# IEOR 160: Homework 1

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## Question 1

Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = x^2(x-1)(x-2)$. Find all stationary points of this function and determine their types.

$$f'(x) = x^2(x^2 - 3x + 2) = x^4 - 3x^3 + 2x^2$$
$$f(x) = \frac{1}{5}x^5 - \frac{3}{4}x^4 + \frac{2}{3}x^3$$

FoC:

$$x^2(x-1)(x-2) = 0$$
$$x = 0, x = 1, x = 2$$

We now have the stationary points of this equation, now we need to find SoC.

$$f''(x) = 4x^3 - 9x^2 + 4x$$

$$f''(0) = 0, f'(1) = -1, f'(2) = 4$$

Therefore $\boxed{x* = 1 \text{ is a local max, and } x* = 2 \text{ is a local min.}}$
Since $f''(0) = 0$, we need to find another derivative.

$$f'''(x) = 12x^2 - 18x + 4$$
$$f'''(0) = 4$$

Since this is the 3rd derivative, then $\boxed{x* = 0 \text{ is a saddle point}}$

## Question 2

Find the globally optimal solution to

$$\begin{aligned} \max: \quad & x^3 - x \\ \text{s.t}: \quad & -1 \le x \le 2 \end{aligned}$$

$$f(x) = x^3 - x, f'(x) = 3x^2 - 1, f''(x) = 6x$$

FoC:

$$f'(x) = 3x^2 - 1 = 0 \implies x = \pm\frac{1}{\sqrt{3}}$$

SoC:

$$f''(-\frac{1}{\sqrt{3}}) = -\frac{6}{\sqrt{3}}, f''(\frac{1}{\sqrt{3}}) = \frac{6}{\sqrt{3}}$$

We can see that $-\frac{1}{\sqrt{3}}$ is a local max, while $\frac{1}{\sqrt{3}}$ is a local min. Therefore, we only care about the former. We also need to check the curvature at the endpoints ($-1$ and $2$ ).

$$f''(-1) = -6, f''(2) = 12$$

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We only care about the right endpoint because it is a at least a local max.

$$f(-\frac{1}{\sqrt{3}} \approx 0.3849, f(2) = 6$$

Since we know the rate of change of the slope is positive at $x = 2$, then we know that this is the maximum point on this interval.
Therefore, $x^* = 2$

## Question 3

Find all local solutions to :

$$\begin{aligned} \max: \quad & x^3 - 3x^2 + 4x - 1 \\ \text{s.t}: \quad & -2 \le x \le 4 \end{aligned}$$

FoC:

$$f(x) = x^3 - 3x^2 + 4x - 1, f'(x) = 3x^2 - 6x + 4, f''(x) = 6x - 6$$

When we try to solve for $f'(x) = 0$, we notice that there are no real solutions. Therefore, there are no inflection points on the graph.
Looking at endpoints $x = -2$ and $x = 4$

$$f''(-2) = -18, f''(4) = 18$$

Therefore, we notice that the left endpoint is a local min, while the right is a local max.
Thus, $x^* = 4$

## Question 4

Show that $\forall x$, we have $e^x \ge x + 1$.

$$\min_{x \in \mathbb{R}}: \quad f(x) = e^x - x - 1$$

If we are minimizing $f(x) = e^x - x - 1$, this is equivalent to finding the closest vertical point between $e^x$ and $x + 1$

$$f'(x) = e^x - 1, f''(x) = e^x$$

$$f'(x) = e^x - 1 = 0 \quad \implies \quad x = 0$$

$$f''(x) = 1$$

We know that at $x = 0$, the function has a local minimum because of a positive second derivative. If we look at the function of $e^x$, we notice that $f''(x) > \forall$X, indicating that $x^* = 0$ is a strict global min. Therefore, at $x^* = 0$, this is the closest vertical distance between $e^x$ and $x + 1$.
Plugging in $x = 0$ into both equations, we get 1. Thus the inequality is proven that $e^x \ge x + 1 \forall x$

## Question 5

Find all local minima, local maxima, and saddle points of the univariate function $f(x) = 49 \cdot x^{99} - 99 \cdot x^{49} + 1$
FoC:

$$f'(x) = 99 \cdot 49 \cdot x^{98} - 99 \cdot 49 \cdot x^{48} = 4851(x^{98} - x^{48}) = 4851x^{48}(x^{50} - 1)$$

$$f'(x) = 0 \implies x = 0, x = 1, x = -1$$

Looking for k:

$$f''(x) = 99 \cdot 48(98x^{97} - 48x^{47})$$

$$f''(1) = 99 \cdot 48(98 - 48) = 242500, f''(-1) = 99 \cdot 48(48 - 98) = -242500 f''(0) = 0$$

For $x = 1$, we can say that this is a local min because $f''(1) > 0$ . Similarly, for $x = -1$, we can say this is a local max because $f''(-1) < 0$ However, we need to find $k$ for $x = 0$.
In general, for $k > 1$

$$f^{(k)} = 99 \cdot 48 \left( \frac{98!}{(98 - (k-1))!} x^{98-(k-1)} - \frac{48!}{(48 - (k-1))!} x^{48-(k-1)} \right)$$

We would need all the $x$ to be eliminated to get a value $f^{(k)}(x) = 0$, which means that $k = 49$, or the 49th derivative of $f(x)$.
This would equate to $f^{(49)}(x) = 99 \cdot 48(0 - 48!)$, which is a constant. This indicated that $x = 0$ is a saddle point.

$\boxed{\text{x=1 is a local min, x=-1 is a local max, x=0 is a saddle point}}$

## Question 6

Given a natural number $n \in \{1, 2, 3, \ldots\}$, find all local minima, local maxima, saddle points, global minima, and global maximum of a univariate function $f(x)$ over the interval $[-10, 10]$ with the property $f'(x) = (x - 1)^{3n} + (x - 1)^n$