CMPSC 465
Data Structures and Algorithms
Spring 2022

Instructor: Chunhao Wang

Greedy algorithms

Greedy algorithms

Warm-up

Greedy algorithms

Minimum Spanning Tree

Kurskal's Algorithm

def $\textsc{Kruskal\_MST}$*(undirected* $G = (V, E)$*, weights* $w = (w_e)_{e \in E}$*)***:**

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    make_set(v);

- make_set(v): put v into a set containing itself. $v \mapsto \{v\}$

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Set A := { };
for v ∈ V:
    make_set(v);
Sort E in increasing order of edge weights;
for (u, v) ∈ E:
    if find_set(u) ≠ find_set(v):
        A := A ∪ {(u, v)};
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            $A := A \cup \{(u, v)\}$;
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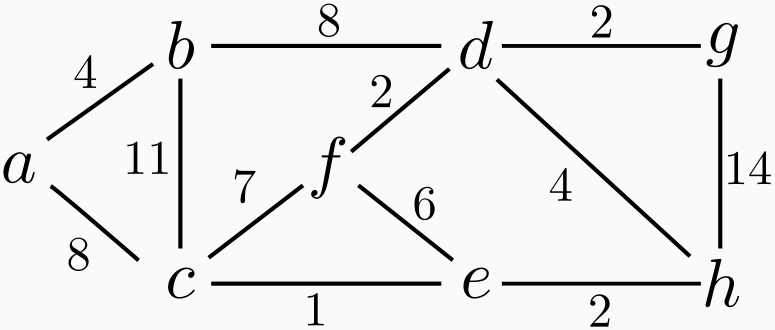
- $\mathrm{make\_set}(v)$: put v into a set containing itself. $v \mapsto \{v\}$
- $\mathrm{find\_set}(u)$: find which set u belongs to
- $\mathrm{union}(u, v)$: merge the sets that u and v are in

Example

A: red edges



$$\{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{g\},\{h\}$$

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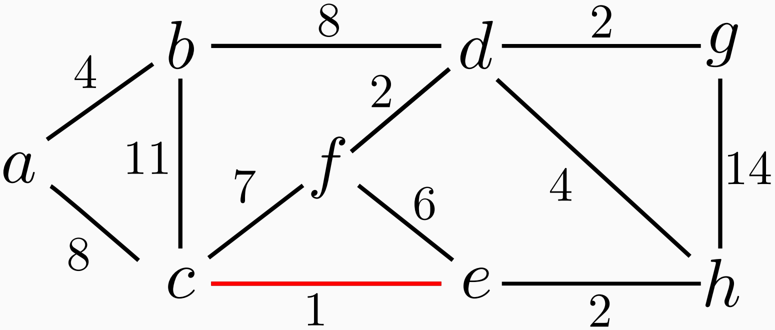

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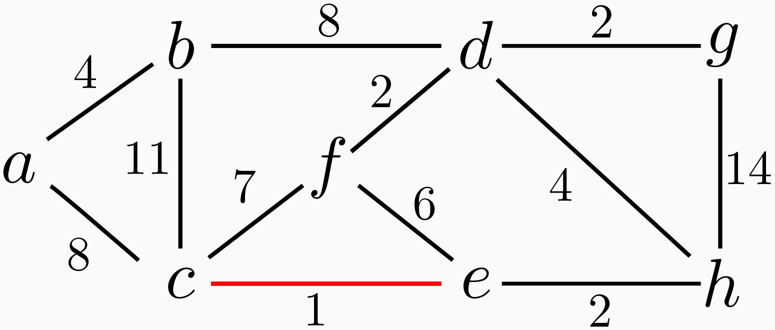

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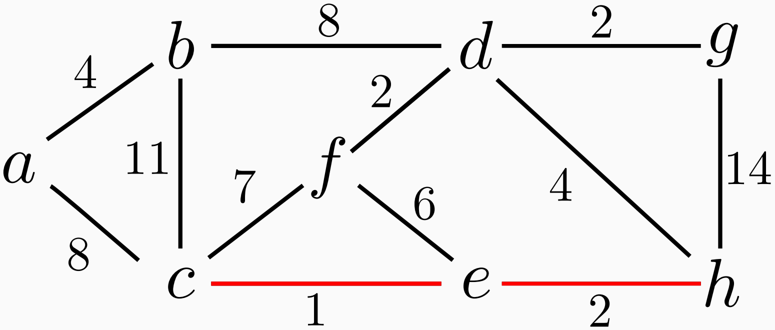

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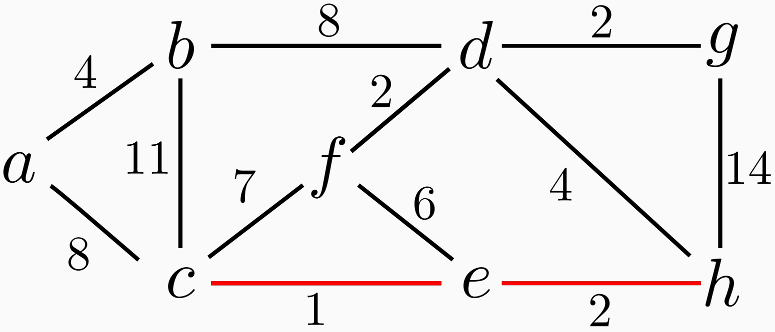

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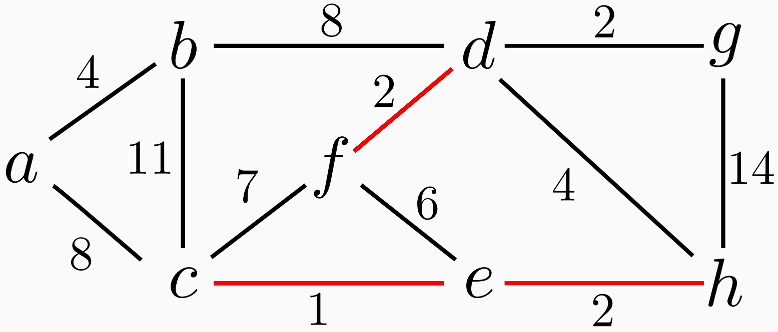

$$\{a\}, \{b\}, \{c, e, h\}, {\color{blue}\{d\}}, {\color{blue}\{f\}}, \{g\}$$

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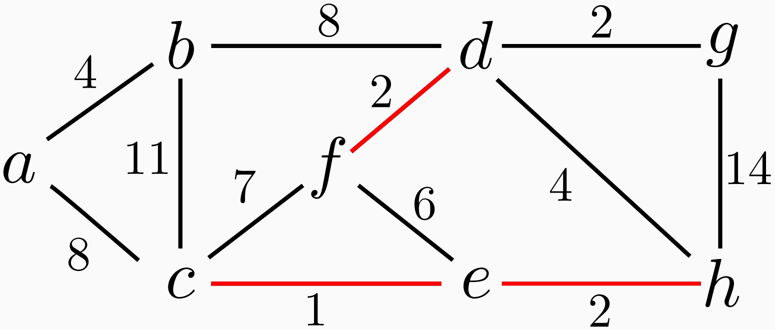

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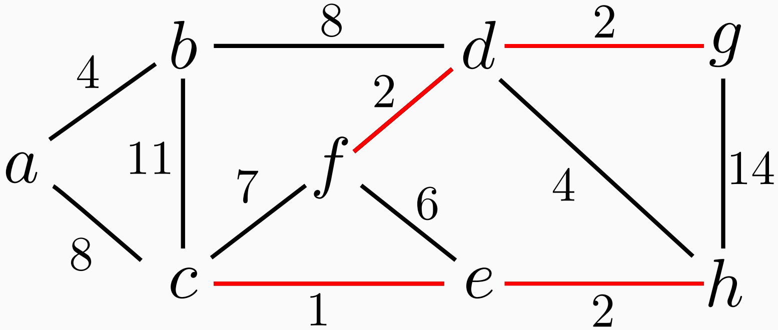

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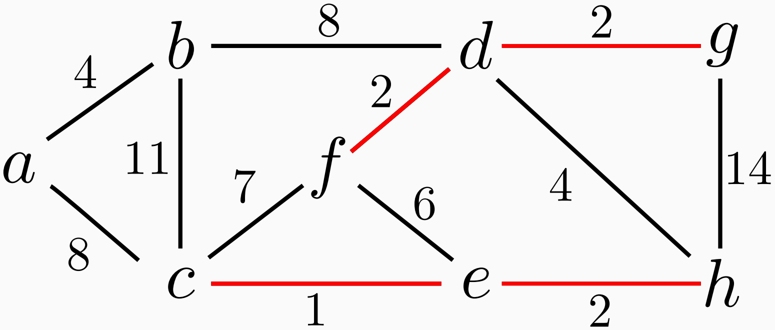

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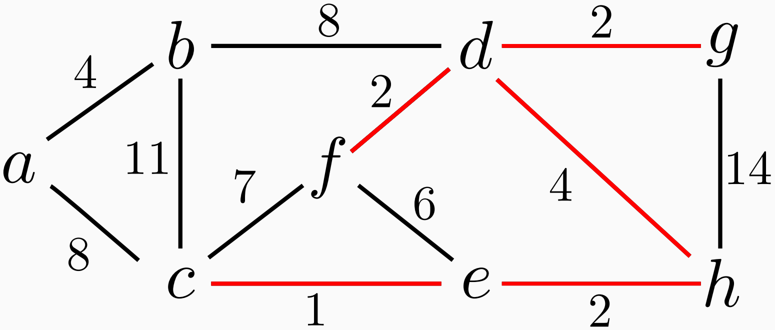

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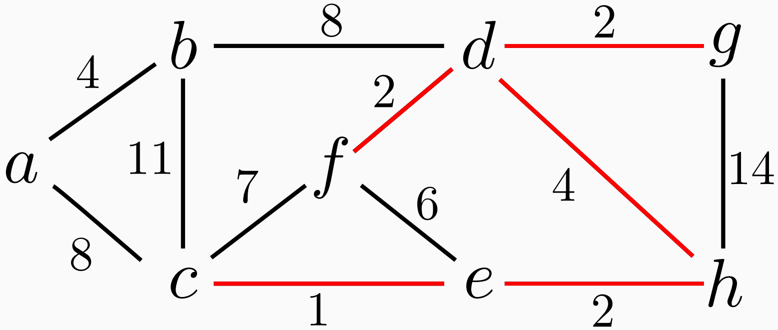

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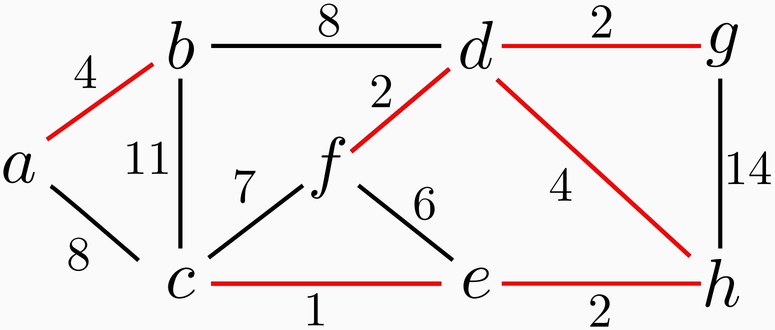

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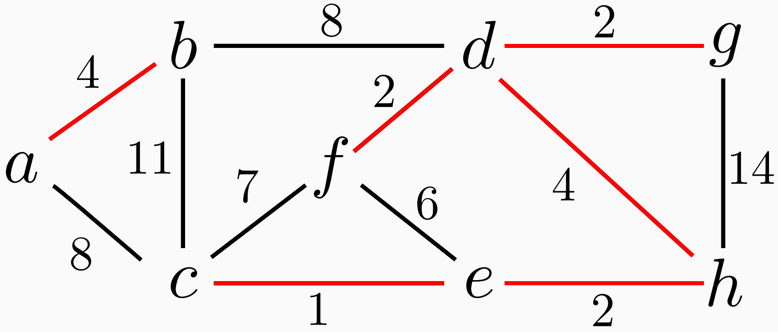

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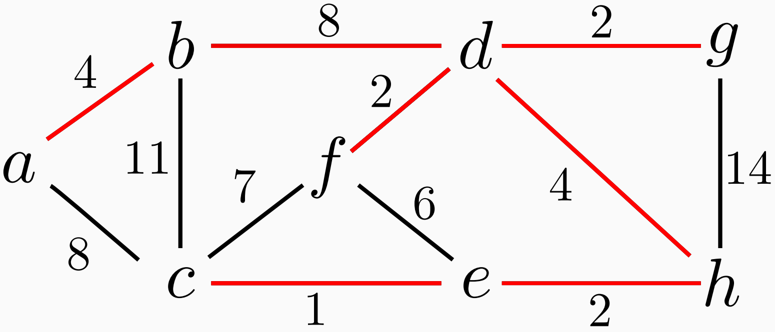

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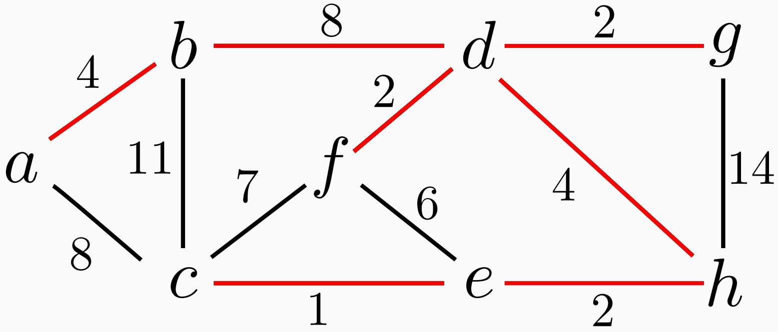

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Theorem (The cut property)

Let A be a subset of edges of some MST of $G = (V, E)$. Let $(S, V - S)$ be a cut that respects A. Let e be the lightest edge across the cut. Then $A \cup \{e\}$ is part of some MST

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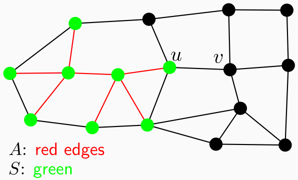

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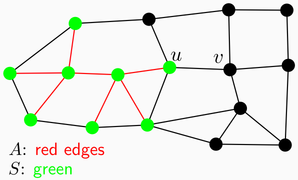

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Then $A \cup \{e\}$ is part of some MST $\square$

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Any connected and undirected graph $G = (V, E)$ is a tree if and only if $|E| = |V| - 1$

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See textbook page 129 $\square$

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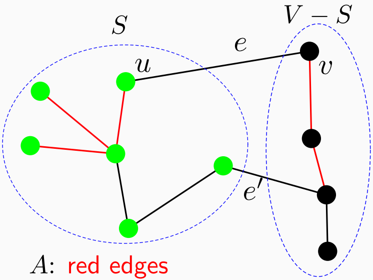

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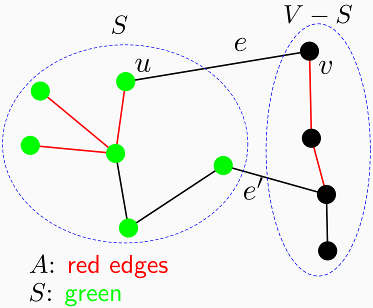

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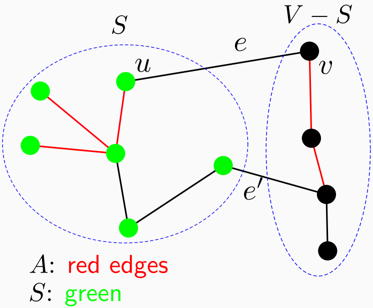

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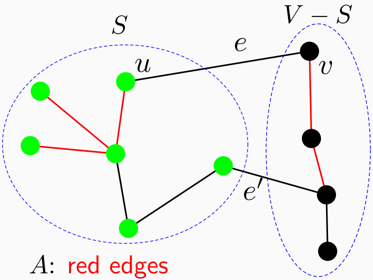

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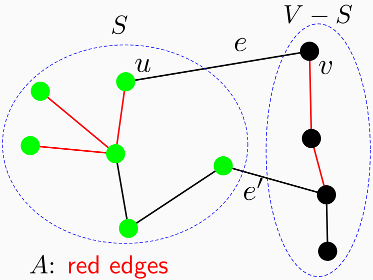

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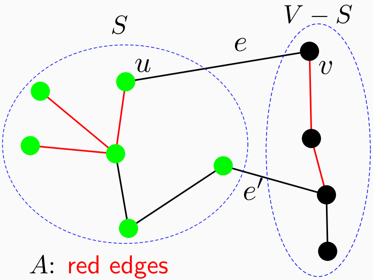

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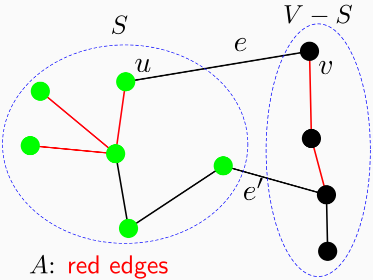

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$$\mathrm{weight}(T') = \mathrm{weight}(T) + w_e - w_{e'}$$

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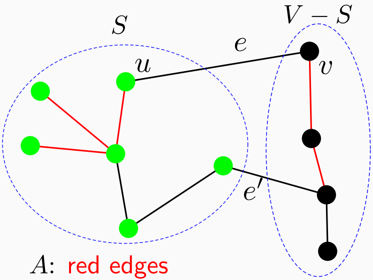

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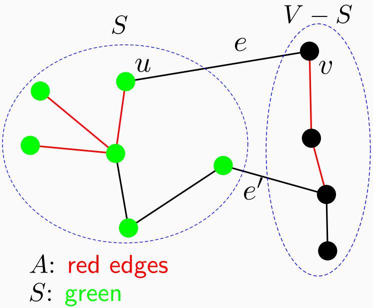

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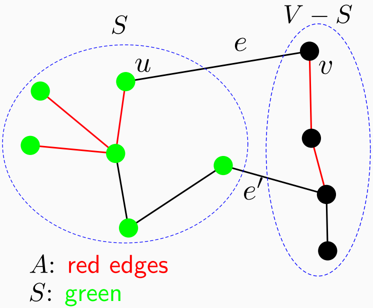

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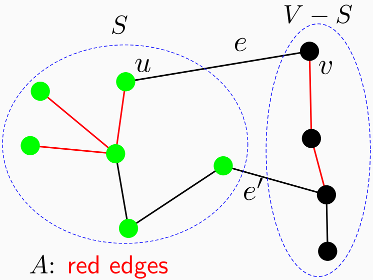

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T is MST $\implies$ $\text{weight}(T') = \text{weight}(T)$ $\implies$ T' is also an MST $\square$

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Depends on how we implement make_set, find_set, and union

Using linked list:

$$\{a, b, c\} \quad \text{head} \to a \to b \to c \qquad \text{find\_set}(b)\text{: } O(1)$$

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

Using linked list:

$\{a, b, c\} \quad \text{head} \to a \to b \to c$

$\text{find\_set}(b)$: $O(1)$

$\text{make\_set}(v)$:

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

Using linked list:

$$\{a, b, c\} \quad \text{head} \to a \to b \to c \qquad \text{find\_set}(b)\text{: } O(1)$$

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$$\{d, e\} \quad \text{head} \to d \to e$$

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

Using linked list:

$\{a, b, c\}$ $\quad \text{head} \to a \to b \to c$ $\qquad \text{find\_set}(b)$: $O(1)$

$\text{make\_set}(v)$: $O(1)$

$\{d, e\}$ $\quad \text{head} \to d \to e$

$\text{union}(a, b)$

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

Using linked list:

$\{a, b, c\}$ $\quad \text{head} \to a \to b \to c$ $\qquad$ find_set(b): $O(1)$

make_set(v): $O(1)$

$\{d, e\}$ $\quad \text{head} \to d \to e$

union(a, b) $\qquad \text{head} \to a \to b \to c \to d \to e$

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

Using linked list:

$\{a, b, c\}$ head $\to a \to b \to c$ find_set(b): $O(1)$

make_set(v): $O(1)$

$\{d, e\}$ head $\to d \to e$

union(a, b) head $\to a \to b \to c \to d \to e$

Cost of union:

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

Using linked list:

$\{a, b, c\}$ $\quad \text{head} \to a \to b \to c$ $\qquad \text{find\_set}(b)$: $O(1)$

$\text{make\_set}(v)$: $O(1)$

$\{d, e\}$ $\quad \text{head} \to d \to e$

$\text{union}(a, b)$ $\qquad \text{head} \to a \to b \to c \to d \to e$

Cost of union: O(length of the shorter list)

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

Using linked list:

$\{a, b, c\}$ $\quad \text{head} \to a \to b \to c$ $\qquad$ $\text{find\_set}(b)$: $O(1)$

$\text{make\_set}(v)$: $O(1)$

$\{d, e\}$ $\quad \text{head} \to d \to e$

$\text{union}(a, b)$ $\qquad \text{head} \to a \to b \to c \to d \to e$

Cost of union: O(length of the shorter list)

Using an array to implement it:

vertex	1	2	3	4	5
head	1	1	1	4	4

$\xrightarrow{\text{union}}$

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

Using linked list:

$\{a, b, c\}$ $\quad \text{head} \to a \to b \to c$ $\qquad$ $\text{find\_set}(b)$: $O(1)$

$\text{make\_set}(v)$: $O(1)$

$\{d, e\}$ $\quad \text{head} \to d \to e$

$\text{union}(a, b)$ $\qquad \text{head} \to a \to b \to c \to d \to e$

Cost of union: O(length of the shorter list)

Using an array to implement it:

$$
\begin{array}{r|c|c|c|c|c|}
\hline
\text{vertex} & 1 & 2 & 3 & 4 & 5 \\
\hline
\text{head} & 1 & 1 & 1 & 4 & 4 \\
\hline
\end{array}
\xrightarrow{\text{union}}
\begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
$$