CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

NP and Computational Hardness

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Polynomial-time reduction
(Kleinberg-Tardos, Section 8.1, 8.2)

Motivation

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Is Problem B really hard? How do we prove hardness?

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Tool: **polynomial-time** reduction

Polynomial-time reduction

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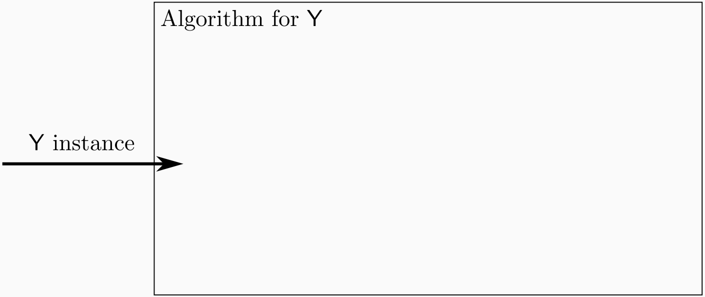

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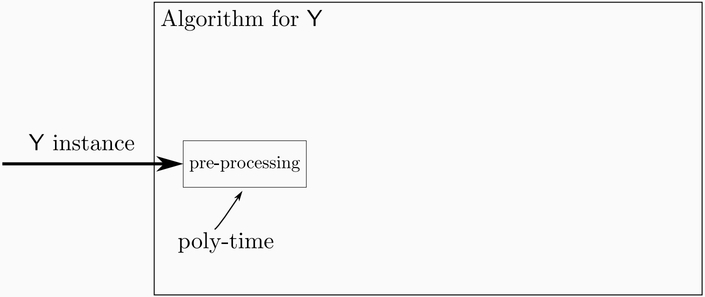

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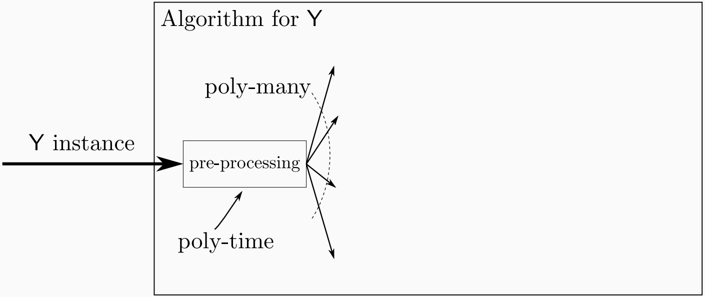

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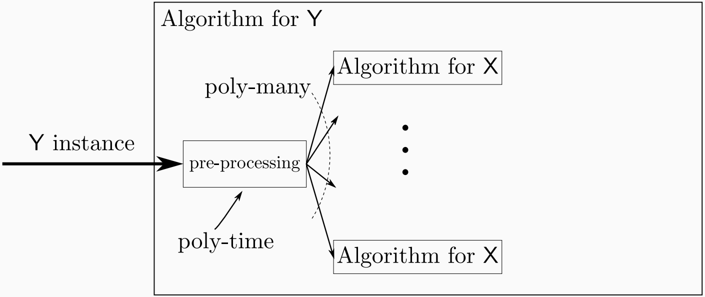

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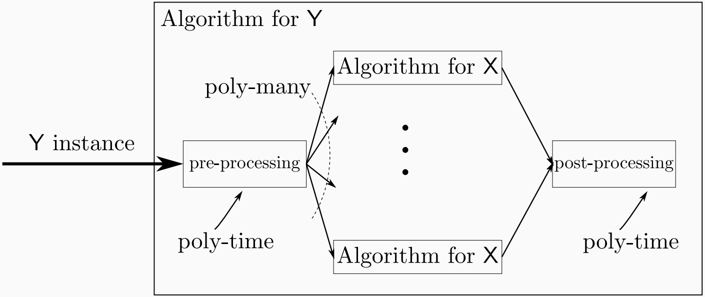

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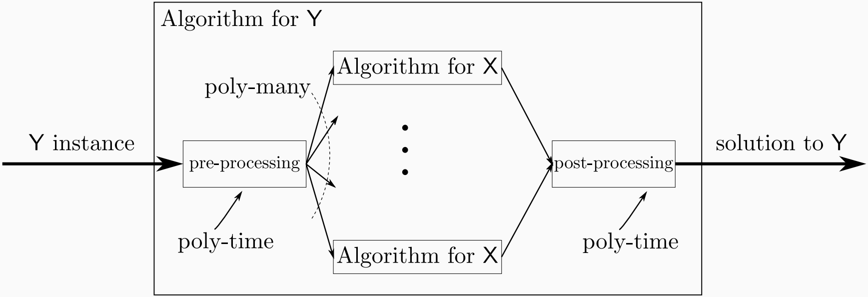

Consequences of polynomial time reductions

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Suppose $\mathsf{Y} \leq_P \mathsf{X}$. *If* Y *cannot be solved in polynomial time, then* X *cannot be solved in polynomial time*

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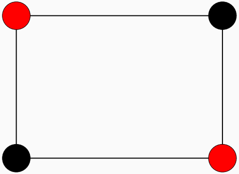

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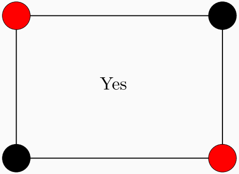

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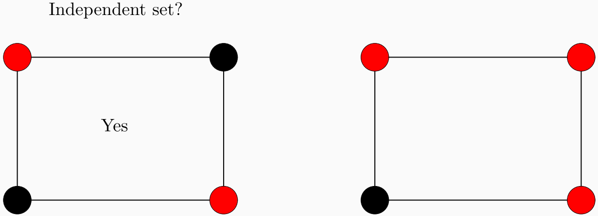

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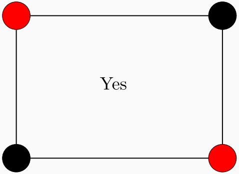

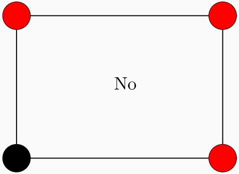

The Maximum Independent Set Problem (Decision version)

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Instance: a graph G, a number k

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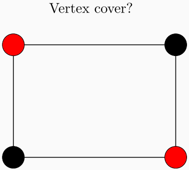

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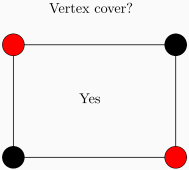

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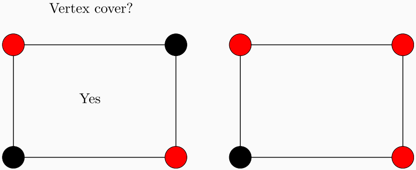

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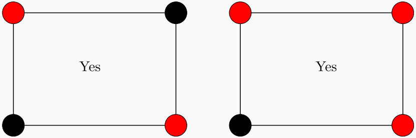

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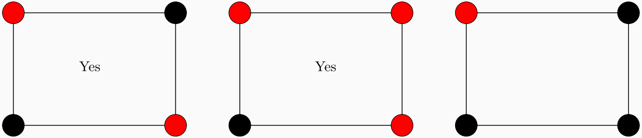

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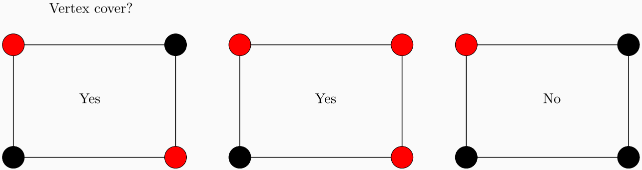

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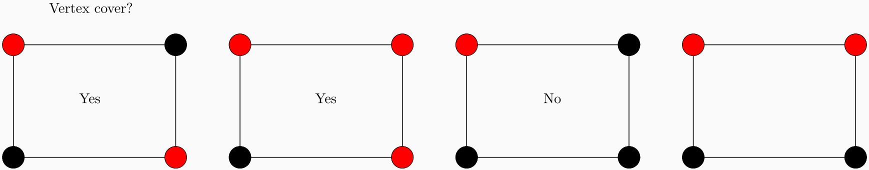

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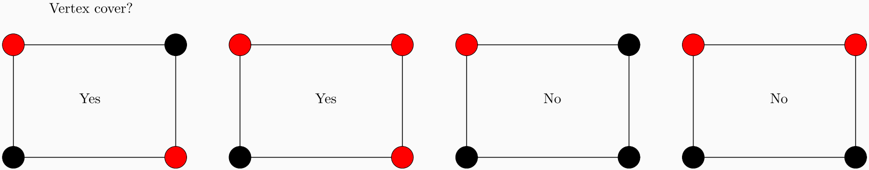

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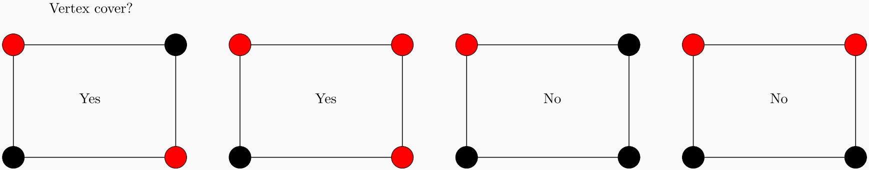

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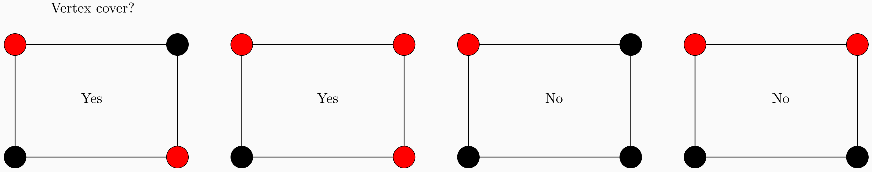

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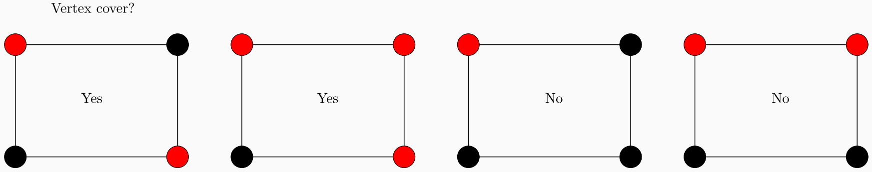

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Lemma

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Proof. $\Rightarrow$

- **"only if":** if S is an IS $\Rightarrow$ $V-S$ is a vertex cover

$\Leftarrow$

"if" $V-S$ is a vertex cover $\Rightarrow$ S is an IS

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Proof.

- **"only if":** Suppose S is an independent set.

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- **"if":** $\Leftarrow$ if $V-S$ is a VC, then S is an IS.

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$\square$

Independent Set vs Vertex Cover (II)

Theorem

- *Independent Set* $\leq_P$ *Vertex Cover*

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Theorem

- *Independent Set* $\leq_P$ *Vertex Cover*
- *Vertex Cover* $\leq_P$ *Independent Set*