球柱坐标下三重积分

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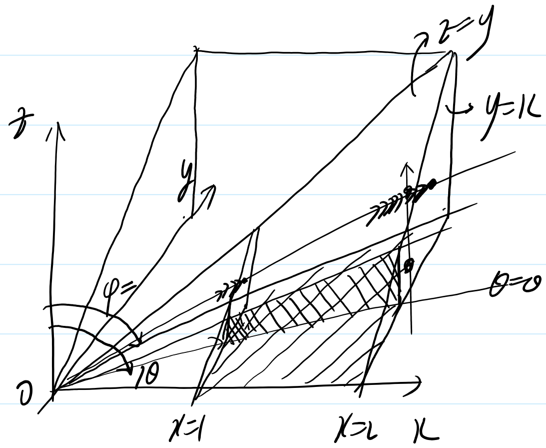
$$\Omega_{\varphi\theta}\begin{cases}\theta_1 \le \theta \le \theta_2 & (\text{左右半平面})\\ \varphi_1(\theta) \le \varphi \le \varphi_2(\theta) & (\text{上下半锥面})\\ r_1(\theta,\varphi) \le r \le r_2(\theta,\varphi) & (\text{内外曲面})\end{cases}$$

$$\iiint_\Omega f(x,y,z)dV = \int_{\theta_1}^{\theta_2} d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} d\varphi \int_{r_1(\theta,\varphi)}^{r_2(\theta,\varphi)} f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)\, r^2\sin\varphi\, dr$$

例 计算 $\iiint_\Omega f(x,y,z)dV$ Ω由 $x=1$, $x=2$, $y=x$, $z=y$ 所围成

$x=1 \quad x=2$

$r\cos\theta=1 \quad r\cos\theta=2$



柱 $\begin{cases}0 \le \theta \le \frac{\pi}{4}\\ \frac{1}{\cos\theta} \le r \le \frac{2}{\cos\theta}\\ 0 \le z \le y\end{cases}$

$$\text{原式} = \int_0^{\frac{\pi}{4}} d\theta \int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} dr \int_0^y f[\text{柱}]\, r\, dz$$

柱 $r=\frac{1}{\cos\theta} \quad r=\frac{2}{\cos\theta}$

球 $r\sin\varphi\cos\theta=1 \quad r\sin\varphi\cos\theta=2$

$z=y$

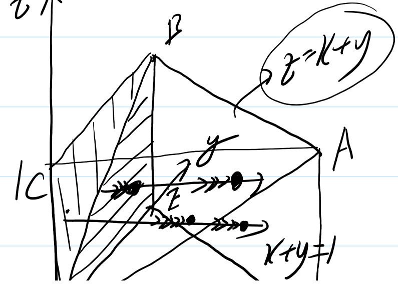
$r\cos\varphi = r\sin\varphi\sin\theta \Rightarrow \tan\varphi = \frac{1}{\sin\theta}$

球 $\begin{cases}0 \le \theta \le \frac{\pi}{4}\\ \arctan\frac{1}{\sin\theta} \le \varphi \le \frac{\pi}{2}\\ \frac{1}{\sin\varphi\cos\theta} \le r \le \frac{2}{\sin\varphi\cos\theta}\end{cases}$

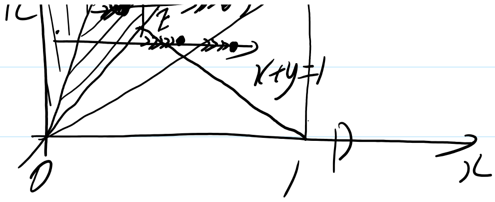
$$\text{原式} = \int_0^{\frac{\pi}{4}} d\theta \int_{\arctan\frac{1}{\sin\theta}}^{\frac{\pi}{2}} d\varphi \int_{\frac{1}{\sin\varphi\cos\theta}}^{\frac{2}{\sin\varphi\cos\theta}} f[\text{球}]\, r^2\sin\varphi\, dr$$

例 $\iiint_\Omega f(x,y,z)dV$ $\quad \Omega: 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le x+y$

1) $\Omega_{xy}\begin{cases}0 \le x \le 1\\ 0 \le y \le 1-x\\ 0 \le z \le x+y\end{cases}$ $\quad$ 原 $= \int_0^1 dx \int_0^{1-x} dy \int_0^{x+y} f[\text{直}]\, dz$



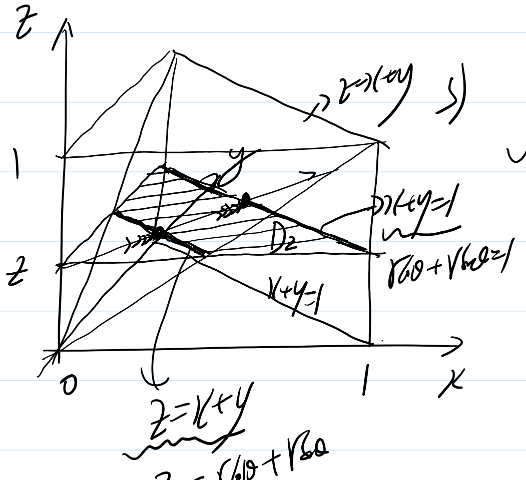
2) $\Omega_{1yz}\begin{cases}0 \le y \le 1\\ y \le z \le 1\\ z-y \le x \le 1-y\end{cases}$ $\quad \Omega_{2yz}\begin{cases}0 \le y \le 1\\ 0 \le z \le y\\ 0 \le x \le 1-y\end{cases}$



$z-y \le x \le 1-y$　　　　$0 \le x \le 1-y$

原式 $=\int_0^1 dy\int_y^1 dz\int_{z-y}^{1-y} f\,dx + \int_0^1 dy\int_0^y dz\int_0^{1-y} f\,dx$

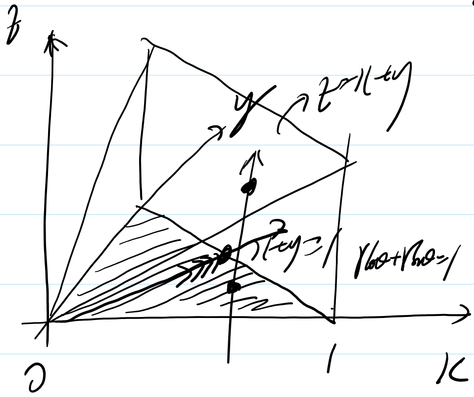
3)



$$\Omega_z \begin{cases} 0 \le z \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \\ D_z\ \frac{z}{\cos\theta+\sin\theta} \le r \le \frac{1}{\cos\theta+\sin\theta} \end{cases}$$

原式 $=\int_0^1 dz\int_0^{\frac{\pi}{2}} d\theta\int_{\frac{z}{\cos\theta+\sin\theta}}^{\frac{1}{\cos\theta+\sin\theta}} f\, r\,dr$

4) 柱



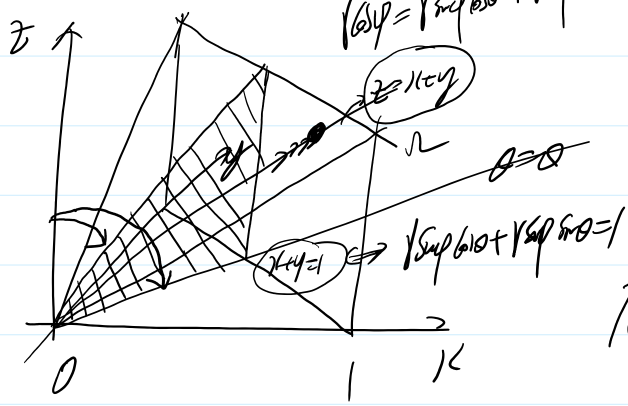
$$\Omega_{r\theta} \begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le \frac{1}{\cos\theta+\sin\theta} \\ 0 \le z \le r\cos\theta + r\sin\theta \end{cases}$$

原式 $=\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{1}{\cos\theta+\sin\theta}} dr\int_0^{r\cos\theta+r\sin\theta} f(\cdots)\, r\,dz$

$\tan\varphi = \frac{1}{\cos\theta+\sin\theta}$　　5) 球

$r\cos\varphi = r\sin\varphi\cos\theta + r\sin\varphi\sin\theta$

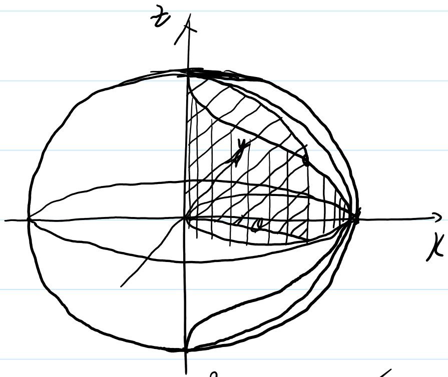
$$\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ \arctan\frac{1}{\cos\theta+\sin\theta} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le \frac{1}{\sin\varphi\cos\theta + \sin\varphi\sin\theta} \end{cases}$$

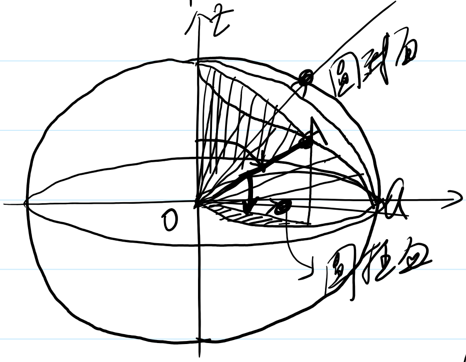


原 $=\int_0^{\frac{\pi}{2}} d\theta\int_{\arctan\frac{1}{\sin\theta+\cos\theta}}^{\frac{\pi}{2}} d\varphi\int_0^{\frac{1}{\sin\varphi\cos\theta+\sin\varphi\sin\theta}} f(\text{球})\, r^2\sin\varphi\,dr$

例　$\iiint_\Omega f(x,y,z)\,dV$　　Ω　$x^2+y^2+z^2 \le a^2$；$x^2+y^2 \le ax$，$z \ge 0$，化为球坐标下三次积分

$z\uparrow$　　$\Omega \begin{cases} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \cdots \end{cases}$　　　　$\begin{cases} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \cdots \end{cases}$





$$\Omega_{1\theta\varphi}\begin{cases}-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\\ 0\le\varphi\le\arcsin\cos\theta\\ 0\le r\le a\end{cases}\qquad \Omega_{2\theta\varphi}\begin{cases}-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\\ \arcsin\cos\theta\le\varphi\le\frac{\pi}{2}\\ 0\le r\le\frac{a\cos\theta}{\sin\varphi}\end{cases}$$

$$\begin{cases}x^2+y^2+z^2=a^2\\ x^2+y^2=ax\end{cases}\quad \begin{aligned}&\Leftrightarrow r=a\\ &\Leftrightarrow r^2\sin^2\varphi\cos^2\theta+r^2\sin^2\varphi\sin^2\theta=ar\sin\varphi\cos\theta\end{aligned}$$

$$\sin^2\varphi=\sin\varphi\cos\theta \rightarrow \varphi=\arcsin\cos\theta$$

$$x^2+y^2=ax$$

$$r^2\sin^2\varphi=ar\sin\varphi\cos\theta \qquad r=\frac{a\cos\theta}{\sin\varphi}$$

$$\text{原式}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{\arcsin\cos\theta}d\varphi\int_0^a f[\ldots]r^2\sin\varphi\,dr+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_{\arcsin\cos\theta}^{\frac{\pi}{2}}d\varphi\int_0^{\frac{a\cos\theta}{\sin\varphi}}f(\ldots)r^2\sin\varphi\,dr$$