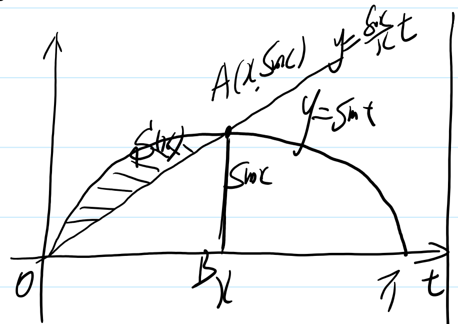
一、2) $A(x,\sin x)$ 为 $y=\sin x\ (0\le x\le \pi)$ 上一个点 设直线OA与 $\sin x$ 所围面积为 $S(x)$

问 $S(x)$ 当 $x\to 0$ 时为 x 的几阶无穷小

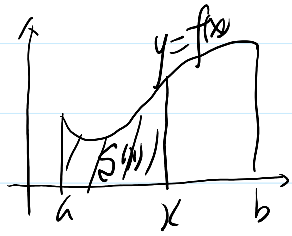


$$S(x)=\int_0^x\left(\sin t-\frac{\sin x}{x}t\right)dt$$

$$=1-\cos x-\frac{\sin x}{x}\cdot\frac{1}{2}t^2\Big|_0^x$$

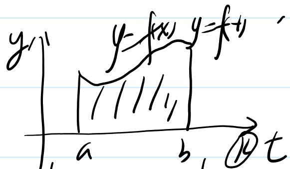
$$=1-\cos x-\frac{1}{2}x\sin x$$

$$S(x)=\int_0^x\left(\sin x-\frac{\sin x}{x}x\right)dx$$ ⊗ 错在积分上限函数定义不规范



$S(x)=\int_0^x f(x)\,dx$ 变量 ×

$S(x)=\int_0^x f(t)\,dt$



定积分的值与积分变量的选取无关

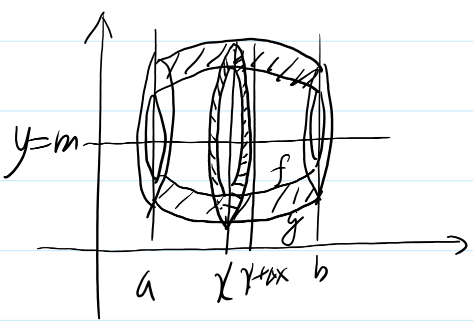
$$S=\int_a^b f(x)dx=\int_a^b f(t)dt$$

$$=1-\left[1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\frac{1}{6!}x^6+o(x^6)\right]-\frac{1}{2}x\left[x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+o(x^5)\right]$$

$$=\left(-\frac{1}{24}+\frac{1}{12}\right)x^4+\square x^5+\square x^6+o(x^4)=\frac{1}{24}x^4+o(x^4)$$

$$\lim_{x\to0}\frac{S(x)}{\frac{1}{24}x^4}=\lim_{x\to0}\left(\frac{\frac{1}{24}x^4}{\frac{1}{24}x^4}+\frac{o(x^4)}{x^4}\right)=1 \qquad S(x)\sim\frac{1}{24}x^4$$

3) 设 $0<g(x)<f(x)<m$ (常) $y=f(x)$, $y=g(x)$, $x=a$, $x=b$ 围成图形绕 $y=m$ 旋转体 V



$$V=V_2-V_1=\int_a^b\pi(g-m)^2dx-\int_a^b\pi(f-m)^2dx$$

$$=\pi\int_a^b\left[(g-m)^2-(f-m)^2\right]dx$$

$$=\cdots\cdots$$

法二: $V=\int_a^b A(x)dx=\int_a^b\left[\pi(f-m)^2-\pi(g-m)^2\right]dx$ ……

$\int_a \qquad \int_a$

5) $f(x)=\begin{cases}\dfrac{1}{(x-1)^{\alpha+1}} & 1<x<e \\ \dfrac{1}{x\ln^{\alpha+1}x} & x\geq e\end{cases}$

$\int_1^{+\infty} f(x)dx$ 收敛 求 α 范围

$=\int_1^e + \int_e^{+\infty}$

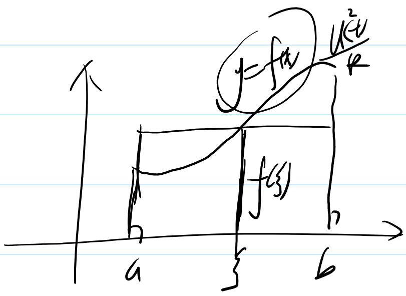
$$\int_1^e \frac{1}{(x-1)^{\alpha+1}}dx = \int_1^e \frac{1}{(x-1)^{\alpha+1}}d(x-1) = \int_0^{e-1}\frac{1}{x^{\alpha+1}}dx$$

$$\int_e^{+\infty}\frac{1}{x\ln^{\alpha+1}x}dx = \int_e^{+\infty}\frac{1}{\ln^{\alpha+1}x}d\ln x = \int_1^{+\infty}\frac{1}{x^{\alpha+1}}dx$$

$\mu=\alpha+1<1$ 收

$\mu=\alpha+1>1$ 收

二 1) 连续函数平均值，定义 $f(\xi)=\dfrac{\int_a^b f(x)dx}{b-a}$ 为连续函数 $f(x)$ 在 $[a,b]$ 上平均值



$$f(\xi)=\frac{\int_1^{\sqrt5}\frac{1}{1+x^2}dx}{\sqrt5-1}$$

$$\int_a^b f(x)dx = f(\xi)(b-a) \Longleftrightarrow f(\xi)=\frac{\int_a^b f(x)dx}{b-a}$$

$$W=\frac{\int_a^b \frac{u^2(t)}{R}dt}{b-a} = \frac{u^2(\xi)}{R}$$

$u(\xi)$

例4 $I=\int_0^{+\infty} xe^{ax^2}dx = \int_0^{+\infty}\frac{1}{2a}e^{ax^2}dax^2 = \frac{1}{2a}e^{ax^2}\Big|_0^{+\infty}$

$=-\frac{1}{2a}$

$\int_0^{+\infty}e^{-x^2}dx=\frac{\sqrt\pi}{2}$

概率积分

另一题 $I=\int_0^{+\infty}e^{ax^2}dx$

$$I=\int_0^{+\infty}e^{ax^2}dx = \int_0^{+\infty}\frac{1}{\sqrt a}e^{-(\sqrt a x)^2}d\sqrt a x$$

$$\overset{u=\sqrt a x}{=\!=\!=} \frac{1}{\sqrt a}\int_0^{+\infty}e^{-u^2}du = \frac{1}{\sqrt a}\cdot\frac{\sqrt\pi}{2}$$

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5) $\int_0^{+\infty}\frac{\ln(1+x)}{(1+x)^2}d(x+1) = \int_0^{+\infty}-\ln(1+x)\,d\frac{1}{1+x}$

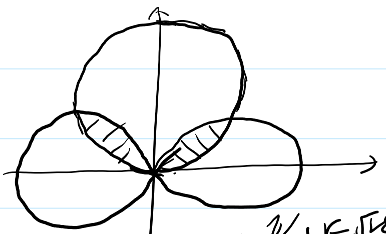
$\frac{1}{(1+x)^2}d(x+1)=-d\frac{1}{x+1}$

5) $\int_0 \frac{}{(1+x)^2}\, d(x+1) = \int_0 -\ln(1+x)\, d\frac{1}{1+x}$ $\quad \frac{1}{(1+x)^2}d(x+1) = -d\frac{1}{x+1}$

分部积分

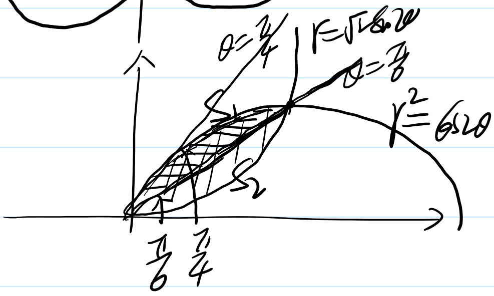
习题6.4 A

1 (4) 求 $r=\sqrt{2}\sin\theta$ 与 $r^2=\cos 2\theta$ 所围面积 $\quad \begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$



$r^2=\sqrt{2}\,r\sin\theta$

$x^2+y^2=\sqrt{2}y \quad x^2+(y-\frac{\sqrt{2}}{2})^2=(\frac{\sqrt{2}}{2})^2$



$S=2(S_1+S_2)$

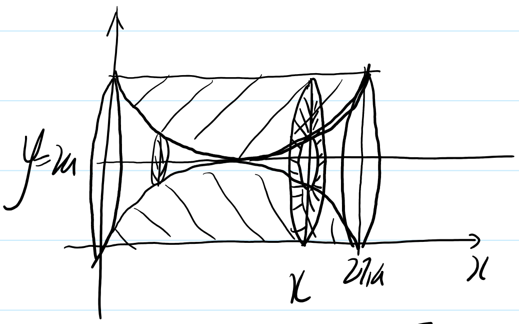
$r^2=\cos2\theta=0 \Rightarrow 2\theta=\frac{\pi}{2} \quad \theta=\pm\frac{\pi}{4}$

$r^2=\cos2\theta = r^2=2\sin^2\theta \quad 1-2\sin^2\theta=2\sin^2\theta \Rightarrow \sin\theta=\pm\frac{1}{2}$

$\theta=\pm\frac{\pi}{6}$

$S_1=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{1}{2}\cos2\theta\, d\theta, \quad S_2=\int_0^{\frac{\pi}{6}}\frac{1}{2}(2\sin^2\theta)\, d\theta$

习题6.4 A.6 求 $\begin{cases}x=a(t-\sin t)\\ y=a(1-\cos t)\end{cases}$ $(0\le t\le 2\pi)$ 与 $y=0$ 围区域绕 $y=2a$ 旋转体 V

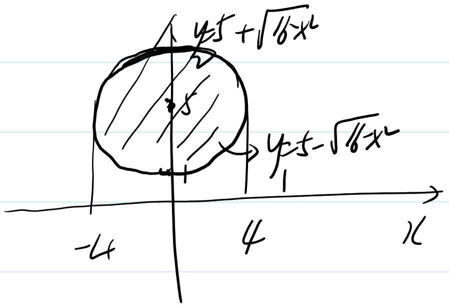


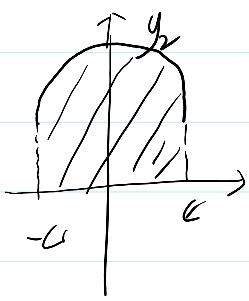
$\int_0^{2\pi a}\pi(2a)^2dx$

法一：$V_{y=2a}=V_2-V_1=\pi(2a)^2\cdot 2\pi a-\int_0^{2\pi a}\pi(y-2a)^2dx$

法二：$V=\int_0^{2\pi a}A(x)dx=\int_0^{2\pi a}[\pi(2a)^2-\pi(y-2a)^2]\,dx$

7.





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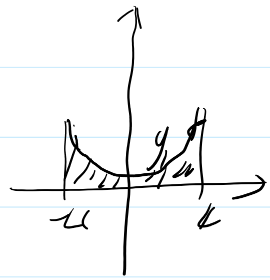
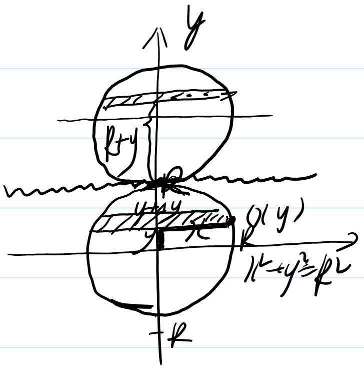


$x^2+(y-5)^2=16$

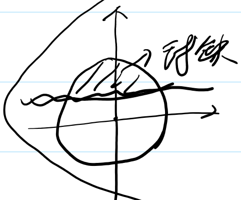
$V=\pi\int_{-4}^{4}(y_2^2-y_1^2)\,dx$

$y=5\pm\sqrt{16-x^2}$

12 半径为R球沉入水中 球密度与水密度相同(密度均为1) 于将球由水中取出至少要作多少功



法一：



变力作功，需正球缺体积公式

$f(x)=\rho V g \quad W=\int f(x)dx$

法二：求$[-R,R]$上功W

1) $[y,y+dy]\subset[-R,R]$ 计算ΔW近似dW

$\Delta W\approx dW=\rho\pi x^2 dy\cdot g\cdot(R+y)$

$=\rho g\pi(R^2-y^2)(R+y)dy$

2) $W=\int_{-R}^{R}dW=\int_{-R}^{R}\rho g\pi(R^2-y^2)(R+y)dy=$ ——

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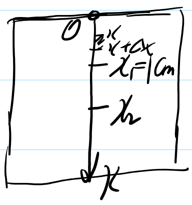
14)



$F_{阻}(x)=kx \qquad W_1=\int_0^1 kx\,dx=\frac{1}{2}kx^2\Big|_0^1=\frac{1}{2}k$

$W_1=\frac{1}{2}k=W_2=\int_{x_1=1}^{x_2}kx\,dx=\frac{1}{2}kx^2\Big|_{x_1=1}^{x_2}$

$=\frac{1}{2}k(x_2^2-1^2)$

$h=x_2-x_1$

$W+rW+r^2W\cdots+r^{n}W$

$W_1+W_2+\cdots+W_n=\int_0^{x_1}kx\,dx+\int_{x_1}^{x_2}kx\,dx+\text{——}+\int_{x_{n-1}}^{x_n}kx\,dx$

$=\int_0^{x_n}kx\,dx$

习题6.5 A 2. $\int_2^{+\infty}\frac{dx}{x\ln^k x}\ \underline{\underline{\frac{1}{x}dx=d\ln x}}\ \int_2^{+\infty}\frac{1}{\ln^k x}d\ln x\ \underline{\underline{u=\ln x}}\ \int_{\ln 2}^{+\infty}\frac{1}{u^k}du$

$k>1$收，$k\leq1$发

A 3. (2) $\lim_{x\to+\infty}x^{n}\frac{\arctan x}{x}\ \underline{\underline{n=1}}\ \frac{\pi}{2}$

(3) $\int^{+\infty}\frac{\sin x}{\ }dx \qquad \left|\frac{\sin x}{\ }\right|<\frac{1}{\sqrt{\ }}$

(3) $\int_1^{+\infty} \frac{\sin x}{\sqrt{x^3}} dx$ $\quad \left|\frac{\sin x}{\sqrt{x^3}}\right| \le \frac{1}{x^{\frac{3}{2}}}$

(4) $\int_0^{+\infty} \frac{x^m}{1+x^n} dx \ (n\ge 0, m\ge 0)$ $\quad \lim_{x\to+\infty} x^\mu \cdot \frac{x^m}{x^n+1} = \lim_{x\to+\infty}\frac{x^{\mu+m}}{x^n+1} \overset{\mu+m=n}{=} 1$

$\mu = n-m>1$ 收

A 4 $\int_{-\infty}^{+\infty} \frac{x}{1+x^2}dx \overset{?}{=} \lim_{a\to+\infty}\int_{-a}^{a}\frac{x}{1+x^2}dx = 0$ ✗ $\to$ Cauch主值收敛

应 $\forall c\in\mathbb{R}$ $\quad \int_{-\infty}^{c}\frac{x}{1+x^2}dx + \int_c^{+\infty}\frac{x}{1+x^2}dx$

$= \lim_{a\to\infty}\int_a^c \frac{x}{1+x^2}dx + \lim_{b\to+\infty}\int_c^b \frac{x}{1+x^2}dx$ ✓

5. $\int_1^{+\infty}\frac{1}{x(1+x)}dx = \int_1^{+\infty}\frac{1}{x} - \frac{1}{1+x}dx$

$= \lim_{b\to+\infty}\left[\ln x - \ln(1+x)\right]\Big|_1^b = \ln 2$ ✓

$\int_1^{+\infty}\frac{1}{x(1+x)}dx \neq \lim_{b\to+\infty}\ln x\Big|_1^b - \lim_{b\to+\infty}\ln(1+x)\Big|_1^b = \infty-\infty$ ✗ 发散

B 3 (1) $\int_0^{\frac{\pi}{2}}\frac{dx}{\sin^p x\cos^q x}\ (p>0, q>0) = \int_0^{\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

记一: $\int_0^{\frac{\pi}{4}}\frac{dx}{\sin^p x\cos^q x}$ $\quad \lim_{x\to 0^+}x^\mu\cdot\frac{1}{\sin^p x\cos^q x}\overset{\mu=p}{=}1$ $\quad \int_0^1\frac{1}{x^\mu}dx$ $\quad \mu=p<1$ 收 $q\in\mathbb{R}$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{\sin^p x\cos^q x}$ $\quad \lim_{x\to\frac{\pi}{2}^-}\left(\frac{\pi}{2}-x\right)^\mu\cdot\frac{1}{\sin^p x\cos^q x} = \lim_{x\to\frac{\pi}{2}^-}\left(\frac{\pi}{2}-x\right)^\mu\frac{1}{\sin^q\left(\frac{\pi}{2}-x\right)}\overset{\mu=q}{=}1$ $\quad \mu=q<1$ 收 $p\in\mathbb{R}$

记二: $B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$ 三叉点 $(a>0, b>0)$

注: $B(a,b) = \int_0 x^{\cdot}(1-x)\,dx \quad \ldots(a>0,\ b>0)$

$$\overset{x=\sin^2 t}{=\!=\!=} \int_0^{\frac{\pi}{2}} 2\sin^{2a-1}t\,\cos^{2b-1}t\,dt$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sin^p x\,\cos^q x}\,dx = \int_0^{\frac{\pi}{2}} \sin^{-p}x\,\cos^{-q}x\,dx$$

$-p = 2a-1 \Rightarrow 1-p = 2a > 0$

$-q = 2b-1 \qquad 1-q = 2b > 0$