CMPSC 465
Data Structures and Algorithms
Spring 2022

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Greedy algorithms

Greedy algorithms

Warm-up

Greedy algorithms

Minimum Spanning Tree

Kurskal's Algorithm

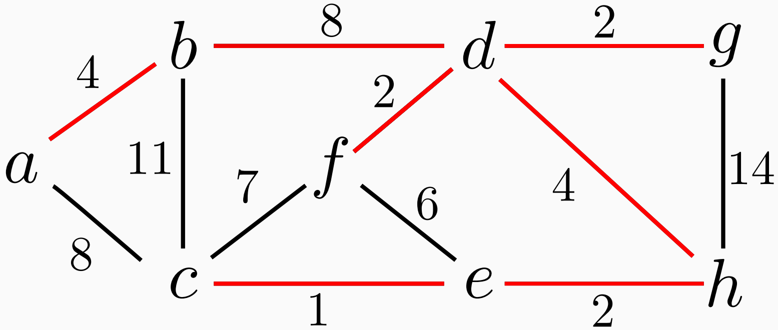
def $\textsc{Kruskal\_MST}$*(undirected $G = (V, E)$, weights $w = (w_e)_{e \in E}$)*:

```
Set A := { };
for v ∈ V:
    make_set(v);
Sort E in increasing order of edge weights;
for (u, v) ∈ E:
    if find_set(u) ≠ find_set(v):
        A := A ∪ {(u, v)};
        union(u, v);
```

- $\text{make\_set}(v)$: put v into a set containing itself. $v \mapsto \{v\}$
- $\text{find\_set}(u)$: find which set u belongs to
- $\text{union}(u, v)$: merge the sets that u and v are in

Example

A: red edges



$$\{a, b, c, e, h, d, f, g\}$$

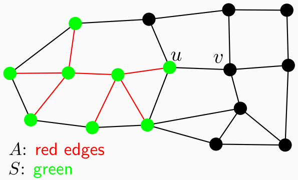
Proof of correctness

We need to show two things:

1. A is a spanning tree
   - it has no cycles
   - if there exists $v \in V$ that is not connected by A
     then there exists $e \in E$ s.t. $A \cup \{e\}$ contains no cycle
     then $\textsc{Kruskal\_MST}$ will add the lightest such edge
2. A has the minimum weight. Use the cut property

Theorem (The cut property)

Let A be a subset of edges of some MST of $G = (V, E)$. Let $(S, V - S)$ be a cut that respects A. Let e be the lightest edge across the cut. Then $A \cup \{e\}$ is part of some MST



if $\textsc{Kruskal\_MST}$ adds $e = (u, v)$. Let S be the vertices reachable from u in A (the partial solution so far)

Then $A \cup \{e\}$ is part of some MST $\square$

Proof of the cut property theorem (I)

To prove the cut property, we need a lemma

Lemma

Any connected and undirected graph $G = (V, E)$ is a tree if and only if $|E| = |V| - 1$

Proof.

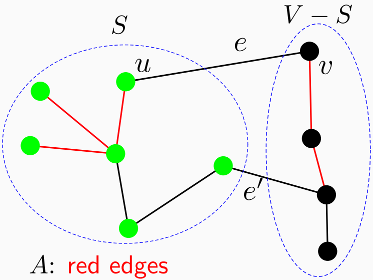
See textbook page 129 $\square$

Proof of the theorem (II)

By assumption, A is a subset of edges of some MST T

If e is part of T, then there is nothing to prove

If $e \notin T$, we will construct a new MST T' with $e \in T'$:



A: red edges

S: green

Add e to T. This will create a cycle, which must have an edge $e' \in T$ across the cut

By definition of e, $w_e \leq w_{e'}$

Let $T' = T \cup \{e\} - \{e'\}$. We didn't change the edge count. Lemma $\implies$ T' is a tree

$$\begin{aligned} \text{weight}(T') &= \text{weight}(T) + w_e - w_{e'} \\ &\leq \text{weight}(T) + w_{e'} - w_{e'} \\ &= \text{weight}(T) \end{aligned}$$

T is MST $\implies$ $\text{weight}(T') = \text{weight}(T)$ $\implies$ T' is also an MST $\square$

Running time of Kruskal's algorithm (I)

Depends on how we implement make_set, find_set, and union

Using linked list:

$\{a, b, c\}$ $\quad \text{head} \to a \to b \to c$ $\qquad$ $\text{find\_set}(b)$: $O(1)$

$\text{make\_set}(v)$: $O(1)$

$\{d, e\}$ $\quad \text{head} \to d \to e$

$\text{union}(a, b)$ $\qquad \text{head} \to a \to b \to c \to d \to e$

Cost of union: O(length of the shorter list)

Using an array to implement it:

vertex	1	2	3	4	5
head	1	1	1	4	4

$\xrightarrow{\text{union}}$

1	2	3	4	5
1	1	1	1	1