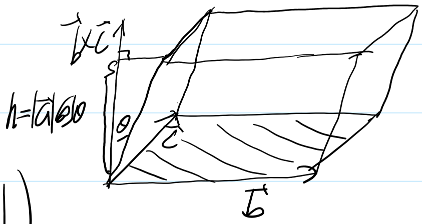
空间解析几何13-1

2022年4月12日 7:45

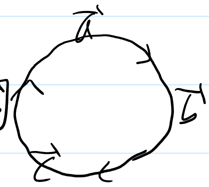


① 混合积 $\vec a\cdot(\vec b\times\vec c)=|\vec a|\,|\vec b\times\vec c|\cos\theta$

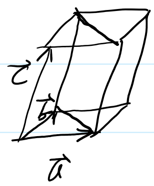
② $\vec a\cdot(\vec b\times\vec c)=(a_x,a_y,a_z)\cdot\left(\begin{vmatrix}b_y&b_z\\c_y&c_z\end{vmatrix},\begin{vmatrix}b_z&b_x\\c_z&c_x\end{vmatrix},\begin{vmatrix}b_x&b_y\\c_x&c_y\end{vmatrix}\right)$ $\quad V=\pm\,\vec a\cdot(\vec b\times\vec c)$

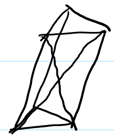
$=a_x\begin{vmatrix}b_y&b_z\\c_y&c_z\end{vmatrix}+a_y\begin{vmatrix}b_z&b_x\\c_z&c_x\end{vmatrix}+a_z\begin{vmatrix}b_x&b_y\\c_x&c_y\end{vmatrix}=\begin{vmatrix}a_x&a_y&a_z\\b_x&b_y&b_z\\c_x&c_y&c_z\end{vmatrix}$ 简记为 $[\vec a\,\vec b\,\vec c]$

③ 轮换性 对换性 $[\vec a\,\vec b\,\vec c]=[\vec b\,\vec c\,\vec a]=[\vec c\,\vec a\,\vec b]=-[\vec b\,\vec a\,\vec c]=-[\vec c\,\vec b\,\vec a]=-[\vec a\,\vec c\,\vec b]$



④ $\left|[\vec a\,\vec b\,\vec c]\right|=V_{\vec a\vec b\vec c\text{平行六面体}}=2V_{\vec a\vec b\vec c\text{三棱柱}}=6V_{\vec a\vec b\vec c\text{四面体}}$

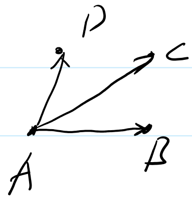


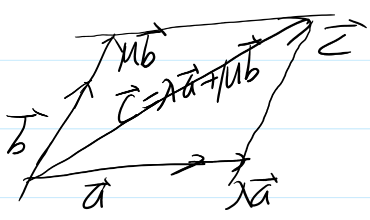


⑤ $\vec a,\vec b,\vec c$ 三向量共面 $\Longleftrightarrow[\vec a\,\vec b\,\vec c]=0$

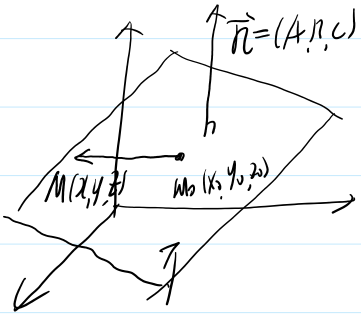
或 空间中四点 A, B, C, D 共面 $\Longleftrightarrow[\vec{AB}\,\vec{AC}\,\vec{AD}]=0$

或 线代中 $\vec a,\vec b,\vec c$ 三向量线性相关





平面及其方程



1) 点法式方程, $M_0(x_0,y_0,z_0)\in\pi$, $\vec n=(A,B,C)$ $\vec n\perp\pi$

定义 称垂直于平面的向量为平面的法向量

$\forall M(x,y,z)\in\pi$, 则 $\vec n\perp\vec{M_0M}\Longleftrightarrow\vec n\cdot\vec{M_0M}=0$

即 $(A,B,C)\cdot(x-x_0,y-y_0,z-z_0)=0$

即 1) 点法式方程 $\boxed{A(x-x_0)+B(y-y_0)+C(z-z_0)=0}$ 称为平面的点法式方程

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$\vec{n}=A,B,C$ $\quad M_0(x_0,y_0,z_0)\in\pi$

$$Ax+By+Cz\overbrace{-Ax_0-By_0-Cz_0}^{D}=0$$

即 2) 一般式方程 $\boxed{Ax+By+Cz+D=0}$

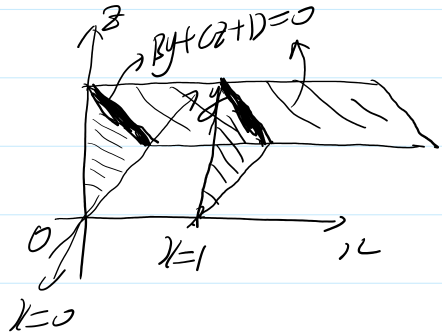
$\vec{n}=(A,B,C)$

① $D=0$, $Ax+By+Cz=0$, 则平面π过原点

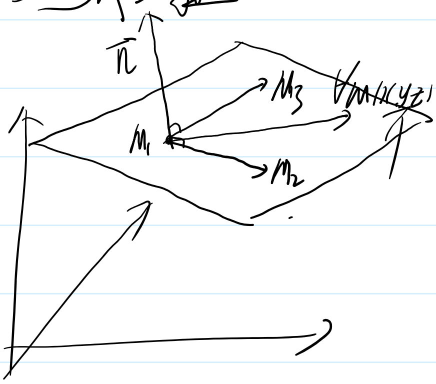
② $A=0$, $By+Cz+D=0$, $\pi\parallel x$轴

③ $A=B=0$, $Cz+D=0$, $\pi\parallel xOy$平面

④ $A=D=0$, $By+Cz=0$, π过x轴



3) 三点式方程 $\quad M_i(x_i,y_i,z_i)\in\pi$



法一：$M_1\ M_2\ M_3$ 代入 $Ax+By+Cz+D=0$

$\Rightarrow A:B:C=\lambda D:\mu D:\gamma D$ 代回即可

法二 $\quad \vec{n}=\overrightarrow{M_1M_2}\times\overrightarrow{M_1M_3}\perp\pi$

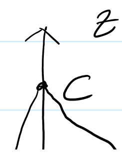
$M_1(x_1,y_1,z_1)\in\pi$

法三，则有 $[\overrightarrow{M_1M}\ \overrightarrow{M_1M_2}\ \overrightarrow{M_1M_3}]=0$

$$\boxed{\text{三点式方程}\quad \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0}$$

4) 截距式方程 $\quad Ax+By+Cz+D=0\quad (D\neq 0)$

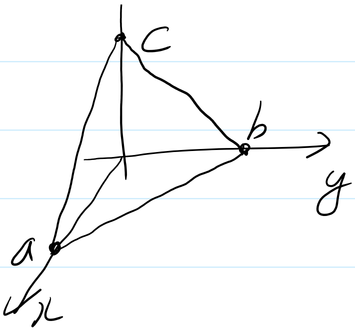
$\Leftrightarrow\quad Ax+By+Cz=-D$



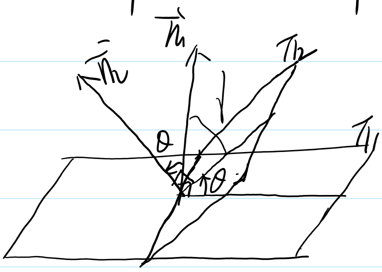
$\Leftrightarrow Ax+By+Cz=-D$

$\Leftrightarrow \frac{x}{-\frac{D}{A}}+\frac{y}{-\frac{D}{B}}+\frac{z}{-\frac{D}{C}}=1$

$\Leftrightarrow \boxed{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1}$



5) 平面间夹角

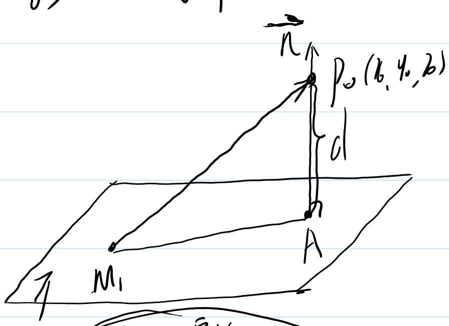


$\pi_i: A_ix+B_iy+C_iz+D_i=0 \quad (i=1,2)$

$\vec{n}_i=(A_i,B_i,C_i) \quad (i=1,2)$

$\cos\theta=\frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} \quad \theta\in(0,\frac{\pi}{2}]$

6) 点到平面的距离



$Prj_{\vec{b}}\vec{a} \overset{数}{=} \vec{a}\cdot\vec{b}^0$

设 $P_0(x_0,y_0,z_0)$ 为平面 π $Ax+By+Cz+D=0$ 外一点

$\vec{n}=(A,B,C)$

$\forall M_1(x_1,y_1,z_1)\in\pi$, $Ax_1+By_1+Cz_1+D=0$

则有 $d=\left|Prj_{\vec{n}}\overrightarrow{M_1P_0}\right|=\left|\overrightarrow{M_1P_0}\cdot\vec{n}^0\right|=\left|\frac{\overrightarrow{M_1P_0}\cdot\vec{n}}{|\vec{n}|}\right|$

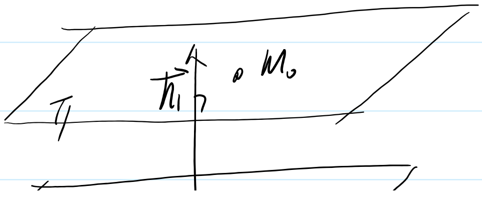
$=\left|\frac{(x_0-x_1,y_0-y_1,z_0-z_1)\cdot(A,B,C)}{|\vec{n}|}\right|$

$=\left|\frac{Ax_0+By_0+Cz_0+\underbrace{(-Ax_1-By_1-Cz_1)}_{=D}}{\sqrt{A^2+B^2+C^2}}\right|$

$=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$

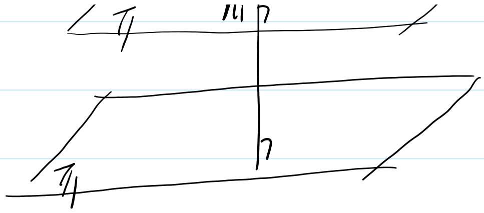
即 $\boxed{d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}}$

例 求过 $M_0(3,0,-1)$ 且与平面 π $3x-7y+5z-12=0$ 平行的平面方程



$\begin{cases}\vec{n}_1=(3,-7,5)\\ M_0\ (3,0,-1)\end{cases}$

$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$



$M_0(3,0,-1)$

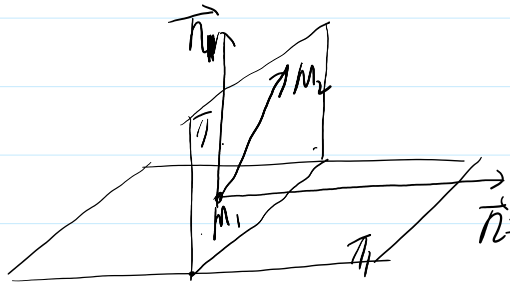
$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

$3(x-3)-(y-0)+5(z+1)=0$

$3x-y+5z-4=0$

例 求过点 $M_1(3,-5,1)$ 及 $M_2(4,1,2)$ 且垂直于 π_1: $x-8y+3z-1=0$ 的平面方程

$\vec{n_1}=(1,-8,3)$



解一：设 π: $Ax+By+Cz+D=0$ $\vec{n}=(A,B,C)$

$$\begin{cases} M_1\in\pi \\ M_2\in\pi \\ \vec{n}\perp\vec{n_1} \end{cases} \Rightarrow \begin{cases} A=\lambda D \\ B=\mu D \\ C=\gamma D \end{cases}$$ 代入 $Ax+By+Cz+D=0$ 即可

$\vec{n}=\overrightarrow{M_1M_2}\times\vec{n_1}$

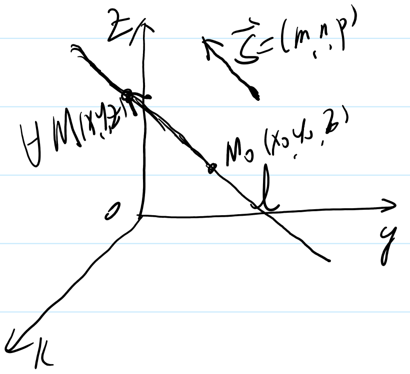
解二：

$$\vec{n}=\overrightarrow{M_1M_2}\times\vec{n_1}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 6 & 1 \\ 1 & -8 & 3 \end{vmatrix}$$

$=(26,\ 4,\ 2)$

$M_1\in\pi$ $\vec{n}\perp\vec{n_1}$ 点法式即可

直线及其方程



1) 已知 $M_0(x_0,y_0,z_0)\in l$, $\vec{s}=\{m,n,p\}$ 且 $\vec{s}\parallel l$

定义 若 $\vec{s}\parallel l$，则称 $\vec{s}$ 为直线 l 的方向向量

$\forall M(x,y,z)\in l$，则有 $\overrightarrow{M_0M}\parallel\vec{s}$ $\Longleftrightarrow$ 坐标成比例 $\Longleftrightarrow$ $\overrightarrow{M_0M}=\lambda\vec{s}$

$\overrightarrow{M_0M}=(x-x_0,\ y-y_0,\ z-z_0)$ $\vec{s}=(m,n,p)$

即 $$\boxed{\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}}$$ $M_0(x_0,y_0,z_0)\in l$, $\vec{s}=(m,n,p)$ 为方向向量

1) 直线的对称式方程或标准式方程

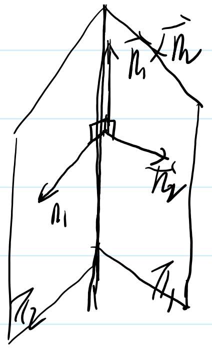
$(x-x_0,\ y-y_0,\ z-z_0)=\lambda(m,n,p)=(\lambda m,\ \lambda n,\ \lambda p)$

即 2) $$\begin{cases} x-x_0=\lambda m \\ y-y_0=\lambda n \\ z-z_0=\lambda p \end{cases}$$ 即 $$\boxed{\begin{cases} x=x_0+\lambda m \\ y=y_0+\lambda n \\ z=z_0+\lambda p \end{cases}}$$ $M_0(x_0,y_0,z_0)$ $\vec{s}=(m,n,p)$

直线参数式方程 $\lambda\in R$

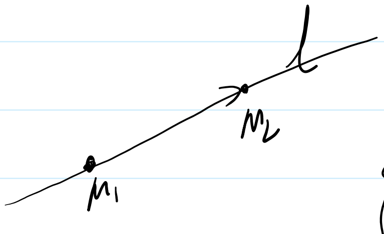
$z = z_0 - \lambda p$ $\quad$ $z = z_0 + \lambda p$

直线参数式方程 $\lambda \in R$



3) 直线的一般式方程 $\begin{cases} \pi_1 & A_1x+B_1y+C_1z+D_1=0 \\ \pi_2 & A_2x+B_2y+C_2z+D_2=0 \end{cases}$

$$\vec{s} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix}$$

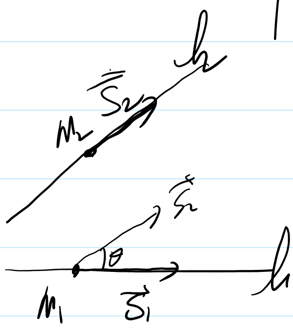


4. 两点式方程 $M_1(x_1,y_1,z_1) \in l$ $\quad M_2(x_2,y_2,z_2) \in l$

$$\begin{cases} \vec{s} = \overrightarrow{M_1M_2} = (x_2-x_1,\ y_2-y_1,\ z_2-z_1) \\ M_1(x_1,y_1,z_1) \in l \end{cases}$$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$ 即两点式方程

直线间关系 $\begin{cases} \text{共面, 异面} \\ \text{夹角} \begin{cases} \text{平行} \\ \text{垂直} \end{cases} \end{cases}$



$$l_i: \frac{x-x_i}{m_i} = \frac{y-y_i}{n_i} = \frac{z-z_i}{p_i} \quad i=1,2$$

$$\vec{s_i} = (m_i, n_i, p_i),\ M_i(x_i,y_i,z_i) \in l_i \quad i=1,2$$

$$\cos\theta = \frac{|\vec{s_1}\cdot\vec{s_2}|}{|\vec{s_1}||\vec{s_2}|} = \frac{|m_1m_2+n_1n_2+p_1p_2|}{\sqrt{m_1^2+n_1^2+p_1^2}\sqrt{m_2^2+n_2^2+p_2^2}} \quad \theta \in [0, \tfrac{\pi}{2}]$$

$$l_1 \parallel l_2 \iff \vec{s_1} \parallel \vec{s_2} \iff \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2}$$

$$l_1 \perp l_2 \iff \vec{s_1} \perp \vec{s_2} \iff \vec{s_1}\cdot\vec{s_2}=0 \iff m_1m_2+n_1n_2+p_1p_2=0$$