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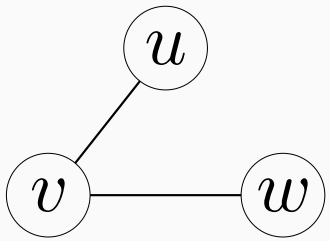
If no, there's such t that $B(s, t) =$ yes. So $s \notin$ X $\square$

Example of such C_s

Decide if there's an IS of size 2



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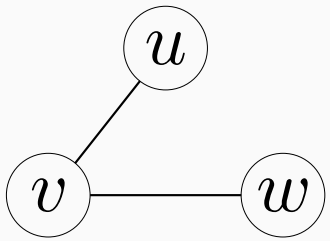

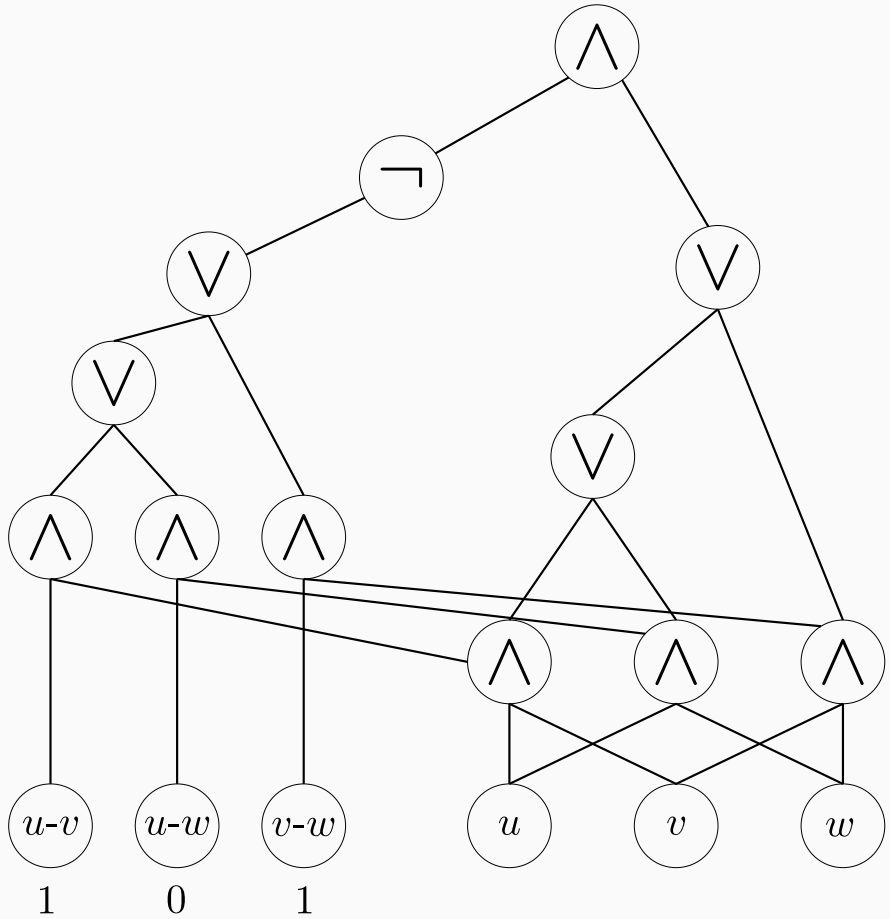

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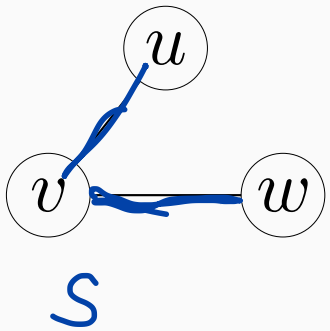

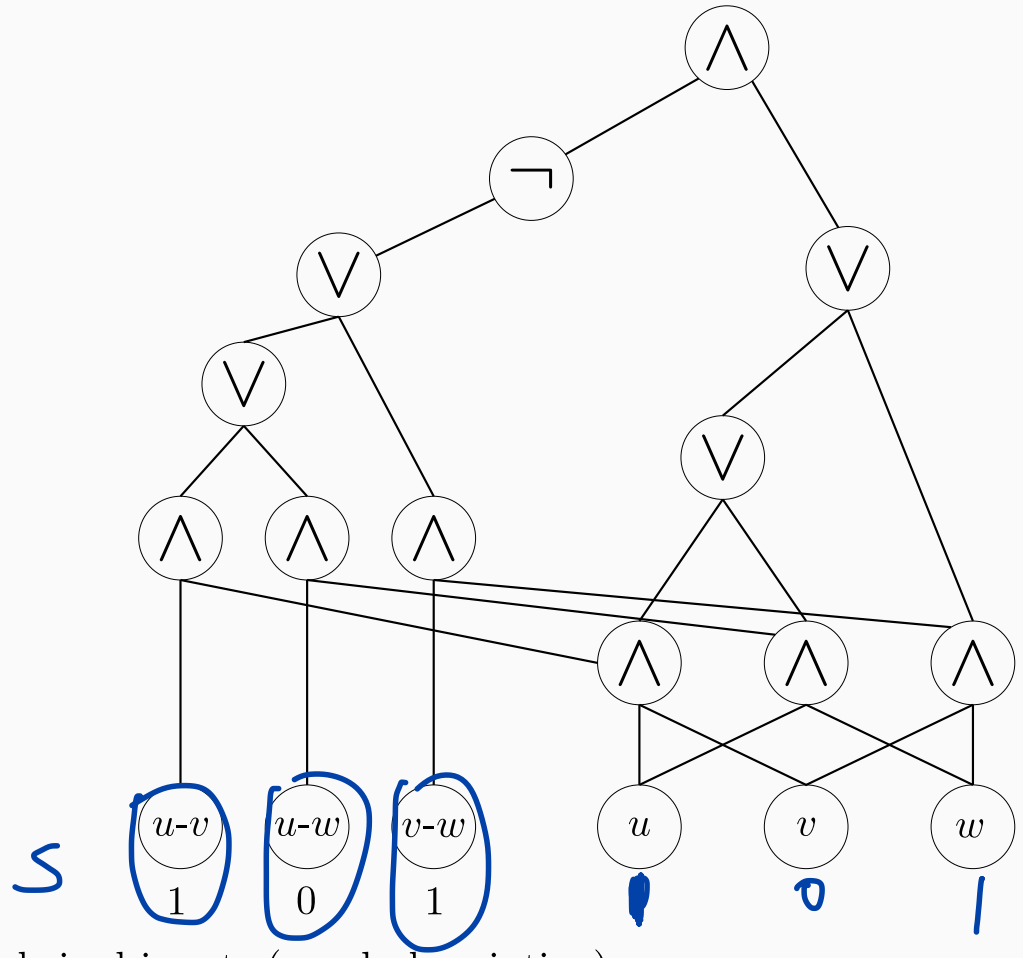

$\binom{n}{2}$ hardwired inputs (graph description)

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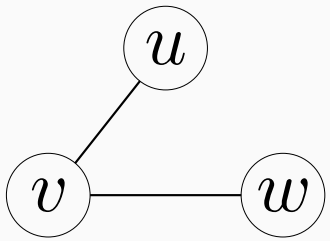

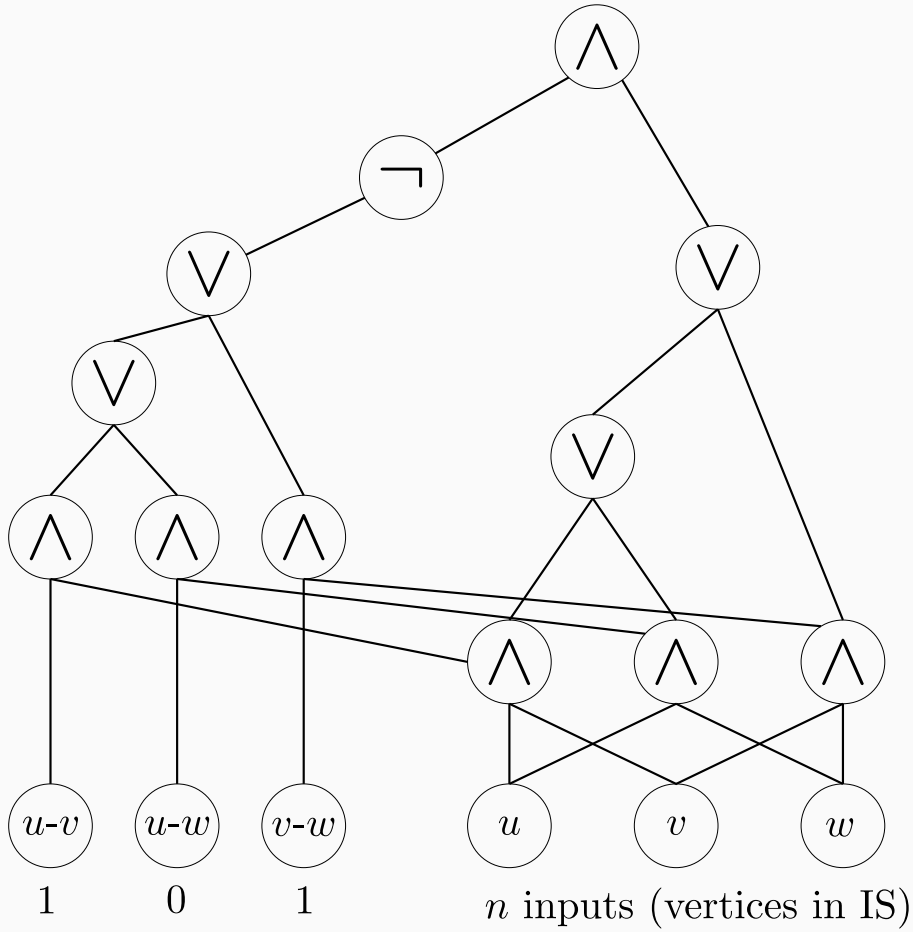

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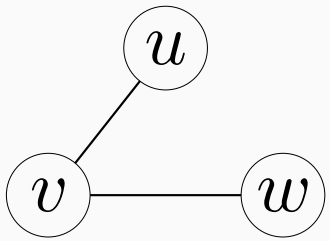

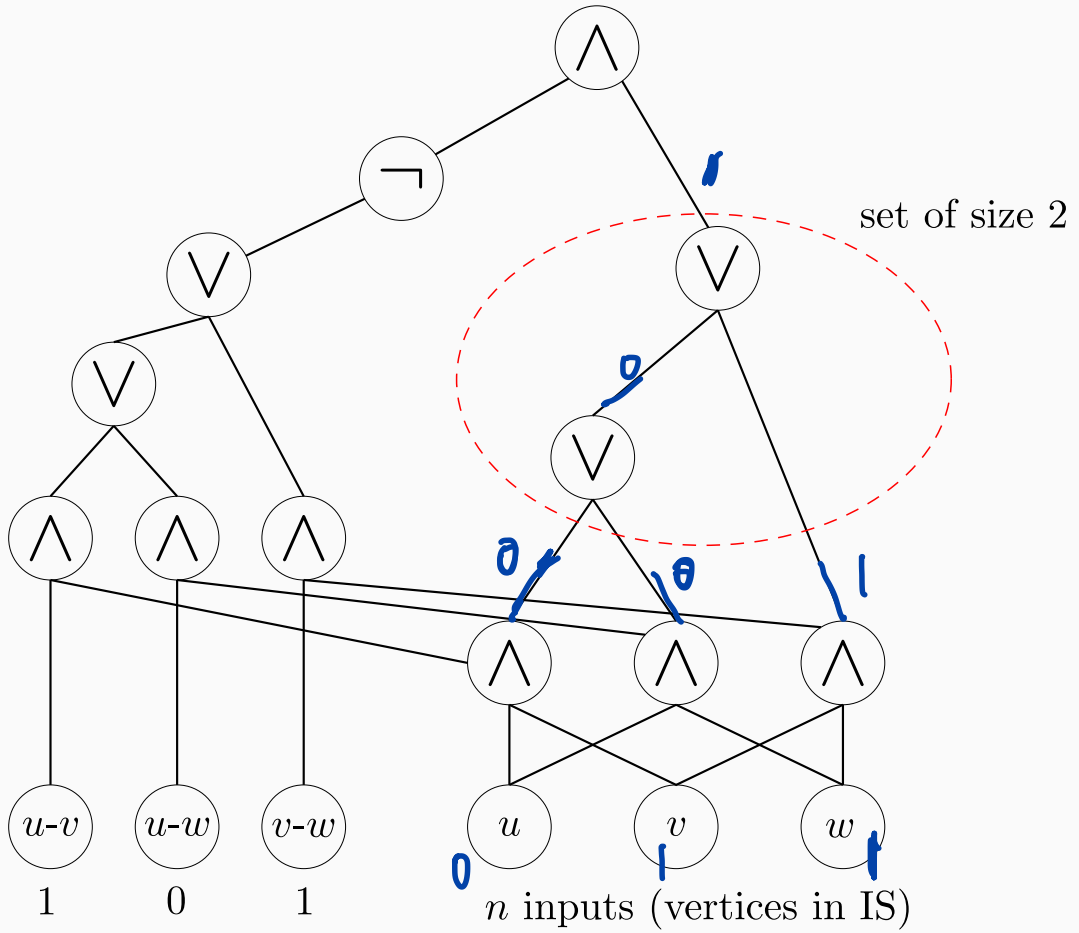

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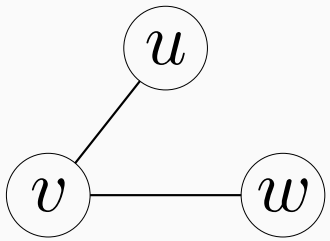

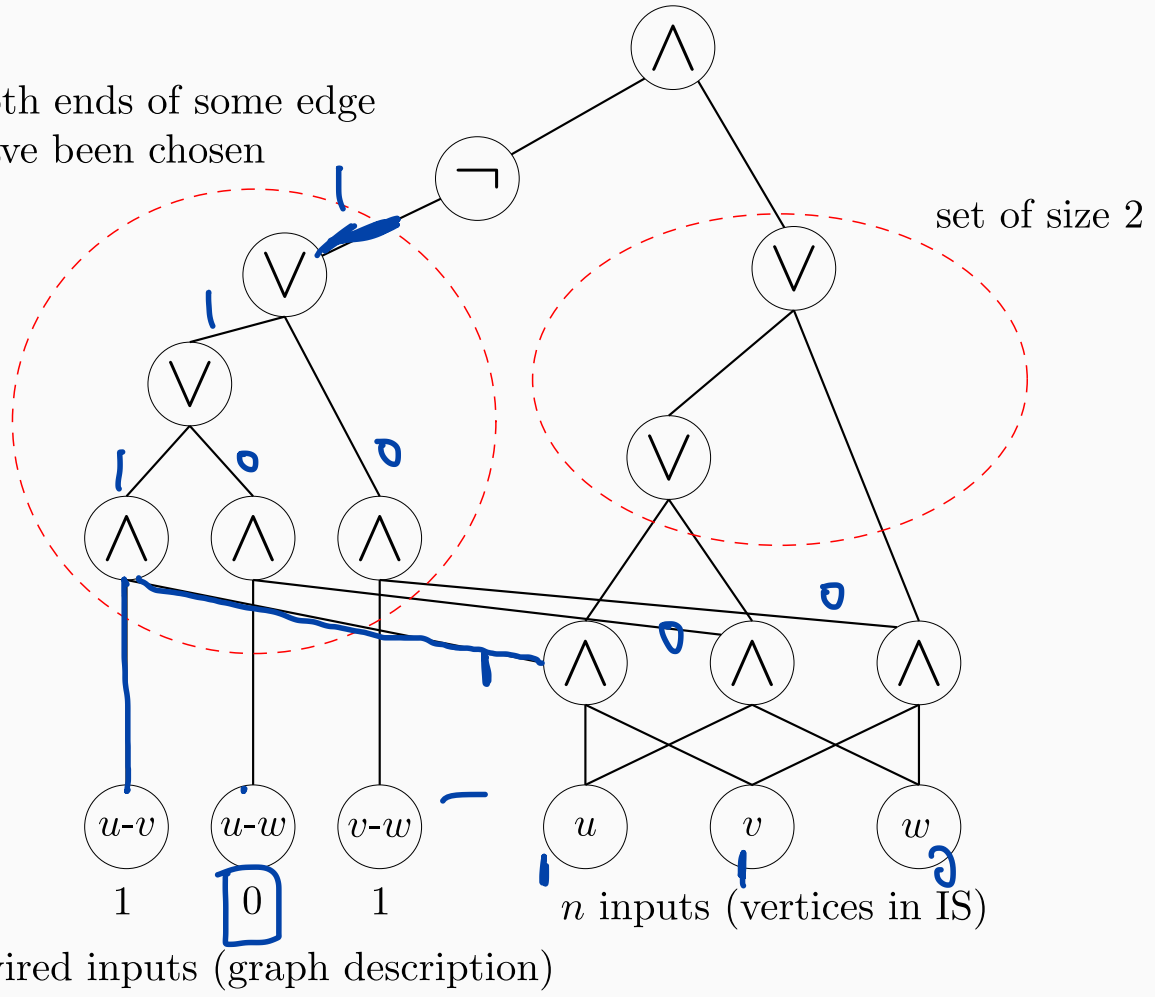

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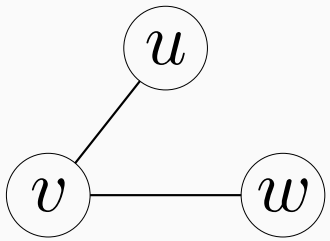

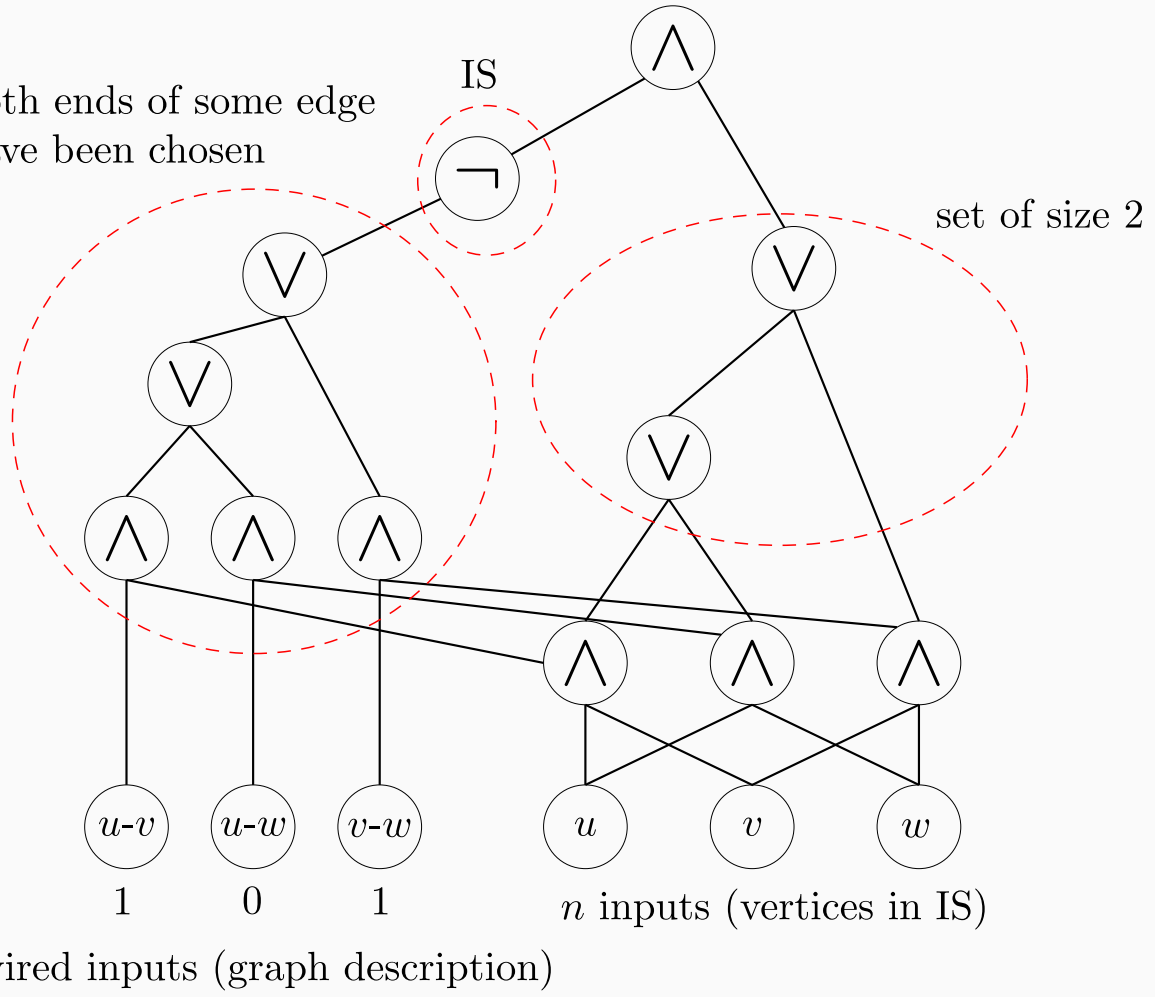

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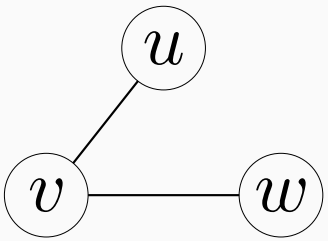

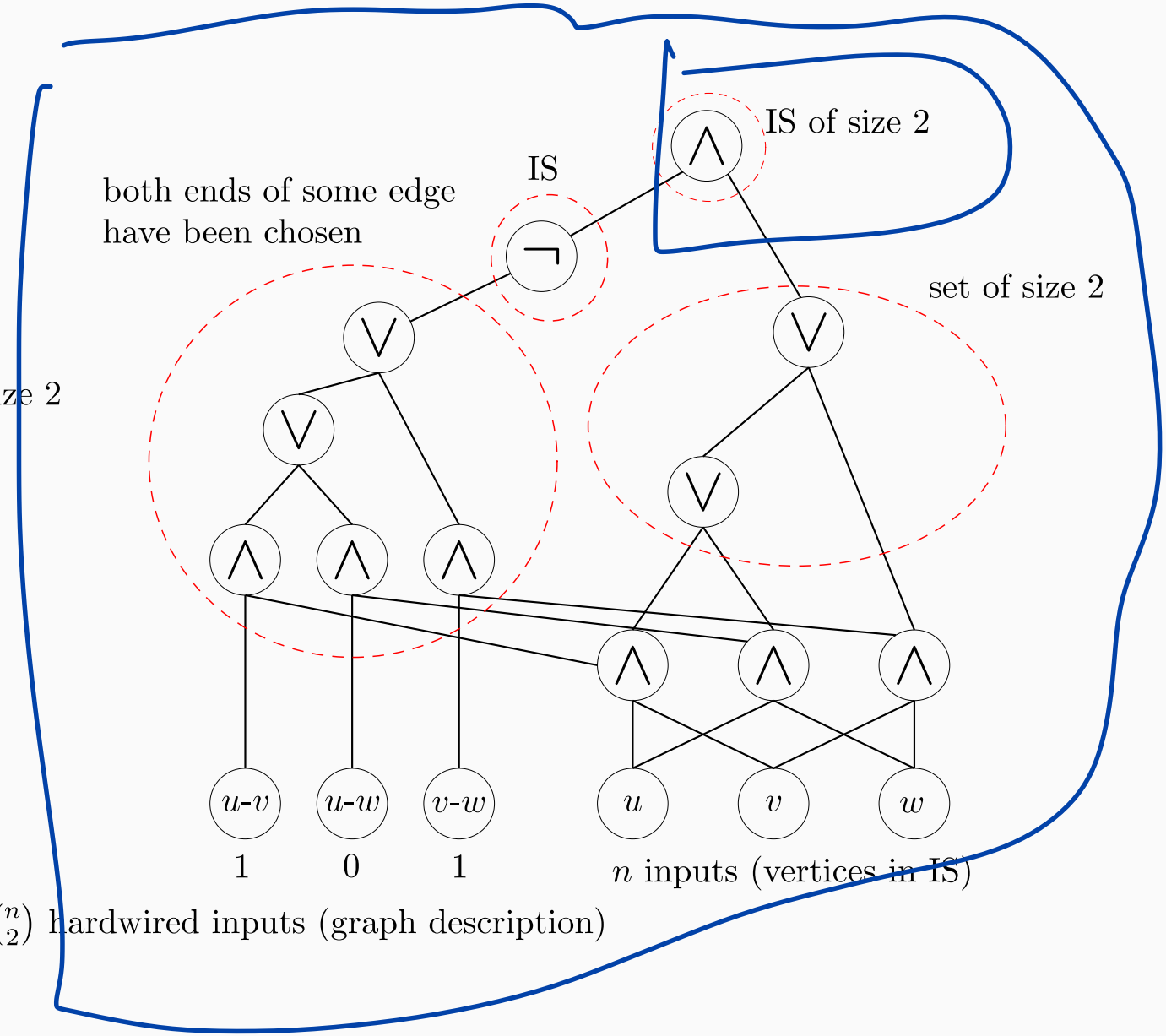

Proving NP-completeness

Recipe for proving Y is **NP**-complete

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Proof.

Let W be any problem in **NP**. Then $W \leq_P X \leq_P Y$ implies that $W \leq_P Y$. Therefore, Y is **NP**-complete □

3-SAT is NP-complete

Theorem

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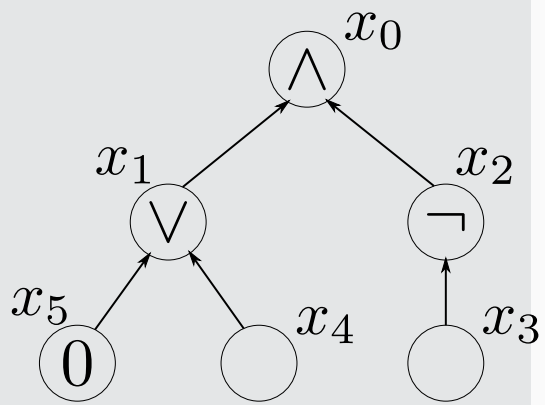

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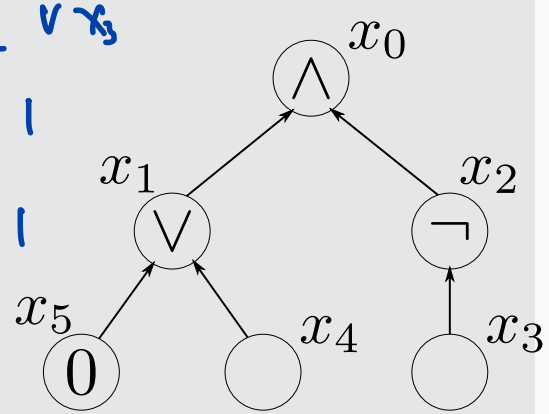
0	1	1	1
1	0	1	1



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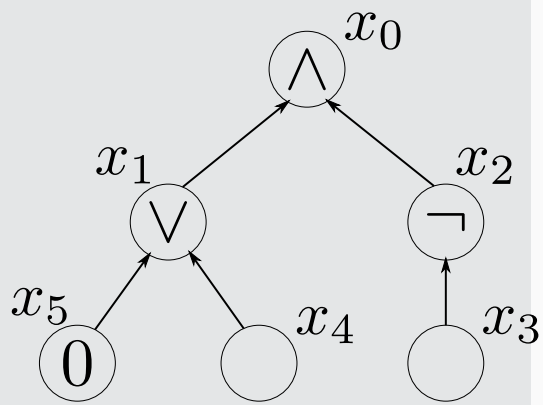

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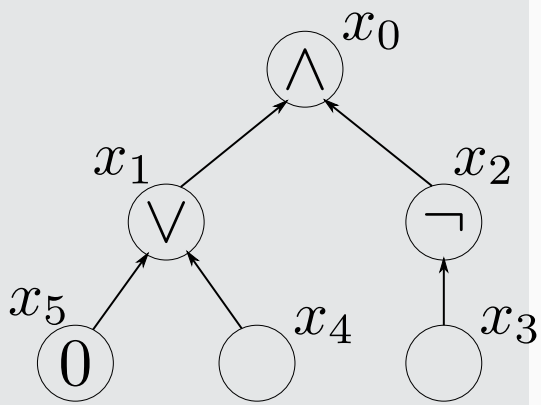

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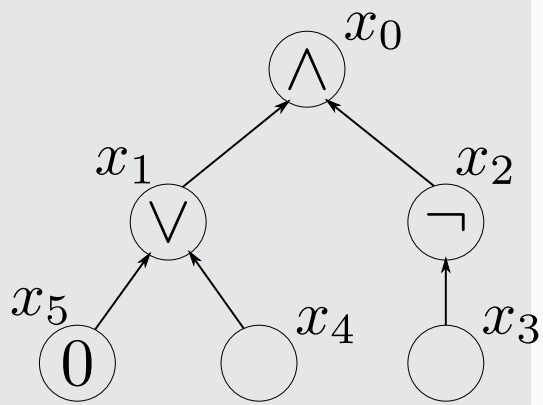

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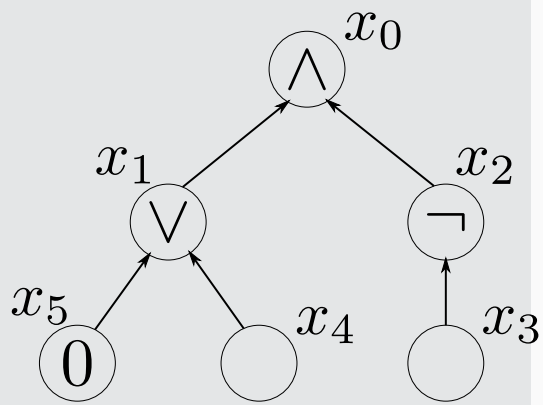

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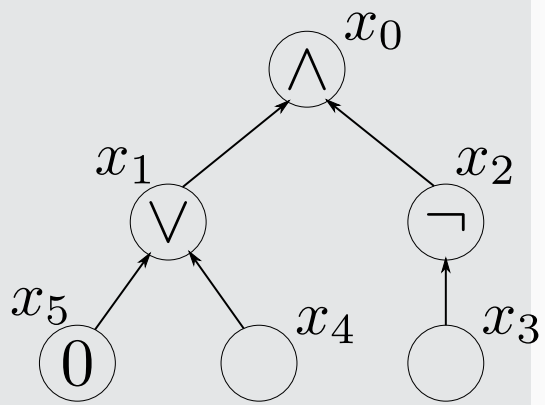

Turn clauses of length < 3 into clauses of length exactly 3 $\square$

Other NP-complete problems

From last lecture:

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