CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

NP and Computational Hardness

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Polynomial-time reduction
(Kleinberg-Tardos, Section 8.1, 8.2)

Motivation

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Is Problem B really hard? How do we prove hardness?

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Tool: **polynomial-time** reduction

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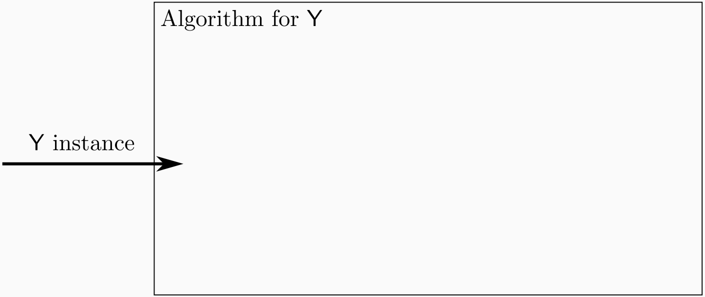

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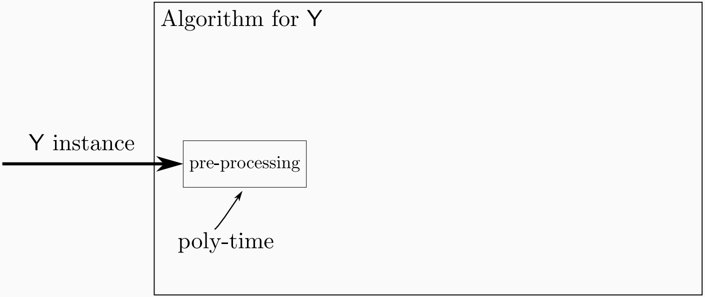

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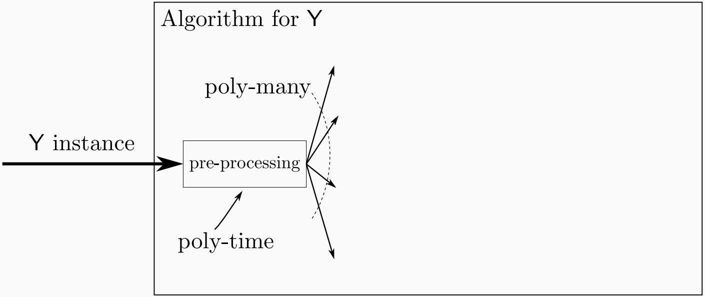

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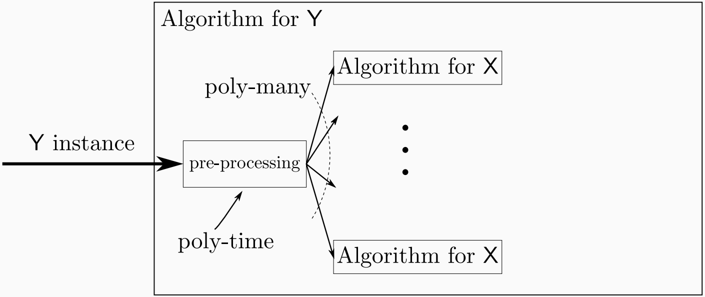

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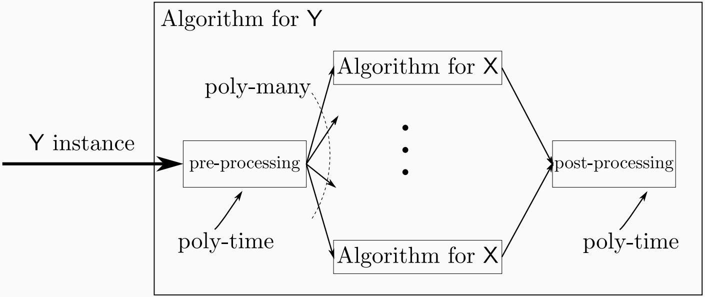

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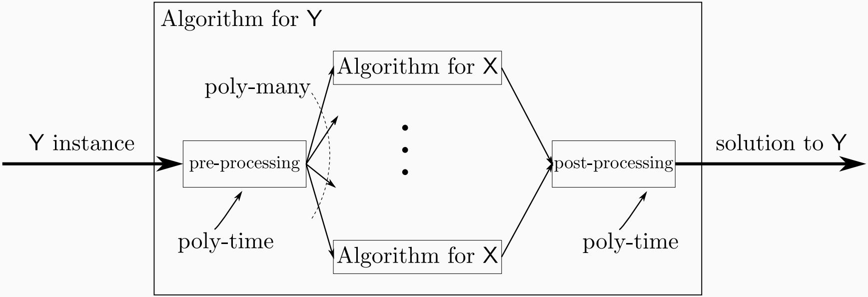

Consequences of polynomial time reductions

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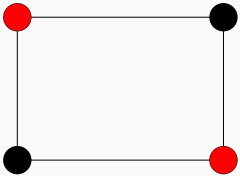

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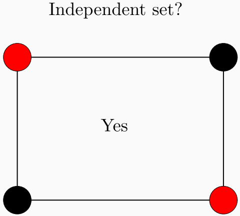

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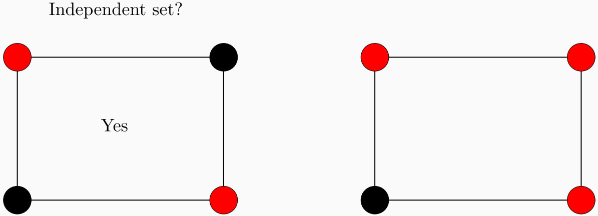

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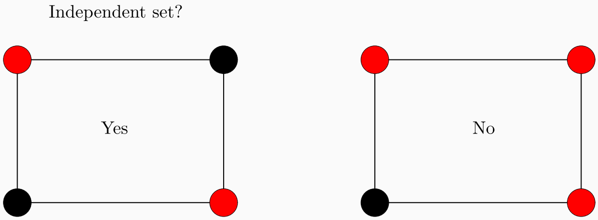

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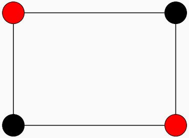

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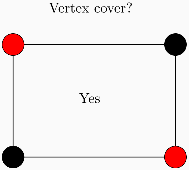

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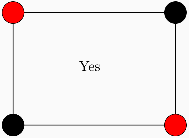

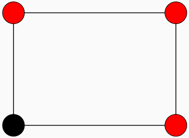

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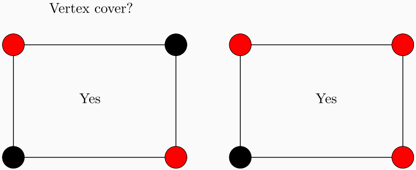

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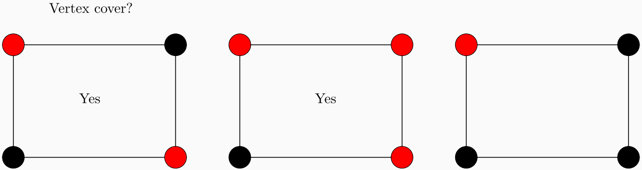

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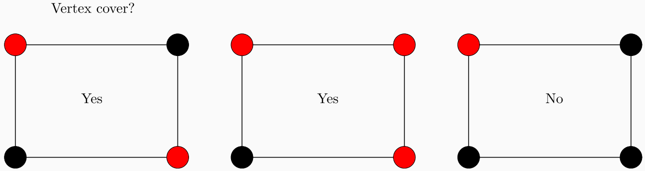

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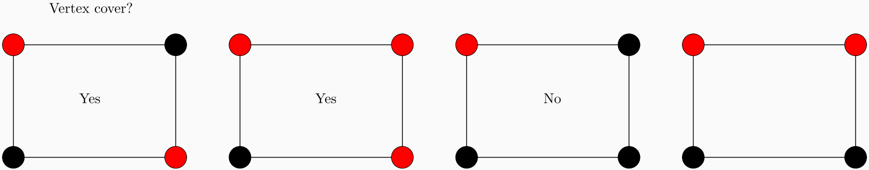

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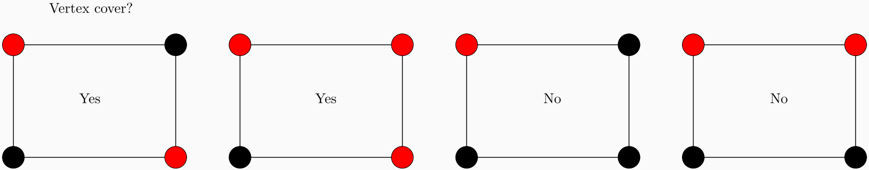

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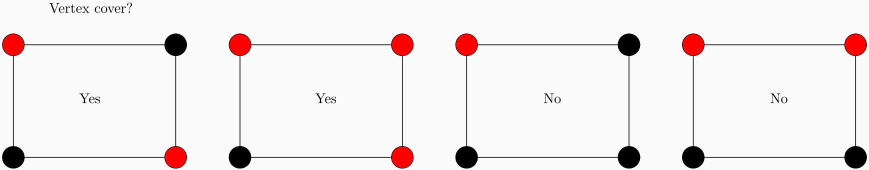

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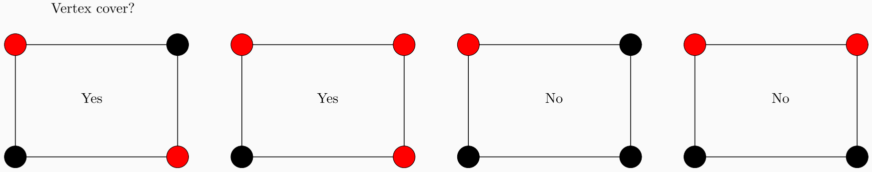

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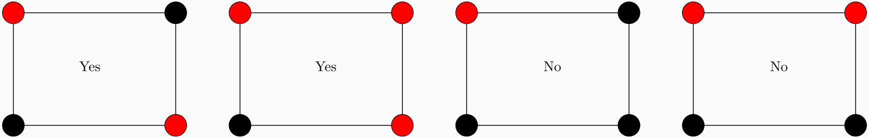

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Independent Set vs Vertex Cover (II)

Theorem

- *Independent Set $\leq_P$ Vertex Cover*

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Theorem

- *Independent Set* $\leq_P$ *Vertex Cover*
- *Vertex Cover* $\leq_P$ *Independent Set*