定识分的应用4-1

2022年3月11日 7:52

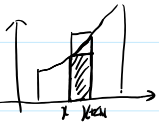
微元法： 求 $[a,b]$ 上面量 A

1) 微分 $\forall [x, x+\Delta x] \subset [a,b]$，计算其上 ΔA 的近似 dA

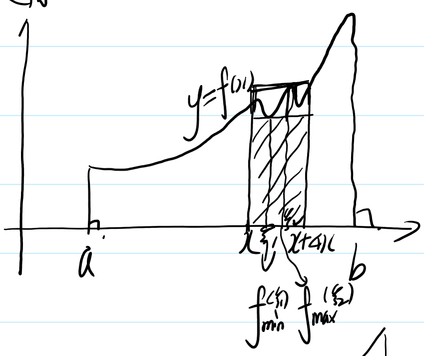
$$\Delta A \approx dA = A'(x)\Delta x = f(x)dx \quad (\text{误差}\ o(\Delta x))$$

2) 积分 $A = \int_a^b dA = \int_a^b f(x)dx$

特殊情形



一般情形

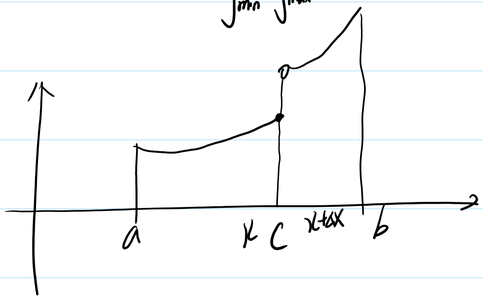


$$f_{min}dx \le \Delta S \approx dS = f(x)\cdot\Delta x \le f_{max}dx$$

$$|\Delta S - f(x)dx| \le (f_{max} - f_{min})dx$$

$$0 \le \left|\frac{\Delta S - f(x)dx}{dx}\right| \le f_{max}(\xi_1) - f_{min}(\xi_2) \xrightarrow[\xi_1\to\xi_2\to x]{\Delta x\to 0} 0$$

需 $f(x)$ 连续

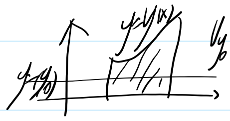


$f(x)$ 有有限多个间断点

旋转体体积

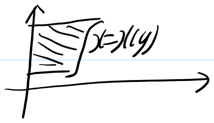
1) 切片法

$$V_x = \pi\int_a^b y^2 dx \quad (y = y(x))$$



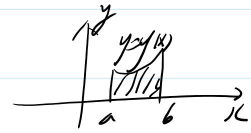
$$V_{y_0} = \pi\int_a^b (y-y_0)^2 dx$$

$$V_y = \pi\int_c^d x^2 dy \quad (x = x(y))$$

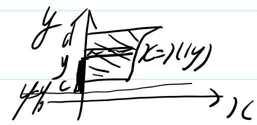


2) 剥皮法（柱壳法）

$$V_y = 2\pi\int_a^b x\, y\, dx \quad (y = y(x))$$

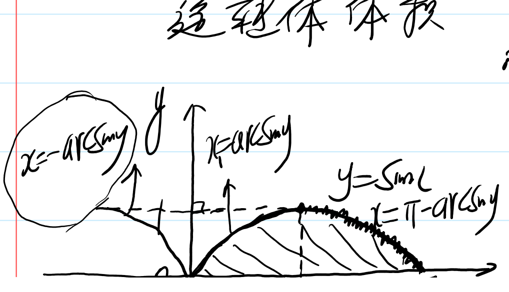


$$V_x = 2\pi\int_c^d y\, x\, dy \quad (x = x(y))$$



$$V_{y=y_0} = 2\pi\int_c^d (y-y_0)x\,dy$$

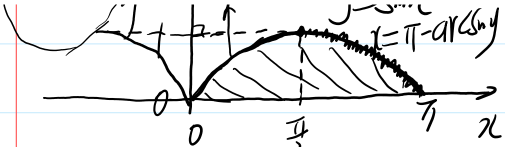
例 求 $y=\sin x$ ($x\in[0,\pi]$) 与 x 轴所围区域分别绕 x 轴及 y 轴旋转一周所得旋转体体积



法一

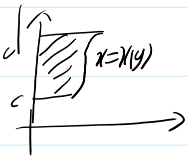
切片 1) $V_x = \pi\int_0^\pi y^2 dx = \pi\int_0^\pi \sin^2 x\, dx = \pi\int_0^\pi \frac{1-\cos 2x}{2}dx = \frac{\pi^2}{2}$

剥皮 2) $V_y = 2\pi\int_0^\pi x\, y\, dx = 2\pi\int_0^\pi x\sin x\, dx = 2\pi\int_0^\pi -x\, d\cos x$

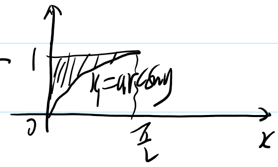


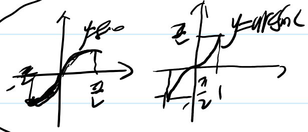
剥皮 (2) $V_y = 2\pi\int_0^{\pi} xy\,dx = 2\pi\int_0^{\pi} x\sin x\,dx = 2\pi\int_0^{\pi} -x\,d\cos x$

$= 2\pi\left[-x\cos x\Big|_0^{\pi} + \int_0^{\pi}\cos x\,dx\right] = \cdots$



$V_x = 2\pi\int_c^d y\,x\,dy$





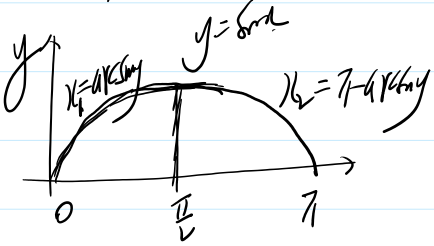
法二：剥皮 (1) $V_x = V_2 - V_1 = 2\pi\int_0^1 y\,x_2\,dy - 2\pi\int_0^1 y\,x_1\,dy$

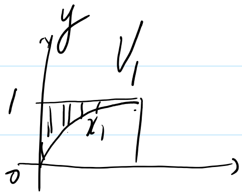
$= 2\pi\int_0^1 y[\pi-\arcsin y - \arcsin y]\,dy = 2\pi\int_0^1 y(\pi-2\arcsin y)\,dy$

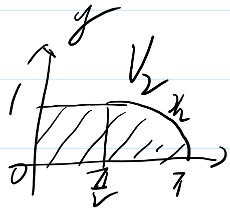
$= 2\pi\int_0^1 y\cdot\pi\,dy - 4\pi\int_0^1 y\arcsin y\,dy$

$= 2\pi^2\int_0^1 y\,dy - 2\pi\int_0^1 \arcsin y\,dy^2 = \cdots$

法二：切片 (2)





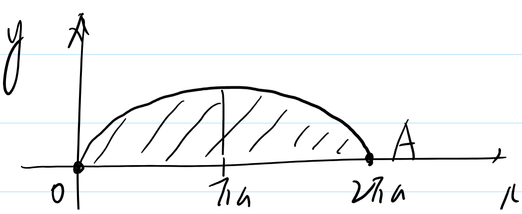


$V_y = V_2 - V_1 = \pi\int_0^1 x_2^2\,dy - \pi\int_0^1 x_1^2\,dy$

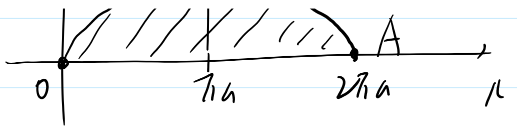
$= \pi\int_0^1 (x_2^2 - x_1^2)\,dy = \pi\int_0^1 \left[(\pi-\arcsin y)^2 - \arcsin^2 y\right]dy$

$= \cdots$

例2 求摆线 $\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}$ $(0\le t\le 2\pi)$ 与x轴围区域 绕x轴旋转一周所成旋转体体积



$V_x = \pi\int_0^{2\pi a} y^2\,dx$



$$= \pi \int_0^{2\pi} [a(1-\cos t)]^2 \, d\, a(t-\sin t)$$

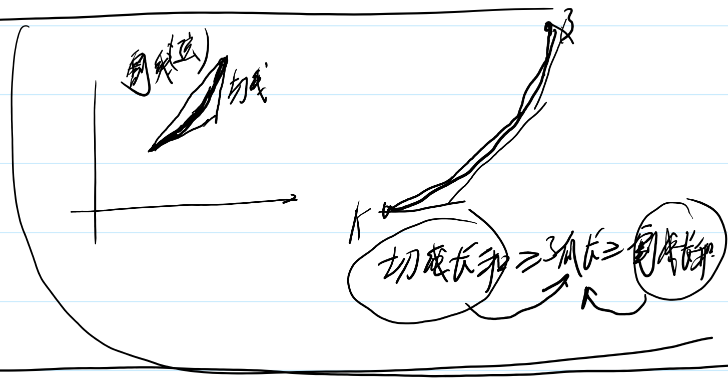
$$= \pi \int_0^{2\pi} a^3 (1-\cos t)^3 \, dt$$

$$= \pi a^3 \int_0^{2\pi} [1 - \cos t - 3\cos t + 3\cos^2 t] \, dt$$

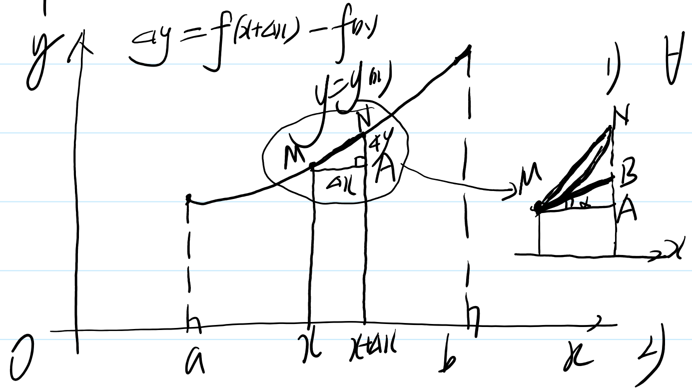
$\cos t\,dt = d\sin t$　　$\frac{1+\cos 2t}{2}$

$= \ldots$

3) $r = r(\theta) \xleftrightarrow{\text{转换为参数式}} \begin{cases} x = r\cos\theta = r(\theta)\cos\theta \\ y = r\sin\theta = r(\theta)\sin\theta \end{cases}$ 其体积公式参照 1) 2) 参数式即可



平面曲线的弧长　　求 [a, b] 上弧长



1) $\forall [x, x+\Delta x] \subset [a, b]$，计算其上 ΔS 的近似值 ds

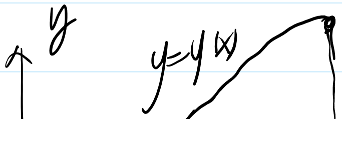
$$\Delta S \approx ds = \sqrt{\Delta x^2 + \Delta y^2} \approx \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{dx^2 + y'^2 dx^2} = \sqrt{1 + y'^2(x)}\, dx$$

2) $S = \int_a^b ds = \int_a^b \sqrt{1+y'^2(x)}\, dx$

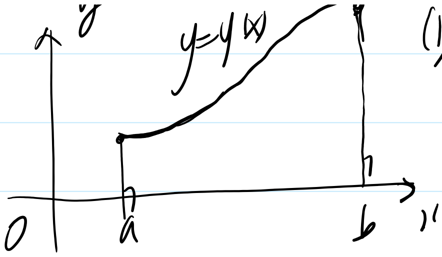
$\begin{cases} MA = \Delta x = dx \\ AB = AM \cdot \tan\alpha = \Delta x \cdot y' = y' dx \end{cases}$　　$MB^2 = MA^2 + AB^2 = dx^2 + (y'dx)^2$

$MB = \sqrt{1 + y'^2(x)}\, dx$



(1) $y = y(x)$

$S = \int^b ds = \int^b \sqrt{1+y'^2(x)}\, dx$　弧长



(1)
$$\boxed{\begin{aligned} &S=\int_a^b ds=\int_a^b\sqrt{1+y'^2(x)}\,dx \quad \text{弧长}\\ &\text{即 } ds=\sqrt{1+y'^2(x)}\,dx \quad \text{弧微分} \quad \Delta s\approx ds\end{aligned}}$$

(2) $\begin{cases} x=x(t)\\ y=y(t)\end{cases}$ $(\alpha\le t\le\beta)$ $\begin{cases} x(\alpha)=a\\ x(\beta)=b\end{cases}$ $ds=\sqrt{dx^2+dy^2}=\sqrt{[x'(t)dt]^2+[y'(t)dt]^2}=\sqrt{x'^2(t)+y'^2(t)}\,dt$

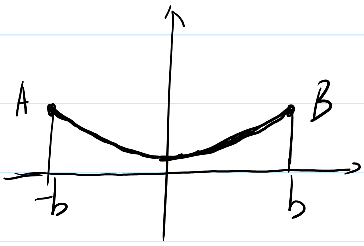
$$\boxed{\begin{aligned} &S=\int_a^b ds=\int_\alpha^\beta\sqrt{x'^2(t)+y'^2(t)}\,dt\\ &\text{即 } ds=\sqrt{x'^2(t)+y'^2(t)}\,dt\end{aligned}}$$

(3) $r=r(\theta)$ $(\alpha\le\theta\le\beta)$ $\begin{cases} x=r(\theta)\cdot\cos\theta\\ y=r(\theta)\cdot\sin\theta\end{cases}$

$$\begin{aligned} ds&=\sqrt{x'^2(\theta)+y'^2(\theta)}\,d\theta\\ &=\sqrt{[r'\cos\theta+r(\theta)(-\sin\theta)]^2+[r'\sin\theta+r(\theta)\cos\theta]^2}\,d\theta\\ &=\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta\end{aligned}$$

$$\boxed{\begin{aligned} &S=\int_a^b ds=\int_\alpha^\beta ds=\int_\alpha^\beta\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta\\ &ds=\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta\end{aligned}}$$

例 求悬链线 $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ 在 $[-b,b]$ 上弧长



$$\begin{aligned} s&=\int_{-b}^b ds=\int_{-b}^b\sqrt{1+y'^2(x)}\,dx\\ &=\int_{-b}^b\sqrt{1+\left[\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)\right]^2}\,dx\\ &=\int_{-b}^b\sqrt{\frac{1}{4}e^{\frac{2x}{a}}+\frac{1}{4}e^{-\frac{2x}{a}}+\frac{1}{2}}\,dx\\ &=\int_{-b}^b\frac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)dx=\cdots=a\left(e^{\frac{b}{a}}-e^{-\frac{b}{a}}\right)\end{aligned}$$

例 $\begin{cases} x=a(t-\sin t)\\ y=a(1-\cos t)\end{cases}$ $(0\le t\le 2\pi)$ 求弧长

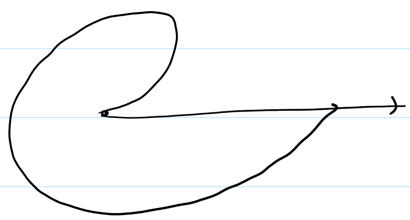
例 $\begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases}$ $(0 \le t \le 2\pi)$ 求弧长

$$s = \int_0^{2\pi a} ds = \int_0^{2\pi} \sqrt{x'^2(t) + y'^2(t)}\, dt = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\, dt$$

$$= a\int_0^{2\pi} \sqrt{2-2\cos t}\, dt$$

$$= a\int_0^{2\pi} \sqrt{2 \cdot 2\sin^2\frac{t}{2}}\, dt = 8a$$

例 求阿基米德螺线 $r = a\theta$ $\theta \in [0, 2\pi]$ 的弧长



$$s = \int_0^{2\pi} ds = \int_0^{2\pi} \sqrt{r^2(\theta) + r'^2(\theta)}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2\theta^2 + a^2}\, d\theta$$

$$= a\int_0^{2\pi} \sqrt{1+\theta^2}\, d\theta$$

$$= \frac{a}{2}\left[2\pi\sqrt{1+4\pi^2} + \ln\left(2\pi + \sqrt{1+4\pi^2}\right)\right]$$

$\int \sqrt{a^2+x^2}\, dx$ $\begin{cases} x = a\tan t & 换元 \\ x = & 分部 \end{cases}$

$$\int \sqrt{a^2+x^2}\, dx = x\sqrt{a^2+x^2} - \int x \frac{x^2+a^2-a^2}{\sqrt{a^2+x^2}}\, dx$$

$$= x\sqrt{a^2+x^2} - \int \sqrt{x^2+a^2}\, dx + a^2 \int \frac{1}{\sqrt{a^2+x^2}}\, dx$$

$$\int \frac{1}{\sqrt{a^2+x^2}}\, dx = \ln\left|x + \sqrt{a^2+x^2}\right| + C$$

$$\Rightarrow \int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + C$$