Math 486 Midterm

Summer 2022, June 30–July 5

Please read the following guidelines.

- Complete all of the following problems.
- Please use adequate space for each problem. Do not try to put solutions to more than one problem on a single page. You can include multiple parts of a single problem on one page, but keep problems readable.
- The work that you submit must be your own and should reflect your own understanding of the material.
- The problems are all equally weighted but they are not equal in length. Some problems may be more involved than others.
- Midterm Solutions are Due on Gradescope at 11:59pm on Tuesday July 5th (EDT UTC-4).

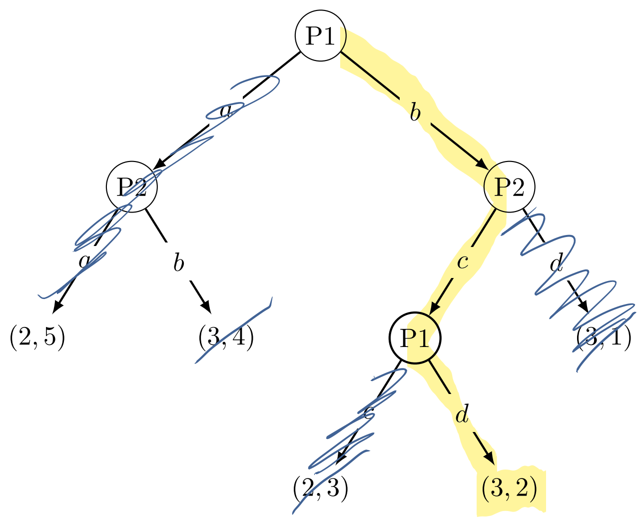
1. A 2-player extensive form game with the players $P1$ and $P2$ is shown below. Payoffs are given in the usual order $(P1, P2)$.

   (a) Analyze the game using backward induction and indicate the path through the game tree that will be taken (according to your analysis). It is not necessary to provide explanations for each step of the backward induction.

   (b) Create a normal form game corresponding to the extensive form game.

   (c) Find any Nash equilibria in the normal form game.

   (d) Which Nash equilibria from the normal form game, if any, correspond to the outcome found through backward induction? Briefly explain.



P2

		ac	ad	bc	bd
P1	ac	2,5	2,5	3,4	3,4
	ad	2,5	2,5	3,4	3,4
	bc	2,3	3,1	2,3	3,1
	bd	3,2	3,1	3,2	3,1

Nash equ. (bd,ac)
(bd, bc)

payoff is (3,2)

both Nash Equ. from normal form correspond to the outcome found through backward induction.

2. Consider the two player game shown below where Player 1 has the strategies a and b and Player 2 has the strategies c and d, and where x is a positive parameter, $x \in (0, \infty)$.

		Player 2 c	d
Player 1	a	$x+2, 5$	$4, 0$
	b	$x, 7$	x, x

(a) Find the set of values for x where player 1 has a strictly dominated strategy. State which strategy is strictly dominated.

(b) Find the set of values for x where player 2 has a strictly dominated strategy. State which strategy is strictly dominated.

(c) Find the set of values for x where neither player has a strictly dominated strategy.

(d) Choose a value of x where one of the players has a weakly, but not strictly, dominated strategy. What is the rational outcome of the game if that player does not play their weakly dominated strategy (assuming common knowledge of rationality)?

a) when $x \in (0,4)$, strategy b has been strictly dominated by strategy a

b) when $x \in (0,7)$, strategy d has been strictly dominated by strategy c

c) when $x \in [7,\infty)$, neither player has a strictly dominated strategy.

d) when $x = 7$, strategy d has been weakly dominated by strategy c

		P2 c	d
P1	a	9,5	4,0
	b	7,7	7,7

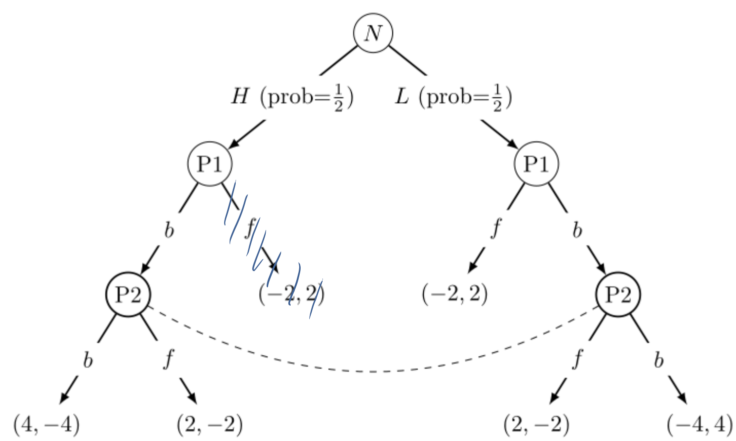
rational cell : (a, c)

So p2 will play d, player will choose b, with out come (7,7)

3. In the Lesson 5 Homework we considered a one-card, one-round poker game. We also considered a similar game with two rounds in the Lesson 6 lectures.

   Here are the details for the one-card, one-round poker game. The extensive form game is shown below.

   - There are two players, Player 1 and Player 2 and a deck with two cards: H (high) and L (low).
   - The players each put \$2 into the pot. One card is dealt at random to player 1. Player 1 knows their card, player 2 does not get a card and does not know player 1's card.
   - Player 1 can bet \$2 or fold. If player 1 folds, the game ends.
   - If Player 1 bets, Player 2 can then bet \$2 or fold. If player 2 folds, the game ends.
   - If both players bet, and player 1 has H, then Player 1 wins the pot (\$4).
   - If both players bet, and player 1 has L, then Player 2 wins the pot (\$4)



   In the homework, we created a Bayesian normal form game where Player 1 has four strategies, $S_1 = \{bb, bf, fb, ff\}$ and we demonstrated that this normal form game has no pure strategy Nash equilibria.

   (a) Argue that player 1 should never fold when they have the high card, H. You can make this argument directly in the extensive form game or you can use elimination of dominated strategies in the Bayesian normal form.

   (b) Following from part (a), write out the resulting Bayesian normal form game where player 1 has just two strategies.

   (c) As we saw in the homework, there should be no pure strategy Nash equilibria. Find a mixed strategy Nash equilibrium.

   (d) Provide an interpretation of each player's mixed strategy in the equilibrium.

   (e) In the equilibrium, when player 1 bids, player 2 should assume there is a 3/4 chance that player 1 has H. Use this to show that in the equilibrium, when player 1 bids, player 2's expected payoff from bidding is equal to their expected payoff from folding.

   Remark: You can use player 1's mixed strategy and Bayes Rule to show that when player 1 bids, player 2 should believe there is a 3/4 chance that player 1 has the high card (but I am not asking you to show this - you can take it as given). This is an example of the way players perform Bayesian updating in response to the actions of other players.

P2

H (P1)	b	f
bb	4, -4	2, -2
bf	4, -4	2, -2
fb	-2, 2	-2, 2
ff	-2, 2	-2, 2

P2

L (P1)	b	f
bb	-4, 4	2, -2
bf	-2, 2	-2, 2
fb	-4, 4	2, -2
ff	-2, 2	-2, 2

a) For player 1, fb&ff is strongly dominated by bb&bf during H, So player 1 should never fold if he have H.

P2

P1	b	f
bb	0, 0	2, -2
bf	1, -1	0, 0
fb	-3, 3	0, 0
ff	-2, 2	-2, 2

P2

H (P1)	b	f
bb	4, -4	2, -2
bf	4, -4	-2, 2

P2

L (P1)	b	f
bb	-4, 4	2, -2
bf	-2, 2	-2, 2

$(p, 1-p)$

$4p + 4(1-p) = -2p - 2(1-p)$

$\cancel{4p} + 4 - \cancel{4p} = -2p$

$p = -2$

P2

P1	b	f
bb	0, 0	2, -2
bf	1, -1	0, 0

No pure Nash equ.

$(q, 1-q)$ $(p, 1-p)$ $\pi_1(FF, \sigma_2) = -2 = \pi_1(FB, \sigma_2)$

So Mixed strategy Nash Equ.

$$2 - 2q = q$$
$$q = \frac{2}{3}$$
$$\sigma_2 = \left(\frac{2}{3}, \frac{1}{3}\right)$$

$$-1 + p = -2p$$
$$p = \frac{1}{3}$$
$$\sigma_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$$

for player 1, he will select bb at $\frac{1}{3}$ time and select bf for other $\frac{2}{3}$ time

For player 2, he will select b for $\frac{2}{3}$ time and f for $\frac{1}{3}$ time.

$\frac{3}{4}$ chance to get high for player1

		P2	
	H	b	f
P1	bb	4, -4	2, -2

		P2	
	L	b	f
P1	bb	-4, 4	2, -2

		P2	
		b	f
P1	bb	2, -2	2, -2

$$\pi_2(bb, b) = \pi_2(bb, f) = -2$$

So it is equal

4. Consider the following two player game:

		Player 2 x	y
Player 1	a	$4, 1$	$5, 2$
	b	$4, 3$	$1, 2$

(a) Find any pure strategy Nash equilibria.

(b) Show that there is no mixed strategy Nash equilibrium where player 2 has two active strategies.

(c) Assume there is a mixed strategy Nash equilibrium where Player 1 has two active strategies and player 2 has a single active strategy, x. Show that if $\sigma_1 = (p, 1-p)$, then an equality condition holds for all $p \in (0, 1)$.

(d) Use an inequality condition to find a range of values of p such that (σ_1, x) is a mixed strategy Nash equilibrium.

(e) Is there a mixed strategy Nash equilibrium where player 1 has two active strategies and player 2 has only y as an active strategy? Explain.

		Player 2 x	y
Player 1	a	$4, 1$	$5, 2$
	b	$4, 3$	$1, 2$

a) 2 pure (b,x) & (a,y)
$(4,3)$ $(5,2)$

$\alpha_1 = (p, 1-p)$

$\alpha_2 = (q, 1-q)$

$\pi_1(a, \sigma_2) = \pi_1(b, \sigma_2)$

$4 \cdot q + 5 \cdot (1-q) = 4 \cdot q + 1 \cdot (1-q)$

$5 - 5q = 1 - q$

$4 = 4q$

$q = 1$

$\sigma_1 = (1, 0)$

$(p, 1-p)$

$\pi_1(a, \sigma_2) = 4 = \pi_1(b, \sigma_2)$

So equ condition holds.

$\cancel{1q} + 3(1-q) \geq \cancel{2}q + 2(1-q)$

$3 - 3q \geq \cancel{q} + 2 - \cancel{2}q$

$-2q \geq -1$

$q \geq \frac{1}{2}$

$p \in (0, \frac{1}{2}]$ $\quad$ (σ_1, x) is a mixed strategy Nash equ.

p1 have 2 active strategies and p2 have y

$\sigma_1 = (p, 1-p) \quad \sigma_2(0, 1)$

$\pi_1(\sigma, \sigma_2) = 4p + 1$

$$\frac{\partial \pi_1(\sigma, \sigma_2)}{\partial p} = 4 \neq 0$$ So no mixed Nash equ under this cond.

5. Early in the course we discussed the hawk-dove game, a two-player game used to analyze strategies that animals of a single species use in territorial conflicts with other animals of the same species.

   There are two strategies available to each player: The factor of $\frac{1}{2}$ when both players use the same strategy indicates an expected payoff assuming each player has an equal chance of prevailing. **Here we will assume** $w > v$.

		Player 2 h	d
Player 1	h	$\frac{v-w}{2}, \frac{v-w}{2}$	$v, 0$
	d	$0, v$	$\frac{v}{2}, \frac{v}{2}$

   (a) Find any pure strategy Nash equilibria.

   (b) Find a mixed strategy Nash equilibrium where each player has two active strategies. Note that the game is symmetric and you only need to check the condition for one player.

   (c) Briefly describe how the probability of playing h changes as the cost of injury w increases.

   Remark: Imagine that instead of two individual players we consider an entire population of individuals from the same species, where the mixed strategy in the Nash equilibrium represents the probability an individual in the population uses each behavioral strategy. This is the perspective we will adopt in evolutionary games.

$W > V$

a)

		Player 2 h	d
Player 1	h	$\frac{v-w}{2}, \frac{v-w}{2}$	$\underline{v}, \underline{0}$
	d	$\underline{0}, \underline{v}$	$\frac{v}{2}, \frac{v}{2}$

2 pure Nash eqn. (d, h) & (h, d)
$(0, V)$ $(V, 0)$

$(P, 1-P)$

$$\frac{V-W}{2} p + V(1-p) = 0 + \frac{V}{2}(1-p)$$

$$\cancel{Vp} - Wp + \cancel{2}V - \cancel{2Vp} = \cancel{V} - \cancel{Vp}$$

$$V = Wp$$

$$p = \frac{V}{W}$$

$$\sigma_1 = \left(\frac{V}{W}, 1 - \frac{V}{W}\right) = \sigma_2$$ So (σ_1, σ_2) is Mixed Nash Equ.

As W increase, P1 & P2 will play d more since $\frac{V}{W}$ will decrease and $1-\frac{V}{W}$ will increase.

6. Suppose there is a national election with two candidates, A and B. Assume (unrealistically) that each candidate has the ability to choose their location along a one-dimensional political spectrum represented by the interval [0,1]. We assume voters are uniformly distributed in the interval [0,1].

   In this simplified view of elections, the candidates, A and B are the players. They simultaneously choose real numbers in the interval $[0,1]$ representing their chosen positions along the political spectrum. We assume each voter votes for the candidate nearest to themselves on the political spectrum. In this game, the voters are not players. The payoff for each candidate is that candidate's share of the total vote. Here are the details:

   - Candidate A chooses a real number $a \in [0,1]$. Candidate B simultaneously chooses a real number $b \in [0,1]$.
   - The payoffs can be written as follows:

$$\pi_A(a,b) = \begin{cases} \frac{a+b}{2}, & \text{if } a < b \\ \frac{1}{2}, & \text{if } a = b \\ 1 - \frac{a+b}{2}, & \text{if } a > b \end{cases} \qquad \pi_B(a,b) = \begin{cases} \frac{a+b}{2}, & \text{if } b < a \\ \frac{1}{2}, & \text{if } a = b \\ 1 - \frac{a+b}{2}, & \text{if } b > a \end{cases}$$

   - For each candidate, the payoff is their vote share. For Candidate A, the payoff is the length of the portion of the interval $[0,1]$ that is nearer to a than to b.

   Determine the answers to each of the following questions. In each case, show why your answer is correct.

   (a) Consider strategies profiles (a,b) where $a \le b < \frac{1}{2}$. Are there any Nash equilibria of this form?

   (b) Are there any Nash equilibria where $a < \frac{1}{2}$ and $b \ge \frac{1}{2}$?

   (c) Is the profile $(a,b) = (\frac{1}{2}, \frac{1}{2})$ a Nash equilibrium?

a)

$$\pi_A(a,b) = \begin{cases} \frac{a+b}{2} & a<b \\ \frac{1}{2} & a=b \end{cases} \qquad \pi_B(a,b) = \begin{cases} \frac{1}{2} & a=b \\ 1-\frac{a+b}{2} & b>a \end{cases}$$

$a \le b < \frac{1}{2}$

$a+b<1$

$\frac{a+b}{2} < \frac{1}{2}$ $\rightarrow$ $1-\frac{a+b}{2} > \frac{1}{2}$

if $a<b$, $\pi_A(a',b) = \frac{1}{2} > \frac{a+b}{2} < \frac{1}{2}$

if $a=b$ $\pi_A(a',b) = 1-\frac{a-b}{2} > \frac{1}{2} = \pi_A(a,b)$

So No Nash Equ.

$a < \frac{1}{2}$ $b \geq \frac{1}{2}$

if $a+b<1$

$\pi_A(a,b) = \frac{a+b}{2}$ $\pi_B(a,b) = 1-\frac{a+b}{2}$

Player A choose bigger, payoff $= \pi_A(a',b) > \pi_A(a,b)$

if $a+b=1$

$\pi_A(a,b) = \frac{1}{2}$

$\pi_B(a,b) = \frac{1}{2}$

player A choose bigger makes $\pi_A(a',b) > \pi_A(a,b)$

player B choose smaller makes $\pi_B(a,b') > \pi_B(a,b)$

if $a+b>1$ $\pi_A(a,b) = \frac{a+b}{2}$ $\pi_B(a,b) = 1-\frac{a+b}{2}$

player B can choose smaller to get $\pi_B(a,b') > \pi_B(a,b)$

So there is No Nash Equ under $a<\frac{1}{2}$ and $b \geq \frac{1}{2}$

given $(a,b) = (\frac{1}{2}, \frac{1}{2})$

$\pi_A(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$

$\pi_B(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$

if a or b choose to go bigger, they will lower their payoff

if a or b choose to go smaller, they will also lower their payoff

So it is a Nash equ.

7. Suppose there are 100 students participating in a Game Theory course (for credit). The professor makes the following offer:

> You may request up to 10 points of extra credit. The total amount of extra credit available is 450 points. If the total number of points requested does not exceed the total amount available, then everyone will receive the extra credit they requested. If the total requested exceeds the total available, then each student receives 0 extra credit points.

Assume that each student prefers to maximize their own extra credit. Is it a Nash equilibrium for every student to request 10 points? Show that conclusion is correct.

other

one student	$x<4.5$	4.5	$y>4.5$
$x<4.5$	x, x	$x, 4.5$	x, y
4.5	$4.5, x$	$4.5, 4.5$	$0, 0$
$y>4.5$	y, x	$0, 0$	$0, 0$

both $(4.5, 4.5)$, (y, x) (x, y) are Nash equ.

So it is a Nash equilibrium for every student to request 10 points.