likelihood estimator for θ is $\widehat{\theta} = (1/n)\sum_{i=1}^{n}(X_i - \mu)^2$ and that this estimator is an unbiased estimator of θ.

6.4-3. A random sample $X_1, X_2, \ldots, X_n$ of size n is taken from a Poisson distribution with a mean of λ, $0 < \lambda < \infty$.

(a) Show that the maximum likelihood estimator for λ is $\widehat{\lambda} = \overline{X}$.

(b) Let X equal the number of flaws per 100 feet of a used computer tape. Assume that X has a Poisson distribution with a mean of λ. If 40 observations of X yielded 5 zeros, 7 ones, 12 twos, 9 threes, 5 fours, 1 five, and 1 six, find the maximum likelihood estimate of λ.

6.4-4. For determining half-lives of radioactive isotopes, it is important to know what the background radiation is in a given detector over a specific period. The following data were taken in a γ-ray detection experiment over 98 ten-second intervals:

58	50	57	58	64	63	54	64	59	41	43	56	60	50
46	59	54	60	59	60	67	52	65	63	55	61	68	58
63	36	42	54	58	54	40	60	64	56	61	51	48	50
60	42	62	67	58	49	66	58	57	59	52	54	53	53
57	43	73	65	45	43	57	55	73	62	68	55	51	55
53	68	58	53	51	73	44	50	53	62	58	47	63	59
59	56	60	59	50	52	62	51	66	51	56	53	59	57

Assume that these data are observations of a Poisson random variable with mean λ.

(a) Find the values of $\overline{x}$ and s^2.

(b) What is the value of the maximum likelihood estimator of λ?

(c) Is S^2 an unbiased estimator of λ?

(d) Which of $\overline{x}$ and s^2 would you recommend for estimating λ? Why? You could compare the variance of $\overline{X}$ with the variance of S^2, which is

$$\mathrm{Var}(S^2) = \frac{\lambda(2\lambda n + n - 1)}{n(n-1)}.$$

6.4-5. Let $X_1, X_2, \ldots, X_n$ be a random sample from distributions with the given probability density functions. In each case, find the maximum likelihood estimator $\widehat{\theta}$.

(a) $f(x;\theta) = (1/\theta^2)\,x\,e^{-x/\theta}, \quad 0 < x < \infty, \quad 0 < \theta < \infty.$

(b) $f(x;\theta) = (1/2\theta^3)\,x^2\,e^{-x/\theta}, \quad 0 < x < \infty, \quad 0 < \theta < \infty.$

(c) $f(x;\theta) = (1/2)\,e^{-|x-\theta|}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$

HINT: Finding θ involves minimizing $\sum |x_i - \theta|$, which is a difficult problem. When $n = 5$, do it for $x_1 = 6.1$, $x_2 = -1.1$, $x_3 = 3.2$, $x_4 = 0.7$, and $x_5 = 1.7$, and you will see the answer. (See also Exercise 2.2-8.)

6.4-6. Find the maximum likelihood estimates for $\theta_1 = \mu$ and $\theta_2 = \sigma^2$ if a random sample of size 15 from $N(\mu, \sigma^2)$ yielded the following values:

31.5	36.9	33.8	30.1	33.9
35.2	29.6	34.4	30.5	34.2
31.6	36.7	35.8	34.5	32.7

6.4-7. Let $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta \in \Omega = \{\theta : 0 < \theta < \infty\}$. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from this distribution.

(a) Sketch the pdf of X for **(i)** $\theta = 1/2$, **(ii)** $\theta = 1$, and **(iii)** $\theta = 2$.

(b) Show that $\widehat{\theta} = -n/\ln\left(\prod_{i=1}^{n} X_i\right)$ is the maximum likelihood estimator of θ.

(c) For each of the following three sets of 10 observations from the given distribution, calculate the values of the maximum likelihood estimate and the method-of-moments estimate of θ:

(i)	0.0256	0.3051	0.0278	0.8971	0.0739
	0.3191	0.7379	0.3671	0.9763	0.0102
(ii)	0.9960	0.3125	0.4374	0.7464	0.8278
	0.9518	0.9924	0.7112	0.2228	0.8609
(iii)	0.4698	0.3675	0.5991	0.9513	0.6049
	0.9917	0.1551	0.0710	0.2110	0.2154

6.4-8. Let $f(x;\theta) = (1/\theta)x^{(1-\theta)/\theta}$, $0 < x < 1$, $0 < \theta < \infty$.

(a) Show that the maximum likelihood estimator of θ is $\widehat{\theta} = -(1/n)\sum_{i=1}^{n}\ln X_i$.

(b) Show that $E(\widehat{\theta}) = \theta$ and thus that $\widehat{\theta}$ is an unbiased estimator of θ.

6.4-9. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the exponential distribution whose pdf is $f(x;\theta) = (1/\theta)e^{-x/\theta}$, $0 < x < \infty$, $0 < \theta < \infty$.

(a) Show that $\overline{X}$ is an unbiased estimator of θ.

(b) Show that the variance of $\overline{X}$ is θ^2/n.

(c) What is a good estimate of θ if a random sample of size 5 yielded the sample values 3.5, 8.1, 0.9, 4.4, and 0.5?

6.4-10. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a geometric distribution for which p is the probability of success.

(a) Use the method of moments to find a point estimate for p.

(b) Explain intuitively why your estimate makes good sense.

(c) Use the following data to give a point estimate of p:

3 34 7 4 19 2 1 19 43 2

22 4 19 11 7 1 2 21 15 16

6.4-11. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution having finite variance σ^2. Show that

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$$

is an unbiased estimator of σ^2. HINT: Write

$$S^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right)$$

and compute $E(S^2)$.

6.4-12. Let $X_1, X_2, \ldots, X_n$ be a random sample from $b(1,p)$ (i.e., n Bernoulli trials). Thus,

$$Y = \sum_{i=1}^{n} X_i \text{ is } b(n,p).$$

(a) Show that $\overline{X} = Y/n$ is an unbiased estimator of p.

(b) Show that $\text{Var}(\overline{X}) = p(1-p)/n$.

(c) Show that $E[\overline{X}(1-\overline{X})/n] = (n-1)[p(1-p)/n^2]$.

(d) Find the value of c so that $c\overline{X}(1-\overline{X})$ is an unbiased estimator of $\text{Var}(\overline{X}) = p(1-p)/n$.

6.4-13. Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(\theta-1, \theta+1)$.

(a) Find the method-of-moments estimator of θ.

(b) Is your estimator in part (a) an unbiased estimator of θ?

(c) Given the following $n = 5$ observations of X, give a point estimate of θ:

6.61 7.70 6.98 8.36 7.26

(d) The method-of-moments estimator actually has greater variance than the maximum likelihood estimator of θ, *namely* $[\min(X_i) + \max(X_i)]/2$. Compute the value of the latter estimator for the $n = 5$ observations in (c).

6.4-14. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a normal distribution.

(a) Show that an unbiased estimator of σ is cS, where

$$c = \frac{\sqrt{n-1}\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}.$$

HINT: Recall that the distribution of $(n-1)S^2/\sigma^2$ is $\chi^2(n-1)$.

(b) Find the value of c when $n = 5$; when $n = 6$.

(c) Graph c as a function of n. What is the limit of c as n increases without bound?

6.4-15. Given the following 25 observations from a gamma distribution with mean $\mu = \alpha\theta$ and variance $\sigma^2 = \alpha\theta^2$, use the method-of-moments estimators to find point estimates of α and θ:

6.9 7.3 6.7 6.4 6.3 5.9 7.0 7.1 6.5 7.6 7.2 7.1 6.1

7.3 7.6 7.6 6.7 6.3 5.7 6.7 7.5 5.3 5.4 7.4 6.9

6.4-16. An urn contains 64 balls, of which N_1 are orange and N_2 are blue. A random sample of $n = 8$ balls is selected from the urn without replacement, and X is equal to the number of orange balls in the sample. This experiment was repeated 30 times (the 8 balls being returned to the urn before each repetition), yielding the following data:

3 0 0 1 1 1 1 3 1 1 2 0 1 3 1

0 1 0 2 1 1 2 3 2 2 4 3 1 1 2

Using these data, guess the value of N_1 and give a reason for your guess.

6.4-17. Let the pdf of X be defined by

$$f(x) = \begin{cases} \left(\dfrac{4}{\theta^2}\right)x, & 0 < x \le \dfrac{\theta}{2}, \\[2ex] -\left(\dfrac{4}{\theta^2}\right)x + \dfrac{4}{\theta}, & \dfrac{\theta}{2} < x \le \theta, \\[2ex] 0, & \text{elsewhere,} \end{cases}$$

where $\theta \in \Omega = \{\theta : 0 < \theta \le 2\}$.

(a) Sketch the graph of this pdf when $\theta = 1/2, \theta = 1$, and $\theta = 2$.

(b) Find an estimator of θ by the method of moments.

(c) For the following observations of X, give a point estimate of θ:

0.3206 0.2408 0.2577 0.3557 0.4188

0.5601 0.0240 0.5422 0.4532 0.5592

6.4-18. Let independent random samples, each of size n, be taken from the k normal distributions with means $\mu_j = c + d[j - (k+1)/2]$, $j = 1, 2, \ldots, k$, respectively, and common variance σ^2. Find the maximum likelihood estimators of c and d.

6.4-19. Let the independent normal random variables $Y_1, Y_2, \ldots, Y_n$ have the respective distributions $N(\mu, \gamma^2 x_i^2)$, $i = 1, 2, \ldots, n$, where $x_1, x_2, \ldots, x_n$ are known but not all the same and no one of which is equal to zero. Find the maximum likelihood estimators for μ and γ^2.

If we are not able to assume that the underlying distribution is normal, but μ and σ are both unknown, approximate confidence intervals for μ can still be constructed with the formula

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}},$$

which now only has an approximate t distribution. Generally, this approximation is quite good (i.e., it is robust) for many nonnormal distributions; in particular, it works well if the underlying distribution is symmetric, unimodal, and of the continuous type. However, if the distribution is highly skewed, there is great danger in using that approximation. In such a situation, it would be safer to use certain nonparametric methods for finding a confidence interval for the median of the distribution, one of which is given in Section 7.5.

There is one other aspect of confidence intervals that should be mentioned. So far, we have created only what are called **two-sided confidence intervals** for the mean μ. Sometimes, however, we might want only a lower (or upper) bound on μ. We proceed as follows.

Say $\overline{X}$ is the mean of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$, where, for the moment, assume that σ^2 is known. Then

$$P\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \le z_\alpha\right) = 1 - \alpha,$$

or equivalently,

$$P\left[\overline{X} - z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right) \le \mu\right] = 1 - \alpha.$$

Once $\overline{X}$ is observed to be equal to $\overline{x}$, it follows that $[\overline{x} - z_\alpha(\sigma/\sqrt{n}), \infty)$ is a $100(1-\alpha)\%$ **one-sided confidence interval** for μ. That is, with the confidence coefficient $1-\alpha$, $\overline{x} - z_\alpha(\sigma/\sqrt{n})$ is a lower bound for μ. Similarly, $(-\infty, \overline{x} + z_\alpha(\sigma/\sqrt{n})]$ is a one-sided confidence interval for μ and $\overline{x} + z_\alpha(\sigma/\sqrt{n})$ provides an upper bound for μ with confidence coefficient $1-\alpha$.

When σ is unknown, we would use $T = (\overline{X} - \mu)/(S/\sqrt{n})$ to find the corresponding lower or upper bounds for μ, namely,

$$\overline{x} - t_\alpha(n-1)(s/\sqrt{n}) \quad \text{and} \quad \overline{x} + t_\alpha(n-1)(s/\sqrt{n}).$$

Exercises

7.1-1. A random sample of size 16 from the normal distribution $N(\mu, 25)$ yielded $\overline{x} = 73.8$. Find a 95% confidence interval for μ.

7.1-2. A random sample of size 8 from $N(\mu, 72)$ yielded $\overline{x} = 85$. Find the following confidence intervals for μ:

(a) 99%. **(b)** 95%. **(c)** 90%. **(d)** 80%.

7.1-3. To determine the effect of 100% nitrate on the growth of pea plants, several specimens were planted and then watered with 100% nitrate every day. At the end of two weeks, the plants were measured. Here are data on seven of them:

17.5	14.5	15.2	14.0	17.3	18.0	13.8

Assume that these data are a random sample from a normal distribution $N(\mu, \sigma^2)$.

(a) Find the value of a point estimate of μ.

(b) Find the value of a point estimate of σ.

(c) Give the endpoints for a 90% confidence interval for μ.

7.1-4. Let X equal the weight in grams of a "52-gram" snack pack of candies. Assume that the distribution of X is $N(\mu, 4)$. A random sample of $n = 10$ observations of X yielded the following data:

55.95	56.54	57.58	55.13	57.48
56.06	59.93	58.30	52.57	58.46

(a) Give a point estimate for μ.

(b) Find the endpoints for a 95% confidence interval for μ.

(c) On the basis of these very limited data, what is the probability that an individual snack pack selected at random is filled with less than 52 grams of candy?

7.1-5. As a clue to the amount of organic waste in Lake Macatawa (see Example 7.1-4), a count was made of the number of bacteria colonies in 100 milliliters of water. The number of colonies, in hundreds, for $n = 30$ samples of water from the east basin yielded

93	140	8	120	3	120	33	70	91	61
7	100	19	98	110	23	14	94	57	9
66	53	28	76	58	9	73	49	37	92

Find an approximate 90% confidence interval for the mean number (say, μ_E) of colonies in 100 milliliters of water in the east basin.

7.1-6. To determine whether the bacteria count was lower in the west basin of Lake Macatawa than in the east basin, $n = 37$ samples of water were taken from the west basin and the number of bacteria colonies in 100 milliliters of water was counted. The sample characteristics were $\bar{x} = 11.95$ and $s = 11.80$, measured in hundreds of colonies. Find an approximate 95% confidence interval for the mean number of colonies (say, μ_W) in 100 milliliters of water in the west basin.

7.1-7. Thirteen tons of cheese, including "22-pound" wheels (label weight), is stored in some old gypsum mines. A random sample of $n = 9$ of these wheels yielded the following weights in pounds:

21.50	18.95	18.55	19.40	19.15
22.35	22.90	22.20	23.10	

Assuming that the distribution of the weights of the wheels of cheese is $N(\mu, \sigma^2)$, find a 95% confidence interval for μ.

7.1-8. Assume that the yield per acre for a particular variety of soybeans is $N(\mu, \sigma^2)$. For a random sample of $n = 5$ plots, the yields in bushels per acre were 37.4, 48.8, 46.9, 55.0, and 44.0.

(a) Give a point estimate for μ.

(b) Find a 90% confidence interval for μ.

7.1-9. During the Friday night shift, $n = 28$ mints were selected at random from a production line and weighed. They had an average weight of $\bar{x} = 21.45$ grams and a standard deviation of $s = 0.31$ grams. Give the lower endpoint of a 90% one-sided confidence interval for μ, the mean weight of all the mints.

7.1-10. A leakage test was conducted to determine the effectiveness of a seal designed to keep the inside of a plug airtight. An air needle was inserted into the plug, and the plug and needle were placed under water. The pressure was then increased until leakage was observed. Let X equal the pressure in pounds per square inch. Assume that the distribution of X is $N(\mu, \sigma^2)$. The following $n = 10$ observations of X were obtained:

3.1 3.3 4.5 2.8 3.5 3.5 3.7 4.2 3.9 3.3

Use the observations to

(a) Find a point estimate of μ.

(b) Find a point estimate of σ.

(c) Find a 95% one-sided confidence interval for μ that provides an upper bound for μ.

7.1-11. Students took $n = 35$ samples of water from the east basin of Lake Macatawa (see Example 7.1-4) and measured the amount of sodium in parts per million. For their data, they calculated $\bar{x} = 24.11$ and $s^2 = 24.44$. Find an approximate 90% confidence interval for μ, the mean of the amount of sodium in parts per million.

7.1-12. In nuclear physics, detectors are often used to measure the energy of a particle. To calibrate a detector, particles of known energy are directed into it. The values of signals from 15 different detectors, for the same energy, are

260	216	259	206	265	284	291	229
232	250	225	242	240	252	236	

(a) Find a 95% confidence interval for μ, assuming that these are observations from a $N(\mu, \sigma^2)$ distribution.

(b) Construct a box-and-whisker diagram of the data.

(c) Are these detectors doing a good job or a poor job of putting out the same signal for the same input energy?

7.1-13. A study was conducted to measure (1) the amount of cervical spine movement induced by different methods of gaining access to the mouth and nose to begin resuscitation of a football player who is wearing a helmet and (2) the time it takes to complete each method. One method involves using a manual screwdriver to remove the side clips holding the face mask in place and then flipping