Math 486 - Summer 2020

Summary Notes and Questions from Lessons 1 through 6

Here I have included some notes and questions that you can ask yourself while reviewing the material in Lessons 1 through 6 (Chapters 1–5 in the textbook) I hope this will help you focus on which material to prioritize from the first half of the course..

Chapter 1

Extensive form games

We considered extensive form games with complete and perfect information when players have a finite number of moves at each decision node. We introduced imperfect information and information sets.

Backward induction

We learned how to use backward induction to analyze extensive form games with complete and perfect information. We made a (necessary) assumption that at each decision node a player has a unique move that will determine the highest payoff going forward from that node. Without this assumption, backward induction will fail to produce a unique result. When we find a backward induction solution, this corresponds to a Nash equilibrium.

Continuous strategies/actions

In the second half of Chapter 1 we extended backward induction to cases where players can choose an action from an interval of choices. Our main example was the Stackelberg Duopoly, but we also considered the double marginalization problem in the homework. In these problems we were able to find a best response function for player 2 (a unique best response for every choice that player 1 made.

Player 1 then assumes player 2 will play this best response. This creates an optimization problems for player 1 that depends only on their own strategy.

Continuous strategy examples

- Stackelberg Duopoly (Problem 1 on HW 2)
- Double Marginalization (Problem 2 on HW 2)
- Textbook problems 1.14.6, 1.4.7 , and 1.14.8 (Grocery Store and Gas Station 1 and 2, and Tax Rate)

Use of Parameters

In many cases we have analyzed games that depended on unspecified parameters (and bounds on these parameters).

Useful Chapter 1 examples

Backward induction in finite games:

- Homework 1, Exercise 1,
- Homework 1, Problem 1
- Textbook problems 1.14.2 and 1.14.5 (Chain Stores, Three Pirates)

Continuous strategy examples

- Stackelberg Duopoly (Problem 1 on HW 2)
- Double Marginalization (Problem 2 on HW 2)
- Textbook problems 1.14.6, 1.4.7 , and 1.14.8 (Grocery Store and Gas Station 1 and 2, and Tax Rate)

Chapter 2

The Prisoner's Dilemma Game

A prisoner's dilemma game has a specific payoff structure. There are also two key features of the Prisoner's dilemma.

(i) There is a single Nash equilibrium (when we first introduced this game, we did not call this a Nash equilibrium). This Nash equilibrium results from eliminating dominated strategies.

(ii) There is another outcome that would be better for both players, but the players would need to cooperate to achieve this outcome. This is what makes the game a dilemma.

Normal Form Games

Strategies.

A player's strategy is a **complete contingency plan** for the play of the game. In an extensive form game with complete and perfect information, a strategy for player i specifies an action/move at every decision node for player i. When there is imperfect information, a strategy indicates a move/action at each information set. We include every decision node even when specific actions at earlier nodes will exclude later node of the game. There are multiple reasons that we define strategies this was: concise definition (we can make reductions in individual games as appropriate), we are interested in game structure off the path of play (this will be important in later solution concepts), and some solution concepts allow for the possibility of mistakes.

We can build a normal form game from an extensive form game. We have to understand how the strategy in the normal form corresponds to moves at decision nodes or information sets in the extensive form.

Questions to ask yourself:

- Strategies
  - What is a strategy? How does this differ from a "move" in a game where a player has more than one decision node or information set in the extensive form game?
  - What is a **strategy set**?
  - Whas is a **strategy profile**?
  - What is an **information set**?
  - Notation: In a game with n players, what is meant by the notation $s = (s_i, s_{-i})$ (where s is a strategy profile)?
  - What is meant by describing a strategy as "a complete contingency plan"?
- Payoff functions:
  - Suppose that π_i is a payoff function for player i. What is meant by the following notation (for example)?

$$\pi_i(s_i, s_{-i}) = 3$$

Best Response

We defined a best response. If s_{-i} represents the strategies played by every player except player i, then s_i^* is a best response to s_{-i} if

$$\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i, s_{-i})$$

for all $s_i \in S_i$.

Dominated Strategies

We defined what it means for one strategy to be either strictly or weakly dominated by another. For player i, the strategy s_i is **strictly dominated** by s_i' if for all s_{-i}, we have

$$\pi_i(s_i, s_{-i}) < \pi_i(s_i', s_{-i})$$

For player i, the the strategy s_i is **weakly dominated** by s_i' if for all s_{-i}, we have

$$\pi_i(s_i, s_{-i}) \leq \pi_i(s_i', s_{-i})$$

and for at least one s_{-i}, we have

$$\pi_i(s_i, s_{-i}) < \pi_i(s_i', s_{-i})$$

A strictly dominated strategy is also weakly dominated. That is to say, if a strategy satisfies the definition of being strictly dominated, then it also satisfies the definition of being weakly dominated.

Questions on dominated strategies and best responses

- Can you explain the definition of a best response in words?
- Is it possible for a player to have more than one best response to a particular s_{-i}?
- Do you understand the meaning of the notation s_{-i}?
- Can you explain the definition of strictly dominated and weakly dominated in words?
- Can you apply each definition to a normal form game given in payoff matrix form?
- Can you apply each definition to games with more than two players where players have continuous strategies?
- Can you interpret/explain the definition in terms of specific applications?

Useful Chapter 2 examples

Dominated Strategies:

- Homework 2, Exercise 2 (hawk-dove game)
- Homework 3, Problem 1 (tragedy of the commons)

- Homework 3, Problem 2 (location game)
- Textbook section 2.5 (global warming game)
- Textbook section 2.7 (second-price auction)

Iterated Elimination of Dominated Strategies:

- Homework 3, Exercise 1 (finite three player game)
- Homework 3, Problem 2 (location game)
- The Pick a Number Game
- Homework 3, exercises 2
- Textbook problems 2.14.5, 2.14.6 (Football, Traveler's Dilemma)

Chapter 3 Nash Equilibria

In Chapter 3, we covered Nash equilibria of many different examples.

Nash equilibrium definition

A given strategy profile is a Nash equilibrium if each player is playing a best response to the others: $(s_1^*, s_2^*, \ldots, s_n^*)$ is a Nash equilibrium if for each player i,

$$\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i, s_i^*)$$

for all $s_i \in S_i$.

Questions to ask yourself:

- Can you summarize the definition of a Nash equilibrium in words?
- If a particular strategy profile in an N-person game is a Nash equilibrium, what would you need to do to show that the strategy profile is a Nash equilibrium? Can you summarize what needs to be shown in words? Can you summarize what needs to be shown using mathematical notation?
- If a particular strategy profile in an N-person game is not a Nash equilibrium, what would you need to do to show that the strategy profile is not a Nash equilibrium?
- What is the difference between a strict Nash equilibrium and a Nash equilibrium that is not strict?
- Is it true that a Nash equilibrium will guarantee a player the best possible payoff available to them in the game?
- How do you identify Nash equilibria in normal form games given by a payoff matrix?

Useful examples of two player, finite games

- Big Monkey, Little Monkey
- Prisoner's Dilemma
- Stag Hunt
- Chicken
- Coordination games

Useful examples with more than 2 players, finite games

- Water pollution (Lecture)
- Homework 4, Problem 1 (Pick a number game)
- Homework 4, Problem 2 (First-price auction)

Useful examples with continuous strategies

- Arguing over marbles (Lecture)
- Homework 4, Exercise 2
- Cournot Duopoly (lecture)
- The Tobacco Market (continuous strategies, lecture)
- Price competition (textbook problems 3.12.1 and 3.12.2)
- Problem 3.12.14 (Braess's Paradox)
- Homework 4, Problem 2 (first price auction)

Additional Notes

- A Nash equilibrium is the most general **solution concept** we have considered up to this point. We've had a steady progression toward more general solutions concepts toward Nash equilibria.
- When we can use backward induction, the resulting solution is a Nash equilibrium of the game.
- When we can use iterated elimination of dominated strategies to arrive at a single strategy profile, the resulting profile is a Nash equilibrium.
- Many games cannot be analyzed through backward induction. Many games cannot be reduced to a single strategy profile by iterated elimination of dominated strategies. But these are natural starting points for analysis.
- A game is symmetric if the players can be swapped without impacting their role in the game or the payoffs.

  In games with many players where the game is **symmetric** it is sometimes easier to look for Nash equilibria where every player is playing the same strategy (a symmetric Nash equilibrium). The Pick a number game is symmetric and we found many symmetric Nash equilibria in this game. However, not all equilibria are of this form. For an example, see the Water Pollution Game in the textbook and lectures.

Chapter 4 - Extensive form games with incomplete information

In Chapter 4 we considered games where there is uncertainty or where some players have more information than others.

Extensive form games with incomplete information

We reviewed the idea of an **information set**. In games with incomplete information a player may have multiple nodes that belong to the same information set. A player cannot distinguish between these nodes at the time they are taking a move/action. This leads to two important ideas

(i) The player must have identical actions from every node in the same information set (otherwise the player knows which node they are at)

(ii) The player's strategy must indicate one move/action for each information set.

In games with incomplete information we introduce chance nodes (nodes for Nature) to transform these games into games with complete but imperfect information.

Bayesian Normal Form

We limited our discussion to cases where there are only two players and where each player has a finite set of strategies.

We create the Bayesian normal form by considering the normal form game for each probabilitic outcome (due to Nature). We then combine the resulting payoff matrices for each probabilistic outcome.

Our text calls this just the normal form - and in complete treatment we need to consider the beliefs players have about the probabilities of uncertain events. Players use the actions of other players to update these beliefs as the game proceeds. An accessible example is the simple poker game. At the start of the game there is a 50% that player 1 holds the high card. If player 1 bets, then player 2 should update their belief about the probability that player 1 holds the high card.

Suppose that nature determines the players are in one of k possible games with the probabilities $\$\$\mathrm{p}_{1,\mathrm{p}2,\ldots,\mathrm{p}k}\$$, where $p_1 + p_2 + \cdots + p_k = 1$.

If the payoff matrices are denoted by $A_1, A_2, \ldots, A_k$, then the payoff matrix for the Bayesian norml form game is given by

$$A = p_1 A_1 + p_2 A_2 + \cdots + p_k A_k$$

Useful Examples

- Homework 5, Exercise 1 and Problems 1.
- Homework 5, Exercise 2.
- Buying a used car (lecture, textbook)

- The Travails of Boss Gorilla (lecture, textbook)
- Cuban Missile Crisis (lecture, textbook)
- Textbook problem 4.7.3 (travails of boss gorilla 2)
- Textbook problem 4.7.4 (expert opinion)

Properties of Utility Functions

We also introduced utility functions and their key properties. The idea that we used is how utility to a player depends on financial gain (as an example). The two key assumptions are

(i) Utility is increasing (higher amount of money corresponds to a higher payoff)

(ii) The utility function is concave (concave down in the language of first semester calculus). This means that a jump from $10,000 to $20,000 represents a larger increase in utility than a jump from $100,000 to $110,000.

Both of these assumptions are standard in economics. However, there are interesting and important outcomes beyond this course in decision theory that deal with the idea that losses and gains are experienced differently by most individuals (see Prospect Theory).

Buying Insurance

We used the concavity of utility functions to demonstrate how a lower expected payoff in terms of strict financial gain/loss may lead to a higher expected utility.

1 Chapter 5 - Mixed Strategies

In chapter 5 we introduced mixed strategies and discussed mixed strategy Nash equilibria and how to find them.

1.1 Mixed strategy:

1. What is a mixed strategy?
2. How do you compute the expected payoffs when mixed strategies are used?
3. What does Nash's Existence Theorem (Theorem 5.1 in our text) say about the existence of mixed strategies Nash equilibria?

1.2 Mixed Strategy Nash Equilibria

1. How did we extend the definition of a Nash equilbrium to to a mixed strategy Nash equilibrium?
2. What are the mixed strategy Nash equilibrium conditions (The Fundamental Theorem of Nash Equilibria in the textbook)?
3. How do you use the conditions to find Nash equilibria?

1.3 Key Examples:

1. Lesson 6 Homework (all exercises and Problems)
2. Coordination games (lecture example)
3. Tennis (lecture)
4. Matching Pennies (lecture)
5. Water Pollution (textbook problem)
6. One-card-two-round Poker (Lecture - mixed strategies in a Bayesian Normal Form Game and interpretation of the results)