STAT/MATH 415 HW#2

September 15, 2016

EXERCISES

5.4.4 Generalize exercise 5.4.3 by showing that the sum of n independent Poisson random variables with respective means $\mu_1, \mu_2, ..., \mu_n$ is Poisson with mean $\mu_1 + \mu_2 + ... + \mu_n$

Answer: $X_i \sim Pois(\mu_i)$, thus the moment generating function is $M_{X_i}(t) = e^{\mu_i(e^t-1)}$. Then for Y

$$M_Y(t) = E(e^{ty}) = E(e^{t(x_1+x_2+...+x_n)}) = E(e^{tx_1})E(e^{tx_2})...E(e^{tx_n})$$
$$= e^{\mu_1(e^t-1)}e^{\mu_2(e^t-1)}...e^{\mu_n(e^t-1)} = e^{(\mu_1+\mu_2+...+\mu_n)(e^t-1)}$$

Based on mgf, we can get $Y \sim Pois(\mu_1 + \mu_2 + ... + \mu_n)$, and $E(Y) = \mu_1 + \mu_2 + ... + \mu_n$

5.4.5 Let $Z_1, Z_2, ..., Z_7$ be a random sample from the standard normal distribution N(0, 1). Let $W = {Z_1}^2 + {Z_2}^2 + ... + {Z_7}^2$. Find $P(1.69 < W < 14.07)$

Answer: Since $Z_i \sim N(0,1)$, then ${Z_i}^2 \sim \chi^2(1)$, then $W \sim \chi^2(7)$. From chi-square table, we can get

$$\chi^2_{0.975} = 1.690 \qquad \chi^2_{0.050} = 14.067$$
$$P(1.69 < W < 14.07) = P(W > 1.690) - P(W > 14.07) = 0.95 - 0.025 = 0.925$$

5.4.14 The number of accidents in a period of one week follows a Poisson distribution with mean 2. The numbers of accidents from week to week are independent. What is the probability of exactly seven accidents in a given three weeks?

Answer: Let $X_i, i = 1, 2, 3$ denote the number of accidents in the given three weeks. Since $X_i \sim Pois(2)$ and independent, then from 5.4.4 we can get $Y = X_1 + X_2 + X_3 \sim Pois(6)$

$$P(Y = 7) = \frac{6^7 e^{-6}}{7!} = 0.1377$$

5.5.7 Suppose the distribution of the weight of a prepackaged "1-pound bag" of carrots is $N(1.18, 0.07^2)$ and the distribution of the weight of a prepackaged "3-pound bag" of carrots is $N(3.22, 0.09^2)$. Select bags at random, and find the probability that the sum of three 1-pound bag exceeds the weight of one 3-pound bag.

Answer: Let $X_i, i = 1, 2, 3$ denote the weight of three selected 1-pound bag, W denote the weight of selected 3-pound bag. $Y = X_1 + X_2 + X_3 - W$

$$X_i \sim N(1.18, 0.07^2) \qquad W \sim N(3.22, 0.09^2)$$
$$Y \sim N(1.18 \times 3 - 3.22, 0.07^2 \times 3 + 0.09^2) = N(0.32, 0.0228)$$
$$P(X_1 + X_2 + X_3 > W) = P(Y > 0) = P(Z > \frac{0 - 0.32}{\sqrt{0.0228}}) = P(Z > -2.12) = P(Z < 2.12) = 0.9830$$

5.6.2 Let $Y = X_1 + X_2 + ... + X_{15}$ be the sum of a random sample of size 15 from the distribution whose pdf is $f(x) = (3/2)x^2, \ -1 < x < 1$. Using the pdf of Y, we find that $P(-0.3 \le Y \le 1.5) = 0.22788$. Use central limit theorem to approximate this probability.

Answer: To estimate the sum of X_i, let $\bar{X} = \dfrac{X_1 + ... + X_{15}}{15} = \dfrac{Y}{15}$

$$E(X) = \mu = \int_{-1}^{1} xf(x)dx = 0 \qquad E(X^2) = \int_{-1}^{1} x^2 f(x)dx = \frac{3}{5} \qquad Var(X) = \sigma^2 = E(X^2) - E(X)^2 = \frac{3}{5}$$

$$\bar{X} \text{ approximately} \sim N(\mu, \frac{\sigma^2}{15}) = N(0, \frac{1}{25})$$

$$Y = 15\bar{X} \text{ approximately} \sim N(0, \frac{15^2}{25}) = N(0, 3^2)$$

$$P(-0.3 \le Y \le 1.5) = P(\frac{-0.3 - 0}{3} \le Z \le \frac{1.5 - 0}{3}) = P(-0.1 \le Z \le 0.5) = 0.6915 - (1 - 0.5398) = 0.2313$$

Based on central limit theorem, our estimate of $P(-0.3 \le Y \le 1.5) = 0.2313$

5.6.7 Let X equal the maximal oxygen intake of a human on a treadmill, where the measurements are in ml of oxygen per minute per kilogram of weight. Assume that, for a particular population, the mean is $\mu = 54.030$ and the standard deviation is $\sigma = 5.8$. Let $\bar{X}$ be the sample mean of a random sample of size $n = 47$. Find $P(52.761 \le \bar{X} \le 54.453)$, approximately.

Answer: Based on CLT, we can get

$$E(\bar{X}) = \mu = 54.030 \qquad Var(\bar{X}) = \frac{\sigma^2}{n} = 0.7157 \qquad \bar{X} \text{ approximately} \sim N(54.030, 0.7157)$$

$$P(52.761 \le \bar{X} \le 54.453) = P(\frac{52.761 - 54.030}{\sqrt{0.7157}} \le Z \le \frac{54.453 - 54.030}{\sqrt{0.7157}}) = P(-1.5 \le Z \le 0.5) = \Phi(0.5) - \Phi(-1.5)$$

$$= 0.6915 - (1 - 0.9332) = 0.6247$$

So the probability is approximately 0.6247