二重积分对称性

积分区间关于原点对称 $[-a,a]$

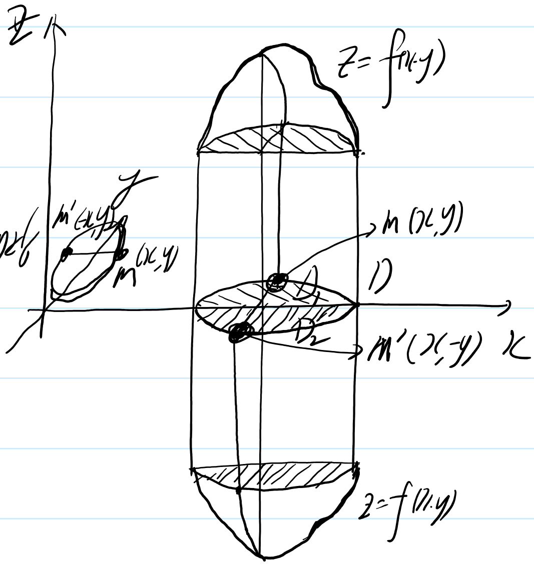
$$\begin{cases} 1)\ f(-x)=-f(x) \Rightarrow \int_{-a}^{a} f(x)dx=0 \\ 2)\ f(-x)=f(x) \Rightarrow \int_{-a}^{a} f(x)dx=2\int_{0}^{a} f(x)dx \end{cases}$$

$-a \quad -x \quad 0 \quad x \quad a$

$D \subset \mathbb{R}^2$ 有界 $\quad \iint_D f(x,y)\,d\sigma$

1) D关于x轴对称 $(D(x,y)=D(x,-y))$

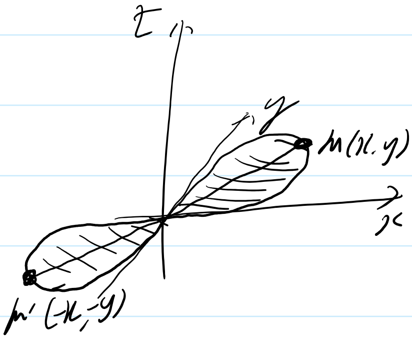
$$\begin{cases} f(x,-y)=-f(x,y) \Rightarrow \iint_D f(x,y)\,d\sigma=0 \\ f(x,-y)=f(x,y) \Rightarrow \iint_D f(x,y)\,d\sigma=2\iint_{D_1} f(x,y)\,d\sigma \end{cases}$$



2) D关于y轴对 $(D(x,y)=D(-x,y))$

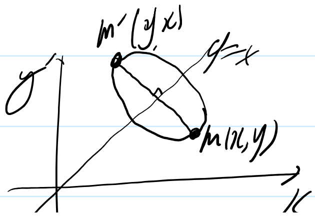
$$\begin{cases} f(-x,y)=-f(x,y) \Rightarrow \iint_D f(x,y)\,d\sigma=0 \\ f(-x,y)=f(x,y) \Rightarrow \iint_D f\,d\sigma=2\iint_{D_1} f\,d\sigma \end{cases}$$

3) D关于原点对称 $(D(x,y)=D(-x,-y))$

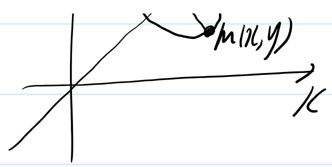


$$\begin{cases} f(-x,-y)=-f(x,y) \Rightarrow \iint_D f(x,y)\,d\sigma=0 \\ f(-x,-y)=f(x,y) \Rightarrow \iint_D f\,d\sigma=2\iint_{D_1} f\,d\sigma \end{cases}$$

4) D关于 $y=x$ 对称 $(D(y,x)=D(x,y))$

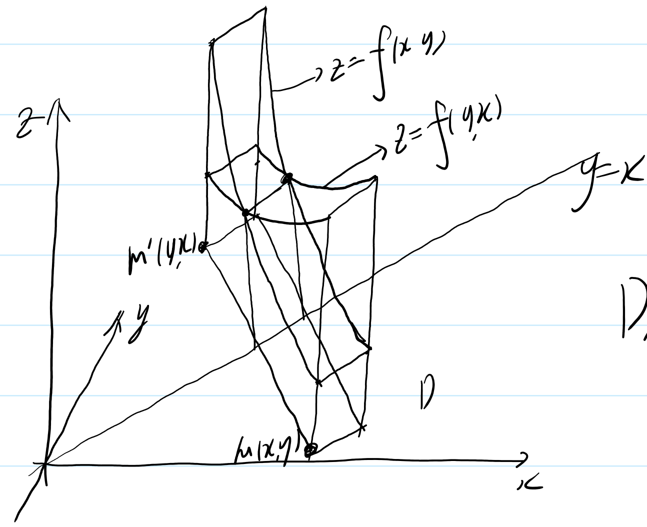


4) D 关于 $y=x$ 对称 $(D(y,x)=D(x,y))$



$$\begin{cases} f(y,x) = -f(x,y) \Rightarrow \iint\limits_D f(x,y)\,d\sigma = 0 \\ f(y,x) = f(x,y) \Rightarrow \iint\limits_D f(x,y)\,d\sigma = 2\iint\limits_{D_1} f(x,y)\,d\sigma \end{cases}$$

5) D 关于 $y=x$ 对称，则有 $\iint\limits_D f(x,y)\,d\sigma = \iint\limits_D f(y,x)\,d\sigma$



$D_{xy} = D_{yx}$

6) $\iint\limits_{D_{xy}} f(x,y)\,dxdy = \iint\limits_{D_{yx}} f(y,x)\,dxdy$ $\quad x \longleftrightarrow y$

例 (05-2) $D=\{(x,y) \mid x^2+y^2\le 4,\ x\ge 0,\ y\ge 0\}$ $f(x)$ 为 D 上正值连续 a、b 常数

则 $\iint\limits_D \dfrac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}\,d\sigma =$ ______

(A) πab (B) $\frac{\pi}{2}ab$ (C) $\pi(a+b)$ (D) $\frac{\pi}{2}(a+b)$

原式 $= \dfrac{1}{2}\left[\iint\limits_D \dfrac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}\,dxdy + \iint\limits_D \dfrac{a\sqrt{f(y)}+b\sqrt{f(x)}}{\sqrt{f(y)}+\sqrt{f(x)}}\,dxdy\right]$

$$\ldots 2\left(\iint_D \frac{\ldots}{\sqrt{f(x)}+\sqrt{f(y)}} \ldots \iint_D \frac{\ldots}{\sqrt{f(x)}+\sqrt{f(y)}} \ldots\right)$$

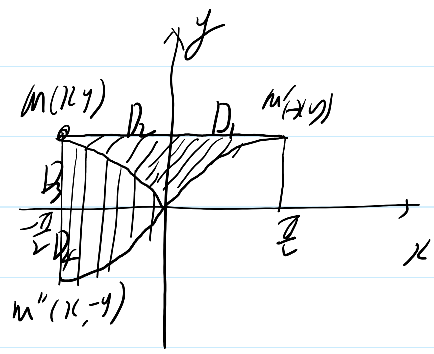
$$= \frac{1}{2}\iint_D \frac{a\sqrt{f(x)}+b\sqrt{f(x)}+a\sqrt{f(y)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}\,dxdy = \frac{1}{2}\iint_D (a+b)\,dxdy$$

$$= \frac{a+b}{2}\cdot\frac{1}{4}\pi\cdot 2^2$$

$$= \frac{\pi}{2}(a+b)$$

(12-2) 1) 由 $y=\sin x$，$x=\pm\frac{\pi}{2}$，$y=1$ 围成 则 $\iint_D (xy^5-1)\,dxdy=$ ______

(A) π　(B) 2　(C) -2，(D) $-\pi$



$$\text{原式} = \iint_{D_1\cup D_2}(xy^5-1)\,d\sigma + \iint_{D_3\cup D_4}(xy^5-1)\,d\sigma$$

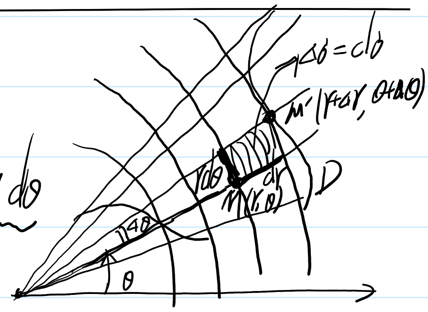
$$= \iint_{D_1\cup D_2}(-1)\,d\sigma + \iint_{D_3\cup D_4}(-1)\,d\sigma$$

$$= \iint_D -1\,d\sigma = -S_D$$

极坐标下二重积分

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

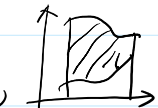
$\iint_D f(x,y)\,d\sigma$ 存在 $\xrightarrow[\Delta r,\ \Delta\theta \text{划分} D]{\text{对} D \text{任划分}}$ $d\sigma = dr\cdot r\,d\theta$



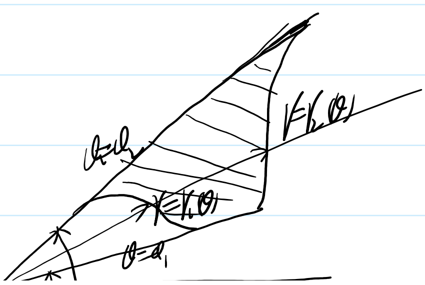
$$\boxed{d\sigma = r\,dr\,d\theta}$$

$$\iint_{D_{xy}} f(x,y)\,d\sigma = \iint_{D_{\theta r}} f[r\cos\theta,\ r\sin\theta]\,r\,dr\,d\theta$$

D_x: $\begin{cases} a\le x\le b \\ y_1(x)\le y\le y_2(x) \end{cases}$



D_y: $\begin{cases} c\le y\le d \\ x_1(y)\le x\le x_2(y) \end{cases}$



1) D_θ: $\begin{cases} \theta_1\le\theta\le\theta_2 \\ r_1(\theta)\le r\le r_2(\theta) \end{cases}$

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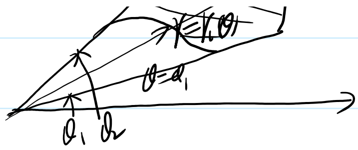
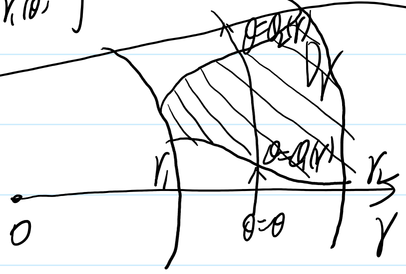


$$\iint_{D_{\theta r}} f(x,y)\,d\sigma=\int_{\theta_1}^{\theta_2}d\theta\int_{r_1(\theta)}^{r_2(\theta)}f(r\cos\theta,\ r\sin\theta)\,r\,dr$$

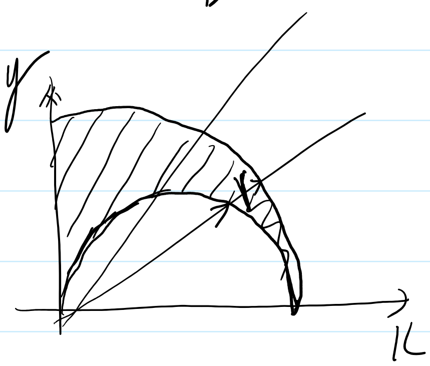
2) $D_r:\begin{cases}r_1\le r\le r_2\\ \theta_1(r)\le\theta\le\theta_2(r)\end{cases}$



不要求

$$\iint_{D_{\theta r}} f(x,y)\,d\sigma=\int_{r_1}^{r_2}dr\int_{\theta_1(r)}^{\theta_2(r)}f(r\cos\theta,\ r\sin\theta)\,r\,d\theta$$

例 计算 $\iint_D(x^2+y^2)\,d\sigma$ 其中 D 由 $x^2+y^2=2x$ 与 $x^2+y^2=4$ 及 y 轴一象限围成



$x^2+y^2=2x$

$r^2=2r\cos\theta\Rightarrow r=2\cos\theta$

$r^2=4\Rightarrow r=2$

$D_\theta\begin{cases}0\le\theta\le\frac{\pi}{2}\\ 2\cos\theta\le r\le 2\end{cases}$

$$\iint_D(x^2+y^2)\,d\sigma=\int_0^{\frac{\pi}{2}}d\theta\int_{2\cos\theta}^{2}(r^2\cos^2\theta+r^2\sin^2\theta)\,r\,dr$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_{2\cos\theta}^{2}r^3\,dr$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{4}r^4\Big|_{2\cos\theta}^{2}d\theta$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{2}}(16-16\cos^4\theta)\,d\theta$$

$$=4\int_0^{\frac{\pi}{2}}(1-\cos^4\theta)\,d\theta$$

$\left(\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\int_0^{\frac{\pi}{2}}\right)$

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots$$

$$\cos^4\theta = \left(\frac{1+\cos 2x}{2}\right)^2 = \cdots$$