

Math 486

Lesson 4 Homework

Due Tues, June 14 at 11:59 on Gradescope

Instructions

Please refer to the solution guidelines posted on Canvas under Course Essentials.

Exercise 1.

Suppose we have a game with N players ($N \geq 3$) and suppose that the strategy profile $s^* = (s_1^*, s_2^*, \dots, s_N^*)$ is a Nash equilibrium.

- (a) Describe in words what it means for the strategy profile s^* to be a Nash equilibrium. This description can be informal—it should reflect the way that you think about the Nash equilibrium concept—but it also must be conceptually correct.
- (b) State, mathematically, what it means for s^* to be a Nash equilibrium. You need to use mathematical notation involving payoffs in an N -player game here.

Remark: The purpose here is not memorization of a mathematical definition, but to connect the mathematical definition with the conceptual idea expressed in words in part (a). The overall goal is to strengthen your understanding of what a Nash Equilibrium is, so that you can approach problems requiring you to show that a particular strategy profile is or is not a Nash equilibrium.

a) make s^* to be a Nash equilibrium, it means for the player, there is no better strategy while others choose this strategy, vice versa, and at this time, it is a Nash equilibrium.

$$b) \pi_i(s_i^*, s_{-i}^*) - \pi_i(s_i, s_{-i}^*) \geq 0 \quad \text{for all } s_i \in S_i$$

Exercise 2.

Consider the following two player game. Each player has 1 (divisible) unit of resource.

Player 1 chooses some amount of the resource $s_1 \in [0, 1]$. Player 2 simultaneously chooses some amount $s_2 \in [0, 1]$.

The player that chooses the higher amount wins the total of what remains: $(1 - s_1) + (1 - s_2)$. The player that chooses the lower amount gets a payoff of zero (they forfeit the remaining amount to the other player). In the case of a tie, they split what remains.

The payoff for each player can be written as follows:

$$\pi_1(s_1, s_2) = \begin{cases} (1 - s_1) + (1 - s_2), & \text{if } s_1 > s_2, \\ 1 - s_1, & \text{if } s_1 = s_2, \\ 0, & \text{if } s_1 < s_2 \end{cases}, \quad \pi_2(s_1, s_2) = \begin{cases} (1 - s_1) + (1 - s_2), & \text{if } s_2 > s_1, \\ 1 - s_2, & \text{if } s_1 = s_2, \\ 0, & \text{if } s_2 < s_1 \end{cases}$$

- (a) Show that when $s_1 = s_2 = 1$, this is a Nash equilibrium of the game.
- (b) Show that there are no Nash equilibria where $s_2 < s_1 < 1$.
- (c) Show that there are no Nash equilibria where $s_2 < s_1$ with $s_1 = 1$.
- (d) Show that there are no Nash equilibria where $s_1 = s_2 < 1$.

a) $s_1 = s_2 = 1$

$(0, 0)$ is Nash Equ

$$\pi_1(1, 1) = 1 - 1 = 0$$

$$\pi_2(1, 1) = 0$$

b) $s_2 < s_1 < 1$

$$\pi_1(s_1, s_2) = (1 - s_1) + (1 - s_2)$$

$$\pi_2(s_1, s_2) = 0$$

No Nash Equ (s_1, s_2)
where $s_2 < s_1 < 1$

c) $s_2 < s_1 = 1$

$$\pi_1(s_1, s_2) = 1 - s_2$$

$$\pi_2(s_1, s_2) = 0$$

No Nash Equ (s_1, s_2)
where $s_2 < s_1 = 1$

d) $s_1 = s_2 < 1$

$$\pi_1(s_1, s_2) = 1 - s_1$$

$$\pi_2(s_1, s_2) = 1 - s_2$$

No Nash Equ (s_1, s_2)
where $s_2 = s_1 < 1$

Problem 1.

Recall the Pick a Number Interactive game. Players simultaneously choose an integer in the set $\{1, 2, \dots, 100\}$. The player whose choice is nearest to $3/4$ of the average choice gets a payoff of 4. Other players get a payoff of zero. In the event of a tie, any players who are equally close to $3/4$ of the average receive a payoff of 4.

Assume that the number of players is large enough that the effect on the average by any single player is negligible. Denote the payoff to player i as $\pi_i(s_i, \bar{s})$ where \bar{s} is the average. Under our assumption \bar{s} does not change if player i changes s_i (with the other players' strategies fixed).

- (a) Show that for any $k \in \{1, \dots, 100\}$, if $s_i = k$ for all i (if all players choose the same amount), then the resulting strategy profile is a Nash Equilibrium.
- (b) For which values of k is the Nash equilibrium from part (a) a **strict Nash equilibrium**. Show that the Nash equilibrium is strict.

$$\{1, 2, \dots, 100\}$$

most is 100, $100 \cdot \frac{3}{4} = 75$ so every player will choose

from $\{1, 2, \dots, 75\}$

$$\begin{cases} \pi_i(s_i, \bar{s}) = 4 & \text{if choose correctly} \\ \pi_i(s_i, \bar{s}) = 0 & \text{if choose incorrectly} \end{cases}$$

$$b) \quad \text{if } k \text{ is correct, } \pi_i(s_i^*, \bar{s}) > \pi_i(s_i, \bar{s})$$

$$4 > 0$$

Problem 2.

In the Lesson 3 lectures we considered a second-price auction. We showed that the strategy of bidding your value weakly dominates any other strategy. Here we consider a first-price auction. You will show that one particular strategy profile is a Nash equilibrium and that bidding above your value is weakly dominated.

Suppose there are N players participating in a sealed-bid first-price auction ($N \geq 3$). Each player i has a value, $v_i > 0$ for the object being auctioned. Player i makes a sealed-bid $b_i \in [0, \infty)$. You can assume the bids are made simultaneously or that players only know their own bid, so that player i 's strategy set is $S_i = [0, \infty)$. The player with the highest bid will win the object. The winner pays the amount of their bid, and so has the payoff $v_i - b_i$ (the value gained minus the cost). The other players pay nothing and have a payoff of zero.

We assume that the values are ranked in the following way (the players know this):

$$v_1 \geq v_2 \geq v_3 \geq \dots \geq v_N > 0$$

In the case of a tie, where two players have the same highest bid, the winner is the player with the lower index. For example if player 2 and player 4 both have the same highest bid, player 2 will win the object and player 4 will get a payoff of zero.

For player i , let h_i be the maximum bid of the other players, $h_i = \max_{j \neq i} b_j$. Then the payoff function for player i can be written as

$$\pi_i(b_i, b_{-i}) = \begin{cases} v_i - b_i, & \text{if } b_i > h_i \\ v_i - b_i, & \text{if } b_i = h_i, \text{ and } i \text{ is the lowest index with bid equal to } h_i, \\ 0, & \text{otherwise} \end{cases}$$

- (a) Show that if $b_1 = v_2$ and $b_i = v_i$ for all $i \geq 2$, then this is a Nash equilibrium of the game.
- (b) Show that any strategy $b_i > v_i$ is weakly dominated by the strategy $b'_i = v_i$
- (c) Show that the strategy $b_i = v_i$ is weakly dominated by some $b'_i < v_i$.

Note: In part (b), you must demonstrate or explain why both conditions in the definition of a weakly dominated strategy are satisfied.

a) $b_1 = v_2 \quad b_i = v_i \quad i \geq 2$

$$\pi_i(b_i, b_{-i}) = 0$$

$$\pi_1(b_1, b_{-1}) = v_1 - v_2 > 0$$

either condition,
player 1 will win

and for i , either b_i goes higher or lower, there will be no improvement

So too for player 1, either b goes higher or lower, there will be no improvement

So it is a Nash

b) $b_i > v_i$ is weakly dominated by the strategy $b_i' = v_i$

When $b_i' = v_i$, $\pi_i(b_i, b_{-i})$ will be smaller or equal to $\pi_i(b_i', b_{-i})$ or $\pi_i(b_i, v_{-i})$ and exist one b_{-i} such $\pi_i(b_i, b_{-i})$ smaller to $\pi_i(b_i', b_{-i})$

c) $b_i = v_i$ is weakly dominated by some $b_i' < v_i$ when $\pi_i(b_i', b_{-i}') > \pi_i(b_i, b_{-i})$ and exist b_{-i}' that $\pi_i(b_i', b_{-i}')$ will be bigger than $\pi_i(b_i, b_{-i})$