Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

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Using linked list:

$$\{a, b, c\} \quad \text{head} \to a \to b \to c \qquad \text{find\_set}(b):$$

Running time of Kruskal's algorithm (I)

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Using linked list:

$$\{a, b, c\} \quad \text{head} \to a \to b \to c \qquad \text{find\_set}(b)\text{: } O(1)$$

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Using linked list:

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$\text{make\_set}(v)\text{:}$

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Using linked list:

$\{a, b, c\}$ $\quad \text{head} \to a \to b \to c$ $\qquad$ $\text{find\_set}(b)$: $O(1)$

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Using linked list:

$$\{a, b, c\} \quad \text{head} \to a \to b \to c \qquad \text{find\_set}(b)\text{: } O(1)$$

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$$\{d, e\} \quad \text{head} \to d \to e$$

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union(a, b)

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union(a, b) $\qquad$ $\text{head} \to a \to b \to c \to d \to e$

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$$\{d, e\} \quad \text{head} \to d \to e$$

$$\text{union}(a, b) \qquad \text{head} \to a \to b \to c \to d \to e$$

Cost of union:

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

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$\text{union}(a, b)$ $\qquad \text{head} \to a \to b \to c \to d \to e$

Cost of union: O(length of the shorter list)

Running time of Kruskal's algorithm (I)

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$\text{union}(a, b) \qquad \text{head} \to a \to b \to c \to d \to e$

Cost of union: O(length of the shorter list)

Using an array to implement it:

vertex	1	2	3	4	5	$\xrightarrow{\text{union}}$
head	1	1	1	4	4	

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

Using linked list:

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head → d → e → a → b → c

$\text{union}(a, b)$ $\qquad$ $\text{head} \to a \to b \to c \to d \to e$

Cost of union: O(length of the shorter list)

Using an array to implement it:

vertex	1	2	3	4	5
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$\xrightarrow{\text{union}}$

vertex	1	2	3	4	5
head	1	1	1	1	1

Running time of Kruskal's algorithm (II)

def KRUSKAL_MST(*undirected* $G = (V, E)$, *weights* $w = (w_e)_{e \in E}$)**:**

  Set $A := \{\,\}$;

  for $v \in V$**:**

    make_set(v)

  Sort E in increasing order of edge weights

  for $(u, v) \in E$**:**

    if $\mathrm{find\_set}(u) \neq \mathrm{find\_set}(v)$**:**

      $A := A \cup \{(u, v)\}$;

      $\mathrm{union}(u, v)$;

Running time of Kruskal's algorithm (II)

def KRUSKAL_MST(*undirected* $G = (V, E)$, *weights* $w = (w_e)_{e \in E}$):

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    Sort E in increasing order of edge weights

    for $(u, v) \in E$:

        if $\text{find\_set}(u) \neq \text{find\_set}(v)$:

            $A := A \cup \{(u, v)\}$;

            $\text{union}(u, v)$;

$|E| = O(|V|^2)$

$O(|E| \log |E|) = O(|E| \log |V|)$

$\log |V|^2 = 2 \log |V|$

Running time of Kruskal's algorithm (II)

def KRUSKAL_MST(*undirected* $G = (V, E)$, *weights* $w = (w_e)_{e \in E}$)**:**

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    for $v \in V$**:**
        make_set(v) ; // $O(|V|)$
    Sort E in increasing order of edge weights ; // $O(|E| \log |V|)$
    for $(u, v) \in E$**:**
        if $\mathrm{find\_set}(u) \neq \mathrm{find\_set}(v)$**:**
            $A := A \cup \{(u, v)\}$;
            union(u, v); worst case? $O(|V|)$

} $O(|E| \cdot |V|)$

Running time of Kruskal's algorithm (II)

```
def KRUSKAL_MST(undirected G = (V, E), weights w = (w_e)_{e∈E}):
    Set A := { };
    for v ∈ V:
        make_set(v) ;                                         // O(|V|)
    Sort E in increasing order of edge weights ;              // O(|E| log |V|)
    for (u, v) ∈ E:
        if find_set(u) ≠ find_set(v):
            A := A ∪ {(u, v)};
            union(u, v);
```

Worst-case cost for union:

Running time of Kruskal's algorithm (II)

```
def KRUSKAL_MST(undirected G = (V, E), weights w = (w_e)_{e∈E}):
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Worst-case cost for union: $O(|V|)$.

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Worst-case cost for union: $O(|V|)$. What about the cost for lines 6-9?

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Worst-case cost for union: $O(|V|)$. What about the cost for lines 6-9?

Consider a single $v \in V$. Once it's touched in some union operation, the size of the set at least doubles.

Running time of Kruskal's algorithm (II)

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1 def Kruskal_MST(undirected G = (V, E), weights w = (w_e)_{e∈E}):
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Worst-case cost for union: $O(|V|)$. What about the cost for lines 6-9? Consider a single $v \in V$. Once it's touched in some union operation, the size of the set at least doubles. Since the maximum size of a set can be $|V|$, each v is touched at most $O(\log |V|)$ times

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At most $|V|$ vertices are involved in union operations, so the total cost of lines 6-9:

Running time of Kruskal's algorithm (II)

1 **def** KRUSKAL_MST(*undirected* $G = (V, E)$, *weights* $w = (w_e)_{e \in E}$):
2   Set $A := \{\}$;
3   **for** $v \in V$:
4   make_set(v) ; // $O(|V|)$
5   Sort E in increasing order of edge weights ; // $O(|E| \log |V|)$
6   **for** $(u, v) \in E$:
7   **if** $\text{find\_set}(u) \neq \text{find\_set}(v)$:
8             $A := A \cup \{(u, v)\}$;
9   union(u, v);

$O(|V| \log |V|)$

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At most $|V|$ vertices are involved in union operations, so the total cost of lines 6-9: $O(|V|\log|V|)$

Total cost of the algorithm:

Running time of Kruskal's algorithm (II)

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2     Set A := { };
3     for v ∈ V:
4         make_set(v) ;                                      // O(|V|)
5     Sort E in increasing order of edge weights ;           // O(|E| log |V|)
6     for (u, v) ∈ E:
7         if find_set(u) ≠ find_set(v):    worst case O(|E|+n)
8             A := A ∪ {(u, v)};
9             union(u, v);
```

Worst-case cost for union: $O(|V|)$. What about the cost for lines 6-9?

Consider a single $v \in V$. Once it's touched in some union operation, the size of the set at least doubles. Since the maximum size of a set can be $|V|$, each v is touched at most $O(\log |V|)$ times

At most $|V|$ vertices are involved in union operations, so the total cost of lines 6-9: $O(|V| \log |V|)$

Total cost of the algorithm: $O(|E| \log |V|)$

Alternative data structure

The linked-list implementation is good enough, but there exist better data structures to improve the worst-case cost for union

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Directed tree disjoint set:

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Directed tree disjoint set:

$$\{a\}$$

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Directed tree disjoint set:

$\{a\}$ $\quad a$

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Directed tree disjoint set:

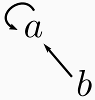
$$\{a\} \quad a \quad \{a, b\}$$

Alternative data structure

The linked-list implementation is good enough, but there exist better data structures to improve the worst-case cost for union

Directed tree disjoint set:

$\{a\}$ 

$\{a, b\}$ 

Alternative data structure

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Directed tree disjoint set:

$$\{a\} \quad \begin{tikzcd} a \arrow[loop left] \end{tikzcd} \qquad \{a, b\} \quad \begin{tikzcd} a \arrow[loop left] & \\ & b \arrow[ul] \end{tikzcd} \qquad \{a, b, c\}$$

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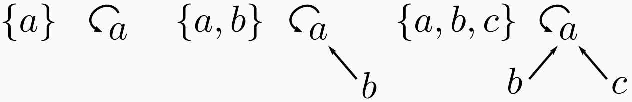

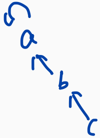

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Directed tree disjoint set:

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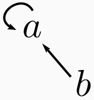
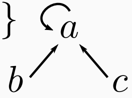
$\{a, b\}$ 

$\{a, b, c\}$ 

$\{a, b, c, d, e\}$

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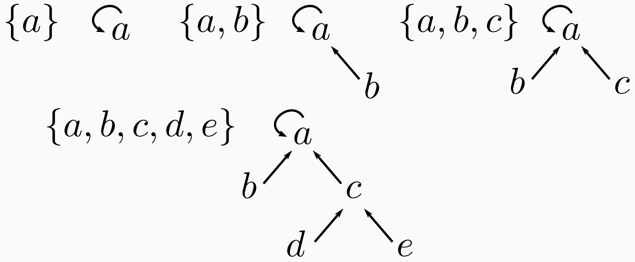

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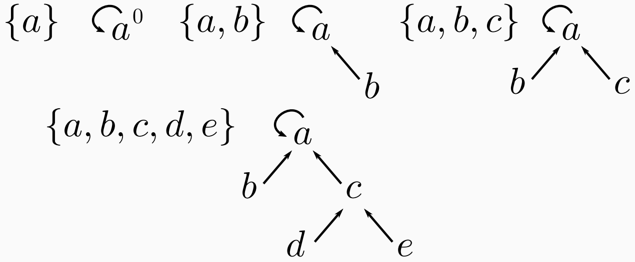

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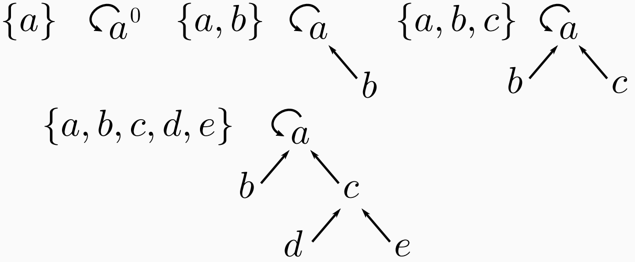

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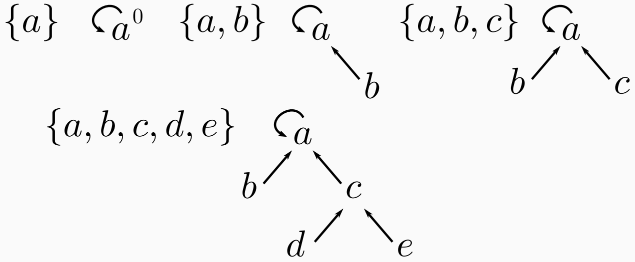

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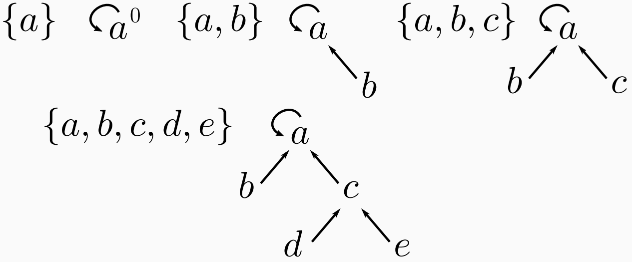

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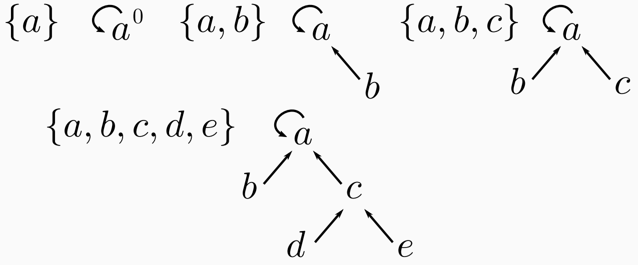

Definition

$\pi(x)$: parent of x

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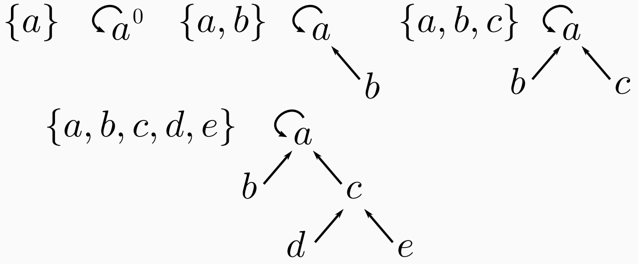

Definition

$\pi(x)$: parent of x

root node: x s.t. $\pi(x) = x$

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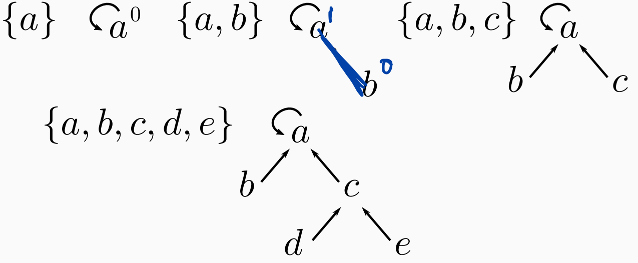

Definition

$\pi(x)$: parent of x

root node: x s.t. $\pi(x) = x$

$\mathrm{rank}(x)$: number of the edges in the longest simple path from x to a leaf

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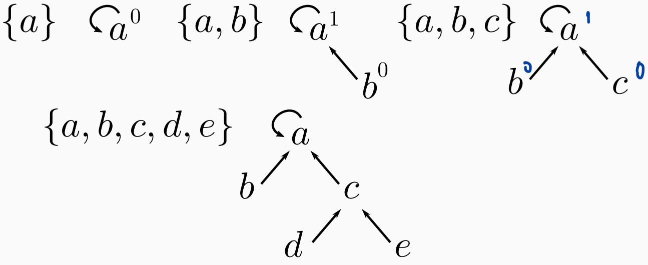

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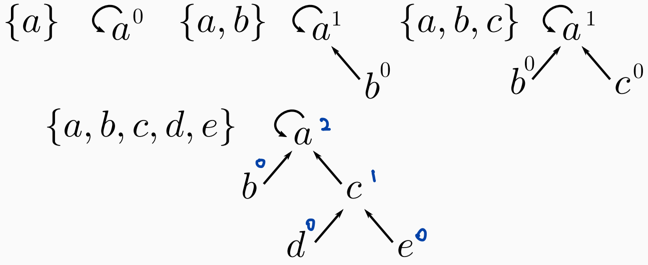

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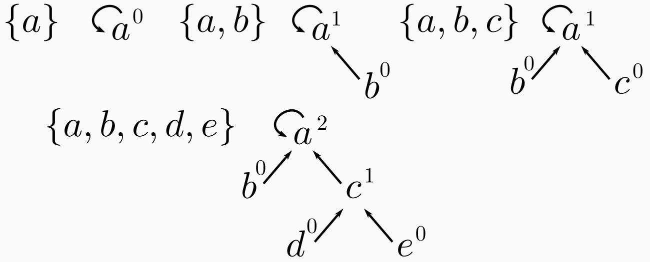

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Operations of direct tree disjoint set (I)

- $\mathrm{make\_set}(v)$

Operations of direct tree disjoint set (I)

- make_set(v)

  def make_set(v)**:**

  $\pi(v) := v$;

  $\text{rank}(v) = 0$;

v^0

Operations of direct tree disjoint set (I)

- $\text{make\_set}(v)$

  def $\text{make\_set}(v)$**:**

  $\pi(v) := v$;
  $\text{rank}(v) = 0$;

  Cost: $O(1)$

Operations of direct tree disjoint set (I)

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  def $\text{make\_set}(v)$**:**

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  $\quad \text{rank}(v) = 0;$

  Cost: $O(1)$

- $\text{find\_set}(v)$

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  $\quad \text{rank}(v) = 0;$

  Cost: $O(1)$

- $\text{find\_set}(v)$

  def ~~make~~ find $\text{\_set}(v)$**:**

  $\quad$ **while** $v \neq \pi(v)$**:**

  $\quad\quad v := \pi(v);$

  $\quad$ return v

Operations of direct tree disjoint set (I)

- $\mathrm{make\_set}(v)$

  def $\mathrm{make\_set}(v)$**:**

  $\pi(v) := v$;

  $\mathrm{rank}(v) = 0$;

  Cost: $O(1)$

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  def $\mathrm{make\_set}(v)$**:**

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  $v := \pi(v)$;

  Cost: $O(\text{depth of the node in the tree})$

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- $\mathrm{make\_set}(v)$

  def $\mathrm{make\_set}(v)$**:**

  $\pi(v) := v$;

  $\mathrm{rank}(v) = 0$;

  Cost: $O(1)$

- $\mathrm{find\_set}(v)$

  def $\mathrm{make\_set}(v)$**:**

  while $v \neq \pi(v)$**:**

  $v := \pi(v)$;

  Cost: O(depth of the node in the tree)

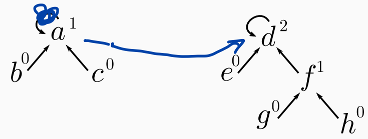
- what about union?

Operations of direct tree disjoint set (II)

- union:

Operations of direct tree disjoint set (II)

- union:



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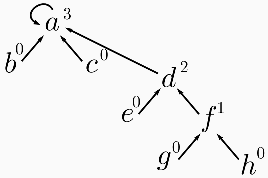

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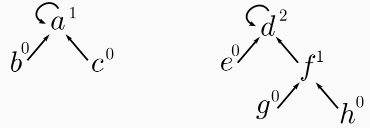

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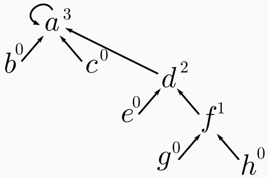
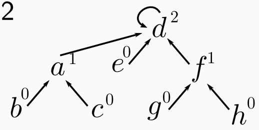


Option 2



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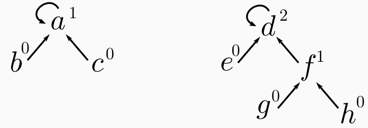

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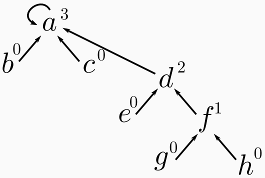

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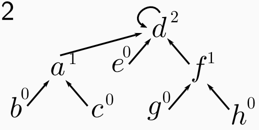

better!

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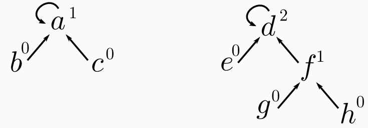

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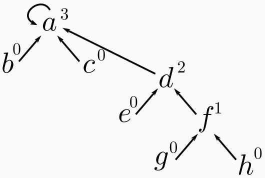

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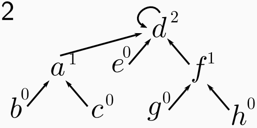

better!

Basic idea: attach the smaller ranked tree to a larger one

Operations of direct tree disjoint set (II)

def union(x, y)**:**

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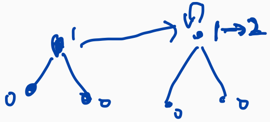

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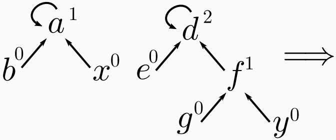

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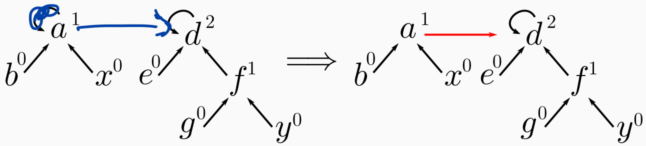

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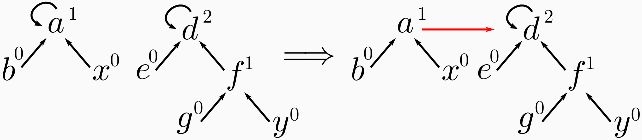

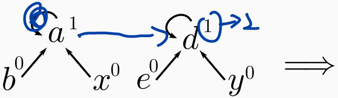

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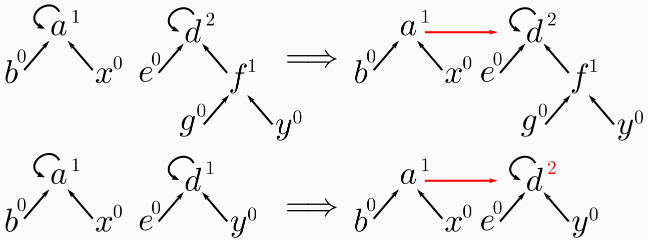

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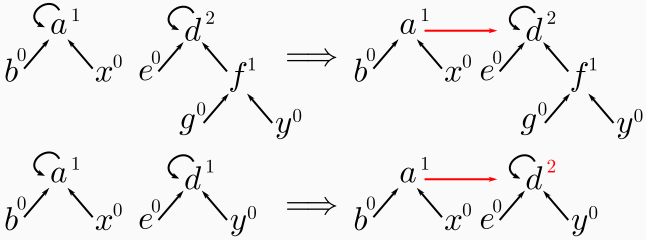
Cost: dominated by find_set



Cost of find_set using directed tree disjoint set

Observation

Root note with rank k is formed by the merge of two rank $k-1$ trees

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Lemma

Any root node of rank k has at least 2^k nodes in it

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Proof.

By induction: base case has $k = 0$ and $2^0 = 1$.

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$\le \log|V|$ $\le |V|$

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$|V|$ nodes, max rank ?

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By the lemma, if we have $|V|$ nodes, the maximum rank is $\log|V|$. So

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Total running time of Kruskal using directed tree disjoint set

def KRUSKAL_MST(*undirected* $G = (V, E)$, *weights* $w = (w_e)_{e \in E}$)**:**

  Set $A := \{\}$;

  for $v \in V$**:**

    make_set(v) ; // $O(|V|)$

  Sort E in increasing order of edge weights ; // $O(|E| \log |V|)$

  for $(u, v) \in E$**:**

    if find_set(u) $\neq$ find_set(v)**:** $\log|V|$

      $A := A \cup \{(u, v)\}$;

      union(u, v); $\log|V|$

$O(|E| \log |V|)$

Total running time of Kruskal using directed tree disjoint set

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2     Set A := { };
3     for v ∈ V:
4         make_set(v) ;                                        // O(|V|)
5     Sort E in increasing order of edge weights ;             // O(|E| log |V|)
6     for (u, v) ∈ E:
7         if find_set(u) ≠ find_set(v):
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Lines 6-9:

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Prim's algorithm

Intuition: iteratively grows the tree

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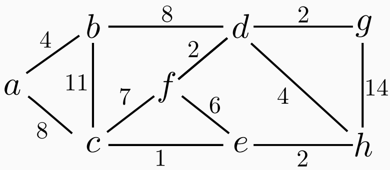
Kruskal



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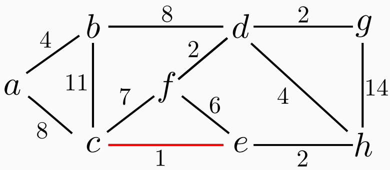

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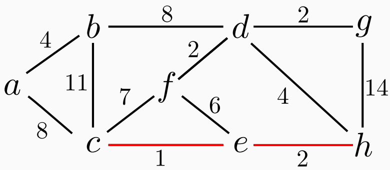

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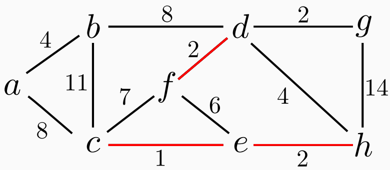

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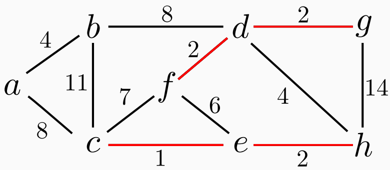

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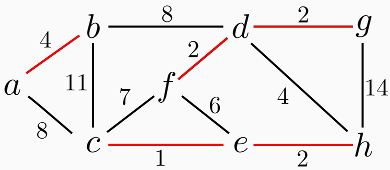

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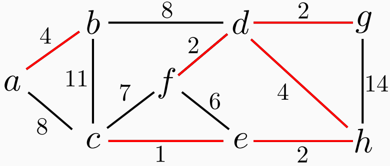

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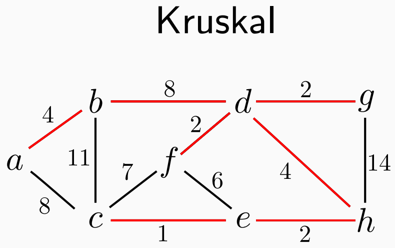

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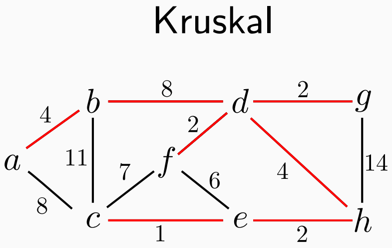

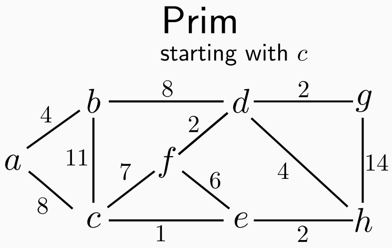

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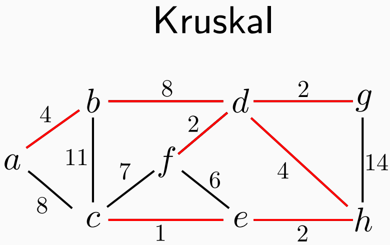

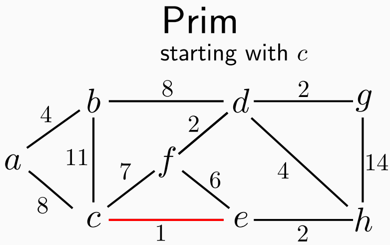

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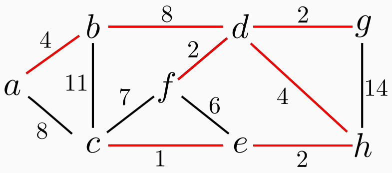
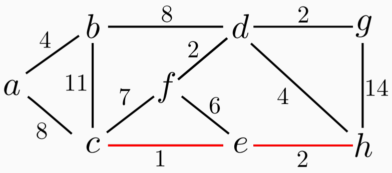


Prim

starting with c



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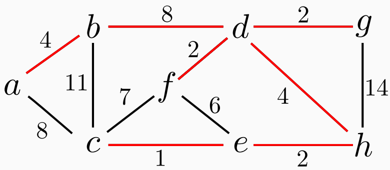

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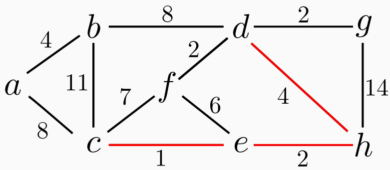

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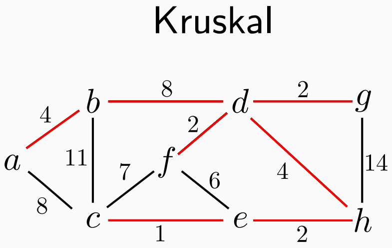

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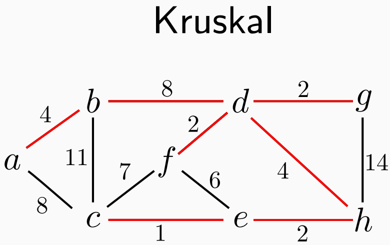
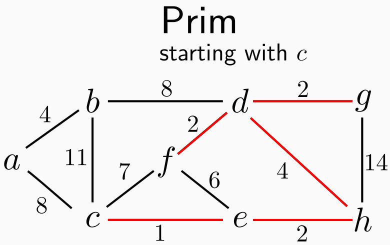
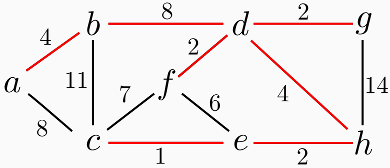




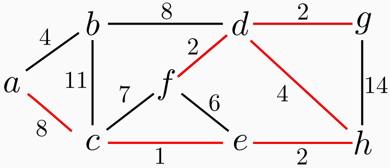
Intuition: iteratively grows the tree

Kruskal



Prim

starting with c



Prim's algorithm

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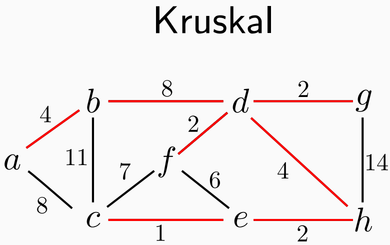

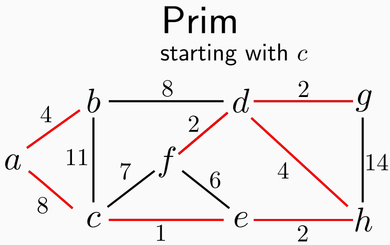

Prim's algorithm: pseudocode

Let S be the set included in the tree so far

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$$\mathrm{cost}(v) := \min_{e=(u,v) \text{ s.t. } u \in S} w_e$$

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$\mathrm{cost}(v) := \min_{e=(u,v) \text{ s.t. } u \in S} w_e$ and $\mathrm{prev}(\cdot)$ is used to keep track of the tree

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def $\textsc{Prim\_MST}$*(undirected $G = (V, E)$, weights $w = (w_e)_{e \in E}$)***:**

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def $\textsc{Prim\_MST}$*(undirected $G = (V, E)$, weights $w = (w_e)_{e \in E}$)***:**

    for $v \in V$**:**

        $\mathrm{cost}(v) := \infty;$

        $\mathrm{prev} := \mathrm{nil};$

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  Pick any initial vertex u_0;

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    for $v \in V$**:**

        $\mathrm{cost}(v) := \infty$;

        $\mathrm{prev} := \mathrm{nil}$;

    Pick any initial vertex u_0;

    $\mathrm{cost}(u_0) := 0$;

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for v ∈ V:
    cost(v) := ∞;
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Pick any initial vertex u_0;
cost(u_0) := 0;
H := make_queue(V) ;                                  // keys are cost(v)
while H is not empty:
    v = delete_min(H);
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- $\mathrm{cost}(u_0) := 0$;
- $H := \mathrm{make\_queue}(V)$; // keys are $\mathrm{cost}(v)$
- **while** H *is not empty*:
  - $v = \mathrm{delete\_min}(H)$;
  - **for** $e := (v, z) \in E$:

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def PRIM_MST(undirected G = (V, E), weights w = (w_e)_{e∈E}):
    for v ∈ V:
        cost(v) := ∞;
        prev := nil;
    Pick any initial vertex u_0;
    cost(u_0) := 0;
    H := make_queue(V) ;                         // keys are cost(v)
    while H is not empty:
        v = delete_min(H);
        for e := (v, z) ∈ E:
            if cost(z) > w_e:
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    $H := \mathrm{make\_queue}(V)$;   // keys are $\mathrm{cost}(v)$
    while *H is not empty*:
        $v = \mathrm{delete\_min}(H)$;
        for $e := (v, z) \in E$:
            if $\mathrm{cost}(z) > w_e$:
                $\mathrm{cost}(z) := w_e$;

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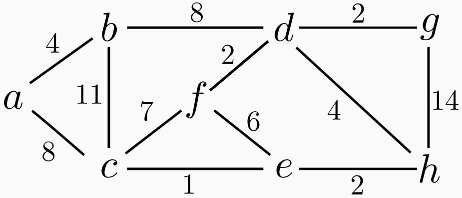
    for v ∈ V:
        cost(v) := ∞;
        prev := nil;
    Pick any initial vertex u_0;
    cost(u_0) := 0;
    H := make_queue(V) ;                                    // keys are cost(v)
    while H is not empty:
        v = delete_min(H);
        for e := (v, z) ∈ E:
            if cost(z) > w_e:
                cost(z) := w_e;
                prev(z) := v;
```

Prim's algorithm: a running example

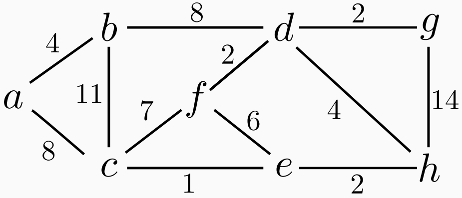
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil

Prim's algorithm: a running example

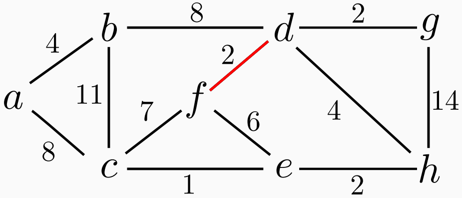
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil

Prim's algorithm: a running example

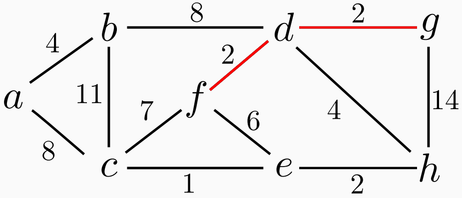
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$

Prim's algorithm: a running example

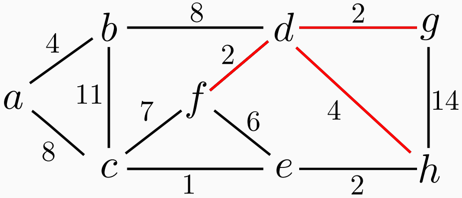
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$

Prim's algorithm: a running example

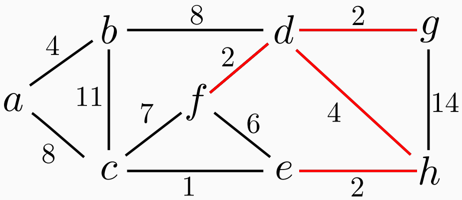
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$
f, d, g, h	∞/nil	$8/d$	$7/f$		$2/h$			

Prim's algorithm: a running example

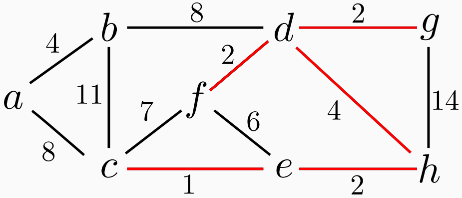
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$
f, d, g, h	∞/nil	$8/d$	$7/f$		$2/h$			
f, d, g, h, e	∞/nil	$8/d$	$1/e$					

Prim's algorithm: a running example

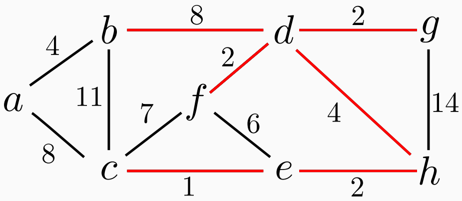
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$
f, d, g, h	∞/nil	$8/d$	$7/f$		$2/h$			
f, d, g, h, e	∞/nil	$8/d$	$1/e$					
f, d, g, h, e, c	$8/c$	$8/d$						

Prim's algorithm: a running example

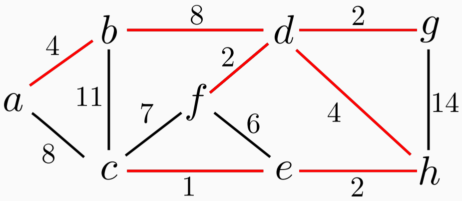
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$
f, d, g, h	∞/nil	$8/d$	$7/f$		$2/h$			
f, d, g, h, e	∞/nil	$8/d$	$1/e$					
f, d, g, h, e, c	$8/c$	$8/d$						
f, d, g, h, e, c, b	$4/b$							

Prim's algorithm: a running example

Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$
f, d, g, h	∞/nil	$8/d$	$7/f$		$2/h$			
f, d, g, h, e	∞/nil	$8/d$	$1/e$					
f, d, g, h, e, c	$8/c$	$8/d$						
f, d, g, h, e, c, b	$4/b$							
f, d, g, h, e, c, b, a								

Greedy algorithms

Huffman Encoding (Textbook Section 5.2)