Team 6: LU Factorization

Optimizations targeting towards multicore processors

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Linear Algebra

- The quintessential problem in linear algebra is solving a linear system of equations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- We want to find values of x_1, x_2, and x_3 such that
    1. $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$
    2. $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$
    3. $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

Impact

- Linear algebra comes up in a lot of professions:
  - Physics
  - Partial differential equations
  - Graph theory
  - Statistics / Curve Fitting
  - Sports Ranking

Solving Linear Systems

- If A is an $n \times n$ matrix, solving a system of the form $Ax = b$ takes $O(n^3)$ time.
- If A is a triangular matrix, then solving the system takes $O(n^2)$ time

LU Factorization Background

- LU Factorization works by decomposing a square matrix A into a lower triangular matrix, L, and an upper triangular matrix, U:

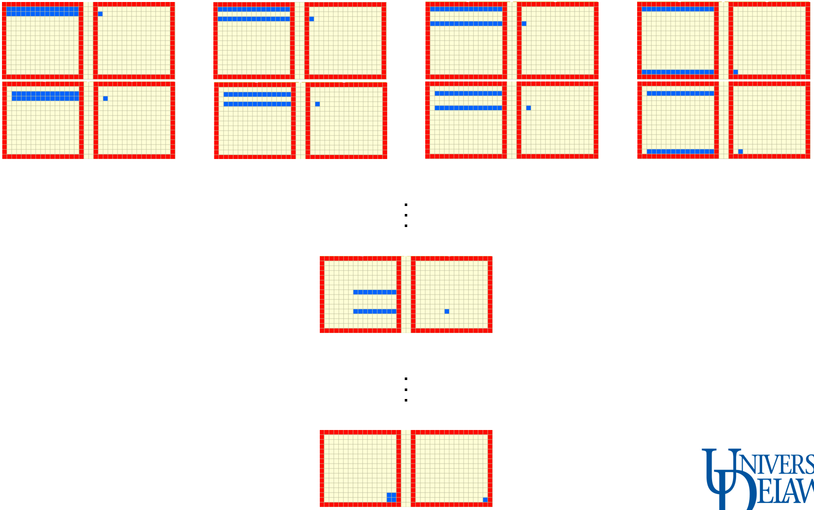
$$A = LU$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- With L and U, we can solve $Ax = LUx = b$ in $O(n^2)$.

Algorithm Description / Access Pattern

LU factorization is an $O(n^3)$ algorithm:



Implementation

```
void lu(double **A, double **L, double **U, int n) {
    zero (L, n);
    copy (U, A, n);
    init (L, n);
    for(int j=0; j < n; j++) {
        for(int i=j+1; i < n; i++) {
            double m = U[i][j] / U[j][j];
            L[i][j] = m;
            for(int k=j; k < n; k++)
                U[i][k] -= m*U[j][k];
        }
    }
}
```

Architecture

- Intel i7-5930K CPU @ 3.50 GHz
- 6 cores and a total of 12 available threads
- Cache sizes: L1I: 32K, L1D: 32K, L2: 256K, L3: 15,360K.

Approach

1. Generate random matrices up to 6400 x 6400.
2. Run and time 4 trials of the factorization algorithm for each matrix size.
3. Repeat for every optimization configuration.

Optimizations

- -O1, -O2, -O3
- loop unrolling
- vectorization
- native (architecture specific) optimizations
- openMP

Compiler

Both gcc and icc used in program compilation

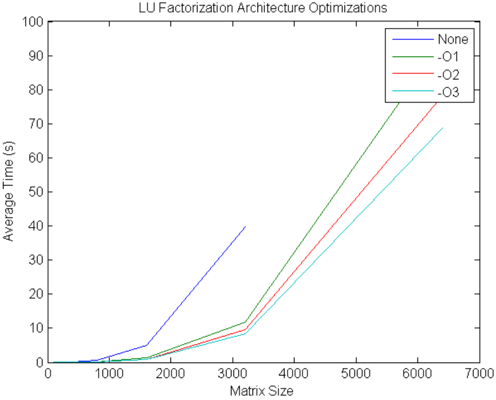
- gcc -O3 -march=native -fopenmp -funroll-loops
- icc -O3 -xHOST -openmp -funroll-loops

Sequential Results

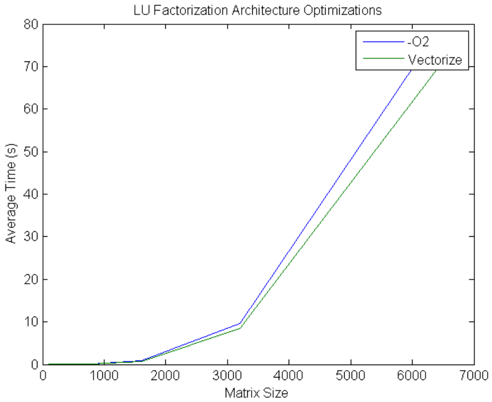
	Optimization / Time (s)						
n	None	-O1	-O2	Vector	-O3	Unroll	Native
100	0.0028	0.0010	0.0008	0.0005	0.0005	0.0004	0.0003
200	0.0101	0.0034	0.0026	0.0017	0.0018	0.0025	0.0014
400	0.07902	0.0219	0.0189	0.0158	0.01561	0.0151	0.0124
800	0.6222	0.1575	0.1070	0.0850	0.0821	0.0800	0.0662
1600	4.9364	1.3342	0.9388	0.7737	0.7900	0.7465	0.6732
3200	39.7114	11.7545	9.5557	8.4604	8.4156	8.1363	8.0514
6400	316.3	94.23	78.12	69.35	68.70	66.54	66.76

- Going from no optimizations to -O1 gave nearly 400% speedup.
- Vectorization yielded an 11% speedup.
- Loop unrolling gave a 3 % speedup in the largest case.
- Native optimizations yieled a 15 % speedup in the best case ($n = 1600$).

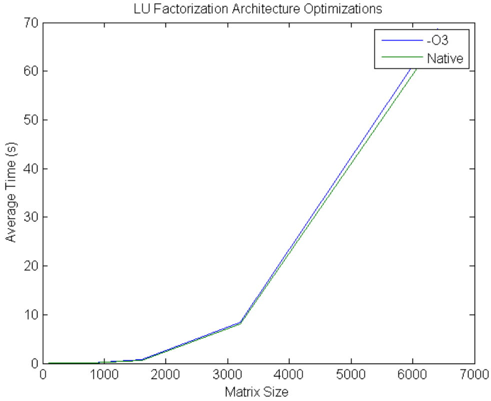
Sequential Results



Sequential Results



Sequential Results



Parallel Implementation

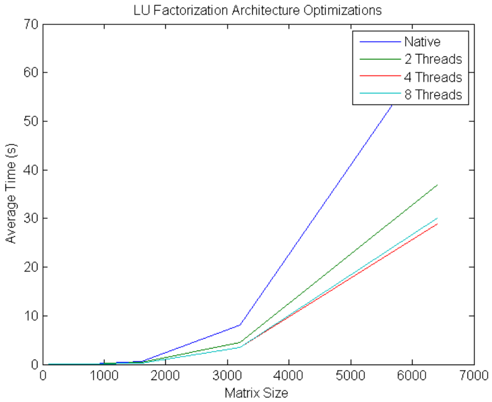
```
void lu(double **A, double **L, double **U, int n) {
    zero (L, n);
    copy (U, A, n);
    init (L, n);
    for(int j=0; j < n; j++) {
        #pragma omp parallel for schedule(static,8)
        for(int i=j+1; i < n; i++) {
            double m = U[i][j] / U[j][j];
            L[i][j] = m;
            for(int k=j; k < n; k++)
                U[i][k] -= m*U[j][k];
        }
    }
}
```

Parallel Results

n	Native	2 Threads	4 Threads	8 Threads
100	0.000338	0.000337	0.001492	0.000409
200	0.001417	0.0014	0.000963	0.000725
400	0.012435	0.006679	0.003321	0.002973
800	0.066184	0.036134	0.02161	0.021725
1600	0.6732	0.354768	0.221714	0.219362
3200	8.05138	4.457911	3.46599	3.567894
6400	66.760086	36.836628	28.870047	30.108377

- From 1 core to 2 cores, we achieved a scalability of 1.81.
- Past 2 cores, we saw diminishing returns.

Parallel Results



Conclusion

- Free 4.75x speedup with just compiler flags
- 10 x speedup by targeting multi-core machines
- Don't need to sacrifice code readability for performance
- Vectorization didn't help as much as expected