多元微分24-1

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一元函数

$\xi = x_0+\theta(x-x_0)\quad \theta\in(0,1)$

定理1 设 $f(x)\in C^{(n+1)}U(x_0)$，则 $f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}[x_0+\theta(x-x_0)]}{(n+1)!}(x-x_0)^{n+1}\quad \theta\in(0,1)$

设 $x=x_0+\Delta x$　$dx=\Delta x$

$dy=f'(x_0)dx$
$d^2y=f''(x_0)dx^2$
$\vdots$
$d^ny=f^{(n)}(x_0)dx^n$

或 $\boxed{f(x_0+\Delta x)=f(x_0)+f'(x_0)\Delta x+\frac{f''(x_0)}{2!}\Delta x^2+\cdots+\frac{f^{(n)}(x_0)}{n!}\Delta x^n+\frac{f^{(n+1)}(x_0+\theta\Delta x)}{(n+1)!}\Delta x^{n+1}}$ Lagrange

或 $f(x_0+\Delta x)=f(x_0)+df(x_0)+\frac{1}{2!}d^2f(x_0)+\cdots+\frac{1}{n!}d^nf(x_0)+\frac{1}{(n+1)!}d^{n+1}f(x_0+\theta\Delta x)$

定理2 设 $f^{(n)}(x_0)$ 存在，则 $f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n)$

或 $\boxed{f(x_0+\Delta x)=f(x_0)+f'(x_0)\Delta x+\frac{f''(x_0)}{2!}\Delta x^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(\Delta x)^n+o(\Delta x^n)}$ Peano型

二元函数

$z=f(x,y)\quad \begin{matrix}dx=\Delta x\\ dy=\Delta y\end{matrix}\quad dz=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f$

定理1 设 $f(x,y)\in C^{(n+1)}U(P_0)$，则

$$f(x_0+\Delta x,y_0+\Delta y)=f(x_0,y_0)+\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2f(x_0,y_0)+\cdots+\frac{1}{n!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^nf(x_0,y_0)+\frac{1}{(n+1)!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^{n+1}f(x_0+\theta\Delta x,y_0+\theta\Delta y)$$

$\theta\in(0,1)$

或 $\boxed{f(P_0+\Delta P)=f(P_0)+df(P_0)+\frac{1}{2!}d^2f(P_0)+\cdots+\frac{1}{n!}d^nf(P_0)+\frac{1}{(n+1)!}d^{n+1}f(P_0+\theta\Delta P)\quad \theta\in(0,1)}$

证明 记 $\varphi(t)=f(x_0+t\Delta x,y_0+t\Delta y)$，则 $\varphi(0)=f(x_0,y_0)$，$\varphi(1)=f(x_0+\Delta x,y_0+\Delta y)$，$\varphi^{(n)}(t)=\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^nf(x_0+t\Delta x,y_0+t\Delta y)$

Maclaurin 即 $t=0$ 处 Taylor

$\varphi(t)=\varphi(0)+\varphi'(0)t+\frac{1}{2!}\varphi''(0)t^2+\cdots+\frac{1}{n!}\varphi^{(n)}(0)t^n+\frac{1}{(n+1)!}\varphi^{(n+1)}(\theta t)t^{n+1}\quad \theta\in(0,1)$

令 $t=1$　$\varphi(1)=\varphi(0)+\varphi'(0)+\frac{1}{2!}\varphi''(0)+\cdots+\frac{1}{n!}\varphi^{(n)}(0)+\frac{1}{(n+1)!}\varphi^{(n+1)}(\theta)\quad \theta\in(0,1)$

则 $f(x_0+\Delta x,y_0+\Delta y)=f(x_0,y_0)+\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2f(x_0,y_0)+\cdots+\frac{1}{n!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^nf(x_0,y_0)+\frac{1}{(n+1)!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^{n+1}f(x_0+\theta\Delta x,y_0+\theta\Delta y)$

定理2 设 $f(x,y)\in C^{(n)}U(P_0)$，则 $f(x_0+\Delta x,y_0+\Delta y)=f(x_0,y_0)+\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2f(x_0,y_0)+\cdots+\frac{1}{n!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^nf(x_0,y_0)+o(\rho^n)\quad(\rho=\sqrt{\Delta x^2+\Delta y^2})$

$\varphi(t)=f(\underbrace{x_0+t\Delta x}_{x},\underbrace{y_0+t\Delta y}_{y})\qquad \varphi(0)=f(x_0,y_0)\qquad \varphi(1)=f(x_0+\Delta x,y_0+\Delta y)$

$z=f(x,y)$

$\varphi'(t)=\frac{\partial f}{\partial x}\cdot\Delta x+\frac{\partial f}{\partial y}\Delta y=\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f$

$\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)$

$\varphi'(t) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = \left(\Delta x\frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) f$

$\varphi''(t) = \left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)\varphi'(t) = \left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 f$

$\vdots$

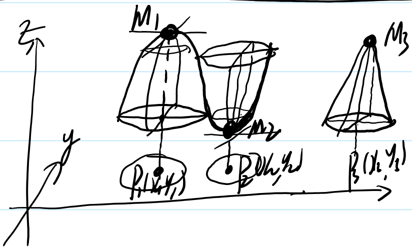
$\varphi^{(n)}(t) = \left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^n f(x_0+t\Delta x,\ y_0+t\Delta y)$

$\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)$

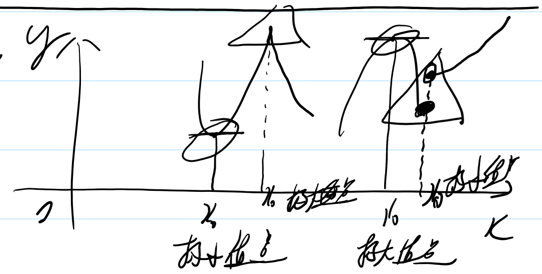
$u=f(x,y,z)$

$\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}+\Delta z\frac{\partial}{\partial z}\right)$

多元函数极值



一元



定义

$z=f(x,y)$ $\begin{cases} \forall P\in \mathring{U}(P_0) \Rightarrow f(P)>f(P_0),\ f(P_0)\text{极小值} \\ \forall P\in \mathring{U}(P_0) \Rightarrow f(P)>f(P_0),\ f(P_0)\text{极大值} \end{cases}$

定理 可微函数 $f(x)$ 在 x_0 点取极值，则 $f'(x_0)=0$

称 $x=x_0$ 为 $f(x)$ 驻点或稳定点

可能取极值的点：驻点及不可导点

定理 可微函数 $f(x,y)$ 在 $P_0(x_0,y_0)$ 处取极值

则 $\Rightarrow \begin{cases} f'_x(P_0)=0 \\ f'_y(P_0)=0 \end{cases}$ 或 $\nabla f(P_0)=0$ 此时称 $P=P_0(x_0,y_0)$ 为 $z=f(x,y)$ 驻点（稳定点）　$y=|x|$

驻点及不可偏导的点处 $f(x,y)$ 可能取极值　$z=\sqrt{x^2+y^2}$

函数取极值充分条件（只有二阶导数）

$f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0) + \left(\frac{\partial}{\partial x}\Delta x+\frac{\partial}{\partial y}\Delta y\right)f(x_0,y_0) + \frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) + o(\rho^2)$,　$(\rho=\sqrt{\Delta x^2+\Delta y^2})$

定理 $z=f(x,y)\in C^{(2)}U(P_0)$ 且 $\begin{cases} f'_x(P_0)=0 \\ f'_y(P_0)=0 \end{cases}$，$P_0(x_0,y_0)$，

记 $A=f''_{xx}(P_0)$　$B=f''_{xy}(P_0)$　$C=f''_{yy}(P_0)$　则有

1) $AC-B^2>0$ 时 取极值 $\begin{cases} A>0,\ f(P_0)\text{取极小值} \\ A<0,\ f(P_0)\text{取极大值} \end{cases}$

2) $AC-B^2<0$ 时 不取极值

3) $AC-B^2=0$ 时 不确定 $f(P_0)$ 是否取极值（需用其它方法）

一元判别 $\begin{cases} \text{一阶导数} \\ \text{二阶导数} \\ n\text{阶导数} \end{cases}$

定理：$f''(x_0)$ 存在 $f'(x_0)=0$，$f''(x_0)=\begin{cases} + & f(x_0)\text{极小值} \\ - & f(x_0)\text{极大值} \\ 0 & \text{不确定} \end{cases}$

$f(x_0+\Delta x)=f(x_0)+f'(x_0)\Delta x+f''(x_0)\Delta x^2+o(\Delta x^2)$

$\boxed{f(x_0+\Delta x)-f(x_0) = f''(x_0)\Delta x^2+o(\Delta x^2)}$

$\Delta x\to 0$ 时 上式正负由等号右侧 $f''(x_0)$ 决定

Thm: $f(x,y)\in C^{(2)}U(P_0)$ 且 $\left.\frac{\partial f}{\partial x}\right|_{P_0}=\left.\frac{\partial f}{\partial y}\right|_{P_0}=0$，则由Taylor式

$$f(x_0+\Delta x,\ y_0+\Delta y)=f(x_0,y_0)+\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+o(\rho^2)\quad \left(\rho=\sqrt{\Delta x^2+\Delta y^2}\right)$$

$$\Rightarrow f(x_0+\Delta x,\ y_0+\Delta y)-f(x_0,y_0)=\frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+o(\rho^2)$$

$\rho\to 0$ 时 上式正负由等号右侧第一项决定

而此项为 $g(\Delta x,\Delta y)=\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$

$$=\underbrace{\frac{\partial^2}{\partial x^2}f(x_0,y_0)}_{A}\Delta x^2+2\underbrace{\frac{\partial^2}{\partial x\partial y}f(x_0,y_0)}_{B}\Delta x\Delta y+\underbrace{\frac{\partial^2}{\partial y^2}f(x_0,y_0)}_{C}\Delta y^2$$

$$=A\Delta x^2+2B\Delta x\Delta y+C\Delta y^2=(\Delta x,\Delta y)\begin{pmatrix}A & B\\ B & C\end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}$$ 为二次型

Sylvester 西尔维斯特定理

- $\begin{pmatrix}A & B\\ B & C\end{pmatrix}$ 正定 $\Leftrightarrow$ $A>0$ 且 $\begin{vmatrix}A & B\\ B & C\end{vmatrix}=AC-B^2>0$ $\Rightarrow f(x_0,y_0)$为极小值
- $\begin{pmatrix}A & B\\ B & C\end{pmatrix}$ 负定 $\Leftrightarrow$ $A<0$ 且 $\begin{vmatrix}A & B\\ B & C\end{vmatrix}=AC-B^2>0$ $\Rightarrow f(x_0,y_0)$为极大值
- $\begin{pmatrix}A & B\\ B & C\end{pmatrix}$ 不定 $\Leftrightarrow$ $\begin{vmatrix}A & B\\ B & C\end{vmatrix}<0$ $\Rightarrow f(x_0,y_0)$不是极值
- $\begin{pmatrix}A & B\\ B & C\end{pmatrix}$ 半正定 或 半负定，$f(x_0,y_0)$不确定

$$u=f(x,y,z)\longleftrightarrow\begin{pmatrix}f''_{xx} & f''_{xy} & f''_{xz}\\ f''_{xy} & f''_{yy} & f''_{yz}\\ f''_{xz} & f''_{zy} & f''_{zz}\end{pmatrix}$$

- 正定 极小值（顺序主子式全大于0）
- 负定 极大值（顺序主子式负正相间）
- 不定 不是极值（不定阵）