# 空间解析几何14-2

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## 直线及其方程

（直线 $\vec{s}=(m,n,p)$ ①＊， $M_0(x_0,y_0,z_0)\in l$ ②） （平面 $\vec{n}=(A,B,C)$ ①＊ $M_0(x_0,y_0,z_0)$ ②）

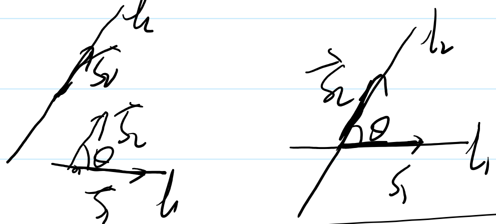
1) 对称式（标准式） $\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}=t$ $\vec{s}=(m,n,p)$ $M_0(x_0,y_0,z_0)\in l$

2) 参数式 $\begin{cases} x=x_0+mt \\ y=y_0+nt \\ z=z_0+pt \end{cases}$ $\vec{s}=(m,n,p)$， $M_0(x_0,y_0,z_0)\in l$

3) 一般式 $\begin{cases} \pi_1\ A_1x+B_1y+C_1z+D_1=0 \\ \pi_2\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$ $\vec{s}=\vec{n}_1\times\vec{n}_2$ 令 $z=z_0$ 代入方程解 $\begin{cases} x=x_0 \\ y=y_0 \end{cases}$ $M_0(x_0,y_0,z_0)\in l$

4) 两点式 $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

5) 直线间夹角 $l_i$ $\frac{x-x_i}{m_i}=\frac{y-y_i}{n_i}=\frac{z-z_i}{p_i}$ $i=1,2$ $M_i$， $\vec{s}_i$，



$$\cos\theta=\frac{|\vec{s}_1\cdot\vec{s}_2|}{|\vec{s}_1|\,|\vec{s}_2|}=\frac{|m_1m_2+n_1n_2+p_1p_2|}{\sqrt{m_1^2+n_1^2+p_1^2}\sqrt{m_2^2+n_2^2+p_2^2}}$$

$\theta\in[0,\frac{\pi}{2}]$

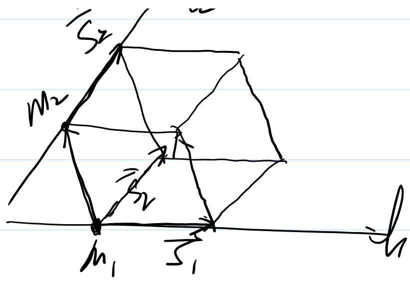
$l_1\parallel l_2 \Leftrightarrow \vec{s}_1\parallel\vec{s}_2 \Leftrightarrow \frac{m_1}{m_2}=\frac{n_1}{n_2}=\frac{p_1}{p_2}$

$l_1\perp l_2 \Leftrightarrow \vec{s}_1\perp\vec{s}_2 \Leftrightarrow m_1m_2+n_1n_2+p_1p_2=0$

6) 异面直线判别 $l_i$ $\frac{x-x_i}{m_i}=\frac{y-y_i}{n_i}=\frac{z-z_i}{p_i}$ $i=1,2$

$M_i(x_i,y_i,z_i)\in l_i$ $\vec{s}_i=(m_i,n_i,p_i)\parallel l_i$

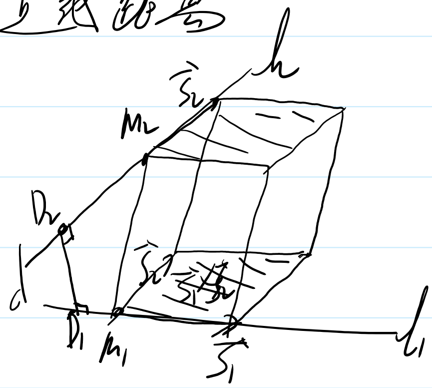
$M_2(x_2, y_2, z_2) \in l_2$ $\vec{s_2}=(m_2,n_2,p_2)//l_2$




$l_1$与$l_2$异面 $\Leftrightarrow [\overrightarrow{M_1M_2}\ \vec{s_1}\ \vec{s_2}] = \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1\\ m_1 & n_1 & p_1\\ m_2 & n_2 & p_2\end{vmatrix}\neq 0$

$l_1$与$l_2$共面 $\Leftrightarrow [\overrightarrow{M_1M_2}\ \vec{s_1}\ \vec{s_2}] = \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix} = 0$

7) 异面直线的距离




$\overrightarrow{P_1P_2}//\vec{s_1}\times\vec{s_2}$

$$d=\frac{V}{底面积}=\frac{|[\overrightarrow{M_1M_2}\ \vec{s_1}\ \vec{s_2}]|}{|\vec{s_1}\times\vec{s_2}|}$$

或 $d=\left|\mathrm{Prj}_{\overrightarrow{P_1P_2}}\overrightarrow{M_1M_2}\right| = \left|\overrightarrow{M_1M_2}\cdot\overrightarrow{P_1P_2}^0\right| = \left|\overrightarrow{M_1M_2}\cdot(\vec{s_1}\times\vec{s_2})^0\right|$

$$=\frac{|\overrightarrow{M_1M_2}\cdot(\vec{s_1}\times\vec{s_2})|}{|\vec{s_1}\times\vec{s_2}|}$$

8) 异面直线公垂线方程

$l_2$ $\frac{x-x_2}{m_2}=\frac{y-y_2}{n_2}=\frac{z-z_2}{p_2}$

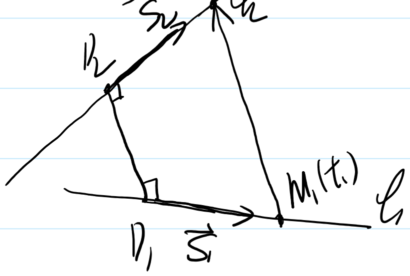

法一

$\pi_1$: $M_1(x_1,y_1,z_1)$, $\vec{n_1}=\vec{s_1}\times(\vec{s_1}\times\vec{s_2})$

$\pi_2$: $M_2(x_2,y_2,z_2)$ $\vec{n_2}=\vec{s_2}\times(\vec{s_1}\times\vec{s_2})$

联立 $\begin{cases}\pi_1\\ \pi_2\end{cases}$ 即为 $P_1P_2$ 一般式方程

法二 $M(t)\begin{cases}x=x_0+mt\\ y=y_0+nt\\ z=z_0+pt\end{cases}$


$M_1(t_1)\begin{cases}x=x_1+m_1t_1\\ y=y_1+n_1t_1\\ z=z_1+p_1t_1\end{cases}$ $(t_1\in R)$, $M_2(t_2)=\begin{cases}x=x_2+m_2t_2\\ y=y_2+n_2t_2\\ z=z_2+p_2t_2\end{cases}$ $t_2\in R$

$\overrightarrow{M_1(t_1)M_2(t_2)}\perp\vec{s_1}$ 且 $\overrightarrow{M_1(t_1)M_2(t_2)}\perp\vec{s_2}$

$\begin{cases}\overrightarrow{M_1(t_1)M_2(t_2)}\cdot\vec{s_1}=0\\ \overrightarrow{M_1(t_1)M_2(t_2)}\cdot\vec{s_2}=0\end{cases}$

2个方程 2个未知量 $t_1$与$t_2$，解出$t_1$与$t_2$代回

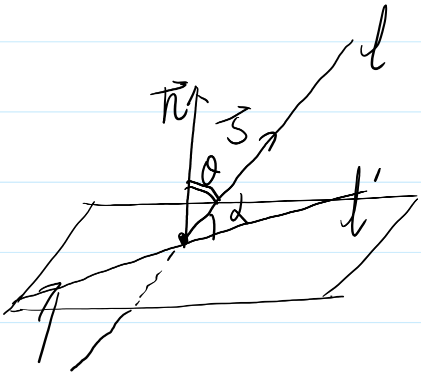
$M_1(t_1)$, $M_2(t_2)$ 即为$P_1P_2$坐标

去掉 $D_1 D_2$ 两式即可

## 直线与平面位置关系（夹角）

$l$：$\dfrac{x-x_0}{m}=\dfrac{y-y_0}{n}=\dfrac{z-z_0}{p}$ $M_0(x_0,y_0,z_0)\in l$，$\vec{s}=(m,n,p)\parallel l$

$\pi$：$Ax+By+Cz+D=0$ $\vec{n}=(A,B,C)\perp\pi$




$\alpha\in[0,\frac{\pi}{2}]$

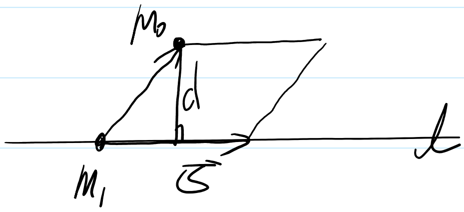
$$\sin\alpha=\cos\theta=\frac{|\vec{s}\cdot\vec{n}|}{|\vec{s}|\,|\vec{n}|}=\frac{|mA+nB+pC|}{\sqrt{m^2+n^2+p^2}\sqrt{A^2+B^2+C^2}}$$

$l\perp\pi \iff \vec{s}\parallel\vec{n} \iff \vec{s}\times\vec{n}=0 \iff \dfrac{m}{A}=\dfrac{n}{B}=\dfrac{p}{C}$

$l\parallel\pi \iff \vec{s}\perp\vec{n} \iff \vec{s}\cdot\vec{n}=0 \iff mA+nB+pC=0$

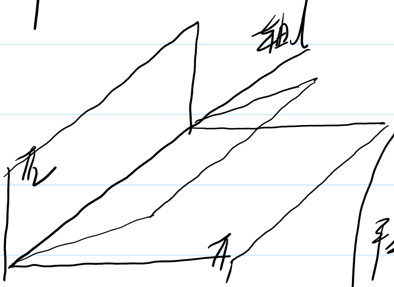
## 点到直线距离

设 $M_0(x_0,y_0,z_0)\notin l$，$l$：$\dfrac{x-x_1}{m}=\dfrac{y-y_1}{n}=\dfrac{z-z_1}{p}$




$$d=\frac{S_{▱}}{底边长}=\frac{|\vec{s}\times\overrightarrow{M_1M_0}|}{|\vec{s}|}$$

## 平面束(*)




轴 $l$：$\begin{cases}\pi_1 & A_1x+B_1y+C_1z+D_1=0\\ \pi_2 & A_2x+B_2y+C_2z+D_2=0\end{cases}$ $\pi_1\nparallel\pi_2$ $\forall M(x,y,z)\in l$

平面束（以 $l$ 为轴）$\pi_\lambda$（$\pi_1+\lambda\pi_2=0$）即

平面束 $\pi_\lambda$：$A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0$

$\pi_\lambda$：$(A_1+\lambda A_2)x+(B_1+\lambda B_2)y+(C_1+\lambda C_2)z+D_1+\lambda D_2=0$

$\pi_\lambda$ 中除了 $\pi_2$ 外，其包含了过轴 $l$ 的所有平面

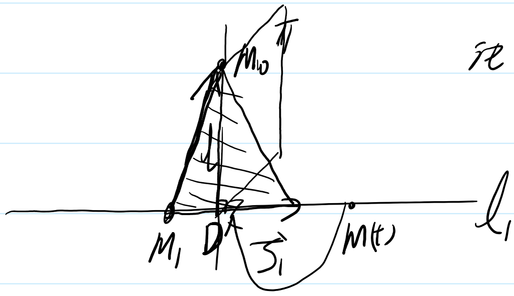
$\pi_\lambda$中除了$\pi_2$外，其包含了过轴$l$的所有平面

平面束 ① $\pi_1+\lambda\pi_2=0$ $A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0$ (不含$\pi_2$)

② $\mu\pi_1+\lambda\pi_2=0$ $\mu(A_1x+B_1y+C_1z+D_1)+\lambda(A_2x+B_2y+C_2z+D_2)=0$ (包含了所有以$l$为轴的平面)

例 求过$M_0(2,1,3)$且与直线$l_1$ $\frac{x+1}{3}=\frac{y-1}{2}=\frac{z}{-1}=t$ ~~垂直~~相交的直线$l$的方程

$M_1(-1,1,0)$ $\vec{s_1}=(3,2,-1)$




法一：一般式

$l\begin{cases}\pi_1 & M_0(2,1,3),\ \vec{n}=\vec{s_1}=(3,2,-1) \quad 点法式\\ \pi_2 & M_0(2,1,3)\ \ \vec{n_2}=\vec{s_1}\times\overrightarrow{M_1M_0} \quad 点法式\end{cases}$

法二：两点式 $M_0D$

$M(t)\begin{cases}x=-1+3t\\ y=1+2t\\ z=-t\end{cases}(t\in R)$ $\overrightarrow{M(t)M_0}\perp\vec{s_1}$ $\overrightarrow{M(t)M_0}\cdot\vec{s_1}=0$

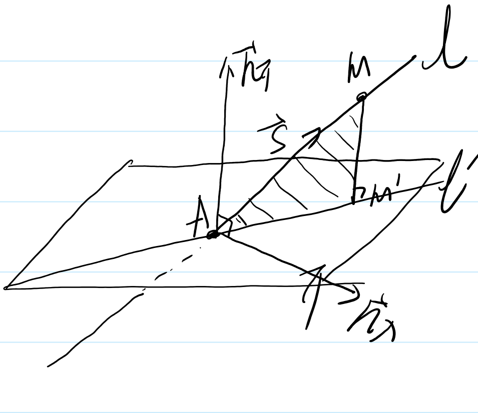
$(2-(-1+3t),\ 1-(1+2t),\ 3-(-t))\cdot(3,2,-1)=0$

$\Rightarrow t=t_0$

$M(t_0)=D(t_0)$

两点式即可

例 求直线 $l:\begin{cases}\pi_1: x+y-z-1=0\\ \pi_2: x-y+z+1=0\end{cases}$ 在平面$\pi: x+y+z=0$上的投影$l'$的方程




法一 $\begin{cases}\pi & x+y+z=0\\ \pi_2 & \begin{cases}\pi_1\\ \pi_2\\ \pi\end{cases}\Rightarrow A(x_0,y_0,z_0) \quad \vec{n_2}=\vec{s}\times\vec{n}\\ & \qquad =(\vec{s_1}\times\vec{s_2})\times\vec{n}\end{cases}$

法二 以$l$为轴的$\pi_\lambda$: $(x+y-z-1)+\lambda(x-y+z+1)=0$

$\pi_\lambda: (1+\lambda)x + (1-\lambda)y + (\lambda-1)z + \lambda-1 = 0$

$\vec{n}_\lambda = (1+\lambda, 1-\lambda, \lambda-1)$

$\vec{n}_\lambda \cdot \vec{n}_{\pi} = 0$, $(1+\lambda, 1-\lambda, \lambda-1)\cdot(1,1,1) = 0$

$1+\lambda + 1-\lambda + \lambda-1 = 1+\lambda = 0$

$\Rightarrow \lambda = -1$

代回 $\pi_\lambda$

$\Rightarrow \pi_\Delta: y - z - 1 = 0$

$l' \begin{cases} \pi & x+y+z=0 \\ \pi_\Delta & y-z-1=0 \end{cases}$