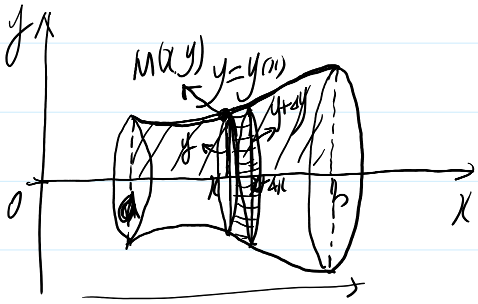
旋转体的侧面积



***微元法：求$[a,b]$上的某个量A

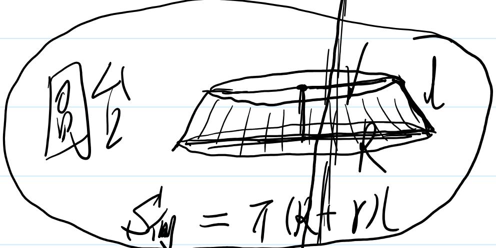
1) 微分 $\forall [x,x+\Delta x]\subset[a,b]$ 计算其上ΔA近似dA

$$\underbrace{\Delta A}_{误差为o(\Delta x)} \approx dA = A'(x)\Delta x = f(x)dx$$

2) 积分 $A=\int_a^b dA = \int_a^b f(x)dx$

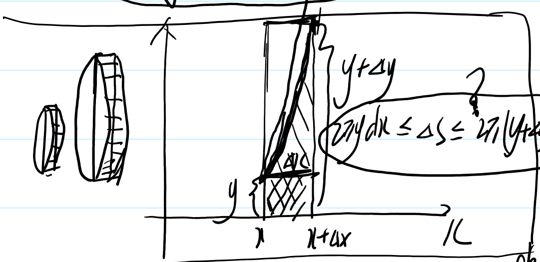
问题 求$[a,b]$上区域 $D:\begin{cases}0\le y\le y(x)\\ a\le x\le b\end{cases}$ 绕x轴旋转一周所生成旋转体的侧面积S

1) 微分 $\forall [x,x+\Delta x]\subset[a,b]$ 其上$\Delta S_侧$的近似值为dS 则有



$l=ds=\sqrt{1+y'^2(x)}\,dx$; $r=y$　$R=y+\Delta y$

则有 $\Delta S\approx dS=\pi(y+\Delta y+y)ds = 2\pi y\,ds + \underbrace{\pi\Delta y\,ds}_{误差=o(\Delta x)}$

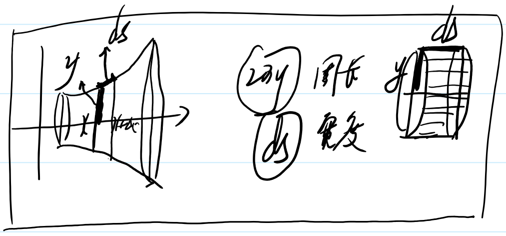


$2\pi y\,dx\le\Delta S\le 2\pi(y+\Delta y)dx$?

其中 $\pi\Delta y\,ds \approx \pi dy\,ds = \pi y'(x)\Delta x\sqrt{1+y'^2(x)}\,\Delta x$
$=\pi y'\sqrt{1+y'^2}\,\Delta x^2 = o(\Delta x)$

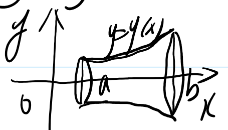
即 $dS=2\pi y\,ds$

2) 积分 $S_侧=\int_a^b dS=\int_a^b 2\pi y\,ds$ → 记



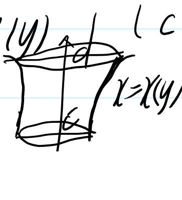
即有

$y=y(x)$ $(a\le x\le b)$ 绕x轴旋转一周所生成的旋转体侧面积为

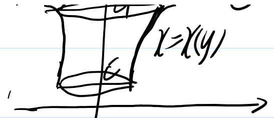


$$S_{x侧}=2\pi\int_a^b y\,ds \qquad (y=y(x),\ ds=\sqrt{1+y'^2(x)}\,dx)$$

$x=x(y)$ $(c\le y\le d)$ 绕y轴旋转一周生成旋转体侧面积为



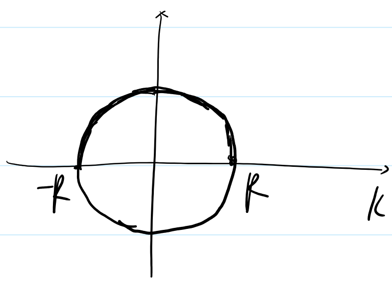
$$S_{y侧}=2\pi\int_c^d x\,ds \qquad (x=x(y),\ ds=\sqrt{1+x'^2(y)}\,dy)$$



$$S_{y轴} = 2\pi\int_c^d x\,ds \quad (x=x(y) \quad ds=\sqrt{1+x'^2(y)}\,dy)$$

参数方程 $\begin{cases} x=x(t) \\ y=y(t) \end{cases} \alpha\le t\le\beta$ $S_{x轴}=2\pi\int_\alpha^\beta y\,ds$ 其 $y=y(t)$ $ds=\sqrt{x'^2(t)+y'^2(t)}\,dt$	极坐标 $r=r(\theta)$ ⟶ 化成参数式 $\begin{cases} x=r(\theta)\cos\theta \\ y=r(\theta)\sin\theta \end{cases} \alpha\le\theta\le\beta$ 参照参数方程即可 $ds=\sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$

例 求 $x^2+y^2=R^2$ 绕 x 轴旋转一周生成的旋转体的侧面积



$y_上=\sqrt{R^2-x^2}$ $\left(\begin{cases} x=R\cos t \\ y=R\sin t \end{cases} (0\le t\le\pi), \quad r=R_常 \right)$

$$S_{x轴}=2\pi\int_{-R}^{R} y\,ds = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\,\sqrt{1+y'^2(x)}\,dx$$

$$=2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\,\sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2}\,dx$$

$$=2\pi\int_{-R}^{R}\cancel{\sqrt{R^2-x^2}}\,\frac{R}{\cancel{\sqrt{R^2-x^2}}}\,dx = 2\pi R\cdot 2R = 4\pi R^2$$

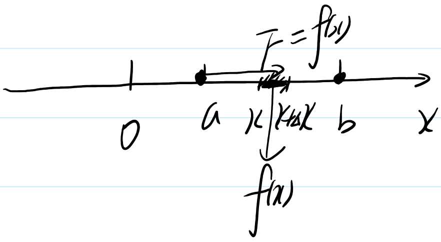
例 $r=a(1+\cos\theta)$ $\theta\in[0,\pi]$ 绕极轴旋转一周生成旋转体的侧面积

$\left(\frac{32}{5}\pi a^2\right)$

三 定积分的物理应用 $\begin{cases} ①变力作功 \\ ②液体静压力 \\ ③引力 \\ ④重心、质心、形心 \end{cases}$

→ $\begin{cases} 古尔丁第一定理 \\ 古尔丁第二定理 \end{cases}$

变力作功： 求 $F=f(x)$ 在 $x=a$ 到 $x=b$ 上所作的功.



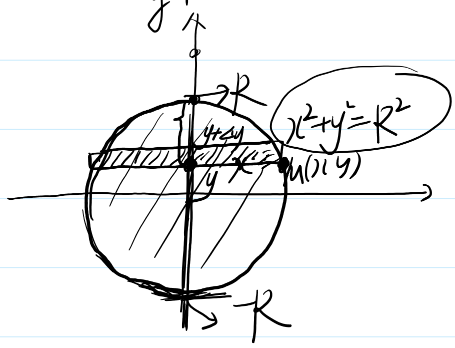
问题：求 $[a,b]$ 上的功 W

1) 微分 $\forall[x,x+dx]\subset[a,b]$，计算其上 ΔW 近似 dW

$$\Delta W\approx dW=f(x)\cdot dx$$

2) 积分 $W=\int_a^b dW=\int_a^b f(x)\,dx$

例　半径为R的球形容器中，充满了液体，其密度为ρ，求将液体由球的顶部抽出，至少要作多少功.



问题，求$[-R,R]$上的功W

1) 微分 $\forall[y,y+dy]\subset[-R,R]$，计算其上$\Delta W$近似$dW$

$$\Delta W\approx dW=\rho\cdot\pi x^2dy\cdot g(R-y)$$

2) 积分 $W=\int_{-R}^{R}dW=\int_{-R}^{R}\rho g\pi x^2(R-y)dy$

$$=\int_{-R}^{R}\rho g\pi(R^2-y^2)(R-y)dy=\cdots\cdots$$

例(课外)(99年)井底清除淤泥，用缆绳将抓斗放入井底，抓起污泥提至井口，井深30m，抓斗自重400N，缆绳重50N/m，一斗泥重2000N，提速3m/s，漏速20N/s，现将一斗泥由井底提至井口，问克服重力作功多少？

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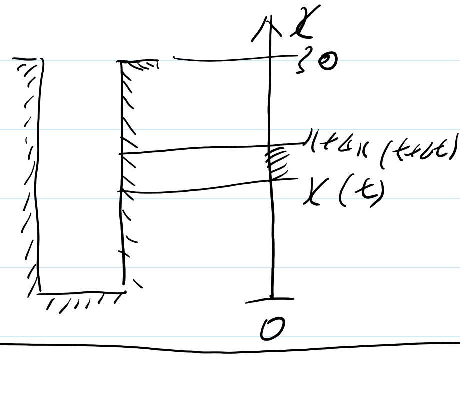
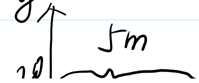
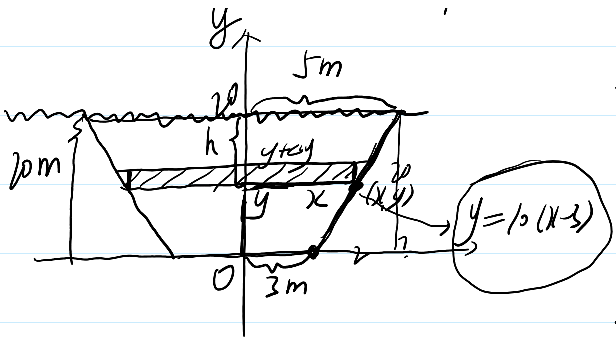


液体静压力

例　有一梯形闸门垂直立于水中，且闸门上边缘与水平齐平，闸门上边长5m，下边长为6m，闸门高20m，求闸门一侧所受液体静压力



求$[0,20]$上的压力F



求[0,20]上的压力F

1) 微分 $\forall [y, y+dy] \subset [0, 20]$ 计算其上ΔW近似dW

$$\Delta W \approx dW = \rho g(20-y)\cdot 2x\,dy$$

2) 积分 $W=\int_0^{20} dW = \int_0^{20} 2\rho g(20-y)x\,dy = \int_0^{20} 2\rho g(20-y)(\frac{y}{10}+3)\,dy \approx 1.4373\times 10^{7}\,N$

引力问题 $\left(F_{引} = k\frac{m_1 m_2}{r^2}\right)$

$\vec{F}_{引} = |\vec{F}|\vec{F^0}$

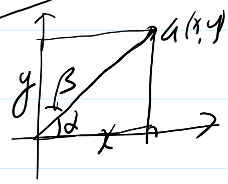
模长

$\vec{F} = |\vec{F}|\vec{F^0} = |\vec{F}|\left(\frac{x}{|\vec{r}|}, \frac{y}{|\vec{r}|}\right)$

$= |\vec{F}|(\cos\alpha, \cos\beta)$

$= (|\vec{F}|\cos\alpha, |\vec{F}|\cos\beta)$

水平 垂直



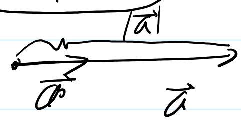
$\vec{a} = (x, y)$

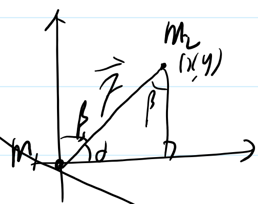
$\vec{a^0} = \left(\frac{x}{|\vec{a}|}, \frac{y}{|\vec{a}|}\right)$

$= (\cos\alpha, \cos\beta)$

方向余弦

$\vec{a} = |\vec{a}|\vec{a^0}$





$\vec{a} = |\vec{a}|\vec{a^0} = |\vec{a}|(\cos\alpha, \cos\beta) = (\vec{a}\cos\alpha, \vec{a}\cos\beta)$

$\vec{F} = |\vec{F}|\cdot\vec{F^0} = |\vec{F}|(\cos\alpha, \cos\beta) = (|\vec{F}|\cos\alpha, |\vec{F}|\cos\beta)$

$= (\vec{F}_{水平}, \vec{F}_{垂直})$