Orthogonal polynomials.

let $w(x) \geq 0$ on $[a,b]$

let $\phi_0, \phi_1, \cdots \phi_n$ be polynomials

with $\deg(\phi_j) = j$ they are

Orthogonal polynomials if

$$\int_a^b \phi_i(x)\phi_j(x)\,w(x)\,dx = \begin{cases} 0, & i \neq j \\ \alpha_j > 0, & i = j \end{cases}$$

Example:

$a = 0$, $b = 1$ $\qquad w(x) = \sqrt{x}$

$$\phi_0 = 1$$

$$\phi_1 = x + c$$

$$\int_0^1 \phi_0(x)\phi_1(x)\,w(x)\,dx = 0$$

$$\Rightarrow \int_0^1 (x+c)\sqrt{x}\,dx = 0$$

$$= \int_0^1 x^{\frac{3}{2}}\,dx + c\int_0^1 x^{\frac{1}{2}}\,dx = 0$$

$$= \frac{2}{5} + c \cdot \frac{2}{3} = 0$$

$$c = -\frac{3}{5}$$

Set $\phi_2 = x^2 + Cx + D$

$$\Rightarrow \int_0^1 (x^2 + Cx + D)\sqrt{x}\, dx = 0$$

$$\int_0^1 (x^2 + Cx + D)(x - \tfrac{3}{5})\sqrt{x}\, dx = 0$$

We can get C & D

Alternatively

set $\phi_2 = x^2 + C_1\phi_1 + C_0\phi_0$

$$\int_0^1 (x^2 + C_1\phi_1 + C_0\phi_0)\,\phi_0\sqrt{x}\, dx = 0$$

$$\int_0^1 x^{\frac{5}{2}} dx + 0 + C_0\int_0^1 x^{\frac{1}{2}} dx = 0$$

$$\frac{2}{7} + C_0\,\frac{2}{3} = 0$$

$$C_0 = -\frac{3}{7}$$

$$\int_0^1 (x^2 + c_1\phi_1 + c_0\phi_0)\,\phi_1(x)\sqrt{x}\,dx = 0$$

$$\phi_1 = x - \frac{3}{5}$$

$$\int_0^1 x^2\phi_1(x)\sqrt{x}\,dx + c_1\int_0^1 \phi_1^2\sqrt{x}\,dx + 0 = 0$$

$$\int_0^1 x^{\frac{7}{2}} - \frac{3}{5}x^{\frac{5}{2}}\,dx$$

$$\int_0^1 x^{\frac{7}{2}}\,dx - \frac{3}{5}\int_0^1 x^{\frac{5}{2}}\,dx$$

$$= -c_1\left(\int_0^1 x^{\frac{5}{2}}\,dx - \frac{6}{5}\int_0^1 x^{\frac{3}{2}}\,dx + \frac{9}{25}\int_0^1 x^{\frac{1}{2}}\,dx\right)$$

So we get c_1

Example 2 le gendre Polynomials.

$a=-1$ $b=1$ $W(x)=1$

$\phi_0=1$ $\phi_1=x$

$$\phi_{k+1}=\frac{2k+1}{k+1}x\phi_k-\frac{k}{k+1}\phi_{k-1} \quad k\geq 2$$

3-term recurrence relation

$$\int_{-1}^{1}\phi_k^2\,dx=\frac{2}{2k+1}$$

Matlab

input $\vec{x}=(x_1 \cdots x_m)$

$\phi_0=\text{ones}(\text{size}(\vec{x}))$; $\phi_1=\vec{x}$

for $k=1$ to $n-1$

$$\phi_{k+1}=\frac{2k+1}{k+1}x\,\phi_k-\frac{k}{k+1}\phi_{k-1}$$

end

Output $(\phi_k(x_1),\phi_k(x_2),\cdots\cdots\phi_k(x_m))$ $k=0,1\cdots n$

L.S.P. $\min \int_{-1}^{1} (f(x) - P_n(x))^2 \, dx$

$$P_n(x) = c_0 \phi_0 + c_1 \phi_1 + \cdots + c_n \phi_n$$

$$c_k = \frac{\int_{-1}^{1} f(x) \phi_k(x) \, dx}{\left(\int_{-1}^{1} \phi_k(x)^2 \, dx \right) = \frac{2}{2k+1}}$$

Algorithm (L.S.P.)

$\vec{x} = (x_1, x_2, \cdots x_m)$ quadrature point

$A_1, A_2, \cdots A_m$ quadrature weights.

$[\phi_0, \phi_1, \cdots \phi_n] = \Phi(\vec{x})$

for $k = 0, 1 \cdots n$

$$c_k = \frac{2k+1}{2} \sum_{i=1}^{m} A_i \phi_k(x_i) f(x_i)$$

end