一、单项选择题

1. 设有空间区域 $\Omega_1=\{(x,y,z)|x^2+y^2+z^2\leqslant R^2,z\geqslant 0\}$ 及 $\Omega_2=\{(x,y,z)|x^2+y^2+z^2\leqslant R^2,x\geqslant 0,y\geqslant 0,z\geqslant\}$, 则 (C).

(A) $\iiint\limits_{\Omega_1} x\mathrm{d}V=4\iiint\limits_{\Omega_2} z\mathrm{d}V$; (B) $\iiint\limits_{\Omega_1} y\mathrm{d}V=4\iiint\limits_{\Omega_2} z\mathrm{d}V$;

(C) $\iiint\limits_{\Omega_1} z\mathrm{d}V=4\iiint\limits_{\Omega_2} z\mathrm{d}V$; (D) $\iiint\limits_{\Omega_1} xyz\mathrm{d}V=4\iiint\limits_{\Omega_2} xyz\mathrm{d}V$.

2. 设 Ω 由平面 $x+y+z+1=0,x+y+z+2=0,x=0,y=0,z=0$ 围成, $I_1=\iiint\limits_{\Omega}[\ln(x+y+z+3)]^2\mathrm{d}V, I_2=\iiint\limits_{\Omega}(x+y+z)^2\mathrm{d}V$, 则 (A).

(A) $I_1<I_2$; (B) $I_1>I_2$; (C) $I_1\leqslant I_2$; (D) $I_1\geqslant I_2$.

3. 曲面 $z=\sqrt{x^2+y^2}$ 与 $z=2-x^2-y^2$ 所围成的立体体积为(B).

(A) $\dfrac{\pi}{2}$; (B) $\dfrac{5\pi}{6}$; (C) $\dfrac{2\pi}{3}$; (D) π.

4. 设空间区域 $\Omega=\{(x,y,z)|\sqrt{x^2+y^2}\leqslant z\leqslant\sqrt{2-x^2-y^2}\}$, $f(x,y,z)$ 为连续函数, 则三重积分$\iiint\limits_{\Omega}\mathrm{d}V=($ D).

(A) $\int_{-1}^{1}\mathrm{d}x\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\mathrm{d}y\int_{\sqrt{2-x^2-y^2}}^{\sqrt{x^2+y^2}}f(x,y,z)\mathrm{d}z$;

(B) $4\int_{0}^{1}\mathrm{d}x\int_{0}^{\sqrt{1-x^2}}\mathrm{d}y\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}}f(x,y,z)\mathrm{d}z$;

(C) $\int_{0}^{2\pi}\mathrm{d}\theta\int_{0}^{1}\mathrm{d}r\int_{r}^{2-r^2}f(r\cos\theta,r\sin\theta,z)\mathrm{d}z$;

(D) $\int_{0}^{2\pi}\mathrm{d}\theta\int_{0}^{\frac{\pi}{4}}\mathrm{d}\varphi\int_{0}^{\sqrt{2}}f(r\sin\varphi\cos\theta,r\sin\varphi\sin\theta,r\cos\varphi)r^2\sin\varphi\mathrm{d}r$.

5. 设空间区域 $\Omega=\{(x,y,z)|0\leqslant x\leqslant 1,0\leqslant y\leqslant 1-x,0\leqslant z\leqslant x+y\}$, $f(x,y,z)$为连续函数, 则三重积分$\iiint\limits_{\Omega}f(x,y,z)\mathrm{d}V=($ A).

(A) $\int_0^1 \mathrm{d}y \int_0^y \mathrm{d}z \int_0^{1-y} f(x,y,z)\mathrm{d}x + \int_0^1 \mathrm{d}y \int_y^1 \mathrm{d}z \int_{z-y}^{1-y} f(x,y,z)\mathrm{d}x$;

(B) $\int_0^1 \mathrm{d}z \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\frac{1}{\cos\theta+\sin\theta}}^{\frac{z}{\cos\theta+\sin\theta}} f(r\cos\theta, r\sin\theta, z)r\mathrm{d}r$;

(C) $\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\sin\theta+\cos\theta} \mathrm{d}r \int_0^{r(\sin\theta+\cos\theta)} f(r\cos\theta, r\sin\theta, z)r\mathrm{d}z$;

(D) $\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{\frac{1}{\sin\varphi\cos\theta+\sin\varphi\sin\theta}} f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)r^2\sin\varphi\mathrm{d}r$.

二、填空题

1. 设 Ω 为 $x^2+y^2+z^2 \leqslant R^2, z \geqslant 0$, 则 $\iiint\limits_{\Omega}(x+y+z)\mathrm{d}V =$ ______________.

答案 $\dfrac{\pi R^4}{4}$.

2. 设 Ω 是由曲面 $z=\sqrt{2-x^2-y^2}$ 及 $z=x^2+y^2$ 所围成的空间闭区域, 则三重积分 $\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}V$ 化为柱面坐标下的先 z 再 r 后 θ 顺序的三次积分为______________.

答案 $\int_0^{2\pi}\mathrm{d}\theta \int_0^1 r\mathrm{d}r \int_{r^2}^{\sqrt{2-r^2}} f(r\cos\theta, r\sin\theta, z)\mathrm{d}z$.

3. 设 Ω 为 $x^2+y^2+z \leqslant 1, z \geqslant 0$, 则 $\iiint\limits_{\Omega}(x+1)(y+z)(z+1)\mathrm{d}V =$ ______________.

答案 $\dfrac{2\pi}{3}$.

4. 设 $F(t) = \iiint\limits_{\Omega_t} f(x^2+y^2+z^2)\mathrm{d}V$, 其中 $\Omega_t = \{(x,y,z) | x^2+y^2+z^2 \leqslant t^2\}$, f 为连续函数, 则 $F'(t) =$ ______________.

答案 $4\pi t^2 f(t^2)$.

5. 设 $\varphi(y) = \int_0^y \dfrac{\ln(1+xy)}{x}\mathrm{d}x (y \neq 0)$, 则 $\varphi'(1) =$ ______________.

答案 $2\ln 2$.

三、计算题

1. 设 Ω 是由 $x+y=1, y=x, y=0, z=0$ 和 $z=\pi$ 所围成的空间闭区域, 计算 $\iiint\limits_{\Omega}(x+y)\sin z\mathrm{d}V$.

解 $\iiint\limits_{\Omega}(x+y)\sin z\mathrm{d}V=\int_0^{\frac{1}{2}}\mathrm{d}y\int_y^{1-y}(x+y)\mathrm{d}x\int_0^{\pi}\sin z\mathrm{d}z=\frac{1}{3}.$

2. 设 Ω 为 $x^2+y^2+(z-1)^2\leqslant 1$ 所确定的空间闭区域, 计算 $\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}\mathrm{d}V.$

解 $\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{2\cos\varphi}r\cdot r^2\sin\varphi\mathrm{d}r=\frac{8\pi}{5}.$

3. 设 Ω 由旋转抛物面 $x^2+y^2=2z$ 与平面 $z=1,z=2$ 所围成的空间闭区域, 计算 $\iiint\limits_{\Omega}(x^2+y^2)\mathrm{d}V.$

解 $\Omega=\{(x,y,z)|(x,y)\in D_z 1\leqslant z\leqslant 2\},D_z=\{(x,y)|x^2+y^2\leqslant 2z\}$. 从而

$$\begin{aligned}\iiint\limits_{\Omega}(x^2+y^2)\mathrm{d}V&=\int_1^2\mathrm{d}z\iint\limits_{D_z}(x^2+y^2)\mathrm{d}\sigma\\&=\int_1^2\mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{2z}}r^2\cdot r\mathrm{d}r\\&=2\pi\int_1^2 z^2\mathrm{d}z=\frac{14\pi}{3}.\end{aligned}$$

4. 计算 $\iiint\limits_{\Omega}|z-x^2-y^2|\mathrm{d}V$, 其中 $\Omega:0\leqslant z\leqslant 1,x^2+y^2\leqslant 1$.

解 用曲面 $z=x^2+y^2$ 将 Ω 分为两部分, 记 Ω 中 $z\geqslant x^2+y^2$ 的部分为 $\Omega_1,z\leqslant x^2+y^2$ 的部分为 Ω_2. 在柱面坐标系下

$$\Omega_1=\{(\theta,r,z)|0\leqslant\theta\leqslant 2\pi,0\leqslant r\leqslant 1,r^2\leqslant z\leqslant 1\},$$

$$\Omega_2=\{(\theta,r,z)|0\leqslant\theta\leqslant 2\pi,0\leqslant r\leqslant 1,0\leqslant z\leqslant r^2\},$$

$$\begin{aligned}\iiint\limits_{\Omega}|z-x^2-y^2|\mathrm{d}V&=\iiint\limits_{\Omega_1}|z-x^2-y^2|\mathrm{d}V+\iiint\limits_{\Omega_2}|z-x^2-y^2|\mathrm{d}V\\&=\iiint\limits_{\Omega_1}(z-x^2-y^2)\mathrm{d}V+\iiint\limits_{\Omega_2}(x^2+y^2-z)\mathrm{d}V\\&=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\mathrm{d}r\int_{r^2}^1(z-r^2)\mathrm{d}z+\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\mathrm{d}r\int_0^{r^2}(r^2-z)\mathrm{d}z\\&=\frac{\pi}{6}+\frac{\pi}{6}=\frac{\pi}{3}.\end{aligned}$$

5. 计算 $\iiint\limits_{\Omega}\left(x+\frac{y}{2}+\frac{z}{3}\right)^2\mathrm{d}V$, 其中 $\Omega=\left\{(x,y,z)\left|\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1,\ a>0,b>0,c>0\right.\right\}$.

解

$$\begin{aligned}\iiint\limits_{\Omega}\left(x+\frac{y}{2}+\frac{z}{3}\right)^2\mathrm{d}V&=\iiint\limits_{\Omega}\left[\left(x^2+\frac{y^2}{4}+\frac{z^2}{9}\right)+\left(xy+\frac{2}{3}xy+\frac{yz}{3}\right)\right]\mathrm{d}V\\&=\iiint\limits_{\Omega}\left(x^2+\frac{y^2}{4}+\frac{z^2}{9}\right)\mathrm{d}V.\end{aligned}$$

方法一　作广义球坐标变换 $\begin{cases}x=ar\sin\varphi\cos\theta & 0\leqslant\theta\leqslant 2\pi,\\ y=br\sin\varphi\sin\theta & 0\leqslant\varphi\leqslant\pi,\\ z=cr\cos\varphi, & 0\leqslant r<+\infty\end{cases}$ $J=abcr^2\sin\varphi$.

$$\iiint\limits_{\Omega}x^2\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^1 a^2r^2\sin^2\varphi\cos^2\theta\cdot abcr^2\sin\varphi\mathrm{d}r=\frac{4\pi a^3bc}{15},$$

$$\iiint\limits_{\Omega}y^2\mathrm{d}V=\frac{4\pi ab^3c}{15},\iiint\limits_{\Omega}z^2\mathrm{d}V=\frac{4\pi abc^3}{15},$$

因此

$$\iiint\limits_{\Omega}\left(x+\frac{y}{2}+\frac{z}{3}\right)^2\mathrm{d}V=\frac{4\pi abc}{15}\left(a^2+\frac{b^2}{4}+\frac{z^2}{9}\right).$$

方法二(先二后一法)Ω 可以写成

$$\{(x,y,z)|(y,z)\in D_x,-a\leqslant x\leqslant a\},\ D_x=\left\{(y,z)|\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1-\frac{x^2}{a^2}\right\}.$$

$$\iiint\limits_{\Omega}x^2\mathrm{d}V=\int_{-a}^{a}x^2\mathrm{d}x\iint\limits_{D_x}\mathrm{d}\sigma=\int_{-a}^{a}x^2\pi bc\left(1-\frac{x^2}{a^2}\right)\mathrm{d}x=\frac{4\pi a^3bc}{15},$$

$$\iiint\limits_{\Omega}y^2\mathrm{d}V=\frac{4\pi ab^3c}{15},\iiint\limits_{\Omega}z^2\mathrm{d}V=\frac{4\pi abc^3}{15},$$

因此

$$\iiint\limits_{\Omega}\left(x+\frac{y}{2}+\frac{z}{3}\right)^2\mathrm{d}V=\frac{4\pi abc}{15}\left(a^2+\frac{b^2}{4}+\frac{z^2}{9}\right).$$

6. 利用 $\boldsymbol{\Gamma}$ 函数, $\mathbf{B}$ 函数计算积分 $\int_0^1\frac{\mathrm{d}x}{\sqrt{1-x^{\frac{1}{4}}}}$.

解 令 $x^{\frac{1}{4}}=u$, 则 $x=u^4, \mathrm{d}x=4u^3\mathrm{d}u$,

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^{\frac{1}{4}}}} &= \int_0^1 \frac{4u^3}{\sqrt{1-u}}\mathrm{d}u = 4\int_0^1 u^{4-1}(1-u)^{\frac{1}{2}-1}\mathrm{d}u \\ &= 4\mathbf{B}\left(4,\frac{1}{2}\right) = 4\frac{\boldsymbol{\Gamma}(4)\boldsymbol{\Gamma}\left(\frac{1}{2}\right)}{\boldsymbol{\Gamma}\left(\frac{9}{2}\right)} \\ &= \frac{4\times 3!\sqrt{\pi}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \frac{128}{35}.\end{aligned}$$