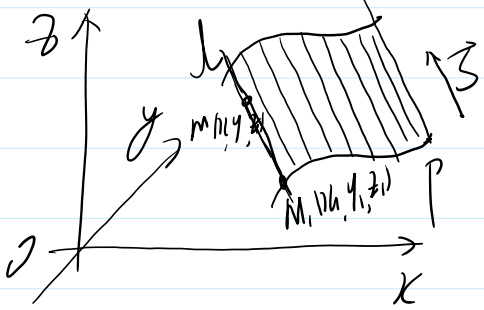
空间解析几何15-2

2022年4月19日 7:56

空间曲面（柱面，旋转曲面，锥面，二次曲面）

柱面

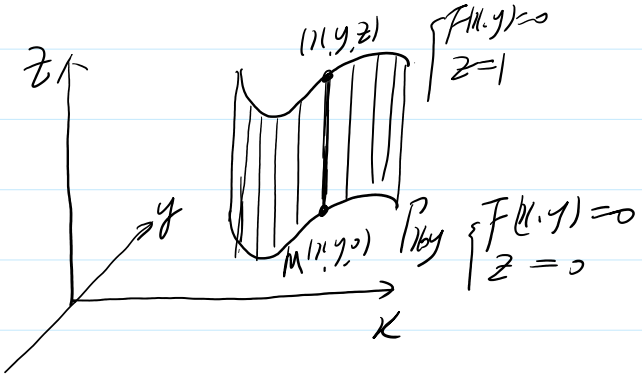
准线 $\Gamma\begin{cases}F_1(x,y,z)=0\\F_2(x,y,z)=0\end{cases}$ 运动方向 $\tau=(m,n,p)$



$$\begin{cases}L:\ \dfrac{x-x_1}{m}=\dfrac{y-y_1}{n}=\dfrac{z-z_1}{p} & ①②\\ \Gamma:\ F_1(x_1,y_1,z_1)=0 & ③\\ \quad F_2(x_1,y_1,z_1)=0 & ④\end{cases}$$

①～④中消去 x_1,y_1,z_1 $\Rightarrow$ $F(x,y,z)=0$ 即为所求柱面方程

母线平行于坐标轴柱面（准线在坐标面上平面曲线）

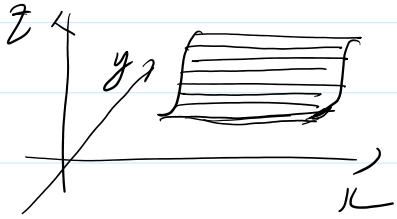


1) $\Gamma\begin{cases}F(x,y)=0\\z=0\end{cases}$ 为准线，母线平行于 z 轴

柱面为 $F(x,y)=0$

2) $\Gamma\begin{cases}H(y,z)=0\\x=0\end{cases}$ 为准线，母线平行于 x 轴

柱面为 $H(y,z)=0$



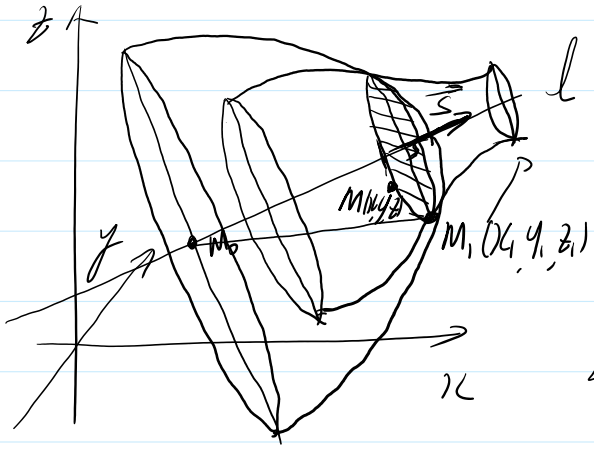
3) $\Gamma\begin{cases}G(z,x)=0\\y=0\end{cases}$ 为准线，母线平行于 y 轴

柱面 $G(z,x)=0$

旋转曲面

曲线 $\Gamma\begin{cases}F(x,y,z)=0\\\end{cases}$

旋转曲面



母线 $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$

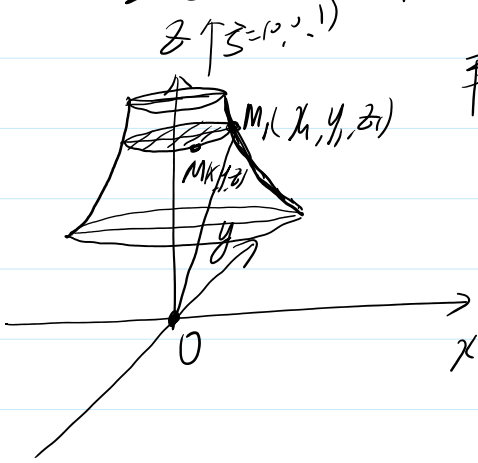
旋转轴 l $\dfrac{x-x_0}{m}=\dfrac{y-y_0}{n}=\dfrac{z-z_0}{p}$

$M_0\in l$ $\vec{s}=(m,n,p)$

$$\begin{cases} \text{纬圆} \begin{cases} \text{过}M_1\text{垂直于}l\text{的平面}\ m(x-x_1)+n(y-y_1)+p(z-z_1)=0 \quad ① \\ \text{以}M\text{为球心过}M_1\text{的球面}\ (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2 \quad ② \end{cases} \\ \text{母线} \begin{cases} F(x_1,y_1,z_1)=0 \quad ③ \\ G(x_1,y_1,z_1)=0 \quad ④ \end{cases} \end{cases}$$

①②③④中消去x_1,y_1,z_1 $\Rightarrow$ $F(x,y,z)=0$ 即为所求

坐标面上的平面曲线绕坐标轴旋转曲面



平面曲线 $\begin{cases} F(x,z)=0 \\ y=0 \end{cases}$ 为母线，以z轴为轴

$$\begin{cases} \text{纬圆} \begin{cases} z=z_1 \\ x^2+y^2+z^2=x_1^2+y_1^2+z_1^2 \end{cases} \\ \text{母线} \begin{cases} F(x_1,z_1)=0 \\ y_1=0 \end{cases} \end{cases}$$

$z_1=z$，$x_1=\pm\sqrt{x^2+y^2}$

$\Rightarrow$ $F(\pm\sqrt{x^2+y^2},z)=0$ 即为所求

由 $\begin{cases} F(x,z)=0 \\ y=0 \end{cases}$

- 绕z轴旋转曲面 $F(\pm\sqrt{x^2+y^2},z)=0$
- 绕x轴旋转曲面 $F(x,\pm\sqrt{y^2+z^2})=0$
- $G(\pm\sqrt{x^2+z^2},z)=0$

2) $\begin{cases} G(y,z)=0 \\ x=0 \end{cases}$ 绕 z —— $G(\pm\sqrt{x^2+y^2}, z)=0$

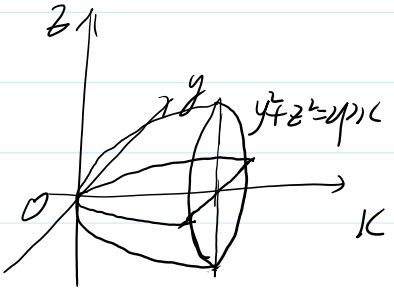
绕 y —— $G(y, \pm\sqrt{x^2+z^2})=0$

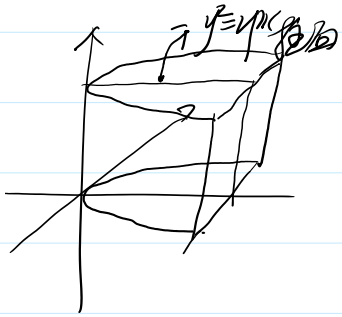
3) $\begin{cases} H(x,y)=0 \\ z=0 \end{cases}$ 绕 x —— $H(x, \pm\sqrt{y^2+z^2})=0$

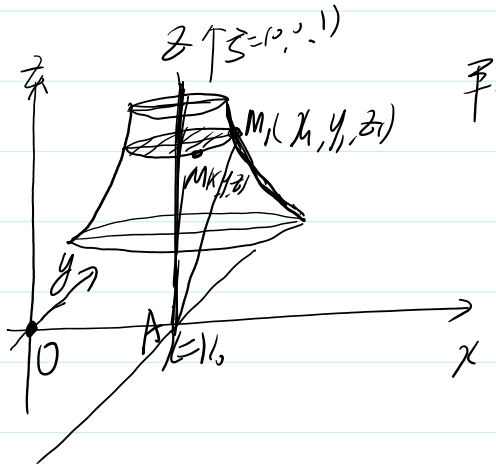
绕 y —— $H(\pm\sqrt{x^2+z^2}, y)=0$

4) 求以曲线 $y^2=2px$ 绕 x 轴旋转曲面，即 $\left(\pm\sqrt{y^2+z^2}\right)^2=2px$

$y^2+z^2=2px$







平面曲线 $\begin{cases} F(x,z)=0 \\ y=0 \end{cases}$ 为母线，以 $\begin{cases} x=x_0 \\ y=0 \end{cases}$ 轴为轴

$\begin{cases} \text{绕} & z=z_1 \\ & (x-x_0)^2+y^2+z^2=(x_1-x_0)^2+y_1^2+z_1^2 \\ \text{即} & \begin{cases} y_1=0 \\ F(x_1,z_1)=0 \end{cases} \end{cases}$ $\begin{cases} z_1=z \\ x_1=x_0\pm\sqrt{(x-x_0)^2+y^2} \end{cases}$

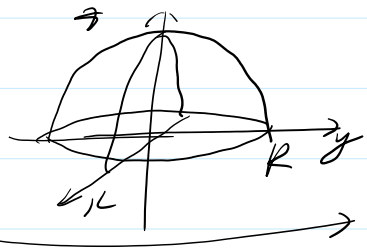
$\Rightarrow F(x_0\pm\sqrt{(x-x_0)^2+y^2}, z)=0$ 即为所求

曲面方程三：$\begin{cases} \text{显式} & z=f(x,y) \\ \text{隐式} & F(x,y,z)=0 \\ \text{参数式} & \begin{cases} x=x(u,v) \\ \cdots \end{cases} \Rightarrow \begin{cases} u=u(x,y) \\ \cdots \end{cases} \end{cases}$

参数式 $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \Rightarrow \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$

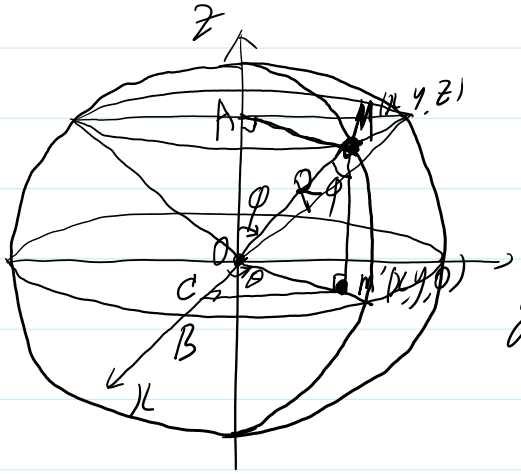
$z = \sqrt{R^2 - x^2 - y^2}$

$\Leftrightarrow x^2+y^2+z^2 = R^2 \quad (z \geq 0)$



$F(x,y,z) = x^2+y^2+z^2-R^2 = 0$

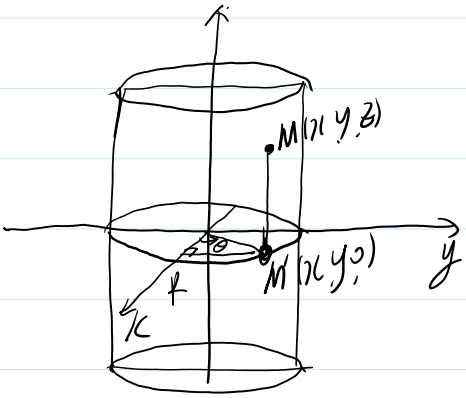
球面参数方程 $x^2+y^2+z^2 = R^2$



$OM' = R\sin\varphi$

$\begin{cases} x = OM'\cos\theta = R\sin\varphi\cos\theta \\ y = OM'\sin\theta = R\sin\varphi\sin\theta \\ z = R\cos\varphi \end{cases} \quad \begin{pmatrix} 0 \leq \varphi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{pmatrix}$

柱面参数方程 $x^2+y^2 = R^2$



$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{cases} \quad \begin{pmatrix} 0 \leq \theta \leq 2\pi \\ z \in \mathbb{R} \end{pmatrix}$

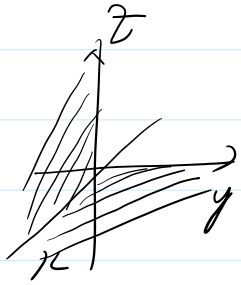
空间曲线及其方程

一般式 $\Gamma: \begin{cases} z = z_1(x,y) & \Sigma_1 \\ z = z_2(x,y) & \Sigma_2 \end{cases}$

隐式 $\Gamma: \begin{cases} F_1(x,y,z) = 0 & \Sigma_1 \\ F_2(x,y,z) = 0 & \Sigma_2 \end{cases}$

$\begin{cases} F_1(x,y,z)=0 \\ F_2(x,y,z)=0 \end{cases}$ 空间曲线

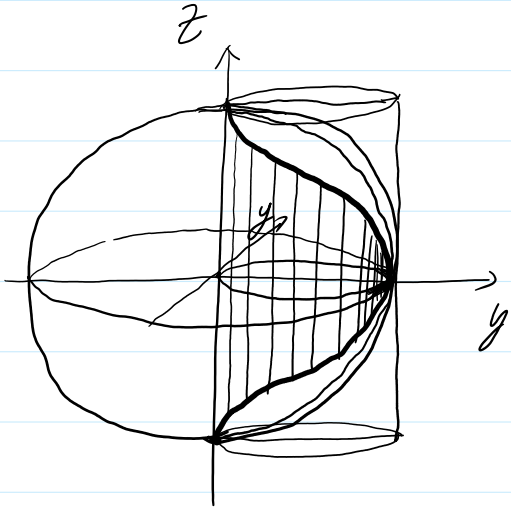
参数式：$\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases}$



例 x 轴 $\begin{cases} z=0 \\ y=0 \end{cases}$

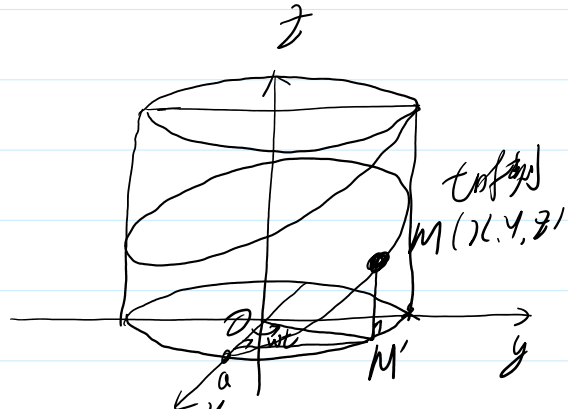
例 $\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p} \Leftrightarrow \begin{cases} \frac{x-x_0}{m}=\frac{z-z_0}{p} \\ \frac{y-y_0}{n}=\frac{z-z_0}{p} \end{cases}$ → $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$

例 Viviani（维维安尼）曲线 $\begin{cases} x^2+y^2+z^2=R^2 \\ x^2+y^2=Rx \end{cases}$ $(x-\frac{R}{2})^2+y^2=(\frac{R}{2})^2$



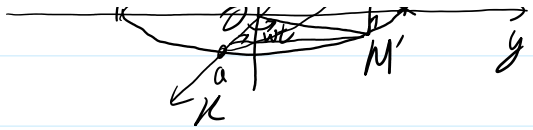
参数式：$\begin{cases} x=\frac{R}{2}+\frac{R}{2}\cos t \\ y=\frac{R}{2}\sin t \\ z=\pm\sqrt{R^2-(\frac{R}{2}+\frac{R}{2}\cos t)^2-(\frac{R}{2}\sin t)^2} \end{cases}$

例 螺旋线 动点 $M(x,y,z)$ 由 $M_0(a,0,0)$ 沿柱面匀速运动 $\begin{cases} \text{角速度 } w \\ \text{沿 } z \text{ 轴正向线速 } v \end{cases}$



$\begin{cases} x=a\cos wt \\ y=a\sin wt \\ z=vt \end{cases}$ 设 $wt=\theta$ $\begin{cases} x=a\cos\theta \\ y=a\sin\theta \\ z=\frac{v}{w}\theta \end{cases}$ $t=\frac{1}{w}\theta$

$\begin{cases} x=a\cos\theta \\ y=a\sin\theta \end{cases}$



$$\text{令} \begin{cases} x = a\cos\theta \\ y = a\sin\theta \\ z = b\theta \end{cases}$$