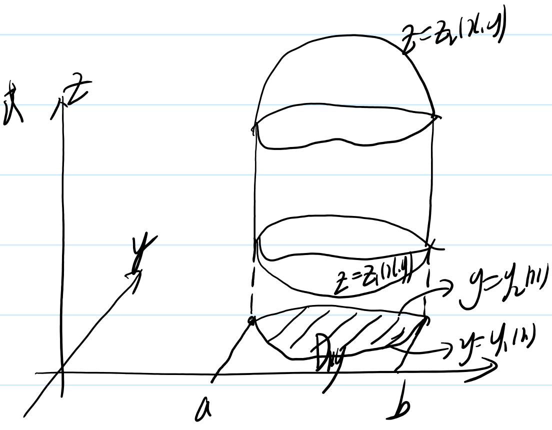
三重积分及习题

2022年6月2日 18:33

公式



$$\Omega_{xy}\begin{cases} a\le x\le b & \text{表示}\Omega\text{在}x=a\text{与}x=b\text{两平面之间} \\ y_1(x)\le y\le y_2(x) & \text{表示}\Omega\text{在}y=y_1(x)\text{与}y=y_2(x)\text{两柱面之间} \\ z_1(x,y)\le z\le z_2(x,y) & \text{表示}\Omega\text{在两曲面}z=z_1(x,y)\text{与}z=z_2(x,y)\text{之间} \end{cases}$$

$\Omega \in \mathbb{R}^4$ $\quad \iiiint_{\Omega} f(x,y,z,t)\,dv$

$$\begin{cases} a\le x\le b \\ y_1(x)\le y\le y_2(x) \\ z_1(x,y)\le z\le z_2(x,y) \\ t_1(x,y,z)\le t\le t_2(x,y,z) \end{cases}$$

例 计算 $\iiint_{\Omega}\frac{xy}{\sqrt{z}}\,dV$，$\Omega$ 由 $z=2\sqrt{x^2+y^2}$ 与 $z=2$ 在一卦限所围成

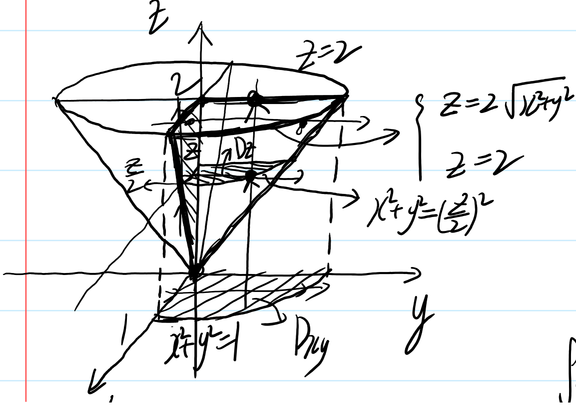
法一：

$$\begin{cases} 0\le x\le 1 \\ 0\le y\le\sqrt{1-x^2} \\ 2\sqrt{x^2+y^2}\le z\le 2 \end{cases}$$

$$\iiint_{\Omega}\frac{xy}{\sqrt{z}}\,dV=\int_0^1 dx\int_0^{\sqrt{1-x^2}}dy\int_{2\sqrt{x^2+y^2}}^{2}\frac{xy}{\sqrt{z}}\,dz$$

$$=\int_0^1 dx\int_0^{\sqrt{1-x^2}}2xy\cdot\sqrt{z}\Big|_{2\sqrt{x^2+y^2}}^{2}$$

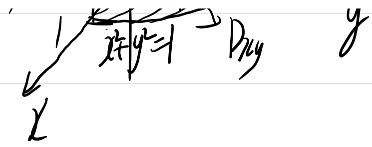
$$=\int_0^1 dx\int_0^{\sqrt{1-x^2}}2xy\left(\sqrt{2}-\sqrt{2\sqrt{x^2+y^2}}\right)dy=\cdots\cdots$$



法二：

$$\Omega_z\begin{cases} 0\le z\le 2 \\ D_z\begin{cases} 0\le x\le\frac{z}{2} \\ 0\le y\le\sqrt{\frac{z^2}{4}-x^2} \end{cases} \end{cases}$$

$$\iiint\frac{xy}{\sqrt{z}}\,dV=\int_0^2 dz\int_0^{\frac{z}{2}}dx\int_0^{\sqrt{\frac{z^2}{4}-x^2}}\frac{xy}{\sqrt{z}}\,dy=\cdots\cdots$$



$$\iiint_\Omega \frac{xy}{\sqrt z}\,dV=\int_0^2 dz\int_0^{\frac z2}dx\int_0^{\sqrt{\frac{z^2}{4}-x^2}}\frac{xy}{\sqrt z}\,dy=\cdots$$

$\frac{z^2}{4}=x^2+y^2$

记 Ω_{zx}

$$\begin{cases}0\le x\le 1\\ 2x\le z\le 2\\ 0\le y\le\sqrt{\frac{z^2}{4}-x^2}\end{cases}$$

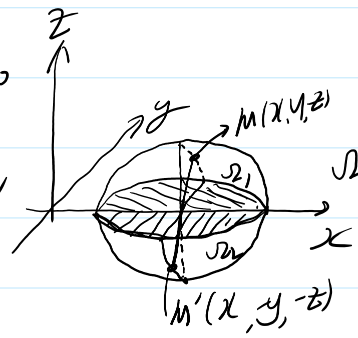
$$\iiint_\Omega \frac{xy}{\sqrt z}\,dV=\int_0^1 dx\int_{2x}^{2}dz\int_0^{\sqrt{\frac{z^2}{4}-x^2}}\frac{xy}{\sqrt z}\,dy$$

三重积分对称性 $\iiint_\Omega f(x,y,z)\,dV$

1) Ω 关于 x 轴对称

$\Omega(x,-y,-z)=\Omega(x,y,z)$

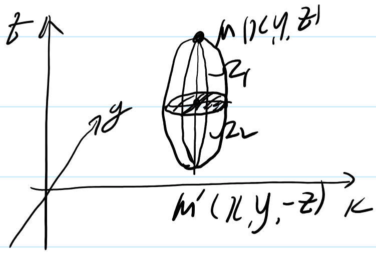
$$\begin{cases}f(x,-y,-z)=-f(x,y,z)\Rightarrow\iiint_\Omega f(x,y,z)\,dV=0\\ f(x,-y,-z)=f(x,y,z)\Rightarrow\iiint_\Omega f\,dV=2\iiint_{\Omega_1}f\,dV\end{cases}$$



$\Omega=\Omega_1\cup\Omega_2$

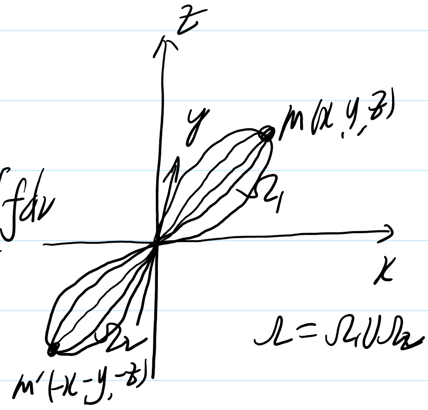
2) Ω 关于 xoy 面对称

$$\begin{cases}f(x,y,-z)=-f(x,y,z)\Rightarrow\iiint_\Omega f\,dV=0\\ f(x,y,-z)=f(x,y,z)\Rightarrow\iiint_\Omega f\,dV=2\iiint_{\Omega_1}f\,dV\end{cases}$$



3) Ω 关于原点 O 对称

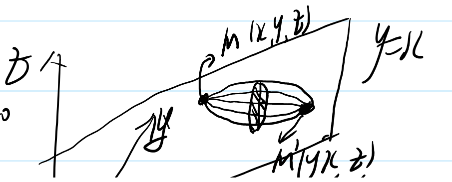
$$\begin{cases}f(-x,-y,-z)=-f(x,y,z)\Rightarrow\iiint_\Omega f\,dV=0\\ f(-x,-y,-z)=f(x,y,z)\Rightarrow\iiint_\Omega f\,dV=2\iiint_{\Omega_1}f\,dV\end{cases}$$



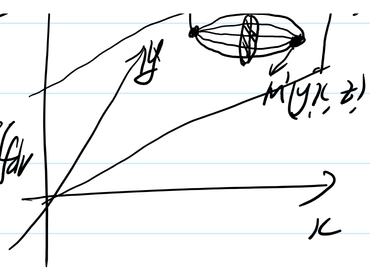
$\Omega=\Omega_1\cup\Omega_2$

4) Ω 关于 $y=x$ 平面对称

$$\begin{cases}f(y,x,z)=-f(x,y,z)\Rightarrow\iiint_\Omega f\,dV=0\end{cases}$$



4) Ω 关于 $y=x$ 平面对称 $\left| \begin{array}{l} f(y,x,z)=-f(x,y,z) \Rightarrow \iiint_{\Omega} f\,dv=0 \\ f(y,x,z)=f(x,y,z) \Rightarrow \iiint_{\Omega} f\,dv=2\iiint_{\Omega_1} f\,dv \end{array} \right.$



习题 B组 1. $f(x,y)$ 在 D 上连续, $f(x,y)\geq 0$, $f(x,y)\not\equiv 0$, 证明 $\iint_D f(x,y)d\sigma>0$

证明 由 $f(x,y)\geq 0$ 且 $f(x,y)\not\equiv 0$, 则至少存在 $M_0(x_0,y_0)\in D$, 使得 $f(x_0,y_0)>0$

$f(x,y)$ 在 M_0 连续 $\Leftrightarrow \lim\limits_{M\to M_0} f(M)=f(M_0) \Leftrightarrow$

$\forall \varepsilon>0$, $\exists \delta>0$ 当 $|MM_0|<\delta \Rightarrow |f(M)-f(M_0)|<\varepsilon$

即 $f(M_0)-\varepsilon<f(M)<f(M_0)+\varepsilon$

不妨取 $\varepsilon=\frac{1}{2}f(M_0) \Rightarrow f(M)>f(M_0)-\frac{1}{2}f(M_0)=\frac{1}{2}f(M_0)>0$

2 (二重积分的第一积分中值定理)

$f(x,y)$ 与 $g(x,y)$ 在有界闭域 D 上连续, 且 $g(x,y)\geq 0$, 证明 $\exists(\xi,\eta)\in D$ 使得

$$\iint_D f(x,y)g(x,y)d\sigma=f(\xi,\eta)\iint_D g(x,y)d\sigma$$

证明 只需证 $\mu=\dfrac{\iint_D f(x,y)g(x,y)d\sigma}{\iint_D g(x,y)d\sigma}=f(\xi,\eta)$

$f(x,y)$ 在 D 上连续, 则 $\exists\, m, M$, 使得 $m\leq f(x,y)\leq M$, 则有

$$m\iint_D g(x,y)d\sigma=\iint_D m\,g(x,y)d\sigma\leq\iint_D f(x,y)g(x,y)d\sigma\leq\iint_D M g(x,y)d\sigma=M\iint_D g(x,y)d\sigma$$

即 $m\leq\mu=\dfrac{\iint_D f(x,y)g(x,y)d\sigma}{\iint_D g(x,y)d\sigma}\leq M$ 由 $f(x,y)$ 在 D 上介值定理

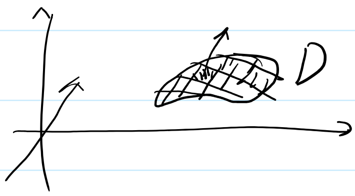
y $\quad m = \mu = \frac{\quad}{\iint_D g(x,y)d\sigma}$ …

$\exists(\xi,\eta)\in D$，使得 $f(\xi,\eta) = \mu = \dfrac{\iint_D f(x,y)g(x,y)d\sigma}{\iint_D g(x,y)d\sigma}$

3. 证明 Dirichlet (狄利克雷) 函数 $D(x,y)=\begin{cases}1, & x,y \text{ 均为有理数} \\ 0, & x,y \text{ 至少有一个为无理数}\end{cases}$ 在区域D上不可积

① 分割 $\forall$ 将D分为n份 ——

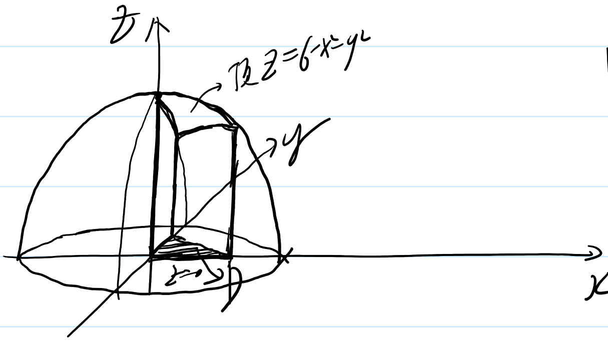
② 近似 $\forall(\xi_i,\eta_i)\in\Delta D_i$ $\quad \Delta m_i \approx D(\xi_i,\eta_i)\Delta\sigma_i$



③ 求和 $m = \sum_{i=1}^{n}\Delta m_i = \sum_{i=1}^{n} D(\xi_i,\eta_i)\Delta\sigma_i$

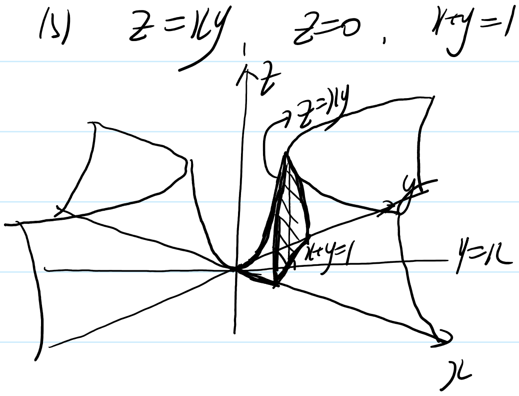
④ 取极限 $\lim_{\lambda\to 0}\sum D(\xi_i,\eta_i)\Delta\sigma$ $\begin{cases}① \ \xi_i,\eta_i \text{ 均为有理数 } D(\xi_i,\eta_i)=1 \Rightarrow \lim_{\lambda\to 0}\sum_{i=1}^{n} 1\cdot\Delta\sigma = S_D>0 \\ ② \ \xi_i,\eta_i \text{ 至少有一个无理数 } D(\xi_i,\eta_i)=0 \Rightarrow \lim\sum_{i=1}^{n} 0\cdot\Delta\sigma = 0\end{cases}$

习题 9.2 A.7 (2) $x=0$ $y=0$ $x+y=1$ $z=0$ $z=6-x^2-y^2$ 围 Ω 求 V_Ω

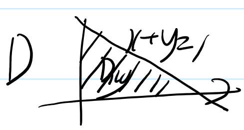


$$V = \iint_D (6-x^2-y^2)dxdy$$

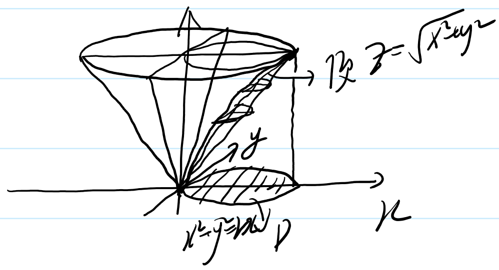
(3) $z=xy$，$z=0$，$x+y=1$ 围 Ω 求 V_2



$$V_2 = \iint_D xy\,dxdy$$



(4) $z=\sqrt{x^2+y^2}$，$x^2+y^2=2x$，$z=0$



$$V_\Omega = \iint_{D:\,x^2+y^2\le 2x} \sqrt{x^2+y^2}\, dV$$

B组 3　$f(x)$在$[a,b]$连续，证明 $\left(\int_a^b f(x)dx\right)^2 \le (b-a)\int_a^b f^2(x)dx$ ← $g(x)=1$ 即可

Cauchy不等式　$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b f^2(x)dx \int_a^b g^2(x)dx$

$f=f(x)$　$g=g(x)$

$(f+\lambda g)^2 \ge 0$ 对$\forall\lambda$成立

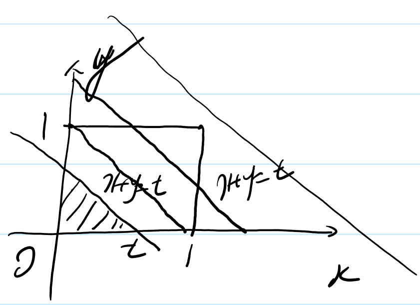
$\lambda^2 g^2 + 2fg\lambda + f^2 \ge 0$

$\lambda^2\int_a^b g^2(x)dx + 2\int_a^b f(x)g(x)dx\cdot\lambda + \int_a^b f^2(x)dx \ge 0$ 对$\lambda\in\mathbb{R}$成立

$\Delta = \left(2\int_a^b f(x)g(x)dx\right)^2 - 4\int_a^b g^2(x)dx\cdot\int_a^b f^2(x)dx \le 0$

证毕

6) $f(x,y)=\begin{cases}2x & 0\le x\le 1,\ 0\le y\le 1\\ 0 & 其它\end{cases}$　求 $F(t)=\iint_{x+y\le t} f(x,y)\,dxdy$



$F(t)=\begin{cases} \text{——} & t\le 0\\ \text{——} & 0\le t\le 1\\ \text{——} & 1\le t\le 2\\ \text{——} & t\ge 2\end{cases}$