CISC 360 Project

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1 Introduction

LU Factorization is the most common technique used to solve systems of linear equations. It is most useful when solving a dense linear system and is only appropriate when the system is square. It works by decomposing a square matrix A into an lower triangle matrix, L and an upper triangular matrix U such that

$$A = LU$$

1.1 Impact

Triangular matrices have a number of nice properties that make them easier to work with. For example, if T is a triangular matrix, then you can solve the equation

$$Tx = b$$

in $O(n^2)$ time, as opposed to $O(n^3)$ for full matrices. Computationally, this means we can solve k equations of the form

$$Ax = b_i$$

for $1 \leq i \leq k$ in $O(n^3 + kn^2)$ as opposed to $O(kn^3)$.

1.2 Use Cases

Dense linear algebra is very important for mathematicians, scientists, and engineers alike. Linear algebra comes up in so many situations:

- Physics
- Partial differential equations
- Graph theory
- Statistics / Curve Fitting
- Sports Ranking

Solving linear systems is a key component of linear algebra, and LU factorization is the best known way to solve these systems. Having a highly optimized LU Factorization algorithm gives you the ability to (1) solve systems faster, and (2) solve bigger systems.

2 Background

2.1 Triangular Matrices

Triangular matrices have a number of nice properties. Most importantly, if A is a triangular matrix, then you can solve an equation of the form $Ax = b$ directly using back substitution. The time complexity of back substitution is $O(n^2)$ which is much faster than the $O(n^3)$ time complexity to solve full systems. To see this, consider the general system of equations which we want to solve for $\mathbf{x}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We can see immediately that

$$x_3 = \frac{b_3}{a_{33}}$$

We can take this result, and substitute it through column three of the matrix. Now we can immediately solve for x_2 and repeat until all x_i have been solved for.

2.2 Algorithm Description

As we've seen from the previous section, back substitution is relatively cheap compared to triangularization. The LU factorization algorithm is an iterative algorithm that iteratively zeroes out columns of A until it becomes an upper triangular matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

we start by zeroing out everything below the diagonal in row 1 by replacing row i with a *l*inear combination of row i and row 1 such that the first element is 0. Thus, after one iteration of the algorithm, U would look like:

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix}$$

After the next iteration, we would zero out everything below the diagonal in row 2 by replacing row i with a linear combination of row i with row 2 such that the second element is 0. After two iterations of the algorithm, U would look like:

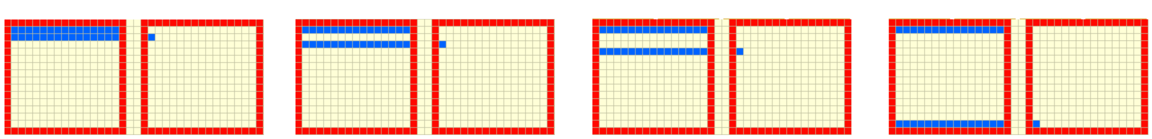
$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

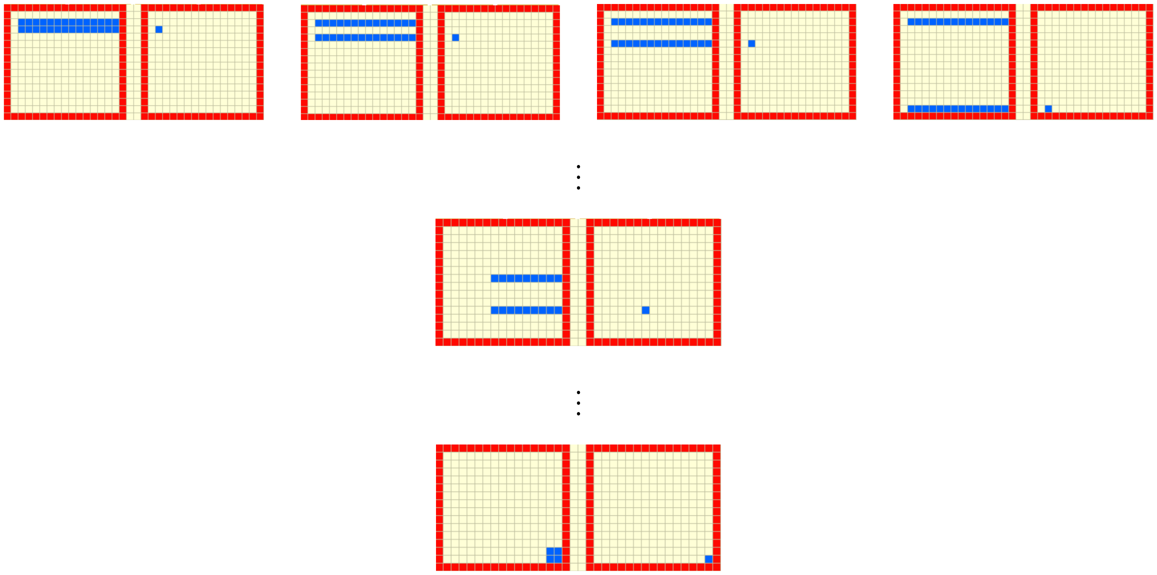
The values of L (the lower triangular matrix) can be trivially filled in as the multiplication factor used in the linear combination of the rows.

This process repeats until U is a upper triangular matrix, meaning that all elements below the main diagonal are 0.

2.3 Access Pattern

In the standard implementation of LU factorization, the access pattern for L and U respectively is:





3 Existing Work

4 Approach

We will start off with a simple implementation of LU factorization that is not optimized for any particular architecture which will serve as a starting point to measure performance improvements.

4.1 Architecture

Our initial optimizations target a multicore CPU architecture, such as the current Intel Haswell series i7 4970K processor. By utilizing OpenMP, our code will be accessible on a wide variety of platforms. This project may also utilize CUDA in order to implement parallel optimizations on the GPU. GPU parallelization is advantageous as it allows us to take advantage of a large number of cores + threads. Code optimizations will target the most recent nvidia GTX 970 GPU.

4.2 Optimizations

TO-DO: cache, loop unrolling, openMP, vector operations, cuda

5 Results

5.1 Sequential Optimizations

5.2 Parallel Optimizations

6 Conclusion