Greedy algorithms

Matroid, Task Scheduling
(Cormen et al. 16.4, 16.5)

Warning

Very abstract!

"Computer Science is a science of **abstraction** — creating the right model for a problem and devising the appropriate mechanizable techniques to solve it."

— Alfred Aho

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Many problems for which a greedy approach provides optimal solution can be formulated as some problems involves matroids

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A more abstract view of graph vs. matroid

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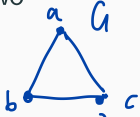

$V = \{a, b, c\}$

$E = \{(a,b), (b,c), (a,c)\}$

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For a matroid $M = (S, \mathcal{I})$, each $A \in \mathcal{I}$ is called an **independent subset**

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  A is a forest collection of trees.

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  $B \in \mathcal{I}$, $A \subseteq B$, then $A \subseteq \mathcal{I}$

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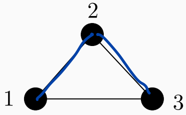
- Exchange: demonstrate by example

$S = \{(1,2), (2,3), (1,3)\}$

$\mathcal{I} = \{ \{(1,2), (1,3)\}, \{(1,2),(2,3)\}, \{(1,3),(2,3)\},$
$\{(1,2)\}, \{(2,3)\}, \{(1,3)\}, \phi \}$

$\cancel{\{(1,2), (1,3), (2,3)\}}$



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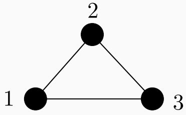

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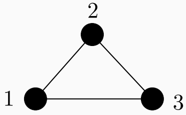

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$A = \{(2,3)\}$, $B = \{(1,3), (2,3)\}$

$\exists x \in B - A$, $x = \{(1,3)\}$ $A \cup \{x\} \in \mathcal{I}$

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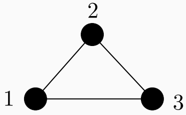

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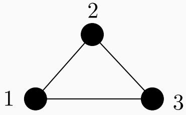

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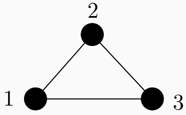

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Say $A = \{(2,3)\}$, $B = \{(1,3), (1,2)\}$

$x \in B - A$, for example, $x = (1,3)$

then $A \cup \{x\} = \{(2,3), (1,3)\} \subseteq \mathcal{I}$

Connection to spanning tree

Definition

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For connected undirected G, every maximal independent subset of M_G must be a tree with $|V| - 1$ edges. Hence it is a spanning tree

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Note: for graphic matroids, weight of M_G is corresponding to edge weights

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$\rightarrow M$ a assign weights

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- G with $w(e) \rightarrow M_G$ with $\boxed{w'(e) = c - w(e)}$ where c is a constant larger than the largest $w(e)$
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- For max-weighted independent subset A
  $w'(A) = (|V| - 1)c - w(A)$, so $w(A)$ is minimized

Hence a max-weighted indep. subset of M_G corresponds to an MST of G

Pseudocode for finding max-weighted independent subset

def GREEDY($M = (S, \mathcal{I})$, *weights w*)**:**

Pseudocode for finding max-weighted independent subset

def $\textsc{Greedy}$(*$M = (S, \mathcal{I})$, weights w*)**:**

  Set $A := \{\,\}$;

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def GREEDY(M = (S, I), weights w):
    Set A := { };
    Sort S in decreasing order of w ;            // O(n log n)
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    for x ∈ S:
        if A ∪ {x} ∈ I:
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def $\textsc{Greedy}(M = (S, \mathcal{I}),\ \textit{weights } w)$**:**

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    Sort S in decreasing order of w ; // $O(n \log n)$

    for $x \in S$**:**

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            $A := A \cup \{x\}$;

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    for $x \in S$**:**

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    return A;

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            $A := A \cup \{x\}$;

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Proof of correctness: Cormen et al. 16.4

Pseudocode for finding max-weighted independent subset

def $\textsc{Greedy}(M = (S, \mathcal{I}), \textit{weights } w)$**:**
- Set $A := \{\,\}$;
- Sort S in decreasing order of w ; // $O(n \log n)$
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Running time: let $n = |S|$

Pseudocode for finding max-weighted independent subset

```
def GREEDY(M = (S, I), weights w):
    Set A := { };
    Sort S in decreasing order of w ;          // O(n log n)
    for x ∈ S:
        if A ∪ {x} ∈ I:
            A := A ∪ {x};
    return A;
```

Proof of correctness: Cormen et al. 16.4

Running time: let $n = |S|$

Assume checking if $A \cup \{x\} \in \mathcal{I}$ takes $O(f(n))$.

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Total running time: $O(n \log n + n \cdot f(n))$

Application: task scheduling

Problem (Task scheduling)

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b d c a
✓ ✓ ✓ ✗

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task	a	b	c	d
deadline	1	1	4	2
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1	2	3	4	
b	d	a	c	penalty: 5
✓	✓	✗	✓	

1	2	3	4	
a	b	d	c	penalty: 13
✓	✗	✗	✓	

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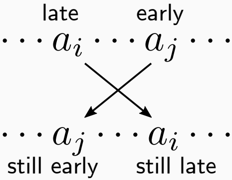

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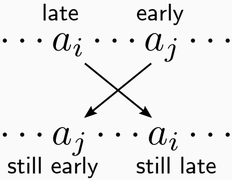

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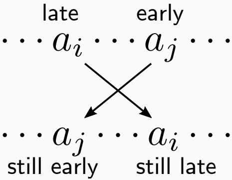

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Finding optimal schedule using matroid

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Finding an optimal schedule $\equiv$ finding max-weighted indep. subset of M