

15-16

一, B C B A D B

二, π $\pi/6$ $x^2+xz+z^2=2$ $3\pi/10$ $x+y+z+1=0$ $\frac{1}{2}(1-e^4)$

三、

1. $dz = (f'_1 + f'_2)dx + (f'_1 - f'_2)dy$ $\frac{\partial^2 z}{\partial x \partial y} = f''_{11} - f''_{12} + f''_{21} - f''_{22} = f''_{11} - f''_{22}$

2. $\begin{cases} \frac{dy}{dx} = \frac{1-x}{2y-1}, & \frac{dz}{dx} = \frac{x-2y}{2y-1} \end{cases}$

3. $\frac{x+1}{1} = \frac{y-2}{-2} = \frac{z-3}{1}$

4. $\iint_D \frac{1}{1+x^2+y^2} dx dy = 2 \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \frac{r}{1+r^2} dr = \frac{\pi \ln 2}{2}$ $\iint_D \frac{xy}{1+x^2+y^2} dx dy = 0$,

$$\iint_D \frac{1+xy}{1+x^2+y^2} dx dy = \iint_D \frac{1}{1+x^2+y^2} dx dy + \iint_D \frac{xy}{1+x^2+y^2} dx dy = \frac{\pi \ln 2}{2}$$

四、按要求解答下列各题（共4道小题，每小题8分，满分32分）.

1. $ds = \sqrt{r'^2 + r'^2} d\theta = 4 \left| \cos \frac{\theta}{2} \right| d\theta$ $s = \int_0^{2\pi} 4 \left| \cos \frac{\theta}{2} \right| d\theta = \int_0^{\pi} 4 \cos \frac{\theta}{2} d\theta - \int_{\pi}^{2\pi} 4 \cos \frac{\theta}{2} d\theta = 16$

2. 方向导数为 $u_x \cos \alpha + u_y \cos \beta + u_z \cos \gamma = \frac{6}{\sqrt{14}} \times \frac{2}{\sqrt{14}} + \frac{8}{\sqrt{14}} \times \frac{3}{\sqrt{14}} - \sqrt{14} \times \frac{1}{\sqrt{14}} = \frac{6}{7}$

3. (1) Σ 的方程 $z = 1 + x^2 + y^2, (1 \leq z \leq 3)$;

(2) $\iiint_{\Omega} e^z dx dy dz = \int_1^3 e^z dz \iint_{D_z} dx dy = \pi \int_1^3 e^z (z-1) dz = e + e^3$

4. $\int_a^b |x| dx = \int_a^0 -x dx + \int_0^b x dx = \frac{1}{2}(a^2 + b^2) = \frac{1}{2} \Rightarrow a^2 + b^2 = 1$, $s = \int_0^{b-a} (bx - x^2 - ax) dx = \frac{1}{6}(b-a)^3$

$$F(a, b, \lambda) = \frac{1}{6}(b-a)^3 + \lambda(a^2 + b^2 - 1), \quad \begin{cases} F_a = -\frac{1}{2}(b-a)^2 + 2\lambda a = 0, \\ F_b = \frac{1}{2}(b-a)^2 + 2\lambda b = 0, \\ a^2 + b^2 - 1 = 0 \end{cases}, \quad \text{驻点} \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),$$

$s = \frac{\sqrt{2}}{3}$, $a=0, b=1, s=\frac{1}{6}$, $a=-1, b=0, s=\frac{1}{6}$, 面积最大值为 $s = \frac{\sqrt{2}}{3}$, 最小值为 $s = \frac{1}{6}$.

16-17

一 1. $\underline{x^2 + y^2 = 1}$. 2. $\underline{\begin{cases} x + y = 1 \\ z = 0 \end{cases}}$. 3. $\underline{dx + dy}$. 4. $\underline{0}$. 5. $\underline{2x - 4y - z - 3 = 0}$.

二 1. D 2. C 3. B 4. C 5. A

三

1. $A = \int_{-1}^3 (2x + 3 - x^2) dx = \left[x^2 + 3x - \frac{x^3}{3} \right]_{-1}^3 = \frac{32}{3}$

2. $2x + z - 2 + \lambda(y - 1) = 0$, $\vec{n} \cdot \vec{s} = 0 \Rightarrow 4 - \lambda - 2 = 0$, $\lambda = 2$ 所求平面为 $2x + 2y + z - 4 = 0$.

3. $u_x' = f_1' + 2xf_2'$, $u_{xx}'' = f_{11}'' + 4xf_{12}'' + 4x^2f_{22}'' + 2f_2'$

4. $L(x, y, z, \lambda) = x - 2y + 2z + \lambda(x^2 + y^2 + z^2 - 1)$, $L_x' = 1 + 2\lambda x = 0$, $L_y' = -2 + 2\lambda y = 0$, $L_z' = 2 + 2\lambda z = 0$,

$L_\lambda' = x^2 + y^2 + z^2 - 1 = 0$, $x = \pm \frac{1}{3}$, $y = \mp \frac{2}{3}$, $z = \pm \frac{2}{3}$, $u = \pm 3$

四、

1. $\iint_D |x^2 + y^2 - 1| d\sigma = \iint_{D_1} (1 - x^2 - y^2) d\sigma + \iint_{D_2} (x^2 + y^2 - 1) d\sigma = \int_0^{\frac{\pi}{4}} d\theta \int_0^1 (1 - r^2) r dr + \int_0^{\frac{\pi}{4}} d\theta \int_1^{\frac{1}{\cos \theta}} (r^2 - 1) r dr = \frac{\pi}{8} - \frac{1}{6}$

2. $V = \iiint_{\Omega} dV = 8 \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 abc r^2 \sin \varphi dr = \frac{4\pi}{3} abc$

3. $f_x'(0, 0) = \lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, 0) - f(0, 0)}{x} = 0$ 同理 $f_y'(0, 0) = 0$

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - f(0, 0) - f_x'(0, 0)x - f_y'(0, 0)y}{\sqrt{x^2 + y^2}} = \lim_{(x, y) \rightarrow (0, 0)} \sqrt{\frac{x^4 + y^4}{x^2 + y^2}} = 0$$

4. (1) $\iiint_{\Omega} \frac{dV}{x^2 + y^2} = \int_0^{\frac{\pi}{4}} d\theta \int_{\frac{1}{\cos \theta}}^{\frac{2}{\cos \theta}} r dr \int_0^{\sin \theta} \frac{dz}{r^2} = \frac{1}{2} \ln 2$; (2) $\iiint_{\Omega} \frac{dV}{x^2 + y^2} = \int_0^{\frac{\pi}{4}} d\theta \int_{\arccot \sin \theta}^{\frac{\pi}{2}} d\varphi \int_{\frac{1}{\sin \varphi \cos \theta}}^{\frac{2}{\sin \varphi \cos \theta}} \frac{dr}{\sin \varphi} = \frac{1}{2} \ln 2$

一、 B B C D C B

$$\text{二、 } \ln 3 - \frac{1}{2} \quad \frac{1}{2} \quad -\frac{4}{3} \quad f(\pm\sqrt{x^2+y^2}, z) = 0 \quad f'_1 + yf'_2 \quad \sqrt{3}$$

$$13. I = \int_0^1 dx \int_0^{x^2} \frac{ye^y}{1-\sqrt{y}} dy = \int_0^1 \frac{ye^y}{1-\sqrt{y}} dy \int_{\sqrt{y}}^1 dx = \int_0^1 y de^y = ye^y \Big|_0^1 - \int_0^1 e^y dy = e - e^y \Big|_0^1 = 1.$$

$$14. \frac{x}{-2} = \frac{y-2}{3} = \frac{z-4}{1}. \quad \begin{cases} x = -2t, \\ y = 2+3t, \\ z = 4+t. \end{cases}$$

$$15. dz|_{(e,0)} = \frac{1}{2e} dx - \frac{1}{2} dy, \quad \frac{\partial^2 z}{\partial y \partial x} = -\frac{\frac{\partial z}{\partial x}(1+z) - z \frac{\partial z}{\partial x}}{(1+z)^2} = -\frac{z}{x(1+z)^3}, \quad \frac{\partial^2 z}{\partial y \partial x} \Big|_{(e,0)} = -\frac{1}{8e}.$$

$$16. I = \iiint_{\Omega} \sqrt{(r \cos \theta)^2 + (r \sin \theta)^2} r dr d\theta dz = \int_0^{2\pi} d\theta \int_0^1 r^2 dr \int_r^1 dz = 2\pi \int_0^1 r^2(1-r) dr = \frac{\pi}{6}.$$

$$I = \int_0^1 dz \iint_{D_z} \sqrt{x^2+y^2} dx dy = \int_0^1 dz \int_0^{2\pi} d\theta \int_0^z r^2 dr = 2\pi \int_0^1 \frac{z^3}{3} dz = \frac{\pi}{6}.$$

$$17. V = \pi \int_{-4}^4 (5 + \sqrt{16-x^2})^2 dx - \pi \int_{-4}^4 (5 - \sqrt{16-x^2})^2 dx = \pi \int_{-4}^4 20\sqrt{16-x^2} dx = 40\pi \int_0^4 \sqrt{16-x^2} dx \\ = 40\pi \cdot \frac{1}{4} \pi \cdot 4^2 = 160\pi.$$

$$18. L = 2x - y + 1 + \lambda(x^2 + y^2 - 5), \quad \begin{cases} L_x = 2 + 2\lambda x = 0, \\ L_y = -1 + 2\lambda y = 0, \\ L_\lambda = x^2 + y^2 - 5 = 0, \end{cases} \text{ 解得驻点 } (-2, 1), (2, -1) \quad \text{又}$$

$f(-2, 1) = -4, f(2, -1) = 6$, 所以函数 $f(x, y)$ 满足约束条件 $x^2 + y^2 = 5$ 下的最大值为 $f(2, -1) = 6$, 最小值为

$f(-2, 1) = -4$.

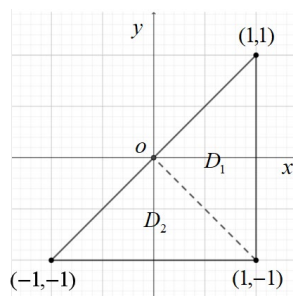
$$19. \int_0^{x^2} f(x^2-t) dt = \int_0^{x^2-t=u} f(u) du, \text{ 故 } f(x) = x^2 + x \int_0^{x^2} f(u) du + \iint_D f(xy) dx dy.$$

$$f(xy) = (xy)^2 + xy \int_0^{(xy)^2} f(u) du + \iint_D f(xy) dx dy$$

$$\text{令 } \iint_D f(xy) dx dy = k, \quad k = \iint_D (xy)^2 dx dy + \iint_D \left[xy \int_0^{(xy)^2} f(u) du \right] dx dy + \iint_D k dx dy.$$

$$\iint_D (xy)^2 dx dy = \int_{-1}^1 dx \int_{-1}^x x^2 y^2 dy = \frac{2}{9}. \quad k = \frac{2}{9} + 0 + 2k, \quad k = -\frac{2}{9}. \quad \text{于是 } f(x) = x^2 + x \int_0^{x^2} f(u) du - \frac{2}{9}.$$

$$\text{令 } x=1, \text{ 得 } \int_0^1 f(x) dx = -\frac{7}{9}.$$



18-19 A2

一、 $D \ A \ D \ B \ C \ B$

二、 $e^{xy} \sin e^{xy} (ydx + xdy)$; $\frac{1}{4} \ln 17$; $-\frac{1}{2}$; $\frac{3\pi}{10}$; 3 ; $\begin{cases} x^2 + 2z^2 = 4 \\ y = 0 \end{cases}$

三、

$$8x - 9y - 22z - 59 = 0$$

$$z'_x = 3x^2 f + x^3 y f'_1 - xy f'_2, \quad z''_{xy} = 4x^3 f'_1 + 2x f'_2 + x^4 y f''_{11} - y f''_{22}$$

$$5\pi + \frac{32}{5}$$

$$\operatorname{grad} u(P_0) = (1, -3, -3), \quad \frac{\partial u}{\partial l}(P_0) = -\frac{1}{3}$$

$$z_{\max} = 25, \quad z_{\min} = 0$$

$$(x-1)+4(y-2)+6(z-2)=0 \text{ 或 } (x+1)+4(y+2)+6(z+2)=0$$

$$\frac{4}{5} \pi abc$$

19-20

C A A D C C

$$4. \quad x+3y+z-3=0, \quad \sqrt{3}, \quad 1, \quad (t-1)f(t), \quad \frac{2\pi}{3}$$

13. 求 $y=2x-x^2$ 与 $y=0$ 所围的封闭区域绕 x 轴旋转一周生成旋转体的体积.

$$V = \pi \int_0^2 (2x-x^2)^2 dx = \frac{16\pi}{15}$$

14. 求过直线 $L_1: \frac{x-1}{1} = \frac{y-1}{0} = \frac{z-1}{-2}$ 且平行于直线 $L_2: \frac{x+2}{2} = \frac{y-1}{-1} = \frac{z}{-2}$ 的平面方程.

$$\bar{n} = \bar{s}_1 \times \bar{s}_2 = -(2, 2, 1), \quad 2(x-1) + 2(y-1) + (z-1) = 0, \quad 2x + 2y + z - 5 = 0$$

15. 求椭圆面 $2x^2 + 3y^2 + z^2 = 9$ 上点 $M(1, 1, 2)$ 处的切平面方程与法线方程.

$$\bar{n} = \{2x, 3y, z\}_M = (2, 3, 2) \quad 2(x-1) + 3(y-1) + 2(z-2) = 0 \quad \frac{x-1}{2} = \frac{y-1}{3} = \frac{z-2}{2}.$$

16. 设 $u = f(x, xy, xyz)$, f 具有二阶连续偏导数, 求 $\frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}$.

$$\frac{\partial u}{\partial y} = xf'_2 + xzf'_3 \quad \frac{\partial^2 u}{\partial y^2} = x^2(f''_{22} + 2zf''_{23} + z^2f''_{33})$$

17. 求函数 $z = x^2 + y^2 - 2x - y$ 在 $D = \{(x, y) | 2x + y \leq 4, x \geq 0, y \geq 0\}$ 上的最值.

$$\begin{cases} z'_x = 2x - 2 = 0 \\ z'_y = 2y - 1 = 0 \end{cases}, \quad z_{\min}\left(1, \frac{1}{2}\right) = -\frac{5}{4}$$

$$y = 0, \quad z = x^2 - 2x = 0 \quad (0 \leq x \leq 2), \quad z(1, 0) = -1$$

$$x = 0, \quad z = y^2 - y = 0 \quad (0 \leq y \leq 4), \quad z\left(0, \frac{1}{2}\right) = -\frac{1}{4}$$

$$2x + y = 4, \quad z = 5x^2 - 16x + 12 = 0 \quad (0 \leq x \leq 2), \quad z\left(\frac{8}{5}, \frac{4}{5}\right) = -\frac{4}{5}$$

$$z(0, 0) = 0, \quad z(2, 0) = 0, \quad z_{\max}(0, 4) = 12$$

18. 求在上半球体 $x^2 + y^2 + z^2 \leq 1$ ($z \geq 0$) 除去柱体 $x^2 + y^2 \leq x$ 的空间立体的体积.

$$V = \frac{2\pi}{3} - 2 \int_0^{\frac{\pi}{2}} d\theta \int_0^{\cos\theta} \sqrt{1-r^2} r dr = \frac{3\pi+4}{9} \quad \text{或} \quad D = \{(x, y) | x^2 + y^2 \leq 1, x^2 + y^2 \geq x\}$$

$$V = \iint_D \sqrt{1-x^2-y^2} dx dy = 2 \left[\int_0^{\frac{\pi}{2}} d\theta \int_{\cos\theta}^1 \sqrt{1-r^2} r dr + \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^1 \sqrt{1-r^2} r dr \right] = \frac{3\pi+4}{9}$$

19. 已知 $\Omega = \left\{ (x, y, z) \left| \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^4}{c^4} \leq 1 \right. \right\}$, 计算 $I = \iiint_{\Omega} \left(\frac{x}{a} + \frac{y}{b} + \frac{z^2}{c^2} \right)^2 dV$.

$$I = \iiint_{\Omega} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^4}{c^4} \right) dV = 8 \int_0^{\frac{\pi}{2}} d\theta \int_0^1 dr \int_0^{c(1-r^2)^{\frac{1}{4}}} ab r \left(r^2 + \frac{z^4}{c^4} \right) dz = \frac{8\pi}{9} abc \quad \text{或}$$

$$I = \iiint_{\Omega} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^4}{c^4} \right) dV = 8 \int_0^c dz \int_0^{\frac{\pi}{2}} d\theta \int_0^1 ab \left(1 - \frac{z^4}{c^4} \right) \left[r^2 \left(1 - \frac{z^4}{c^4} \right) + \frac{z^4}{c^4} \right] r dr = \frac{8\pi}{9} abc$$

20-21

一、 1. (A) 2. (C). 3. (B). 4. (D). 5. (C) . 6. (A).

二、 7. $\frac{3}{10}\pi$. 8. $\frac{1}{2}$. 9. $x^2 + y^2 + z = 1$. 10. $\frac{2}{5}$, 11. $\frac{1}{y}(3\sin y^3 - 2\sin y^2)$. 12. $\frac{1}{2}$.

三、

13. $S = \int_{-1}^2 (2x - x^2 - x + 2) dx = \frac{9}{2}$

14. $x - y + z = 0$

15. $\frac{\partial u}{\partial x} = yf_1' + 2xf_2' \quad \frac{\partial^2 u}{\partial x \partial y} = xyf_{11}'' + 2(x^2 + y^2)f_{12}'' + 4xyf_{22}'' + f_1'$

16. $(x, y, z) = \left(\pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{2}{3} \right)$
 $f_{\max} = 3, f_{\min} = -3$

17. $I = \int_0^1 \frac{\sin y}{y} dy \int_{y^2}^y dx = \int_0^1 (1 - y) \sin y dy = 1 - \sin 1$

18 .

$$I = \iiint_{\Omega_1} + \iiint_{\Omega_2} = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_{\cos \varphi}^1 r^6 \sin \varphi dr + \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{1}{\sin \varphi}} r^6 \sin \varphi dr$$
$$= \frac{\pi}{84} + \frac{\sqrt{2}\pi}{28} + \frac{2\pi}{15} - \frac{\sqrt{2}\pi}{28} = \frac{61\pi}{420}$$

19.

$$I = \iiint_{\Omega_1} + \iiint_{\Omega_2} = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{\sin(\theta + \frac{\pi}{4})} r dr \int_0^r \left(\sqrt{2} \frac{\cos \theta + \sin \theta}{r} - 2 \right) z dz$$
$$+ \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_{\sin(\theta + \frac{\pi}{4})}^1 r dr \int_0^r \left(2 - \sqrt{2} \frac{\cos \theta + \sin \theta}{r} \right) z dz$$
$$= \frac{\pi}{32} + \left(\frac{\pi}{4} - \frac{2}{3} + \frac{\pi}{32} \right) = \frac{5\pi}{16} - \frac{2}{3}$$