多元微分22-1

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定理 $F(x,y,u,v)$ 与 $G(x,y,z,u,v)$. $P_0(x_0,y_0,u_0,v_0)$

若 1) $F,G\in C^{(1)}U(P_0)$; 2) $F(P_0)=0$ $G(P_0)=0$, 3) $\left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{P_0}\neq 0$

则 $\begin{cases}F(x,y,u,v)=0\\G(x,y,u,v)=0\end{cases}$ 在 $U(P_0)$ 内确定了唯一一组且有连续偏导的隐函数组 $\begin{cases}u=u(x,y)\\v=v(x,y)\end{cases}$

且 $\begin{cases}u_0=u(x_0,y_0)\\v_0=v(x_0,y_0)\end{cases}$ 及 $\frac{\partial u}{\partial x}=-\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$ $\frac{\partial u}{\partial y}=-\frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$ $\frac{\partial v}{\partial x}=-\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}$ $\frac{\partial v}{\partial y}=-\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}}$

例 $\begin{cases}F(x,y,\underline{u,v,w},t)=0\\G(x,y,\underline{u,v,w},t)=0\\H(x,y,\underline{u,v,w},t)=0\end{cases}$ 且 $\begin{cases}(1)\ F,G,H\in C^{(1)}\\(2)\ F,G,H \text{ 在 } P_0 \text{ 点为 } 0\\(3)\ \left.\frac{\partial(F,G,H)}{\partial(u,v,w)}\right|_{P_0}\neq 0\end{cases}$ $\Rightarrow$ $\begin{cases}u=u(x,y,t)\\v=v(x,y,t)\\w=w(x,y,t)\end{cases}$

$$\frac{\partial u}{\partial t}=-\frac{\frac{\partial(F,G,H)}{\partial(t,v,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}\quad \cdots\cdots$$

例 $y=f(x,t)$ t 是由 $F(x,y,t)=0$ 确定的关于 x,y 的函数 $f,F\in C^{(1)}$, 求 $\frac{dy}{dx}$

法一: $\begin{cases}y-f(x,t)=0\\F(x,y,t)=0\end{cases}\Rightarrow\begin{cases}y'_x-(f'_x+f'_t\cdot t'_x)=0 & ①\\F'_x+F'_y y'_x+F'_t t'_x=0 & ②\end{cases}$

$y<\begin{matrix}x\\t<\begin{matrix}x\\y<\begin{matrix}x\\t<y\cdots\end{matrix}\end{matrix}\end{matrix}$

$F<\begin{matrix}x\\y-x\\t-x\end{matrix}$

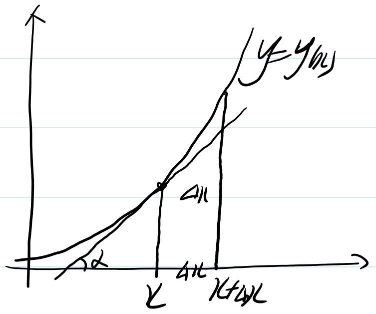
在①②中消去 t'_x $\Rightarrow y'_x=\frac{f'_xF'_t-f'_tF'_x}{F'_t-f'_tF'_y}$

法二 $\begin{cases}dy-(f'_xdx+f'_tdt)=0 & ①\\F'_xdx+F'_ydy+F'_tdt=0 & ②\end{cases}$

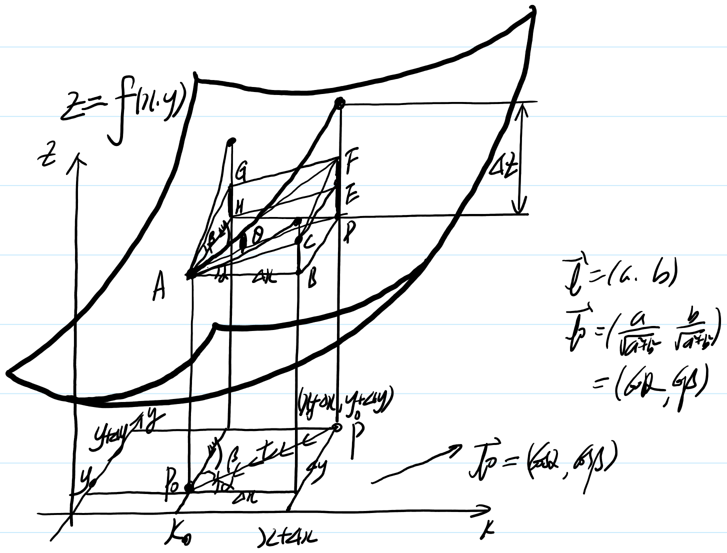
由①②消去 dt $\Rightarrow$ y'_x $dy=y'_xdx$

$$dy=\frac{f'_xF'_t-f'_tF'_x}{F'_t-f'_tF'_y}dx$$

方向导数、梯度



$$P(x,y)\begin{cases} x = x_0 + t\cos\alpha \\ y = y_0 + t\cos\beta \end{cases}$$ 当 $P_0 P$



$z=f(x,y)$ 在 $P_0(x_0,y_0)$ 沿 $\vec{l^0}=(\cos\alpha,\cos\beta)$ 方向的导数 $P(x_0+t\cos\alpha,\ y_0+t\cos\beta)$

$$\left.\frac{\partial z}{\partial l}\right|_{P_0}=\lim_{t\to0^+}\frac{\Delta z}{t}=\lim_{t\to0^+}\frac{f(P)-f(P_0)}{t}=\lim_{t\to0^+}\frac{f(x_0+t\cos\alpha,\ y_0+t\cos\beta)-f(x_0,y_0)}{t}=\tan\theta$$

$$\boxed{\left.\frac{\partial z}{\partial l}\right|_{P_0}=\lim_{t\to0^+}\frac{f(x_0+t\cos\alpha,\ y_0+t\cos\beta)-f(x_0,y_0)}{t}}$$

* 定理 $z=f(x,y)$ 在 $P_0(x_0,y_0)$ 可微，$\vec{l^0}=(\cos\alpha,\cos\beta)$ 则 $z=f(x,y)$ 在 P_0 沿方向 $\vec{l}$ 的方向导数存在且

$$\left.\frac{\partial z}{\partial l}\right|_{P_0}=\left(z'_x\cos\alpha+z'_y\cos\beta\right)_{P_0}$$

证明. $z=f(x,y)$ 可微 (P_0) 则 $\Delta z=f(x_0+\Delta x,\ y_0+\Delta y)-f(x_0,y_0)=z'_x\Delta x+z'_y\Delta y+o(\rho)$ 其中 $\begin{cases}\Delta x=t\cos\alpha\\ \Delta y=t\cos\beta\\ \rho=\sqrt{\Delta x^2+\Delta y^2}=t\end{cases}$

$$\left.\frac{\partial z}{\partial l}\right|_{P_0}=\lim_{t\to0^+}\frac{f(x_0+t\cos\alpha,\ y_0+t\cos\beta)-f(x_0,y_0)}{t}=\lim_{t\to0^+}\frac{z'_x t\cos\alpha+z'_y t\cos\beta+o(t)}{t}$$

$$=\lim_{t\to0^+}\left(z'_x\cos\alpha+z'_y\cos\beta\right)+\lim_{t\to0^+}\frac{o(t)}{t}=z'_x\cos\alpha+z'_y\cos\beta$$

例 $u=f(x,y,z)$ 在 $P(x,y,z)$ 沿 $\vec{l^0}=(\cos\alpha,\cos\beta,\cos\gamma)$ 的方向导 $\begin{cases}\Delta x=t\cos\alpha\\ \Delta y=t\cos\beta\\ \Delta z=t\cos\gamma\end{cases}$

设 $u=f(x,y,z)$ 在 $P(x,y,z)$ 沿 $\vec{l^0}=(\cos\alpha,\cos\beta,\cos\gamma)$ 的方向导 $\begin{cases}\Delta x=t\cos\alpha\\ \Delta y=t\cos\beta\\ \Delta z=t\cos\gamma\end{cases}$

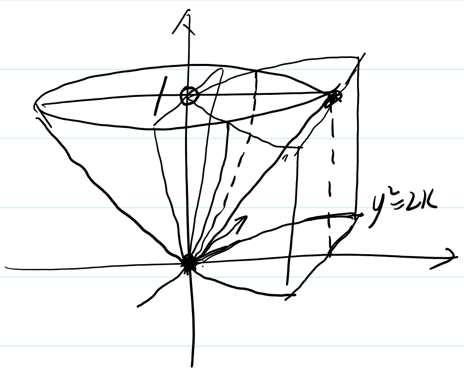
$$\frac{\partial u}{\partial l}=\lim_{t\to0^+}\frac{f(x_0+t\cos\alpha,\ y_0+t\cos\beta,\ z_0+t\cos\gamma)-f(x_0,y_0,z_0)}{t}$$

定理 $u=f(x,y,z)$ 在 $P(x,y,z)$ 处可微 $\vec{l^0}=(\cos\alpha,\cos\beta,\cos\gamma)$，则 $\frac{\partial u}{\partial l}$ 存在，且

$$\frac{\partial u}{\partial l}=u'_x\cos\alpha+u'_y\cos\beta+u'_z\cos\gamma$$

结论 $z=f(x,y)$ 在 $P(x,y)$ 处沿 $\forall$ 方向式方向导存在 $\begin{cases}\not\Rightarrow\ z=f(x,y)\text{ 在 }P(x,y)\text{ 处连续}\\ \not\Rightarrow\ z=f(x,y)\text{ 在 }P(x,y)\text{ 处可偏导}\\ \not\Rightarrow\ \text{可微}\end{cases}$

例



$$f(x,y)=\begin{cases}\sqrt{x^2+y^2} & y^2\neq 2x\\ 1 & y^2=2x\end{cases}$$

$\forall\ \vec l=(\cos\alpha,\cos\beta)$ $\left.\frac{\partial z}{\partial l}\right|_{P(0,0)}=\lim_{t\to0^+}\frac{f(0+t\cos\alpha,\ 0+t\cos\beta)-f(0,0)}{t}=\lim_{t\to0^+}\frac{\sqrt{t^2\cos^2\alpha+t^2\cos^2\beta}-0}{t}=1$

显然 $z=f(x,y)$ 在 $P(0,0)$ 处不连续 因 $\lim\limits_{\substack{x\to0\\ y^2=2x\to0}}f(x,y)=\lim\limits_{\substack{x\to0\\ y^2=2x\to0}}1=1\neq f(0,0)=0$

例 $u=x^2y^2+y^2z^2+z^2x^2$ $\vec l=(1,1,1)$ 求 $\left.\frac{\partial u}{\partial l}\right|_{P(1,1,1)}$ $\vec{l^0}=\left(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right)$

$$\left.\frac{\partial u}{\partial l}\right|_{P(1,1,1)}=\left(u'_x\cos\alpha+u'_y\cos\beta+u'_z\cos\gamma\right)_{P(1,1,1)}$$
$$=\left[(2xy^2+2xz^2)\frac{1}{\sqrt3}+(2yz^2+2yx^2)\frac{1}{\sqrt3}+(2zy^2+2zx^2)\frac{1}{\sqrt3}\right]_{P(1,1,1)}$$
$$=3\cdot\frac{4}{\sqrt3}$$

(***) 梯度 $\begin{cases} z=f(x,y)\quad \nabla z=\frac{\partial z}{\partial x}\vec i+\frac{\partial z}{\partial y}\vec j=\left(\frac{\partial}{\partial x}\vec i+\frac{\partial}{\partial y}\vec j\right)z \quad \text{或记为 } \mathrm{grad}\,z=\frac{\partial z}{\partial x}\vec i+\frac{\partial z}{\partial y}\vec j\\ \nabla=\frac{\partial}{\partial x}\vec i+\frac{\partial}{\partial y}\vec j \quad \text{“}\nabla\text{”(Nabla) 读 为 Hamilton 算子 ……}\end{cases}$

(八) 梯度

$z=f(x,y)$ $\quad \nabla z=\frac{\partial z}{\partial x}\vec{i}+\frac{\partial z}{\partial y}\vec{j}=\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}\right)z$ 或记为 $\operatorname{grad} z=\frac{\partial z}{\partial x}\vec{i}+\frac{\partial z}{\partial y}\vec{j}$

$\nabla=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}$ "∇" 读 Nabla 为 Hamilton 哈密顿算符，也称梯度算子

$=\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ 读 del $\quad \nabla z$ 也记为 $\operatorname{grad} z$

$u=f(x,y,z)$ $\quad \nabla u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}+\frac{\partial u}{\partial z}\vec{k}=\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)z$

$\nabla=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}$ $\quad \nabla u$ 也记为 $\operatorname{grad} u$

$=\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$\operatorname{Prj}_{\vec{b}}\vec{a}=\vec{a}\cdot\vec{b}^{\circ}=|\vec{a}||\vec{b}^{\circ}|\cos\theta=|\vec{a}|\cos\theta$

(1) 定理 $z=z(x,y)$ 可微，$\vec{l}^{\circ}=(\cos\alpha, \cos\beta)$，$P(x,y)$

$$\frac{\partial z}{\partial l}=\frac{\partial z}{\partial x}\cos\alpha+\frac{\partial z}{\partial y}\cos\beta=\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\cdot(\cos\alpha, \cos\beta)=\nabla z\cdot\vec{l}^{\circ}=\operatorname{Prj}_{\vec{l}^{\circ}}\nabla z$$

(2) 定理 $u=u(x,y,z)$ 可微 $\vec{l}^{\circ}=(\cos\alpha, \cos\beta, \cos\gamma)$ $P(x,y,z)$

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma=\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)\cdot(\cos\alpha, \cos\beta, \cos\gamma)=\nabla u\cdot\vec{l}^{\circ}=\operatorname{Prj}_{\vec{l}^{\circ}}\nabla u$$

梯度含义

$z=z(x,y)$ (等高线) $\quad \nabla z=z'_x\vec{i}+z'_y\vec{j}$

$u=f(x,y,z)$ (等值面) $\quad \nabla u=u'_x\vec{i}+u'_y\vec{j}+u'_z\vec{k}$