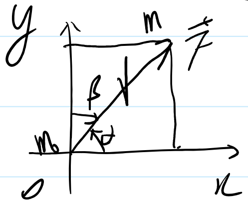
定积分的应用6-1

2022年3月18日 5:19

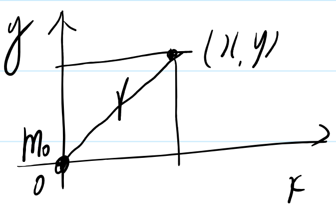
引力：$\boxed{\vec F = |\vec F|\,\vec F^0 = |\vec F|\left(\frac{x}{|\vec r|}, \frac{y}{|\vec r|}\right)}$

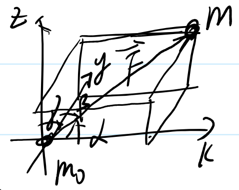
$\vec a=(x,y)$
$\vec a = |\vec a|\,\vec a^0 = |\vec a|\left(\frac{x}{|\vec a|}, \frac{y}{|\vec a|}\right)$
$= |\vec a|(\cos\alpha, \cos\beta)$



$= |\vec F|(\cos\alpha, \cos\beta)$
$= (|\vec F|\cos\alpha, |\vec F|\cos\beta) = (\vec F_x, \vec F_y)$

故 $\boxed{\vec F_{平面} = |\vec F|\left(\frac{x}{|\vec r|}, \frac{y}{|\vec r|}\right) = G\frac{m_0 m}{r^2}\left(\frac{x}{r}, \frac{y}{r}\right) = \left(G\frac{m_0 m x}{r^3}, G\frac{m_0 m y}{r^3}\right)}$

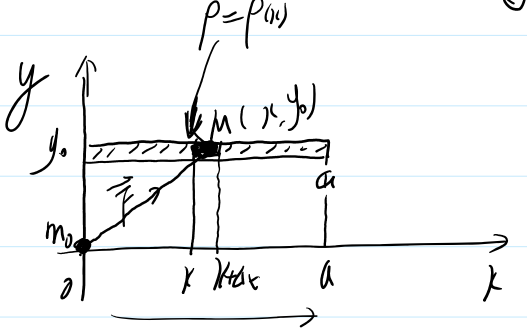




$\vec F_{空} = \left(G\frac{m_0 m x}{r^3}, G\frac{m_0 m y}{r^3}, G\frac{m_0 m z}{r^3}\right)$
$= (|\vec F|\cos\alpha, |\vec F|\cos\beta, |\vec F|\cos\gamma)$
$(\vec F_x, \vec F_y, \vec F_z)$

例 求密度不均匀的细棒 $\left(\begin{matrix}0\le x\le a\\ y=y_0\end{matrix}\right)$ 对 $(0,0)$ 处的质量为 m_0 的质点的引力

$\rho=\rho(x)$



求 $[0,a]$ 上引力 $\vec F$

1) 微分 $\forall [x, x+dx]\subset[0,a]$ 计算其上 $\Delta\vec F$ 近似 $d\vec F$

$$\Delta\vec F \approx d\vec F = \left(G\frac{m_0 \overset{\rho(x)dx}{m}\, x}{r^3}, G\frac{m_0 \overset{\rho(x)dx}{m}\, y_0}{r^3}\right)$$

(其中 $r=\sqrt{x^2+y_0^2}$)

$$= \left(G\frac{m_0\rho x}{\sqrt{x^2+y_0^2}^3}dx, G\frac{m_0\rho y_0 dx}{\sqrt{x^2+y_0^2}^3}\right)$$

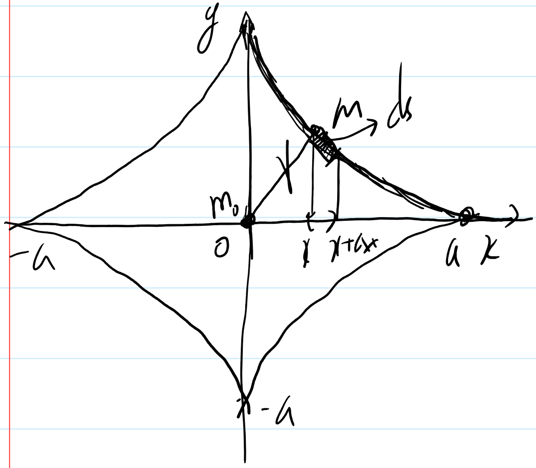
如 $\rho=\rho(x)=\sqrt{x^2+y_0^2}$

2) $\vec F = \left(\int_0^a G\frac{m_0\rho(x)x}{\sqrt{x^2+y_0^2}^3}dx, \int_0^a G\frac{m_0\rho(x)y_0}{\sqrt{x^2+y_0^2}^3}dx\right)$

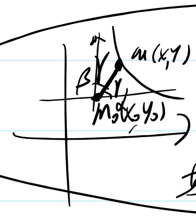
例 求星形线 $\begin{cases}x=a\cos^3\theta\\ y=a\sin^3\theta\end{cases}$ (参数 θ 非曲线上点的向径与 x 轴正向夹角) $(0\le\theta\le\frac{\pi}{2})$，对位于

星形线处质量为 m_0 的质点引力（设曲线上 M 点密度与 M 点到坐标原点的距离立方成正比）

$\rho = k|OM|^3$



1) $d\vec{F} \approx d\vec{F} = \left(\frac{G m_0 m x}{r^3}, \frac{G m_0 m y}{r^3} \right)$



$\left(\frac{G m_0 m (x-x_0)}{r^3}, G\frac{m_0 m (y-y_0)}{r^3} \right)$

其中 $r = \sqrt{(x-x_0)^2+(y-y_0)^2}$

1) $d\vec{F} \approx d\vec{F} = \left(\frac{G m_0 \rho ds\, x}{r^3}, G\frac{m_0 \rho ds \cdot y}{r^3} \right)$ 其中 $r = \sqrt{x^2+y^2}$

$= \left(\frac{G m_0 k r^3 ds\, x}{r^3}, \frac{G m_0 k r^3 ds\, y}{r^3} \right)$

$\begin{cases} x = a\cos^3\theta \\ y = a\sin^3\theta \end{cases}$

2) $\vec{F} = \left(\int_0^a G m_0 k x\, ds \quad \int_0^a G m_0 k y\, ds \right)$

$ds = \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta$

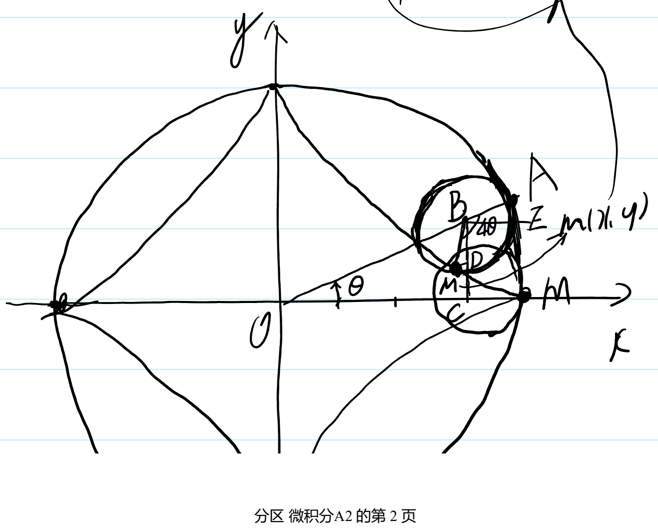
$= \sqrt{[3a\cos^2\theta(-\sin\theta)]^2 + [3a\sin^2\theta\cos\theta]^2}\, d\theta$

$= 3a\cos\theta\sin\theta\, d\theta$

$= \left(\int_0^{\frac{\pi}{2}} G m_0 k \cdot a\cos^3\theta\; 3a\cos\theta\sin\theta\, d\theta, \int_0^{\frac{\pi}{2}} G m_0 k\, a\sin^3\theta\; 3a\cos\theta\sin\theta\, d\theta \right)$

$= 3a^2 G m_0 k \left(\int_0^{\frac{\pi}{2}} \cos^4\theta \sin\theta\, d\theta, \int_0^{\frac{\pi}{2}} \sin^4\theta \cos\theta\, d\theta \right) = \left(\frac{3}{5}a^2 k G m_0, \frac{3}{5}a^2 k G m_0 \right)$

例 四尖点星形线 $\begin{cases} x = a\cos^3\theta \\ y = a\sin^3\theta \end{cases}$（圆的内摆线）



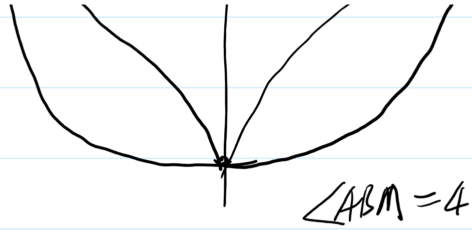
大小圆半径比为 4:1

用了 3 倍角公式

$\cos 3\theta = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2(1-\cos^2\theta)\cos\theta$



$\angle ABM = 4\theta$

因为 $\overset{\frown}{AD} = \overset{\frown}{AM}$

$\angle ABZ = \theta \quad \angle ZBM = 3\theta$

$= \cdots$

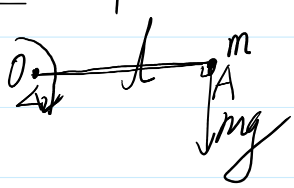
$= (2\cos^2\theta - 1)\cos\theta - 2(1-\cos^2\theta)\cos\theta$

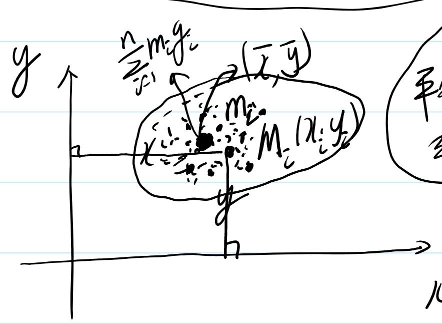
$= 4\cos^3\theta - 3\cos\theta$

因为 $\cos 3\theta =$ ______

重心、质心、形心

力矩：重力矩 $H_{重} = F_{重} \cdot l_{距离}$ $\begin{cases} 重心 \\ 质心 \\ 形心 \end{cases}$　静力矩 $= m_{质量} \cdot l_{距离}$ $\begin{cases} 质心 \\ 形心 \end{cases}$





平面薄板
平面曲线

单质点 $\begin{cases} H_{力矩y} = mgx \\ H_{力矩x} = mgy \end{cases}$

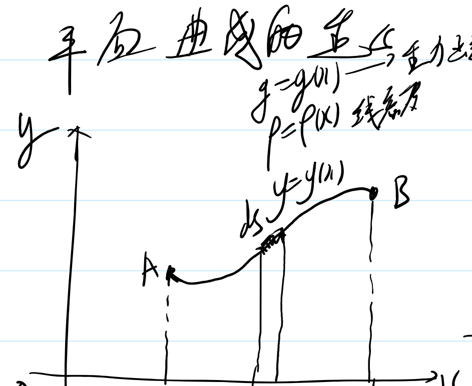
质点系 $\begin{cases} H_y = \sum_{i=1}^{n} m_i g_i x_i = \left(\sum_{i=1}^{n} m_i g_i\right) \bar{x}_{重心} \\ H_x = \sum_{i=1}^{n} m_i g_i y_i = \left(\sum_{i=1}^{n} m_i g_i\right) \bar{y}_{重心} \end{cases}$

$\Omega \begin{cases} H_y = \lim_{\lambda\to 0} \sum_{i=1}^{n} m_i g_i x_i = \left(\lim_{\lambda\to 0} \sum_{i=1}^{n} m_i g_i\right) \bar{x}_{重心} \\ H_x = \lim_{\lambda\to 0} \sum_{i=1}^{n} m_i g_i y_i = \left(\lim_{\lambda\to 0} \sum_{i=1}^{n} m_i g_i\right) \bar{y}_{重心} \end{cases}$

1) 平面曲线的重心

$g = g(x)$ → 重力函数

$\rho = \rho(x)$ 线密度



$$H_y = \lim_{\lambda\to 0} \sum_{i=1}^{n} \underbrace{m_i}_{=\rho(x_i)\Delta s_i} \underbrace{g_i}_{=g(x_i)} x_i = \left(\lim_{\lambda\to 0} \sum_{i=1}^{n} \underbrace{m_i}_{\rho(x_i)\Delta s_i} \underbrace{g_i}_{g(x_i)}\right) \bar{x}$$

$$* \quad H_y = \int_a^b \rho g x \, ds = \left(\int_a^b \rho g \, ds\right) \bar{x}$$

$$H_x = \lim_{\lambda\to 0} \sum_{i=1}^{n} \underbrace{m_i}_{=\rho(x_i)\Delta s_i} \underbrace{g_i}_{=g(x_i)} y_i = \left(\lim_{\lambda\to 0} \sum_{i=1}^{n} m_i g_i\right) \bar{y}$$

$* \quad H_x \cdots \int^b \cdots \int^b \cdots$

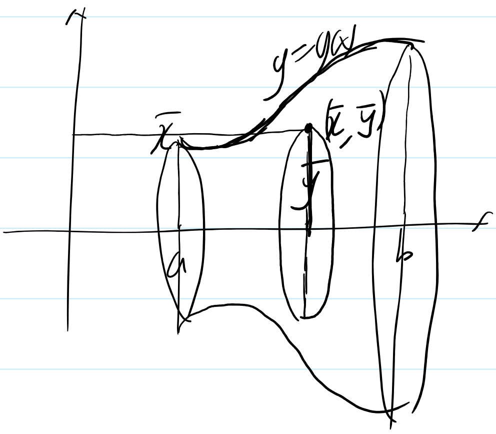
$$* \; M_x = \int_a^b \rho g y \, ds = \left(\int_a^b \rho g \, ds\right) \bar{y}$$

$$\Rightarrow \begin{cases} \bar{x} = \dfrac{\int_a^b \rho \, ds \, g x}{\int_a^b \rho \, ds \, g} ; & \bar{y} = \dfrac{\int_a^b \rho \, ds \, g y}{\int_a^b \rho \, ds \, g} \end{cases}$$

重心 $\rho = \rho(x)$, $g = g(x)$
$$\begin{cases} \bar{x} = \dfrac{\int_a^b \rho g x \, ds}{\int_a^b \rho g \, ds} \\ \bar{y} = \dfrac{\int_a^b \rho g y \, ds}{\int_a^b \rho g \, ds} \end{cases}$$

质心 $g = g_0$
$$\begin{cases} \bar{x} = \dfrac{\int_a^b \rho x \, ds}{\int_a^b \rho \, ds} \\ \bar{y} = \dfrac{\int_a^b \rho y \, ds}{\int_a^b \rho \, ds} \end{cases}$$

形心 $g = g_0$, $\rho = \rho_0$
$$\begin{cases} \bar{x} = \dfrac{\int_a^b x \, ds}{\int_a^b ds} \\ \bar{y} = \dfrac{\int_a^b y \, ds}{\int_a^b ds} \end{cases}$$



Golden定理

$2\pi \bar{x} \cdot \int_a^b ds = 2\pi \int_a^b x \, ds$ 绕y轴~

圆周长 · 弧长

$2\pi \bar{y} \cdot \int_a^b ds = 2\pi \int_a^b y \, ds$

圆周长 · 弧长

绕x轴旋转侧面积

$$\Delta W \approx dW = 400 \, dx + (2000 - 20t) \, dx + 50(30 - 5t) \cdot dx$$

$x = 5t$

$$W = \int_0^{30} dW = \cdots\cdots = 91500.$$