CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

NP and Computational Hardness

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Polynomial-time reduction
(Kleinberg-Tardos, Section 8.1, 8.2)

Satisfiability

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A **k-CNF** is a CNF where each clause contains exactly k literals

The Satisfiability Problem (SAT)

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The k-Satisfiability Problem (k-SAT)

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3-SAT and Independent Set

Theorem

3-SAT $\leq_P$ *Independent Set*

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$$\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \bar{x}_3 \vee x_4) \wedge (x_3 \vee \bar{x}_1 \vee x_5)$$

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We encode a CNF as a graph, and encode an assignment as independent sets (to keep track of the conflicts)

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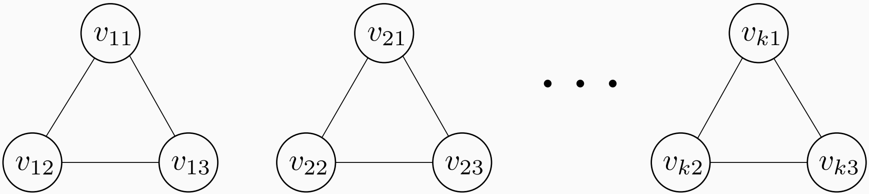

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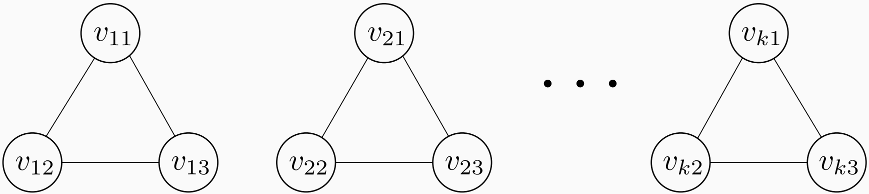

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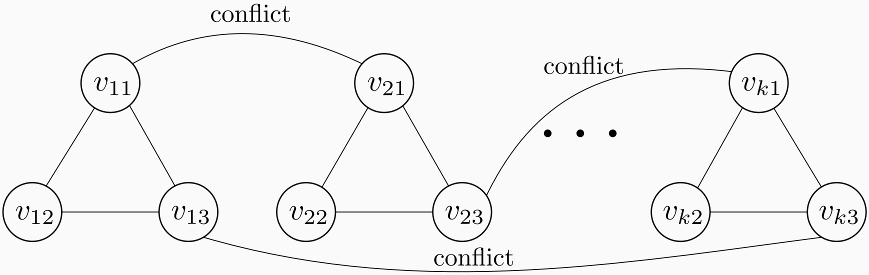

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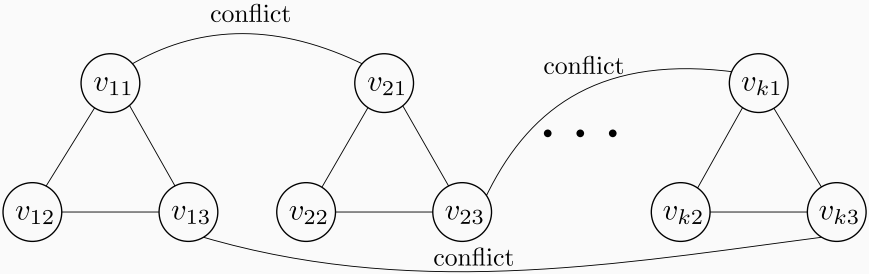

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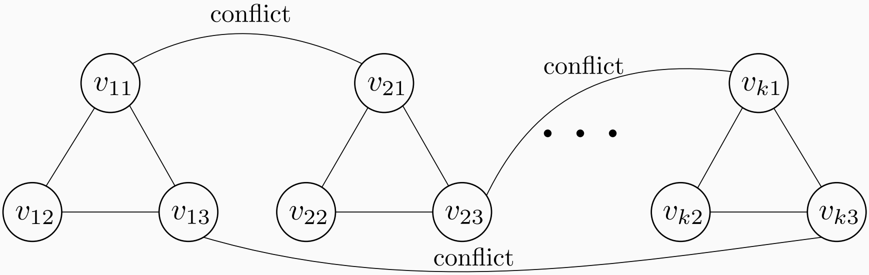

At most one vertex in each triangle can be in an independent set, so the size of an independent set cannot be larger than k

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So the 3-CNF has a satisfying assignment if and only if G has an independent set of size k $\square$

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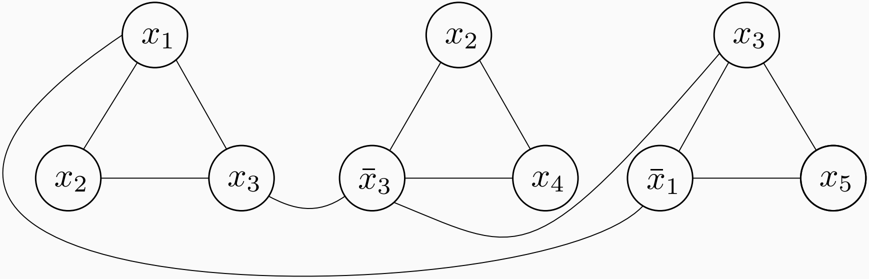

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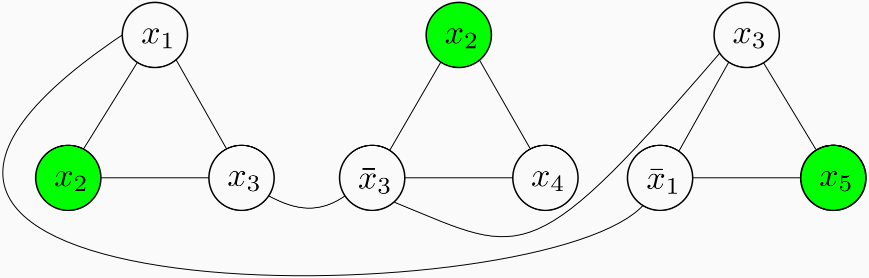

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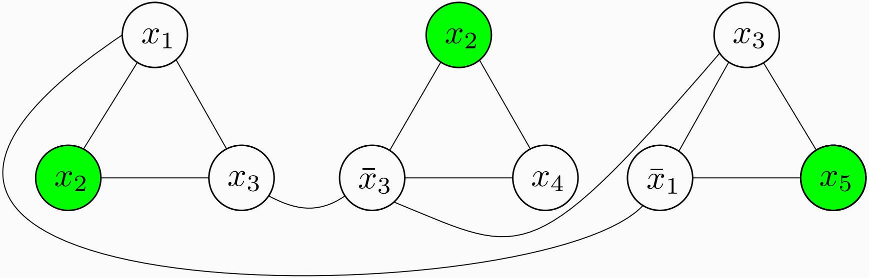

Satisfying assignment: $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 1$

NP and Computational Hardness

P, NP, and NP-completeness
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Problems and algorithms

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Computational class

P : the class of all problems for which there exists a polynomial-time algorithm

Checking vs solving

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But there might be exponentially many possible t's

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Lemma

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Proof.

For any problem in **P** with algorithm A, we construct a certifier B that just returns $A(s)$ with empty certificate t $\square$

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Lemma

If an **NP***-complete problem can be solved in polynomial time, then* $\mathbf{P} = \mathbf{NP}$

Which problems are NP-complete?

A first **NP**-complete problem: Circuit Satisfiability

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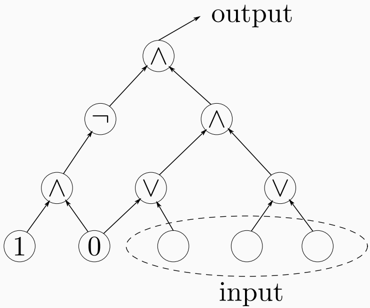

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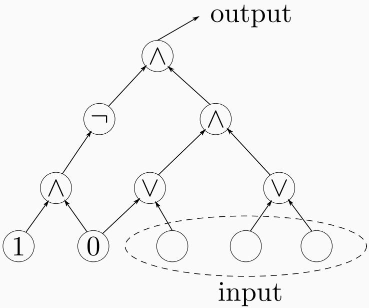

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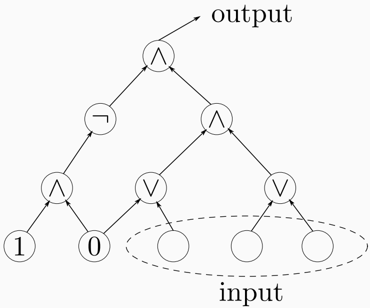

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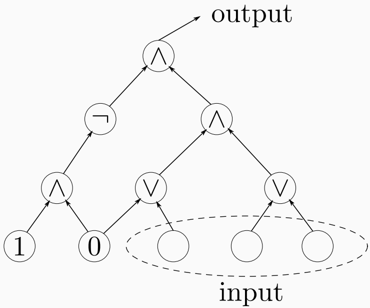

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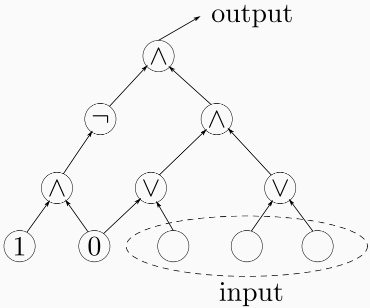

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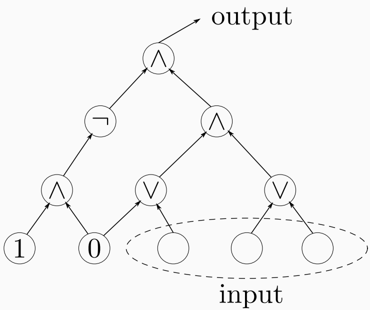

The Circuit Satisfiability Problem (circuit-SAT)

Instance: A circuit C

Objective: Decide if C is satisfiable

The Cook-Levin Theorem

Theorem (Cook-Levin)

circuit-SAT is **NP***-complete*

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Proof sketch.

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If no, there's such t that $B(s, t) =$ yes. So $s \notin$ X $\square$