STAT/MATH 415 HW#1

Sep 8, 2017

EXERCISES

5.1.5 The p.d.f of X is $f(x) = \theta x^{\theta-1},\ 0 < x < 1,\ 0 < \theta < \infty$. Let $Y = -2\theta \ln X$. How is Y distributed?

Answer: From the support of X, we can get $0 < Y < \infty$ and X-Y have 1-1 mapping.

$$Y = u(x) = -2\theta \ln X$$

$$X = v(y) = e^{-y/2\theta} \quad \text{so} \quad v'(y) = -\frac{1}{2\theta}e^{-y/2\theta}$$

$$f_Y(y) = f_X(v(y))\big|v'(y)\big| = \theta(e^{-y/2\theta})^{\theta-1}\frac{1}{2\theta}e^{-y/2\theta} = \frac{1}{2}e^{-y/2} \quad 0 < y < \infty$$

Thus Y follows the exponential distribution with parameter $\theta = 2$.

5.1.15 Let $Y = X^2$

a) Find the p.d.f of Y when the distribution of X is N(0, 1)

Answer: The support of Y is $Y \geq 0$

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad -\infty < x < \infty$$

$$Y = u(x) = x^2 \qquad X = \begin{cases} v_1(y) = \sqrt{y} & x \geq 0 \\ v_2(y) = -\sqrt{y} & x < 0 \end{cases}$$

$$g_1(y) = f(v_1(y))\big|v_1{}'(y)\big| = \frac{1}{\sqrt{2\pi}}e^{-y/2}\frac{1}{2\sqrt{y}} \qquad x \geq 0$$

$$g_2(y) = f(v_2(y))\big|v_2{}'(y)\big| = \frac{1}{\sqrt{2\pi}}e^{-y/2}\Big| - \frac{1}{2\sqrt{y}}\Big| \qquad x < 0$$

$$g(y) = g_1(y) + g_2(y) = \frac{1}{\sqrt{2\pi y}}e^{-y/2} \qquad y \geq 0$$

b) Find the p.d.f of Y when the p.d.f of X is $f(x) = (3/2)x^2,\ -1 < x < 1$

Answer: The support of Y is $0 \leq y < 1$

$$F_X(x) = \int_{-1}^{x} f(t)dt = \frac{t^3}{2}\Big|_{-1}^{x} = \frac{x^3+1}{2} \qquad -1 < x < 1$$

$$F_Y(y) = P(Y \leq y) = P(x^2 \leq y) = P(-\sqrt{y} \leq x \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = y^{3/2}$$

$$f_Y(y) = F_Y{}'(y) = \frac{3}{2}\sqrt{y} \qquad 0 \leq y < 1$$

5.2.1 Let X_1, X_2 denote two independent random variables, each with a $\chi^2(2)$ distribution. Find the joint pdf of $Y_1 = X_1$ and $Y_2 = X_2 + X_1$. Note that the support of Y_1, Y_2 is $0 < y_1 < y_2 < \infty$. Also, find the marginal pdf of each of Y_1 and Y_2. Are Y_1 and Y_2 independent?

Answer: $(X_1, X_2) - (Y_1, Y_2)$ is 1-1 mapping and $\chi^2(2) = \gamma(1, 2)$

$$f(x; 1, 2) = \frac{x^{1-1}e^{-x/2}}{2^1\Gamma(1)} = \frac{e^{-x/2}}{2} \quad \Rightarrow \quad f(x_1, x_2) = f(x_1)f(x_2) = \frac{1}{4}e^{-(x_1+x_2)/2}$$

$$x_1 = v_1(y_1, y_2) = y_1 \qquad x_2 = v_2(y_1, y_2) = y_2 - y_1 \quad \Rightarrow \quad J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$g(y_1, y_2) = f(v_1, v_2)|J| = \frac{1}{4}e^{-y_2/2} \qquad y_2 > 0 \quad \Rightarrow \quad g_1(y_1) = \int_{y_1}^{\infty} g(y_1, y_2)dy_2 = \frac{1}{2}e^{-y_1/2} \qquad y_1 > 0$$

$$\Rightarrow \quad g_2(y_2) = \int_0^{y_2} g(y_1, y_2)dy_1 = \frac{y_2}{4}e^{-y_2/2} \qquad y_2 > 0$$

$$g(y_1, y_2) \neq g_1(y_1)g_2(y_2) \quad \Rightarrow \quad Y_1 \text{ and } Y_2 \text{ are not independent}$$

5.2.5 Let the distribution of W be F(8, 4). Find the following

a) $F_{0.01}(8, 4)$

Answer: That is $0.01 = P(W \geq F_{0.01}(8, 4))$. According to the appendix table, we can get $W = 14.80$

b) $F_{0.99}(8, 4)$

Answer: $F_{1-0.01}(8, 4) = \dfrac{1}{F_{0.01}(4, 8)} \quad \Rightarrow \quad F_{0.99}(8, 4) = \dfrac{1}{7.01} \approx 0.1427$

c) $P(0.198 \leq W \leq 8.98)$

Answer: From the table, we can get $F_{0.025}(8, 4) = 8.98, F_{0.025}(4, 8) = \dfrac{1}{0.198}$

$$0.198 = \frac{1}{F_{0.025}(4, 8)} = F_{1-0.025}(8, 4) \quad \Rightarrow \quad P(W \leq F_{1-0.025}(8, 4)) = 0.025$$

$$8.98 = F_{0.025}(8, 4) \quad \Rightarrow \quad P(W \leq F_{0.025}(8, 4)) = 0.975$$

Thus, $P(0.198 \leq W \leq 8.98) = 0.975 - 0.025 = 0.950$

5.2.6 Let X_1 and X_2 have independent gamma distributions with parameters α, θ and β, θ respectively. Let $W = X_1/(X_1 + X_2)$. Use a method similar to that given in the derivation of F distribution to show that the pdf of W is

$$g(w) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1}(1 - w)^{\beta-1}, \quad 0 < w < 1$$

We say that W has a beta distribution with parameters α and β

Answer: $X_1 \sim \Gamma(\alpha, \theta)$ and $X_2 \sim \Gamma(\beta, \theta)$. Let $U = X_1 + X_2$, then

$$x_1 = v_1(u, w) = uw \qquad x_2 = v_2(u, w) = u(1 - w) \quad \Rightarrow \quad J = \begin{vmatrix} \frac{\partial v_1}{\partial u} & \frac{\partial v_1}{\partial w} \\ \frac{\partial v_2}{\partial u} & \frac{\partial v_2}{\partial w} \end{vmatrix} = \begin{vmatrix} w & u \\ 1 - w & -u \end{vmatrix} = -u$$

The support of W is (0, 1), of U is (0, ∞)

$$f_1(x_1;\alpha,\theta) = \frac{{x_1}^{\alpha-1}e^{-x_1/\theta}}{\theta^\alpha\Gamma(\alpha)} \quad x_1>0 \qquad \text{and} \qquad f_2(x_2;\beta,\theta) = \frac{{x_2}^{\beta-1}e^{-x_2/\theta}}{\theta^\beta\Gamma(\beta)} \quad x_2>0$$

$$f(x_1,x_2) = \frac{{x_1}^{\alpha-1}{x_2}^{\beta-1}e^{-(x_1+x_2)/\theta}}{\theta^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)} \qquad x_1,x_2>0$$

$$g(u,w) = f(v_1(u,w),v_2(u,w))\big|J\big| = \frac{(uw)^{\alpha-1}(u(1-w))^{\beta-1}e^{-u/\theta}}{\theta^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}u$$

$$= \frac{u^{\alpha+\beta-1}w^{\alpha-1}(1-w)^{\beta-1}e^{-u/\theta}}{\theta^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)} \qquad u>0,\ 0<w<1$$

$$g(w) = \int_0^\infty g(u,w)du$$

$$= \frac{w^{\alpha-1}(1-w)^{\beta-1}}{\theta^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty u^{\alpha+\beta-1}e^{-u/\theta}du$$

Note that $\dfrac{u^{\alpha+\beta-1}e^{-u/\theta}}{\Gamma(\alpha+\beta)\theta^{\alpha+\beta}}$ is the form of gamma distribution

$$= \frac{w^{\alpha-1}(1-w)^{\beta-1}}{\theta^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\Gamma(\alpha+\beta)\theta^{\alpha+\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}w^{\alpha-1}(1-w)^{\beta-1}, \quad 0<w<1$$

Thus W has a beta distribution with parameters α and β

5.3.4 Let X_1 and X_2 be a random sample of size $n=2$ from the exponential distribution with pdf $f(x)=2e^{-2x},\ 0<x<\infty$. Find

a) $P(0.5<X_1<1.0,\ 0.7<X_2<1.2)$

Answer: $X_1 \sim Exp(2)$, $X_2 \sim Exp(2)$ and we can assume that X_1, X_2 are independent.

$$F(x) = 1-e^{-2x},\ x\geq 0$$

$$P(0.5<X_1<1.0) = F(1.0)-F(0.5) = e^{-1}-e^{-2} = 0.2325$$

$$P(0.7<X_2<1.2) = F(1.2)-F(0.7) = e^{-1.4}-e^{-2.4} = 0.1559$$

$$P(0.5<X_1<1.0,\ 0.7<X_2<1.2) = P(0.5<X_1<1.0)P(0.7<X_2<1.2) = 0.0362$$

b) $E[X_1(X_2-0.5)^2]$

Answer: $E(X_1)=E(X_2)=\dfrac{1}{2}$. $V(X_1)=V(X_2)=\dfrac{1}{4}$

$$E[X_1(X_2-0.5)^2] = E(X_1)E[(X_2-0.5)^2] = E(X_1)V(X_2)$$

$$= 0.5(0.25) = 0.125$$

5.3.11 Let X_1, X_2, X_3 be three independent random variables with binomial distributions $b(4,1/2), b(6,1/3), b(12,1/6)$ respectively. Find

a) $P(X_1=2, X_2=2, X_3=5)$

Answer: Since X_1, X_2, X_3 are independent, then

$$\begin{aligned}
P(X_1 = 2, X_2 = 2, X_3 = 5) &= P(X_1 = 2)P(X_2 = 2)P(X_3 = 5) \\
&= \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 \times \binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 \times \binom{12}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^7 \\
&= \frac{6}{2^4} \times \frac{15 \times 16}{3^6} \times \frac{792 \times 5^7}{6^{12}} = 0.0035
\end{aligned}$$

b) $E(X_1X_2X_3)$

Answer: Due to their independence, we can get $E(X_1X_2X_3) = E(X_1)E(X_2)E(X_3) = 2 \times 2 \times 2 = 8$

c) The mean and the variance of $Y = X_1 + X_2 + X_3$

Answer: Based on $Var(X) = E(X^2) - E(X)^2$

$$E({X_1}^2) = 1 + 2^2 = 5 \qquad E({X_2}^2) = \frac{4}{3} + 2^2 = \frac{16}{3} \qquad E({X_3}^2) = \frac{5}{3} + 2^2 = \frac{17}{3}$$

$$\begin{aligned}
E(Y^2) &= E({X_1}^2 + {X_2}^2 + {X_3}^2 + 2X_1X_2 + 2X_1X_3 + 2X_2X_3) \\
&= E({X_1}^2) + E({X_2}^2) + E({X_3}^2) + 2E(X_1X_2) + 2E(X_1X_3) + 2E(X_2X_3) \\
&= 5 + \frac{16}{3} + \frac{17}{3} + 8 + 8 + 8 = 40 \\
E(Y) &= E(X_1) + E(X_2) + E(X_3) = 2 + 2 + 2 = 6 \\
Var(Y) &= E(Y^2) - E(Y)^2 = 40 - 6^2 = 4
\end{aligned}$$