CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Greedy algorithms

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Minimum Spanning Tree

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

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Using linked list:

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$$\{a, b, c\} \quad \text{head} \to a \to b \to c \qquad \text{find\_set}(b):$$

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$$\{a, b, c\} \quad \text{head} \to a \to b \to c \qquad \text{find\_set}(b)\text{: } O(1)$$

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$\text{make\_set}(v)$:

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find_set(b): $O(1)$

make_set(v): $O(1)$

Running time of Kruskal's algorithm (I)

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$$\{d, e\} \quad \text{head} \to d \to e$$

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$\text{union}(a, b)$

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Cost of union:

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Cost of union: O(length of the shorter list)

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$\text{union}(a, b)$ $\qquad \text{head} \to a \to b \to c \to d \to e$

Cost of union: O(length of the shorter list)

Using an array to implement it:

vertex	1	2	3	4	5
head	1	1	1	4	4

$\xrightarrow{\text{union}}$

Running time of Kruskal's algorithm (I)

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$\xrightarrow{\text{union}}$

1	2	3	4	5
1	1	1	1	1

Running time of Kruskal's algorithm (II)

```
def KRUSKAL_MST(undirected G = (V, E), weights w = (w_e)_{e∈E}):
    Set A := { };
    for v ∈ V:
        make_set(v)
    Sort E in increasing order of edge weights
    for (u, v) ∈ E:
        if find_set(u) ≠ find_set(v):
            A := A ∪ {(u, v)};
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Worst-case cost for union:

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Worst-case cost for union: $O(|V|)$. What about the cost for lines 6-9?

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Worst-case cost for union: $O(|V|)$. What about the cost for lines 6-9?

Consider a single $v \in V$. Once it's touched in some union operation, the size of the set at least doubles.

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At most $|V|$ vertices are involved in union operations, so the total cost of lines 6-9:

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At most $|V|$ vertices are involved in union operations, so the total cost of lines 6-9: $O(|V| \log |V|)$

Total cost of the algorithm: $O(|E| \log |V|)$

Alternative data structure

The linked-list implementation is good enough, but there exist better data structures to improve the worst-case cost for union

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Directed tree disjoint set:

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Directed tree disjoint set:

$$\{a\}$$

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$$\{a\} \quad a$$

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Directed tree disjoint set:

$$\{a\} \quad a \qquad \{a, b\}$$

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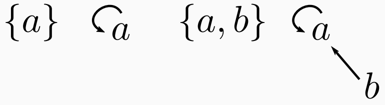
Directed tree disjoint set:



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Directed tree disjoint set:

$$\{a\} \quad \begin{tikzcd} a \arrow[loop left] \end{tikzcd} \qquad \{a, b\} \quad \begin{tikzcd} a \arrow[loop left] & \\ & b \arrow[ul] \end{tikzcd} \qquad \{a, b, c\}$$

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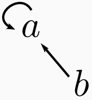

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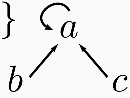

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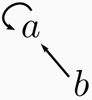

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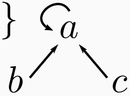

$\{a, b, c, d, e\}$

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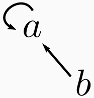

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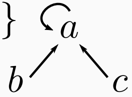

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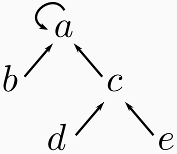

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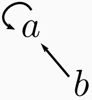

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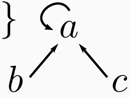

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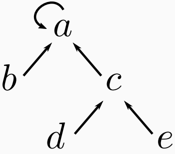

Definition

$\pi(x)$: parent of x

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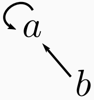
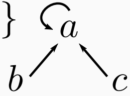
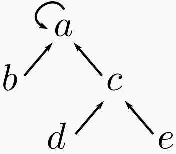
$\{a, b\}$ 

$\{a, b, c\}$ 

$\{a, b, c, d, e\}$ 

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$\pi(x)$: parent of x

root node: x s.t. $\pi(x) = x$

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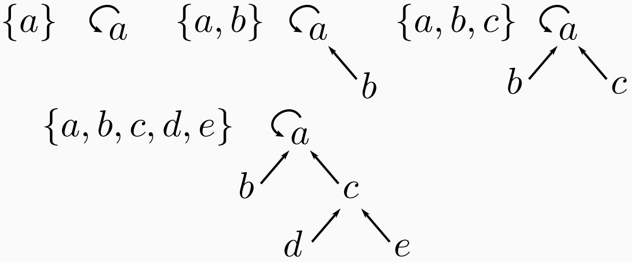

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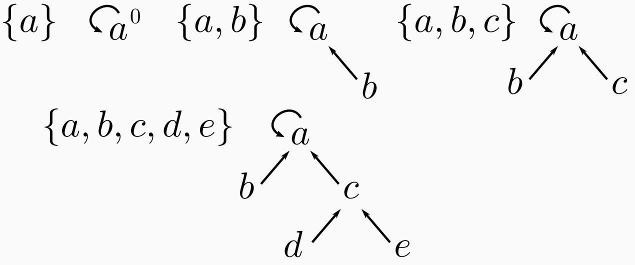

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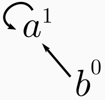

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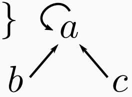

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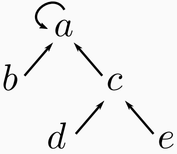

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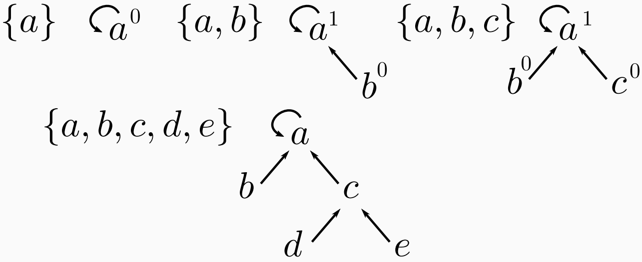

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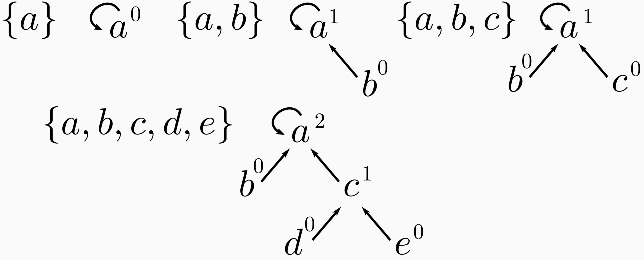

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Operations of direct tree disjoint set (I)

- $\mathrm{make\_set}(v)$

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- make_set(v)

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- $\mathrm{find\_set}(v)$

Operations of direct tree disjoint set (I)

- $\text{make\_set}(v)$

  def $\text{make\_set}(v)$**:**

  $\pi(v) := v$;

  $\text{rank}(v) = 0$;

  Cost: $O(1)$

- $\text{find\_set}(v)$

  def $\text{find\_set}(v)$**:**

  while $v \neq \pi(v)$**:**

  $v := \pi(v)$;

  return v;

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- what about union?

Operations of direct tree disjoint set (II)

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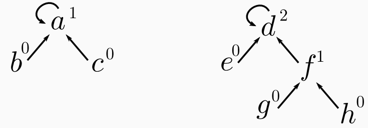

Operations of direct tree disjoint set (II)

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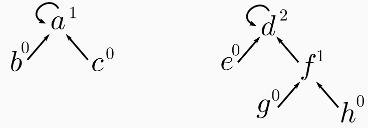
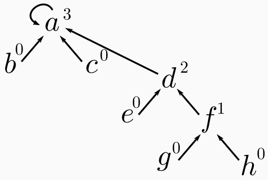


Option 1



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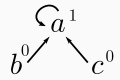

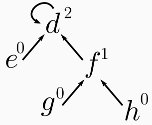

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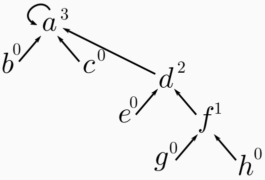
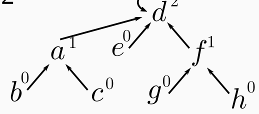


Option 2



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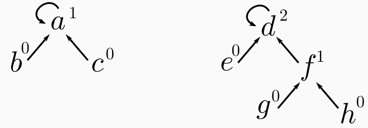

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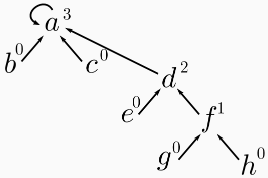

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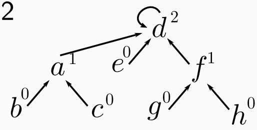

better!

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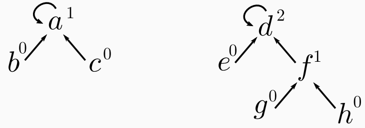

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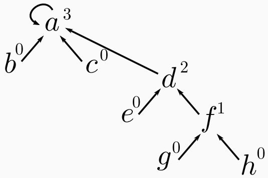

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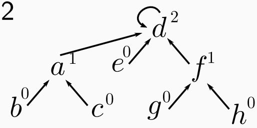

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Basic idea: attach the smaller ranked tree to a larger one

Operations of direct tree disjoint set (II)

def union(x, y)**:**

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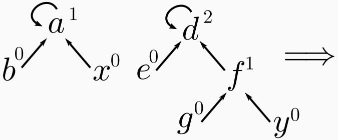

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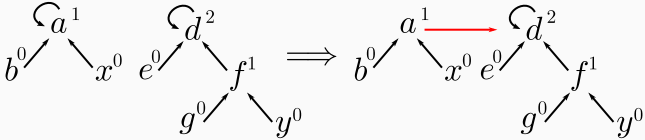

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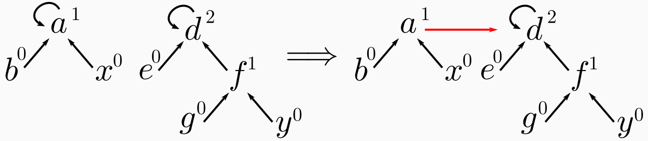

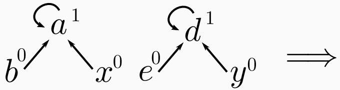

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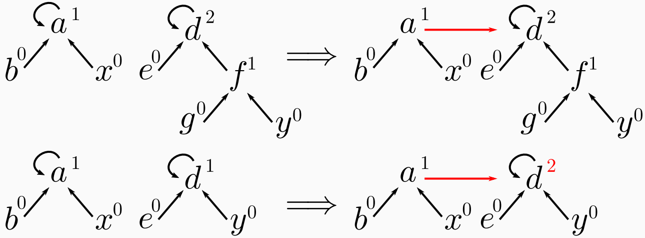

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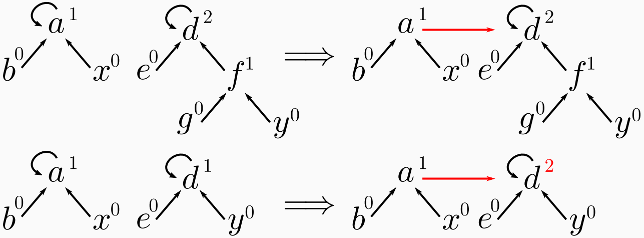
Cost: dominated by find_set



Cost of find_set using directed tree disjoint set

Observation

Root note with rank k is formed by the merge of two rank $k-1$ trees

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By induction: base case has $k = 0$ and $2^0 = 1$.

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Total running time of Kruskal using directed tree disjoint set

```
def KRUSKAL_MST(undirected G = (V, E), weights w = (w_e)_{e∈E}):
    Set A := { };
    for v ∈ V:
        make_set(v) ;                                   // O(|V|)
    Sort E in increasing order of edge weights ;        // O(|E| log |V|)
    for (u, v) ∈ E:
        if find_set(u) ≠ find_set(v):
            A := A ∪ {(u, v)};
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1 **def** KRUSKAL_MST(*undirected* $G = (V, E)$, *weights* $w = (w_e)_{e \in E}$)**:**

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6   **for** $(u, v) \in E$**:**

7   **if** $\text{find\_set}(u) \neq \text{find\_set}(v)$**:**

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Lines 6-9:

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Prim's algorithm

Intuition: iteratively grows the tree

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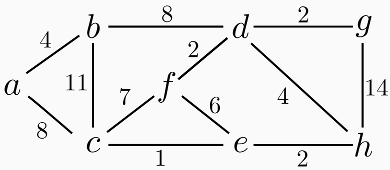

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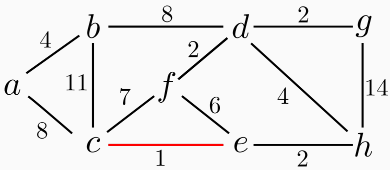

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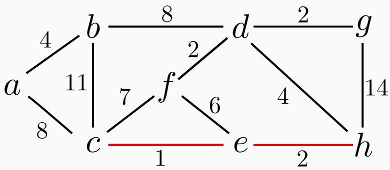

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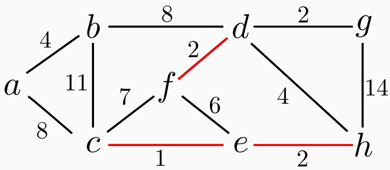

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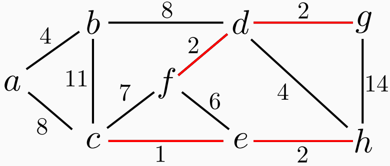

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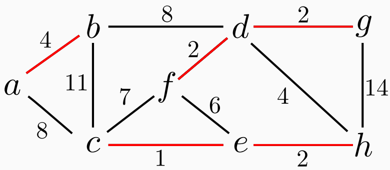

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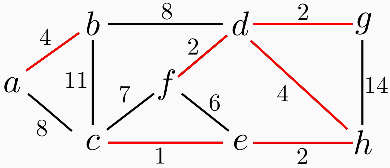

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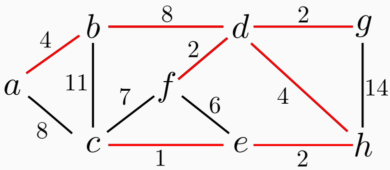

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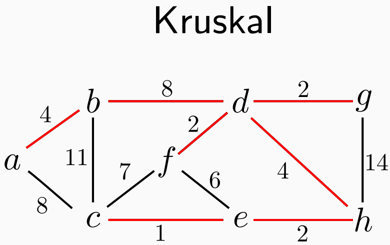

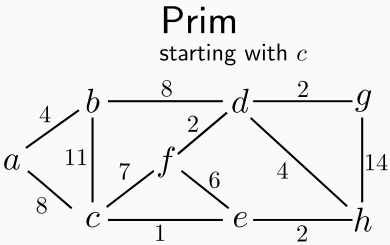

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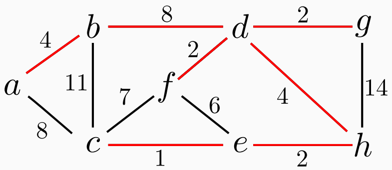
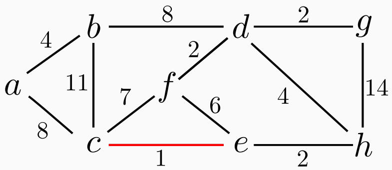


Prim

starting with c



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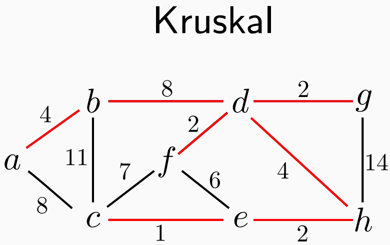
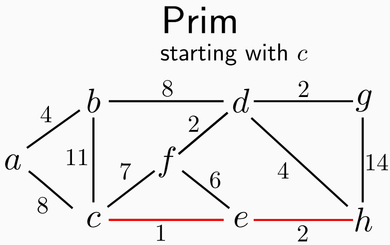
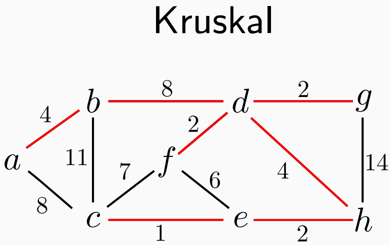
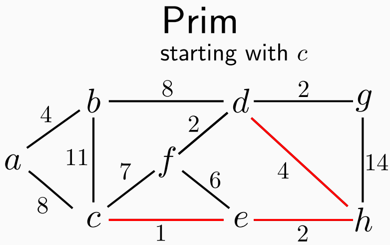




Intuition: iteratively grows the tree

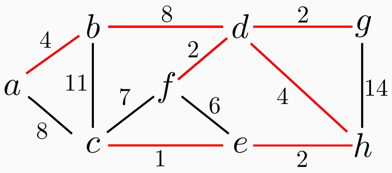




Prim's algorithm

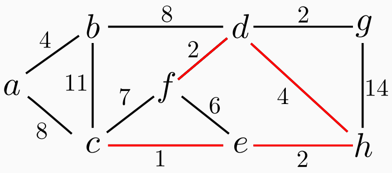
Intuition: iteratively grows the tree

Kruskal



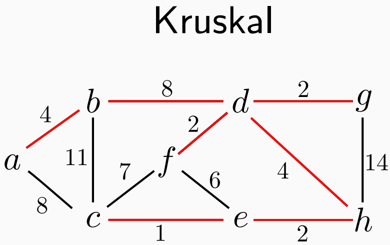
Prim

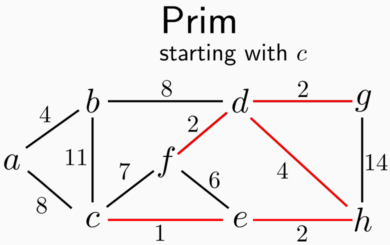
starting with c



Prim's algorithm

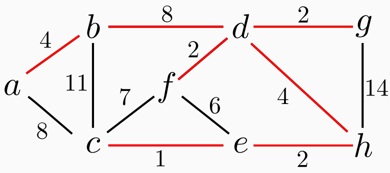
Intuition: iteratively grows the tree





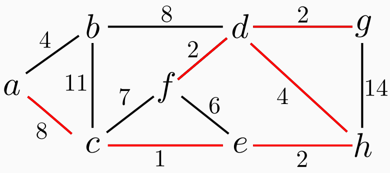
Intuition: iteratively grows the tree

Kruskal



Prim

starting with c



Prim's algorithm

Intuition: iteratively grows the tree

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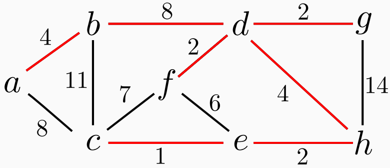
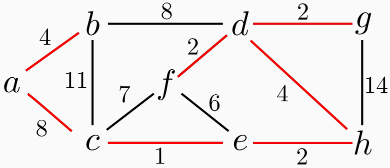


Prim

starting with c



Prim's algorithm: pseudocode

Let S be the set included in the tree so far

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Let S be the set included in the tree so far

$$\mathrm{cost}(v) := \min_{e=(u,v) \text{ s.t. } u \in S} w_e$$

Prim's algorithm: pseudocode

Let S be the set included in the tree so far

$\mathrm{cost}(v) := \min_{e=(u,v) \text{ s.t. } u \in S} w_e$ and $\mathrm{prev}(\cdot)$ is used to keep track of the tree

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def $\textsc{Prim\_MST}$*(undirected $G = (V, E)$, weights $w = (w_e)_{e \in E}$)***:**

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def $\textsc{Prim\_MST}$*(undirected $G = (V, E)$, weights $w = (w_e)_{e \in E}$)***:**

  for $v \in V$**:**

    $\mathrm{cost}(v) := \infty$;

    $\mathrm{prev}(v) := \mathrm{nil}$;

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- **for** $v \in V$**:**
  - $\mathrm{cost}(v) := \infty$;
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- Pick any initial vertex u_0;

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  for $v \in V$:

    $\mathrm{cost}(v) := \infty$;

    $\mathrm{prev}(v) := \mathrm{nil}$;

  Pick any initial vertex u_0;

  $\mathrm{cost}(u_0) := 0$;

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    $H := \mathrm{make\_queue}(V)$; // keys are $\mathrm{cost}(v)$

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    while H *is not empty*:

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- Pick any initial vertex u_0;
- $\text{cost}(u_0) := 0$;
- $H := \text{make\_queue}(V)$; // keys are $\text{cost}(v)$
- **while** *H is not empty*:
  - $v = \text{delete\_min}(H)$;

Prim's algorithm: pseudocode

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```
def PRIM_MST(undirected G = (V, E), weights w = (w_e)_{e∈E}):
    for v ∈ V:
        cost(v) := ∞;
        prev(v) := nil;
    Pick any initial vertex u_0;
    cost(u_0) := 0;
    H := make_queue(V) ;                    // keys are cost(v)
    while H is not empty:
        v = delete_min(H);
        for e := (v, z) ∈ E:
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Prim's algorithm: pseudocode

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  - $\mathrm{cost}(v) := \infty$;
  - $\mathrm{prev}(v) := \mathrm{nil}$;
- Pick any initial vertex u_0;
- $\mathrm{cost}(u_0) := 0$;
- $H := \mathrm{make\_queue}(V)$; // keys are $\mathrm{cost}(v)$
- **while** H *is not empty*:
  - $v = \mathrm{delete\_min}(H)$;
  - **for** $e := (v, z) \in E$:
    - **if** $\mathrm{cost}(z) > w_e$:

Prim's algorithm: pseudocode

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```
def PRIM_MST(undirected G = (V, E), weights w = (w_e)_{e∈E}):
    for v ∈ V:
        cost(v) := ∞;
        prev(v) := nil;
    Pick any initial vertex u_0;
    cost(u_0) := 0;
    H := make_queue(V) ;                          // keys are cost(v)
    while H is not empty:
        v = delete_min(H);
        for e := (v, z) ∈ E:
            if cost(z) > w_e:
                cost(z) := w_e;
```

Prim's algorithm: pseudocode

Let S be the set included in the tree so far

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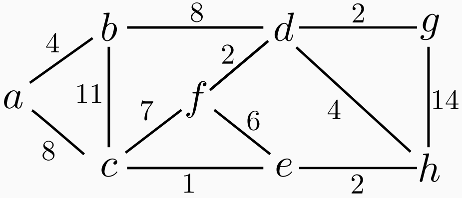
- **for** $v \in V$:
    - $\mathrm{cost}(v) := \infty$;
    - $\mathrm{prev}(v) := \mathrm{nil}$;
- Pick any initial vertex u_0;
- $\mathrm{cost}(u_0) := 0$;
- $H := \mathrm{make\_queue}(V)$; // keys are $\mathrm{cost}(v)$
- **while** *H is not empty*:
    - $v = \mathrm{delete\_min}(H)$;
    - **for** $e := (v, z) \in E$:
        - **if** $\mathrm{cost}(z) > w_e$:
            - $\mathrm{cost}(z) := w_e$;
            - $\mathrm{prev}(z) := v$;

Prim's algorithm: a running example

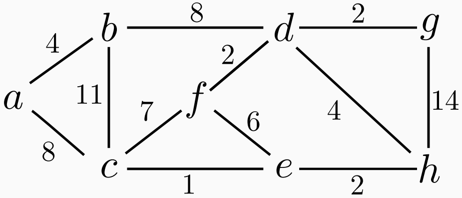
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil

Prim's algorithm: a running example

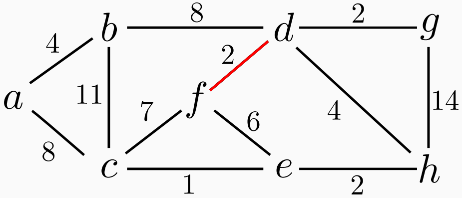
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil

Prim's algorithm: a running example

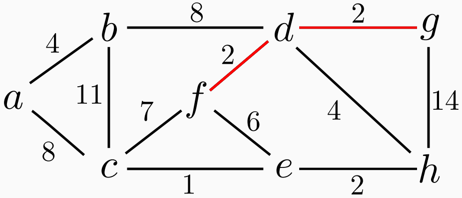
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$

Prim's algorithm: a running example

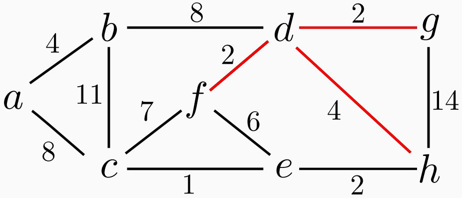
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$

Prim's algorithm: a running example

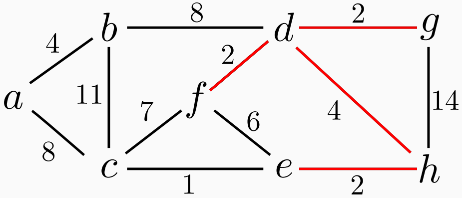
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$
f, d, g, h	∞/nil	$8/d$	$7/f$		$2/h$			

Prim's algorithm: a running example

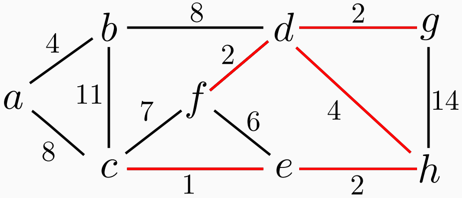
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$
f, d, g, h	∞/nil	$8/d$	$7/f$		$2/h$			
f, d, g, h, e	∞/nil	$8/d$	$1/e$					

Prim's algorithm: a running example

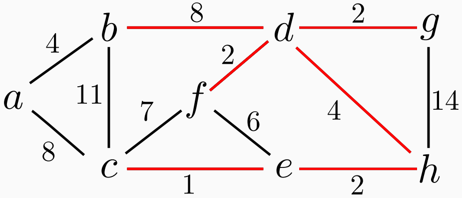
Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$
f, d, g, h	∞/nil	$8/d$	$7/f$		$2/h$			
f, d, g, h, e	∞/nil	$8/d$	$1/e$					
f, d, g, h, e, c	$8/c$	$8/d$						

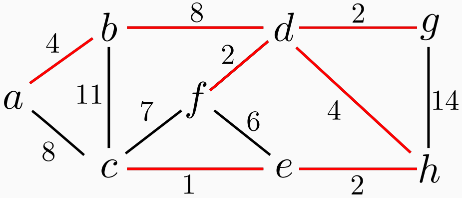
Prim's algorithm: a running example

Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$
f, d, g, h	∞/nil	$8/d$	$7/f$		$2/h$			
f, d, g, h, e	∞/nil	$8/d$	$1/e$					
f, d, g, h, e, c	$8/c$	$8/d$						
f, d, g, h, e, c, b	$4/b$							

Starting with f



Set S	a	b	c	d	e	f	g	h
$\{\}$	∞/nil	∞/nil	∞/nil	∞/nil	∞/nil	0/nil	∞/nil	∞/nil
f	∞/nil	∞/nil	$7/f$	$2/f$	$6/f$		∞/nil	∞/nil
f, d	∞/nil	$8/d$	$7/f$		$6/f$		$2/d$	$4/d$
f, d, g	∞/nil	$8/d$	$7/f$		$6/f$			$4/d$
f, d, g, h	∞/nil	$8/d$	$7/f$		$2/h$			
f, d, g, h, e	∞/nil	$8/d$	$1/e$					
f, d, g, h, e, c	$8/c$	$8/d$						
f, d, g, h, e, c, b	$4/b$							
f, d, g, h, e, c, b, a								