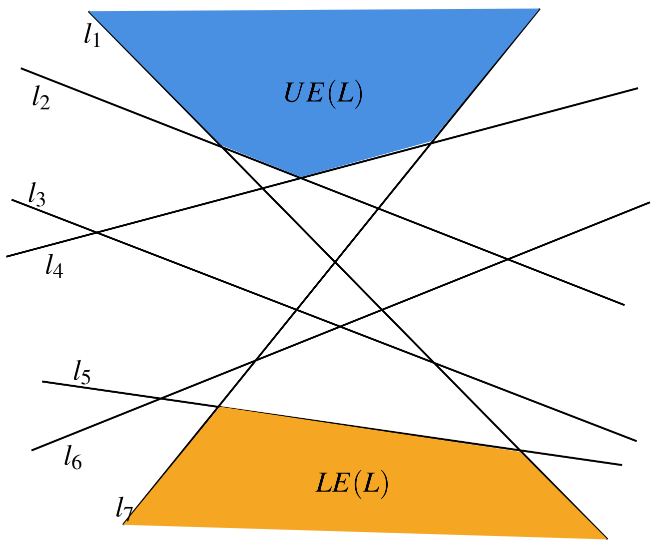

Figure 1: Illustration of upper-envelop and lower-envelop of lines $L = \{l_1, l_2, \cdots, l_7\}$.

We want to design efficient algorithms to calculate the upper- and lower-envelop of a set of lines. In fact, we don't need to design any new algorithm here. Below we will show that, the problem of finding upper- and lower-envolop of a set of lines is equivalent to the problem of finding the convex hull of a set of points. Therefore, the algorithms we've designed for finding the convex hull can be directly used to find the upper- and lower-envelop of lines.

Duality

Definition 3 (dual of a point)**.** Let $p = (p_x, p_y)$ be a point on 2D plane. We define the *dual* of p, denoted as p^*, as a line with function $y = p_x x - p_y$ on 2D plane.

Definition 4 (dual of a line)**.** Let l be a line with function $y = ax - b$ on 2D plane. We define its *dual*, denoted as l^*, as a point with coordinates (a, b) on 2D plane.

The following three properties are direct consequences of above definitions. (Think how to prove them.)

Property 1. For any point p, we have $(p^*)^* = p$. For any line l, we have $(l^*)^* = l$.

Property 2. Point p is on line l if and only if point l^* is on line p^*.

Property 3. Point p is above (resp. below) line l if and only if point l^* is above (resp. below) line p^*.

Half-plane Intersection vs. Convex Hull

Definition 5 (upper- and lower-hull)**.** Let P be a set of points, and let $CH(P)$ be the convex hull of P. Let $p_L \in CH(P)$ be the vertex with smallest x-coordinate, and $p_S \in CH(P)$ be the vertex with largest x-coordinate. Therefore p_L and p_S partition $CH(P)$ into two parts: the list of vertices from p_L to p_S following the counter-clockwise order is called *lower hull* of P, denoted as $LH(P)$; the list of vertices from p_S to p_L following the counter-clockwise order is called *upper hull* of P, denoted as $UH(P)$.

We now show that upper- and lower-envelop of lines is essentially the same with lower- and upper-hull of points. We first prove the connection between upper-envelop and lower-hull; the other one, i.e., lower-envelop and upper-hull, can be proved symmetrically.

Let L be a set of lines, we define $L^* = \{l^* \mid l \in L\}$, i.e., the set of "dual points" of L.

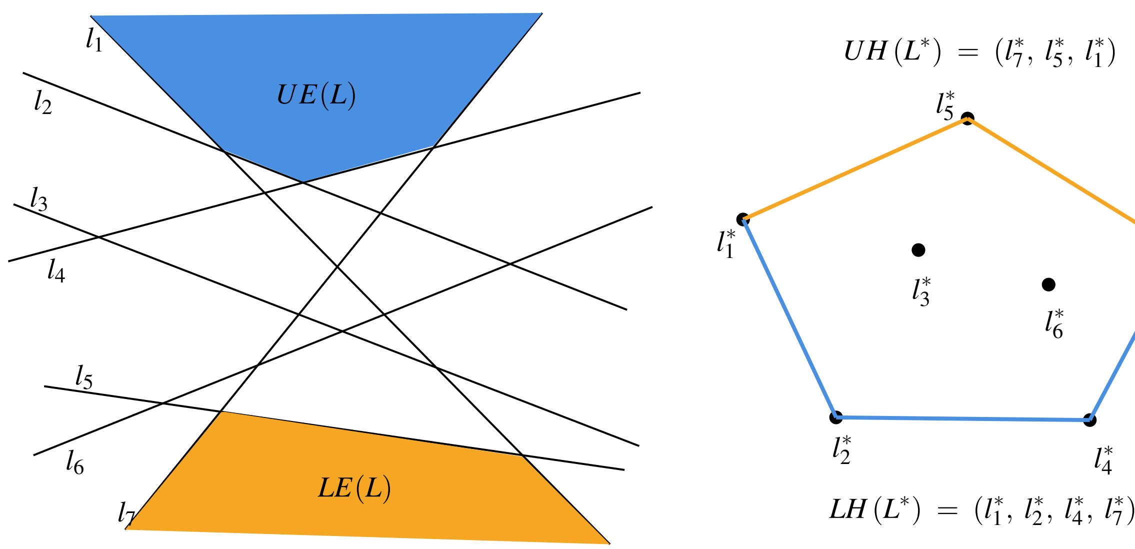

Figure 2: Illustration of duality between upper-/lower-envelop and lower-/upper-hull.

Claim 1. A line $l \in L$ is part of the boundary of $UE(L)$ if and only if l^* is one of the vertices of $LH(L^*)$.

Proof. Line l is part of $UE(L)$, implies that a piece of l is above all other lines. This is equivalent to: there exists a point p, such that p is on l, and that p is above all lines in $L \setminus \{l\}$. This statement is also equivalent to the following statement, by translating everything to their dual counterparts (and applying above Properties of duality): there exists a line p^*, such that l^* is on p^* and that all points in $L^* \setminus \{l^*\}$ are above line p^*. Clearly, this statement is also equivalent to that l^* is one vertex of the lower-hull of L^* (think the Properties of convex hull). □

The above claim shows that lines in $UE(L)$ and vertices in $LH(L^*)$ are in a "dual" relationship. We now show how their ordering are connected. Recall that we represent $UE(L)$ as a list of lines from left to right. Therefore, the *slope* of these lines are in increasing order. As the dual of line $y = ax - b$ is point (a, b), i.e., the slope of a line becomes the x-coordinate of its dual, we know that the corresponing "dual points" of $UE(L)$ are in the increasing order of their x-coordinates.

The above two facts can be combined as the following: $UE(L) = (l_{p_1}, l_{p_2}, \cdots, l_{p_k})$ if and only if $LH(L^*) = (l^*_{p_1}, l^*_{p_2}, \cdots, l^*_{p_k})$. Formally, we can write

Fact 1. $UE(L) = (LH(L^*))^*$.

Symmetrically, with the same reasonging, we can prove that $LE(L) = (l_{p_1}, l_{p_2}, \cdots, l_{p_k})$ if and only if $UE(L^*) = (l^*_{p_1}, l^*_{p_2}, \cdots, l^*_{p_k})$. (Recall that $LE(L)$ is represented as the list of lines from left to right, i.e., their slopes are decreasing, while $UH(L^*)$ is represented as the list of vertices from rightmost vertex to leftmost vertex in couter-clockwise order, i.e., their x-coordinates are also decreasing.) Formally, we can also write

Fact 2. $LE(L) = (UH(L^*))^*$.