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Definition

A **CNF formula** is a conjunction of clauses, where each clause is a disjunction of literals

Example: $(\overset{x_1}{\cancel{x_1}} \vee x_2 \vee \bar{x}_3 \vee x_4) \wedge (x_3 \vee \bar{x}_5 \vee x_6) \wedge (\bar{x}_4 \vee x_7)$

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Definition

A **k-CNF** is a CNF where each clause contains exactly k literals

The Satisfiability Problem (SAT)

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Instance: A CNF Φ

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Objective: Decide if Φ is satisfiable, i.e., is there an assignment so that Φ is true?

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The k-Satisfiability Problem (k-SAT)

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3-SAT and Independent Set

Theorem

3-SAT $\leq_P$ Independent Set

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Proof.

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Proof. First consider an intuition for solving SAT:

$$\Phi = (x_1 \lor x_2 \lor x_3) \land (x_2 \lor \bar{x}_3 \lor x_4) \land (x_3 \lor \bar{x}_1 \lor x_5)$$

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Proof. First consider an intuition for solving SAT:

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$x_1 = 1$, $x_2 = 1$, $x_3 = 1$

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Proof. First consider an intuition for solving SAT:

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We encode a CNF as a graph, and encode an assignment as independent sets (to keep track of the conflicts)

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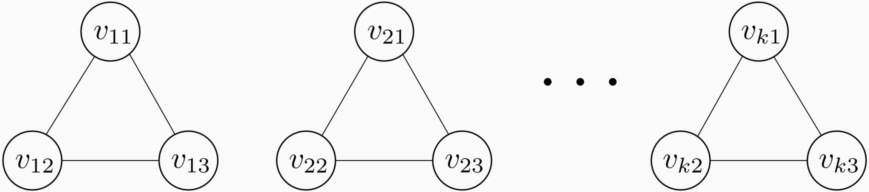

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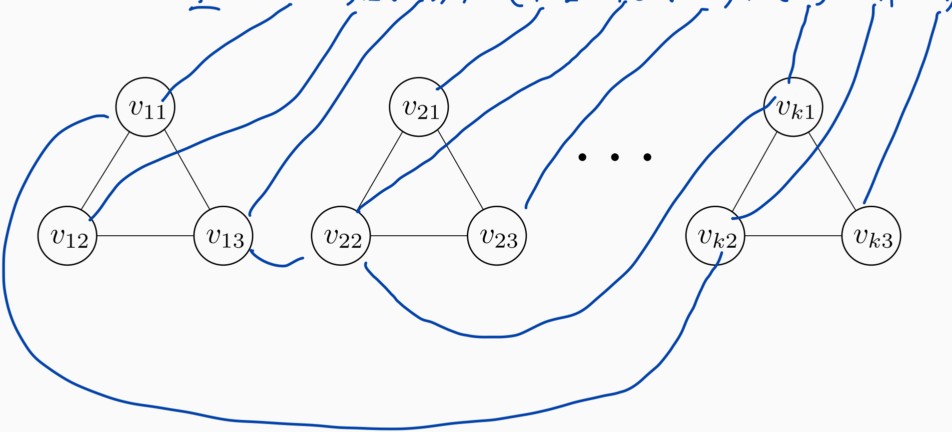

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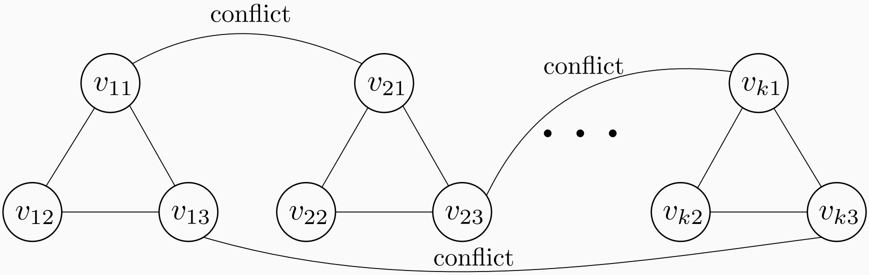

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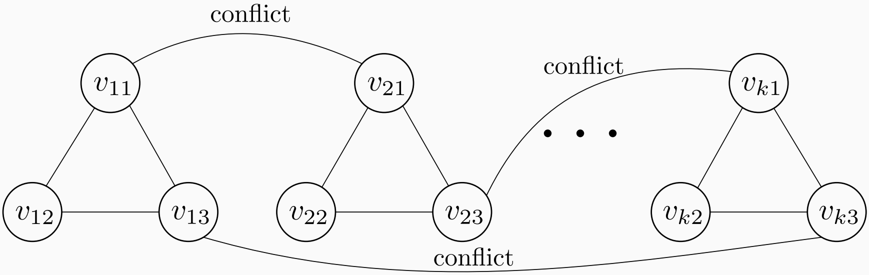

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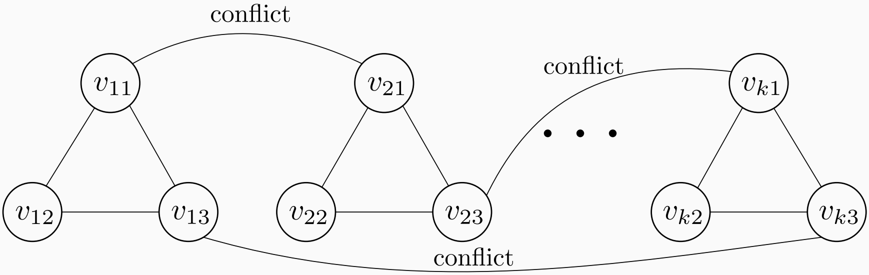

At most one vertex in each triangle can be in an independent set, so the size of an independent set cannot be larger than k

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So the 3-CNF has a satisfying assignment if and only if G has an independent set of size k $\square$

Example of the reduction

Consider $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \bar{x}_3 \vee x_4) \wedge (x_3 \vee \bar{x}_1 \vee x_5)$

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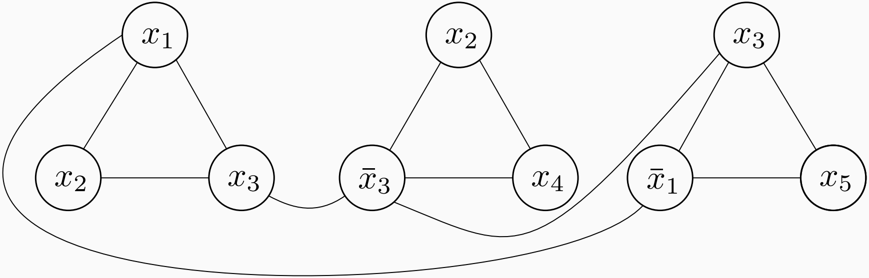

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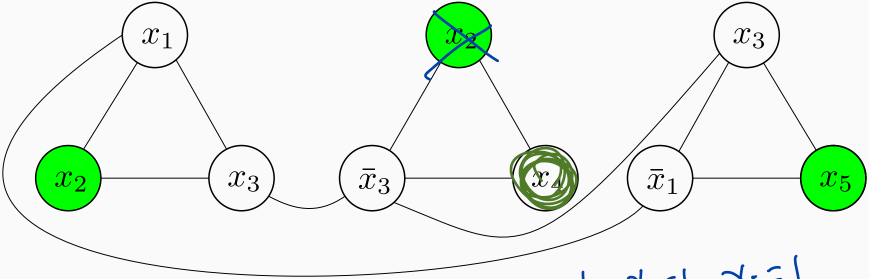

$x_2 = 1$, $x_5 = 1$

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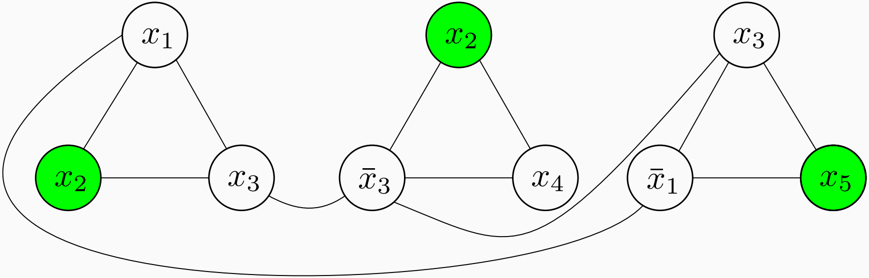

Satisfying assignment: $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 1$

NP and Computational Hardness

P, NP, and NP-completeness
(Kleinberg-Tardos, Section 8.3, 8.4)

Problems and algorithms

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Computational class

P : the class of all problems for which there exists a polynomial-time algorithm

Checking vs solving

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But there might be exponentially many possible t's

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It is easy to see: 3-SAT $\in$ **NP** Independent Set $\in NP$

Vertex Cover $\in NP$

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It is easy to see: 3-SAT $\in$ **NP**

Lemma

P $\subseteq$ **NP**

For any problem in P, there exists an efficient certifier for it.

How to show this?

The computational class NP

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It is easy to see: 3-SAT $\in$ **NP**

Lemma

P $\subseteq$ **NP**

Proof.

For any problem in **P** with algorithm A, we construct a certifier B that just returns $A(s)$ with empty certificate t $\square$

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- for all $Y \in \mathbf{NP}$, $Y \leq_P X$

NP-completeness

Fundamental question in CS: is $\mathbf{P} = \mathbf{NP}$? i.e., does there exist a problem $X \in \mathbf{NP}$ but $X \notin \mathbf{P}$?

We don't know the answer, but we try to find the most difficult problems in **NP**:

Definition

A problem X is **NP-complete** if

- $X \in \mathbf{NP}$ and
- for all $Y \in \mathbf{NP}$, $Y \leq_P X$

Suppose you know

X is NP-complete

and $X \leq_p Y$, then Y is NP-complete

Lemma

If an **NP***-complete problem can be solved in polynomial time, then* $\mathbf{P} = \mathbf{NP}$

Which problems are NP-complete?

A first **NP**-complete problem: Circuit Satisfiability

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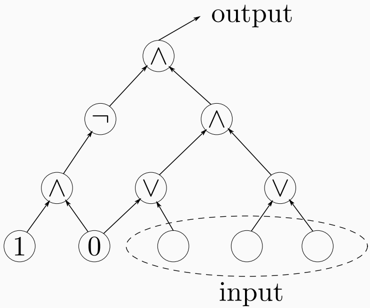
A first **NP**-complete problem: Circuit Satisfiability

A circuit consists of

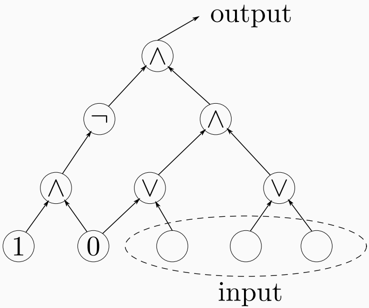


Which problems are NP-complete?

A first **NP**-complete problem: Circuit Satisfiability

A circuit consists of

- inputs



Which problems are NP-complete?

A first **NP**-complete problem: Circuit Satisfiability

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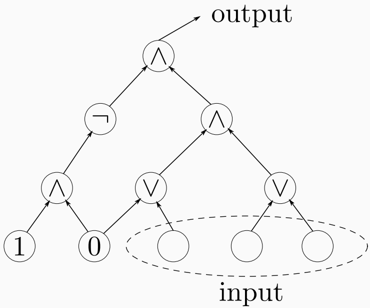

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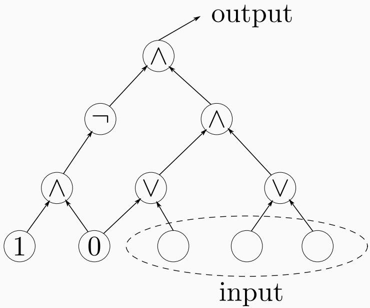

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A first **NP**-complete problem: Circuit Satisfiability

A circuit consists of

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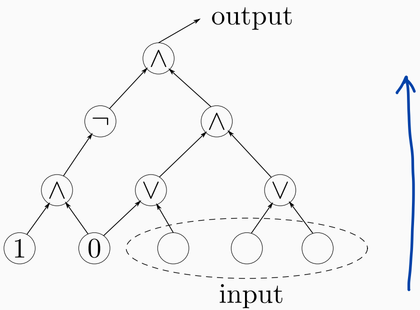

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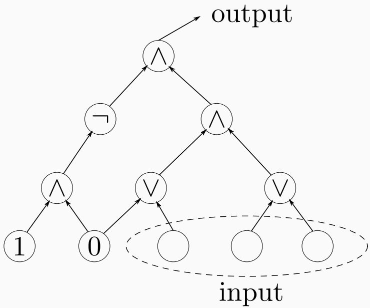

The Circuit Satisfiability Problem (circuit-SAT)

Instance: A circuit C

Objective: Decide if C is satisfiable

The Cook-Levin Theorem

Theorem (Cook-Levin)

circuit-SAT is **NP***-complete*