多元微分19-1

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高阶偏导　$\left(z=x^2y^2 \quad z'_x=2xy^2 \quad z'_y=2yx^2\right) \quad (z'_x)'_y=z''_{xy}=4xy$

$z=f(x,y) \quad f'_x(x,y)=z'_x=\frac{\partial z}{\partial x}=\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x} \qquad f'_y(x,y) \quad z'_y=\frac{\partial z}{\partial y}=\lim\limits_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$

$z'_x < \begin{matrix} z''_{xx} \\ z''_{xy} \end{matrix} \qquad z'_y < \begin{matrix} z''_{yx} \\ z''_{yy} \end{matrix}$

~~二阶混合偏导~~

z''_{xy}

z''_{yx}

$z''_{xx}=(z'_x)'_x=\left\{f'_x(x,y)\right\}'_x=\lim\limits_{\Delta x\to 0}\frac{f'_x(x+\Delta x,y)-f'_x(x,y)}{\Delta x}$

$z''_{xy}=(z'_x)'_y=\left(f'_x(x,y)\right)'_y=\lim\limits_{\Delta y\to 0}\frac{f'_x(x,y+\Delta y)-f'_x(x,y)}{\Delta y}$

$z'_x=\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}z$

$z'_y=\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}z$

$z''_{xx}=(z'_x)'_x=\left(\frac{\partial z}{\partial x}\right)'_x=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}$

$z''_{xy}=\left(\frac{\partial z}{\partial x}\right)'_y=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}$

$z''_{yx}=\left(\frac{\partial z}{\partial y}\right)'_x=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}$

定理：若 z''_{xy} 与 z''_{yx} 在 $P(x,y)$ 处连续，则 $z''_{xy}=z''_{yx}$

例 $f(x,y)=\begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & x^2+y^2\neq 0 \\ 0 & x^2+y^2=0 \end{cases}$

求 $f''_{xy}(0,0)$ 与 $f''_{yx}(0,0)$

$f''_{xy}(0,0)=\lim\limits_{\Delta y\to 0}\frac{f'_x(0,0+\Delta y)-f'_x(0,0)}{\Delta y}$

$f''_{yx}(0,0)=\lim\limits_{\Delta x\to 0}\frac{f'_y(0+\Delta x,0)-f'_y(0,0)}{\Delta x}$

$f'_x(x,y)=\begin{cases} y\frac{x^2-y^2}{x^2+y^2}+xy\frac{2x(x^2+y^2)-2x(x^2-y^2)}{(x^2+y^2)^2} & x^2+y^2\neq 0 \\ \lim\limits_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{0-0}{\Delta x}=0 & x^2+y^2=0 \end{cases}$

$f'_y(x,y)=\begin{cases} x\frac{x^2-y^2}{x^2+y^2}+xy\frac{-2y(x^2+y^2)-2y(x^2-y^2)}{(x^2+y^2)^2} & x^2+y^2\neq 0 \\ \lim\limits_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\frac{0-0}{\Delta y}=0 & x^2+y^2=0 \end{cases}$

$f''_{xy}(0,0)=\lim\limits_{\Delta y\to 0}\frac{f'_x(0,0+\Delta y)-f'_x(0,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\frac{\frac{-\Delta y^3}{\Delta y^2}-0}{\Delta y}=-1$

$f''_{yx}(0,0)=\lim\frac{f'_y(0+\Delta x,0)-f'_y(0,0)}{} \qquad \frac{\frac{\Delta x^3}{\Delta x^2}-0}{\Delta x}=1$

$$f''_{yx}(0,0)=\lim_{\Delta x\to 0}\frac{f'_y(0+\Delta x,0)-f'_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{\Delta x^3}{\Delta x^2}-0}{\Delta x}=1$$

全微分 $z=f(x,y)$

$$\Delta z=A\Delta x+B\Delta y+o(\sqrt{\Delta x^2+\Delta y^2})$$

$$\boxed{\Delta z=A\Delta x+B\Delta y+o(\rho)\quad(\rho=\sqrt{\Delta x^2+\Delta y^2})}\quad \forall\Delta x,\Delta y$$

$\Leftrightarrow z=f(x,y)$ 在 $p(x,y)$ 处可微 ~~且~~ 称

$A\Delta x+B\Delta y$ 为 $z=f(x,y)$ 在 $p(x,y)$ 处全微分

记作 $dz=A\Delta x+B\Delta y$ （$\Delta z\approx dz$ 误差 $o(\rho)$）

一元函数微分 $y=y(x)$

① $\boxed{\Delta y=A\Delta x+o(\Delta x)}\Leftrightarrow y=y(x)$ 在 x 点可微

$\Leftrightarrow \Delta y\approx\boxed{dy=A\Delta x}$

可推出 $A=y'(x)$，$\Delta x=dx$　　$dy=y'(x)dx$

② 式 $\Delta y=A\Delta x+o(|\Delta x|)$

③ $\Delta y=A\Delta x+o(\sqrt{\Delta x^2})$

$o(\Delta x)=o(|\Delta x|)$

1）不妨 $\Delta y=0$　　$\Delta z_x=f(x+\Delta x,y)-f(x,y)=A\Delta x+o(\Delta x)$

$$z'_x(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}=A+\lim_{\Delta x\to 0}\frac{o(\Delta x)}{\Delta x}=A$$

2）令 $\Delta x=0$　　$\Delta z_y=f(x,y+\Delta y)-f(x,y)=B\Delta y+o(\Delta y)$

$$z'_y(x,y)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}=B+\lim_{\Delta y\to 0}\frac{o(\Delta y)}{\Delta y}=B$$

即 $\boxed{dz=z'_x(x,y)dx+z'_y(x,y)dy}$

$y=y(x)$ 连续

① $\lim_{\Delta x\to 0}\Delta y=0$

② $\lim_{x\to x_0}f(x)=f(x_0)$

且有結论 ①可微 $\Rightarrow$ 可偏导且 $A=z'_x$，$B=z'_y$，反之不成立 $\to$ 即 z 可偏导 $\nRightarrow$ 可微

②可微 $\Rightarrow$ 连续 （$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\Delta z=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}(z'_x\Delta x+z'_y\Delta y)=0$）

$\Delta y = A\Delta x + o(\Delta x)$

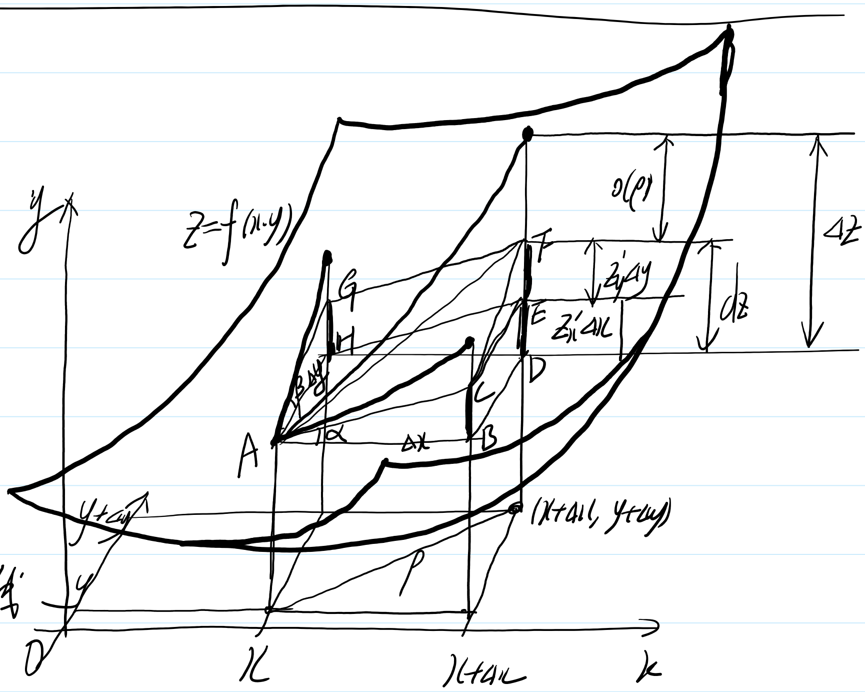
$z=f(x,y)$ 可微定义2 $\Delta z = A\Delta x + B\Delta y + o(\Delta x) + o(\Delta y)$

$\Delta x = dx \quad \Delta y = dy$
x,y自变量

全微分计算 $z = x^2y^2$ 求 dz $\quad dz = z_x' dx + z_y' dy$

$= 2xy^2 dx + 2yx^2 dy$

全微分几何含义



$BC = \tan\alpha\,\Delta x = z_x'\Delta x = DE$

$GH = \tan\beta\,\Delta y = z_y'\Delta y = EF$

AC 与 AG 为过A点的两条切线
x切线 y切线

由 $\Delta x, \Delta y$ 的任意性 可知 平面ACFG为曲面 $z=f(x,y)$ 在A点 (x,y,z) 处的切平面

定理：$z=f(x,y)$ 在 $P(x_0,y_0)$ 处一阶偏导 $f_x'(x,y)$ 与 $f_y'(x,y)$ 连续 则 $z=f(x,y)$ 在 $P(x_0,y_0)$ 处可微

证：$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x,y)$

$\lim f(x) = A \Leftrightarrow f(x) = A+\alpha$, $\lim \alpha = 0$

$= f_x'(\xi, y+\Delta y)(x+\Delta x - x) + f_y'(x,y)(y+\Delta y - y)$

$= f_x'(\xi, y+\Delta y)\Delta x + f_y'(x,y)\Delta y \quad \xi\in(x, x+\Delta x)$, $y\in(y, y+\Delta y)$

$\lim_{\substack{\Delta x\to0\\ \Delta y\to0}} f_x'(\xi, y+\Delta y) = f_x'(x,y)$

$\lim_{\Delta y\to0} f_y'(x,y) = f_y'(x,y)$

$\eta \in (y, y+\Delta y)$ $\left(\lim\limits_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} f'_y(x, \eta) = f'_y(x,y) \right)$

$$= \left[f'_x(x, y+\Delta y) + \alpha \right] \Delta x + \left[f'_y(x,y) + \beta \right] \Delta y$$

$$= \underbrace{f'_x(x, y)}_{A} \Delta x + \underbrace{f'_y(x,y)}_{B} \Delta y + \boxed{\alpha \Delta x + \beta \Delta y}$$

只须证 $\alpha \Delta x + \beta \Delta y = o(\rho)$

$$\boxed{\Delta z = A\Delta x + B\Delta y + o(\rho)}$$

而 $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \dfrac{\alpha \Delta x + \beta \Delta y}{\rho} = \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\overset{\to 0}{\alpha} \overset{有界}{\dfrac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}}} + \overset{\to 0}{\beta} \overset{有界}{\dfrac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}} \right] = 0 \Leftrightarrow \alpha \Delta x + \beta \Delta y = o(\rho)$