MATH 455: HOMEWORK 10

Problem 1. Find the (infinity norm) condition number of

$$A = \begin{bmatrix} 1 & 2.01 \\ 3 & 6 \end{bmatrix}$$

Problem 2. Find the (infinity norm) of the relative forward and backward errors and the error magnification factor for the following approximate solutions of the system $x_1+2x_2 = 3, 2x_1+4.01x_2 = 6.01, (a)[-10, 6]$, $(b)[-600, 301]$, (c) What is the (infinity norm) condition number of the coefficient matrix?

Problem 3. Solve $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{b} = (2, 2, -1)^T$ and

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

by Gassian elimination with partial pivoting.

Problem 4, text book, page 101, Exercises 2 (b) (d). Find the $PA = LU$ factorization (using partial pivoting) of the following matrices:

$$\text{(b)} \quad \begin{bmatrix} 0 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\text{(d)} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix}$$

Problem 5, text book, page 101, Exercises 3 (b). Solve the system using the $PA = LU$ factorization

$$\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \tag{1}$$

Problem 1. Find the (infinity norm) condition number of

$$A = \begin{bmatrix} 1 & 2.01 \\ 3 & 6 \end{bmatrix}$$

$$\|A\|_\infty = \max \sum_{j=1}^{n} |a_{ij}|$$

$$= \max(1+2.01,\ 3+6)$$

$$= 9$$

$$A^{-1} = \frac{1}{6-6.03} \begin{bmatrix} 6 & -2.01 \\ -3 & 1 \end{bmatrix}$$

$$= -\frac{1}{0.03} \begin{bmatrix} 6 & -2.01 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -200 & 67 \\ 100 & -33.33 \end{bmatrix}$$

$$\|A^{-1}\|_\infty = \max(200+67,\ 100+33.33)$$

$$= 267$$

$$K_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$$

$$= 9 \cdot 267$$

$$= 2403$$

Problem 2. Find the (infinity norm) of the relative forward and backward errors and the error magnification factor for the following approximate solutions of the system $x_1+2x_2 = 3, 2x_1+4.01x_2 = 6.01$, $(a)[-10, 6]$ $(b)[-600, 301]$, (c) What is the (infinity norm) condition number of the coefficient matrix?

$x_1 + 2x_2 = 3$

$2x_1 + 4.01x_2 = 6.01$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4.01 & 6.01 \end{array}\right]$$

$\|A\|_\infty = \max(1+2, 2+4.01)$

$= 6.01$

$\begin{bmatrix} -10 \\ 6 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -10 \\ 6 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$

forward error $= 11$

backward $= \dfrac{\frac{11}{1}}{\left(\begin{bmatrix} 3 \\ 6.01 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4.01 \end{bmatrix}\begin{bmatrix} -10 \\ 6 \end{bmatrix}\right)}$

$= \dfrac{\frac{11}{1}}{\begin{bmatrix} 1 \\ 1.95 \end{bmatrix}}$

$= \dfrac{\frac{11}{1}}{\frac{1.95}{6.01}}$

$= 11 \cdot \dfrac{6.01}{1.95}$

$= 33.90$

$\begin{bmatrix} -600 \\ 301 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -600 \\ 301 \end{bmatrix} = \begin{bmatrix} 601 \\ -300 \end{bmatrix}$

forward $= 601$

back $= \begin{bmatrix} 3 \\ 6.01 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4.01 \end{bmatrix}\begin{bmatrix} 601 \\ -300 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\dfrac{601}{1} = \dfrac{1}{601}$

$\dfrac{\frac{601}{1}}{\frac{1}{6.01}} = 3612.01$

$$Cond = \|A\| \|A^{-1}\|$$

$$\|A\| = 6.01$$

$$A^{-1} = \frac{1}{4.01-4} \begin{bmatrix} 4.01 & -2 \\ -2 & 1 \end{bmatrix} = \frac{1}{0.01} \begin{bmatrix} 4.01 & -2 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4.01 & -200 \\ -200 & 100 \end{bmatrix}$$

$$\|A^{-1}\| = 601$$

$$Cond(A) = 6.01 \cdot 601$$

$$= 3612.01$$

Problem 3. Solve $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{b} = (2, 2, -1)^T$ and

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

by Gassian elimination with partial pivoting.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 2 & 1 & -1 & 2 \\ -1 & 1 & -1 & -1 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{P\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & 1 & 0 & 2 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow[R_3 + \frac{1}{2} R_1]{R_2 - \frac{1}{2} R_1} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{P\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[R_3 - \frac{1}{3} R_2]{} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 & 1 \end{array}\right] \qquad x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$2x_1 + x_2 - x_3 = 2$

$\frac{3}{2}x_2 - \frac{3}{2}x_3 = 0 \quad x_2 = x_3$

$x_3 = 1$

Problem 4, text book, page 101, Exercises 2 (b) (d). Find the $PA = LU$ factorization (using partial pivoting) of the following matrices:

(b) $\begin{bmatrix} 0 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow[\substack{R_1 \to R_3 \\ R_2 \to R_1 \\ R_3 \to R_2}]{P\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[R_2 + \frac{1}{2}R_1]{(-\frac{1}{2})} \begin{bmatrix} 2 & 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$

$\xrightarrow[R_3 + \frac{1}{2}R_2]{(-\frac{1}{2})} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ -\frac{1}{2} & -\frac{1}{2} & 4 \end{bmatrix}$

$$\begin{array}{cccc} P & A & L & U \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix} = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ -1 & -1 & 2 \end{bmatrix}$$

d)

$$\underset{P}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & \frac{1}{2} & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} -2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

Problem 5, text book, page 101, Exercises 3 (b). Solve the system using the $PA = LU$ factorization

$$\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \tag{1}$$

$$\underset{A}{} \quad \underset{X}{} \quad \underset{B}{}$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \xrightarrow{P\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 6 & 3 & 4 \\ 3 & 1 & 2 \\ 3 & 1 & 5 \end{bmatrix} \xrightarrow[(3)-\frac{1}{2}(1)]{(2)-\frac{1}{2}(1)} \begin{bmatrix} 6 & 3 & 4 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 3 \end{bmatrix} \xrightarrow{(3)-(2)} \begin{bmatrix} 6 & 3 & 4 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 6 & 3 & 4 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$Ly = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$y_1 = 1$

$y_2 = -\frac{1}{2}$

$y_3 = 3$

$$Ux = y$$

$$\begin{bmatrix} 6 & 3 & 4 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 3 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 &= -1 \\ x_2 &= 1 \\ x_3 &= 1 \end{aligned}}$$