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Section: 03

Math 455, Sample Exam I
September 27, 2021

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 50 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 10 pages of the test.

Please do NOT write in this box.

1.	
2.	
3.	
4.	
Total	

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Partial Credit

You must show your work on the partial credit problems to receive credit!

1.(5 pts.) Explain how to evaluate the polynomial for a given input x, using as few operation as possible. How many multiplications and how many additions are required?

$$P(x) = a_0 + a_4x^4 + a_8x^8 + a_{12}x^{12} + a_{16}x^{16}$$

$a_0 + x^4(a_4 + x^4(a_8 + x^4(a_{12} + a_{16}x^4)$

$4+$ $6\times$

$x \cdot x = x^2$ 1

$x^2 \cdot x^2 = x^4$ 1

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2.(50pts.) (a). Convert binary number to decimal number: $(11011.101)_2$.

$$1\times2^0+1\times2^1+0\times2^2+1\times2^3+1\times2^4+1\times2^{-1}+0\times2^{-2}+1\times2^{-3}$$

$$=1+2+0+8+16+\frac{1}{2}+\frac{0}{4}+\frac{1}{8}$$

$$=10+17+\frac{5}{8}$$

$$=27+\frac{5}{8}$$

$$=\frac{221}{8}$$

(b). Convert decimal number to binary number: $(1023)_{10}$.

$$1023=1024-1$$

$$=1\times2^{10}-1$$

$$=1111111111$$

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(c). Convert decimal number to binary number: $(7.3)_{10}$.

$7.3 = 7 + 0.3$

$7 = 111$

$2\,|\,7 \quad 1$
$2\,|\,3 \quad 1$
$2\,|\,1 \quad 1$

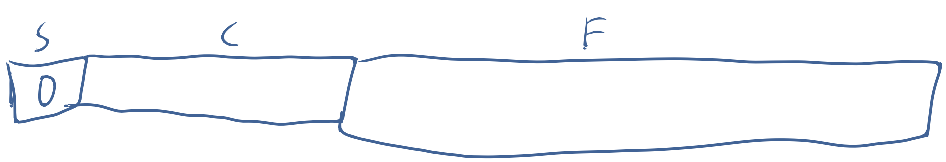
0.3×2
0.6
1.2
0.4
0.8
1.6
1.2

$0.0\overline{1001}$

$7.3_{10} = 111.0\overline{1001}_2$

(d). Express the floating number fl(7.3) by using the Rounding to Nearest Rule.

$7.3 \qquad -1^{s} \times 1.f \times 2^{(1025-1023)}$



$fl(7.3) = 1.\boxed{1101\overline{001} \cdots\cdots 0011}001 \times 2^{2}$ (52)

round down

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(e). Comput the difference 7.3−fl(7.3) and check that the relative error is no more than $\frac{1}{2}\epsilon_{\text{mach}}$.

$7.3 - fl(7.3) = 0.001\overline{1001} \times 2^{-52} \times 2^{2}$

$= 0.\overline{11001} \times 2^{-52}$

$= 0.1_2 \times 2^{-52} + 0.0\overline{1001}_2 \times 2^{-52}$

$= 0.8_{10} \times 2^{-52}$

$0.0\overline{1001}_2$

$= 2^{-2} \times 1 + 2^{-5} \times 1$

$= \frac{1}{4} + -$

$$\frac{|7.3 - fl(7.3)|}{|7.3|} = \frac{0.8_{10} \times 2^{-52}}{7.3} = \frac{8}{73} \times 2^{-52}$$

$\epsilon_{mach} = 2^{-52}$

$$\frac{8}{73} \times 2^{-52} < \frac{1}{2}\epsilon_{mach} = \frac{1}{2} \times 2^{-52}$$

(f). Write out the hex machine number representation of 7.3.

1025 $\quad$ $52/4 = 13$

0	10000000001	1101 0011 0011 - - - - - - - - 10011

4 0 1 14 3 3

$$4\,0\,1\,D\,\overbrace{3\,3 \cdots 3}^{12}$$

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(g) Do the operations $(8.3 - 7.3) - 1$ by hand in IEEE double precision computer arithmetic, using the Rounding to Nearest Rule.

$fl(fl(8.3) - fl(7.3)) - fl(1)$

$8.3 = 1000.0\overline{1001}$

$fl(8.3) = 1.\boxed{000\,0\overline{1001}\,10\ldots\ldots 1001}\,1001$

$= \ldots\ldots\ldots\ldots 1010$

$= 8.3$

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3.(15 pts.) Consider the function

$$f(x) = \sqrt{x^4 + 2x^2 + 2} - x^2 - 1.$$

(a)For what values of x would this function be difficult to compute in a computer? Please explain what difficulty and why.

$$= \sqrt{x^4 + 2x^2 + 1 + 1} - x^2 - 1$$

$$= \sqrt{(x^2+1)^2 + 1} - x^2 - 1$$

$$= x^2 + 1 + 1 - (x^2+1)$$

(b) Could you find a way to avoid this difficulty? Explain in detail.

$$\frac{\left(\sqrt{(x^2+1)^2+1} - (x^2+1)\right)\left(\sim + (x^2+1)\right)}{\left(\sim + (x^2+1)\right)}$$

$$= \frac{(x^2+1)^2 + 1 - (x^2+1)^2}{\sqrt{(x^2+1)+1} + x^2 + 1}$$

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4.(30 pts.)

Consider a function $f(x) = x^5 + x^4 + 1/8$.

(a) Prove that there exists at least one root of $f(x) = 0$ on $[-2, -1]$.

$-32 + 16 + \frac{1}{8} < 0$

$-1 + 1 + \frac{1}{8} > 0$ ✓

By IVT

(b) Starting with $[-2, -1]$, how many steps of the Bisection Method are required to calculate the solution within 10^{-8}? Answer with an integer.

$\frac{1}{2}^n \cdot |b-a| < 10^{-8}$

$\frac{1}{2}^n |-1+2| < 10^{-8}$

$2^{-n} < 10^{-8}$

$\log_2 2^{-n} = \log_2 10^{-8}$

$-n = -8 \log_2 10$

$n = 8 \log_2 10 \quad = 8 \cdot 3.32 = 26.56$

≈ 27

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(c) Consider a fixed point iteration

$$x_{n+1} = g_1(x_n), \qquad \text{where} \qquad g_1(x) = f(x) + x,$$

with the starting point $x_0 = -1$. Does this scheme converge?

$$g(x) = x^5 + x^4 + \frac{1}{8} + x$$

$$g'(x) = 5x^4 + 4x^3 + 1$$

$$g'(-1) = 5 - 4 + 1 = 2 > 1$$

do not converge.

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(d) Consider a fixed point iteration

$$x_{n+1} = g_2(x_n), \qquad \text{where} \qquad g_2(x) = -(x^4 + 1/8)^{1/5},$$

with the starting point $x_0 = -1$. Does this scheme converge? Why? $((\frac{15}{8})^{1/4} = 0.8544)$.

$g_2'(x) = -\frac{1}{5}(x^4+\frac{1}{8})^{-\frac{4}{5}} \cdot (4x^3)$

$g''(x) = -\frac{4}{5}(x^4+\frac{1}{8})^{-\frac{9}{5}}$