# 习题 空间解析几何12-1

2022年4月8日 8:00

习题6.5

B 1. $\int_0^{+\infty} e^{-x^2} = \frac{\sqrt{\pi}}{2}$. 且 $\int_{-\infty}^{+\infty} A e^{-x^2-x} dx = 1$ 求A

$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$, $1 = \int_{-\infty}^{+\infty} A e^{-(x+\frac{1}{2})^2} \cdot e^{\frac{1}{4}} d(x+\frac{1}{2}) = A \cdot e^{\frac{1}{4}} \cdot \sqrt{\pi}$

B 2. $\int_1^{+\infty} \cos x \sin\frac{1}{x^2} dx$ $x \to +\infty$, $|\cos x \sin\frac{1}{x^2}| \le \sin\frac{1}{x^2} \le \frac{1}{x^2}$

B 3. (1) $\int_0^{\frac{\pi}{2}} \frac{\ln\sin x}{\sqrt{x}} dx$ $\lim_{x\to 0^+} x^{\mu} \cdot \frac{\ln\sin x}{\sqrt{x}} \overset{\mu=\frac{2}{3}}{=\!=} 0$

B 3. (3) $\int_0^{+\infty} \frac{\ln(1+x)}{x^n} dx = \int_0^1 + \int_1^{+\infty}$

$\int_0^1 \frac{\ln(1+x)}{x^n} dx$ $\lim_{x\to 0^+} x^{\mu} \frac{\ln(1+x)}{x^n} \overset{\mu=n-1}{=\!=} 1$ $\mu=n-1<1$ 收 $\Rightarrow n<2$ 收; $\mu=n-1\ge 1$ 散 $\Rightarrow n\ge 2$ 散

$\int_1^{+\infty} \frac{\ln(1+x)}{x^n} dx$ $\lim_{x\to+\infty} x^{\mu} \cdot \frac{\ln(1+x)}{x^n} = \lim_{x\to+\infty} \frac{\ln(1+x)}{x^{n-\mu}}$ $\begin{cases} \overset{n-\mu>0}{=\!=} 0 \Rightarrow n>\mu>1 \text{ 收} \\ \overset{n-\mu\le 0}{=\!=} \infty \Rightarrow n\le\mu\le 1 \text{ 散} \end{cases}$

B 3. (5) $\int_1^{+\infty} \frac{dx}{x^p \ln^q x}$ $(p, q>0) = \int_1^2 + \int_2^{+\infty}$

$\int_1^2 \frac{1}{x^p \ln^q x} dx$ $\lim_{x\to 1^+} (x-1)^{\mu} \frac{1}{x^p \ln^q(1+x-1)} = \lim_{x\to 1^+} \frac{(x-1)^{\mu}}{(x-1)^q} \overset{\mu=q}{=\!=} 1$ $\begin{cases} \mu=q<1 \text{ 收} \\ \mu=q\ge 1 \text{ 散} \end{cases}$

$\int_2^{+\infty} \frac{1}{x^p \ln^q x} dx$ $\lim_{x\to+\infty} x^{\mu} \frac{1}{x^p \ln^q x} = \lim_{x\to+\infty} \frac{x^{\mu-p}}{\ln^q x}$ $\begin{cases} \overset{\mu-p\le 0}{=\!=} 0 & 1<\mu\le p \text{ 收} \\ \overset{\mu-p>0}{=\!=} \infty & 1\ge\mu>p \text{ 散} \end{cases}$

B 4 $\int_1^{+\infty} \frac{x^p \sin x}{1+x^2} dx$ $q>0$ 的绝对条件收敛性

$\int_a^{+\infty} |f(x)| dx$ 收 $\Rightarrow \int_a^{+\infty} f(x) dx$ 收

绝对收敛

$\int^{+\infty}$

原式 $= \int_0^{+\infty} = \int_0^1 + \int_1^{+\infty}$

绝对收敛

$\int_a^{+\infty}|f(x)|dx$ 发散，但 $\int_a^{+\infty} f(x)dx$ 收敛

称 $\int_a^{+\infty} f(x)dx$ 条件收敛

Cauchy收敛准则　Abel判别法，Dirichlet判别法

$\int_0^1 \frac{x^p \sin x}{1+x^q}dx$ $\begin{cases} p\geq 0 & \text{正常积分} \\ p<0 & x=0\ \text{瑕点} \end{cases}$　$\lim_{x\to 0^+} x^\mu \frac{x^p \sin x}{1+x^q} = \lim_{x\to 0^+} x^{\mu+p+1} \xlongequal{\mu=-p-1} 1$　$\begin{cases} -p-1<1\ \text{收} \Rightarrow p>-2\ \text{收} \\ -p-1\geq 1\ \text{散} \Rightarrow p\leq -2\ \text{散} \end{cases}$

$\sin x>0$　$\sin x \sim x$　$1+x^q \to 1$

$\int_1^{+\infty} \frac{x^p \sin x}{1+x^q}dx \longrightarrow \int_1^{+\infty}\left|\frac{x^p \sin x}{1+x^q}\right|dx$　　$\left|\frac{x^p \sin x}{1+x^q}\right| \leq \frac{x^p}{1+x^q}$

又由判别 $\int_1^{+\infty}\frac{x^p}{1+x^q}dx$　　$\lim_{x\to+\infty} x^\mu \frac{x^p}{1+x^q} \xlongequal{\mu+p=q} 1$　$\mu=q-p>1$ 收 $\Rightarrow$ 原式绝对收敛

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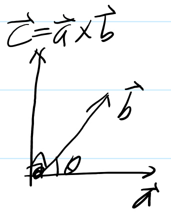
外积　$\vec{a}\times\vec{b}=\vec{c}$　　$\vec{c}\begin{cases} |\vec{c}|=|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta \quad \theta=(\widehat{\vec{a},\vec{b}}) \\ \vec{c}\perp\vec{a},\ \vec{c}\perp\vec{b},\ \text{右手系} \end{cases}$




例　力矩 $\vec{M}=\vec{OA}\times\vec{F}$　　　线速度 $\vec{v}=\vec{\omega}\times\vec{OA}$

性质 1) $\vec{a}\times\vec{a}=0$

2) $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}\parallel\vec{b} \Leftrightarrow \vec{a}=\lambda\vec{b}$

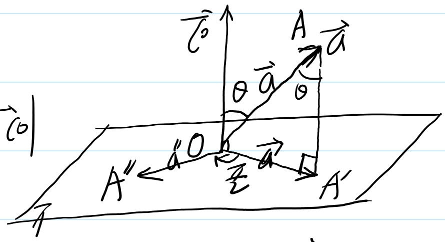
3) 运算律 $\begin{cases} ① & \vec{a}\times\vec{b}=-\vec{b}\times\vec{a} \\ ② & (\lambda\vec{a})\times\vec{b}=\lambda(\vec{a}\times\vec{b})=\vec{a}\times(\lambda\vec{b}) \\ ③ & (\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c} \end{cases}$

$\vec{c}=|\vec{c}|\vec{c_0}$

证明：③ 只须证 $(\vec{a}+\vec{b})\times\vec{c_0}=\vec{a}\times\vec{c_0}+\vec{b}\times\vec{c_0}$

原式

先将 $\vec{a}\times\vec{c}$ 的几何表示展现出来



$\vec{a}$投影到平面$\vec{a}'$
再将$\vec{a}'$逆时针
转$\frac{\pi}{2}$至$a''$
$a''=\vec{a}\times\vec{c}$

$$|\overrightarrow{OA''}| = |\overrightarrow{OA'}| = |\vec{a}|\sin\theta = |\vec{a}||\vec{c}|\sin\theta = |\vec{a}\times\vec{c}|$$

$$\begin{cases} \overrightarrow{OA''}\perp\vec{c} \\ \left.\begin{array}{l}\overrightarrow{OA''}\perp OA' \\ \overrightarrow{OA''}\perp\vec{c}\end{array}\right\} \Rightarrow \overrightarrow{OA''}\perp OAA'\text{面} \Rightarrow \overrightarrow{OA''}\perp\vec{a} \end{cases}$$

则 $OA''=\vec{a}\times\vec{c}$

~~证毕~~

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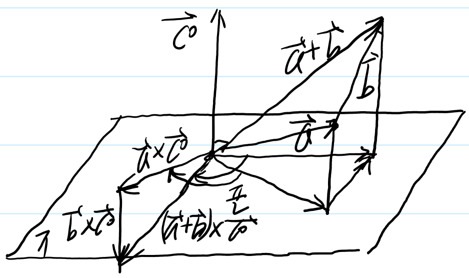
要证 $(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$



由左图知 ~~要先上试又之~~

旋转再乘$|\vec{c}|$

则 $(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$ 证毕

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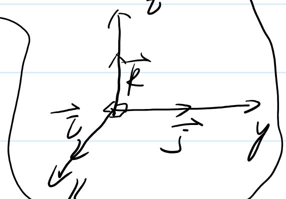
性质④ 外积坐标

$\vec{a}=(a_x,a_y,a_z)$ $\quad\vec{b}=(b_x,b_y,b_z)$

$\vec{i}\ \vec{j}\ \vec{k}$为基本单位向量



$$\vec{a}\times\vec{b}=(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})\times(b_x\vec{i}+b_y\vec{j}+b_z\vec{k})$$

$$=a_xb_x\underbrace{\vec{i}\times\vec{i}}_{=\vec{0}}+a_yb_y\underbrace{\vec{j}\times\vec{j}}_{=\vec{0}}+a_zb_z\underbrace{\vec{k}\times\vec{k}}_{=\vec{0}}$$

$$+(a_xb_y-a_yb_x)\underbrace{\vec{i}\times\vec{j}}_{=\vec{k}}+(a_yb_z-a_zb_y)\underbrace{\vec{j}\times\vec{k}}_{=\vec{i}}+(a_zb_x-a_xb_z)\underbrace{\vec{k}\times\vec{i}}_{=\vec{j}}$$

$\vec{i}\times\vec{j}$, $\vec{j}\times\vec{k}$, $\vec{k}\times\vec{i}$

$\vec{j}\times\vec{i}$, $\vec{k}\times\vec{j}$, $\vec{i}\times\vec{k}$

$$=(a_yb_z-a_zb_y)\vec{i}+(a_zb_x-a_xb_z)\vec{j}+(a_xb_y-a_yb_x)\vec{k}$$

$$=\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\vec{i}+\begin{vmatrix}a_z & a_x\\ b_z & b_x\end{vmatrix}\vec{j}+\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\vec{k}$$

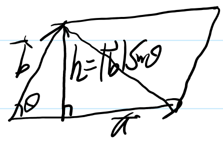
$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

$$=(-1)^{1+1}\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\vec{i}+(-1)^{1+2}\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}\vec{j}+(-1)^{1+3}\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\vec{k}$$

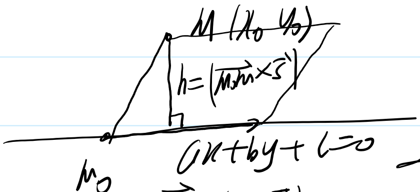
$$= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}\vec{i} + \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix}\vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\vec{k}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \left( \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}, \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix}, \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \right)$$

性质5 $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin(\vec{a},\vec{b}) = S_{\vec{a}\vec{b}\text{▱}} = 2S_{\vec{a}\vec{b}\triangle}$




性质6 点到直线距离




方向（方向多维中）

$\vec{s}=(a, b)$

例 $\vec{a}=(2,1,-1)$ $\vec{b}=(1,-1,2)$ 求 $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

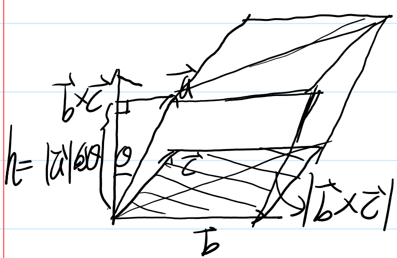
例 $|\vec{a}|=1$ $|\vec{b}|=4$ 且 $\vec{a}\perp\vec{b}$ 求 $|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|$

$$|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})| = |\vec{a}\times\vec{a} - \vec{a}\times\vec{b} + \vec{b}\times\vec{a} - \vec{b}\times\vec{b}| = |2\,\vec{b}\times\vec{a}| = 2|\vec{b}\times\vec{a}| = 2|\vec{b}||\vec{a}|\sin\theta =$$

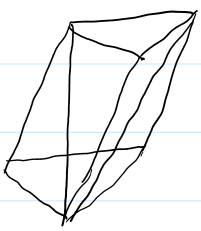
混合积 (1) $\vec{a}\cdot(\vec{b}\times\vec{c}) = |\vec{a}|\,|\vec{b}\times\vec{c}|\cos\theta$ $\theta=(\vec{a}, \vec{b}\times\vec{c})$

$$= \underbrace{|\vec{b}\times\vec{c}|}_{\text{底面积}}\ \underbrace{|\vec{a}|\cos\theta}_{h}$$

$$= \pm V_{\vec{a}\vec{b}\vec{c}\text{平行六面体}}$$




$\vec{a}\cdot(\vec{b}\times\vec{c})$ ✓
$(\vec{a}\times\vec{b})\cdot\vec{c}$ ✓ } 混合积
$\vec{a}\cdot\vec{b}\cdot\vec{c}$ ✗
$\vec{a}\times\vec{b}\times\vec{c}$ { $(\vec{a}\times\vec{b})\times\vec{c}$, $\vec{a}\times(\vec{b}\times\vec{c})$
双重外积



即 $|\vec{a}\cdot(\vec{b}\times\vec{c})| = V_{\vec{a}\vec{b}\vec{c}\text{六面体}} = 2V_{\vec{a}\vec{b}\vec{c}\text{三棱柱}} = 6V_{\vec{a}\vec{b}\vec{c}\text{四面体}}$

(2) 坐标 $\vec{a}\cdot(\vec{b}\times\vec{c}) = (a_x, a_y, a_z)\left( \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix}, \begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix}, \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \right)$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ b_x&b_y&b_z\\ c_x&c_y&c_z\end{vmatrix}=\begin{vmatrix}b_y&b_z\\c_y&c_z\end{vmatrix}a_x+\begin{vmatrix}b_z&b_x\\c_z&c_x\end{vmatrix}a_y+\begin{vmatrix}b_x&b_y\\c_x&c_y\end{vmatrix}a_z$$

$$=\left(\begin{vmatrix}b_y&b_z\\c_y&c_z\end{vmatrix},\begin{vmatrix}b_z&b_x\\c_z&c_x\end{vmatrix},\begin{vmatrix}b_x&b_y\\c_x&c_y\end{vmatrix}\right)=\begin{vmatrix}a_x&a_y&a_z\\ b_x&b_y&b_z\\ c_x&c_y&c_z\end{vmatrix}$$

$$=\begin{vmatrix}b_y&b_z\\c_y&c_z\end{vmatrix}\vec{i}+\begin{vmatrix}b_z&b_x\\c_z&c_x\end{vmatrix}\vec{j}+\begin{vmatrix}b_x&b_y\\c_x&c_y\end{vmatrix}\vec{k}$$

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3) 记 $\vec{a}\cdot(\vec{b}\times\vec{c})=[\vec{a}\,\vec{b}\,\vec{c}]$

则 $[\vec{a}\,\vec{b}\,\vec{c}]=[\vec{b}\,\vec{c}\,\vec{a}]=[\vec{c}\,\vec{a}\,\vec{b}]=-[\vec{b}\,\vec{a}\,\vec{c}]=-[\vec{c}\,\vec{b}\,\vec{a}]=-[\vec{a}\,\vec{c}\,\vec{b}]$

$[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}a_x&a_y&a_z\\ b_x&b_y&b_z\\ c_x&c_y&c_z\end{vmatrix}$ $\quad[\vec{b}\,\vec{c}\,\vec{a}]=\begin{vmatrix}b_x&b_y&b_z\\ c_x&c_y&c_z\\ a_x&a_y&a_z\end{vmatrix}$

轮换性 对换性