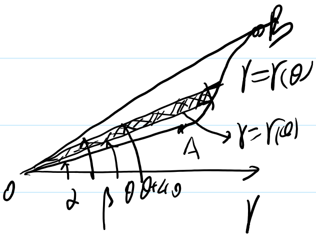
# 定识分的应用

2022年2月27日 16:54

微元法： 求$[a,b]$上的某个量$A$

1) 微分 $\forall$取"点"$[x, x+\Delta x]\subset[a,b]$ 计算其上$\Delta A$近似值，$\Delta A\approx \underset{\sim}{dA}=A'(x)\Delta x=f(x)dx$ 误差 $o(\Delta x)$

2) 积分 $A=\int_a^b dA=\int_a^b f(x)dx$

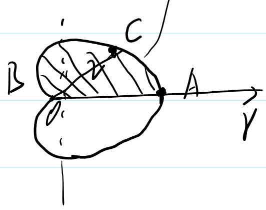



求$[\alpha,\beta]$上面积量$A$

1) $\forall[\theta,\theta+\Delta\theta]\subset[\alpha,\beta]$ $\Delta A\approx dA=\frac{1}{2}\cdot r\Delta\theta\cdot r=\frac{1}{2}r^2(\theta)d\theta$

2) $A=\int_\alpha^\beta dA=\int_\alpha^\beta\frac{1}{2}r^2(\theta)d\theta$

例 求 $r=a(1+\cos\theta)$ 围面积

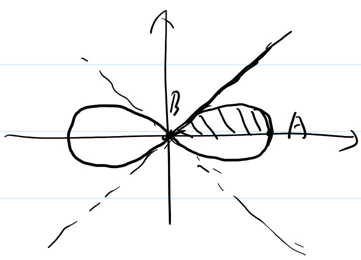



$0=a(1+\cos\theta)\Rightarrow\theta=\pi$ $\theta\in[0,\pi]$

$S=2\cdot\frac{1}{2}\int_0^\pi r^2(\theta)d\theta=2\cdot\frac{1}{2}\int_0^\pi a^2(1+\cos\theta)^2d\theta=a^2\int_0^\pi\left[1+2\cos\theta+\frac{1+\cos2\theta}{2}\right]d\theta$

$=\frac{3}{2}a^2\pi$

例 求 $r^2=a^2\cos2\theta$ 所围的面积



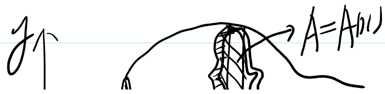

$0=a^2\cos2\theta\Rightarrow2\theta=\pm\frac{\pi}{2}\Rightarrow\theta=\pm\frac{\pi}{4}$

$\theta\in[0,\frac{\pi}{4}]$

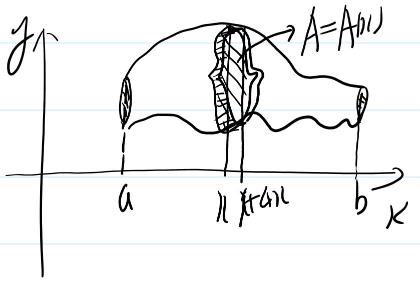
$S=4\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}}r^2(\theta)d\theta=2\int_0^{\frac{\pi}{4}}a^2\cos2\theta\, d\theta=\cdots$

体积 {
- 截面面积已知的立体体积
- 旋转体体积

1) 截面积已知的立体体积




求$[a,b]$上体积量$V$



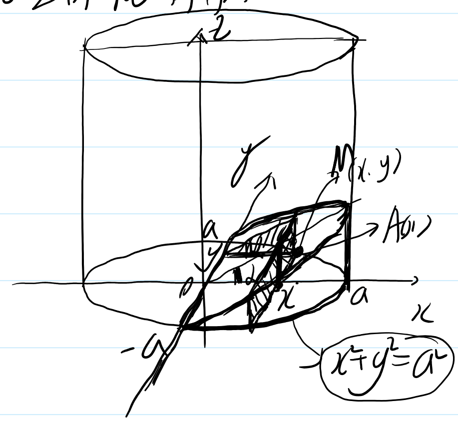

求$[a,b]$上体积量$V$

1) $\forall [x, x+\Delta x] \subset [a,b]$,

$$\Delta V \approx dV = A(x)\Delta x = A(x)dx$$

2) $$V=\int_a^b dV = \int_a^b A(x)dx$$

例 一平面过半径为R的圆柱体的底圆圆心，并与底面成α角，求此平面所截得的立体的体积




$$A(x) = 2y\cdot x\tan\alpha = 2\sqrt{a^2-x^2}\cdot x\tan\alpha$$

$$V=\int_0^a A(x)dx = \int_0^a 2\sqrt{a^2-x^2}\cdot x\cdot\tan\alpha\, dx$$

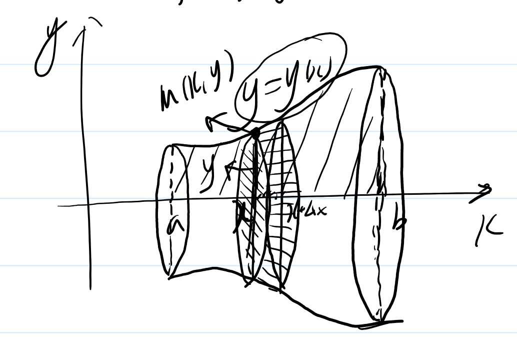
$$= \tan\alpha\int_0^a 2\sqrt{a^2-x^2}\cdot\left(-\frac{1}{2}\right)d(a^2-x^2)$$

$$= -\tan\alpha\int_0^a (a^2-x^2)^{\frac{1}{2}}d(a^2-x^2) = \frac{2}{3}a^3\tan\alpha$$

或 $$A(y) = \frac{1}{2}\cdot x\cdot x\tan\alpha = \frac{1}{2}(a^2-y^2)\tan\alpha$$

$$V=\int_{-a}^a A(y)dy = \int_{-a}^a \frac{1}{2}(a^2-y^2)\tan\alpha\, dy = \text{——}$$

旋转体体积 (1) (切片法)

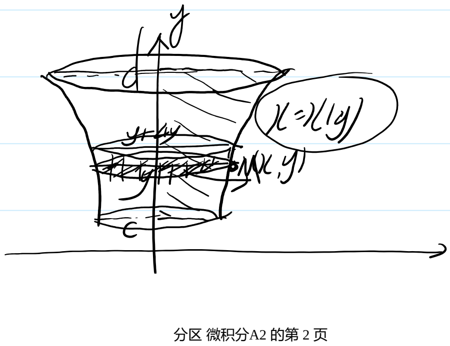



求$[a,b]$上体积量$V$

1) $\forall [x, x+\Delta x] \subset [a,b]$ 误差为$o(\Delta x)$

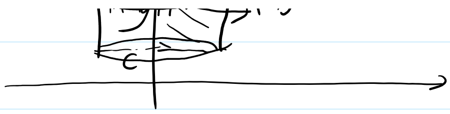
$$\Delta V \approx dV = \pi y^2\Delta x = \pi y^2 dx$$

2) $$V_x = \int_a^b dV = \int_a^b \pi y^2 dx = \int_a^b \pi y^2(x)dx$$



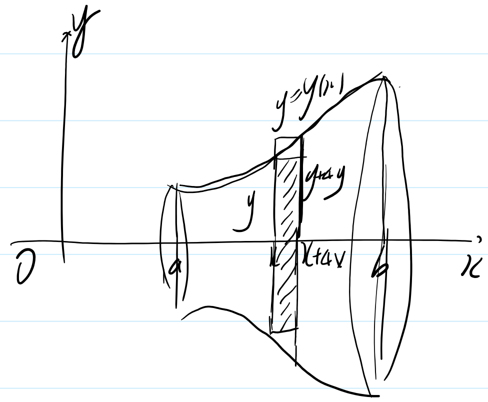

1) $$\Delta V \approx dV = \pi x^2 dy$$

2) $$V_y = \int_c^d dV = \int_c^d \pi x^2 dy = \int_c^d \pi x^2(y)dy$$



2) $V_y = \int_c^d dV = \int_c^d \pi x^2 dy = \int_c^d \pi x^2(y) dy$

简要证明:



例 $\Delta V_{min} \approx \pi y^2 dx$

$\Delta V_{max} \approx \pi (y+\Delta y)^2 dx$

$\Delta V_{min} < \Delta V < \Delta V_{max}$ $o(\Delta x)$

误差 $|\Delta V - \pi y^2 dx| \le \Delta V_{max} - \Delta V_{min}$

$= \pi (y+\Delta y)^2 dx - \pi y^2 dx$

$= (2\pi y \cdot \Delta y + \pi \Delta y^2) dx$

$\alpha = o(\beta) \iff \lim_{\Delta x \to 0} \frac{\alpha}{\beta} = 0$

$\Delta y = dy + o(\Delta x)$
$= y'(x) dx + o(\Delta x)$

$0 \le \left| \frac{\Delta V - \pi y^2 dx}{\Delta x} \right| \le \frac{(2\pi y \Delta y + \pi \Delta y^2)\cancel{dx}}{\cancel{\Delta x}}$

$= \underbrace{2\pi y[y'dx + o(\Delta x)] + \pi(y'dx + o(\Delta x))^2}$

$= 2\pi y\left[y'\Delta x + \frac{o(\Delta x)}{\Delta x}\Delta x\right] + \pi\left[y'\Delta x + \frac{o(\Delta x)}{\Delta x}\Delta x\right]^2$

$\to 0$

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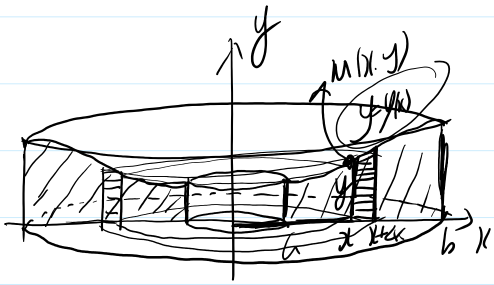
旋转体体积 (2) 剥皮法 (柱壳法)

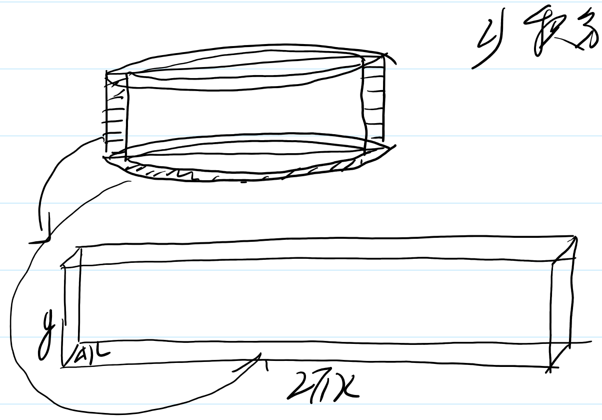
求 $[a,b]$ 上体积为 $V$



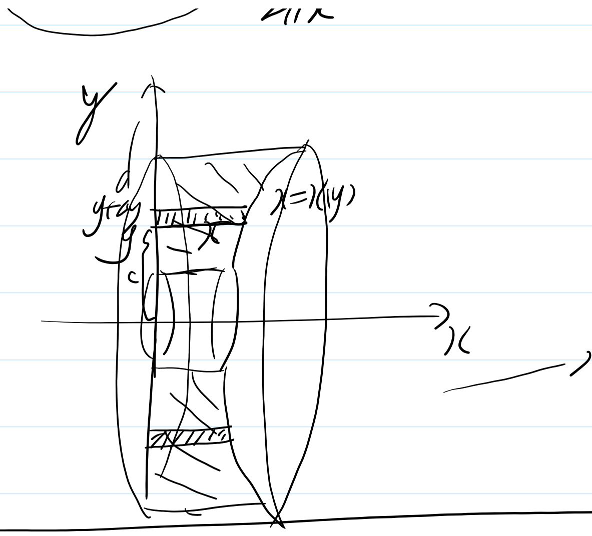
1) 微分 $\forall [x, x+\Delta x]$, 计算其上 $\Delta V$ 近似值 $dV$

$\Delta V \approx dV = \underset{\text{周长}}{2\pi x}\ \underset{\text{高}}{y} \cdot \underset{\text{厚度}}{dx}$

2) 积分 $V_y = \int_a^b dV = 2\pi \int_a^b x y\, dx = 2\pi \int_a^b x y(x) dx$



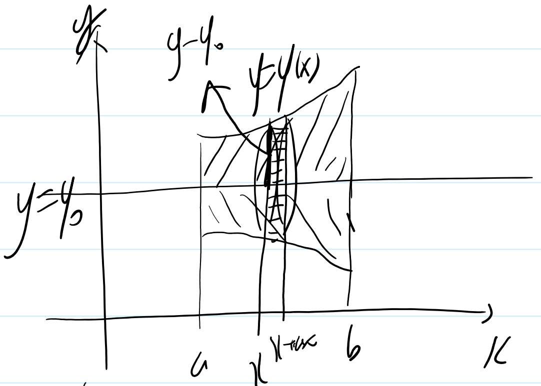
$V_y = 2\pi \int x y(x) dx$



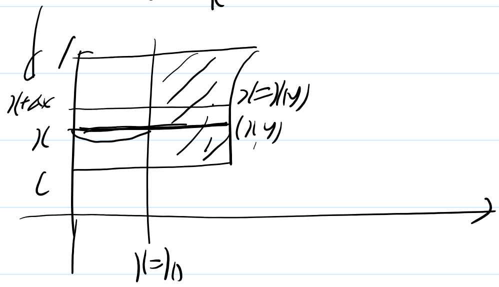
$$V_y = 2\pi\int x\,y(x)\,dx$$

$$\downarrow$$

$$V_x = 2\pi\int y\,x(y)\,dy$$

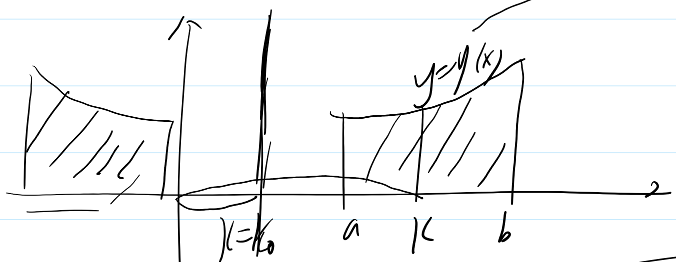


$$V_{y=y_0} = \pi\int_a^b (y-y_0)^2\,dx = \pi\int_a^b (y(x)-y_0)^2\,dx$$

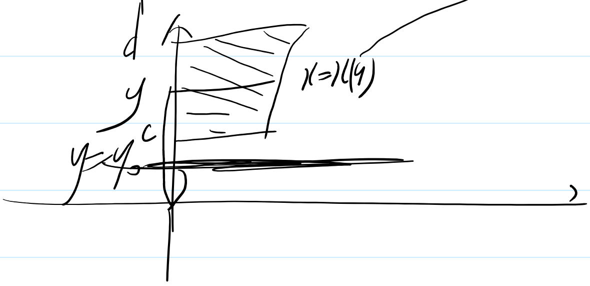


$$V_{x=x_0} = \pi\int_c^d (x-x_0)^2\,dy$$

$$= \pi\int_c^d [x(y)-x_0]^2\,dy$$



$$V_{x=x_0} = 2\pi\int_a^b (x-x_0)\,y\,dx$$



$$V_{y=y_0} = \int_c^d 2\pi\,(y-y_0)\,x\,dy$$