多元微分23-1

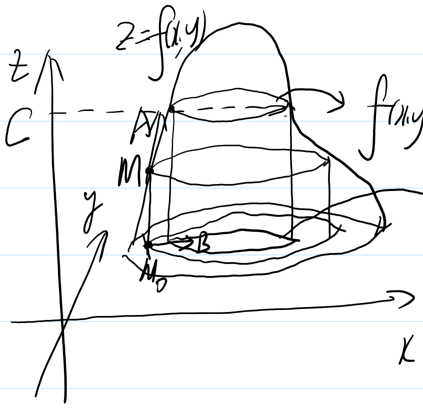
2022年5月17日 8:00

$z=f(x,y)$ 可微，则 $\frac{\partial z}{\partial l}=\frac{\partial z}{\partial x}\cos\alpha+\frac{\partial z}{\partial y}\cos\beta=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)\cdot(\cos\alpha,\cos\beta)=\nabla z\cdot\vec{l}=|\nabla z||\vec{l}|\cos(\nabla z,\vec{l})=\text{Prj}_{\vec{l}}\nabla z$

$\theta=(\nabla z,\vec{l})=0$ 时 即 $\vec{l}$ 取 ∇z 方向

$\cos\theta=1\Rightarrow\frac{\partial z}{\partial l}\Big|_{max}=|\nabla z|$

$\nabla z=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)=(f_x',f_y')$ 与梯度同向



$\frac{\partial z}{\partial l}=|\nabla z|\cos(\nabla z,\vec{l})$

$f(x,y)=C$ ($z=C$上)

$\begin{cases}f(x,y)=0\\z=0\end{cases}$

M_0B 为等高线 $\begin{cases}f(x,y)=0\\z=0\end{cases}$ 在 M_0 点法向

且 M_0B 由值低的等高线指向值比较高等高线

在 xoy 面上 $f(x,y)=0$. $f_x'+f_y'\cdot y_x'=0$

$\Rightarrow y_x'=-\frac{f_x'}{f_y'}$ 即 M_0 点切线斜率

即过 M_0 切向 $(f_y',-f_x')$

梯度 $\nabla f(x,y)=(f_x',f_y')$

内积为零

$u=f(x,y,z)$, $\nabla u=(f_x',f_y',f_z')$ 为等值面上 $M(x,y,z)$ 处对应的法向量

且 ∇u 为值较低的等值面指向值较高等值面

方向导数高阶导. $z=f(x,y)\in C^{(2)}(D)$, 则 $z=f(x,y)$ 在D内存在二阶方向导数

记 $\vec{l}=(a,b)$

$\frac{\partial f}{\partial l}=f_x'a+f_y'b$

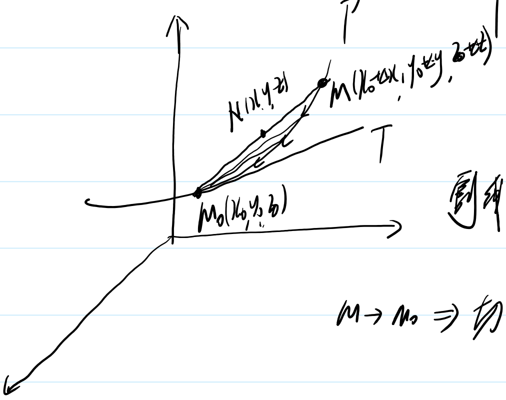
$\frac{\partial}{\partial l}\left(\frac{\partial f}{\partial l}\right)=\frac{\partial^2 f}{\partial l^2}=\frac{\partial}{\partial l}(f_x'a+f_y'b)$

$=\frac{\partial}{\partial l}f_x'a+\frac{\partial}{\partial l}f_y'b$

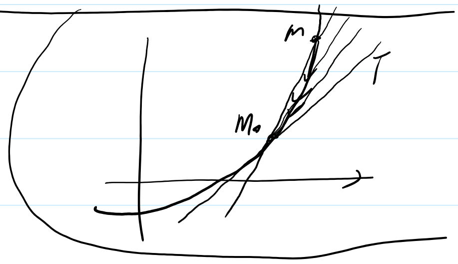
$=(f_{xx}''a+f_{xy}''b)a+(f_{yx}''a+f_{yy}''b)b$

$$= f''_{xx}a^2 + 2f''_{xy}ab + f''_{yy}b^2$$

空间曲线切线



$\Gamma: \begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases}$ $\quad M_0 \begin{cases} x_0=x(t_0) \\ y_0=y(t_0) \\ z_0=z(t_0) \end{cases}$



割线 M_0M 的： $\dfrac{x-x_0}{\frac{\Delta x}{\Delta t}} = \dfrac{y-y_0}{\frac{\Delta y}{\Delta t}} = \dfrac{z-z_0}{\frac{\Delta z}{\Delta t}}$ $\quad \begin{cases} \Delta x = x(t_0+\Delta t) - x(t_0) \\ \Delta y = y(t_0+\Delta t) - y(t_0) \\ \Delta z = z(t_0+\Delta t) - z(t_0) \end{cases}$

$M \to M_0 \Rightarrow$ 切线 M_0T： $\boxed{\dfrac{x-x_0}{x'(t_0)} = \dfrac{y-y_0}{y'(t_0)} = \dfrac{z-z_0}{z'(t_0)}}$

即 Γ 在 $M_0(x_0,y_0,z_0)$ 切方向为 $\vec{s}_{M_0} = (x'(t_0), y'(t_0), z'(t_0))$

或 $\boxed{\Gamma \text{ 在 } M(x,y,z) \text{ 切方向为 } \vec{s} = (x'(t), y'(t), z'(t))}$

过 Γ 上 M_0 之法平面 $x'(t_0)(x-x_0) + y'(t_0)(y-y_0) + z'(t_0)(z-z_0) = 0$

① ※ $\boxed{\text{即 } \Gamma \begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases} \text{ 在 } M(x,y,z) \text{ 处切方向(方向量) } \vec{s} = (x'(t), y'(t), z'(t))}$

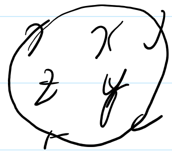
② $\boxed{\Gamma \begin{cases} x=x \\ y=y(x) \\ z=z(x) \end{cases} \quad \vec{s} = (1, y'_x, z'_x)}$

③ $\Gamma \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ $\quad \dfrac{\partial(F,G)}{\partial(y,z)} \neq 0 \Rightarrow$ 隐函数组 $\begin{cases} y=y(x) \\ z=z(x) \end{cases}$ 且有 $y'_x = -\dfrac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}}$ $\quad z'_x = -\dfrac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}}$

取 $\vec{s} = (1, y'_x, z'_x) = \left(1, -\dfrac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}}, -\dfrac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}}\right)$

再取 $\vec{s} = \left(\dfrac{\partial(F,G)}{\partial(y,z)}, -\dfrac{\partial(F,G)}{\partial(x,z)}, -\dfrac{\partial(F,G)}{\partial(y,x)}\right) = \left(\dfrac{\partial(F,G)}{\partial(y,z)}, \dfrac{\partial(F,G)}{\partial(z,x)}, \dfrac{\partial(F,G)}{\partial(x,y)}\right)$

$\boxed{\text{即 } \Gamma \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \text{ 切方向 } \vec{s} = \left(\dfrac{\partial(F,G)}{\partial(y,z)}, \dfrac{\partial(F,G)}{\partial(z,x)}, \dfrac{\partial(F,G)}{\partial(x,y)}\right)}$



$\left(= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \end{vmatrix} = \nabla F \times \nabla G \right.$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F'_x & F'_y & F'_z \\ G'_x & G'_y & G'_z \end{vmatrix} = \nabla F \times \nabla G$$

例 $\Gamma: \begin{cases} x^2+y^2+z^2=6 \\ x+y+z=0 \end{cases}$，求 Γ 上 $M_0(1,-2,1)$ 处切线方程

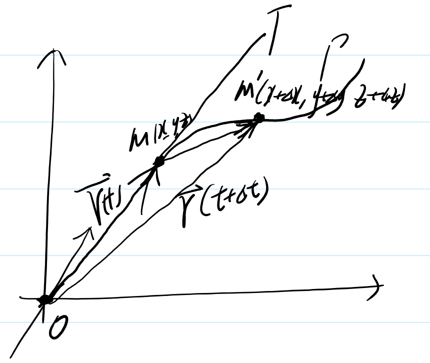
解 $F(x,y,z)=x^2+y^2+z^2-6$

$G(x,y,z)=x+y+z$

$$\vec{S}_{M_0} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F'_x & F'_y & F'_z \\ G'_x & G'_y & G'_z \end{vmatrix}_{M_0} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}_{(1,-2,1)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 2 \\ 1 & 1 & 1 \end{vmatrix} = (-6, 0, 6)$$

$$\frac{x-1}{-6} = \frac{y+2}{0} = \frac{z-1}{6}$$

$\Gamma \begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases}$ $\qquad \vec{r} = \vec{r}(t) = \{x(t), y(t), z(t)\}$

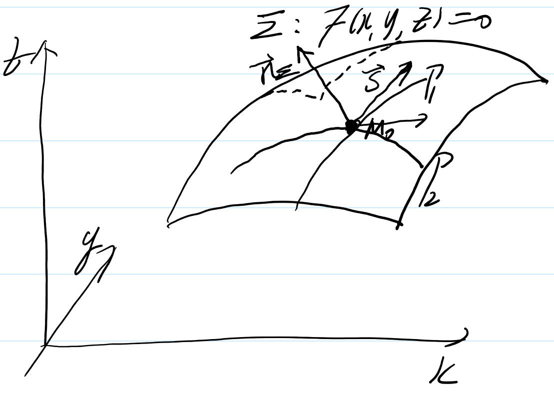


$$k_{\overrightarrow{MT}} = \vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$= \left(\lim_{\Delta t \to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{y(t+\Delta t)-y(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{z(t+\Delta t)-z(t)}{\Delta t} \right)$$

$$= \left(x'(t), y'(t), z'(t) \right)$$

空间曲面的切平面、法线



曲面 Σ $F(x,y,z)=0$ $\quad M_0(x_0,y_0,z_0) \in \Sigma$

任取一条 Σ 上的过 M_0 之曲线 $\Gamma_1 \begin{cases} x=x_1(t) \\ y=y_1(t) \\ z=z_1(t) \end{cases}$

$F[x_1(t), y_1(t), z_1(t)] = 0 \Rightarrow F_x' \cdot x_t' + F_y' y_t' + F_z' z_t' = 0$

$\Leftrightarrow (F_x', F_y', F_z') \cdot (x_t', y_t', z_t') = 0$

$\Leftrightarrow \nabla F \perp \vec{s}_{切}$

$\Gamma_2 \begin{cases} x = x_2(t) \\ y = y_2(t) \\ z = z_2(t) \end{cases}$ $F[x_2(t), y_2(t), z_2(t)] = 0$ $F_x' x_2'(t) + F_y' y_2'(t) + F_z' z_2'(t) = 0$

$\Leftrightarrow \nabla F \cdot (x_2'(t), y_2'(t), z_2'(t)) = 0$

$\Leftrightarrow \nabla F \perp \vec{s}_{切}$

即 $\boxed{\begin{cases} \Sigma \quad F(x, y, z) = 0 \\ \vec{n}_{\Sigma_M} = \nabla F = (F_x', F_y', F_z') \end{cases}}$

$\boxed{\begin{array}{l} \Sigma \quad z = f(x, y) \\ \vec{n}_{\Sigma_M} = (-f_x', -f_y', 1) \end{array}}$

$F(x, y, z) = z - f(x, y) = 0$

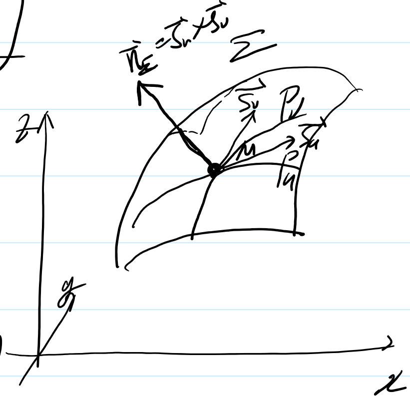
$\vec{n} = (F_x', F_y', F_z') = (-f_x', -f_y', 1)$

$\boxed{\begin{array}{l} \Sigma \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \\ \vec{n}_{\Sigma_M} = \left(\dfrac{\partial(y, z)}{\partial(u, v)}, \dfrac{\partial(z, x)}{\partial(u, v)}, \dfrac{\partial(x, y)}{\partial(u, v)} \right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u' & y_u' & z_u' \\ x_v' & y_v' & z_v' \end{vmatrix} = \vec{s}_u \times \vec{s}_v \end{array}}$

图解:

$\Sigma \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$ $\Gamma_u: \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$ $\Gamma_v = \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$

$\vec{s}_u = (x_u', y_u', z_u')$ $\vec{s}_v = (x_v', y_v', z_v')$



例 $\Sigma \quad x^2 + 2y^2 + 3z^2 = 6$ 在 $P(1,1,1)$ 处切平面、法线

$\vec{n} = (F_x', F_y', F_z')_P = (2x, 4y, 6z)_P = (2, 4, 6)$

Σ 上过 P 切平面 $2(x-1) + 4(y-1) + 6(z-1) = 0$

Σ 上过 P 法线 $\dfrac{x-1}{2} = \dfrac{y-1}{4} = \dfrac{z-1}{6}$

$\begin{cases} 1 = u\cos v \\ 0 = u\sin v \\ 0 = v \end{cases}$

$$\frac{\ }{2}-\frac{\ }{4}-\frac{\ }{6}$$

$$\begin{cases} 0=u\sin v \\ 0=v \end{cases}$$

例 螺旋面 $\begin{cases} x=u\cos v \\ y=u\sin v \\ z=v \end{cases}$ $(u\geq 0,\ v\in\mathbb{R})$ 在 $P(1,0,0)$ 处法向 $P(1,0,0)\to\begin{cases} u=1 \\ v=0 \end{cases}$

$$\vec{n}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u' & y_u' & z_u' \\ x_v' & y_v' & z_v' \end{vmatrix}_P=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}_{\substack{u=1\\v=0}}=\text{——}$$