Duality of LP (I)

Consider

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 \\
\text{subject to} \quad & x_1 \leq 20 \\
& x_2 \leq 30 \\
& x_1 + x_2 \leq 40 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Duality of LP (I)

Consider

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \leq 20 \\ & x_2 \leq 30 \\ & x_1 + x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Can we show the optimal solution is at least 60?

Duality of LP (I)

Consider

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \leq 20 \\ & x_2 \leq 30 \\ & x_1 + x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Can we show the optimal solution is at least 60? Check (0, 30)

Duality of LP (I)

Consider

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \le 20 \\ & x_2 \le 30 \\ & x_1 + x_2 \le 40 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Can we show the optimal solution is at least 60? Check (0, 30)

Can we show that optimal solution is at most 90?

Duality of LP (I)

Consider

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \leq 20 \\ & x_2 \leq 30 \\ & x_1 + x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Can we show the optimal solution is at least 60? Check (0, 30)

Can we show that optimal solution is at most 90? Use linear combinations constraints

Duality of LP (II)

Define a variable for each constraint

$$
\begin{array}{rl}
\text{maximize} & x_1 + 2x_2 \\
\text{subject to} & x_1 \leq 20 \\
& x_2 \leq 30 \\
& x_1 + x_2 \leq 40 \\
& x_1, x_2 \geq 0
\end{array}
$$

Duality of LP (II)

Define a variable for each constraint

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 \\
\text{subject to} \quad y_1 \;& x_1 \le 20 \; y_1 \\
y_2 \;& x_2 \le 30 \; y_2 \\
y_3 \;& x_1 + y_3 x_2 \le 40 \; y_3 \\
& x_1, x_2 \ge 0
\end{aligned}
\qquad\qquad
\begin{aligned}
& y_1 \ge 0 \\
& y_2 \ge 0 \\
& y_3 \ge 0
\end{aligned}
$$

Duality of LP (II)

Define a variable for each constraint

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 \\
\text{subject to} \quad & y_1 \; x_1 \le 20 \; y_1 & \color{red}{y_1} \ge 0 \\
& y_2 \; x_2 \le 30 \; y_2 & \color{red}{y_2} \ge 0 \\
& y_3 \; x_1 + y_3 x_2 \le 40 \; y_3 & \color{red}{y_3} \ge 0 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Adding them together: $\underline{y_1 x_1} + y_2 x_2 + \underline{y_3} x_1 + y_3 x_2 \le 20 y_1 + 30 y_2 + 40 y_3$

$(y_1 + y_3) x_1 + (y_2 + y_3) x_2 \le 20 y_1 + 30 y_2 + 40 y_3$

Duality of LP (II)

Define a variable for each constraint

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 & \\
\text{subject to} \quad & x_1 \leq 20 & \qquad y_1 \\
& x_2 \leq 30 & \qquad y_2 \\
& x_1 + x_2 \leq 40 & \qquad y_3 \\
& x_1, x_2 \geq 0 &
\end{aligned}
$$

Adding them together:

$$\underbrace{\boxed{(y_1 + y_3)}}_{\substack{=1 \\ \geq 1}} x_1 + \underbrace{(y_2 + y_3)}_{\substack{=2 \\ \geq 2}} x_2 \leq 20y_1 + 30y_2 + 40y_3$$

$$\underline{x_1 + 2x_2} \leq (y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq \boxed{20y_1 + 30y_2 + 40y_3}$$

Duality of LP (II)

Define a variable for each constraint

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 & \\
\text{subject to} \quad & x_1 \le 20 & \qquad y_1 \\
& x_2 \le 30 & \qquad y_2 \\
& x_1 + x_2 \le 40 & \qquad y_3 \\
& x_1, x_2 \ge 0 &
\end{aligned}
$$

Adding them together:

$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \le 20y_1 + 30y_2 + 40y_3$

We let $y_1 + y_3 \ge 1$ and $y_2 + y_3 \ge 2$ to get an upper bound on $x_1 + 2x_2$:

Duality of LP (II)

Define a variable for each constraint

$$\begin{array}{lll} \text{maximize} & x_1 + 2x_2 & \\ \text{subject to} & x_1 \le 20 & \color{red}{y_1} \\ & x_2 \le 30 & \color{red}{y_2} \\ & x_1 + x_2 \le 40 & \color{red}{y_3} \\ & x_1, x_2 \ge 0 & \end{array}$$

Adding them together:

$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \le 20y_1 + 30y_2 + 40y_3$

We let $y_1 + y_3 \ge 1$ and $y_2 + y_3 \ge 2$ to get an upper bound on $x_1 + 2x_2$:

$x_1 + 2x_2 \le (y_1 + y_3)x_1 + (y_2 + y_3)x_2 \le 20y_1 + 30y_2 + 40y_3$

Duality of LP (III)

Primal LP

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \leq 20 \\ & x_2 \leq 30 \\ & x_1 + x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Duality of LP (III)

Primal LP

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \le 20 \\ & x_2 \le 30 \\ & x_1 + x_2 \le 40 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \le \quad \text{minimize} \quad & 20y_1 + 30y_2 + 40y_3 \\ \text{subject to} \quad & y_1 + y_3 \ge 1 \\ & y_2 + y_3 \ge 2 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

Duality of LP (III)

Primal LP

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \leq 20 \\ & x_2 \leq 30 \\ & x_1 + x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Optimal solution: $(x_1, x_2) = (10, 30) \implies x_1 + 2x_2 = 70$

Dual LP

$$\begin{aligned} \text{minimize} \quad & 20y_1 + 30y_2 + 40y_3 \\ \text{subject to} \quad & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Duality of LP (III)

Primal LP

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \leq 20 \\ & x_2 \leq 30 \\ & x_1 + x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Optimal solution: $(x_1, x_2) = (10, 30) \implies x_1 + 2x_2 = 70$

Dual LP

$$\begin{aligned} \text{minimize} \quad & 20y_1 + 30y_2 + 40y_3 \\ \text{subject to} \quad & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Optimal solution: $(y_1, y_2, y_3) = (0, 1, 1) \implies 20y_1 + 30y_2 + 40y_3 = 70$

Duality of LP(IV)

More generally

Primal LP

$$\begin{aligned}
\max \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\
& \quad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\
& x_1, x_2, \ldots, x_n \ge 0
\end{aligned}$$

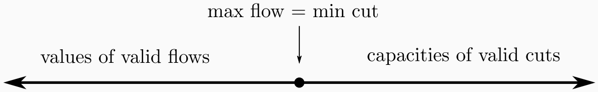
More generally

Primal LP	Dual LP
$\max \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n$	$\min \quad b_1y_1 + b_2y_2 + \cdots + b_my_m$
s.t. $\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$	s.t. $\quad a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$	$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2$
$\vdots$	$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$	$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n$
$x_1, x_2, \ldots, x_n \ge 0$	$y_1, y_2, \ldots, y_m \ge 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

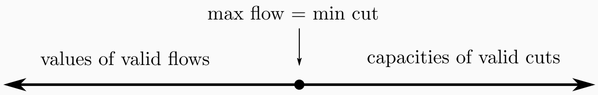
Duality of LP (V)

Duality of flow and cut



Duality of LP (V)

Duality of flow and cut



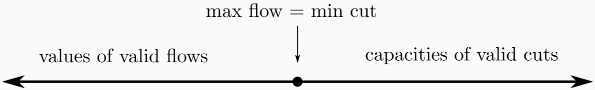
For LP we have:

Theorem (Weak Duality)

A feasible solution to the dual LP is an upper bound on any feasible solution to the primal LP

Duality of LP (V)

Duality of flow and cut



For LP we have:

Theorem (Weak Duality)

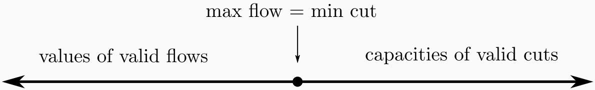
A feasible solution to the dual LP is an upper bound on any feasible solution to the primal LP

Theorem (Strong Duality)

The optimal solution to the dual LP is equal to the optimal solution to the primal LP

Duality of LP (V)

Duality of flow and cut



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