8.1 lease - Square Problems.

x_i	y_i
4	5
5	7
9	13

$y = ax + b$

(1) $A\vec{x} = \vec{b}$

$$\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

(2) minimization.

$$E = \frac{1}{2}(ax_1 + b - y_1)^2 + \cdots$$

$$\frac{\partial E}{\partial a} = 0 \Rightarrow (ax_1 + b - y_1)x_1 + \cdots + (ax_3 + b - y_3)x_3 = 0$$

$$\frac{\partial E}{\partial b} = 0 \Rightarrow (ax_1 + b - y_1)\cdot 1 + \cdots + (ax_3 + b - y_3)\cdot 1 = 0$$

$$\Rightarrow (x_1^2 + x_2^2 + x_3^2)a + (x_1 + x_2 + x_3)b = x_1y_1 + x_2y_2 + x_3y_3$$

$$(x_1 + x_2 + x_3)a + 3b = y_1 + y_2 + y_3$$

$$\Rightarrow \begin{bmatrix} \sum_i x_i^2 & \sum_i x_i \\ \sum_i x_i & \sum_i 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_i x_i y_i \\ \sum_i y_i \end{bmatrix}$$

$$A^T A = \begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{bmatrix} = \begin{bmatrix} \sum_i x_i^2 & \sum_i x_i \\ \sum_i x_i & \sum_i 1 \end{bmatrix}$$

$$A^T A \vec{x} = A^T \vec{b} = \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix}$$

Example 2

$$y = ax + b$$

x_i	y_i
-1	1
0	0
1	0
2	2

$$\rightarrow \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}}_{b}$$

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A^T A$$

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$

$$A^T \quad b$$

$$A^T A \vec{x} = A^T b \qquad \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$

$$a = \left(\frac{9}{10}\right) \qquad b = \left(\frac{1}{5}\right)$$

Square Error:

$SE = |\vec{r}|^2$

$$\vec{r} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{9}{10} \\ \frac{1}{5} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} \\ \frac{1}{5} \\ -\frac{1}{10} \\ \frac{2}{5} \end{bmatrix}$$

$SE = 0.01 + 0.04 + 0.49 + 0.16 = 0.7$

e.g.3. $y = ax^2 + bx + c$.

(-1)	1
0	0
1	0
2	-2

$a - b + c = 1$

$c = 0$

$a + b + c = 0$

$4a + 2b + c = -2$