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Math 455, Sample Final Exam
December 6, 2021

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 110 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 17 pages of the test.

Please do NOT write in this box.

1.	________
2.	________
3.	________
4.	________
5.	________
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9.	________
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11.	________
12.	________
13.	________
Total	________

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Partial Credit

You must show your work on the partial credit problems to receive credit!

1.(12 pts) Write out the hex machine number representation of 3.4.

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2.(13 pts)

Consider a function $f(x) = x^3 + x^2 + 1/4$.

(a) (3 pts) Prove that there exists at least one root of $f(x) = 0$ on $[-2, -1]$.

(b) (5 pts) Consider a fixed point iteration

$$x_{n+1} = g_1(x_n), \qquad \text{where} \qquad g_1(x) = f(x) + x,$$

with the starting point $x_0 = -1$. We have the following sequence

$$x_1 = -0.75,\ x_2 = -0.3593,\ x_3 = -0.026,\ x_4 = 0.2240, \cdots$$

Does this scheme converge? Why?

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(c) (5 pts) Consider a fixed point iteration

$$x_{n+1} = g_2(x_n), \qquad \text{where} \qquad g_2(x) = -\frac{x^2 + 1/4}{x^2},$$

with the starting point $x_0 = -1$. We have the following sequence

$$x_1 = -1.25,\ x_2 = -1.16,\ x_3 = -1.185,\ x_4 = -1.177, \cdots$$

Does this scheme converge? Why?

$$g_2'(x) = -\frac{x^2}{x^2} - \frac{1}{4x^2}$$

$$= -\frac{1}{4x^2} - 1$$

$$= \frac{1}{4}x^{-2} - 1$$

$$= \frac{-2}{4}x^{-3}$$

$$= \frac{1}{2x^3}$$

$x_0 = -1$

$$|g_2'(x)| = \left|\frac{1}{2 \cdot -1}\right|$$

$$= \left|-\frac{1}{2}\right| < 1$$

Converge

inverse 3x3

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 - R_3$

$$\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & -1 \end{array}$$

$2 \cdot R_1 + R_2$

$$\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & -1 \end{array}$$

$3R_2 - 4R_3$

$$\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$R_1 + R_3$

$$\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 3 & 4 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$R_3 + R_2$

$$\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 3 & 4 \\ 0 & 4 & 0 & 4 & 4 & -1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$$\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 3 & 4 \\ 0 & 1 & 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$R_1 - 2R_2$

$$\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 4\frac{1}{2} \\ 0 & 1 & 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$22 - \frac{1}{2}$

⋮

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3.(7 pts.)

(a) Compute the condition number of the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

$\text{Cond} = \|A\|_2 \cdot \|A^{-1}\|_2$

by using 2 norm.

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - A \longrightarrow \begin{bmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-4 & -1 \\ 0 & -1 & \lambda+2 \end{bmatrix}$$

$$(\lambda-2)\left((\lambda-4)(\lambda+2)-(-1\cdot -1)\right)$$

$$=(\lambda-2)(\lambda^2-2\lambda-8-1)=0$$

$\lambda_1=2 \qquad \lambda^2-2\lambda-9$

$\dfrac{-b\pm\sqrt{4ac}}{2a} \qquad \dfrac{2\pm\sqrt{b^2+4ac}}{2}$

$2\pm 2\sqrt{10}$

$1\pm\sqrt{10}$

$\dfrac{\max}{\min} \qquad \dfrac{1+\sqrt{10}}{2}$

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4.(10pts.)

Consider the following linear system

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

.

- Does the Gauss-Seidel method converge for solving this linear system? If so, prove $\rho(T_{GS}) < 1$.

$A^T = A$ ✓ Symmetric

$\det(A_{11}) = 1 > 0$

$\det(A_{22}) = \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = (1 \cdot 3) - (-1 \cdot -1)$

$= 3 - 1$

$= 2 > 0$

$\det(A_{33}) =$

$1 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} - (-1) \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} + 0$

$= (9-4) + (-1-2)$

$= 5 - 3 = 2 > 0$

$-(D+L)^{-1} \cdot U$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

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5.(5 pts.)

The polynomial $p(x) = x^4 - 2x^3 + 4x^2 - x + 5$ has the values shown.

x	-2	-1	0	1	2
p(x)	55	13	5	7	19

Find a polynomial $q(x)$ that takes these values (you don't need expand it):

x	-2	-1	0	1	2
q(x)	55	13	5	7	10

$$p(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) + y_3 l_3(x) + y_4 l_4(x)$$

$$q(x) - p(x) = 9 l_4(x)$$

$$l_4 = \frac{(x+2)(x+1)(x)(x-1)}{(2+2)(2+1)(2)(2-1)} = \frac{(x^3-1)(x+2)}{24} = \frac{x^4 - x + 2x^3 - 2}{24}$$

$$q(x) = x^4 - 2x^3 + 4x^2 - x + 5 + \frac{9(x^4 - x + 2x^3 - 2)}{24}$$

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6.(15 pts.)

Given the data set

x_i	1	2	3
y_i	3	2	4

(a) Write down the linear system in matrix form for solving the coefficients a_i ($i = 0, \cdots, n$) of the polynomial $p_n(x)$ (Do NOT solve).

$n=2 \quad a_2 x^2 + a_1 x + a_0$

$$\begin{matrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{matrix} \qquad \begin{matrix} a_2 \\ a_1 \\ a_0 \end{matrix} \qquad \begin{matrix} 3 \\ 2 \\ 4 \end{matrix}$$

(b) Find a polynomial $p(x)$ that interpolate the data by using Lagrange polynomial interpolation. Please simplify the polynomial.

$l_1 \ \dfrac{(x-2)(x-3)}{(1-2)(1-3)} \qquad l_2 \ \dfrac{(x-1)(x-3)}{(2-1)(2-3)} \qquad l_3 \ \dfrac{(x-1)(x-2)}{(3-1)(3-2)}$

$-1 \quad -2$

$$Y_1 \frac{(x-2)(x-3)}{2} + Y_2 \, l_2 + Y_3 \, l_3$$

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7.(10 pts.)

Find a, b and c such that $s(x)$ is a cubic spline, where

$$S(x) = \begin{cases} s_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3 & 1 \le x \le 2 \\ s_1(x) = a + b(x-2) + c(x-2)^2 + (x-2)^3 & 2 \le x \le 3 \end{cases}$$

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8.(13pts.)

(a) Find the best line to fit the data points $(0,0),(1,3),(2,3),(5,6)$;

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(b)Fit the data to the periodic model $y = F_3(t) = c_1 + c_2 \cos 2\pi t + c_3 \sin 2\pi t + c_4 \cos 4\pi t$.

t	0	1/6	1/3	1/2	2/3	5/6
y	4	2	0	-5	-1	3

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9.(10 pts.) Use Householder reflectors to find the QR factorization of

$$A = \begin{pmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{pmatrix}$$

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10.(10 pts.) Use the two-point forward-difference formula to approximate $f'(1)$ and find the approximation error, where $f(x) = \ln x$, for $h = 0.01$.

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11.(10 pts.) The error estimate for the Trapezoidal rule with $n+1$ uniform grid points yields

$$E_T(f;h) = \frac{b-a}{12} h^2 \max_x |f''(x)|, \quad h = \frac{b-a}{n}.$$

Consider $\int_0^1 (\cos(x) + x^6) dx$ by using the Trapezoidal rule. If we wish the absolute value of the error to be smaller or equal than 10^{-6}, how many points would be needed for trapezoidal rule?

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12.(20 pts.)

(a). Determine constants a, b, c and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has degree of precision 3.

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(b) If the following quadrature formula is given

$$\int_{-1}^{1} f(x)dx = \frac{1}{3}f(-1) + \frac{1}{3}f(1) + \frac{4}{3}f(0),$$

what's the degree of precision?

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13.(15 pts.)

Write a Matlab function for solving linear systems by the Jacobi method. Your function should be used by the following command in Matlab command window:

```
>> v=Jacobi(A,b,x0)
```

where `A` is a matrix, `b` is a vector, and `x0` is the initial guess.

Answer:

```
function x=Jacobi(A,b,x0)
%A-- a nXn matrix
%b-- a nX1 vector
%x-- a solution of Ax=b
D=diag(A);
L=tril(A)-diag(D);
U=triu(A)-diag(D);
CurIter=0;
MaxIter=100;
Tol=1e-5;
while 1
    x=-(L+U)*x0./D+b./D;
    CurIter=CurIter+1;
    if CurIter>MaxIter
        break
    end
    if norm(x-x0)<Tol
         break
    end
    if norm(A*x-b)<Tol
        break
    end
    x0=x;
end
```