CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Flow network
(Textbook, Section 7.2
Kleinberg & Tardos Section 7.1)

Analysis

We need to show the following three things:

- Running time
- FORD-FULKERSON outputs a flow
- FORD-FULKERSON outputs the max flow

Running time analysis (I)

For simplicity, we assume the capacities are all integers

Fact (Fact 1)

In every step of the algorithm, the flow and the residual capacities are all integers

Fact (Fact 2)

Let f be a flow in G and P be a simple s-t path in G_f. Then

$$v(f') = v(f) + \mathrm{bottleneck}(P, f)$$

Proof.

The first edge of P leaves s, and P doesn't revisit s again. Moreover, it's a forward edge. So

$$v(f') = v(f) + \mathrm{bottleneck}(P, f)$$

□

Running time analysis (II)

Since $\mathrm{bottleneck}(P, f) \geq 1$,

$$v(f') \geq v(f) + 1$$

Let $C = \sum_{e \text{ out of } s} c_e$. We have

Corollary

The Ford-Fulkerson algorithm performs $\leq C$ iterations

Proof.

All capacities are integers. Every iteration increase the value by ≥ 1 $\square$

Finding an s-t path takes $O(|V| + |E|) = O(|E|)$ time (BFS)
Augmentation takes $O(|V|)$ time

So total running time is $O(C \cdot |E|)$

Ford-Fulkerson outputs a flow

Lemma

Let f' be the function obtained after augmenting. Then f' is a flow

Proof.

- **Capacity constraint.** It suffices to consider edges of P
  Let $e = (u, v) \in P$. $\text{bottleneck}(P, f)$ is at most the residual capacity of e
  - if e is a forward edge, then

    $$0 \leq f(e) \leq f'(e) = f(e) + \text{bottleneck}(P, f) \leq f(e) + (c_e - f(e)) = c_e$$

    So $0 \leq f'(e) \leq c_e$
  - if e is a backward edge, then

    $$c_e \geq f(e) \geq f'(e) = f(e) - \text{bottleneck}(P, f) \geq f(e) - f(e) = 0$$

    So $0 \leq f'(e) \leq c_e$

- **Conservation condition.** It suffices to observe that for every vertex, additional amount of flow, 0, or $\mathrm{bottleneck}(P, f)$ entering this vertex equals the additional amount of flow, 0, or $\mathrm{bottleneck}(P, f)$ leaving it

$\square$

Correctness of Ford-Fulkerson (I)

Flow and Cut

Definition

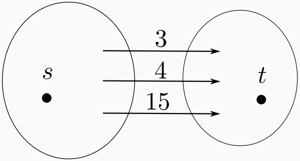
An **s-t cut** is a partition of V, (A, B) where $s \in A$ and $t \in B$

Definition

The **capacity** of the cut is

$$c(A, B) = \sum_{e \text{ out of } A} c_e$$

How does a cut help?



The flow must have a value ≤ 22
Capacity of a cut put a bound on the flow value

Correctness of Ford-Fulkerson (II)

Lemma

Let f be an s-t flow, (A, B) be an s-t cut. Then $v(f) \leq c(A, B)$

Proof. Notation:

$$f^{\text{out}}(A) = \sum_{e \text{ out of } A} f(e)$$
$$f^{\text{in}}(A) = \sum_{e \text{ into } A} f(e)$$

So, $v(f) = \sum_{e \text{ out of } s} f(e) = f^{\text{out}}(s) = f^{\text{out}}(s) - f^{\text{in}}(s)$ (no edge into s)
Also, for all $v \in A - \{s, t\}$, $f^{\text{out}}(v) = f^{\text{in}}(v)$ (flow conservation)
$\implies f^{\text{out}}(v) - f^{\text{in}}(v) = 0$ for all $v \neq s, t$

So

$$v(f) = \sum_{v \in A} \left(f^{\text{out}}(v) - f^{\text{in}}(v) \right)$$

For this expression

$$v(f) = \sum_{v \in A} \Big(f^{\text{out}}(v) - f^{\text{in}}(v) \Big)$$

consider every edge (v, w)

- if $v, w \in A$, this edge contributes 0 in the summation
- if $v \in A, w \notin A$, this edge contributes $f(e)$
- if $v \notin A, w \in A$, this edge contributes $-f(e)$

We rewrite the summation as

$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \le \sum_{e \text{ out of } A} f(e) = c(A, B)$$

$\square$

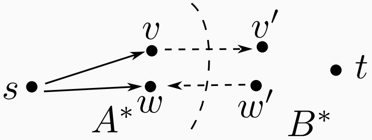
The upper bound $c(A, B)$ is achievable by Ford-Fulkerson

Lemma

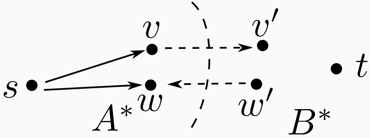
Let f be a flow s.t. there's no s-t path in G_f. Then there exists an s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$

Proof. Let A^* be the set of vertices reachable from s in G_f. Let B^* be $V - A^*$. We have the follow facts

- (A^*, B^*) is an s-t cut: $s \in A^*$, $t \in B^*$ (no s-t path in G_f)
- for all edge $e = (v, v') \in E$ with $v \in A^*, v' \in B^*$, we have $f(e) = c_e$ Otherwise, (v, v') is an edge in G_f with capacity $c_e - f(e) \neq 0$. Forward edge. So v' is reachable from s in G_f (contradiction)



- for all edge $e = (w', w)$ with $w' \in B^*, w \in A^*$, we have $f(e) = 0$
  Otherwise, (w, w') is a backward edge in G_f, and w' would be reachable from s (contraction)



Then

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A^*} f(e) - \sum_{e \text{ into } A^*} f(e) \quad \text{(from the proof of } v(f) \le c(A, B)) \\
&= \sum_{e \text{ out of } A^*} f(e) - 0 \\
&= \sum_{e \text{ out of } A^*} c_e = c(A^*, B^*)
\end{aligned}
$$

$\square$

Summary of Max flow

Some consequences:

- The flow returned by Ford-Fulkerson is a maximum flow
- In every flow network, maximum value of a flow = minimum capacity of a cut
- Given a flow of max value, can compute a cut of minimum capacity in $O(|E|)$ time
- If all capacities of a flow network are integers, then there is a max flow f s.t. $f(e)$ is an integer for all e