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Running time analysis (I)

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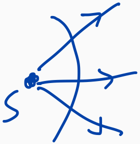
$$\underline{v(f)} = \sum_{e \text{ out of } s} v(f)$$



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e

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□

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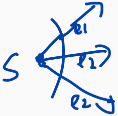

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Finding an s-t path takes $O(|V| + |E|) = O(|E|)$ time (BFS)

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So total running time is $O(C \cdot |E|)$

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residual capacity : $f(e)$

$\boxed{\text{bottleneck}(P, f) \le f(e)}$

$c_e \geq f(e) \geq f'(e) = f(e) - \text{bottleneck}(P, f) \geq f(e) - f(e) = 0$

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$\forall v \in V - \{s, t\}$ $\quad \sum_{e \text{ out of } v} f(e) = \sum_{e \text{ into } v} f(e)$

- **Conservation condition.** It suffices to observe that for every vertex, additional amount of flow, 0, or $\mathrm{bottleneck}(P, f)$ entering this vertex equals the additional amount of flow, 0, or $\mathrm{bottleneck}(P, f)$ leaving it

□

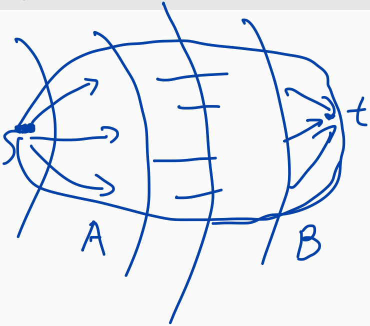
Correctness of Ford-Fulkerson (I)

Flow and Cut

Definition

An **s-t cut** is a partition of V, (A, B) where $s \in A$ and $t \in B$

$\{s\}$, $V - \{s\}$



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An **s-t cut** is a partition of V, (A, B) where $s \in A$ and $t \in B$

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The **capacity** of the cut is

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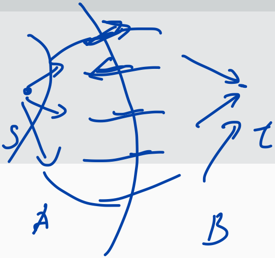

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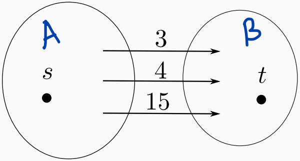
capacity: 22



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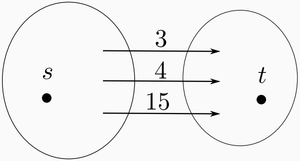

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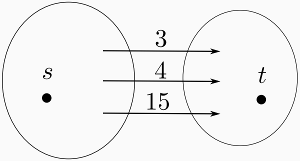

The flow must have a value ≤ 22
Capacity of a cut put a bound on the flow value

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$\implies \boxed{f^{\text{out}}(v) - f^{\text{in}}(v) = 0}$ for all $v \neq s, t$

$v(f) = f^{out}(s) - f^{in}(s) + 0 + 0 + 0 + 0 + 0 + 0 \cdots + 0$

$f^{out}(v) - f^{in}(v)$

$= \sum_{v \in A} (f^{out}(v) - f^{in}(v))$

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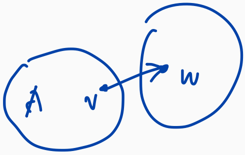

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- if $v, w \in A$, this edge contributes 0 in the summation
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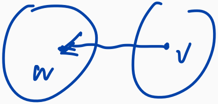

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Proof. Let A^* be the set of vertices reachable from s in G_f. Let B^* be $V - A^*$. We need to show (A^*, B^*) is an s-t cut

need to show $s \in A^*$, $t \in B^*$

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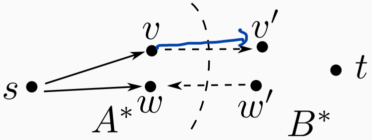


Correctness of Ford-Fulkerson (III)

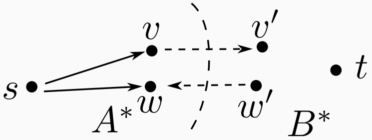
The upper bound $c(A, B)$ is achievable by Ford-Fulkerson

Lemma

Let f be a flow s.t. there's no s-t path in G_f. Then there exists an s-t cut (A^*, B^*) s.t. $v(f) = c(A^*, B^*)$

Proof. Let A^* be the set of vertices reachable from s in G_f. Let B^* be $V - A^*$. We have the follow facts

- (A^*, B^*) is an s-t cut: $s \in A^*$, $t \in B^*$ (no s-t path in G_f)
- for all edge $e = (v, v') \in E$ with $v \in A^*, v' \in B^*$, we have $f(e) = c_e$ Otherwise, (v, v') is an edge in G_f with capacity $c_e - f(e) \neq 0$. Forward edge. So v' is reachable from s in G_f (contradiction)



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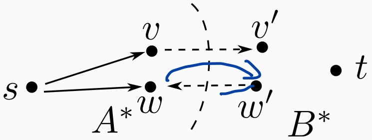

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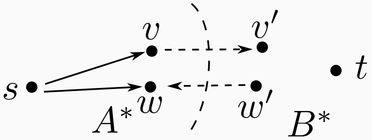

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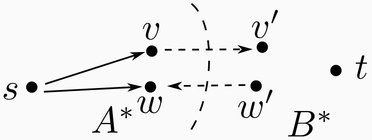

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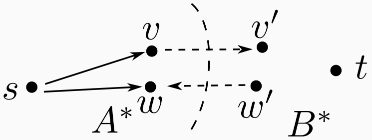

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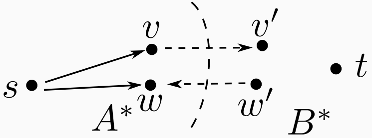

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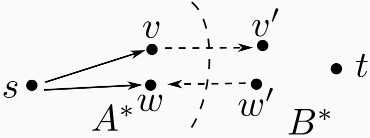

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&= \sum_{e \text{ out of } A^*} c_e = c(A^*, B^*)
\end{aligned}
$$

□

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- Given a flow of max value, can compute a cut of minimum capacity in $O(|E|)$ time
- If all capacities of a flow network are integers, then there is a max flow f s.t. $f(e)$ is an integer for all e