定积分的应用7-1

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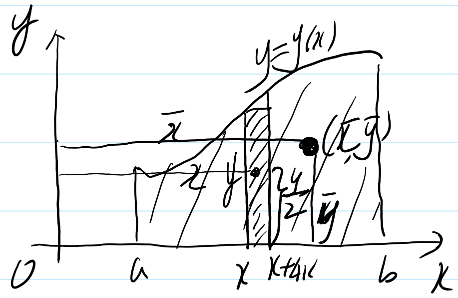
例 平面薄板的重心 质心 形心 (其密度为 $\rho=\rho(x)$, $g=g(x)$)



$$M_y=\int_a^b \rho\cdot y\,dx\cdot g\cdot x=\int_a^b \rho\cdot y\,dx\cdot g\cdot \bar{x}$$

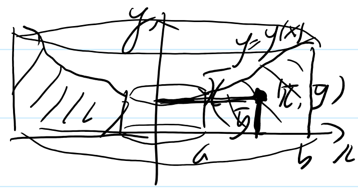
$$M_x=\int_a^b \rho\cdot y\cdot dx\cdot g\cdot \frac{y}{2}=\int_a^b \rho y\,dx\,g\cdot \bar{y}$$

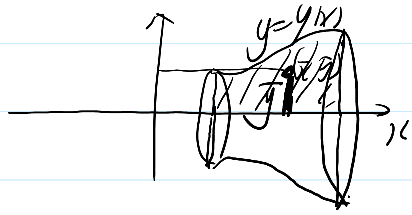
$\Rightarrow$ 重心
$$\begin{cases}\bar{x}=\dfrac{\int_a^b \rho y\,dx\,g\,x}{\int_a^b \rho y\,dx\,g}=\dfrac{\int_a^b \rho g y x\,dx}{\int_a^b \rho g y\,dx}\\[2ex] \bar{y}=\dfrac{\int_a^b \rho y\,dx\,g\cdot\frac{y}{2}}{\int_a^b \rho y\,dx\,g}=\dfrac{\int_a^b \rho g y\frac{y}{2}\,dx}{\int_a^b \rho g y\,dx}\end{cases}$$

质心 $g=g_0$
$$\begin{cases}\bar{x}=\dfrac{\int_a^b \rho y x\,dx}{\int_a^b \rho y\,dx}\\[2ex] \bar{y}=\dfrac{\int_a^b \rho\frac{y^2}{2}\,dx}{\int_a^b \rho y\,dx}\end{cases}$$

形心 $g=g_0$ $\rho=\rho_0$
$$\begin{cases}\bar{x}=\dfrac{\int_a^b yx\,dx}{\int_a^b y\,dx}\\[2ex] \bar{y}=\dfrac{\int_a^b \frac{y^2}{2}\,dx}{\int_a^b y\,dx}\end{cases}$$

Guldin第二定理
$$\begin{cases}2\pi\bar{x}\cdot\int_a^b y\,dx=2\pi\int_a^b yx\,dx\\[2ex] 2\pi\bar{y}\cdot\int_a^b y\,dx=\cancel{2}\pi\int_a^b \frac{y^2}{\cancel{2}}\,dx\end{cases}$$

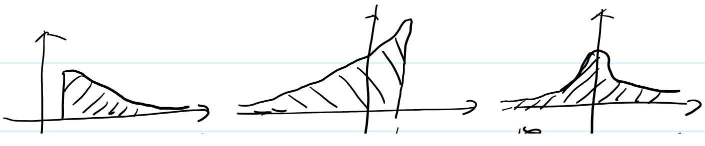




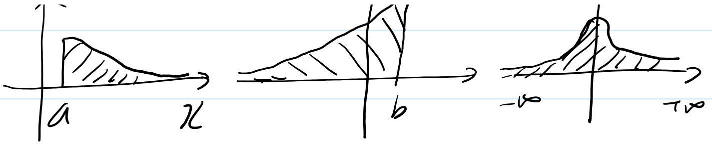
广义积分(反常积分)
$$\begin{cases}\text{无穷积分} & \int_a^{+\infty} f(x)\,dx \quad \int_{-\infty}^{b} f(x)\,dx \quad \int_{-\infty}^{+\infty} f(x)\,dx\\ \text{瑕积分} & \int_{a\,(\text{瑕点})}^{b} f(x)\,dx \quad \int_a^{b\,(\text{瑕点})} f(x)\,dx \quad \int_a^{b} f(x)\,dx\ (c\ \text{瑕点})\end{cases}$$

无穷积分 (本质是极限)

$$\left.\begin{aligned}\int_a^{+\infty} f(x)\,dx&=\lim_{b\to+\infty}\int_a^b f(x)\,dx\\ \int_{-\infty}^{b} f(x)\,dx&=\lim_{a\to-\infty}\int_a^b f(x)\,dx\end{aligned}\right\}$$ 存在 则称积分收敛，否则称其发散



$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$



$\int_{-\infty}^{+\infty} f(x)dx \xlongequal{\forall c\in\mathbb{R}} \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx$，当且仅当右侧二个积分皆收敛时，称 $\int_{-\infty}^{+\infty} f(x)dx$ 收敛，否则称 $\int_{-\infty}^{+\infty} f(x)dx$ 发散

例 $\int_1^{+\infty}\frac{1}{x^\mu}dx \begin{cases} \xlongequal{\mu\neq1} \lim_{b\to+\infty}\frac{1}{1-\mu}x^{1-\mu}\Big|_1^b = \lim_{b\to+\infty}\frac{1}{1-\mu}(b^{1-\mu}-1) = \begin{cases}\frac{1}{\mu-1} & 1-\mu<0 \text{ 即 } \mu>1 \text{ 收} \\ \text{不存在} & 1-\mu>0 \text{ 即 } \mu<1 \text{ 散}\end{cases} \\ \underline{\mu=1} \text{ 散}: \int_1^{+\infty}\frac{1}{x}dx = \lim_{b\to+\infty}\ln|x|\Big|_1^b = \infty \text{ 不存在} \end{cases}$

即 $\boxed{\int_1^{+\infty}\frac{1}{x^\mu}dx \begin{cases}\text{收} & \mu>1 \\ \text{散} & \mu\leq1\end{cases}}$ 背下来

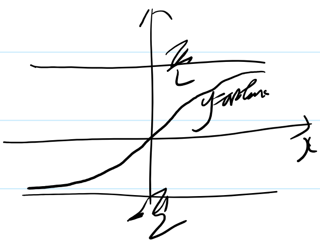
分部积分口诀
对 反 幂 三 指　前者u 后者dv
反 对 幂 三 指

例 $\int_0^{+\infty} t e^{-pt}dt \ (p>0) = -\frac{1}{p}\int_0^{+\infty} t\, de^{-pt} = -\frac{1}{p}\Big[te^{-pt}\Big|_0^{+\infty} - \int_0^{+\infty}e^{-pt}dt\Big]$

$\underset{u}{\text{幂}}$　$\underset{dv}{\text{指}}$

$$= -\frac{1}{p}\Big[\underbrace{\lim_{b\to+\infty}\Big(\frac{b}{e^{pb}}\Big)}_{0} + \frac{1}{p}\underbrace{\lim_{b\to+\infty}e^{-pt}\Big|_0^b}_{\lim_{b\to+\infty}(e^{-pb}-1)}\Big] = -\frac{2}{p}$$

例 $\int_{-\infty}^{+\infty}\frac{1}{1+x^2}dx = \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_0^{+\infty}\frac{1}{1+x^2}dx$



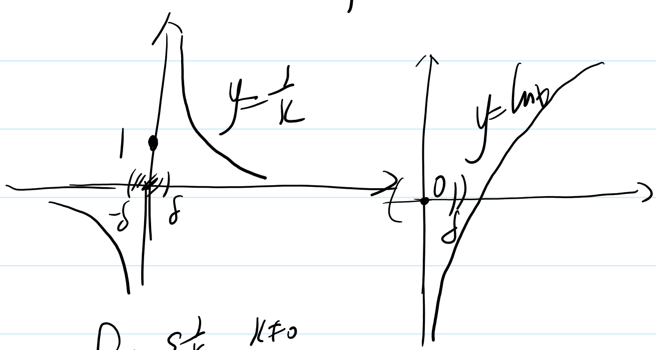
$= \arctan x\Big|_{-\infty}^{0} + \arctan x\Big|_0^{+\infty}$

$= \lim_{b\to-\infty}\arctan x\Big|_b^0 + \lim_{a\to+\infty}\arctan x\Big|_0^a$

$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$

瑕积分：即积分区间内含有瑕点的积分

瑕点：若 $f(x)$ 在点 x_0 的任一邻域内无界，则称 $x=x_0$ 为 $f(x)$ 的瑕点



$\forall(-\delta,\delta)$ 内 有 $y=\frac{1}{x}$ 无界，$x=0$ 为 $y=\frac{1}{x}$ 瑕点

$\forall(0,\delta)$ 内 有 $y=\ln x$ 无界 $x=0$ 为 $y=\ln x$ 瑕点

$$f(x)=\begin{cases}\frac{1}{x} & x\neq 0\\ 1 & x=0\end{cases}$$

例 $f(x)=\frac{\sin x}{x}$ $x=0$ 不是瑕点 $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$，由局部有界性

$\Rightarrow (-\delta,\delta)$ 内 $\frac{\sin x}{x}$ 有界

定义
$$\int_{a(\text{瑕})}^{b} f(x)\,dx=\lim_{d\to a^+}\int_d^b f(x)\,dx$$
$$\int_{a(\text{瑕})}^{b} f(x)\,dx=\lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^{b} f(x)\,dx$$
$$\int_{a}^{b(\text{瑕})} f(x)\,dx=\lim_{\varepsilon\to 0^+}\int_{a}^{b-\varepsilon} f(x)\,dx$$
极限存在时称瑕积分收敛 否则称其发散

$$\int_a^b f(x)\,dx \xlongequal{c\in(a,b)\text{瑕点}} \int_a^c f(x)\,dx+\int_c^b f(x)\,dx$$
$$=\lim_{\varepsilon_1\to 0^+}\int_a^{c-\varepsilon_1} f(x)\,dx+\lim_{\varepsilon_2\to 0^+}\int_{c+\varepsilon_2}^{b} f(x)\,dx$$
当且仅当右侧二式皆收敛时，称 $\int_a^b f(x)\,dx$ 收敛，否则称其发散

例 $$\int_0^1\frac{1}{x^\mu}\,dx \xlongequal{\mu\neq 1} \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 x^{-\mu}\,dx=\lim_{\varepsilon\to 0^+}\left(\frac{1}{1-\mu}x^{1-\mu}\Big|_\varepsilon^1\right)=\begin{cases}\text{存在}\ (\text{收})\ 1-\mu>0 & \mu<1\\ \text{不存在}\ 1-\mu<0 & \mu>1\end{cases}$$
$$\xlongequal{\mu=1}\lim_{\varepsilon\to 0^+}\int_\varepsilon^1\frac{1}{x}\,dx=\lim_{\varepsilon\to 0^+}\ln|x|\Big|_\varepsilon^1\ \text{不存在}\ (\text{散})$$

即 $\int_0^1\frac{1}{x^\mu}dx$ $\begin{cases}\mu<1\ \text{收敛}\end{cases}$ | $\int^b\frac{1}{(x-a)^\mu}dx$ 同左 | $\int^b\frac{1}{(b-x)^\mu}dx$ 同左 | 若不是

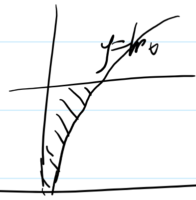
即

$\int_0^1 \frac{1}{x^\mu}dx \begin{cases} \mu<1 \text{ 收敛} \\ \mu\geq 1 \text{ 发散} \end{cases}$	$\int_a^b \frac{1}{(x-a)^\mu}dx$ 同左	$\int_a^b \frac{1}{(b-x)^\mu}dx$ 同左

背下来

$\varepsilon\ln\varepsilon = \frac{\ln\varepsilon}{\frac{1}{\varepsilon}}$

例 $\int_0^1 \underset{u}{\ln x}\, \underset{v}{dx} = \lim_{\varepsilon\to 0^+}\left(x\ln x\Big|_\varepsilon^1 - \int_\varepsilon^1 x\cdot\frac{1}{x}dx\right) = \lim_{\varepsilon\to 0^+}(-\underbrace{\varepsilon\ln\varepsilon}_{\to 0} + \underbrace{\varepsilon}_{\to 0} - 1) = -1$



无穷积分与瑕积分关系

$\lim_{x\to a^+} f(x) = +\infty$

$\int_a^b f(x)dx$ 瑕 $\xrightarrow[dx = d(\frac{1}{t}+a)]{x=\frac{1}{t}+a} \int_{+\infty}^{\frac{1}{b-a}} f(\frac{1}{t}+a)(-\frac{1}{t^2})dt = \int_{\frac{1}{b-a}}^{+\infty} f(\frac{1}{t}+a)\frac{1}{t^2}dt$

$x\to a^+ \Leftrightarrow x-a\to 0^+ \Leftrightarrow t=\frac{1}{x-a}\to +\infty$

$\Leftrightarrow x = \frac{1}{t}+a$