HW6 revision
score: 16/20

2 Quesetions

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Math 486

Lesson 6 Homework

Due Tues, June 28 at 11:59 on Gradescope

Instructions

Please refer to the solution guidelines posted on Canvas under Course Essentials.

Exercise 1.

(a) For the following game, use the fundamental theorem of Nash Equilibria discussed in the textbook and lectures to compute a mixed strategy Nash equilibrium.

		Player 2	
		x	y
Player 1	a	$2, 4$	$0, 0$
	b	$1, 6$	$3, 7$

(b) What is each player's expected payoff in the mixed strategy Nash equilibrium?

		Player 2 ($\frac{3}{4}$)	
		q x	$(1-q)$ y
$\frac{1}{5}$ **Player 1**	p a	$2, 4$	$0, 0$
	$(1-p)$ b	$1, 6$	$3, 7$

2 pure streategy Nash Equa

(a, x) and (b, y)

$\alpha_1 = (p, 1-p)$

$\alpha_2 = (q, 1-q)$

$\pi_1(a, \sigma_2) = \pi_1(b, \sigma_2)$

$2 \cdot q + 0 \cdot (1-q) = 1q + 3(1-q)$

$q = 3 - 3q$

$q = \frac{3}{4}$

$4p + 6(1-p) = 0p + 7(1-p)$

$4p + 6 - 6p = 7 - 7p$

$5p = 1$

$p = \frac{1}{5}$

$\sigma_1 = (\frac{1}{5}, \frac{4}{5})$ $\qquad$ $\sigma_2 = (\frac{3}{4}, \frac{1}{4})$

b) $\pi_1(p, q) = 2 \cdot \frac{1}{5} \cdot \frac{3}{4} + 0 \cdot \frac{1}{5} \cdot \frac{1}{4} + 1 \cdot \frac{4}{5} \cdot \frac{3}{4} + 3 \cdot \frac{4}{5} \cdot \frac{1}{4}$

$= 1.5$

$\pi_2(p, q) = 4 \cdot \frac{1}{5} \cdot \frac{3}{4} + 0 \cdot \frac{1}{5} \cdot \frac{1}{4} + 6 \cdot \frac{4}{5} \cdot \frac{3}{4} + 7 \cdot \frac{4}{5} \cdot \frac{1}{4}$

$= 5.6$

Exercise 2

Consider the two-player game shown below.

		Player 2		
		a	b	c
Player 1	a	$2,2$	$6,3$	$7,4$
	b	$3,6$	$3,3$	$9,4$
	c	$4,7$	$4,9$	$1,1$

Use the Fundamental Theorem of Nash Equilibria to calculate a mixed strategy Nash equilibrium, (σ, σ), where both players are using the same mixed strategy σ, and where all three strategies are active in σ.

$\pi_2(\sigma_1, a) = \pi_2(\sigma_1, b) = \pi_2(\sigma_1, c)$

$\pi_2(\sigma_1, a) = p_1(2) + p_2(6) + p_3(7) = 2p_1 + 6p_2 + 7p_3$

$\pi_2(\sigma_1, b) = 3p_1 + 3p_2 + 9p_3$

$\pi_2(\sigma_1, c) = 4p_1 + 4p_2 + 1p_3$

$p_1 = \frac{11}{2}p_3$

$p_2 = \frac{5}{2}p_3$

$\pi_1(a, \sigma_2) = 2q_1 + 6q_2 + 7q_3$

$\pi_1(b, \sigma_2) = 3q_1 + 3q_2 + 9q_3$

$\pi_1(c, \sigma_2) = 4q_1 + 4q_2 + 1q_3$

$\frac{11}{2}p_3 + \frac{5}{2}p_3 + p_3 = 1$

$\frac{18}{2}p_3 = 1$

$p_3 = \frac{1}{9}$

$p_1 = \frac{11}{18}$

$p_2 = \frac{5}{18}$

$\sigma_1 = (\frac{11}{18}, \frac{5}{18}, \frac{1}{9})$ $\sigma_2 = (\frac{11}{18}, \frac{5}{18}, \frac{1}{9})$

Problem 1

Consider the game below, where player 1 has three strategies and player 2 has two strategies.

		Player 2	
		x	y
Player 1	a	$2,5$	$6,4$
	b	$5,5$	$3,8$
	c	$3,7$	$4,9$

(a) Use the equality condition of the Fundamental Theorem of Nash Equilibria to show that there is no mixed strategy Nash equilibrium where player 1 has three active strategies.

(b) Find a mixed strategy Nash Equilibrium (σ_1, σ_2), where a and b are active in σ_1, c is not active in σ_1, and x and y are active in σ_2. Note that since c is not active you will need to check the inequality condition of the Fundamental Theorem of Nash Equilibria..

(c) Show that there is no mixed strategy Nash equilibrium (σ_1, σ_2) where a and c are active in σ_1 and b is inactive. One way to do this is to first look for a mixed strategy Nash equilibrium with a and c active to find a candidate for σ_2. You can then show that the inequality condition from the Fundamental Theorem of Nash Equilibria does not hold. It is not necessary to find a candidate σ_1 to conclude there is no mixed strategy Nash equilibrium of this type.

			Player 2	
			q	$1-q$
			x	y
Player 1	p	a	$2,5$	$6,4$
	$1-p$	b	$5,5$	$3,8$
	0	c	$3,7$	$4,9$

a) $\sigma_2 = (q, 1-q)$

$\pi_1(a, \sigma_2) = \pi_1(b, \sigma_2) = \pi_1(c, \sigma_2)$

$2q + 6(1-q) = 5q + 3(1-q) = 3q + 4(1-q)$

$q_1 = \frac{1}{2}$ $q_2 = \frac{1}{3}$

So No solution.

b)

$\sigma_1 = (p, 1-p)$

$\pi_2(\sigma_1, x) = \pi_2(\sigma_1, y)$

$5p + 5(1-p) = 4p + 8(1-p)$

$p = 3 - 3p$

$p = \frac{3}{4}$

(σ_1, σ_2) is a MSNE candidate.

$\sigma_1 = (\frac{3}{4}, \frac{1}{4}, 0)$ $\sigma_2 = (\frac{1}{2}, \frac{1}{2})$

$\pi_1(\sigma_1, \sigma_2) = \pi_1(a, \sigma_2) = \pi_1(b, \sigma_2)$

$= 2 \cdot \frac{1}{2} + 6 \cdot \frac{1}{2} = 5 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 4$

$\pi_1(c, \sigma_1) = 3 \cdot \frac{1}{2} + 4 \frac{1}{2} = 3.5$

since $\pi_1(c, \sigma_1) \leq \pi_1(\sigma_1, \sigma_2)$

So (σ_1, σ_2) is MSNE

c) $\pi_1(a, \sigma_2) = \pi_1(c, \sigma_2)$

$\sigma_1 = (p, 0, 1-p)$
$\sigma_2 = (q, 1-q)$

$2 \cdot q + 6 \cdot (1-q) = 3 \cdot q + 4 \cdot (1-q)$

$2 - 2q = q$

$q = \frac{2}{3}$

$\sigma_1 = (\frac{2}{3}, 0, \frac{1}{3})$

$\pi_2(\sigma_1, x) = \pi_2(\sigma_1, y)$

$\sigma_2 = (\frac{2}{3}, \frac{1}{3})$

$5p + 7(1-p) = 4p + 9(1-p)$

$p = 2 - 2p$

$p = \frac{2}{3}$

$\pi_1(\sigma_1, \sigma_2) = \pi_1(a, \sigma_2) = \pi_1(c, \sigma_2)$

$= 2 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} = 3 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} = \frac{10}{3}$

$\pi_1(b, \sigma_2) = 5 \cdot \frac{2}{3} + \frac{1}{3} \cdot 3 = \frac{13}{3}$

$\pi_1(\sigma_1, \sigma_2) \leq \pi_1(b, \sigma_2)$ So (σ_1, σ_2) is not MSNE.

Problem 2.

Consider the following prisoner's dilemma game where each player can choose to cooperate (C) or defect (D).

		Player 2	
		C	D
Player 1	C	$3, 3$	$-1, 5$
	D	$5, -1$	$0, 0$

In this problem we will extend this Prisoner's Dilemma to a game played over "two stages". Players engage in the prisoner's dilemma shown above twice and the total payoff is the sum of the payoffs for each stage. Players know the outcome of the first stage before the second stage commences.

Because the game is played over two stages, we will consider the choices C and D as **moves** that can be taken in each stage. A strategy for the two-stage game would need to indicate what move should be taken in each stage. Furthermore, the move chosen for the second stage could depend on the outcome of the first stage.

We will consider the following three strategies for the two-stage game (more strategies are possible):

- Always cooperate: (F)
- Always defect: (S)
- Tit-for-tat: (P) Choose C in stage 1. In stage 2, copy the other player's move from stage 1.

Restricting the players to these 3 strategies gives us the following normal form game:

		Player 2		
		F	S	P
Player 1	F	$6, 6$	$-2, 10$	$6, 6$
	S	$10, -2$	$0, 0$	
	P	$6, 6$		$6, 6$

(a) The payoffs for the strategy profiles (S, P) and (P, S) have been left out of the payoff matrix. Determine these payoffs and include them in the payoff matrix.

(b) Use best responses to find any pure strategy Nash equilibria.

(c) Suppose there is a mixed strategy Nash equilibrium (σ_1, σ_2) where for each player the strategies F and P are active and the strategy S is inactive, so that $\sigma_1 = (p, 0, 1-p)$ and $\sigma_2 = (q, 0, 1-q)$, with $0 < p, q < 1$. Show that the equality condition from the Fundamental theorem of Nash equilibria holds for any values of p and q in $(0, 1)$.

Remark: You will be able to show that the equality condition holds for any p and q, but, by assumption, we restrict to valid probabilities that indicate both F and P are active.

(d) Use the inequality condition from the Fundamental Theorem of Nash Equilibria to determine the range of p and q where (σ_1, σ_2) is a mixed strategy Nash equilibrium.

a)

		Player 2		
		F	S	P
Player 1	F	$6, 6$	$-2, \underline{10}$	$\underline{6}, 6$
	S	$\underline{10}, -2$	$\underline{0}, \underline{0}$	$5, -1$
	P	$6, \underline{6}$	$-1, 5$	$\underline{6}, \underline{6}$

b) 2 pure Nash Equ (S, S) and (P, P)

c) $(p, 0, 1-p)$

$\pi_1(F, \sigma_2) = 6q + 6(1-q) = 6$ $\quad$ $\pi_2(\sigma_1, F) = 6p + 6(1-p) = 6$

$\pi_1(P, \sigma_2) = 6q + 6(1-q) = 6$ $\quad$ $\pi_2(\sigma_1, P) = 6p + 6(1-p) = 6$

they equal each other

so exist a mixed strategy Nash equ. (σ_1, σ_2)

d) S is inactive

so $\pi_1(S, \sigma_2) = 5q + 5 \leq 6$ $\quad$ $\pi_1(\sigma_1, \sigma_2) = 6$

$5q \leq 1$

$\underline{q \leq \frac{1}{5}}$

So (σ_1, σ_2) is a MSNE $\quad$ $\sigma_1 = (p, 0, 1-p)$ $\quad$ $\sigma_2 = (q, 0, 1-q)$

with $0 < p \leq \frac{1}{5}$ $\quad$ $0 \leq q \leq \frac{1}{5}$