The Definite Integral

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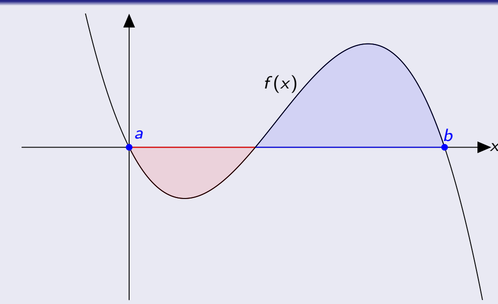
What is the Definite Integral?

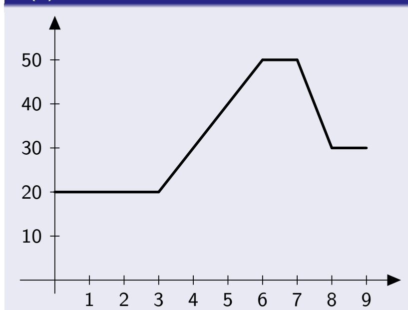
Roughly speaking, the definite integral is the area between the curve and the x-axis on an interval $[a, b]$.



Why do we care about the area under the curve?

Below is a graph of $v(t)$, the velocity (in cm/s) of an ice puck being pushed across a surface as a function of time (in sec).

$v(t)$



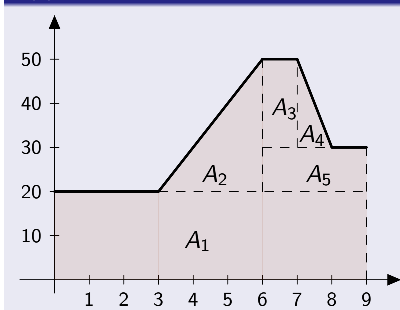
Question

Calculate the area under the "curve" on the interval $[0, 9]$. What does this area tell us about the ice puck?

Why do we care about the area under the curve?

Below is a graph of $v(t)$, the velocity (in cm/s) of an ice puck being pushed across a surface as a function of time (in sec).

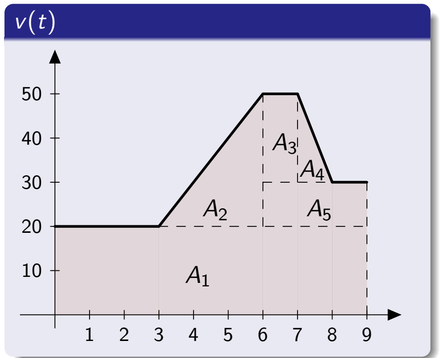
$v(t)$



$$
\begin{aligned}
A &= \sum_{i=1}^{5} A_i \\
&= A_1 + A_2 + A_3 + A_4 + A_5 \\
&= (180 + 45 + 20 + 10 + 30) \\
&= 285 \left(\frac{sec}{1} \cdot \frac{cm}{sec} \right) \\
&= 285 \left(\frac{\cancel{sec}}{1} \cdot \frac{cm}{\cancel{sec}} \right) \\
&= 285cm
\end{aligned}
$$

Why do we care about the area under the curve?

Below is a graph of $v(t)$, the velocity (in cm/s) of an ice puck being pushed across a surface as a function of time (in sec).



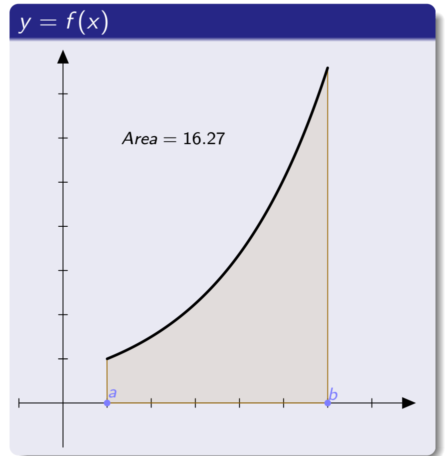
Question

Calculate the area under the "curve" on the interval $[0, 9]$. What does this area tell us about the ice puck?

Answer

The puck traveled a total distance of 285 cm in 9 seconds.

How would we find the area under this curve?



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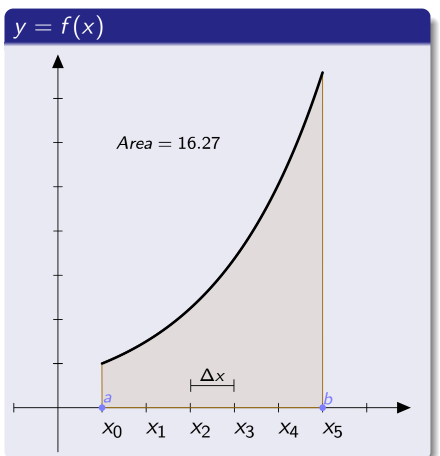


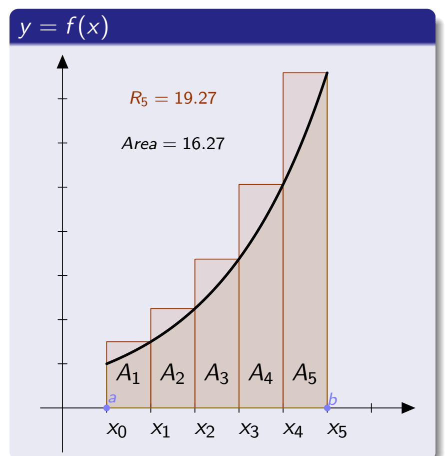
Cut the interval in n subintervals.

The length of each subinterval is:

$$\Delta x = \frac{b-a}{n}$$

Here, the graph shows $n = 5$ subintervals.

Area approximation using right endpoints (R_n)



$$A \approx A_1 + A_2 + \cdots + A_5$$

$$= f(x_1)\Delta x + \cdots + f(x_5)\Delta x$$

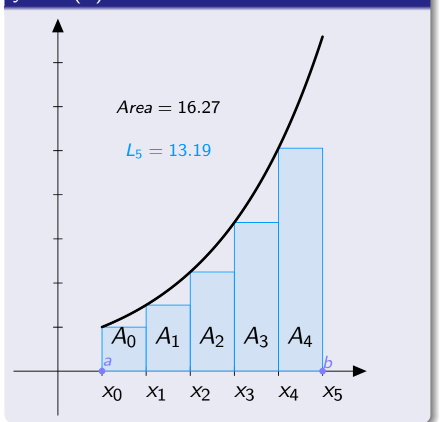
$$A \approx R_5 = \sum_{i=1}^{5} f(x_i)\Delta x$$

In General:

$$A \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

Area approximated using left endpoints (L_n)

$y = f(x)$



$$A \approx A_0 + A_1 + \cdots + A_4$$

$$= f(x_0)\Delta x + \cdots + f(x_4)\Delta x$$

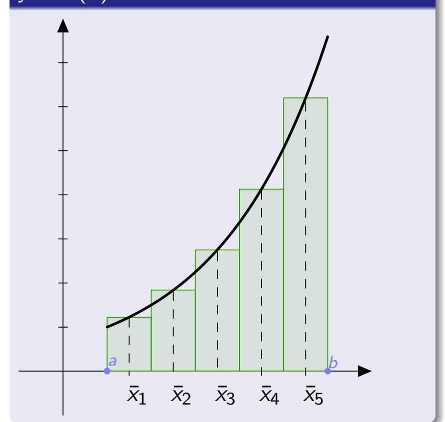
$$A \approx L_5 = \sum_{i=1}^{5} f(x_{i-1})\Delta x$$

In general:

$$A \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

Area approximated using midpoints (M_n)

$y = f(x)$

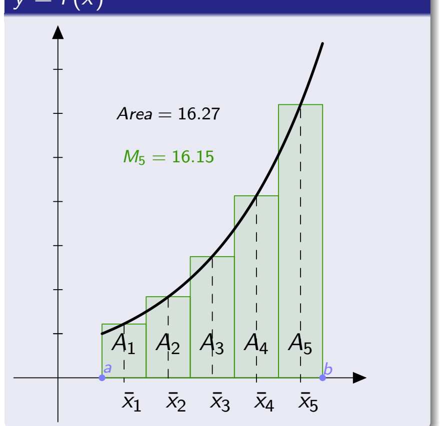


The midpoint of each subinterval is:

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2}$$

Area approximated using midpoints (M_n)

$y = f(x)$



$$A \approx A_1 + A_2 + \cdots + A_5$$

$$= f(\bar{x}_1)\Delta x + \cdots + f(\bar{x}_5)\Delta x$$

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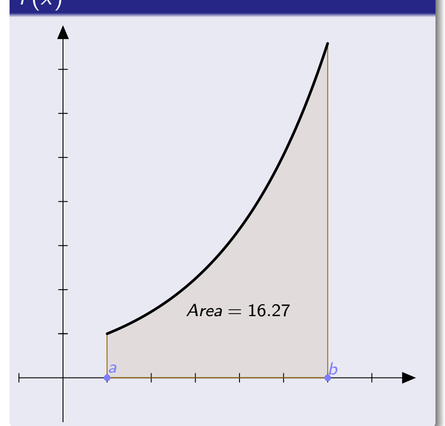
In general:

$$A \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$

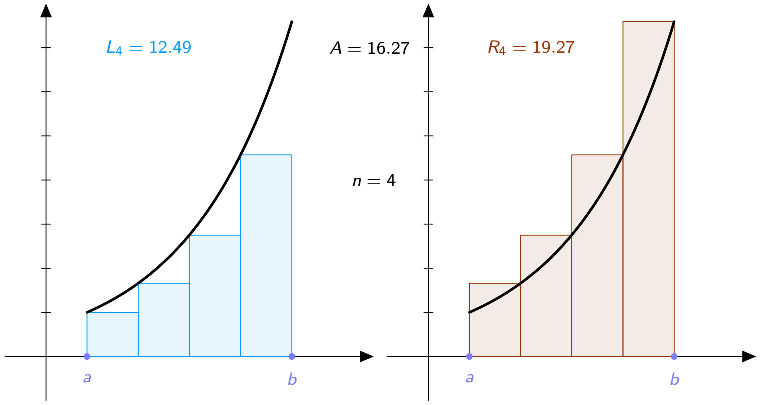
How can we use this idea to find the exact area?

$f(x)$



Instead of cutting the interval $[a, b]$ into a finite number of subintervals, cut the interval up into n subintervals and let the $n \to \infty$.

The approximate area converges to the exact area as the $n \to \infty$



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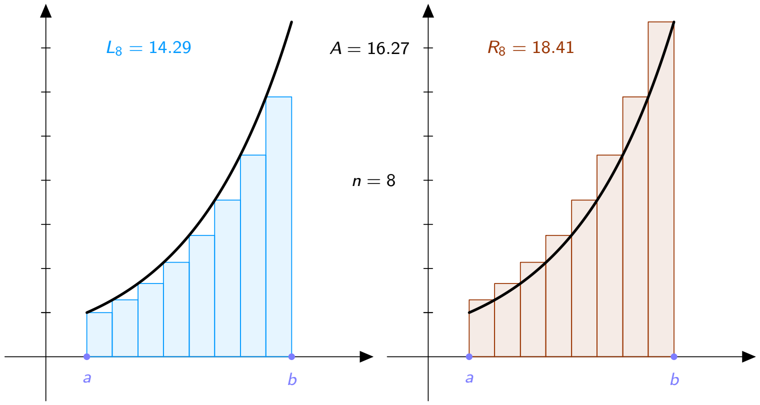

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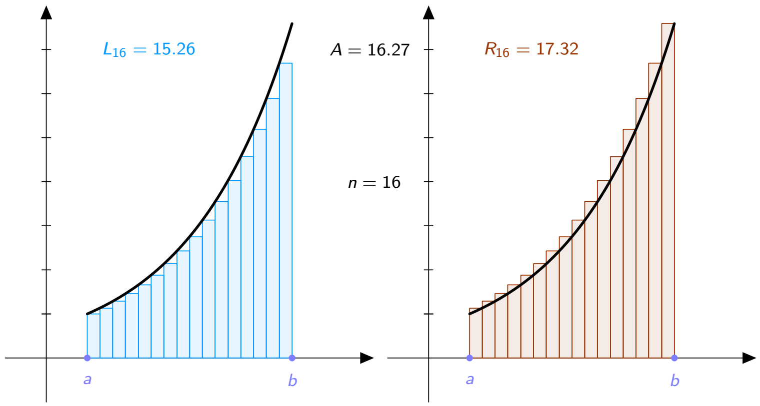

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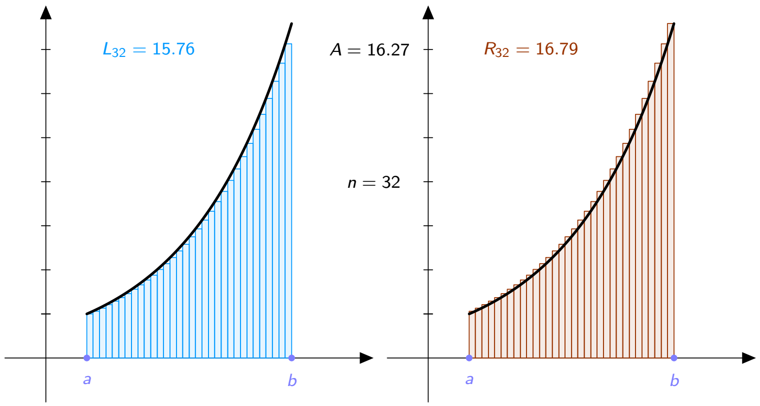

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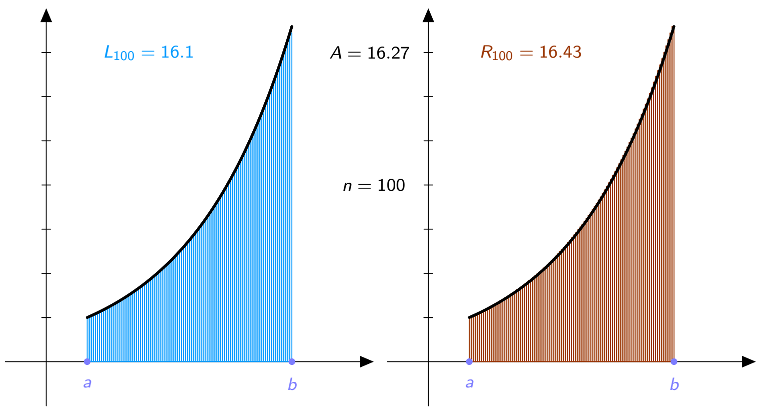
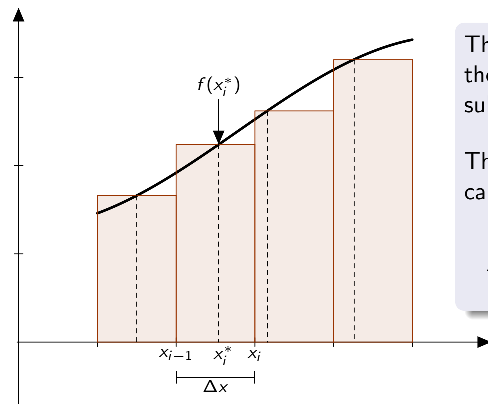


Discussion about the symbol x_i^*



The symbol, x_i^*, denotes the **sample point** in a subinterval $[x_{i-1}, x_i]$.

The exact area can be calculated by:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Definition of the definite integral

Definite Integral

The **definite integral of** f **on the interval** $[a, b]$ is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Where, f, is a continuous function defined for $[a, b]$.

The quantity Δx is the width of the subintervals found by:

$$\Delta x = \frac{b-a}{n}.$$

Every x_i^* is the **sample point** that lies in the i^{th} subinterval $[x_{i-1}, x_i]$.

Sum Formulas

You will use these in your bookwork:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$