多元微分20-1

2022年5月6日 7:43

全微分 $z=f(x,y)$ $\boxed{\Delta z = A\Delta x + B\Delta y + o(\rho)}$ 则 $\Delta z = A\Delta x + B\Delta y + o(\Delta x) + o(\Delta y)$

$$dz = A\Delta x + B\Delta y = z'_x dx + z'_y dy \quad (\Delta x = dx,\ \Delta y = dy)$$

几何意义 $z=f(x,y)$ 在 $P(x,y)$ 处可微 则 曲面 $z=f(x,y)$ 在 $P(x,y)$ 处存在切平面

$$\Delta z \approx dz = z'_x dx + z'_y dy$$

$$\underbrace{z+\Delta z}_{z_{曲}} \approx \underbrace{z + z'_x dx + z'_y dy}_{z_{切}}$$ 以切平面上的z值 $z_{切}$ ≈ 曲面上对应点 $z_{曲}$

全微分 $u=f(x,y,z)$ $\Delta u = \underbrace{A\Delta x + B\Delta y + C\Delta z} + o(\rho)$

$$\begin{cases} \Delta u = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z) \\ A = u'_x \quad B = u'_y \quad C = u'_z \\ \rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} \end{cases}$$

$$du = A\Delta x + B\Delta y + C\Delta z$$
$$= u'_x dx + u'_y dy + u'_z dz$$

高阶微分

$\Delta x = dx$

一元, $y=y(x)$, $dy = y'(x)dx$, $d(dy)$, $d^2y = d(dy) = d(y'(x)dx) = (y'(x)dx)'_x dx = y''(dx)^2 = y''dx^2$

$\vdots$

$$d^n y = y^{(n)}(x)\, dx^n$$

$z=f(x,y) \in C^{(2)}$, $dz = z'_x dx + z'_y dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)z = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)z$

$$d^2z = d(dz) = d(z'_x dx + z'_y dy) = (z'_x dx + z'_y dy)'_x dx + (z'_x dx + z'_y dy)'_y dy$$

$$= (z''_{xx} dx + z''_{yx} dy)dx + (z''_{xy} dx + z''_{yy} dy)dy$$

$$= z''_{xx} dx^2 + 2 z''_{xy} dx dy + z''_{yy} dy^2$$

$$= \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx dy + \frac{\partial^2 z}{\partial y^2}dy^2$$

$$= \left(\frac{\partial^2}{\partial x^2}dx^2 + 2\frac{\partial^2}{\partial x\partial y}dx dy + \frac{\partial^2}{\partial y^2}dy^2\right)z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^2 z$$

$z=f(x,y)$ $\quad dz=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)z$

$d^2z=d(dz)=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^2 z$

$=d\left[\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right]z=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}\right)\left[\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)z\right]$

$\vdots$

$d^n z=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)$ $d^n z=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^n z$

计算 $d^2z=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^2 z=z''_{xx}dx^2+2z''_{xy}dxdy+z''_{yy}dy$

$u=f(x,y,z)$ $\quad du=u'_x dx+u'_y dy+u'_z dz=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy+\frac{\partial}{\partial z}dz\right)u$

$d^2u=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy+\frac{\partial}{\partial z}dz\right)^2 u$

$\vdots$

$d^n u=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy+\frac{\partial}{\partial z}dz\right)^n u$

复合函数微分法

$y-u-x$

$y=f(u)$ $\quad u=u(x)$ $\quad y'_x=y'_u\cdot u'_x$

定理： $z=f(u,v)$ 在 (u,v) 处可微 $u=u(t)$，$v=v(t)$，可导，则 $z=f[u(t),v(t)]$ 在 t 处可导，且

有 $\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}$ $\quad (z'_t=z'_u u'_t+z'_v v'_t)$ $\quad z<\begin{matrix}u-t\\v-t\end{matrix}$

全导数

证明： $\Delta z=z'_u\Delta u+z'_v\Delta v+o(\rho)$ $\quad \rho=\sqrt{\Delta u^2+\Delta v^2}$

$\lim\limits_{\Delta t\to0}\frac{\Delta z}{\Delta t}=\lim\limits_{\Delta t\to0}z'_u\frac{\Delta u}{\Delta t}+\lim\limits_{\Delta t\to0}z'_v\frac{\Delta v}{\Delta t}+\lim\limits_{\Delta t\to0}\frac{o(\rho)}{\Delta t}$

$=z'_u\underbrace{\lim\limits_{\Delta t\to0}\frac{\Delta u}{\Delta t}}_{\frac{du}{dt}}+z'_v\underbrace{\lim\limits_{\Delta t\to0}\frac{\Delta v}{\Delta t}}_{\frac{dv}{dt}}+\lim\limits_{\Delta t\to0}\frac{o(\rho)}{\rho}\frac{\rho}{\Delta t}$

$\lim\limits_{\Delta t\to0}\frac{o(\rho)}{\rho}\sqrt{\frac{\Delta u^2}{\Delta t^2}+\frac{\Delta v^2}{\Delta t^2}}$

$\lim\limits_{\Delta t\to0}\frac{o(\rho)}{\rho}\cdot\sqrt{u'(t)^2+v'(t)^2}=0$

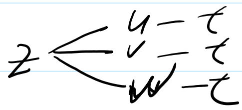
$=z'_u\frac{du}{dt}+z'_v\frac{dv}{dt}$

$$= z'_u \frac{du}{dt} + z'_v \frac{dv}{dt}$$

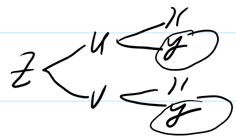
$$\lim_{\substack{\Delta t\to 0\\ \Delta u\to 0\\ \Delta v\to 0}} \frac{o(\rho)}{\rho}\cdot\sqrt{u'(t)^2+v'(t)^2} = 0$$

推论1. $z=f(u,v,w)$ 可微，$u=u(t)$, $v=v(t)$, $w=w(t)$ 可导，则 $z=f[u(t),v(t),w(t)]$ 可导

且 $$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial w}\frac{dw}{dt}$$



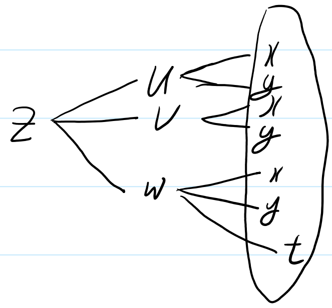
推论2. $z=f(u,v)$ 可微，$u=u(x,y)$ 及 $v=v(x,y)$ 可偏导，则 $z=f[u(x,y),v(x,y)]$ 可偏导



且 $$\begin{cases} z'_x = z'_u u'_x + z'_v v'_x \\ z'_y = z'_u u'_y + z'_v v'_y \end{cases}$$

推论2. $z=f[u,v,w]$ 可微，$u=u(x,y)$, $v=v(x,y)$, $w=w(x,y,t)$ 可偏导，则

$z=f[u(x,y),v(x,y),w(x,y,t)]$ 可偏导 且

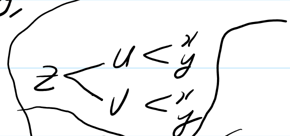


$$z'_x = z'_u u'_x + z'_v v'_x + z'_w w'_x$$

$$z'_y = z'_u u'_y + z'_v v'_y + z'_w w'_y$$

$$z'_t = z'_w w'_t$$

例 $z=f(u,v)\in C^{(2)}$，$u=u(x,y)$, $v=v(x,y)$ 二阶可偏导，求 $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\partial y}$

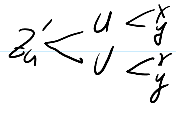


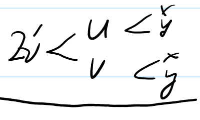
$z=u^2v^2$ $z'_u=2uv^2$ $z'_v=2vu^2$

$$z'_x = z'_u u'_x + z'_v v'_x$$

$$z''_{xx} = (z'_u u'_x)'_x + (z'_v v'_x)'_x = \underline{(z'_u)'_x} u'_x + z'_u u''_{xx} + \underline{(z'_v)'_x} v'_x + z'_v v''_{xx}$$

$$= (z''_{uu} u'_x + z''_{uv} v'_x) u'_x + z'_u u''_{xx} + (z''_{vu} u'_x + z''_{vv} v'_x) v'_x + z'_v v''_{xx}$$





$$z''_{xy} = (z'_u u'_x)'_y + (z'_v v'_x)'_y$$

$$= (z''_{uu} u'_y + z''_{uv} v'_y) u'_x + z'_u u''_{xy} + (z''_{vu} u'_y + z''_{vv} v'_y) v'_x + z'_v v''_{xy}$$

例 $u = f(x, y, z)$, $y = \varphi(x, t)$, $t = \psi(x, z)$ $\in C''$ 求 u'_x u'_z

$$u \begin{cases} x \\ y \quad \varphi \begin{matrix} x \\ t \quad \psi \begin{matrix} x \\ z \end{matrix} \end{matrix} \\ z \end{cases}$$

$$u'_x = f'_x + f'_y y'_x = f'_x + f'_y (\varphi'_x + \varphi'_t t'_x)$$

$$= f'_x + f'_y [\varphi'_x + \varphi'_t \psi'_x]$$

$$u'_z = u'_y y'_t t'_z + u'_z = f'_y \varphi'_t \psi'_z + f'_z$$

例 $z = f(xy, \frac{y}{x})$ 设 $u = xy$ 为 f 第一变量，则有 $v = \frac{y}{x}$ 为 f 第二变量，可标记 xy 为"1"，$\frac{y}{x}$ 为"2"

求 z''_{xy}

$$z = f \begin{cases} 1 = xy < \begin{matrix} x \\ y \end{matrix} \\ 2 = \frac{y}{x} < \begin{matrix} x \\ y \end{matrix} \end{cases}$$

$$f'_1 < \begin{matrix} 1 < \begin{matrix} x \\ y \end{matrix} \\ 2 < \begin{matrix} x \\ y \end{matrix} \end{matrix} \qquad f'_2 < \begin{matrix} 1 < \begin{matrix} x \\ y \end{matrix} \\ 2 < \begin{matrix} x \\ y \end{matrix} \end{matrix}$$

$$z'_x = f'_1 \cdot y + f'_2 \cdot (-\frac{y}{x^2})$$

$$z''_{xy} = (f'_1)'_y y + f'_1 \cdot 1 - (f'_2)'_y \frac{y}{x^2} - f'_2 \cdot \frac{1}{x^2}$$

$$= (f''_{11} x + f''_{12} \frac{1}{x}) y + f'_1 - (f''_{21} x + f''_{22} \frac{1}{x}) \frac{y}{x^2} - f'_2 \frac{1}{x^2}$$

一阶微分形式不变性

1) u 为自变量 $y = f(u)$ $dy = f'(u) du \to d^2y = (f'(u) du)'_u du = f''(u) du^2$

2) u 为中间变量 $y = f(u)$, $u = u(x)$, $\Rightarrow y = f[u(x)]$

$$dy = d f[u(x)] = f'_u \underbrace{u'_x dx}_{=du} = f'_u du$$

即无论 u 为自变量还是中间变量，皆有其微分皆为 $f'_u du$ 相同

$$d^2y = d\,dy = d(f'_u du)$$

$$= (f'_u du)'_x dx = [f''_u u'_x du + f'_u (du)'_x] dx$$

$$= f''_u du^2 + f'_u u''_{xx} dx^2$$

1) u, v 为自变量 $z = f(u, v)$ 可微 $dz = z'_u du + z'_v dv$

2) u, v 为中间变量 $z = f(u, v)$ 可微 $u = u(x, y)$, $v = v(x, y)$ 可微，则 $z = f[u(x,y)\ v(x,y)]$ 可微

$f - u < \begin{matrix} x \\ y \end{matrix}$

$f<{}^{u<{}^{x}_{y}}_{v<{}^{x}_{y}}$

$$dz = df[u(x,y),v(x,y)] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= (f_u' u_x' + f_v' v_x')dx + (f_u' u_y' + f_v' v_y')dy$$

$$= f_u'(\underbrace{u_x' dx + u_y' dy}_{\text{即 } du}) + f_v'(\underbrace{v_x' dx + v_y' dy}_{\text{即 } dv}) = f_u' du + f_v' dv$$

$d(u\cdot v) = v du + u dv$

$$d^2 z = d(dz) = d(z_u' du + z_v' dv) = d(z_u')du + z_u' d(du) + d(z_v')dv + z_v' d(dv)$$

$$= (z_{uu}'' du + z_{uv}'' dv)du + (z_{vu}'' du + z_{vv}'' dv)dv + z_u' d^2u + z_v' d^2v$$

多?

法则 $(u=u(x,y)\quad v=v(x,y))$

$$\begin{cases} d(u\pm v) = du \pm dv \\ d(u\cdot v) = v du + u dv \\ d\left(\frac{u}{v}\right) = \frac{v du - u dv}{v^2} \end{cases}$$

例 $u=f(x,y,z)$ $y=\varphi(x,t)$ $t=\psi(x,z)$ 求 du

$u<{}^{x}_{y<t<{}^{x}_{z}}_{z}$ $\longrightarrow$ $u=u(x,z)$

$$du = f_x' dx + f_y' dy + f_z' dz$$

$$= f_x' dx + f_y'[\varphi_x' dx + \varphi_t' dt] + f_z' dz$$

$$= f_x' dx + f_y'[\varphi_x' dx + \varphi_t'(\psi_x' dx + \psi_z' dz)] + f_z' dz$$

$$= [f_x' + f_y'\varphi_x' + f_y'\varphi_t'\psi_x']dx + (f_y'\varphi_t'\psi_z' + f_z')dz$$

$$\doteq u_x' \cdot dx + u_z' dz$$