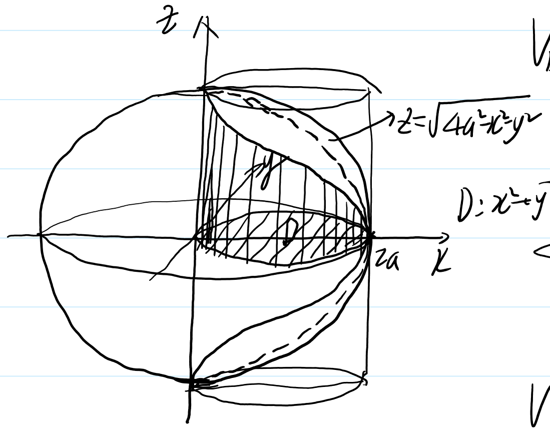
例 求 $x^2+y^2+z^2\le 4a^2$ 被 $x^2+y^2=2ax$ $(a>0)$ 所截得的含在其内部分立体体积



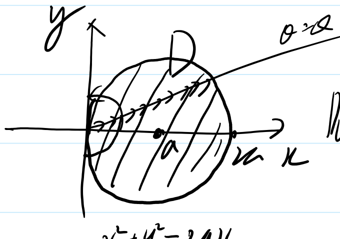
$$V_{曲顶柱体}=\iint_{D_{底}}\underbrace{f(x,y)}_{顶}\,d\sigma=\iint_D\sqrt{4a^2-x^2-y^2}\,d\sigma$$

$D: x^2+y^2\le 2ax$

用极坐标：在被积函数 $f(x,y)$ 中 或 D 的边界方程中 含 x^2+y^2 多用于极坐标

$$V=2\iint_D\sqrt{4a^2-x^2-y^2}\,d\sigma=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}\sqrt{4a^2-r^2}\,r\,dr$$

$r\,dr=-\frac{1}{2}d(4a^2-r^2)$



$D_\theta\begin{cases}-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\\0\le r\le 2a\cos\theta\end{cases}$

$x^2+y^2=2ax$

$r^2=2ar\cos\theta$

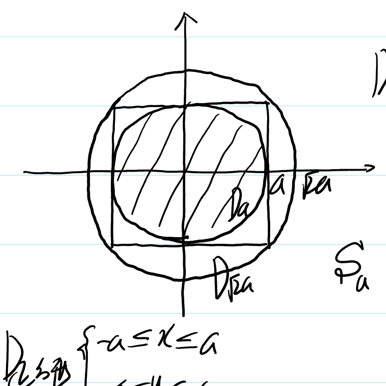
$r=2a\cos\theta$

$$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[-\frac{1}{2}\cdot\frac{1}{\frac{1}{2}+1}(4a^2-r^2)^{\frac{3}{2}}\Big|_0^{2a\cos\theta}\right]d\theta$$

$$=-\frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[(4a^2\sin^2\theta)^{\frac{3}{2}}-(4a^2)^{\frac{3}{2}}\right]d\theta$$

$$=-\frac{4}{3}\int_0^{\frac{\pi}{2}}\left[(2a\sin\theta)^3-(2a)^3\right]d\theta=\cdots\cdots$$

例(***) 计算 $\iint_{D_a}e^{-x^2-y^2}dxdy$ $D_a: x^2+y^2\le a^2$ $(a>0)$，并推导概率积分 $\int_{-\infty}^{+\infty}e^{-x^2}dx=\sqrt{\pi}$



$D_a\begin{cases}0\le\theta\le 2\pi\\0\le r\le a\end{cases}$

$$I_a=\iint_{D_a}e^{-x^2-y^2}dxdy=\iint_{D_{r\theta}}e^{-r^2}r\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^a e^{-r^2}r\,dr=2\pi\int_0^a e^{-r^2}\left(-\frac{1}{2}\right)d(-r^2)$$

$$=2\pi\cdot\left(-\frac{1}{2}\right)\left(e^{-r^2}\Big|_0^a\right)=\pi\left[1-e^{-a^2}\right]$$

$S_a\le S_{D_{正方形}}\le S_{D_{\sqrt{2}a}}$

$D_{正方形}\begin{cases}-a\le x\le a\\-a\le y\le a\end{cases}$

$$I_a=\iint_{D_a}e^{-x^2-y^2}dxdy\le\iint_{D_{正方形}}e^{-x^2-y^2}dxdy\le\iint_{D_{\sqrt{2}a}}e^{-x^2-y^2}dxdy=I_{\sqrt{2}a}$$

$$\pi[1-e^{-a^2}]=I_a\le\int_{-a}^{a}dx\int_{-a}^{a}e^{-x^2}e^{-y^2}dy=\left(\int_{-a}^{a}e^{-x^2}dx\right)^2\le I_{\sqrt{2}a}=\pi(1-e^{-2a^2})$$

$\downarrow$ π $\qquad\downarrow$ $\qquad\downarrow$ π

令 $a \to +\infty$ $\quad \downarrow \pi \qquad \downarrow (\int_{-\infty}^{+\infty} e^{-x^2}dx)^2 \qquad \downarrow \pi$

则有 $\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2 = \lim\limits_{a\to+\infty}\left(\int_{-a}^{a} e^{-x^2}dx\right)^2 = \pi \Rightarrow \int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$

$$\int_{0}^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

二重积分换元法

$$\int_a^b f(x)dx \xlongequal[\varphi'(t)\neq 0]{x=\varphi(t)\in C^1} \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$$

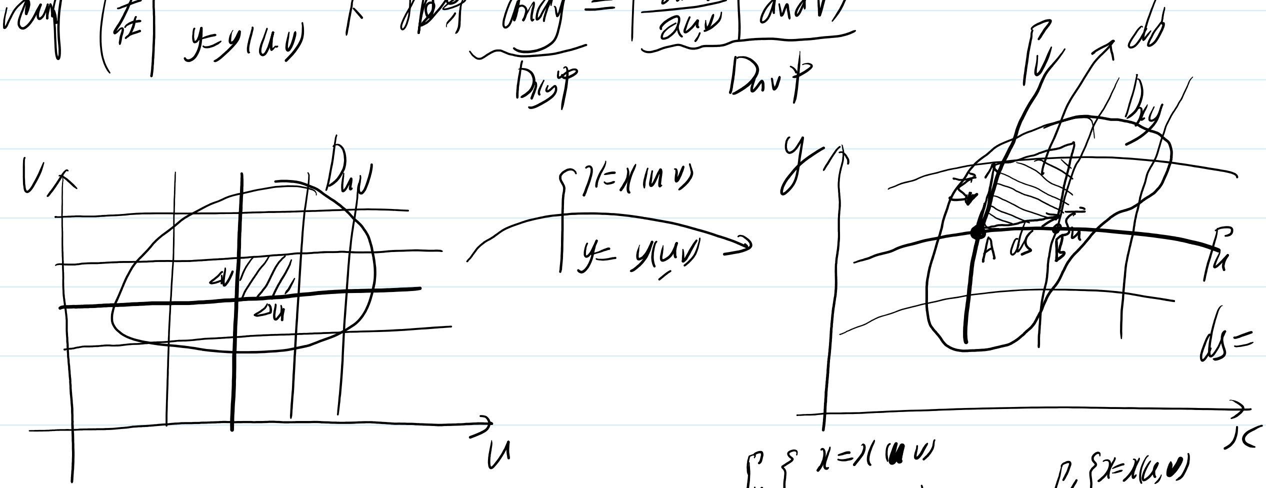
$\begin{cases} a=\varphi(\alpha) \\ b=\varphi(\beta) \end{cases}$ 一元函数换元法

定理 $\iint_D f(x,y)dxdy$ 存在 $f(x,y)$ 在D上连续，$\begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$，$\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ 则有

$$\iint_{D_{xy}} f(x,y)dxdy = \iint_{D_{uv}} f[x(u,v),y(u,v)]\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

只需证 $dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$

证明 (在 $\begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$ 下 推导 $\underbrace{dxdy}_{D_{xy}中} = \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv}_{D_{uv}中}$)



$ds = \overset{\frown}{AB} = |\vec{S_u}|$

$\Gamma_u \begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$ $\qquad \Gamma_v \begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$

$$\vec{S_u} = \left(\frac{x_u'}{\sqrt{x_u'^2+y_u'^2}}, \frac{y_u'}{\sqrt{x_u'^2+y_u'^2}}\right) ds = \left(\frac{x_u'}{\sqrt{x_u'^2+y_u'^2}}, \frac{y_u'}{\sqrt{x_u'^2+y_u'^2}}\right)\sqrt{x_u'^2+y_u'^2}\,du = (x_u', y_u')du$$

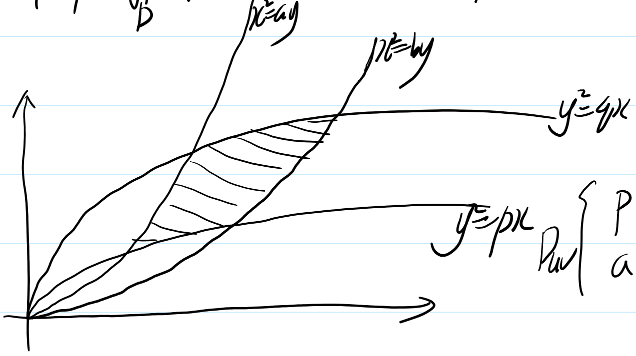
同理 $\vec{S_v} = \cdots\cdots = (x_v', y_v')dv$

$$dxdy = d\sigma = \left\|\vec{S_u}\times\vec{S_v}\right\| = \left\|\begin{vmatrix} x_u'du & y_u'du \\ x_v'dv & y_v'dv \end{vmatrix}\right\| = \left\|\begin{vmatrix} x_u' & x_v' \\ y_u' & y_v' \end{vmatrix}\right\| dudv$$

$$= \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$= \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

例 计算 $\iint_D \frac{y^2}{x} d\sigma$　D由 $y^2=px$，$y^2=qx$，$x^2=ay$　$x^2=by$　$(0<p<q,\ 0<a<b)$ 所围区域



$$\begin{cases} y^2=ux \\ x^2=vy \end{cases} \Rightarrow \begin{cases} x=\sqrt[3]{uv^2} \\ y=\sqrt[3]{u^2v} \end{cases}$$

$$D_{uv}\begin{cases} p<\frac{y^2}{x}=u<q \\ a<\frac{x^2}{y}=v<b \end{cases} \quad \frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix}=\begin{vmatrix} \frac13 u^{-\frac23}v^{\frac23} & \frac23 u^{\frac13}v^{-\frac13} \\ \frac23 u^{-\frac13}v^{\frac13} & \frac13 u^{\frac23}v^{-\frac23} \end{vmatrix}=-\frac13$$

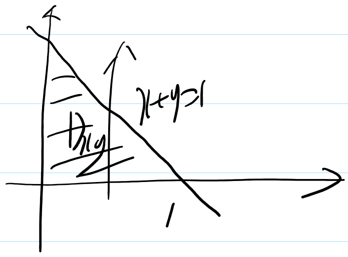
即 $$\iint_{D_{xy}} \frac{y^2}{x} d\sigma = \iint_{D_{uv}} \frac{y^2}{x}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_a^b dv\int_p^q u\,\frac13\,du = \cdots$$

$$e^{\frac{y+x-x}{x+y}} = e^{1-\frac{x}{x+y}}$$

例 计算 $\iint_D e^{\frac{y}{x+y}} d\sigma$　D由x轴、y轴、$x+y=1$ 围区　$\left(\int e^{x^2}dx \text{ 积不出}\quad \int e^{\frac1y} \text{ 积不出}\right)$

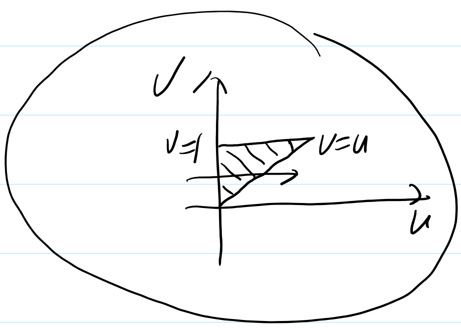
$$\begin{cases} u=y \\ v=x+y \end{cases} \Rightarrow \begin{cases} x=v-u \\ y=u \end{cases} \qquad \frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix}=\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}=-1$$

$$\iint_{D_{xy}} e^{\frac{y}{x+y}} dx\,dy = \iint_{D_{uv}} e^{\frac{u}{v}}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{D_{uv}} e^{\frac{u}{v}} du\,dv$$ (先积u 为D_v型)

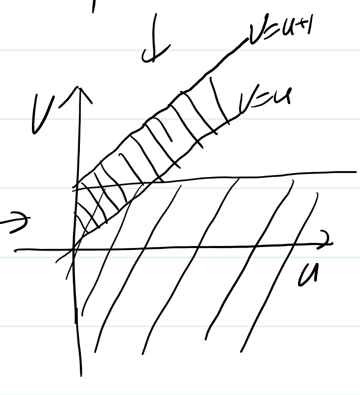


$$\begin{cases} 0\le x\le 1 \\ 0\le y\le 1-x \end{cases} \Leftrightarrow \begin{cases} 0\le v-u\le 1 \\ 0\le u\le 1-v+u \end{cases} \Rightarrow \begin{cases} u\le v\le u+1 \\ u\ge 0,\ v\le 1 \end{cases}$$

$\Downarrow$



$\longleftrightarrow$



$$D_v = \begin{cases} 0\le v\le 1 \\ 0\le u\le v \end{cases}$$

$$D_{uv} = \begin{cases} 0 \le v \le 1 \\ 0 \le u \le v \end{cases}$$

则 $$\iint_{D_{uv}} e^{\frac{u}{v}} = \int_0^1 dv \int_0^v e^{\frac{u}{v}} du = \int_0^1 \left(v e^{\frac{u}{v}} \Big|_0^v \right) dv$$

$$= \int_0^1 (ve - v) dv = \_\_\_\_$$