CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Linear Programming
(Textbook, Section 7.1)

Advertisement

Please consider taking

CMPSC 497 — Quantum Computation in Fall 2022

if you are interested in learning **Quantum Computing**

Background

Optimization: we want to maximize some function $f(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$, subject to constraints

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How to allocate your time?

LP formulation

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& S + P + E & \leq & 168
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How to solve an LP

Consider a simpler LP:

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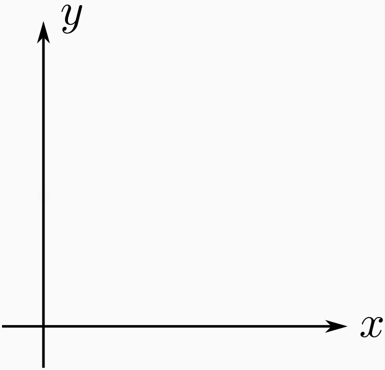

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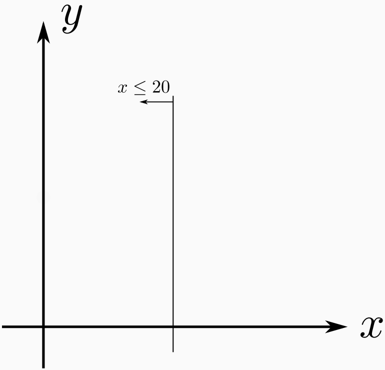

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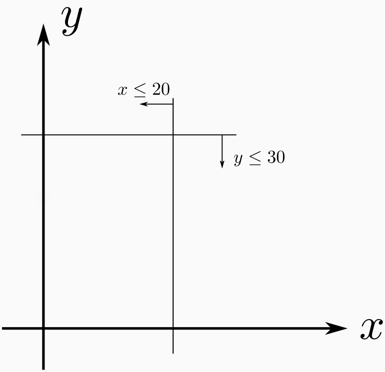

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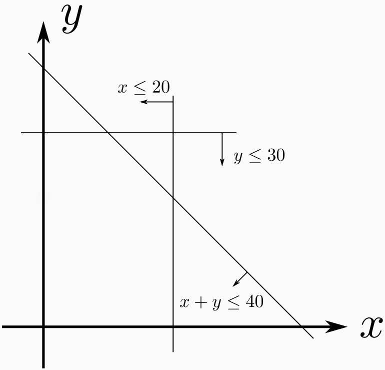

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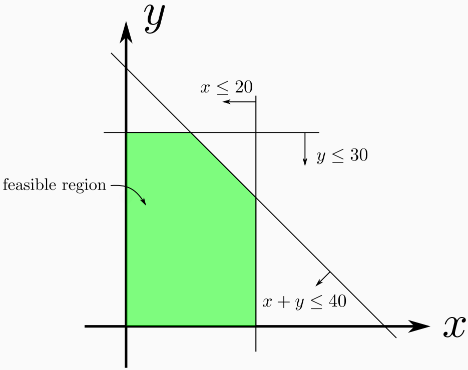

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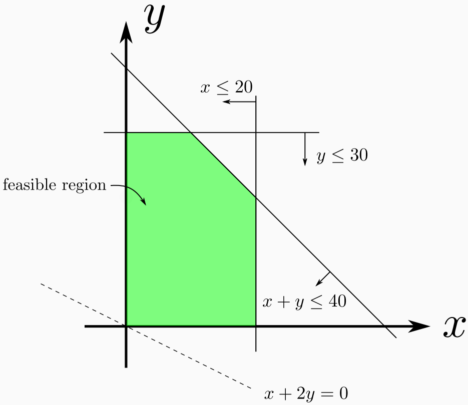

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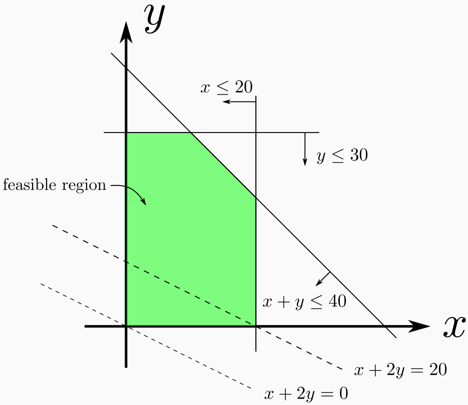

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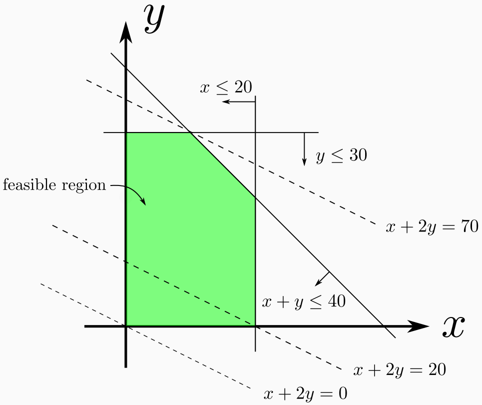

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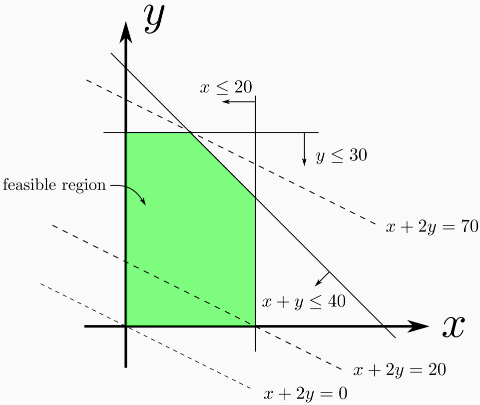

Optimal solution: $x + 2y = 70$

Algorithm for solving LP

Observation: (search for an optimal solution)

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Theorem

For an LP with bounded, nonempty feasible region, the maximum value will be attained at some vertex of the feasible region

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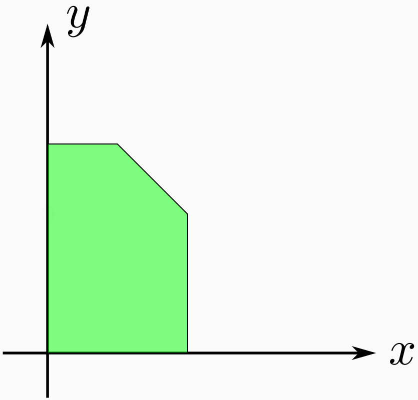

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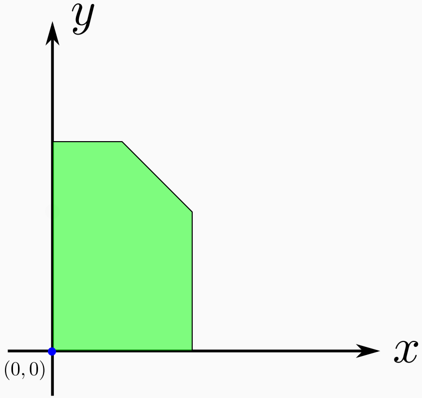

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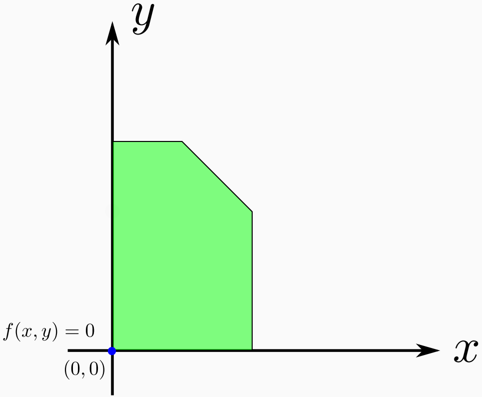

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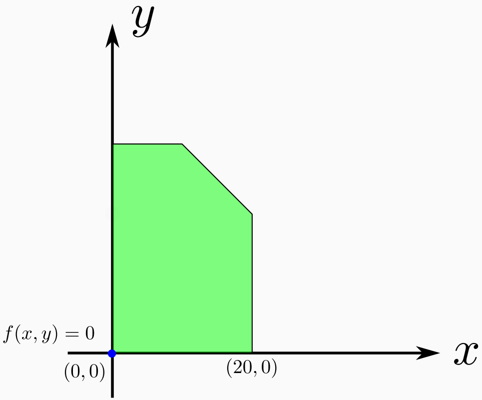

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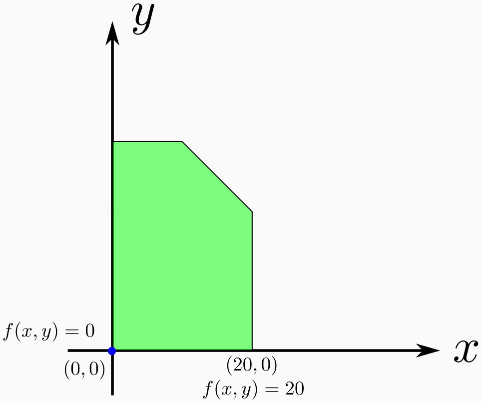

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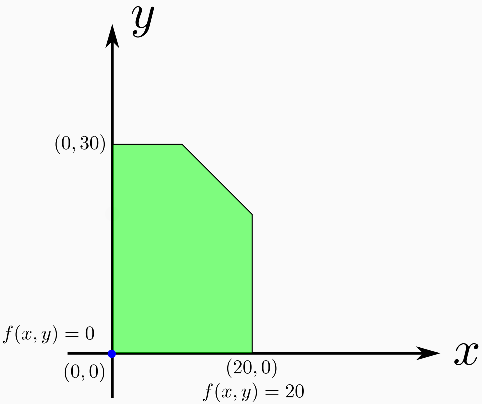

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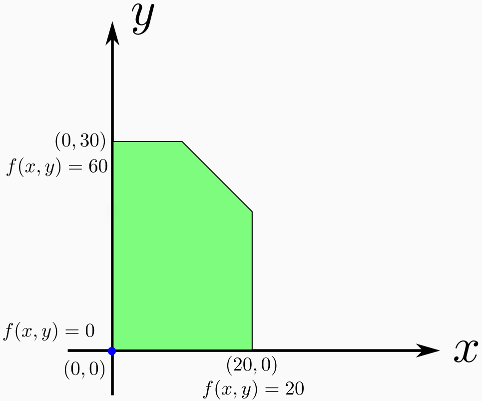

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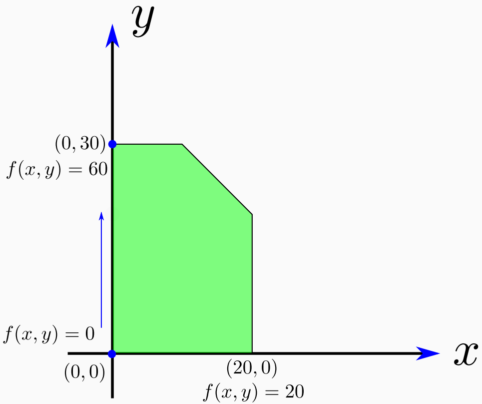

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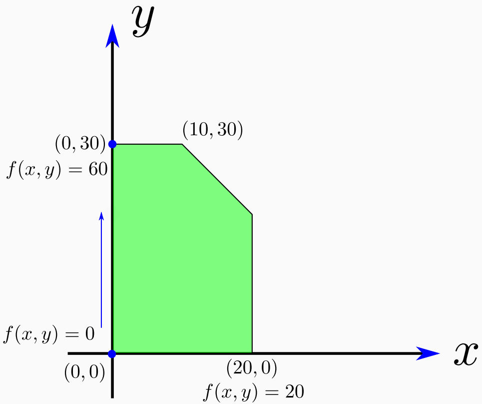

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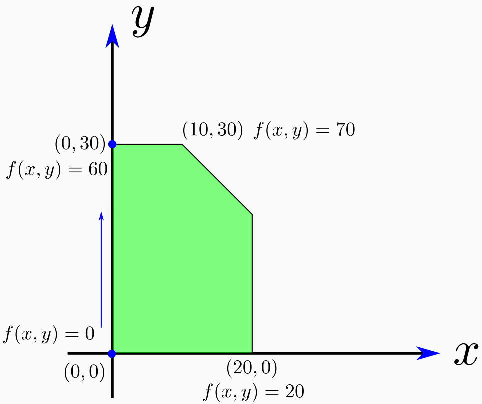

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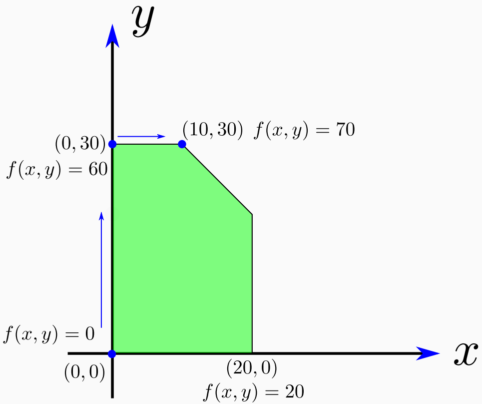

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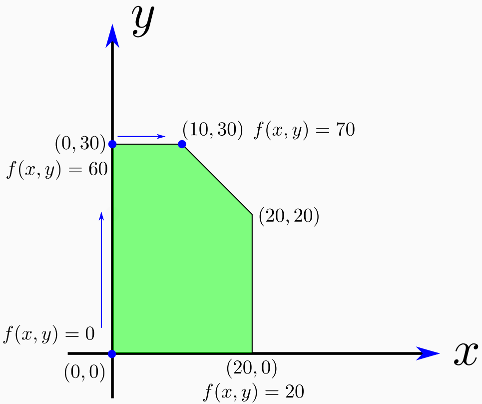

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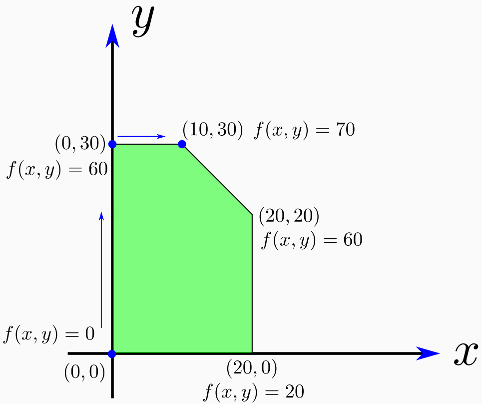

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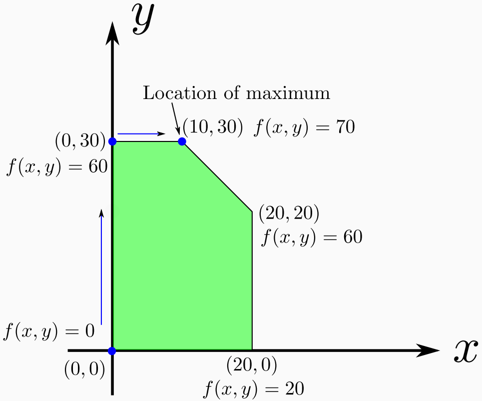

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Standard form 1

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\text{subject to} & A\mathbf{x} & \leq & \mathbf{b} \\
 & \mathbf{x} & \geq & 0 \\
 & \mathbf{x}, \mathbf{c}, \mathbf{b} & \in & \mathbb{R}^n, A \in \mathbb{R}^{m\times n}
\end{array}
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Standard forms

LP solvers, such as MOSEK, Gurobi, CVX, and COIN are implementations of the simplex method. They require the LP to be in certain standard form

Standard form 1

$$
\begin{aligned}
\text{maximize} \quad & \mathbf{c}^T\mathbf{x} & & \\
\text{subject to} \quad & A\mathbf{x} & \leq & \ \mathbf{b} \\
& \mathbf{x} & \geq & \ 0 \\
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\end{aligned}
$$

Example:

$$
\begin{aligned}
\text{maximize} \quad & x + 2y & & \\
\text{subject to} \quad & x & \leq & \ 20 \\
& y & \leq & \ 30 \\
& x + y & \leq & \ 40 \\
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Convert to the standard form

- Minimization to maximization

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} \quad & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{aligned} \qquad \equiv$$

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- Equality to inequality

$$\begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & x_1 + x_2 & = & 7 \end{array} \quad \equiv$$

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- Wrong inequality direction

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$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s. t.} \quad & x_1 \leq 20 \\ & x_1 + x_2 \leq 40 \\ & x_1 \geq 0 \end{aligned} \equiv$$

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\quad \equiv \quad
\begin{array}{lrcl}
 & \text{rewrite } x_2 = x_2^+ - x_2^- & & \\
\max & x_1 + 2(x_2^+ - x_2^-) & & \\
\text{s. t.} & x_1 & \leq & 20 \\
 & x_1 + (x_2^+ - x_2^-) & \leq & 40 \\
 & x_1 & \geq & 0 \\
 & x_2^+ & \geq & 0 \\
 & x_2^- & \geq & 0
\end{array}
$$

Another Standard form

Standard form 2

$$\begin{array}{lrcl} \text{maximize} & \mathbf{c}^T\mathbf{x} & & \\ \text{subject to} & A\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \\ & \mathbf{x}, \mathbf{c}, \mathbf{b} & \in & \mathbb{R}^n, A \in \mathbb{R}^{m\times n} \end{array}$$

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The new variable s is call the *slack variable*

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We are given $G = (V, E), w : E \to \mathbb{R}$

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We just **reduced** shortest_path to LP

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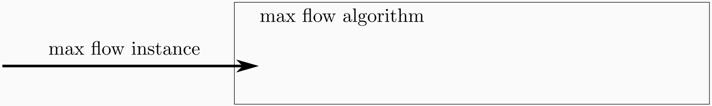

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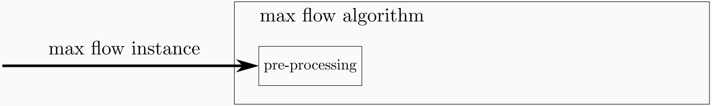

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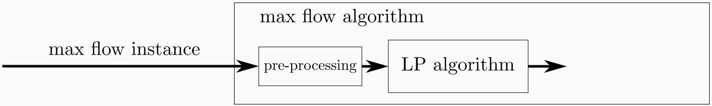

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