应用曲线向坐标面的 { 投影柱面
                        { 投影线

$\Gamma \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$
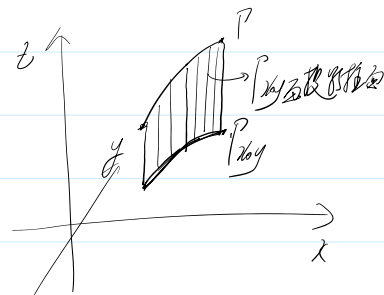
1) Γ 向 xoy 面投影柱面，即在 $\Gamma \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ 消去 z

$\Rightarrow H(x,y)=0$, 即为 Γ 向 xoy 面的投影柱面的方程 向 xoy 面投影的
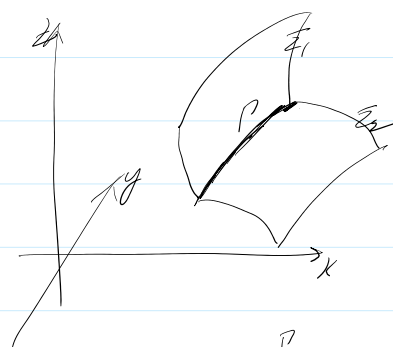
$\forall M(x,y,z) \in \Gamma \Rightarrow M(x,y,z)$ 适合 $H(x,y)=0$

Γ 向 xoy 面的投影线 $\begin{cases} H(x,y)=0 \\ z=0 \end{cases}$ 向 xoy 面投影线

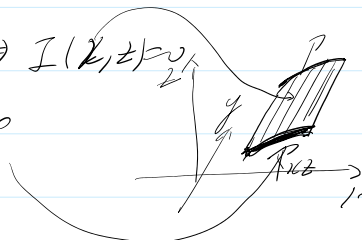2) $\Gamma \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ 向 yoz 面投影 $\begin{cases} (1) \text{投影柱面 (消去} x) \Rightarrow K(y,z)=0 \\ (2) \text{投影线} \begin{cases} K(y,z)=0 \\ x=0 \end{cases} \end{cases}$
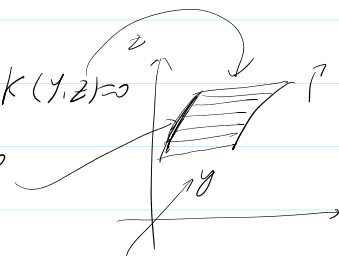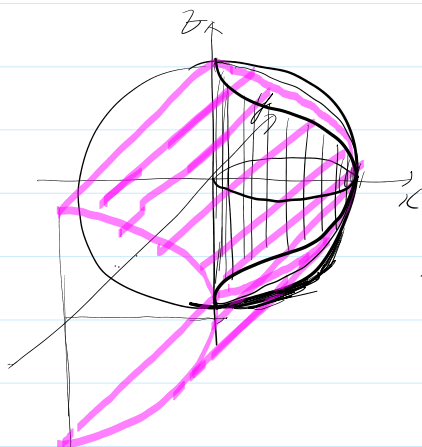

3) $\Gamma \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ 向 zox 面投影 $\begin{cases} (1) \text{投影柱面 (消去} y) \Rightarrow I(x,z)=0 \\ (2) \text{投影线} \begin{cases} I(x,z)=0 \\ y=0 \end{cases} \end{cases}$

4) 求 $\Gamma \begin{cases} x^2+y^2+z^2=4a^2 \\ x^2+y^2=2ax \end{cases}$ 将 Γ 向 xoy 面，xoz 面 投影柱面，及投影线

1) $\Gamma \begin{cases} x^2+y^2+z^2=4a^2 \\ x^2+y^2=2ax \end{cases}$ 向 xoy 面，消去 z $\Rightarrow x^2+y^2=2ax$ 为向 xoy 面投影柱面

$\begin{cases} x^2+y^2=2ax \\ z=0 \end{cases}$ 为向 xoy 面投影线

2) $\Gamma \begin{cases} x^2+y^2+z^2=4a^2 \\ x^2+y^2=2ax \end{cases}$ 向 xoz 面 $\Rightarrow z^2+2ax=4a^2$ 即为向 xoz 面投影柱面
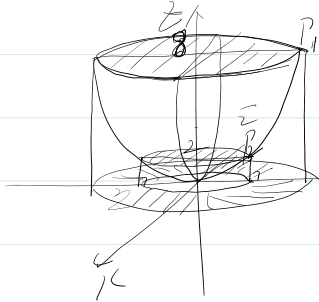
$\begin{cases} z^2+2ax=4a^2 \\ y=0 \end{cases}$ 即为向 xoz 面投影线

$z^2=4a^2-2ax$

$$z^2 = 4a^2 - 2ax$$
$$= 2a(2a-x)$$

圆周 $z^2 = -2a(x-2a)$ 的面积

例 $\begin{cases} y^2 = 2z \\ x = 0 \end{cases}$ 绕z轴旋转一周生成的曲面 ($z=2,8$) 为Σ, 求Σ在xoy面投影域

$P_1 \begin{cases} x^2+y^2=2z \\ z=8 \end{cases}$ → $\begin{cases} x^2+y^2=16 \\ z=0 \end{cases}$

$P_2 \begin{cases} x^2+y^2=2z \\ z=2 \end{cases}$ → $\begin{cases} x^2+y^2=4 \\ z=0 \end{cases}$

xoy面上的 $x^2+y^2=16$

投影区域组成 $\begin{cases} 4 \le x^2+y^2 \le 16 \\ z=0 \end{cases}$

常见的二次曲面
1) 椭球面 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

2) 双曲面 $\begin{cases} \text{单叶双曲面} \quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1 \\ \text{双叶双曲面} \quad \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1 \end{cases}$

3) 抛物面 $\begin{cases} \text{椭圆抛物面} \quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2z \\ \text{双曲抛物面} \quad \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=2z \end{cases}$

4) 二次锥面 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0$

1) 椭球面 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$
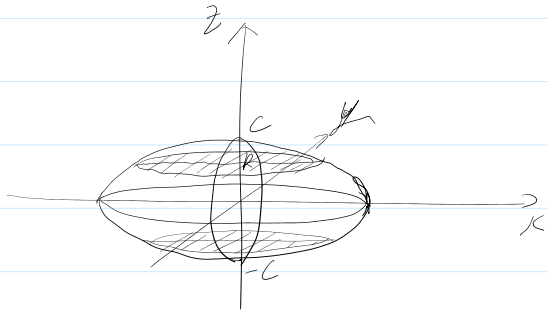
主截口线 $\begin{cases} ① \begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1 \\ z=0 \end{cases} \\ ② \begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1 \\ y=0 \end{cases} \\ ③ \begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1 \\ x=0 \end{cases} \end{cases}$

平行截 $\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1 \\ z=k \quad (|k|\le c) \end{cases}$
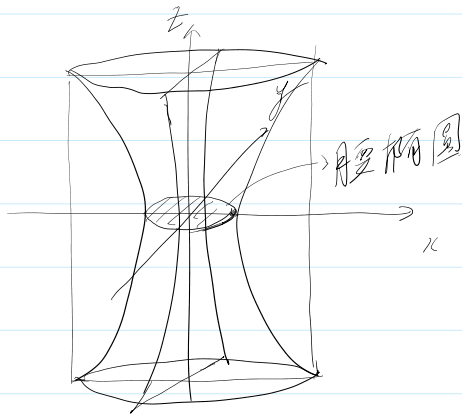
即在平面上 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\left(1-\dfrac{k^2}{c^2}\right)$

$y=kx$

$a=b=c=R \Rightarrow x^2+y^2+z^2=R^2$

$$\frac{x^2}{a^2\left(1-\frac{k^2}{c^2}\right)}+\frac{y^2}{b^2\left(1-\frac{k^2}{c^2}\right)}=1$$

2) 单叶双曲面 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

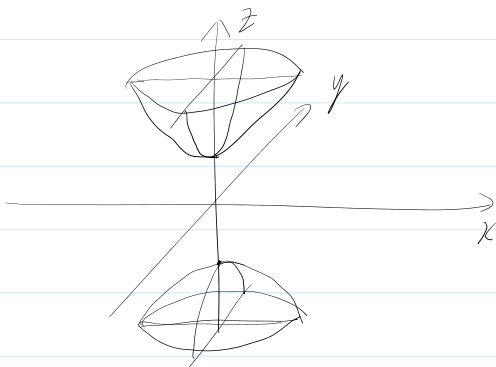双叶双曲面 $-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

→ 腰椭圆

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\ z=0 \end{cases} \quad \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1 \\ y=0 \end{cases} \quad \begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 \\ x=0 \end{cases}$$

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 \\ z=k \end{cases} \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \boxed{1 + \dfrac{k^2}{c^2}}$$

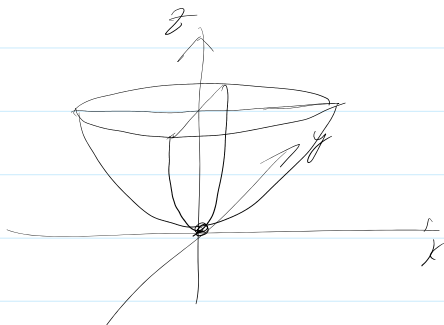双叶 $-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$$\begin{cases} \dfrac{z^2}{c^2} - \dfrac{y^2}{b^2} = 1 \\ x=0 \end{cases} \quad \begin{cases} \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} = 1 \\ y=0 \end{cases}$$

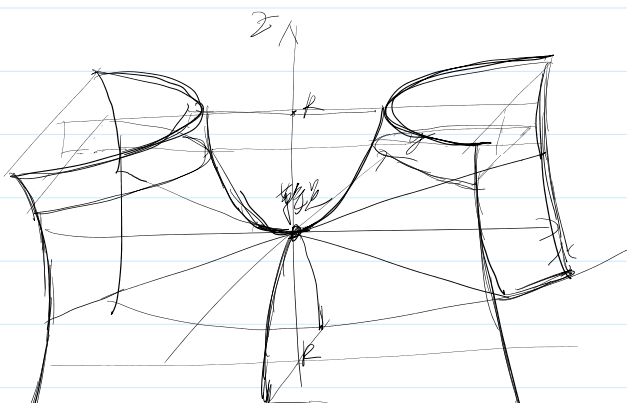$$\begin{cases} -\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \\ z=k \end{cases} \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \boxed{\dfrac{k^2}{c^2} - 1}$$

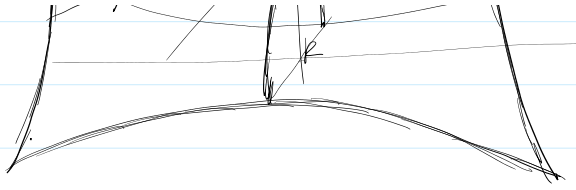3) 抛物面 '椭圆抛物面 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 4pz$

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 4pk \\ z=k \end{cases}$$

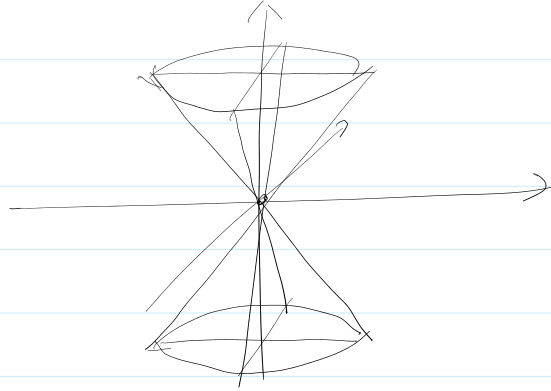4) 双曲抛物面 $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 4pz$ （马鞍面）

$$\begin{cases} -\dfrac{y^2}{b^2} = 4pz \\ x=0 \end{cases} \quad \begin{cases} \dfrac{x^2}{a^2} = 4pz \\ y=0 \end{cases} \quad \begin{cases} \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0 \\ z=0 \end{cases}$$

$$\begin{cases} \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 4pz \\ z=k \end{cases} \Rightarrow \dfrac{x^2}{a^2 4pk} - \dfrac{y^2}{b^2 4pk} = 1$$

$(k>0)$

二次锥面 $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$