Exercises in Fundamental Physics

(Undergraduate L3 – Graduate M1 Level)

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Contents

Abstract

This document gathers a selection of original exercises in fundamental physics, designed with a transversal and progressive perspective, from the third year of undergraduate studies to the first year of a Master's degree. Each exercise is accompanied by a detailed solution (when available), and is embedded in a rigorous historical, theoretical, or practical context. Topics covered include special relativity, quantum mechanics, statistical physics, electrodynamics, and incursions into mathematical physics. A classification by level is proposed to guide the reader's progression.

Chapter 1

Introduction

This document is a compilation of exercises in Fundamental Physics that I designed with passion, in the spirit of an end-of-L3 / M1 course, and beyond. The aim is twofold: to provide rigorous, inspiring problems that highlight the formal and conceptual beauty of physics, and to offer a solid foundation for students wishing to deepen their understanding of major classical and modern theories. I hope to share my enthusiasm for physics that goes beyond what is typically covered in class, drawing on concepts that span multiple areas of physics.

Each exercise involves specific concepts (indicated in parentheses, such as **(SR)** for Special Relativity, **(QM)** for Quantum Mechanics, etc.) and is gradually supplemented with a detailed correction, accessible by clicking on the "(Correction)" link. Exercises are rated with stars (see 2.1), and you are free to start with the one that intrigues you the most.

As a first-year Master's student in Fundamental Physics at Sorbonne University (Pierre and Marie Curie campus), I want this collection to remain dynamic: solutions will be added regularly. Lastly, in the correction section, by clicking on the exercise titles (either in the heading or at the beginning of the solution), you can return to the corresponding exercise.

I hope that by reading and working through these exercises, you will find as much enjoyment as I had in writing them.

Chapter 2

Information

2.1 Notations

1. Vector quantities are written in bold, except for the operator ∇, which is never in bold. Four-vector quantities (in relativity) are written with a Greek letter, in superscript for contravariant components, and in subscript for covariant ones.
   Example: $\mathbf{v}$ for velocity, ∇p for the pressure gradient (which is a vector!), and x^μ for space-time position in contravariant form. Conversely, in Quantum Mechanics, vectors are denoted using kets, and operators in bold.
   Example: $|\psi\rangle$ for a state vector ψ and $\mathbf{H}$ for the Hamiltonian.

2. The notation d denotes the differential operator.

3. The notation ∂_u implicitly means $\frac{\partial}{\partial u}$ if u is a variable, and $\partial_\mu = \frac{\partial}{\partial x^\mu}, \partial^\mu = \frac{\partial}{\partial x_\mu}$ in relativity.

4. $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$ in Cartesian coordinates, is an operator that properly defines the gradient, divergence, and curl. Indeed, ∇f is the gradient of f, $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$, and $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$. The operator $\partial_\mu \partial^\mu = \Box$ is the d'Alembertian, invariant under Lorentz transformations.

5. The notation $\dot{x}$ denotes a time derivative: $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$. In a relativity exercise, the preferred notation will be $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$, where τ is the proper time, and $\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$.

6. The notation f' denotes a derivative with respect to the variable x, i.e., $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$.

7. The notation $[A]$ indicates the physical unit of the quantity A.

8. The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}$ denote the sets of real, complex, and natural numbers, respectively.

9. The metric used in special relativity is $g_{\mu\nu} = (-,+,+,+)$. We also recall that $a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu = g^{\mu\nu} a_\mu b_\nu$.

10. Stars indicate the difficulty level of the exercises, ranging from 1: ★ to 5 stars: ★★★★★. The difficulty assessment is based on the length, technical and mathematical complexity, and the academic level (L3, M1, M2) needed to be comfortable with the concepts involved.

11. A dagger † means the exercise is either taken from or inspired by an existing one. A footnote will also appear in such cases.

12. The symbol $\triangle$ indicates that the solution is still being written.

2.2 Fundamental Constants

Constant	Exact value	Units
Planck constant	$h = 6.62607015 \times 10^{-34}$	$\mathrm{J\,s}$
Dirac constant	$\hbar = \frac{h}{2\pi} = 1.054571817 \times 10^{-34}$	$\mathrm{J\,s}$
Speed of light	$c = 299792458$	$\mathrm{m\,s^{-1}}$
Elementary charge	$e = 1.602176634 \times 10^{-19}$	C
Electron mass	$m_e = 9.1093837015 \times 10^{-31}$	kg
Proton mass	$m_p = 1.67262192369 \times 10^{-27}$	kg
Neutron mass	$m_n = 1.675 \times 10^{-27}$	kg
Vacuum permittivity	$\varepsilon_0 = 8.854187817 \times 10^{-12}$	$\mathrm{F\,m^{-1}}$
Vacuum permeability	$\mu_0 = 4\pi \times 10^{-7}$	$\mathrm{N\,A^{-2}}$
Gravitational constant	$G = 6.67430 \times 10^{-11}$	$\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Boltzmann constant	$k_B = 1.380649 \times 10^{-23}$	$\mathrm{J\,K^{-1}}$
Avogadro number	$\mathcal{N}_A = 6.02214076 \times 10^{23}$	$\mathrm{mol^{-1}}$
Ideal gas constant	$R = 8.314462618$	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Reference temperature (0°C)	$T_0 = 273.15$	K

Table 2.1: Fundamental physical constants with their exact values.

2.3 Formulary

2.3.1 Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \text{(Gauss's law)} \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Absence of magnetic monopoles)} \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's law)} \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampère-Maxwell law)} \tag{2.4}$$

$$\nabla \times \mathbf{A} = \mathbf{B}, \quad -\partial_t \mathbf{A} - \nabla \varphi = \mathbf{E} \quad \text{(Relation between the vector potential and the EM field)} \tag{2.5}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \text{(Electric displacement)} \tag{2.6}$$

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \quad \text{(Polarization)} \tag{2.7}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad \text{(Auxiliary magnetic field)} \tag{2.8}$$

$$\mathbf{M} = \chi_m \mathbf{H} \quad \text{(Magnetization)} \tag{2.9}$$

$$\varepsilon_r = 1 + \chi_e, \quad \mu_r = 1 + \chi_m \quad \text{(Relations to susceptibilities)} \tag{2.10}$$

$$v = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{c}{\sqrt{\varepsilon_r \mu_r}} = \frac{c}{n} \quad \text{(Wave speed in the medium)} \tag{2.11}$$

$$\Box \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0 \quad \text{(Wave equation in vacuum)} \tag{2.12}$$

$$\gamma = \sigma + i\omega\varepsilon \quad \text{(Complex conductivity)} \tag{2.13}$$

$$P = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3} \quad \text{(Larmor power)} \tag{2.14}$$

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad \text{(Poynting vector)} \tag{2.15}$$

2.3.2 Special Relativity

$$E = \gamma mc^2 = \sqrt{p^2c^2 + m^2c^4} \quad \text{(Relativistic energy)} \tag{2.16}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{(Lorentz factor)} \tag{2.17}$$

$$x' = \gamma(x - vt) \quad \text{(Lorentz transformation)} \tag{2.18}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad \text{(Time transformation)} \tag{2.19}$$

$$\beta = \frac{v}{c} \tag{2.20}$$

$$\mathbf{p} = \gamma m\mathbf{v} \quad \text{(Relativistic momentum vector)} \tag{2.21}$$

$$\mathbf{p} = \hbar\mathbf{k} \quad \text{(Photon momentum vector)} \tag{2.22}$$

2.3.3 Quantum Mechanics

$$\mathbf{P} = -i\hbar\nabla \quad \text{(Momentum operator)} \tag{2.23}$$

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \mathbf{H}\left|\psi\right\rangle \quad \text{(Schrödinger equation)} \tag{2.24}$$

$$[\mathbf{X}_i, \mathbf{P}_j] = i\hbar\delta_{ij} \quad \text{(Canonical commutation relation)} \tag{2.25}$$

$$\left\langle\psi\right|\mathbf{A}\left|\psi\right\rangle = \langle A\rangle \quad \text{(Expectation value)} \tag{2.26}$$

$$(\Delta A)^2 = \left\langle\psi\right|(\mathbf{A} - \langle A\rangle)^2\left|\psi\right\rangle \quad \text{(Variance of an observable)} \tag{2.27}$$

$$\Delta A\,\Delta B \geq \frac{1}{2}|\left\langle\psi\right|[\mathbf{A}, \mathbf{B}]\left|\psi\right\rangle| \quad \text{(Heisenberg uncertainty inequality)} \tag{2.28}$$

$$\mathbf{U}(t) = e^{-i\mathbf{H}t/\hbar} \quad \text{(Unitary time evolution)} \tag{2.29}$$

$$\mathbf{H}\left|E_n\right\rangle = E_n\left|E_n\right\rangle \quad \text{(Stationary eigenstates)} \tag{2.30}$$

$$\mathbb{P}(a_n) = |\left\langle a_n|\psi\right\rangle|^2 \quad \text{(Born rule probability)} \tag{2.31}$$

$$\mathbf{X} = \sqrt{\frac{\hbar}{2m\omega}}(\mathbf{a} + \mathbf{a}^\dagger), \quad \mathbf{P} = i\sqrt{\frac{\hbar m\omega}{2}}(\mathbf{a}^\dagger - \mathbf{a}) \quad \text{(Harmonic oscillator)} \tag{2.32}$$

$$[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{1} \quad \text{(Commutator)} \tag{2.33}$$

$$\mathbf{H} = \hbar\omega\left(\mathbf{N} + \frac{1}{2}\right) \quad \text{(Oscillator Hamiltonian)} \tag{2.34}$$

$$\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}, \quad \mathbf{N}\left|n\right\rangle = n\left|n\right\rangle \quad \text{(Number operator)} \tag{2.35}$$

$$\mathbf{L}_i = \varepsilon_{ijk}\mathbf{X}_j\mathbf{P}_k \quad \text{(Orbital angular momentum)} \tag{2.36}$$

$$[\mathbf{J}_i, \mathbf{J}_j] = i\hbar\varepsilon_{ijk}\mathbf{J}_k \quad \text{(Lie algebra of } SU(2)\text{)} \tag{2.37}$$

$$[\mathbf{H}, \mathbf{A}] = 0 \Rightarrow \mathbf{A} = \text{constant of motion} \quad \text{(Symmetry and conservation)} \tag{2.38}$$

2.3.4 Statistical Physics

Canonical ensemble (system in contact with a thermostat, with fixed number of particles)

$$\beta = \frac{1}{k_B T} \qquad \text{(Temperature energy)} \qquad (2.39)$$

$$Z = \sum_n e^{-\beta E_n} \qquad \text{(Partition function)} \qquad (2.40)$$

$$P_n = \frac{e^{-\beta E_n}}{Z} \qquad \text{(Occupation probability of level } n\text{)} \qquad (2.41)$$

$$\langle E \rangle = \sum_n E_n P_n = -\frac{\partial \ln Z}{\partial \beta} \qquad \text{(Mean energy)} \qquad (2.42)$$

$$\Delta E^2 = \langle E^2 \rangle - \langle E \rangle^2 = \frac{\partial^2 \ln Z}{\partial \beta^2} \qquad \text{(Energy fluctuation)} \qquad (2.43)$$

$$S = -k_B \sum_n P_n \ln P_n \qquad \text{(Shannon statistical entropy)} \qquad (2.44)$$

$$F = -k_B T \ln Z \qquad \text{(Helmholtz free energy)} \qquad (2.45)$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V \qquad \text{(Link with thermodynamics)} \qquad (2.46)$$

Grand canonical ensemble (system in contact with a particle and heat reservoir):

$$\mathcal{Z} = \sum_{N=0}^{\infty} \sum_n e^{-\beta(E_{n,N} - \mu N)} \qquad \text{(Grand partition function)} \qquad (2.47)$$

$$\mathcal{Z} = \prod_i \xi_i \qquad \text{(Factorization over states)} \qquad (2.48)$$

$$\xi_i = \sum_{n_i} e^{-\beta(\varepsilon_i - \mu) n_i} \qquad \text{(Partition function for state } i\text{)} \qquad (2.49)$$

$$\mathcal{J} = -k_B T \ln \mathcal{Z} \qquad \text{(Grand potential)} \qquad (2.50)$$

$$\mathcal{J} = -k_B T \sum_i \ln \xi_i \qquad \text{(Grand potential, factorized form)} \qquad (2.51)$$

$$P = -\left(\frac{\mathcal{J}}{V}\right) \qquad \text{(Pressure)} \qquad (2.52)$$

$$\langle N \rangle = \frac{1}{\beta} \frac{\partial \ln \mathcal{Z}}{\partial \mu} \qquad \text{(Mean number of particles)} \qquad (2.53)$$

$$\langle E \rangle = -\frac{\partial \ln \mathcal{Z}}{\partial \beta} + \mu \langle N \rangle \qquad \text{(Mean energy)} \qquad (2.54)$$

$$S = -\left(\frac{\partial \mathcal{J}}{\partial T}\right)_{V,\mu} \qquad \text{(Entropy)} \qquad (2.55)$$

$$F = \langle E \rangle - TS = \mathcal{J} + \mu \langle N \rangle \qquad \text{(Link with free energy)} \qquad (2.56)$$

2.3.5 Analytical Mechanics

$$\mathcal{L} = T - V \tag{Lagrangian}$$
(2.57)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \tag{Lagrange equations}$$
(2.58)

$$\mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}F(q,t)}{\mathrm{d}t} \tag{Non-uniqueness of the Lagrangian}$$
(2.59)

$$\mathcal{S}[q(t)] = \int_{t_1}^{t_2} \mathcal{L}(q,\dot{q},t)\,\mathrm{d}t \tag{Action}$$
(2.60)

$$\delta\mathcal{S} = 0 \tag{Principle of least action}$$
(2.61)

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \tag{Conjugate momentum}$$
(2.62)

$$\frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \Rightarrow p_\alpha = \text{const.} \tag{Cyclic variable}$$
(2.63)

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \sum_\alpha \dot{q}_\alpha p_\alpha - \mathcal{L} = \text{const.} \tag{Beltrami identity}$$
(2.64)

$$\mathcal{H}(q,p,t) = \sum_\alpha p_\alpha \dot{q}_\alpha - \mathcal{L} \tag{Hamiltonian}$$
(2.65)

$$\dot{q}_\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial \mathcal{H}}{\partial q_\alpha} \tag{Hamilton's equations}$$
(2.66)

$$\{f,g\} = \sum_\alpha \left(\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right) \tag{Poisson bracket}$$
(2.67)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f,\mathcal{H}\} + \frac{\partial f}{\partial t} \tag{Time evolution}$$
(2.68)

$$Q_\alpha = \frac{\partial F}{\partial P_\alpha}, \quad p_\alpha = \frac{\partial F}{\partial q_\alpha} \tag{Canonical transformation via $F_2(q,P,t)$}$$
(2.69)

$$\mathcal{K}(Q,P,t) = \mathcal{H}(q,p,t) + \frac{\partial F}{\partial t} \tag{New Hamiltonian}$$
(2.70)

2.3.6 Subatomic Physics

$$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \quad \text{(Elementary solid angle)} \tag{2.71}$$

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega \quad \text{(Total cross section)} \tag{2.72}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta,\varphi)|^2 \quad \text{(Differential cross section)} \tag{2.73}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\sin\theta}\left|\frac{\mathrm{d}b^2}{\mathrm{d}\theta}\right| \quad \text{(Classical differential cross section)} \tag{2.74}$$

$$B = \left[Zm_p + (A-Z)m_n - M(A,Z)\right]c^2 \quad \text{(Nuclear binding energy)} \tag{2.75}$$

$$Q = \left[m_{\text{initial}} - m_{\text{final}}\right]c^2 \quad \text{(Energy released in a reaction)} \tag{2.76}$$

$$N(t) = N_0 e^{-\lambda t} \quad \text{(Mean radioactive decay law)} \tag{2.77}$$

$$\tau = \frac{1}{\lambda}, \quad T_{1/2} = \frac{\ln 2}{\lambda} \quad \text{(Mean lifetime and half-life)} \tag{2.78}$$

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = -\lambda_i N_i + \lambda_{i-1}N_{i-1} \quad \text{(Decay chain)} \tag{2.79}$$

2.3.7 Wave Optics

$$j^2 = -1 \quad \text{(Imaginary unit)} \tag{2.80}$$

$$\psi(x) = \psi_0 e^{jk\delta} \quad \text{(Monochromatic plane wave)} \tag{2.81}$$

$$\mathrm{d}\psi = \psi_0 e^{j\varphi(x)}\mathrm{d}x \quad \text{(Diffracted field element)} \tag{2.82}$$

$$\psi(M) = \int \psi_0(x)e^{j\varphi(x)}\mathrm{d}x \quad \text{(Diffracted field – Fresnel integral)} \tag{2.83}$$

$$\varphi(x) = \frac{(x-x')^2}{2z} \quad \text{(Phase in the Fresnel approximation)} \tag{2.84}$$

$$I(x) = I_0\left[1 + \cos\left(\frac{2\pi}{\lambda}\delta(x)\right)\right] \quad \text{(Two-wave interference)} \tag{2.85}$$

$$I = \left|\int_A^B \mathrm{d}\psi\right|^2 \quad \text{(Superposition principle – intensity)} \tag{2.86}$$

$$L_{AB} = \int_A^B n(\mathbf{r})\mathrm{d}s \quad \text{(Optical path)} \tag{2.87}$$

$$\varphi = k\delta = \frac{2\pi}{\lambda}L_{AB} \quad \text{(Associated phase shift)} \tag{2.88}$$

$$i = \frac{\lambda D}{a} \quad \text{(Fringe spacing in Fraunhofer approximation)} \tag{2.89}$$

2.3.8 Thermodynamics

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N \qquad \text{(First law)} \tag{2.90}$$

$$\mathrm{d}S \geq \frac{\delta Q}{T} \qquad \text{(Second law)} \tag{2.91}$$

$$F = U - TS \qquad \text{(Helmholtz free energy)} \tag{2.92}$$

$$G = U + pV - TS = \mu N \qquad \text{(Gibbs free energy)} \tag{2.93}$$

$$H = U + pV \qquad \text{(Enthalpy)} \tag{2.94}$$

$$pV = Nk_BT = nRT \qquad \text{(Ideal gas law)} \tag{2.95}$$

$$U = \frac{f}{2}Nk_BT \qquad \text{(Internal energy, } f \text{ degrees of freedom)} \tag{2.96}$$

$$= \frac{3}{2}Nk_BT \qquad \text{(Monoatomic ideal gas)} \tag{2.97}$$

$$= \frac{5}{2}Nk_BT \qquad \text{(Diatomic ideal gas at high } T\text{)} \tag{2.98}$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{f}{2}Nk_B \qquad \text{(Heat capacity at constant volume)} \tag{2.99}$$

$$C_P = C_V + Nk_B = \frac{f+2}{2}Nk_B \qquad \text{(Heat capacity at constant pressure)} \tag{2.100}$$

$$\gamma = \frac{C_P}{C_V} = \frac{f+2}{f} \qquad \text{(Adiabatic index)} \tag{2.101}$$

$$\mu = \left(\frac{\partial G}{\partial N}\right)_{T,p} \qquad \text{(Chemical potential)} \tag{2.102}$$

$$p = -\left(\frac{\partial F}{\partial V}\right)_T \qquad \text{(Pressure from free energy)} \tag{2.103}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V \qquad \text{(Entropy from free energy)} \tag{2.104}$$

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V \qquad \text{(Maxwell relation)} \tag{2.105}$$

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V \qquad \text{(Maxwell relation)} \tag{2.106}$$

$$\left(\frac{\partial U}{\partial S}\right)_V = T \qquad \text{(Definition of temperature)} \tag{2.107}$$

$$\left(\frac{\partial U}{\partial V}\right)_S = -p \qquad \text{(Definition of pressure)} \tag{2.108}$$

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g \qquad \text{(Hydrostatic equilibrium)} \tag{2.109}$$

$$p(z) = p_0 e^{-\frac{mgz}{k_BT}} \qquad \text{(Isothermal atmosphere)} \tag{2.110}$$

Cylindrical coordinates:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{2.111}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \tag{2.112}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\mathbf{e}_r \tag{2.113}$$

$$+ \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{e}_\theta \tag{2.114}$$

$$+ \left(\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\mathbf{e}_z \tag{2.115}$$

Spherical coordinates:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi \tag{2.116}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{2.117}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right)\mathbf{e}_r \tag{2.118}$$

$$+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\mathbf{e}_\theta \tag{2.119}$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\mathbf{e}_\phi \tag{2.120}$$

2.3.9 Trigonometric identities

$$\sin^2(\theta) + \cos^2(\theta) = 1, \quad 1 + \tan^2(\theta) = \frac{1}{\cos^2(\theta)}. \tag{2.121}$$

Addition formulas

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b), \tag{2.122}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b). \tag{2.123}$$

Double-angle formulas

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \tag{2.124}$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta). \tag{2.125}$$

These formulas are very useful for variable changes in integration.

Expressions of $\sin(x)$, $\cos(x)$, and $\tan(x)$ in terms of $t = \tan\left(\frac{x}{2}\right)$

$$\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad \tan(x) = \frac{2t}{1-t^2}. \tag{2.126}$$

Variable substitution $t = \tan\left(\frac{x}{2}\right)$

This change of variable is often used to simplify trigonometric integrals. We also have:

$$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t. \tag{2.127}$$

2.4 Legend of thematic notations

- **(SR)**: Special Relativity
- **(QM)**: Quantum Mechanics
- **(EM)**: Electromagnetism
- **(AM)**: Analytical Mechanics
- **(SM)**: Statistical Mechanics
- **(SP)**: Subatomic Physics
- **(WO)**: Wave Optics
- **(TD)**: Thermodynamics

2.5 Suggested paths depending on your level

To help readers navigate this dense collection of exercises, here are a few suggested paths based on your level and goals. Of course, every student is free to explore the problems that inspire them.

Level	**Recommended exercises**
Early Bachelor Year 3	3.1 – Two-body problem 3.2 – Rutherford cross section 3.4 – Pulsed magnetic field machine 3.13 – Electrodynamic instability of the classical atom
End of Bachelor / Beginning of Master 1	3.3 – Cherenkov effect 3.5 – Metric on a sphere 3.6 – Blackbody radiation 3.10 – Hydrogen atom and radial equation 3.12 – Pöschl–Teller potential 3.14 - Geodesics in a Dispersive Optical Medium 3.15 - Bose-Einstein Condensation 3.16 - Decay Chain
Advanced Master 1	3.7 – Minimization of gravitational potential 3.8 – Relativistic charged particle 3.9 – Relativistic hydrodynamics 3.11 – Towards a relativistic formalism

Chapter 3

Exercises

This collection of exercises was designed with the ambition to go beyond mere mechanical practice of methods. Each problem aims to highlight a certain form of mathematical elegance or physical depth — a careful eye will discover, behind the equations and techniques, a subtle coherence, sometimes even a formal beauty. Some exercises are demanding, both in their length and structure: they are sometimes inspired by competitive exams or realistic physical situations, and may require several hours of reflection. Their goal is not only to reinforce technical skills, but to make one feel, through progressive resolution, the deep unity between mathematical rigor and the physical reality it describes. This chapter is dynamic: new problems will be regularly added in the same spirit of elegance, clarity, and depth.

3.1 Two-body problem and quantization of the Bohr atom† (AM) ★★★

[1] (Solution)

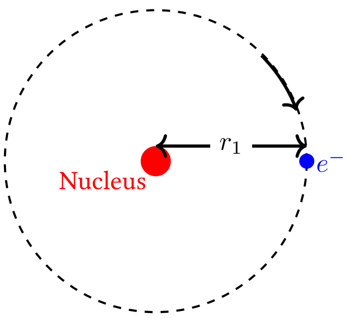

Figure 3.1: Diagram of the Bohr atom.

Consider a system of two particles with masses m_1 and m_2 interacting via a central potential $V(r) = -\frac{C}{r} = -\frac{\vartheta^2}{r}$[2], where r is the distance between the two particles and C is a real constant. Here we use the Coulomb potential, but one could just as well use a gravitational potential. We

[1]† Inspired by Claude Aslangul, *Mécanique Quantique 1*, Chapter 7.

[2]We define $\vartheta^2 = \frac{e^2}{4\pi\varepsilon_0}$.

will study in detail the bound states of the hydrogen atom according to the old quantum theory and, in particular, derive the energy associated with a given trajectory of the electron of mass m_1 around the nucleus of mass m_2.

3.1.1 Center of mass

Let $\mathbf{r}_1, \mathbf{r}_2$ be the position vectors of the electron and the nucleus relative to an arbitrary reference frame, and $\mathbf{v}_1, \mathbf{v}_2$ their respective velocities.

1. Write the Lagrangian $\mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2)$.
2. Let $\mathbf{R}$ be the center-of-mass position vector and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Show that the Lagrangian can be written as:
$$\mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2) = \mathcal{L}_G(\mathbf{V}) + \mathcal{L}_r(\mathbf{r}, \mathbf{v}) \tag{3.1}$$
3. Explain why the total angular momentum about the center of mass G, denoted $\mathbf{J}$, is a conserved quantity. Deduce a conclusion about the trajectory.

From here on, we examine only the internal motion through $\mathcal{L}_r$ in polar coordinates (r, θ) in the plane perpendicular to $\mathbf{J}$.

3.1.2 Integration of the equations of motion

1. Write the Hamiltonian $\mathcal{H}$ for the internal motion and derive Hamilton's equations. Recover the conservation of angular momentum and interpret the equation involving only $\mathbf{r}, \dot{\mathbf{r}}$.
2. Determine the relationship between r and θ, i.e., the trajectory. To do so, eliminate time from the previous equations by setting $u = \frac{1}{r}$, and show that:
$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = K, \quad K = \frac{\mu \vartheta^2}{J^2} \tag{3.2}$$
3. Finally, deduce that the trajectory is a conic, whose equation can always be written in the form:
$$r(\theta) = \frac{p}{1 + \varepsilon \cos\theta} \tag{3.3}$$
Give the expression of p, the conic parameter, and ε, the eccentricity. Check how the value of ε relative to 1 determines the nature of the corresponding state (bound or unbound).

3.1.3 Bohr quantization

In this part, we consider only bound states ($E < 0$) and apply Bohr's rules to select among all classically possible trajectories. These rules involve the action variables J_θ, J_r and are written:

$$J_\theta := \oint p_\theta \, \mathrm{d}\theta = n_\theta h \tag{3.4}$$

$$J_r := \oint p_r \, \mathrm{d}r = n_r h \tag{3.5}$$

$$n_\theta, n_r \in \mathbb{Z} \tag{3.6}$$

1. Determine the possible values of the angular momentum J as a consequence of the quantization of J_θ. Specify the possible values of the integer n_θ.

2. Quantize J_r and deduce the relation between ε and the integers n_r, n_θ[3]. Given:

$$\int_0^{2\pi} \frac{1}{1+\varepsilon\cos\theta}\,\mathrm{d}\theta = \frac{2\pi}{\sqrt{1-\varepsilon^2}} \tag{3.7}$$

3. Deduce that the energy E is quantized, with $n \in \mathbb{N}^*$ depending on n_θ, n_r, and that:

$$E_n = -\frac{\mu\vartheta^4}{2n^2\hbar^2} \tag{3.8}$$

[3] At first glance, one might say that $J_r = 0$; an integration by parts is necessary.

3.2 Rutherford Scattering Cross Section† (SP) ★★

[4] (Solution)

We consider the same situation as in the previous exercise: two particles, one of which is fixed, interacting through a potential of the form $V(r) = \frac{C}{r}$. Here, $C = \frac{Qq}{4\pi\varepsilon_0}, Q = Ze, q = 2e$. We will use some results from the previous exercise, so it is recommended to complete that one first.

3.2.1 Deflection of a charged particle by an atomic nucleus

We work in the polar coordinate system (r, φ), perpendicular to the angular momentum, since the motion is planar. The alpha particle arrives with initial velocity $\mathbf{v}_0$. We assume $\lim_{t\to-\infty} \varphi(t) = \pi$.

1. Determine the non-zero component of $\mathbf{J}$ as a function of r, φ. Also determine it in terms of b, v_0, where b is the impact parameter.
2. Write the equation of motion. Decompose $\mathbf{v} = \dot{\mathbf{r}}$ into a vector parallel and one perpendicular to the polar axis. Deduce that:
$$m\dot{v}_\perp = \frac{C}{r^2}\sin\varphi \tag{3.9}$$
3. We want to introduce the deflection angle θ. By integrating the equation, show that:
$$v_0 \sin\theta = \frac{C}{mbv_0}(\cos\theta + 1) \tag{3.10}$$
4. Using some trigonometric identities, deduce that:
$$\tan\frac{\theta}{2} = \frac{C}{2E_0 b} \tag{3.11}$$
where $E_0 = \frac{1}{2}mv_0^2$.

3.2.2 Rutherford Scattering Cross Section

1. Recall the formula for the differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$.
2. Deduce that:
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{C^2}{16E_0^2\sin^4\frac{\theta}{2}} \tag{3.12}$$
3. Deduce that this model is invalid for small deflection angles.
4. Explain why this experiment demonstrates the existence of atomic nucleus.

[4]† Inspired by Claude Aslangul, *Mécanique Quantique 1*, Chapter 3.

3.3 Cherenkov Effect† (SR, NP) ★★★

[5] (Solution)

The Cherenkov effect occurs when a charged particle travels through a dielectric medium at a speed v greater than the speed of light in that medium c/n, where n is the refractive index of the medium.

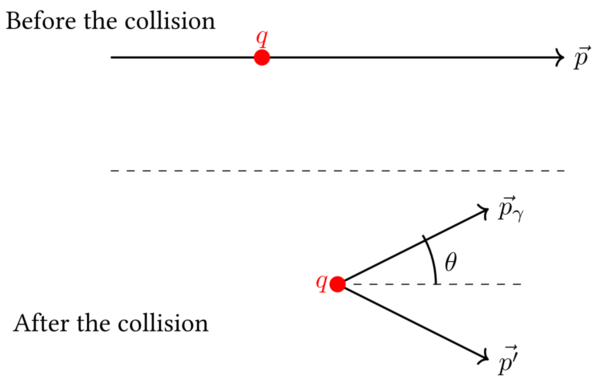

Figure 3.2: Diagram of the Cherenkov effect.

The momentum of the charged particle is $\mathbf{p}$ before the collision, $\mathbf{p}_\gamma$ is the photon momentum after the collision, and $\mathbf{p}'$ is the particle's momentum after the collision, (c.f fig 3.2). The angle θ is the angle between $\mathbf{p}$ and $\mathbf{p}_\gamma$. Recall that $\lambda = \frac{c}{n\nu}$.

1. Express p_γ in terms of h, ν, c, n. Deduce the relation between p_γ and E_γ in a medium with refractive index n.
2. Write the momentum conservation equation for the elementary process.
3. Using the previous question, express ${\mathbf{p}'}^2$ in terms of the magnitudes of the momenta p, p_γ, and the angle θ.
4. Write the energy conservation equation.
5. Deduce that:
$$p'^2 = p^2 - 2\frac{E}{c^2}h\nu + \frac{p_\gamma^2}{n^2} \tag{3.13}$$
where E is the initial energy of the electron.
6. Express $\cos\theta$ in terms of $p, p_\gamma, E, h, n, c, \nu$.
7. Show that:
$$\cos\theta = \frac{c}{nv}\left[1 + \frac{h\nu}{2E}(n^2 - 1)\right] \tag{3.14}$$
8. What is the condition for the Cherenkov effect to occur?
9. In which frequency range are the photons emitted?
10. In which direction are the highest-energy photons emitted?

[5]† Inspired by Claude Aslangul, *Mécanique Quantique 1*, Chapter 5.

11. All photons are emitted within a cone; what is the half-apex angle ϕ of this cone? Estimate ϕ for $n = \frac{4}{3}$ and $v = \frac{4}{5}c$.

12. Compare the minimum kinetic energy required for the particle to produce Cherenkov radiation in the cases of an electron and a proton, for $n = \frac{4}{3}$.

3.4 Pulsed Magnetic Field Machine (EM) ★★

(Solution)

The magnetic stimulation machine is a non-invasive technology used in physiotherapy and rehabilitation. It works by generating pulsed magnetic fields using a circular coil. In practice, the machine sends current pulses through the coil, which creates a time-varying magnetic field. According to Faraday's law, this variation automatically induces an electric field in the surrounding tissues.

This induced electric field acts directly on the cellular membranes of muscles by activating ion channels. As a result, an action potential is triggered, leading to muscle contraction. This mechanism allows not only for the stimulation of weakened or atrophied muscles, but also improves blood circulation and reduces pain. Moreover, the absence of direct skin contact makes the treatment comfortable and safe for the patient.

The magnetic stimulation machine is especially used to:

- Support muscle rehabilitation after injury or surgery. - Relieve chronic pain linked to musculoskeletal disorders. - Improve muscle tone and prevent atrophy. - Stimulate blood and lymph circulation to speed up recovery.

In summary, thanks to an approach based on fundamental principles of electromagnetic induction, this technology provides an effective treatment for various muscular and nervous conditions, serving as a complementary solution to conventional rehabilitation therapies.

To modelize this phenomen, we consider a circular coil of radius R carrying a time-varying current:

$$I(t) = I_0 e^{-t/\tau} \sin(\omega t), \tag{3.15}$$

where I_0 is the current amplitude, τ is the damping time constant, and ω is the oscillation frequency. The coil's axis is assumed to coincide with the z-axis. The coil is considered thin and modeled as a single loop.

1. **Magnetic field of the coil**

   (a) Assuming the coil behaves like a magnetic dipole, express the magnetic field $\mathbf{B}$ along the central axis (at a distance z from the center) in terms of $I(t)$, R, z, and physical constants.

   (b) Show that for $z \gg R$, the field approximates that of a magnetic dipole and give its asymptotic expression.

2. **Induced electric field in biological tissue** We model the tissue as a thin conducting disk of radius a, placed under the coil.

   (a) Starting from the local Faraday law:

   $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.16}$$

   express the induced electric field $\mathbf{E}$ in terms of $\frac{\mathrm{d}B}{\mathrm{d}t}$.

   (b) Assuming cylindrical symmetry (purely azimuthal field), derive the expression for the induced electric field $E_\theta(r,t)$ in the plane of the disk, distinguishing the cases $r < R$ and $r > R$.

3. **Effect on motor neurons** A motor neuron is assumed to be activated when the induced voltage exceeds a threshold V_{thresh}.

   (a) Express V in terms of the parameters of the problem.

   (b) Determine a condition on I_0, τ, ω, and the geometric parameters to ensure neuron activation.

   (c) Using realistic numerical values ($R = 5\,\text{cm}$, $a = 2\,\text{cm}$, $I_0 = 100\,\text{A}$, $\tau = 1\,\text{ms}$, $\omega = 10^4\,\text{rad/s}$, and $V_{\text{thresh}} = 10\,\text{mV}$), check whether neuron activation is possible.

4. **Effect of pulsed magnetic field on muscles** Explain why a pulsed magnetic field, by inducing an electric field in tissues, can provoke muscle contraction. Briefly describe the physiological mechanism (activation of ion channels, generation of an action potential, muscle contraction).

3.5 Metric on a Sphere (AM) ★★★

(Solution)

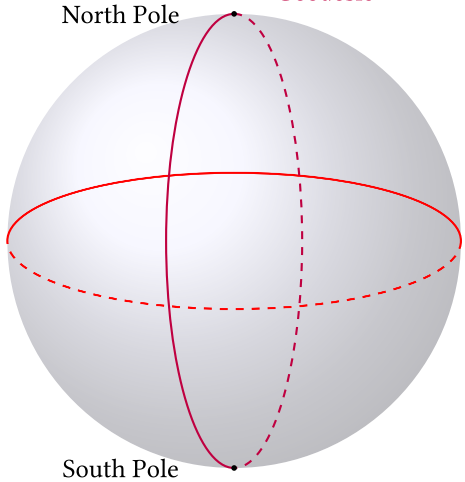

Figure 3.3: Diagram of a sphere and its geodesics.

Our goal is to determine the metric on a sphere and its geodesics. This will help us understand the optimal flight paths for an airplane. Recall that in spherical coordinates, for a fixed radius R,

$$x = R\cos\varphi\sin\theta \tag{3.17}$$

$$y = R\sin\varphi\sin\theta \tag{3.18}$$

$$z = R\cos\theta \tag{3.19}$$

1. Calculate the line element $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$ as a function of R, θ, and φ.
2. Using the action $S = \int \mathrm{d}s = \int \mathcal{L}\mathrm{d}\lambda$, where λ is a suitably chosen parameter, and the variational principle, determine the geodesic equations.
3. Solve the equations by using symmetries. One may use that

$$\int \frac{\mathrm{d}\alpha}{\sin^2\alpha\sqrt{1-\frac{\lambda^2}{\sin^2\alpha}}} \quad \text{set } u = \cot\alpha, \tag{3.20}$$

$$\int -\frac{\mathrm{d}t}{\sqrt{1-t^2}} = \arccos t + C. \tag{3.21}$$

Show that the geodesics have the following form:

$$(x, y, z) \in S^2, \quad ax + by + cz = 0, \tag{3.22}$$

that is, the geodesics are intersections between planes passing through the origin and the sphere, or in other words, arcs of great circles.

3.6 Blackbody Radiation (PS) ★★★★

(Solution)

We seek to obtain the spectral energy density, that is the function

$$u(\nu, T) = \frac{\mathrm{d}^2 W}{\mathrm{d}\nu \mathrm{d}\mathcal{V}} = \frac{\mathrm{d}N}{\mathrm{d}\nu}\frac{\langle W\rangle}{\mathcal{V}}, \tag{3.23}$$

with W the energy and $\langle W\rangle$ the mean energy. We will also work in a historical framework, without using quantum mechanics, which was partly discovered thanks to the results we are about to demonstrate.

3.6.1 Number of Modes Excited per Frequency Unit

1. Consider a blackbody represented by a cubic cavity of side length L and volume $\mathcal{V}$. Write down the wave equation for the electric field $\mathbf{E}$ inside the cavity.

2. Solve the wave equation. Explain why the field $\mathbf{E}$ depends on three modes $n_x, n_y, n_z \in \mathbb{N}^*$.

3. Show that

$$n_x^2 + n_y^2 + n_z^2 = r^2 = \left(\frac{2L}{\lambda}\right)^2. \tag{3.24}$$

4. By counting unit cubes stacked along the axes n_x, n_y, n_z, we can enumerate the total number N of excited modes.

   Each cube can be represented as $\mathbf{r} = n_\mu \mathbf{e}^\mu$. When the cubes are very numerous, that is, when L is much larger than λ, it suffices to calculate the volume of a sphere of radius r.

   However, since the integers are strictly positive, only $1/8$ of the total sphere volume is taken. Also, a factor of 2 must be considered due to the two possible polarization planes of the electric field $\mathbf{E}$.

   Using these data, deduce that

$$\frac{\mathrm{d}N}{\mathrm{d}\nu} = \frac{8\pi\nu^2}{c^3}\mathcal{V}. \tag{3.25}$$

3.6.2 Ultraviolet Catastrophe

1. Explain why the ensemble associated with this problem — the calculation of u — corresponds to the canonical ensemble.

2. Calculate the Hamiltonian of a harmonic oscillator.

3. Give the probability of being in an energy state W. Deduce the partition function Z of a gas of harmonic oscillators.

4. Show that

$$\langle W\rangle = k_B T. \tag{3.26}$$

5. Deduce that

$$u(\nu, T) = 8\pi\frac{\nu^2}{c^3}k_B T, \tag{3.27}$$

   and explain the title of this subsection.

3.6.3 Planck's Law

The revolutionary idea is to estimate that the photon energy is quantized. Thus, we move from the idea of a continuous energy distribution to a discrete one. This idea arose from the fact that the average energy of an oscillator did not depend on the frequency ν. Planck suspected a simple proportionality relation between W and ν:

$$W_n = nh\nu. \tag{3.28}$$

Then came the idea of quanta, that energy is not continuous but distributed in indivisible packets called **quanta**[6].

1. Recalculate the partition function Z.
2. Deduce that
$$u(\nu, T) = 8\pi \frac{\nu^2}{c^3} \frac{h\nu}{e^{\beta h\nu} - 1} \tag{3.29}$$
with $\beta = \frac{1}{k_B T}$.

Thus, the ultraviolet catastrophe was resolved, and this result agreed perfectly with experiments. This function became integrable, which later led to Stefan's law.

3.6.4 Energy Flux Emitted by a Blackbody

Consider a cavity in thermal equilibrium filled with a photon gas at temperature T. The radiation is **isotropic** and characterized by a volumetric spectral energy density $u(\nu)$, such that

$$u(\nu)\,\mathrm{d}\nu = \text{electromagnetic energy per unit volume between frequencies } \nu \text{ and } \nu + \mathrm{d}\nu. \tag{3.30}$$

Let I be the total intensity (energy flux per unit surface perpendicular to it, integrated over all directions) emitted by the blackbody.

1. Recall the expression of the monochromatic energy flux emitted in a direction making an angle θ with respect to the surface normal, in terms of the directional spectral intensity I_ν and the solid angle $\mathrm{d}\Omega$.
2. Show that the total energy flux emitted at frequency ν per unit surface is given by
$$I(\nu) = \int_{\Omega_+} I_\nu \cos\theta \,\mathrm{d}\Omega, \tag{3.31}$$
where Ω_+ denotes the outgoing hemisphere ($0 \le \theta \le \pi/2$).
3. Assuming the radiation is isotropic, i.e., I_ν is independent of direction, show that
$$I(\nu) = \pi I_\nu. \tag{3.32}$$
4. By integrating over all frequencies, deduce that the total emitted intensity is
$$I = \int_0^\infty \pi I_\nu \,\mathrm{d}\nu. \tag{3.33}$$

[6] Albert Einstein used Planck's idea in his annus mirabilis of 1905 to explain the photoelectric effect, which earned him the Nobel Prize in 1921.

5. Show that the volumetric spectral energy density $u(\nu)$ is given by

$$u(\nu) = \frac{1}{c}\int_{S^2} I_\nu(\mathbf{n})\,\mathrm{d}\Omega. \tag{3.34}$$

Assuming isotropic radiation, deduce that

$$u(\nu) = \frac{4\pi}{c} I_\nu. \tag{3.35}$$

6. Deduce that

$$I = \frac{c}{4}\int_0^\infty u(\nu)\,\mathrm{d}\nu. \tag{3.36}$$

3.6.5 Stefan's Law

Stefan's law states that for a blackbody,

$$I(T) = \sigma T^4, \tag{3.37}$$

where σ is the Stefan–Boltzmann constant. We will prove it.

1. Using the previous parts, show that

$$I = \frac{2\pi k_B^4}{h^3c^2}T^4\int_0^\infty \frac{x^3}{e^x-1}\mathrm{d}x. \tag{3.38}$$

2. Verify the convergence of the integral and express it as a series.
3. Finally, prove that

$$I(T) = \frac{2\pi^5 k_B^4}{15h^3c^2}T^4. \tag{3.39}$$

This is recognized as Stefan's law[7]

$$I = \sigma T^4. \tag{3.40}$$

3.6.6 Application: Solar Mass Loss by Electromagnetic Radiation

Assuming the Sun is a blackbody, determine $\dot{m}$, the mass loss per unit time. What is this mass loss rate in $\mathrm{kg}\cdot\mathrm{s}^{-1}$? Knowing that our Sun is approximately 4.6×10^9 years old, estimate how many Earth masses the Sun has lost so far.

Data: $R = 6.96\times10^8\,\mathrm{m}$, $T = 5775\,\mathrm{K}$, $m = 1.98\times10^{30}\,\mathrm{kg}$, $m_\oplus = 6\times10^{24}\,\mathrm{kg}$.

[7] Hence, $\sigma = \frac{2\pi^5k_B^4}{15h^3c^2}$, which is rather unexpected.

3.7 Minimization of the Gravitational Potential by a Ball (AM) ★★★★★

(Solution)

This exercise involves notions of differential calculus.

We consider the following variational problem: among bounded open domains $\Omega \subset \mathbb{R}^3$ of fixed volume, find the one minimizing the **internal gravitational interaction** defined by the functional:

$$\mathcal{F}[\Omega] = \iint_{\Omega\times\Omega} \frac{1}{|x - x'|}\, \mathrm{d}^3x\, \mathrm{d}^3x' \tag{3.41}$$

Note that this expression is proportional to the gravitational self-interaction potential of a body with uniform density. Indeed, for $x \in \mathbb{R}^3$,

$$U(x) = -G \int_\Omega \frac{\rho}{|x - x'|}\, \mathrm{d}^3x' \tag{3.42}$$

The total gravitational potential energy of the system is then:

$$E[\Omega] = \frac{1}{2}\int_\Omega \rho U(x)\, \mathrm{d}^3x = -\frac{G}{2}\rho^2 \int_\Omega\int_\Omega \frac{1}{|x - x'|}\, \mathrm{d}^3x\, \mathrm{d}^3x'. \tag{3.43}$$

- We consider a domain $\Omega \subset \mathbb{R}^3$, i.e., a bounded open set of class $\mathcal{C}^2$, with boundary $\partial\Omega$.
- The volume of Ω is defined by:
$$V := \int_\Omega \mathrm{d}^3x \tag{3.44}$$
- We consider an infinitesimal normal deformation of the boundary of Ω, parametrized by $\varepsilon \in \mathbb{R}$, given by:
$$x \mapsto x + \varepsilon f(x) n(x), \quad \text{for } x \in \partial\Omega \tag{3.45}$$
where $f \in C^\infty(\partial\Omega)$ is a smooth function and $n(x)$ is the outward unit normal vector to $\partial\Omega$.
- The deformed domain is denoted Ω_ε, the bounded open set obtained by this deformation:
$$\Omega_\varepsilon := \{x + \varepsilon f(x) n(x) \mid x \in \Omega\} + o(\varepsilon) \tag{3.46}$$
(The deformation is assumed to be smoothly extended inside Ω to rigorously define Ω_ε.)

3.7.1 Hadamard's Formula

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^1$ function, and Ω_ε a smooth deformation of Ω such that for $x \in \partial\Omega$,

$$x \mapsto x + \varepsilon f(x) n(x) \tag{3.47}$$

and assume this deformation extends smoothly to all of Ω.

We want to prove that:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3x = \int_{\partial\Omega} F(x) f(x)\, \mathrm{d}S(x) \tag{3.48}$$

where $\mathrm{d}S$ is the surface element associated to $\partial\Omega$.

1. We will study the function $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$.

   (a) Justify that $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}, M \mapsto \det M$ is differentiable.

   (b) Prove that for all $M \in \mathcal{M}_n(\mathbb{R})$,

   $$\det(I + \varepsilon M) = 1 + \varepsilon \operatorname{Tr}(M) + o(\varepsilon) \tag{3.49}$$

   Deduce that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \det(I + \varepsilon M)\big|_{\varepsilon=0} = \operatorname{Tr}(M)$.

   (c) Let $X \in \mathrm{GL}_n(\mathbb{R})$, $H \in \mathcal{M}_n(\mathbb{R})$. Prove that

   $$\mathrm{d}(\det)(X)(H) = \operatorname{Tr}({}^t \operatorname{Com}(X)\, H) \tag{3.50}$$

2. Set the change of variables $x(u) = u + \varepsilon f(u) n(u)$, and compute the Jacobian $\det\left(\frac{\partial x}{\partial u}\right)$ at first order in ε, i.e., up to $o(\varepsilon)$.

3. Let $F : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$, $v : \mathbb{R}^n \to \mathbb{R}^n$, and $\varepsilon \in U$ a neighborhood of 0. By considering a well-chosen function, prove that for all $x \in \mathbb{R}^n$,

   $$F(x + \varepsilon v(x)) = F(x) + \varepsilon v(x) \cdot \nabla F(x) + o(\varepsilon) \tag{3.51}$$

4. Deduce the desired result using the Divergence Theorem.

3.7.2 Connection with the Gravitational Potential

1. Show that $E[\Omega] < 0$, and that minimizing the energy is equivalent to maximizing the following quantity:

   $$\mathcal{I}[\Omega] := \int_\Omega \int_\Omega \frac{1}{|x - x'|}\, \mathrm{d}^3x\, \mathrm{d}^3x'. \tag{3.52}$$

2. Suppose $\Omega = B(0, R)$ is a ball centered at the origin of radius R such that $\mathrm{Vol}(\Omega) = \frac{4}{3}\pi R^3 = V$. Show that the gravitational potential at the center is given by:

   $$U(0) = -G\rho \int_\Omega \frac{1}{|x'|}\, \mathrm{d}^3x'. \tag{3.53}$$

   Calculate this integral explicitly.

3.7.3 The Sphere?

1. Prove that

   $$\delta\mathcal{F} = 2 \int_{\partial\Omega} \left(\int_\Omega \frac{1}{|x - x'|}\, \mathrm{d}^3x' \right) f(x)\, \mathrm{d}S(x) \tag{3.54}$$

   You may use or prove (for the more courageous) that for all $\Omega \subset \mathbb{R}^n$, for all $\varphi : \Omega \to \mathbb{R}^n$,

   $$\int_{\partial(\Omega^2)} \varphi(x)\, \mathrm{d}\mu(x) = 2 \int_{\Omega \times \partial\Omega} \varphi(x)\, \mathrm{d}\mu(x) \tag{3.55}$$

2. We want to minimize $\mathcal{F}$ under fixed volume constraint V. To do this, consider the Lagrangian:

   $$\mathcal{L}(\lambda) = \mathcal{F} - \lambda V, \quad \lambda \in \mathbb{R}. \tag{3.56}$$

   Deduce that the first variation of $\mathcal{F}$ writes:

   $$\delta\mathcal{L} = \int_{\partial\Omega} \left(2 \int_\Omega \frac{1}{|x - x'|}\, \mathrm{d}^3x' - \lambda \right) f(x)\, \mathrm{d}S(x). \tag{3.57}$$

3. Using spherical symmetry, show that if Ω is a ball of radius R, then for all $x \in \partial\Omega$, the quantity
$$\int_\Omega \frac{1}{|x - x'|} \, \mathrm{d}^3 x' \tag{3.58}$$
is constant. Deduce that the ball satisfies the **stationary condition** $\delta\mathcal{L} = 0$ for all f.

4. (*Bonus*) Show that the ball is indeed a *local minimum* for $\mathcal{F}$ under volume constraint by studying the second variation.

5. Conclude and explain why large objects in the Universe are spherical.

3.8 Relativistic Motion of a Charged Particle (SR, AM, EM, SM) ★★★★★

(Solution)

3.8.1 Relativistic Lagrangian of a Charged Particle in an Electromagnetic Field

1. Show that using the principle of least action and Lorentz invariance, the action of a free particle of mass m can be written as $S = -mc \int \mathrm{d}s$ where $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2$. Deduce that the Lagrangian of the system is

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\,, \tag{3.59}$$

where $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$.

2. By introducing the electromagnetic four-potential $A^\mu = (\phi/c, \mathbf{A})$, propose an interaction term L_{int} corresponding to a particle of charge q in this field. Show that it can be written as

$$\mathcal{L}_{\text{int}} = q\,\mathbf{A}\cdot\mathbf{v} - q\phi\,, \tag{3.60}$$

and deduce the total Lagrangian $\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_{\text{int}}$.

3. Starting from the total Lagrangian, calculate the generalized momentum $P_i = \partial\mathcal{L}_{\text{tot}}/\partial v^i$. Show that it can be expressed as

$$\mathbf{p} = \gamma m\mathbf{v} + q\mathbf{A}\,, \tag{3.61}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

4. Write the Euler–Lagrange equations associated with L_{tot} and show that they lead to the Lorentz equation in 3 dimensions,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = q\big(\mathbf{E} + \mathbf{v}\times\mathbf{B}\big)\,, \tag{3.62}$$

with $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla\times\mathbf{A}$ and, $\nabla(\mathbf{A}\cdot\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A})$.

5. Express the Lagrangian by parameterizing with proper time τ and deduce that,

$$\mathcal{L} = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + qg_{\mu\nu}A^\mu\dot{x}^\nu \tag{3.63}$$

6. Show that

$$m\ddot{x}_\mu = qF_{\mu\nu}\dot{x}^\nu \tag{3.64}$$

Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor.

7. Explicitly write the components of the tensor $F_{\mu\nu}$ and show that $F_{0i} = E_i/c$ and $F_{ij} = -\varepsilon_{ijk}B_k$. Interpret the physical meaning of these components.

8. Calculate the two invariants of the electromagnetic field,

$$I_1 = F_{\mu\nu}F^{\mu\nu}\,, \quad I_2 = \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \tag{3.65}$$

and express them in terms of $\mathbf{E}$ and $\mathbf{B}$. What are the physical cases corresponding to $I_1 = 0$ and $I_2 = 0$?

9. Verify that under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$, the equations of motion remain unchanged. What is the associated symmetry?

3.8.2 Equations of Motion of a Charged Particle in a Plane Electromagnetic Wave

Consider a particle of mass m and charge q subjected to an electromagnetic field described by the tensor $F^{\mu\nu}$. Its motion is governed by the equation:

$$m\ddot{x}^{\mu} = qF^{\mu\nu}\dot{x}_{\nu} \tag{3.66}$$

where the dots denote derivatives with respect to the particle's proper time τ. We use natural units where $c = 1$.

A plane electromagnetic wave is modeled by a four-potential of the form:

$$A^{\mu}(x) = a^{\mu} f(k_{\nu}x^{\nu}) \tag{3.67}$$

where f is a $\mathcal{C}^1$ function, k^{μ} is a lightlike four-vector, hence $k^{\mu}k_{\mu} = 0$, and a^{μ} is a constant four-vector representing the polarization.

1. Show that
$$F^{\mu\nu}(x) = (k^{\mu}a^{\nu} - k^{\nu}a^{\mu})\, f'(k_{\rho}x^{\rho}) \tag{3.68}$$

2. (a) Calculate $\partial_{\mu}A^{\mu}$ for the potential $A^{\mu}(x) = a^{\mu} f(k_{\rho}x^{\rho})$.
   (b) Deduce that the Lorenz gauge condition $\partial_{\mu}A^{\mu} = 0$ implies:
$$a^{\mu}k_{\mu} = 0 \tag{3.69}$$

3. Now consider the motion of a particle in this electromagnetic wave.
   (a) Using the expression for the tensor $F^{\mu\nu}$ found in question 1, show that:
$$F^{\mu\nu}\dot{x}_{\nu} = [k^{\mu}(a_{\rho}\dot{x}^{\rho}) - a^{\mu}(k_{\rho}\dot{x}^{\rho})]\, f'(k_{\rho}x^{\rho}) \tag{3.70}$$
   (b) Deduce the equation of motion in the form:
$$m\ddot{x}^{\mu} = q\,[k^{\mu}(a_{\rho}\dot{x}^{\rho}) - a^{\mu}(k_{\rho}\dot{x}^{\rho})]\, f'(k_{\rho}x^{\rho}) \tag{3.71}$$

4. Now we seek to integrate this equation.
   (a) Show that the scalar $k_{\rho}\dot{x}^{\rho}$ is constant during the motion.
   (b) Deduce that $\phi = k_{\rho}x^{\rho}(\tau)$ is an affine function of τ, which can be used as a new parameter.
   (c) Using the previous relations, integrate the equation of motion and determine the complete expression for the trajectory $\tau \mapsto x^{\mu}(\tau)$[8]

[8] This exercise allows us to analytically determine the trajectory of a charged particle in a plane electromagnetic wave. You can then plot it in Python using the obtained functions.

3.8.3 Field Theory

We define the action,

$$S = \int_{\Omega} -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - A^{\mu} j_{\mu} \mathrm{d}^4 x, \quad \Omega \subset \mathbb{R}^{1,3} \tag{3.72}$$

We can thus easily define a Lagrangian **density**.

1. For an action depending on a field φ (scalar, tensor, etc.):

$$S = \int_{\Omega} \mathcal{L}(\varphi, \partial_{\mu}\varphi, x^{\mu}) \mathrm{d}^4 x \tag{3.73}$$

Prove that the Euler-Lagrange equations remain valid for a field φ.
To do this, we will postulate the principle of least action, meaning that for an infinitesimal transformation $\varphi \mapsto \varphi + \varepsilon\eta$[9], we have,

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon}[\varphi + \varepsilon\eta, \partial_{\mu}(\varphi + \varepsilon\eta), x^{\mu}](0) = 0 \tag{3.74}$$

2. Derive Maxwell's equations in tensor form,

$$\partial_{\mu} F^{\mu\nu} = \mu_0 j^{\nu}, \quad \partial_{\lambda} F_{\mu\nu} + \partial_{\mu} F_{\nu\lambda} + \partial_{\nu} F_{\lambda\mu} = 0\,, \tag{3.75}$$

where $j^{\mu} = (c\rho, \mathbf{j})$ is the four-current (four-current density).

3.8.4 Trajectory of a Charged Particle in a Constant Magnetic Field

Consider a particle of mass m and charge q moving relativistically in an electromagnetic field. In this section, we gradually introduce the effects of a constant magnetic field $\mathbf{B} = B\,\mathbf{e}_z$ (curved sector of a synchrotron) and an average braking force due to synchrotron radiation.

A. Synchrotron Radiation Neglected

1. Calculate $F^{\mu\nu}$.

2. Deduce that the motion is in the Oxy plane. Show that the energy is constant if radiation losses are neglected.

3. Show that, in the absence of energy loss, for $u^{\mu} = (\gamma c, 0, u_0 = \gamma v, 0)$[10],

$$x(t) = R\cos\left(\frac{\omega}{\gamma}t\right), \qquad y(t) = R\sin\left(\frac{\omega}{\gamma}t\right) \tag{3.76}$$

With (synchrotron law)[11]:

$$R = \frac{\gamma v}{\omega} = \frac{\gamma m v}{qB} \tag{3.77}$$

[9] Where η is a $\mathcal{C}^1(\Omega)$ function, and $\forall x \in \partial\Omega, \eta(x) = 0$, i.e., the function vanishes at the boundaries.
[10] It would also be necessary to show that γ and v are constant and that $\tau(t) = \gamma t$.
[11] For this, we will need to switch to the laboratory frame.

B. Study of the Real Motion

1. Synchrotron radiation leads to an average energy loss. Recall the formula for the average radiated power (relativistic Larmor) for a centripetal acceleration $a = v^2/R$,

$$P = -\frac{\mathrm{d}}{\mathrm{d}t}E \;=\; \frac{q^2}{6\pi\varepsilon_0 c^3}\,\gamma^4 a^2 \tag{3.78}$$

Using $E = \gamma mc^2$, show that by expanding, we obtain the differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma = -C\,(\gamma^2 - 1) \tag{3.79}$$

Give the expression for the coefficient C in terms of $q, B, m, c, \varepsilon_0$.

2. Solve the differential equation for γ[12]. We give, $\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$.
3. Deduce the new trajectory of the charged particle. Study the limit as $t \to \infty$.
4. Plot the parametric curve $x(t), y(t)$ in Python. What problem does this generate?

3.8.5 Physics of Relativistic Colliders

Here, we will use natural units where the speed of light $c = 1$.

1. Define the square of the total energy-momentum invariant $s = (p_1 + p_2)^2$ for the collision of two particles with four-momenta p_1 and p_2. Express the total energy available in the center-of-mass frame (CMS) in terms of s.
2. For a head-on collision of two identical particles of mass m and energy E (each) in the laboratory frame, show that the CMS energy is $\sqrt{s} = 2E$ (assuming $E \gg m$).
3. For the case of a collision with a fixed target of mass m, derive the formula

$$s = m^2 + m^2 + 2mE_{\mathrm{lab}}, \tag{3.80}$$

and deduce the threshold energy for the production of two particles of mass m (extreme elastic collision).

4. Calculate the energy required in a fixed-target experiment to produce a new particle of mass M at threshold, and compare it to the energy required in a symmetric collider ($E_{\mathrm{CM}} = M + M$). Why are colliders with counter-propagating beams more efficient for reaching high energies?

[12] It is indeed much simpler to solve the equation for γ than for v, since here v depends on time.

3.9 Relativistic Hydrodynamics and Heavy-Ion Collisions (SR,SM) ★★★★★

(Solution)

3.9.1 Classical Hydrodynamics

1. Write the mass conservation equation (continuity) for a classical fluid:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3.81}$$

Show that for an incompressible fluid ($\rho =$ constant), this reduces to $\nabla \cdot \mathbf{v} = 0$.

2. Write Euler's equation for a perfect (inviscid) fluid subjected to a gravitational field $\mathbf{g}$:

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\Big) = -\nabla p + \rho \mathbf{g}. \tag{3.82}$$

Briefly describe the physical meaning of each term in this equation.

3. Show how adding viscous effects leads to the Navier–Stokes equation:

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\Big) = -\nabla p + \eta \nabla^2 \mathbf{v} + \Big(\zeta + \frac{\eta}{3}\Big)\nabla(\nabla \cdot \mathbf{v}) + \rho \mathbf{g}, \tag{3.83}$$

where η is the dynamic (shear) viscosity and ζ the bulk viscosity. Explain the role of these terms.

4. Explain the difference between the Lagrangian description (tracking fluid particle trajectories) and the Eulerian description (velocity field at a fixed point in space). In particular, show that the total derivative for a fluid is $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ in the Eulerian formalism.

5. Define streamlines in a fluid and show that these curves are tangent to the velocity field vector $\mathbf{v}$ at every point. Interpret the physical meaning of these lines.

6. Prove Bernoulli's theorem for a stationary, incompressible, and inviscid fluid. Show that along a streamline,

$$\frac{1}{2}\rho v^2 + p + \rho \Phi = \text{constant}, \tag{3.84}$$

where Φ is a force potential (e.g., $\Phi = gz$ for a constant gravitational field $\mathbf{g}$).

3.9.2 Introduction to Relativistic Hydrodynamics

Relativistic hydrodynamics describes the evolution of continuous systems with high energy density (such as quark-gluon plasma) by incorporating the principles of special relativity. Here, we focus on perfect fluids (no viscosity or thermal conduction) and their contravariant description.

1. **Energy-momentum tensor.** The energetic and dynamic content of a perfect fluid is encoded in the energy-momentum tensor:

$$T^{\mu\nu} = (\varepsilon + p)\,\frac{u^\mu u^\nu}{c^2} - p\, g^{\mu\nu}, \tag{3.85}$$

where:

- ε is the energy density (in the fluid's rest frame),
- p is the pressure (same units as ε, i.e., J/m^3),
- u^μ is the fluid's four-velocity,
- $\eta^{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric.

(a) Verify that $T^{\mu\nu}$ is symmetric.

(b) Calculate $T^{\mu\nu}$ in the fluid's rest frame ($u^\mu = (c, 0, 0, 0)$).

(c) Interpret the physical components T^{00}, T^{0i}, and T^{ij}.

(d) Show that the trace $T^\mu_\mu = \varepsilon - 3p$.

2. **Energy and momentum conservation.** In any isolated system, the energy-momentum tensor is locally conserved:

$$\partial_\mu T^{\mu\nu} = 0. \tag{3.86}$$

This tensor equation (4 scalar equations) expresses energy conservation ($\nu = 0$) and momentum conservation ($\nu = 1, 2, 3$). It is the fundamental equation of relativistic hydrodynamics.

(a) What are the dynamical unknowns of the problem?

(b) Why is it necessary to supplement this system with an equation of state relating ε, p, and possibly T?

3. **Relativistic thermodynamics.** In the fluid's rest frame, we locally define:

$$T : \text{temperature}, s : \text{entropy density}, \mu : \text{chemical potential}, n : \text{particle number density}. \tag{3.87}$$

The first law of thermodynamics, expressed in local densities (i.e., in a volume element $\mathrm{d}V$), takes the form:

$$\mathrm{d}\varepsilon = T\,\mathrm{d}s + \mu\,\mathrm{d}n. \tag{3.88}$$

(a) Assuming $\mu = 0$, show that $\mathrm{d}p = s\,\mathrm{d}T$.

(b) Deduce the identity $\varepsilon + p = Ts$, called the Euler relation.

4. **Relativistic speed of sound.** The speed of sound c_s is defined by:

$$c_s^2 = \left(\frac{\partial p}{\partial \varepsilon}\right)_s. \tag{3.89}$$

(a) Calculate c_s for an ultra-relativistic fluid where $p = \varepsilon/3$.

(b) Compare it to the speed of light c and comment.

3.9.3 Relativistic Hydrodynamics

Consider a perfect fluid in special relativity. The total number of particles is given by

$$N = \int_\Sigma J^\mu\,\mathrm{d}\Sigma_\mu, \tag{3.90}$$

across a spacelike hypersurface Σ oriented toward the future (e.g., $t = \text{const}$). We assume N is conserved.

1. Show that particle number conservation is locally expressed as

$$\partial_\mu(nu^\mu) = 0, \tag{3.91}$$

where n is the particle density in the comoving frame, and u^μ is the fluid's four-velocity.

2. Show that for $u^\mu = (c, 0, 0, 0)$, we have $T^{00} = \varepsilon$ and $T^{ii} = p$. Interpret.

3. Using $\partial_\mu T^{\mu\nu} = 0$, derive the equation of motion (or *relativistic Euler equation*) for a perfect fluid without sources:

$$(\varepsilon + p)u^\mu \partial_\mu u^\nu + \left(g^{\mu\nu} + \frac{u^\mu u^\nu}{c^2}\right)\partial_\mu p = 0. \tag{3.92}$$

3.9.4 Application to Heavy-Ion Collisions

1. Describe the scenario of a central heavy-ion collision at RHIC or LHC: formation of a quark-gluon plasma (QGP), thermalization, hydrodynamic expansion, freeze-out[13].

2. Introduce Bjorken coordinates: $\tau = \sqrt{t^2 - z^2}$, $\eta = \frac{1}{2}\ln\frac{t+z}{t-z}$. Assuming boost-invariant flow, show that $\partial_\mu T^{\mu\nu} = 0$ leads to:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} + \frac{\varepsilon + p}{\tau} = 0. \tag{3.93}$$

3. For $p = \varepsilon/3$, solve the above equation and deduce:

$$\varepsilon(\tau) \propto \tau^{-4/3}, \quad T(\tau) \propto \tau^{-1/3}. \tag{3.94}$$

4. During the QGP $\to$ hadrons transition, the equation of state can be written:

$$p = \frac{\varepsilon - 4B}{3}. \tag{3.95}$$

Show that $p = 0$ at the transition implies $\varepsilon = 4B$ and deduce the critical temperature T_c.

5. Modeling a nucleus as a sphere of radius R, define the geometric cross-section $\sigma \simeq \pi(2R)^2$. Relate this quantity to the distinction between central and peripheral collisions.

6. Show that the initial energy density ε_0 is larger for a central collision. Assuming $\varepsilon = aT^4$, estimate the initial temperature T_0 reached at RHIC ($\varepsilon_0 \sim 10$ GeV/fm^3).

7. Define the viscosity-to-entropy ratio η/s. Why do values close to $1/4\pi$ indicate a nearly perfect fluid? What is the effect of low viscosity on the elliptic flow v_2?

8. Explain the concepts of chemical freeze-out (inelastic reactions frozen) and kinetic freeze-out (elastic reactions frozen). Why does hydrodynamics cease to be valid at this stage?

9. How does relativistic hydrodynamics connect measured observables (transverse momentum spectrum, anisotropies, etc.) to the initial state of the QGP?

[13] The term "relativistic fluid" refers to any fluid whose constituents have kinetic energies comparable to their mass: $k_B T \gtrsim mc^2$. This can be a plasma (charged), but also a photon or neutrino gas. Thus, relativistic hydrodynamics is more general than plasma physics.

3.10 Hydrogen Atom and Radial Equation (QM) ★★★

(Solution)

In this problem, we study the hydrogen atom (an electron of mass m_e in the Coulomb potential $V(r) = -e^2/r$ of a fixed proton) in non-relativistic quantum mechanics. We use spherical coordinates (r, θ, ϕ) and the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m_e}\left[\frac{1}{r^2}\partial_r\left(r^2\partial_r\right) - \frac{\mathbf{L}^2}{\hbar^2 r^2}\right]\psi(r,\theta,\phi) - \frac{e^2}{r}\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi), \tag{3.96}$$

where $\mathbf{L}^2$ is the orbital angular momentum operator.

3.10.1 Separation of Variables and Radial Equation

1. Show that the wavefunction can be separated as $\psi(r,\theta,\phi) = R(r)\,Y_{\ell m}(\theta,\phi)$, where $Y_{\ell m}$ is a spherical harmonic eigenfunction of $\mathbf{L}^2$ and $\mathbf{L}_z$, with:

$$\mathbf{L}^2 Y_{\ell m} = \hbar^2\ell(\ell+1)\,Y_{\ell m}, \qquad \mathbf{L}_z Y_{\ell m} = \hbar m\,Y_{\ell m}. \tag{3.97}$$

   Deduce that the radial Schrödinger equation for $R(r)$ is:

$$-\frac{\hbar^2}{2m_e}\left[\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}R\right] - \frac{e^2}{r}R = ER. \tag{3.98}$$

2. Let $u(r) = rR(r)$. Show that the equation becomes:

$$-\frac{\hbar^2}{2m_e}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2m_e r^2} - \frac{e^2}{r}\right]u(r) = Eu(r). \tag{3.99}$$

   Define the parameter κ as:

$$\kappa = \sqrt{\frac{2m_e|E|}{\hbar^2}}. \tag{3.100}$$

   Show that introducing the dimensionless variable $\rho = \kappa r$, the equation takes the form:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} = \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{\rho_0}{\rho} + 1\right]u(\rho), \tag{3.101}$$

   where $\rho_0 = \dfrac{m_e e^2}{\hbar^2\kappa}$.

3. Propose the ansatz:

$$u(\rho) = \rho^{\ell+1}e^{-\rho/2}v(\rho), \tag{3.102}$$

   and show that $v(\rho)$ satisfies the differential equation[14]:

$$\rho\frac{\mathrm{d}^2v}{\mathrm{d}\rho^2} + (2\ell+2-\rho)\frac{\mathrm{d}v}{\mathrm{d}\rho} + (\rho_0 - 2\ell - 2)v = 0. \tag{3.103}$$

4. Expanding $v(\rho) = \sum_{k=0}^{\infty} c_k\rho^k$, show that the series generally diverges at infinity unless it terminates at a finite order. Deduce that the termination condition is:

$$\rho_0 = 2n, \qquad \text{where } n = \hat{k} + \ell + 1 \in \mathbb{N}^*. \tag{3.104}$$

[14] This is a confluent hypergeometric equation.

5. Derive the expression for the bound energy levels of the hydrogen atom:

$$\kappa_n = \frac{m_e e^2}{\hbar^2} \cdot \frac{1}{2n} \quad \Rightarrow \quad E_n = -\frac{\hbar^2 \kappa_n^2}{2m_e} = -\frac{m_e e^4}{2\hbar^2} \cdot \frac{1}{n^2}. \tag{3.105}$$

6. What is the degeneracy of each energy level E_n? Show that it is n^2 by considering the possible values of ℓ (from 0 to $n-1$) and m (from $-\ell$ to $+\ell$). Explain why, in this non-relativistic model, the energy depends only on n and not on ℓ.

3.10.2 Ground State ($n = 1$) and Radial Properties

7. For the ground state ($n = 1$, $\ell = 0$), show that the normalized radial wavefunction is:

$$R_{1,0}(r) = \frac{2}{a_0^{3/2}}\, e^{-r/a_0}. \tag{3.106}$$

Deduce the full expression for $\psi_{1,0,0}(r,\theta,\phi)$ and verify its normalization $\int |\psi_{1,0,0}|^2 \mathrm{d}^3x = 1$ (note that $Y_0^0 = 1/\sqrt{4\pi}$).

8. Calculate the radial probability density $P(r) = 4\pi |R_{1,0}(r)|^2 r^2$ and sketch its qualitative profile as a function of r. Interpret the physical meaning of this density (most probable location of the electron).

9. Show that the expectation value of the distance $\langle r \rangle$ between the electron and the nucleus, as well as the variance $(\Delta r)^2$, are given by:

$$\langle r \rangle = \frac{3}{2} a_0, \qquad (\Delta r)^2 = \langle r^2 \rangle - \langle r \rangle^2 = \frac{3}{2} a_0^2 - \left(\frac{3}{2} a_0\right)^2. \tag{3.107}$$

(Hint: Use the integral $\int_0^\infty r^n e^{-2r/a_0} dr = n!(a_0/2)^{\,n+1}$ and verify the results.)

10. (Optional) Introduce the momentum representation. Compute the Fourier transform $\tilde{\psi}_{1,0,0}(\mathbf{p})$ of the ground state and interpret the associated momentum distribution (square modulus). What are the expectation values of the momentum $\langle \mathbf{p} \rangle$ and its square $\langle p^2 \rangle$?

11. *Interpretation*: Briefly discuss how the $1/n^2$ dependence of the energy levels E_n explains the fine structure of hydrogen spectral lines and the concept of the principal quantum number.

3.11 Toward a Relativistic Formalism (QM, SR) ★★★★★

(Solution)
We use the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and natural units $c = \hbar = 1$.

1. Consider a real scalar field $\phi(x)$ of mass m. Define the relativistic Lagrangian density:
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{3.108}$$
Show that $\mathcal{L}$ is invariant under Lorentz transformations.

2. Apply the Euler–Lagrange equations to the scalar field and derive the Klein–Gordon equation $(\Box + m^2)\phi = 0$.

3. Discuss the Klein–Gordon equation, recalling why a second-order equation poses interpretational challenges for a relativistic field (e.g., probability density issues).

4. Motivated by this difficulty, we seek a first-order relativistic wave equation (symmetric in time and space) for a multicomponent object $\psi(x)$ (a spinor). State the general form of such a linear equation in ∂_μ (e.g., $(iA^\mu\partial_\mu - m)\psi = 0$ with matrices A^μ).

5. We propose the Dirac Lagrangian for a 4-component spinor field $\psi_\alpha(x)$ and its Dirac conjugate $\bar\psi = \psi^\dagger\gamma^0$:
$$\mathcal{L}_D = \bar\psi\,(i\gamma^\mu\partial_\mu - m)\,\psi. \tag{3.109}$$
   (a) Show that $\mathcal{L}_D$ is Hermitian (up to a total derivative).
   (b) Verify that $\mathcal{L}_D$ is invariant under Lorentz transformations (spinorial transformations of ψ).

6. Apply the Euler–Lagrange equations for multicomponent fields (varying with respect to $\bar\psi$) to derive the Dirac equation:
$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{3.110}$$
Using the definition of $\bar\psi$, write the adjoint equation satisfied by $\bar\psi$.

7. (a) Multiply the Dirac equation from the left by $(i\gamma^\nu\partial_\nu + m)$ and show that ψ satisfies the Klein–Gordon equation. Explain why the γ^μ matrices must satisfy anticommutation relations.
   (b) Explicitly derive the Dirac matrix anticommutation relations:
$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}. \tag{3.111}$$

8. (a) Give explicit representations of the γ^μ matrices in the Dirac basis (e.g., $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$).
   (b) Verify the Clifford algebra relations derived above for $\mu, \nu = 0, 1, 2, 3$.

9. We seek plane-wave solutions to the Dirac equation of the form:
$$\psi(x) = u(p)\,e^{-ip\cdot x}, \tag{3.112}$$
where $p^\mu = (E, \mathbf{p})$, $p \cdot x = p_\mu x^\mu = -Et + \mathbf{p}\cdot\mathbf{x}$.

(a) Show that $u(p)$ satisfies the algebraic equation:

$$(\gamma^\mu p_\mu - m)\, u(p) = 0. \tag{3.113}$$

(b) Deduce the allowed values of E and interpret the solutions as positive/negative energy states.

(c) How many linearly independent solutions (spinors u and v) exist for a given momentum? Relate this to the spin degrees of freedom of the particle and its antiparticle.

10. Define the Dirac current:

$$j^\mu = \bar{\psi}\,\gamma^\mu \psi. \tag{3.114}$$

(a) Using the Dirac equation (and its adjoint), show that $\partial_\mu j^\mu = 0$ (current conservation).

(b) Verify that the density $j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi$ is positive definite. Interpret j^0 as a probability density and compare with the Klein–Gordon case.

11. We now quantize the Dirac field.

(a) State the canonical quantization conditions for the Dirac field ψ and $\psi^\dagger$ (or ψ and $\bar{\psi}$). Justify why anticommutators $\{\,,\,\}$ must be used instead of commutators.

(b) Explain how quantization leads to the introduction of creation/annihilation operators for particles and antiparticles (Dirac's interpretation of negative-energy states as antiparticles).

12. Show that the Dirac Lagrangian $\mathcal{L}_D$ is invariant under the global phase transformation $\psi \to e^{i\alpha}\psi$. Using Noether's theorem, deduce the conservation of the current j^μ.

13. Introduce minimal coupling of the Dirac field to an electromagnetic potential A_μ by replacing ∂_μ with $D_\mu = \partial_\mu + ieA_\mu$ in $\mathcal{L}_D$. Show that this prescription makes the Lagrangian invariant under local gauge transformations $\psi \to e^{ie\Lambda(x)}\psi$, $A_\mu \to A_\mu - \partial_\mu\Lambda$. Briefly discuss the physical implications (introduction of electromagnetic interactions and charge conservation).

14. Summarize the physical implications of the Dirac equation: existence of a free spin-$1/2$ particle (and its antiparticle), prediction of the magnetic moment $\mu = g\frac{q}{2m}\mathbf{S}, \mu$ vector, with $g = 2$, etc. Conclude by explaining how this construction respects relativistic covariance and the role of Clifford algebra in describing relativistic fermions.

3.12 Pöschl–Teller Potential $V(x) = -\dfrac{V_0}{\cosh^2(\alpha x)}$ (QM) ★★★

(Solution)

We consider a quantum particle subject to an attractive potential of the form

$$V(x) = -\frac{V_0}{\cosh^2(\alpha x)}, \tag{3.115}$$

called the Pöschl–Teller potential, where $\alpha > 0$. This potential admits a finite number of bound states and allows for an exact solution of the Schrödinger equation.

The Hamiltonian of the system is given by

$$\mathbf{H} = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X}) = \frac{\mathbf{P}^2}{2m} - \frac{V_0}{\cosh^2(\alpha \mathbf{X})} \tag{3.116}$$

1. Write the time-independent Schrödinger equation for a wave function $\psi(x)$:

$$-\frac{\hbar^2}{2m}\psi''(x) - \frac{V_0}{\cosh^2(\alpha x)}\,\psi(x) = E\,\psi(x). \tag{3.117}$$

2. Show that the substitution $u = \tanh(\alpha x)$ yields:

$$\psi'(x) = \alpha\,(1-u^2)\,\frac{\mathrm{d}\phi}{\mathrm{d}u}, \quad \psi''(x) = \alpha^2\Big((1-u^2)\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} - 2u\frac{\mathrm{d}\phi}{\mathrm{d}u}\Big), \tag{3.118}$$

   with $\phi(u) = \psi(x(u))$.

3. Deduce that the equation in $u \in]-1,1[$ becomes

$$(1-u^2)\,\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} - 2u\,\frac{\mathrm{d}\phi}{\mathrm{d}u} + \Big[\lambda(\lambda+1) - \frac{\mu^2}{1-u^2}\Big]\phi = 0, \tag{3.119}$$

   and express λ, μ in terms of $V_0, \alpha, m, \hbar, E$.

4. Identify λ, μ. We seek a solution of the form $\phi(u) = (1-u^2)^{\frac{\mu}{2}}P(u) = Q(u)P(u)$.

   Hint: Express $Q'(u)$ in terms of $Q(u)$ to simplify the calculations.

   Show that $P :]-1,1[\to \mathbb{R}$ satisfies the differential equation

$$(1-u^2)P'' - 2(\mu+1)uP' + [\lambda(\lambda+1) - \mu(\mu+1)]P = 0 \tag{3.120}$$

5. By examining the behavior of the solution at the endpoints $u \to \pm 1$, we aim to show that $P(u)$ must be a polynomial for $\phi \in L^2([-1,1])$. We will proceed by contradiction, assuming P is not a polynomial.

   (a) Show that the normalization condition reads

$$\int_{-1}^{1} \frac{|\phi(u)|^2}{1-u^2} < \infty \tag{3.121}$$

   (b) P is analytic on $]-1,1[$. Write $P(u) = \sum_{p=0}^{\infty} a_p u^p$. Derive a recurrence relation between a_{p+2} and a_p and show that

$$a_{p+2} \underset{\infty}{\sim} a_p \tag{3.122}$$

(c) Conclude using the Riemann criterion for integrals[15].

6. It follows that $n = \deg P \in \mathbb{N}$. We therefore write

$$P(u) = \sum_{p=0}^{n} a_p u^p \tag{3.123}$$

Use the recurrence relation between a_{p+2} and a_p to show that $\mu = \lambda - n$[16].

7. Deduce the quantized energy levels E_n as

$$E_n = -\frac{\hbar^2\alpha^2}{2m}(\lambda - n)^2, \quad n = 0, 1, \dots, \lfloor\lambda\rfloor, \tag{3.124}$$

where $\lambda(\lambda+1) = \frac{2mV_0}{\hbar^2\alpha^2}$.

8. Show that the number of bound states is finite: $N = \lfloor\lambda\rfloor + 1$.

9. Provide a physical explanation for why the number of bound states is finite, despite the "bottomless" shape of the potential. Discuss the connection with the asymptotic decay of the potential.

[15] For full mathematical rigor, one should also invoke the theorem on equivalence of divergent series.
[16] We consider only bound states, hence $\mu > 0$.

3.13 Larmor Power and Electrodynamic Instability of the Classical Atom† (EM) ★★

[17] (Solution)

A confined (and thus accelerated) charge emits electromagnetic radiation. We now examine in more detail some consequences of classical Electromagnetic laws combined with those of dynamics (hence: *Electrodynamics*), and in particular, show that the classical atom is fundamentally unstable: the electron localized in the atom emits radiation and, as a result, gradually loses energy.

The description below relies on the assumption that the radiation effect is a minor phenomenon, although it ultimately leads to dramatic conclusions. We will therefore start with an ordinary dynamical description, to which we will add the perturbative effects of the source's (the confined electron's) radiation on its own motion.

3.13.1 Calculation of the radiation reaction force $\mathbf{F}_{\mathrm{rad}}$.

The Larmor power is the power lost by an accelerated charge. We will deduce a radiation reaction force $\mathbf{F}_{\mathrm{rad}}$ from it, which leads to dramatic consequences.

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{2q^2 a^2}{3c^3} \tag{3.125}$$

1. Write the work $\mathrm{d}E_{\mathrm{at}} = \mathrm{d}W$, equal to the change in atomic energy over a time $\mathrm{d}t$ due to the radiation force $\mathbf{F}_{\mathrm{rad}}$.

2. Write the energy variation of the atom over a time $\mathrm{d}t$ due to the radiated power of the electron.

3. By integrating by parts and assuming periodic motion, show that[18],

$$\mathbf{F}_{\mathrm{rad}} = \frac{2\vartheta^2}{3c^3}\ddot{\mathbf{v}} \tag{3.126}$$

4. Apply Newton's second law with the previously calculated $\mathbf{F}_{\mathrm{rad}}$[19] and a restoring force $\mathbf{F} = -m\omega_0^2\mathbf{r}$. Seeking a solution of the form $\mathbf{r}(t) = \mathrm{Re}\{\mathbf{r}_0 e^{i\omega t}\}$, and letting

$$\omega = \omega_0(1 + \alpha(\omega_0\tau) + o(\omega_0\tau)), \alpha \in \mathbb{R} \tag{3.127}$$

show that the solution is a damped oscillator.
N.B. Given $\tau = \frac{2e^2}{3mc^3} \simeq 6.4 \times 10^{-24}$ s, $\omega_0 = 3 \times 10^{15}$ rad.s^{-1}. Comment.

3.13.2 Conceptual issues raised by the radiation reaction force $\mathbf{F}_{\mathrm{rad}}$.

The radiation reaction force $\mathbf{F}_{\mathrm{rad}}$ written above is conceptually pathological, as the following analysis shows. Using the notations of Section 1.5, Volume I, the Abraham-Lorentz equation for a particle of charge e and mass m subjected to a force $\mathbf{F}$ (with $\mathbf{v} = \dot{\mathbf{r}}$) is:

$$m\ddot{\mathbf{r}} = m\tau\,\dddot{\mathbf{r}} + \mathbf{F}, \tag{3.128}$$

[17] † Inspired by Claude Aslangul, *Quantum Mechanics 1*, Chapter 1.

[18] Where $\vartheta^2 = \frac{e^2}{4\pi\varepsilon_0}$

[19] Note the appearance of a force depending on the derivative of the acceleration. We will study in the next part the issues caused by this force.

where $\tau = \frac{2e^2}{3mc^3} \simeq 6.4 \times 10^{-24}$ s is a characteristic time. One oddity of this equation is the appearance of a third derivative of the particle's position (defined by the radius vector $\mathbf{r}$), which is meant to represent the radiation damping effect.

Moreover, the perturbation of motion caused by this effect is fundamentally *singular*, in the sense that it alters the order of the motion equation, which changes from second to third order as soon as the charge is nonzero. In fact, it is precisely because the small parameter τ multiplies the highest derivative that the perturbation is called *singular*, by definition[20].

With these warnings in mind, we now examine the consequences of equation (3.128) as it stands, to highlight the deep conceptual issues it poses.

1. Using the standard method for solving a differential equation like (3.128), write the general expression for the acceleration $\ddot{\mathbf{r}}(t)$, assuming the acceleration at some instant t_0, $\ddot{\mathbf{r}}(t_0)$, is known.

2. Examine the particular case $\mathbf{F} = 0$, and show that the solution is physically aberrant.

3. Returning to the general solution obtained in 1 for $\mathbf{F} \neq 0$, show that the divergent solutions can formally be eliminated by a suitable choice of t_0. Comment on this choice — which, from a technical standpoint, expresses a boundary condition rather than an initial condition.

4. Deduce the regularized expression of the solution obtained in 1. Stepping back, analyze the integral kernel in this expression and verify that, in the limit of zero charge, the motion equation reduces to the standard dynamical equation.

5. To clearly exhibit the violation of a major physical principle, make a simple change of integration variable to obtain:

$$\dot{\mathbf{v}}(t) = \frac{1}{m} \int_0^{+\infty} e^{-s} \times \mathbf{F}(t + \tau s)\, \mathrm{d}s. \tag{3.129}$$

   Comment on this equation and show that it violates a physical principle.

6. To highlight this violation even more spectacularly, treat the case of a particle with zero velocity at $t = -\infty$ and subjected to a step force:

$$\mathbf{F}(t) = \begin{cases} 0 & \text{if } t < 0, \\ \mathbf{F}_0 & \text{if } t > 0. \end{cases} \tag{3.130}$$

   Summarize these results by plotting the time evolution of the acceleration and velocity. Note that the particle starts moving... **before the force is applied**!

[20] The same phenomenon occurs in the Schrödinger eigenvalue equation, where it is Planck's constant that multiplies the highest derivative. A specific perturbation technique is used for such problems, known as the WKB (or BKW) method in the quantum context.

3.14 Geodesics in a Dispersive Optical Medium (WO, AM, EM, TH) ★★★★★

(Correction)
The goal of this exercise is to understand how light propagates in media from the variational principle, taking into account dispersion (dependence of the refractive index on the wavelength λ). This will then allow us to explain various common optical phenomena.

3.14.1 Fermat's Principle and Optical Metric

1. Derive Fermat's principle from the principle of least action. This principle can be interpreted as a geodesic for an effective metric given by:
$$\mathrm{d}s^2 = n(\mathbf{r},\lambda)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{3.131}$$
which is a metric conformal to the Euclidean one: $g_{ij}(\mathbf{r},\lambda) = n(\mathbf{r},\lambda)^2\delta_{ij}$ in Einstein notation.

2. Justify why this metric is suitable for light propagation in an inhomogeneous and dispersive medium.

3. Show that the optical trajectories are the geodesics of this metric.

3.14.2 Calculation of a Refractive Index $n(\mathbf{r},\lambda)$

In any medium, the refractive index depends microscopically on the density through the electric susceptibility. Here, we want to justify, from a realistic electromagnetic model[21], that the index can be expressed in the form, with $\zeta = \frac{\omega}{\omega_0}$:
$$\boxed{n(\mathbf{r},\zeta) \underset{\zeta\to 0}{=} 1 + \frac{1}{2}\frac{N(\mathbf{r})e^2}{m\omega_0^2\varepsilon_0}\left(1+\zeta^2\right) + o(\zeta^2)}. \tag{3.132}$$

1. Recall that in an isotropic linear medium with negligible magnetic field, the refractive index satisfies $n^2 = \varepsilon_r = 1+\chi$.

2. From the Lorentz model for a bound electron subjected to an electric field, with a single resonance ω_0, express the susceptibility $\chi(\mathbf{r},\omega)$ as a function of the local density $N(\mathbf{r})$. In particular, show that for negligible damping γ,
$$\boxed{n^2(\mathbf{r},\zeta) = 1 + \frac{N(\mathbf{r})e^2}{m\omega_0^2\varepsilon_0}\frac{1}{1-\zeta^2}}. \tag{3.133}$$

3. Assuming $\omega \ll \omega_0$, deduce the Taylor expansion of equation (3.132).

This model will then allow the introduction of an optical metric to study the geodesics of light in the drop.

[21] The validity of the expansion depends on the medium considered. In the case of a gas or a liquid, it is usually sufficient to consider a single main resonance located in the ultraviolet. This allows a very precise approximation: the error on the index is typically less than 0.01 % in air, and about 0.1 % in water, within the visible range.

3.14.3 Calculation of $N(\mathbf{r})$ for a Gas and a Liquid

We seek to calculate $N(\mathbf{r})$ for air and water, treating each case independently. Recall that,

$$N(\mathbf{r}) = \frac{\rho(\mathbf{r})}{M}\mathcal{N}_A, \tag{3.134}$$

where ρ is the mass density, M the molar mass, and $\mathcal{N}_A$ Avogadro's number.

1. Assume air is a diatomic ideal gas. It is subject to the gravitational field $\mathbf{g} = -g\mathbf{e}_z$. We suppose hydrostatic equilibrium and that $\delta Q = 0$, i.e., the atmosphere is adiabatic.
   (a) Using the first law of thermodynamics, prove that
   $$\frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{g}{C_p}, \tag{3.135}$$
   and deduce $T(z)$.
   (b) Determine a differential equation for p and prove that
   $$p(z) = p_0\left(\frac{T(z)}{T_0}\right)^{\frac{gM}{R\Gamma}}, \quad \Gamma = \frac{g}{C_p}, \tag{3.136}$$
   (c) Deduce $N(\mathbf{r}) = N(z)$ in this case.
2. In the case of a liquid, one can generally assume $\rho(\mathbf{r})$ is constant. Deduce $N(\mathbf{r})$.

3.14.4 Optical Geodesics in a Spherical Medium

Consider an isotropic medium where the refractive index n depends only on the radial distance r and the wavelength λ, through the function $n(r, \lambda)$. This framework models, for example, spherically symmetric profiles of electron density $N(r)$, relevant for idealized planetary atmospheres.

In this context, light propagation can be described by an optical metric defining the infinitesimal inner product between two vectors $\mathrm{d}x^i = (\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3)$ as:

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j, \tag{3.137}$$

where g_{ij} is the optical spherical metric.

To study geodesics, parametrize the trajectory by an affine parameter s, and consider functions $r(s)$, $\theta(s)$, and $\varphi(s)$.

The associated Lagrangian, using the principle of least action for this metric, is:

$$\mathcal{L} = n^2(r, \lambda)\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right), \tag{3.138}$$

where $\dot{} = \frac{\mathrm{d}}{\mathrm{d}s}$.

For simplicity, one can fix the equatorial plane $\theta = \frac{\pi}{2}$, which simplifies the Lagrangian to:

$$\mathcal{L} = n^2(r, \lambda)\left(\dot{r}^2 + r^2\dot{\varphi}^2\right), \tag{3.139}$$

with $\dot{\theta} = 0$.

Due to azimuthal symmetry, one can also use φ as the independent parameter, allowing the trajectory to be written as $r(\varphi)$ and simplifying the resolution.

1. **Derivative of the squared index:** Explicitly express $\partial_r n^2(r,\lambda)$ as a function of $N(r)$ and its derivative $N'(r) = \frac{\partial N}{\partial r}$, knowing that $n(r,\lambda)$ depends on $N(r)$.

2. **Christoffel symbols:** Compute the Christoffel symbols Γ^k_{ij} for this spherical metric in terms of $\partial_r n^2$, and give their explicit form for coordinates r and θ.

3. **Geodesic equations:** Using the Christoffel symbols, explicitly write the differential equations for $r(s)$ and $\theta(s)$ from the geodesic equation[22]:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\nu\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \frac{\mathrm{d}x^\lambda}{\mathrm{d}s} = 0. \tag{3.140}$$

4. **Lagrangian formalism:** Write the associated optical Lagrangian,

$$\mathcal{L} = n^2(r,\lambda) \left(\dot{r}^2 + r^2\dot{\theta}^2\right), \tag{3.141}$$

with $\dot{} = \frac{\mathrm{d}}{\mathrm{d}s}$, and obtain the Euler–Lagrange equations for $r(s)$ and $\theta(s)$.

5. **Variational principle:** Verify that the equations obtained from the Lagrangian formalism coincide with those derived from the Christoffel symbols.

6. **Symmetry and conservation:** Show that the system is invariant under rotation around the axis (symmetry in φ) and deduce the conservation of an effective angular momentum.

7. **Parameterization by the angle φ:** Propose a change of parameter to describe the trajectory as a function of φ (or θ in the equatorial plane), and rewrite the differential equations accordingly.

8. **Local Snell's law:** Using angular momentum conservation, derive a relation analogous to Snell's law, explicitly depending on λ via $n(r,\lambda)$.

9. **Numerical method:** Propose a method to compute the full trajectory of the light ray for a fixed wavelength, for example by numerically solving the equations obtained in φ.

3.14.5 1st application: The rainbow as a geometric manifestation of dispersion

We model a water drop as a homogeneous sphere of radius R, with refractive index $n(\omega)$ calculated in section 3.14.2, immersed in air with index approximately 1 (see Fig. 3.4).

1. **Refractive index of water.** Recall the expression of $n(\mathbf{r},\omega) = n(\omega)$ for a liquid.

2. **Geometric modeling of a primary rainbow.**

   (a) Consider an incident ray coming from the Sun, entering the drop with an incidence angle α, refracted according to Snell–Descartes law, reflected once inside, then refracted again on exit.
   Justify that the ray remains straight inside the drop (constant index), and that the only deflections occur at the spherical surfaces.

[22] A knowledgeable reader will note that equation (3.140) is the geodesic equation in general relativity. The setting is similar: a geometric curvature term is imposed due to the optical medium.

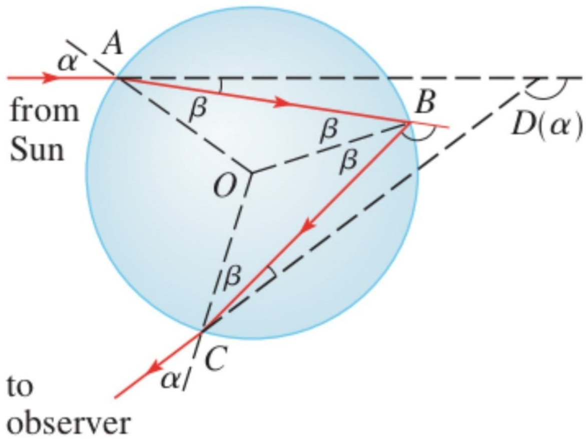

Figure 3.4: Geometric scheme of the angles of incidence, refraction, and deviation of a light ray passing through a spherical drop. This diagram introduces the total deviation angle $\Theta - \pi = D$.

(b) $\Theta(\omega, \alpha)$ is the total deviation angle. Show that

$$\Theta(\omega, \alpha) = 2\alpha - 4\arcsin\left(\frac{\sin\alpha}{n(\omega)}\right) + \pi. \tag{3.142}$$

(c) Show that $\Theta(\omega, \alpha)$ has a minimum at a certain angle $\alpha_c(\omega)$. Demonstrate that

$$\alpha_c(\omega) = \arcsin\left(\sqrt{\frac{4 - n^2(\omega)}{3}}\right). \tag{3.143}$$

Deduce that the exit directions concentrate around a particular angle $\Theta_{\text{min}}(\omega)$, which produces a maximum of luminous intensity observed in this direction (see Fig. 3.5).

(d) **Numerical application:** For $n \simeq 1.33$, calculate the angle D_{min}.

(e) Study the function $\Theta_{\text{min}}(\omega)$, and explain the phenomenon of rainbows.

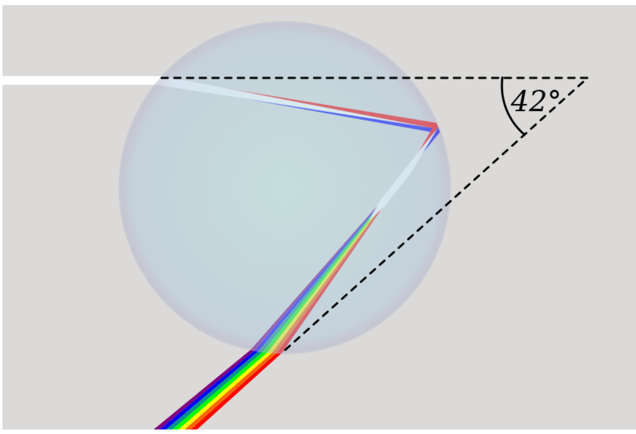

Figure 3.5: Diagram of rainbow formation and the angle $\Theta_{\text{min}}(\omega)$.

3. **Link with the optical metric and continuous modeling.**

(a) Suppose that the density of polarizable particles inside the drop varies radially as $N(r) = N_0\left(1 - \mu r^2\right)$, with $\mu > 0$ and $r \in [0, R]$.
Using equation (3.132), deduce the expression of the refractive index n. Verify that $n(\mathbf{r}, \omega)$ decreases from the center to the edge of the drop.

(b) Working in the equatorial plane of the drop, write the associated optical metric:

$$\mathrm{d}s^2 = n^2(r)\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2\right), \tag{3.144}$$

and recall that light trajectories are geodesics of this metric.

(c) Using the rotational invariance of the system, write the geodesic equation for a trajectory $r(\theta)$ (or its integral equation via conservation of the optical angular momentum). Discuss qualitatively the shape of light trajectories in this decreasing-index medium.

(d) Comparing trajectories for different frequencies (via the ω-dependence of α), qualitatively explain the separation of colors and the formation of a colored arc.

3.14.6 2nd application: colors of soap bubbles (thin-film interference)†

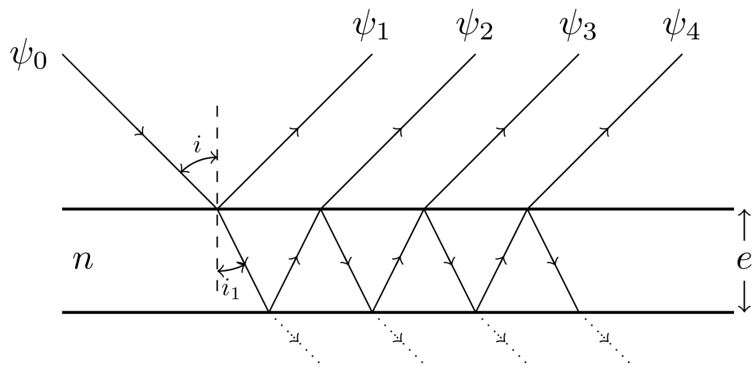

Figure 3.6: Diagram of the soap film.

[23] Here we study the colorful iridescence observed on soap bubbles, explained by interference between waves reflected within a very thin liquid layer modeled as a thin film.

A bubble is represented by a liquid layer with refractive index $n \simeq \frac{4}{3}$ and constant thickness e, enclosed between two air/liquid/air interfaces (see Fig. 3.6)[24]. The air refractive index is taken as 1. The incident light is monochromatic with wavelength λ and is nearly normal incidence[25].

1. **Multiple reflections and interference**

   (a) Explain why the light reflected by the bubble can be decomposed as an infinite sum of waves successively reflected and transmitted at the two interfaces.

   (b) Show that the phase shift between two consecutive reflected waves is given by:

$$\varphi = \frac{4\pi n e}{\lambda}. \tag{3.145}$$

[23] † Exercise inspired by the HAP502P tutorial by Benoît Rufflé.
[24] In the diagram, the angle i is given but can be considered as zero.
[25] This problem illustrates how thin-film interferences, modulated by microscopic variations in thickness and optical index, can produce spectacular visual effects similar to those observed in rainbows.

2. **Fresnel coefficients and reflected amplitude**

   Let $r = \frac{1-n}{1+n}$ and $t = \frac{2}{1+n}$ be the reflection and transmission coefficients at the air-soap interface (normal incidence).

   (a) Compute numerically r and t for $n = \frac{4}{3}$.

   (b) Write the expression for the total reflected amplitude ψ_r as the sum of the geometric series of multiple waves, clearly showing the contribution of each reflection and transmission. Show that the sum can be written as:

$$\psi_r = \psi_0 \left[r + \frac{t^2 r e^{i\varphi}}{1 - r^2 e^{i2\varphi}} \right]. \tag{3.146}$$

3. **Reflected intensity and interference conditions**

   (a) Deduce the expression for the reflected intensity $I_r \propto |\psi_r|^2$.

   (b) Show that this intensity exhibits maxima and minima as a function of φ, and determine the conditions on λ for which the reflection is minimal (destructive interference) and maximal (constructive interference).

4. **Numerical application: thickness** $e = 0.3\,\mu m$

   (a) Determine the values of λ in the interval $[0.4, 0.8]\,\mu m$ for which reflection is minimal.

   (b) What is the dominant hue perceived by the human eye in this case?

5. **Limiting cases**

   (a) Qualitatively study what happens when $e = 0.03\,\mu m$.

   (b) Qualitatively study what happens when $e = 30\,\mu m$.

   (c) Interpret the optical consequences of these two situations.

6. **Extension: position-dependent, frequency-dependent index, and white light**

   Consider that the optical index in the bubble depends both on the radial position $R \in [0, e]$ (where e is the local average thickness of the film, not the bubble radius) and on the frequency ω via the reduced variable $\zeta = \frac{\omega}{\omega_0}$, with ω_0 a characteristic frequency.

   Suppose that the local oscillator density follows a radial variation[26] given by:

$$N(R) = N_0 \left(1 - \lambda R^2\right), \tag{3.147}$$

   with $\lambda > 0$ characterizing the spatial variation.

   (a) Recalling the expanded form of the index for $\zeta \to 0$ (eq. 3.132) and the radial dependence of $N(R)$, explain how the radial variation $N(R)$ locally influences the optical index.

   (b) Taking into account the index variation with R, write the integral expression for the total phase shift $\varphi(\omega)$ across the film, integrating the local phase contribution over the full thickness e, as

$$\boxed{\varphi(\omega) = \frac{2\omega}{c} \int_0^e n\left(R, \frac{\omega}{\omega_0}\right) \mathrm{d}R,} \tag{3.148}$$

   clearly expressing the frequency dependence.

[26] Note that here R denotes the local coordinate across the film thickness (perpendicular to the bubble surface), not the bubble radius. The local oscillator density $N(R)$ thus varies through the soap film thickness, influencing the optical index in this direction.

(c) Deduce the expression of the reflected amplitude $\psi_r(\omega)$ as a function of the phase $\varphi(\omega)$:

$$\psi_r(\omega) = \psi_0 r \left(1 + e^{i\varphi(\omega)}\right), \tag{3.149}$$

taking into account the reflection and transmission coefficients at each interface.

(d) Deduce the spectral reflected intensity $I_r(\omega) \propto |\psi_r(\omega)|^2$, and show that it writes:

$$I_r(\omega) \propto 4\psi_0^2 r^2 \cos^2\left(\frac{\varphi(\omega)}{2}\right). \tag{3.150}$$

(e) Study how the spatial variation of the index $n(R,\omega)$ modifies the interference structure, in particular the angular and spectral distribution of reflected colors.

(f) Explain why illumination by white light (broad spectrum in ω) can generate complex colored patterns (rainbow) on the bubble surface.

(g) For the specific case $N(R) = N_0(1-\lambda R^2)$, explicitly calculate the integral contribution to the phase $\varphi(\omega)$. Discuss qualitatively the effect of λ on the observed colors.

(h) Suppose the local thickness $e(R)$ also varies with position.

i. Express the combined effect of $n(R,\omega)$ and $e(R)$ on the phase $\varphi(\omega)$, in the form

$$\varphi(\omega) = \frac{2\omega}{c}\int_0^{e(R)} n\left(R, \frac{\omega}{\omega_0}\right) \mathrm{d}R, \tag{3.151}$$

specifying the order of integration depending on the geometry.

ii. Discuss how these local variations of index and thickness modulate the interference fringes, leading to the formation of complex patterns on the bubble.

iii. How do these microscopic variations explain the richness of colors and shapes observed in real soap bubbles?

(i) *Bonus*: qualitatively discuss the influence of scattering and absorption in the bubble on the visibility and sharpness of the interferences.

3.14.7 3rd application: mirages†

[27]

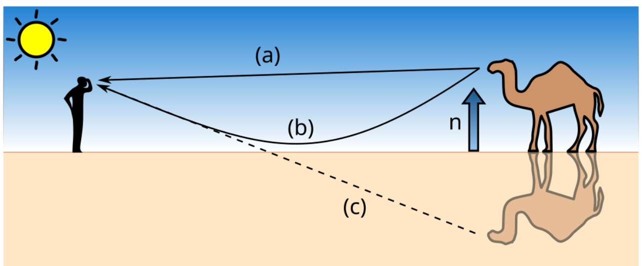

Figure 3.7: Diagram of a mirage.

Fermat's principle states that the time taken by light to travel from one point to another is minimal. Suppose the refractive index of air near the ground depends only on altitude, $n(z)$. Consider a light ray starting at height h and making a downward angle θ. Points in the xOz plane are identified by Cartesian coordinates (x, z) (see Fig. 3.7).

[27]† This section is taken from exercise 3.2 of the tutorial sheet by J. Dorignac.

1. Show that the time taken by light to go from point $(0, h)$ to (x_f, z_f) along a path $z(x)$ can be written as
$$T = \frac{1}{c}\int_0^{x_f} n(z(x))\sqrt{1 + (z'(x))^2}\,\mathrm{d}x, \tag{3.152}$$
where $z'(x)$ is the derivative of z with respect to x.

2. Since T must be minimal, deduce from Beltrami's identity that the light trajectory satisfies
$$n(z(x))^2 = A\big(1 + (z'(x))^2\big), \tag{3.153}$$
where A is a constant.

3. Suppose the ground is hot and the atmosphere colder above. Then the refractive index increases with z. Use the model
$$n(z)^2 = n_0^2 + \alpha z. \tag{3.154}$$
Show that
$$A = \frac{n(h)^2}{1 + \tan^2\theta}. \tag{3.155}$$

4. Show that the light path
$$z(x) = h + x\tan\theta + \frac{\alpha}{4A}x^2 \tag{3.156}$$
is a solution of the problem (just substitute).

5. Suppose an observer's eye is located at point (L, H). Show that there exist two initial angles θ_1 and θ_2 that allow rays leaving $(0, h)$ to be generally observed.

6. Explain the mirage effect.

3.14.8 4th application: study of the sunset phenomenon via $N(z)$ and $n(z, \omega)$

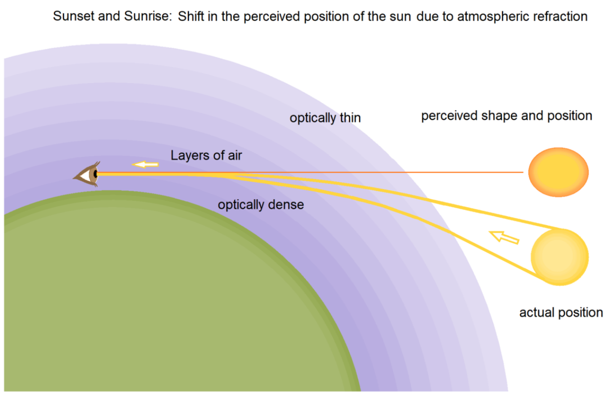

Figure 3.8: Representative diagram of a sunset.

Recall the expression of the refractive index as a function of height z and reduced frequency $\zeta = \frac{\omega}{\omega_0}$:
$$n(z, \zeta) = 1 + \frac{1}{2}\frac{N(z)e^2}{m\omega_0^2\varepsilon_0}\left(1 + \zeta^2\right) + o(\zeta^2). \tag{3.157}$$

1. Show that the variation of $N(z)$ implies a variation of $n(z,\zeta)$ with z. Express $\frac{\partial n}{\partial z}$ in terms of $\frac{\mathrm{d}N}{\mathrm{d}z}$.

2. Write the Lagrangian $\mathcal{L}$ describing the trajectory of a light ray in this isotropic medium where n depends on z and ω, recalling the standard form of $\mathcal{L}$ for a medium with variable index.

3. Assuming a trajectory in the vertical plane (x,z), write the Euler–Lagrange equations for $x(s)$ and $z(s)$, where s is a parameter along the trajectory.

4. Considering that the dependence on ω induces a variation of n, discuss how this modifies the light ray trajectory according to its frequency.

5. Based on these results, explain the sunset phenomenon (reddening of the Sun).

3.15 Bose-Einstein Condensation† (PS) ★★★★

(Correction)

[28] We consider a gas of identical bosonic particles with zero spin in a container of volume V in contact with a thermostat at temperature T. The particles do not interact with each other.

1. The gas is described in the grand canonical ensemble. We aim to express the average number of particles as a function of the temperature T and the chemical potential μ in the form of an integral, without attempting to evaluate it.

   (a) Write the expression of the average number $\langle n_\varepsilon \rangle$ of particles in a state of energy ε according to Bose-Einstein statistics, as a function of μ, T, and k_B.

   (b) Considering a gas in a cubic box of volume V with periodic boundary conditions, express the density of states $g(\varepsilon)$ in the approximation of a non-relativistic free gas of particles of mass m.

   (c) Deduce that the total average number of particles can be written as:

   $$\langle N \rangle = \int_0^{+\infty} \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)} - 1} \mathrm{d}\varepsilon \tag{3.158}$$

   where $\beta = 1/(k_B T)$, then rewrite this expression as an integral depending on T, μ, and m, without solving it.

2. We now assume the system is closed and contains N particles. The chemical potential then becomes a function of temperature and particle density $\rho = N/V$. Using the previous result and the equivalence between canonical and grand canonical ensembles, show that $\mu(T)$ is given by:

   $$\rho = \left(\frac{2mk_B T}{4\pi^2\hbar^2}\right)^{3/2} \int_0^{+\infty} \frac{x^{1/2}}{e^{x/\varphi(T)} - 1} \mathrm{d}x, \tag{3.159}$$

   with $\varphi(T) = e^{\mu(T)/(k_B T)}$.

3. From equation (3.159), justify that $\mu(T)$ increases as the temperature decreases.

4. Recall why the chemical potential must be negative. Conclude that equation (3.159) can only be valid for $T \geq T_{\mathrm{BE}}$, and determine the explicit expression of T_{BE}. Given:

   $$\int_0^{+\infty} \frac{x^{1/2}}{e^x - 1} \mathrm{d}x \simeq 2.612 \times \sqrt{\frac{\pi}{2}}. \tag{3.160}$$

5. For $T \leq T_{\mathrm{BE}}$, $\mu(T) = 0$ and equation (3.159) is not satisfied. Identify the flaw in the reasoning from the previous questions.

6. To fix this issue, the population of the ground state, denoted N_0, is isolated. Justify that

   $$N = N_0 + \left(\frac{2m}{4\pi^2\hbar^2}\right)^{3/2} V \int_0^{+\infty} \frac{\varepsilon^{1/2}}{e^{\beta\varepsilon} - 1} \mathrm{d}\varepsilon. \tag{3.161}$$

   Then compute N_0 as a function of N, T, T_{BE}, and plot N_0/N as a function of T/T_{BE}. Comment.

[28]† Exercise taken from the HAP602P problem set by Benjamin Guiselin, Université de Montpellier.

7. Justify that for $T \leq T_{\text{BE}}$, the grand potential $\mathcal{J}$ is given by

$$\frac{\mathcal{J}}{k_B T} = -\ln(1+N_0) + \left(\frac{2m}{4\pi^2\hbar^2}\right)^{3/2} V \int_0^{+\infty} \varepsilon^{1/2} \ln\left(1-e^{-\beta\varepsilon}\right) \mathrm{d}\varepsilon. \tag{3.162}$$

What does this expression become in the thermodynamic limit? Then compute the pressure of the bosonic gas for $T \leq T_{\text{BE}}$ and $N \gg 1$, and comment on its dependencies. Given:

$$\int_0^{+\infty} \frac{x^{3/2}}{e^x - 1} \mathrm{d}x \simeq 1.341 \times \frac{3\sqrt{\pi}}{4}. \tag{3.163}$$

3.16 Decay Chain (FS) ★★★★★

(Correction)

We consider a radioactive decay chain formed by n isotopes denoted (N_k), which decay successively into one another ($N_1 \to N_2 \to \cdots \to N_n$), the last one being assumed stable. We denote by $N_k(t)$ the number of nuclei of type k at time $t \geq 0$. Each nucleus N_k is unstable for $k \in [\![1, n-1]\!]$ and has a radioactive decay constant $\lambda_k > 0$. The last isotope N_n is stable, which amounts to setting $\lambda_n = 0$.

3.16.1 Physical modeling of the decay chain

1. Justify that the functions $N_k(t)$ satisfy the differential system:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\lambda_1 N_1, \qquad \frac{\mathrm{d}N_k}{\mathrm{d}t} = -\lambda_k N_k + \lambda_{k-1} N_{k-1} \quad \text{for } k \in [\![2, n]\!]. \tag{3.164}$$

2. Solve the case $n = 2$ with initial conditions $N_1(0) = N_0$, $N_2(0) = 0$. Sketch qualitatively the curves $N_1(t)$ and $N_2(t)$.

3. Show that the solution satisfies for all $t \geq 0$:

$$N_1(t) + N_2(t) = N_0. \tag{3.165}$$

   Physically interpret this: it corresponds here to conservation of matter in the system.

4. Discuss the time when the quantity $N_2(t)$ is maximal, and give its expression if $\lambda_1 \neq \lambda_2$.

3.16.2 Mathematical study of the differential system

Let $A \in \mathcal{M}_n(\mathbb{R})$ be the matrix defined by:

$$A = \begin{pmatrix} -\lambda_1 & 0 & 0 & \cdots & 0 \\ \lambda_1 & -\lambda_2 & 0 & \cdots & 0 \\ 0 & \lambda_2 & -\lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1} & 0 \end{pmatrix}. \tag{3.166}$$

Consider the vector system:

$$\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}t} = A\mathbf{N}, \qquad \mathbf{N}(0) = \mathbf{N}_0 \in \mathbb{R}^n. \tag{3.167}$$

1. Show that A is diagonalizable over $\mathbb{R}$ if the λ_k (for $k \in [\![1, n-1]\!]$) are pairwise distinct. Give the eigenvalues.

2. Show that the system admits a unique global solution on $\mathbb{R}_+$ for any initial condition $\mathbf{N}_0$.

3. Let us define $E(t) = \|\mathbf{N}(t)\|^2$. Show that E is differentiable. We then aim to prove that E is a decreasing function.

(a) Show that

$$\forall x \in \mathbb{R}^n, \quad \langle x|Ax\rangle = \langle x|Sx\rangle\,, \quad \text{where } S = \frac{A + A^T}{2}. \tag{3.168}$$

(b) Show that

$$\langle x|Sx\rangle = -\frac{1}{2}\sum_k \lambda_k (x_k - x_{k+1})^2 - \frac{\lambda_1}{2} x_1. \tag{3.169}$$

(c) Discuss the sign of $E'(t)$. Deduce the stability of the system.

4. Assume that $\lambda_k \geq \alpha > 0$ for all $k \in [\![1, n-1]\!]$. We define a norm satisfying:

$$\forall x \in \mathbb{R}^n, \quad \forall M \in \mathcal{M}_n(\mathbb{R}), \quad \|Mx\| \leq C \times \xi \|x\|, \tag{3.170}$$

where ξ satisfies $\xi > \mu_p \in \mathrm{Sp}(M), \quad \forall p$.

Prove that:

$$\|\mathbf{N}(t) - \mathbf{N}_\infty\| \leq C e^{-\alpha t} \|\mathbf{N}_0\|, \quad \text{where } \mathbf{N}_\infty = (0, \ldots, 0, N_\infty). \tag{3.171}$$

5. Assume that $\lambda_k \geq \alpha > 0$ for all $k \in [\![1, n-1]\!]$. We define a norm satisfying:

$$\forall x \in \mathbb{R}^n, \quad \forall M \in \mathcal{M}_n(\mathbb{R}), \quad \|Mx\| \leq C \times \xi \|x\|, \tag{3.172}$$

where ξ satisfies $\xi > \mu_p \in \mathrm{Sp}(M), \quad \forall p$.

Prove that:

$$\|\mathbf{N}(t) - \mathbf{N}_\infty\| \leq C e^{-\alpha t} \|\mathbf{N}_0\|, \quad \text{where } \mathbf{N}_\infty = (0, \ldots, 0, N_\infty). \tag{3.173}$$

6. Verify that the system conserves the total amount of matter:

$$\sum_{k=1}^{n} N_k(t) = \sum_{k=1}^{n} N_k(0), \quad \forall t \geq 0. \tag{3.174}$$

3.16.3 Nonlinear model with saturating dependence of decay constants

We now assume that the decay constants λ_k depend on the population $N_k(t)$ according to the saturating law, physically motivated by an auto-inhibition or nuclear saturation effect:

$$\lambda_k(N_k) = \frac{\lambda_k^0}{1 + a_k N_k}, \quad \lambda_k^0 > 0, \quad a_k > 0, \tag{3.175}$$

where λ_k^0 and a_k are fixed positive real constants for $k \in [\![1, n-1]\!]$. We still set $\lambda_n \equiv 0$.

The system becomes nonlinear:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = -\frac{\lambda_k^0}{1 + a_k N_k} N_k + \frac{\lambda_{k-1}^0}{1 + a_{k-1} N_{k-1}} N_{k-1}, \quad k \in [\![1, n]\!], \tag{3.176}$$

with the convention $\lambda_n^0 = 0$.

1. Show that the system satisfies the assumptions of the Cauchy-Lipschitz theorem and admits a unique local solution on $\mathbb{R}_+$ for any initial condition $\mathbf{N}_0 \in \mathbb{R}_+^n$.

2. Show that this solution is globally defined and remains positive for all $t \geq 0$, i.e.:

$$N_k(t) \geq 0, \quad \forall t \geq 0, \quad \forall k \in [\![1, n]\!]. \tag{3.177}$$

3. Determine the fixed points $\mathbf{N}^* = (N_1^*, \ldots, N_n^*)$ of the system, solutions of the nonlinear system:

$$0 = -\frac{\lambda_1^0}{1 + a_1 N_1^*} N_1^*, \qquad 0 = -\frac{\lambda_k^0}{1 + a_k N_k^*} N_k^* + \frac{\lambda_{k-1}^0}{1 + a_{k-1} N_{k-1}^*} N_{k-1}^*, \quad k \in [\![2, n]\!]. \tag{3.178}$$

4. Show that the only physically relevant fixed point is $\mathbf{N}^* = (0, \ldots, 0, N_\infty)$, where $N_\infty = \sum_{k=1}^{n} N_k(0)$.

5. Calculate the Jacobian matrix $J(\mathbf{N})$ of the system at point $\mathbf{N}$, then $J(\mathbf{0})$.

6. Study the local stability of $\mathbf{0}$ by analyzing the eigenvalues of $J(\mathbf{0})$.

7. Discuss qualitatively the effect of the coefficients a_k on the system dynamics compared to the classical linear model.

Chapter 4

Exercise Solutions

As you may notice, not **all** exercises have been solved yet. Unsolved exercises are marked with the symbol $\triangle$. The remaining solutions will be added progressively. If you would like to submit a solution, please send it to the following email address in LaTeX format:
ryanartero2005@gmail.com.
Moreover, you can return to the exercise you were working on by clicking on its title, either at the top of the page or at the beginning of the exercise.

4.1 Two-Body Problem

4.1.1 Center of Mass

We denote by $\mathbf{r}_1, \mathbf{r}_2$ the position vectors of the electron and the nucleus with respect to an arbitrary reference frame, and by $\mathbf{v}_1, \mathbf{v}_2$ the corresponding velocities.

1. $\mathcal{L} = \frac{1}{2}(m_1\mathbf{v}_1^2 + m_2\mathbf{v}_2^2) - \frac{\vartheta^2}{\|\mathbf{r}_1 - \mathbf{r}_2\|}$.

2.

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \implies \mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \tag{4.1}$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \implies \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \tag{4.2}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{4.3}$$

$$\implies \mathcal{L} = \frac{1}{2}(m_1 + m_2)\mathbf{V}^2 + \frac{1}{2}\mu\mathbf{v}^2 - \frac{\vartheta^2}{r} = \mathcal{L}_G(\mathbf{V}) + \mathcal{L}_r(\mathbf{r}, \mathbf{v}) \tag{4.4}$$

3. The potential is central for the center of mass. This implies that $\mathbf{J}$ is a conserved quantity.

In the following, we focus exclusively on the internal motion described by $\mathcal{L}_r$ in polar coordinates (r, θ) in the plane perpendicular to $\mathbf{J}$.

4.1.2 Integration of the Equations of Motion

1. The expression for the kinetic energy in polar coordinates in $\mathbb{R}^2$ is:

$$\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2), \tag{4.5}$$

which gives the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}, \quad \text{with } k = \vartheta^2. \tag{4.6}$$

The Euler-Lagrange equations are:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu\dot{r}) - \mu r\dot{\theta}^2 + \frac{k}{r^2} = 0, \tag{4.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu r^2\dot{\theta}\right) = 0. \tag{4.8}$$

Conjugate momenta are given by:

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = \mu\dot{r}, \qquad p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \mu r^2\dot{\theta}. \tag{4.9}$$

The Hamiltonian reads:

$$H = p_r\dot{r} + p_\theta\dot{\theta} - \mathcal{L} = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} - \frac{k}{r}. \tag{4.10}$$

Hamilton's equations are then:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu}, \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{\mu r^3} - \frac{k}{r^2}, \tag{4.11}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2}, \qquad \dot{p}_\theta = -\frac{\partial H}{\partial\theta} = 0. \tag{4.12}$$

p_θ is a conserved quantity (since θ is a cyclic variable); thus, $p_\theta = \mu r^2\dot{\theta}$ is constant — the angular momentum J, fixed by the initial conditions.

Indeed, $\mathbf{J} = \mu\mathbf{r}\times\dot{\mathbf{r}} = \mu r\mathbf{u}_r \times (\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta) = \mu r^2\dot{\theta} = p_\theta$.

The first integral of energy is:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} - \frac{k}{r}. \tag{4.13}$$

By differentiating $p_r = \mu\dot{r}$ and substituting:

$$\dot{p}_r = \mu\ddot{r} = \frac{J^2}{\mu r^3} - \frac{k}{r^2}, \tag{4.14}$$

we recover the radial equation of motion:

$$\mu\ddot{r} = \frac{J^2}{\mu r^3} - \frac{k}{r^2}. \tag{4.15}$$

The first term on the right-hand side is the centrifugal force, the second is the attractive Coulomb force.

2. To eliminate time, we differentiate the composite function $r(\theta(t))$:
   Let $r'(\theta) = \frac{\mathrm{d}r}{\mathrm{d}\theta}$ and $r''(\theta) = \frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2}$. Using $p_\theta = \mu r^2\dot{\theta} = J$, we get:
   $$\dot{\theta} = \frac{J}{\mu r^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{J}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}. \tag{4.16}$$
   Thus:
   $$\dot{r} = r'\frac{J}{\mu r^2}, \qquad \ddot{r} = \frac{J}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(r'\frac{J}{\mu r^2}\right). \tag{4.17}$$
   Setting $u = \frac{1}{r}$, we obtain:
   $$\dot{r} = -\frac{J}{\mu}u', \qquad \ddot{r} = -\frac{J^2}{\mu^2}(u'' + u), \tag{4.18}$$
   and substitution into (7.25) gives:
   $$-\frac{J^2}{\mu^2}(u'' + u) = \frac{J^2}{\mu}u^3 - \frac{k}{\mu}u^2. \tag{4.19}$$
   Multiplying both sides by $-\frac{\mu^2}{J^2}$ yields:
   $$u'' + u = \frac{\mu k}{J^2}. \tag{4.20}$$

3. The differential equation in $u(\theta)$:
   $$u'' + u = \frac{\mu k}{J^2} \tag{4.21}$$
   has the general solution:
   $$u(\theta) = A\cos(\theta + \varphi) + \frac{\mu k}{J^2}, \tag{4.22}$$
   hence:
   $$r(\theta) = \frac{1}{A\cos(\theta + \varphi) + \frac{\mu k}{J^2}}. \tag{4.23}$$
   One can always choose the polar axis so that $r(\theta)$ is extremal at $\theta = 0$ (or π), which gives $\varphi = 0$:
   $$r(\theta) = \frac{1}{\frac{\mu k}{J^2}(1 + \varepsilon\cos\theta)}, \tag{4.24}$$
   where $\varepsilon = \dfrac{AJ^2}{\mu k}$ is the eccentricity.
   The constant A (or ε) is determined by the initial conditions or by the energy:
   $$E = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} - \frac{k}{r}. \tag{4.25}$$
   Using $r(\theta)$ and $J = \mu r^2\dot{\theta}$, one can express E as a function of ε:
   $$\varepsilon^2 = 1 + \frac{2EJ^2}{\mu k^2}. \tag{4.26}$$

4. Equation (7.26) defines a family of curves called **conic sections** (intersections of a cone with a plane). Three subfamilies are distinguished according to the value of ε:

- If $\varepsilon < 1$, the trajectory is an **ellipse**, closed, corresponding to energy $E < 0$: a bound and periodic motion (in particular, $\varepsilon = 0$ gives a circle).
- If $\varepsilon = 1$, the trajectory is a **parabola**: a limiting case $E = 0$ separating bound and unbound motions.
- If $\varepsilon > 1$, the denominator in (7.26) can vanish for some angle $\theta_\infty = \arccos\left(-\frac{1}{\varepsilon}\right)$: the trajectory is a (open) **hyperbola** with asymptotes; $E > 0$ corresponds to a particle arriving from infinity with nonzero initial velocity.

In all cases, the origin (center of force) is one of the foci of the conic.

4.1.3 Bohr Quantization

In this section, we consider only bound states with $E < 0$.

1. The quantization condition on the angle θ is immediate since $p_\theta = J$ is a conserved quantity:

$$J_\theta = \int_0^{2\pi} p_\theta \,\mathrm{d}\theta = 2\pi J, \quad \text{which implies} \quad J = n_\theta \hbar \quad \text{with } n_\theta \in \mathbb{N}^*. \tag{4.27}$$

n_θ cannot be zero, as this would correspond to a straight-line trajectory periodically crossing the nucleus. Ultimately:

$$J = n_\theta \hbar, \quad n_\theta \in \mathbb{N}^*. \tag{4.28}$$

2. We have:

$$\int p_r \,\mathrm{d}r = \int \mu \dot{r} \,\mathrm{d}r = \mu \int r'(\theta)\dot{\theta} \,\mathrm{d}r = \mu \int r'(\theta) \frac{J}{\mu r^2} \,\mathrm{d}r = \int \frac{J r'(\theta)}{r^2} \,\mathrm{d}r = \int \frac{J}{r^2} \frac{\mathrm{d}r}{\mathrm{d}\theta} \,\mathrm{d}\theta. \tag{4.29}$$

The quantization condition, taking into account equation (7.26), becomes:

$$\int_0^{2\pi} \frac{J\varepsilon \sin\theta}{(1+\varepsilon\cos\theta)^2} \,\mathrm{d}\theta = n_r h. \tag{4.30}$$

The integral evaluates to:

$$2\pi J \left(\int_0^{2\pi} \frac{\varepsilon \sin\theta}{(1+\varepsilon\cos\theta)^2} \,\mathrm{d}\theta \right) = -2\pi J \left(\int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{1}{1+\varepsilon\cos\theta} \right) \mathrm{d}\theta \right). \tag{4.31}$$

Integration by parts yields:

$$\int_0^{2\pi} \left(\frac{1}{1+\varepsilon\cos\theta} - 1 \right) \mathrm{d}\theta = \int_0^{2\pi} \left(\frac{1-(1+\varepsilon\cos\theta)}{1+\varepsilon\cos\theta} \right) \mathrm{d}\theta = \int_0^{2\pi} \left(\frac{-\varepsilon\cos\theta}{1+\varepsilon\cos\theta} \right) \mathrm{d}\theta. \tag{4.32}$$

The quantization condition thus becomes:

$$2\pi J \left(\frac{1}{\sqrt{1-\varepsilon^2}} - 1 \right) = n_r h. \tag{4.33}$$

Since $2\pi J = n_\theta h$, this also implies:

$$n_\theta \left(\frac{1}{\sqrt{1-\varepsilon^2}} - 1 \right) = n_r, \tag{4.34}$$

which can be rewritten as:

$$\frac{1}{\sqrt{1-\varepsilon^2}} = \frac{n}{n_\theta}, \quad \text{where } n = n_r + n_\theta. \tag{4.35}$$

3. From equation (7.27):

$$1-\varepsilon^2=-\frac{2EJ^2}{\mu\vartheta^4},\quad \text{so}\quad E=-\frac{\mu\vartheta^4}{2J^2}(1-\varepsilon^2). \tag{4.36}$$

Since $J=n_\theta\hbar$ and $1-\varepsilon^2=\left(\frac{n_\theta}{n}\right)^2$, we finally obtain:

$$E_n=-\frac{\mu\vartheta^4}{2\hbar^2n^2}. \tag{4.37}$$

4.2 Rutherford Scattering Cross-Section

4.2.1 Deflection of a Charged Particle by an Atomic Nucleus

We work in a polar coordinate system (r, φ) in the plane of motion.

1. **Angular Momentum:** The angular momentum in polar coordinates is:
$$J = mr^2\dot{\varphi}. \tag{4.38}$$
At past infinity, the particle has speed v_0 and an impact parameter b. The angular momentum is then:
$$J = -mbv_0. \tag{4.39}$$
The negative sign comes from the fact that φ decreases with time.

2. **Equation of Motion:** The central repulsive force is given by:
$$\mathbf{F} = \frac{C}{r^2}\hat{\mathbf{r}}, \quad \text{where } C = \frac{qQ}{4\pi\varepsilon_0}. \tag{4.40}$$
We decompose $\mathbf{v} = \dot{\mathbf{r}}$ into two components. Projecting onto the direction perpendicular to the polar axis, we find:
$$m\dot{v}_\perp = \frac{C}{r^2}\sin\varphi. \tag{4.41}$$

3. **Deflection Angle θ:** Multiplying the equation by $\mathrm{d}t$ and changing variables, we use:
$$r^2\dot{\varphi} = \frac{J}{m} \Rightarrow \mathrm{d}t = \frac{mr^2}{J}\mathrm{d}\varphi. \tag{4.42}$$
Integrating from $t = -\infty$ to $t = +\infty$:
$$v_0 \sin\theta = \int \dot{v}_\perp \mathrm{d}t = \frac{C}{J}(\cos\theta + 1). \tag{4.43}$$

4. **Relation to Kinetic Energy:** The initial energy is $E_0 = \frac{1}{2}mv_0^2$, so:
$$\tan\left(\frac{\theta}{2}\right) = \frac{C}{2E_0 b}. \tag{4.44}$$

4.2.2 Rutherford Scattering Cross-Section

1. **Expression for the Differential Cross-Section:** The general definition is $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$.

2. **Using $\tan(\theta/2)$:** With:
$$b = \frac{C}{2E_0}\cot\left(\frac{\theta}{2}\right), \quad \frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{C}{4E_0}\frac{1}{\sin^2(\theta/2)}, \implies \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{C}{4E_0}\right)^2 \frac{1}{\sin^4(\theta/2)}. \tag{4.45}$$

3. **Limit of the Model:** For $\theta \to 0$, we have $\sin(\theta/2) \to 0$ so $\mathrm{d}\sigma/\mathrm{d}\Omega \to \infty$. The integral over $\theta \in [0, \pi]$ diverges: the total cross-section is infinite. This reflects the infinite range of the Coulomb interaction.

4. **Experimental Interpretation:** This model explains Rutherford's experimental results: alpha particles can be strongly deflected. This implies the existence of a highly concentrated atomic nucleus, as such deflection requires a very intense field in a very localized region[1].

[1]By introducing the minimum approach distance $a_{\min}$ for a head-on collision ($b = 0$), we have:
$$a_{\min} = \frac{C}{E_0}. \tag{4.46}$$

4.3 Cherenkov Effect

1. The energy of a photon is given by the standard relation:

$$E_\gamma = h\nu \tag{4.48}$$

In a medium with refractive index n, the phase velocity of light is reduced to c/n, and the associated wave vector is:

$$k = \frac{2\pi n\nu}{c} \tag{4.49}$$

The momentum of the photon in this medium is therefore:

$$p_\gamma = \hbar k = \frac{hn\nu}{c} \tag{4.50}$$

Thus, we obtain the desired relation:

$$\boxed{p_\gamma = \frac{hn\nu}{c}} \tag{4.51}$$

Combining this with the expression for the energy $E_\gamma = h\nu$, we deduce:

$$\boxed{p_\gamma = \frac{n}{c}E_\gamma} \tag{4.52}$$

2. The components of momentum are:

$$p = p' \cos\varphi + p_z \cos\theta, \quad 0 = -p' \sin\varphi + p_z \sin\theta. \tag{4.53}$$

3. We have:

$$p_z^2 = p^2 - 2pp_z \cos\varphi + p_z^2. \tag{4.54}$$

4. Energy conservation reads:

$$\sqrt{p^2c^2 + m^2c^4} = \sqrt{p_z^2c^2 + m^2c^4} + h\nu, \tag{4.55}$$

or equivalently:

$$\frac{1}{\sqrt{1-\beta^2}}mc^2 = \frac{1}{\sqrt{1-\beta_f^2}}mc^2 + h\nu. \tag{4.56}$$

5. Squaring both sides, we obtain:

$$p_z^2 = p^2 - \frac{2h\nu E}{c^2} + \frac{(h\nu)^2}{c^2}, \quad \text{where } E \text{ denotes the initial energy of the electron.} \tag{4.57}$$

6. Comparing the two expressions for p_z^2, we can write:

$$p^2 - 2pp_z \cos\varphi + p_z^2 = p^2 - \frac{2h\nu E}{c^2} + \frac{(h\nu)^2}{c^2}, \tag{4.58}$$

The differential cross-section can then be rewritten as:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{a_{\min}^2}{16} \cdot \frac{1}{\sin^4(\theta/2)}}. \tag{4.47}$$

from which, after simplification, we get:

$$\cos\varphi = \frac{h\nu}{pc}\left(1 - \frac{E}{pc}\right) + \frac{h\nu}{2pc}, \tag{4.59}$$

with $E = \gamma mc^2$, $p = \gamma mv$, $p_z = \frac{nh\nu}{c}$, so that:

$$\cos\theta = \frac{1}{n\beta}\left(1 - \frac{1}{2}\frac{1}{\gamma^2}\right). \tag{4.60}$$

7. Finally:

$$\cos\theta = \frac{1}{n\beta}\left[1 + (n^2 - 1)\frac{1}{2\gamma^2}\right]. \tag{4.61}$$

Since $E = \gamma mc^2$, this can also be written as:

$$\cos\theta = \frac{1}{n\beta}\left[1 + \frac{n^2 - 1}{2}\left(1 - \beta^2\right)\right]. \tag{4.62}$$

8. We must have:

$$\frac{1}{n\beta}\left[1 + (n^2 - 1)\frac{1}{2\gamma^2}\right] \leq 1. \tag{4.63}$$

Since the bracketed term is clearly greater than 1, it is necessary (though not sufficient) that:

$$\beta > \frac{1}{n}. \tag{4.64}$$

9. Photons are emitted between $v = 0$ and a frequency $\nu_{\max}$ such that $\cos\theta = 1$, i.e.:

$$0 \leq \nu \leq \frac{E}{h}\left(1 - \frac{1}{n\beta}\right), \quad \text{with } E = \nu_{\max} h. \tag{4.65}$$

10. The most energetic photons are emitted in the direction $\theta = 0$.

11. All photons are emitted within a cone of half-angle φ, corresponding to the angle θ for a photon of zero frequency:

$$\varphi = \arccos\left(\frac{1}{n\beta}\right) = \arccos\left(\frac{1}{n}\right) \simeq 20^\circ. \tag{4.66}$$

12. For the effect to occur, one needs $\nu > \frac{1}{n}$, i.e. $\beta > \frac{1}{n}$, hence:

$$E > \frac{1}{\sqrt{1 - \frac{1}{n^2}}} mc^2. \tag{4.67}$$

For an electron, this means $E > 0.77$ MeV, and for a proton, $E > 1.4$ GeV.

4.4 Pulsed Magnetic Field Machine

4.4.1 Magnetic Field of the Coil

(a) For a circular loop of radius R, the Biot–Savart law gives the magnetic field along the z-axis:

$$B_z(z,t) = \frac{\mu_0 I(t) R^2}{2\left(z^2+R^2\right)^{3/2}}. \tag{4.68}$$

This is obtained by integrating over the loop, exploiting the circular symmetry.

(b) For $z \gg R$, we can approximate $\left(z^2+R^2\right)^{3/2} \simeq z^3$. Thus,

$$B_z(z,t) \sim \frac{\mu_0 I(t) R^2}{2z^3}, \tag{4.69}$$

which is the expression for the field of a magnetic dipole of moment $m = I(t)R^2$.

4.4.2 Induced Electric Field in Biological Tissue

The local form of Faraday's law in cylindrical coordinates (assuming the induced electric field is purely azimuthal) is:

$$(\nabla \times \mathbf{E})_z = \frac{1}{r}\frac{\partial(rE_\theta)}{\partial r} = -\frac{\partial B_z}{\partial t}. \tag{4.70}$$

Differentiate B_z with respect to time:

$$\frac{\partial B_z}{\partial t} = \frac{\mu_0 R^2}{2\left(z^2+R^2\right)^{3/2}}\,\dot{I}(t). \tag{4.71}$$

So the local equation becomes:

$$\frac{1}{r}\frac{\partial(rE_\theta)}{\partial r} = -\frac{\mu_0 R^2\,\dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}. \tag{4.72}$$

Integration for $r < R$: Integrate from 0 to r, imposing $E_\theta(0,t) = 0$ (to avoid a singularity):

$$\int_0^r \frac{\partial(r' E_\theta(r',t))}{\partial r'}\frac{\mathrm{d}r'}{r'} = -\frac{\mu_0 R^2\,\dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}\int_0^r \mathrm{d}r'. \tag{4.73}$$

This yields:

$$rE_\theta(r,t) = -\frac{\mu_0 R^2\,\dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}\cdot\frac{r^2}{2}, \tag{4.74}$$

and therefore:

$$E_\theta(r,t) = -\frac{\mu_0 R^2\,\dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}}\,r \quad \text{for } r \le R. \tag{4.75}$$

Integration for $r > R$: For $r > R$, since the magnetic flux remains confined within the coil region, it is more appropriate to use the integral form of Faraday's law. Consider a circular path of radius $r > R$. The integral form gives:

$$\oint \mathbf{E}\cdot d\boldsymbol{\ell} = 2\pi r\,E_\theta = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{4.76}$$

where the flux Φ is that through the area of the coil:

$$\Phi = \pi R^2 B_z(z,t) = \pi R^2 \frac{\mu_0 I(t) R^2}{2\left(z^2+R^2\right)^{3/2}}. \tag{4.77}$$

The time derivative of Φ is:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \pi R^2 \frac{\mu_0 R^2}{2\left(z^2+R^2\right)^{3/2}} \dot{I}(t). \tag{4.78}$$

Thus,

$$2\pi r\, E_\theta = -\pi \frac{\mu_0 R^4 \dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}, \tag{4.79}$$

and so for $r > R$:

$$E_\theta(r,t) = -\frac{\mu_0 R^4 \dot{I}(t)}{4r\left(z^2+R^2\right)^{3/2}}. \tag{4.80}$$

Summary:

$$E_\theta(r,t) = \begin{cases} -\dfrac{\mu_0 R^2 \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}} r, & r \le R, \\ -\dfrac{\mu_0 R^4 \dot{I}(t)}{4r\left(z^2+R^2\right)^{3/2}}, & r \ge R. \end{cases} \tag{4.81}$$

Continuity check: At $r = R$, the inner solution gives

$$E_\theta(R,t) = -\frac{\mu_0 R^3 \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}}, \tag{4.82}$$

and the outer solution gives the exact same result. Continuity is thus ensured.

4.4.3 Effect on Motor Neurons

The induced voltage over a disk of radius a is given by:

$$V = \int_0^a E(r,t)\, dr. \tag{4.83}$$

Using the expression of $E_\theta(r,t)$ for $r \le R$ (assuming $a \le R$ for simplicity), we get:

$$V = -\frac{\mu_0 R^2 \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}} \int_0^a r\, dr = -\frac{\mu_0 R^2 \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}} \cdot \frac{a^2}{2}. \tag{4.84}$$

Therefore,

$$V = -\frac{\mu_0 R^2 a^2 \dot{I}(t)}{8\left(z^2+R^2\right)^{3/2}}. \tag{4.85}$$

To activate the neuron, the condition is:

$$|V| \ge V_{\text{threshold}}. \tag{4.86}$$

That is,

$$\left| -\frac{\mu_0 R^2 a^2 \dot{I}(t)}{8\left(z^2+R^2\right)^{3/2}} \right| \ge V_{\text{threshold}}. \tag{4.87}$$

Taking the absolute value, this gives the condition:

$$\frac{\mu_0 R^2 a^2 \, |\dot{I}(t)|}{8\left(z^2 + R^2\right)^{3/2}} \geq V_{\text{threshold}}. \tag{4.88}$$

Or equivalently:

$$|\dot{I}(t)| \geq \frac{8\left(z^2 + R^2\right)^{3/2} V_{\text{threshold}}}{\mu_0 R^2 a^2}. \tag{4.89}$$

This gives a **lower bound** on the current's time derivative necessary to trigger a **motor neuron activation** via the induced electric field.

4.4.4 Effect of Oscillating Current

Assuming that

$$I(t) = I_0 e^{i\omega t}, \tag{4.90}$$

then $\dot{I}(t) = i\omega I_0 e^{i\omega t}$ and the induced electric field becomes oscillatory:

$$E_\theta(r,t) = E_\theta(r)\, e^{i\omega t}. \tag{4.91}$$

This behavior reflects the presence of electromagnetic waves in the system, with phases and amplitudes modulated by the frequency ω.

4.4.5 Effect of Pulsed Magnetic Field on Muscles

When the magnetic stimulation device delivers rapid pulses, the time variation of the magnetic field induces an electric field in the surrounding tissues. In muscles, this electric field can cause depolarization of cell membranes by activating ion channels, which generates an action potential. This excitation leads to involuntary muscle contraction, exploited in physiotherapy to improve muscle rehabilitation, increase blood circulation, and reduce pain.

4.5 Metric of a Sphere

1. Using that $\mathrm{d}(\cos u) = -\sin u\,\mathrm{d}u$ and $\mathrm{d}(\sin u) = \cos u\,\mathrm{d}u$, we get

$$\frac{\mathrm{d}x^2}{R^2} = [-\sin\theta\sin\varphi\mathrm{d}\varphi + \cos\theta\cos\varphi\mathrm{d}\theta]^2 \tag{4.92}$$
$$= (\sin\theta\sin\varphi\mathrm{d}\varphi)^2 - 2\sin\theta\sin\varphi\mathrm{d}\varphi\cos\theta\cos\varphi\mathrm{d}\theta + (\cos\theta\cos\varphi\mathrm{d}\theta)^2, \tag{4.93}$$
$$\frac{\mathrm{d}y^2}{R^2} = [\sin\theta\cos\varphi\mathrm{d}\varphi + \cos\theta\sin\varphi\mathrm{d}\theta]^2 \tag{4.94}$$
$$= (\sin\theta\cos\varphi\mathrm{d}\varphi)^2 + 2\sin\theta\sin\varphi\mathrm{d}\varphi\cos\theta\cos\varphi\mathrm{d}\theta + (\cos\theta\sin\varphi\mathrm{d}\theta)^2, \tag{4.95}$$
$$\frac{\mathrm{d}z^2}{R^2} = \sin^2\theta\mathrm{d}\theta^2. \tag{4.96}$$

Adding these terms and using $\cos^2 + \sin^2 = 1$, we obtain

$$\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{4.97}$$

2. From equation (4.97), factoring by $\mathrm{d}\theta^2$ inside the square root, we have

$$\mathrm{d}s = R\sqrt{\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2} \tag{4.98}$$
$$= R\sqrt{1 + \sin^2\theta\,\varphi'^2}\mathrm{d}\theta, \qquad \varphi' = \frac{\mathrm{d}\varphi}{\mathrm{d}\theta} \tag{4.99}$$
$$= R\mathcal{L}\mathrm{d}\theta. \tag{4.100}$$

We notice that $\partial_\varphi\mathcal{L} = 0$, so φ is a cyclic variable. Thus,

$$\partial_{\varphi'}\mathcal{L} = \lambda \in \mathbb{R} \tag{4.101}$$

where λ is a constant.

3.

$$\partial_{\varphi'}\mathcal{L} = \lambda \in \mathbb{R} \tag{4.102}$$
$$\implies \frac{\varphi'\sin^2\theta}{\sqrt{1+\sin^2\theta\varphi'^2}} = \lambda \tag{4.103}$$
$$\implies \varphi'^2(\sin^4\theta - \lambda^2\sin^2\theta) = \lambda^2 \tag{4.104}$$
$$\implies \mathrm{d}\varphi = \lambda\frac{\mathrm{d}\theta}{\sin^2\theta\sqrt{1-\frac{\lambda^2}{\sin^2\theta}}}. \tag{4.105}$$

Integrating,

$$\varphi - \varphi_0 = \lambda\int^\theta \frac{\mathrm{d}\alpha}{\sin^2\alpha\sqrt{1-\frac{\lambda^2}{\sin^2\alpha}}} \tag{4.106}$$
$$=_{u=\cot\alpha} -\lambda\int^{\cot\theta}\frac{\mathrm{d}u}{\sqrt{1-\lambda^2(1+u^2)}} \tag{4.107}$$
$$=_{t=\frac{u}{\beta}} -\frac{\lambda}{\beta}\int^{\frac{\cot\theta}{\beta}}\frac{\mathrm{d}t}{\sqrt{1-t^2}}, \qquad \beta^2 = 1-\lambda^2 \tag{4.108}$$
$$= \arccos\left(\frac{\cot\theta}{\beta}\right). \tag{4.109}$$

Thus,

$$\beta \cos(\varphi - \varphi_0) = \cot\theta, \tag{4.110}$$
$$\beta \sin\theta \cos(\varphi - \varphi_0) = \cos\theta. \tag{4.111}$$

Using some trigonometric formulas, we obtain

$$R \times \big(\beta \cos\varphi_0 \cos\varphi \sin\theta + \beta \cos\varphi_0 \sin\varphi \sin\theta\big) = \cos\theta \tag{4.112}$$
$$\implies ax + by - z = 0, \tag{4.113}$$

where we substituted using spherical coordinates, with $a = \beta \cos\varphi_0 = b$.

4.6 Blackbody Radiation

4.6.1 Number of Modes Excited per Frequency Unit

1. This is the D'Alembert equation in vacuum,

$$\Box \mathbf{E} = 0. \tag{4.114}$$

2. The cavity enforces a stationary solution, thus

$$\mathbf{E} = \cos \omega t \sum_{\mu=1}^{3} E^\mu \sin(k_\mu x^\mu) \mathbf{e}_\mu. \tag{4.115}$$

For each μ, the boundary condition is $\mathbf{E}(x^\mu = L) = \mathbf{0}$. Hence,

$$\sin(k_\mu L) = 0, \tag{4.116}$$

$$k_\mu L = n_\mu \pi, \tag{4.117}$$

$$k_\mu = \frac{n_\mu \pi}{L}. \tag{4.118}$$

3. We know that the norm of $\mathbf{k}$ equals the sum over each component,

$$\|\mathbf{k}\|^2 = \sum_\mu \left(\frac{n_\mu \pi}{L}\right)^2, \tag{4.119}$$

$$\left(\frac{2\pi}{\lambda}\right)^2 = \frac{\pi^2}{L^2} \sum_\mu n_\mu^2, \tag{4.120}$$

$$r^2 = \left(\frac{2L}{\lambda}\right)^2 = \sum_\mu n_\mu^2. \tag{4.121}$$

4. The volume of modes up to frequency $\|\mathbf{k}\|$ is

$$V(\|\mathbf{k}\|) = \frac{4}{3}\pi \|\mathbf{k}\|^3. \tag{4.122}$$

The number of modes is the mode volume divided by the volume of a single mode, with some factors. Since $k_\mu = \frac{\pi}{L} n_\mu$ and $n_\mu \in \mathbb{N}^*$ (factor $\times\frac{1}{8}$), and polarization (factor $\times 2$), we have

$$\implies N = \frac{1}{8} \times 2 \times \frac{V(\|\mathbf{k}\|)}{\left(\frac{\pi}{L}\right)^3} = 2 \times \frac{4}{3}\pi r^3 \tag{4.123}$$

$$= \frac{1}{8} \times 2 \times \frac{\frac{4}{3}\pi \|\mathbf{k}\|^3}{\pi^3} L^3 \tag{4.124}$$

$$= \frac{1}{8} \times 2 \times \frac{4}{3}\pi \frac{\left(\frac{2\pi}{\lambda}\right)^3}{\pi^3} L^3 \tag{4.125}$$

$$= \pi \frac{8L^3}{3\lambda^3} \tag{4.126}$$

$$= \frac{8\pi\nu^3}{3c^3} L^3, \tag{4.127}$$

$$\implies \frac{\mathrm{d}N}{\mathrm{d}\nu} = \frac{8\pi\nu^2}{c^3} \mathcal{V} \tag{4.128}$$

4.6.2 Ultraviolet Catastrophe

1. The system is in contact with a thermostat at temperature T, and the system is closed.
2. In 1D,
$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}\omega^2 q^2 \tag{4.129}$$
3. $$p(W = \varepsilon) = \frac{1}{Z}\exp(-\beta\varepsilon) \tag{4.130}$$
We also have in 1D,
$$Z = \frac{1}{h}\int_{\mathbb{R}^2} e^{-\beta\mathcal{H}} \mathrm{d}q\mathrm{d}p \tag{4.131}$$
Hence,
$$\int_{\mathbb{R}} e^{-\beta\frac{p^2}{2m}} dp = \sqrt{\frac{2m\pi}{\beta}} \tag{4.132}$$
And,
$$\int_{\mathbb{R}} e^{-\beta\frac{m\omega^2 q^2}{2}} dq = \sqrt{\frac{2\pi}{m\omega^2\beta}} \tag{4.133}$$
Hence,
$$Z = \frac{1}{h}\frac{2\pi}{\omega\beta} = \frac{1}{h}\frac{T}{\beta} \tag{4.134}$$
4. We use the formula for the average energy,
$$\langle W\rangle = -\partial_\beta \ln Z = \partial_\beta \ln\beta = \frac{1}{\beta} = k_B T \tag{4.135}$$
5. It is then obvious to say that thanks to eq. 4.139 and the previous question,
$$u(\nu, T) = 8\pi\frac{\nu^2}{c^3} k_B T \tag{4.136}$$
Hence $u \propto \nu^2$, which implies, $\int_{\mathbb{R}^+} u\mathrm{d}\nu \propto \int_{\mathbb{R}^+} \nu^2 \mathrm{d}\nu$, which diverges.

4.6.3 Planck's Law

1. The energy levels are discrete, so we sum:
$$Z = \sum_n e^{-\beta W_n} = \frac{1}{1 - e^{-\beta W_1}} \tag{4.137}$$
Thus, the average energy becomes by the same calculation,
$$-\partial_\beta \ln Z = \frac{h\nu}{e^{\beta h\nu} - 1} \tag{4.138}$$
Using, $W_1 = h\nu$.
2. It is then obvious that,
$$u(\nu, T) = 8\pi\frac{\nu^2}{c^3}\frac{h\nu}{e^{\beta h\nu} - 1} \tag{4.139}$$

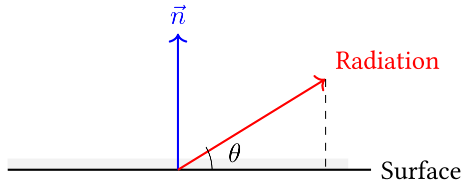

Figure 4.1: The radiation is emitted with an angle θ relative to the normal: only $\cos\theta$ contributes to the flux through the surface. Indeed, it goes out in all directions, so we integrate over $[0, \frac{\pi}{2}]$, and only the contribution of $\cos\theta$ (the projection) matters.

4.6.4 Energy flux emitted by a black body

1. Monochromatic energy flux in a given direction.

   The directional spectral intensity $I_\nu(\theta, \varphi)$ is defined as the energy transported per unit area, time, frequency, and steradian, in direction (θ, φ).

   The monochromatic energy flux emitted in direction (θ, φ) relative to the surface normal is:

$$\mathrm{d}\Phi_\nu = I_\nu(\theta, \varphi)\cos\theta\,\mathrm{d}\Omega, \tag{4.140}$$

   where $\mathrm{d}\Omega$ is the solid angle element around this direction, and $\cos\theta$ comes from the projection of the flux on the normal to the surface (cf. fig 4.1).

2. Total energy flux emitted at frequency ν.

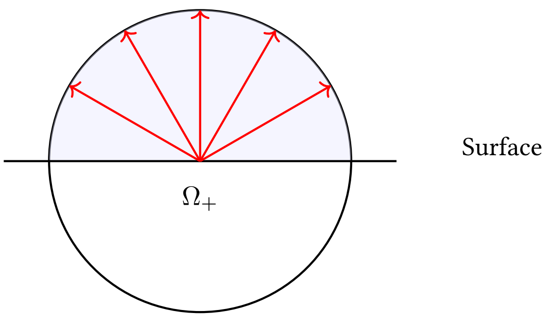

Figure 4.2: The radiation goes out in all directions of the hemisphere Ω_+: we integrate only for $\theta \in [0, \pi/2]$.

   The total energy flux $I(\nu)$ emitted at frequency ν per unit surface is obtained by integrating the elementary flux over the entire outgoing hemisphere (i.e. directions such that $0 \leq \theta \leq \pi/2$, cf. fig 4.2):

$$I(\nu) = \int_{\Omega_+} I_\nu(\theta, \varphi)\cos\theta\,\mathrm{d}\Omega. \tag{4.141}$$

3. Case of isotropic radiation

   If the radiation is isotropic, we have $I_\nu(\theta, \varphi) = I_\nu =$ constant (independent of direction). We can then take I_ν out of the integral:

$$I(\nu) = I_\nu \int_{\Omega_+} \cos\theta\,\mathrm{d}\Omega. \tag{4.142}$$

But:

$$\int_{\Omega_+} \cos\theta \, \mathrm{d}\Omega = \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi. \tag{4.143}$$

Calculating,

$$\int_0^{\pi/2} \cos\theta \sin\theta \, \mathrm{d}\theta = \frac{1}{2}, \quad \text{and} \quad \int_0^{2\pi} \mathrm{d}\varphi = 2\pi. \tag{4.144}$$

Hence,

$$I(\nu) = I_\nu \cdot 2\pi \cdot \frac{1}{2} = \pi I_\nu. \tag{4.145}$$

4. Total emitted intensity (all frequencies combined)

We want to show that the spectral volumetric energy density $u(\nu)$ can be expressed as a function of the directional intensity $I_\nu(\mathbf{n})$ by:

$$u(\nu) = \frac{1}{c} \int_{S^2} I_\nu(\mathbf{n}) \, \mathrm{d}\Omega. \tag{4.146}$$

- $u(\nu)\,\mathrm{d}\nu$ represents the electromagnetic energy contained in a unit volume, for waves whose frequency is between ν and $\nu + \mathrm{d}\nu$.
- $I_\nu(\mathbf{n})$ is the spectral intensity in the direction $\mathbf{n}$, that is, the energy transported per unit time, per unit perpendicular surface, per unit frequency, per unit solid angle.

Consider an elementary surface $\mathrm{d}s$ and a radiation beam incident along a direction $\mathbf{n}$ making an angle θ with the normal to $\mathrm{d}s$.

The volume V swept by the rays in the direction $\mathbf{n}$ during a short time interval $\mathrm{d}t$ is given by:

$$\mathrm{d}V = c\,\mathrm{d}t \cdot \mathrm{d}s \cdot \cos\theta. \tag{4.147}$$

The energy transported through the surface $\mathrm{d}s$ by these rays during this time is:

$$\mathrm{d}E = I_\nu(\mathbf{n}) \cdot \cos\theta \cdot \mathrm{d}s \cdot \mathrm{d}t \cdot \mathrm{d}\Omega. \tag{4.148}$$

We deduce that the energy per unit volume associated with the direction $\mathbf{n}$ is:

$$\frac{\mathrm{d}E}{\mathrm{d}V} = \frac{I_\nu(\mathbf{n}) \cdot \cos\theta \cdot \mathrm{d}s \cdot \mathrm{d}t \cdot \mathrm{d}\Omega}{c\,\mathrm{d}t \cdot \mathrm{d}s \cdot \cos\theta} = \frac{I_\nu(\mathbf{n})}{c} \mathrm{d}\Omega. \tag{4.149}$$

To obtain the total energy density, we sum over all propagation directions on the unit sphere:

$$u(\nu) = \frac{1}{c} \int_{S^2} I_\nu(\mathbf{n}) \, \mathrm{d}\Omega. \tag{4.150}$$

If the radiation is isotropic, then $I_\nu(\mathbf{n}) = I_\nu$ is independent of direction. The integral becomes:

$$u(\nu) = \frac{I_\nu}{c} \int_{S^2} \mathrm{d}\Omega = \frac{I_\nu}{c} \cdot 4\pi. \tag{4.151}$$

Hence,

$$\boxed{u(\nu) = \frac{4\pi}{c} I_\nu} \tag{4.152}$$

5. Relation between total intensity and $u(\nu)$

   We take again the previous expression:

$$I = \int_0^\infty \pi I_\nu \, \mathrm{d}\nu, \tag{4.153}$$

   and substitute $I_\nu = \frac{c}{4\pi} u(\nu)$:

$$I = \int_0^\infty \pi \cdot \frac{c}{4\pi} u(\nu) \, \mathrm{d}\nu = \frac{c}{4} \int_0^\infty u(\nu) \, \mathrm{d}\nu. \tag{4.154}$$

4.6.5 Stefan's Law

1. We previously demonstrated that,

$$I(T) = \frac{c}{4} \int_{\mathbb{R}+} u(\nu, T) \mathrm{d}\nu \tag{4.155}$$

   Replacing with what was obtained in eq. 4.139,

$$I = \frac{c}{4} \frac{8\pi}{c^3} \int_{\mathbb{R}+} \frac{h\nu^3}{e^{\beta h\nu} - 1} \mathrm{d}\nu \tag{4.156}$$

$$\underset{x=\beta h\nu}{=} \frac{2\pi k_B^4}{h^3 c^2} T^4 \int_0^\infty \frac{x^3}{e^x - 1} \mathrm{d}x \tag{4.157}$$

   Recall that $\beta = \frac{1}{k_B T}$.

2. By performing a series expansion, one easily eliminates division by zero. Indeed, near zero,

$$e^x - 1 \underset{0}{=} x + o(x) \implies \frac{x^3}{e^x - 1} \underset{0}{=} x^2 + o(x^2) \tag{4.158}$$

   which converges well at zero. At infinity, the exponential ensures convergence of the integral.

$$\int_{\mathbb{R}+} \frac{x^3}{e^x - 1} \mathrm{d}x = \int_{\mathbb{R}+} \mathrm{d}x \, x^3 e^{-x} \frac{1}{1 - e^{-x}} \tag{4.159}$$

$$=_{\mathrm{DSE}} \int_{\mathbb{R}+} \mathrm{d}x \, x^3 \sum_{n \in \mathbb{N}^*} e^{-nx} \tag{4.160}$$

$$\underset{u=nx}{=} \sum_{n \in \mathbb{N}^*} \frac{1}{n^4} \int_{\mathbb{R}+} \mathrm{d}u \, u^3 e^{-u} \tag{4.161}$$

$$= \zeta(4)\Gamma(4) \tag{4.162}$$

$$= 6\zeta(4) \tag{4.163}$$

3. Thanks to Fourier theory, one can show that $\zeta(4) = \frac{\pi^4}{90}$. We then have,

$$I(T) = \frac{2\pi^5 k_B^4}{15 h^3 c^2} T^4, \tag{4.164}$$

4.6.6 Application: Solar mass loss due to electromagnetic radiation

We consider the Sun as a black body at temperature $T = 5775\,\text{K}$. The total power radiated by the Sun is given by Stefan-Boltzmann law:

$$P = I \cdot S = \sigma T^4 \cdot 4\pi R^2, \tag{4.165}$$

where

$$\sigma = 5{,}67 \times 10^{-8}\ \text{W.m}^{-2}\text{K}^{-4}, \quad R = 6{,}96 \times 10^8\ \text{m} \tag{4.166}$$

is the radius of the Sun.

Let's calculate P:

$$P = 5{,}67 \times 10^{-8} \times (5775)^4 \times 4\pi(6{,}96 \times 10^8)^2. \tag{4.167}$$

We estimate:

$$(5775)^4 \simeq 1{,}11 \times 10^{15}, \tag{4.168}$$

$$4\pi(6{,}96 \times 10^8)^2 = 4\pi \times 4{,}84 \times 10^{17} \simeq 6{,}08 \times 10^{18}. \tag{4.169}$$

Thus,

$$P \simeq 5{,}67 \times 10^{-8} \times 1{,}11 \times 10^{15} \times 6{,}08 \times 10^{18} \simeq 3{,}83 \times 10^{26}\ \text{W}. \tag{4.170}$$

According to Einstein's mass-energy equivalence relation,

$$E = mc^2, \tag{4.171}$$

the mass loss rate $\dot{m}$ per unit time related to this radiated power is

$$\dot{m} = \frac{P}{c^2}, \tag{4.172}$$

with $c = 3{,}00 \times 10^8$ m/s.

Hence,

$$\dot{m} = \frac{3{,}83 \times 10^{26}}{(3{,}00 \times 10^8)^2} = \frac{3{,}83 \times 10^{26}}{9 \times 10^{16}} \simeq 4{,}26 \times 10^9\ \text{kg/s}. \tag{4.173}$$

Knowing that the age of the Sun is about $t = 4{,}6 \times 10^9$ years, i.e.

$$t = 4{,}6 \times 10^9 \times 3{,}15 \times 10^7 \simeq 1{,}45 \times 10^{17}\ \text{s}, \tag{4.174}$$

the total lost mass is

$$\Delta m = \dot{m} \times t = 4{,}26 \times 10^9 \times 1{,}45 \times 10^{17} \simeq 6{,}18 \times 10^{26}\ \text{kg}. \tag{4.175}$$

In number of Earth masses, with $m_T = 6 \times 10^{24}$ kg,

$$\frac{\Delta m}{m_T} = \frac{6{,}18 \times 10^{26}}{6 \times 10^{24}} \simeq 103. \tag{4.176}$$

Thus, the Sun loses about $4{,}3 \times 10^9$ kg/s by radiation. Since its formation, it has lost about 100 times the mass of the Earth.

4.7 Minimization of the gravitational potential by a ball

4.7.1 Hadamard's formula

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^1$ function, and let Ω_ε be a smooth deformation of Ω such that, for $x \in \partial\Omega$,

$$x \;\mapsto\; x + \varepsilon\, f(x)\, n(x), \tag{4.177}$$

extended on all Ω. We want to prove:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3x \;=\; \int_{\partial\Omega} F(x)\, f(x)\, \mathrm{d}S(x), \tag{4.178}$$

where $\mathrm{d}S$ is the surface element on $\partial\Omega$.

1. **Study of the function $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$.**
   (a) *Differentiability of* $\det$.
   Recall that for $M = (m_{ij}) \in \mathcal{M}_n(\mathbb{R})$,

   $$\det(M) \;=\; \sum_{\sigma\in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)}. \tag{4.179}$$

   It is thus a polynomial in the n^2 variables m_{ij}. Any polynomial function $\mathbb{R}^{n^2} \to \mathbb{R}$ is of class $\mathcal{C}^\infty$. In particular, $\det$ is differentiable at every point of $\mathcal{M}_n(\mathbb{R})$, notably near the identity I.

   (b) *Expansion of* $\det(I + \varepsilon M)$.
   We want to show:

   $$\forall M \in \mathcal{M}_n(\mathbb{R}), \quad \det\big(I + \varepsilon M\big) \underset{\varepsilon\to 0}{=} 1 \;+\; \varepsilon\ \mathrm{Tr}(M) \;+\; o(\varepsilon), \tag{4.180}$$

   which implies $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} \det(I + \varepsilon M) = \mathrm{Tr}(M)$.
   It suffices to write M in upper triangular form, then the determinant is the product of the eigenvalues!
   Thus,

   $$\det(I+\varepsilon M) = \prod_{i=1}^{n}(1+\varepsilon\lambda_i) = 1 + \varepsilon\sum_{i=1}^{n}\lambda_i + O(\varepsilon^2) = 1 + \varepsilon\,\mathrm{Tr}\,M + o(\varepsilon) \tag{4.181}$$

   which concludes the proof.

   (c) We reduce to the previous case by factoring out X.

   $$\begin{aligned} \det(X+H) &= \det X \det\big(I + X^{-1}H\big) && (4.182)\\ &= \det X\Big(1 + \mathrm{tr}\big(X^{-1}H\big) + o(\|H\|)\Big) && (4.183)\\ &= \det X + \mathrm{tr}\big({}^t\mathrm{Com}(X)H\big) + o(\|H\|) && (4.184) \end{aligned}$$

   Thus we have,

   $$\mathrm{d}(\det(H))\,(X) = \mathrm{Tr}\big({}^t\mathrm{Com}(X)H\big) \tag{4.185}$$

2. **Change of variables and calculation of the Jacobian.**
We perform the change of variable

$$x = x(u) = u + \varepsilon\, f(u)\, n(u), \qquad u \in \Omega. \tag{4.186}$$

To compute $\det\left(\frac{\partial x}{\partial u}\right)$ at first order in ε, we write

$$x_i(u) = u_i + \varepsilon\, f(u)\, n_i(u), \quad i = 1, \ldots, n. \tag{4.187}$$

Then

$$\frac{\partial x_i}{\partial u_j} = \delta_{ij} + \varepsilon\, \big(\partial_j f(u)\big)\, n_i(u) + \varepsilon\, f(u)\, \partial_j n_i(u). \tag{4.188}$$

Let the matrix $A(u) = \big(\partial_j f\, n_i + f\, \partial_j n_i\big)_{i,j}$. We have $\dfrac{\partial x}{\partial u} = I + \varepsilon\, A(u)$. By the previous expansion,

$$\det\left(\frac{\partial x}{\partial u}\right) = \det\big(I + \varepsilon\, A(u)\big) = 1 + \varepsilon\ \mathrm{Tr}\big(A(u)\big) + o(\varepsilon). \tag{4.189}$$

Noticing that $\mathrm{Tr}(A(u)) = \nabla\cdot\big(f\, n\big)$, we obtain

$$\det\left(\frac{\partial x}{\partial u}\right) = 1 + \varepsilon\, \nabla\cdot\big(f\, n\big)(u) + o(\varepsilon). \tag{4.190}$$

3. **Expansion of $F(x + \varepsilon\, v(x))$.**
Let $F : \mathbb{R}^n \to \mathbb{R} \in \mathcal{C}^1$, $v : \mathbb{R}^n \to \mathbb{R}^n$. Fixing x, define $\varphi(\varepsilon) = F\big(x + \varepsilon\, v(x)\big)$. By the chain rule in dimension 1,

$$\varphi'(\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(x + \varepsilon v(x)) = v(x)\ \cdot \nabla F\big(x + \varepsilon\, v(x)\big). \tag{4.191}$$

In particular, for $\varepsilon \to 0$,

$$\varphi(\varepsilon) = \varphi(0) + \varepsilon\, \varphi'(0) + o(\varepsilon) = F(x) + \varepsilon\, v(x) \cdot \nabla F(x) + o(\varepsilon). \tag{4.192}$$

Hence

$$\forall x \in \mathbb{R}^n, \quad F\big(x + \varepsilon\, v(x)\big) = F(x) + \varepsilon\, v(x) \cdot \nabla F(x) + o(\varepsilon). \tag{4.193}$$

4. **Derivation of Hadamard's formula.**
We perform the change $x(u)$ in $\displaystyle\int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x$. Then

$$\int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x = \int_{\Omega} F\big(x(u)\big)\ \det\left(\frac{\partial x}{\partial u}\right) \mathrm{d}^3 u. \tag{4.194}$$

From the two previous points,

$$F\big(x(u)\big) = F(u) + \varepsilon\, f(u)\, n(u) \cdot \nabla F(u) + o(\varepsilon), \quad \det\left(\tfrac{\partial x}{\partial u}\right) = 1 + \varepsilon\, \nabla\cdot\big(f\, n\big)(u) + o(\varepsilon). \tag{4.195}$$

Multiplying,

$$F\big(x(u)\big) \det\left(\frac{\partial x}{\partial u}\right) = F(u) + \varepsilon\, \Big[f\, n \cdot \nabla F + F\, \nabla\cdot(f\, n)\Big](u) + o(\varepsilon). \tag{4.196}$$

Therefore,

$$\int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x = \int_{\Omega} F(u)\, \mathrm{d}^3 u + \varepsilon \int_{\Omega} \Big[f\, n \cdot \nabla F + F\, \nabla\cdot(f\, n)\Big](u)\, \mathrm{d}^3 u + o(\varepsilon). \tag{4.197}$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{\Omega_\varepsilon} F(x)\,\mathrm{d}^3x = \int_\Omega \nabla\cdot\big(F\,f\,n\big)(u)\,\mathrm{d}^3u, \tag{4.198}$$

using the product rule. Finally, by the divergence theorem,

$$\int_\Omega \nabla\cdot\big(F\,f\,n\big)\,\mathrm{d}^3u = \int_{\partial\Omega} F\,f\,n\cdot n\,\mathrm{d}S(u) = \int_{\partial\Omega} F\,f\,\mathrm{d}S. \tag{4.199}$$

This concludes the proof of Hadamard's formula (3.48).

4.7.2 Connection with the gravitational potential

1. **Sign of $E[\Omega]$ and definition of $\mathcal{I}[\Omega]$.**
   We have

$$E[\Omega] = -\frac{G}{2}\,\rho^2 \iint_{\Omega\times\Omega} \frac{1}{|x-x'|}\,\mathrm{d}^3x\,\mathrm{d}^3x'. \tag{4.200}$$

   Since $G > 0$ and $\rho > 0$, it immediately follows that $E[\Omega] < 0$. Minimizing $E[\Omega]$ is therefore equivalent to *maximizing*

$$\mathcal{I}[\Omega] := \iint_{\Omega\times\Omega} \frac{1}{|x-x'|}\,\mathrm{d}^3x\,\mathrm{d}^3x'. \tag{4.201}$$

2. **Calculation of the potential at the center of a ball.**
   Suppose $\Omega = B(0,R)$ with fixed volume $\frac{4}{3}\pi R^3 = V$. The density is ρ. For $x = 0$,

$$U(0) = -G\,\rho \int_\Omega \frac{1}{|x'|}\,\mathrm{d}^3x' = -G\,\rho \int_0^R \int_{S^2} \frac{1}{r}\,r^2\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r. \tag{4.202}$$

   In spherical coordinates,

$$\int_{S^2} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 4\pi, \quad \text{and} \quad \int_0^R \frac{r^2}{r}\,\mathrm{d}r = \int_0^R r\,\mathrm{d}r = \frac{R^2}{2}. \tag{4.203}$$

   Thus

$$U(0) = -G\,\rho\cdot 4\pi\cdot\frac{R^2}{2} = -2\pi\,G\,\rho\,R^2. \tag{4.204}$$

   Hence the explicit expression of the potential at the center.

4.7.3 The sphere?

1. **First variation of $\mathcal{F}$.**
   We write $\mathcal{F}[\Omega_\varepsilon]$ and apply Hadamard's formula with $F(x) = \int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x'$. Then

$$\delta\mathcal{F} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{1}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.205}$$

   Thus, using Hadamard's formula for Ω^2,

$$\delta\mathcal{F} = 2\int_{\partial\Omega} \left(\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x'\right) f(x)\,\mathrm{d}S(x). \tag{4.206}$$

2. **Introduction of the Lagrange multiplier λ.**
We want to minimize $\mathcal{F}$ under the constraint $V[\Omega] = V$. We define the Lagrangian functional
$$\mathcal{L}[\Omega] := \mathcal{F}[\Omega] - \lambda V[\Omega], \quad \lambda \in \mathbb{R}. \tag{4.207}$$
Its first variation writes
$$\delta\mathcal{L} = \delta\mathcal{F} - \lambda\,\delta V = 2\int_{\partial\Omega}\left(\int_{\Omega}\frac{1}{|x-x'|}\,\mathrm{d}^3x'\right) f(x)\,\mathrm{d}S(x) \;-\; \lambda\int_{\partial\Omega} f(x)\,\mathrm{d}S(x). \tag{4.208}$$
By linearity,
$$\delta\mathcal{L} = \int_{\partial\Omega}\left(2\int_{\Omega}\frac{1}{|x-x'|}\,\mathrm{d}^3x' - \lambda\right) f(x)\,\mathrm{d}S(x). \tag{4.209}$$

3. **Stationary condition for the ball.**
For $\delta\mathcal{L} = 0$ for *all* perturbations f, it is necessary and sufficient that
$$2\int_{\Omega}\frac{1}{|x-x'|}\,\mathrm{d}^3x' - \lambda \;=\; 0, \quad \text{for all } x \in \partial\Omega. \tag{4.210}$$
In other words, the function $x \mapsto \displaystyle\int_{\Omega}\frac{1}{|x-x'|}\,\mathrm{d}^3x'$ is constant on $\partial\Omega$.
If $\Omega = B(0,R)$ is a ball, then by spherical symmetry, for every $x \in \partial B(0,R)$ (i.e. $|x| = R$), the integral $\displaystyle\int_{B(0,R)}\frac{1}{|x-x'|}\,\mathrm{d}^3x'$ depends only on $|x| = R$.
Thus it is *constant* on ∂B. We deduce that the ball satisfies the stationary condition $\delta\mathcal{L} = 0$ for all f.

4. **(*Bonus*) Second variation and local minimum.**
To show that the ball is a *local minimum* of $\mathcal{F}$ under volume constraint V, one must verify that the second variation $\delta^2\mathcal{L}[f]$ is strictly positive for any perturbation $f \neq 0$ satisfying $\int_{\partial\Omega} f\,\mathrm{d}S = 0$.
Without full details here, the second variation can be written as a bilinear form:
$$\delta^2\mathcal{F}[f] = \int_{(\partial\Omega)^2} K(x,x')\,f(x)\,f(x')\,\mathrm{d}S(x)\,\mathrm{d}S(x') \;+\; \int_{\partial\Omega} f(x)^2\kappa(x)\mathrm{d}S(x), \tag{4.211}$$
with kernel $K(x,x') = \frac{1}{|x-x'|}$ and $\kappa(x)$ the mean curvature at x.
For the ball, thanks to the spherical harmonics expansion, one shows this form is strictly positive on $\left\{\, f \mid \int_{\partial\Omega} f\,\mathrm{d}S = 0 \,\right\}$. This proves the ball is a local minimum.

5. **Physical conclusion.**
The ball minimizes the internal gravitational energy for a fixed volume. In physics, this explains that in the approximation of a massive self-gravitating body at rest, the stationary configuration of least energy is spherical. This is why large objects in the Universe (stars, planets in the absence of tidal forces or rapid rotation) tend to a spherical shape.

4.8 Relativistic motion of a charged particle

4.8.1 Relativistic Lagrangian of a Charged Particle in an Electromagnetic Field

1. **Free particle and relativistic action.**
The principle of least action requires that the action be a Lorentz scalar. The simplest scalar is the spacetime interval $\mathrm{d}s$, defined by:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{4.212}$$

The action for a free particle of mass m is thus:

$$S = -mc\int \mathrm{d}s = -mc^2\int \sqrt{1-\frac{\mathbf{v}^2}{c^2}}\,\mathrm{d}t. \tag{4.213}$$

The associated Lagrangian is:

$$\mathcal{L} = -mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}. \tag{4.214}$$

2. **Interaction with an electromagnetic field.**
We introduce the four-potential $A^\mu = (\phi/c, \mathbf{A})$. We seek an interaction term of the scalar form $L_{\text{int}} = qA_\mu \dot{x}^\mu$. In standard coordinates:

$$\mathcal{L}_{\text{int}} = q\mathbf{A}\cdot\mathbf{v} - q\phi. \tag{4.215}$$

The total Lagrangian is therefore:

$$\mathcal{L}_{\text{tot}} = -mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}} + q\mathbf{A}\cdot\mathbf{v} - q\phi. \tag{4.216}$$

3. **Generalized momentum.**
The generalized momentum is defined as:

$$\mathbf{p} = \frac{\partial\mathcal{L}_{\text{tot}}}{\partial\mathbf{v}} = \frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} + q\mathbf{A} = \gamma m\mathbf{v} + q\mathbf{A}. \tag{4.217}$$

4. **Euler–Lagrange equations.**
Applying the Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right) = \frac{\partial\mathcal{L}}{\partial\mathbf{x}}. \tag{4.218}$$

The left-hand side becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) + q\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}. \tag{4.219}$$

The right-hand side yields:

$$q\nabla(\mathbf{A}\cdot\mathbf{v}) - q\nabla\phi. \tag{4.220}$$

Using:

$$\nabla(\mathbf{A}\cdot\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A}), \tag{4.221}$$

and:

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \partial_t\mathbf{A} + (\mathbf{v}\cdot\nabla)\mathbf{A}, \tag{4.222}$$

we find that the $(\mathbf{v} \cdot \nabla)\mathbf{A}$ terms cancel out, giving:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m \mathbf{v}) = q\left[-\partial_t \mathbf{A} - \nabla\phi + \mathbf{v} \times (\nabla \times \mathbf{A})\right]. \tag{4.223}$$

Recognizing the electric and magnetic fields:

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{4.224}$$

we obtain the Lorentz force law:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m \mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{4.225}$$

5. **Covariant formulation.**
We parameterize the worldline using the proper time τ:

$$\mathcal{L} = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + qA_\mu \dot{x}^\mu, \tag{4.226}$$

where the first term represents the free particle, and the second the interaction.

6. **Equation of motion.**
Applying the Euler–Lagrange equations in covariant form, we use the Lagrangian:

$$\mathcal{L} = -mc\sqrt{-\dot{x}^\mu \dot{x}_\mu} + qA_\mu \dot{x}^\mu, \tag{4.227}$$

with $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. Since $\dot{x}^\mu \dot{x}_\mu = -c^2$, we get:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = mc\frac{\dot{x}_\mu}{c} + qA_\mu. \tag{4.228}$$

Differentiating:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = m\ddot{x}_\mu + q\dot{x}^\nu \partial_\nu A_\mu, \tag{4.229}$$

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = q\partial_\mu A_\nu \dot{x}^\nu, \tag{4.230}$$

yielding the equation of motion:

$$m\ddot{x}_\mu + q\dot{x}^\nu \partial_\nu A_\mu = q\dot{x}^\nu \partial_\mu A_\nu, \tag{4.231}$$

$$m\ddot{x}_\mu = q\dot{x}^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{4.232}$$

$$m\ddot{x}_\mu = qF_{\mu\nu}\dot{x}^\nu, \tag{4.233}$$

where the antisymmetric electromagnetic tensor is:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.234}$$

7. **Components of the tensor $F_{\mu\nu}$.**
$F_{\mu\nu}$ is antisymmetric and encodes the electric and magnetic fields. In Cartesian coordinates with $A^\mu = (\phi/c, \mathbf{A})$ and $x^\mu = (ct, \mathbf{x})$:

- For $\mu = 0$, $\nu = i$:

$$F_{0i} = \frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{1}{c}\frac{\partial \phi}{\partial x^i} = -\frac{1}{c}E_i. \tag{4.235}$$

- For $\mu = i$, $\nu = j$:

$$F_{ij} = \partial_i A_j - \partial_j A_i = -\varepsilon_{ijk} B_k. \tag{4.236}$$

Conclusion: the tensor $F_{\mu\nu}$ is:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \tag{4.237}$$

This explicitly shows how $F_{\mu\nu}$ encodes $\mathbf{E}$ and $\mathbf{B}$ in an inertial frame.

8. **Relativistic invariants.**
We compute:

$$I_1 = F_{\mu\nu} F^{\mu\nu} = 2(\mathbf{B}^2 - \frac{\mathbf{E}^2}{c^2}),$$
$$I_2 = \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = -\frac{8}{c} \mathbf{E} \cdot \mathbf{B}.$$

Characteristic cases:

- $\mathbf{E}^2 = c^2 \mathbf{B}^2$ and $\mathbf{E} \cdot \mathbf{B} = 0$: electromagnetic plane wave.
- $I_1 > 0$: magnetic-field dominated; $I_1 < 0$: electric-field dominated.

9. **Gauge transformation.**
Under $A_\mu \to A_\mu + \partial_\mu \Lambda$, we have:

$$F_{\mu\nu} \to F_{\mu\nu}, \tag{4.238}$$

since mixed partial derivatives cancel. This leaves the equations of motion invariant: a **local gauge symmetry** associated with charge conservation via Noether's theorem.

4.8.2 Equations of Motion of a Charged Particle in a Plane Electromagnetic Wave – Solution

We consider a particle of mass m and charge q subjected to an electromagnetic field. Its motion is governed by the equation:

$$m\ddot{x}^\mu = qF^{\mu\nu} \dot{x}_\nu, \tag{4.239}$$

where the dots denote derivatives with respect to the proper time τ, and we work in natural units: $c = 1$.

The potential is given by:

$$A^\mu(x) = a^\mu f(k_\nu x^\nu), \tag{4.240}$$

where a^μ is a constant four-vector, $f \in \mathcal{C}^1$, and k^μ is a lightlike four-vector satisfying $k^\mu k_\mu = 0$.

1. **Computation of the Electromagnetic Tensor.**

By definition:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{4.241}$$

We compute:

$$\partial^\mu A^\nu = a^\nu f'(k\cdot x)k^\mu, \qquad \partial^\nu A^\mu = a^\mu f'(k\cdot x)k^\nu, \tag{4.242}$$

so that:

$$F^{\mu\nu} = (k^\mu a^\nu - k^\nu a^\mu)\, f'(k\cdot x). \tag{4.243}$$

2. **Gauge Condition.**

   (a) We compute the Lorenz gauge condition:

   $$\partial_\mu A^\mu = a^\mu f'(k\cdot x)k_\mu. \tag{4.244}$$

   (b) Hence, the Lorenz condition $\partial_\mu A^\mu = 0$ implies:

   $$a^\mu k_\mu = 0, \tag{4.245}$$

   i.e., the polarization vector is orthogonal to the wave vector. This expresses the transversality of the electromagnetic wave.

3. **Equation of Motion.**

   (a) Using the expression of $F^{\mu\nu}$:

   $$F^{\mu\nu}\dot{x}_\nu = \left[k^\mu(a\cdot\dot{x}) - a^\mu(k\cdot\dot{x})\right] f'(k\cdot x). \tag{4.246}$$

   (b) The equation becomes:

   $$m\ddot{x}^\mu = q\left[k^\mu(a\cdot\dot{x}) - a^\mu(k\cdot\dot{x})\right] f'(k\cdot x). \tag{4.247}$$

4. **Integration of the Equation of Motion.**

   (a) Define $\phi(\tau) = k\cdot x(\tau)$, then:

   $$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = k\cdot\dot{x}, \quad \frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = k\cdot\ddot{x}. \tag{4.248}$$

   Using the motion equation and the identities $k^2 = 0$, $k\cdot a = 0$:

   $$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = 0 \quad \Rightarrow \quad \phi(\tau) = \omega\tau + \phi_0, \tag{4.249}$$

   with $\omega = k\cdot\dot{x} =$ constant.

   (b) Let $u^\mu = \dot{x}^\mu$ and define $\alpha(\phi) = a\cdot u(\phi)$. Then:

   $$m\omega\frac{\mathrm{d}u^\mu}{\mathrm{d}\phi} = q\left[k^\mu\alpha(\phi) - a^\mu\omega\right] f'(\phi). \tag{4.250}$$

(c) Projecting on a_μ and using $a \cdot k = 0$:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\phi} = -\frac{qa^2}{m\omega} f'(\phi), \quad \Rightarrow \quad \alpha(\phi) = \alpha_0 - \frac{qa^2}{m\omega} f(\phi). \tag{4.251}$$

(d) Substituting into the equation:

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\phi} = \frac{q}{m\omega} \left[k^\mu \alpha(\phi) - a^\mu \omega\right] f'(\phi). \tag{4.252}$$

(e) Integration yields:

$$u^\mu(\phi) = u^\mu(\phi_0) + \frac{q}{m\omega} \left[k^\mu \int_{\phi_0}^{\phi} \alpha(\varphi) f'(\varphi) \mathrm{d}\varphi - a^\mu \omega \int_{\phi_0}^{\phi} f'(\varphi) \mathrm{d}\varphi\right] \tag{4.253}$$

$$= u^\mu(\phi_0) + \frac{q}{m\omega} k^\mu \left[\alpha_0 \Delta f - \frac{qa^2}{2m\omega} \Delta(f^2)\right] - \frac{q}{m} a^\mu \Delta f, \tag{4.254}$$

where $\Delta f = f(\phi) - f(\phi_0)$, and $\Delta(f^2) = f(\phi)^2 - f(\phi_0)^2$.

(f) The trajectory is obtained by integrating once more:

$$x^\mu(\phi) = x^\mu(\phi_0) + \frac{1}{\omega} \int_{\phi_0}^{\phi} u^\mu(\varphi) \mathrm{d}\varphi. \tag{4.255}$$

Summary:

$$\boxed{\begin{aligned} &u^\mu(\phi) = u^\mu(\phi_0) + \frac{q}{m\omega} k^\mu \left[\alpha_0 \Delta f - \frac{qa^2}{2m\omega} \Delta(f^2)\right] - \frac{q}{m} a^\mu \Delta f, \\ &x^\mu(\phi) = x^\mu(\phi_0) + \frac{1}{\omega} \int_{\phi_0}^{\phi} u^\mu(\varphi)\, \mathrm{d}\varphi. \end{aligned}} \tag{4.256}$$

5. **Example: Sinusoidal Wave.**

Let

$$f(\phi) = \sin(\phi), \quad \Rightarrow \quad \int \sin(\phi) \mathrm{d}\phi = -\cos(\phi), \quad \int \sin^2(\phi) \mathrm{d}\phi = \frac{\phi}{2} - \frac{\sin(2\phi)}{4}. \tag{4.257}$$

The velocity becomes:

$$u^\mu(\phi) = u^\mu(\phi_0) + \frac{q}{m\omega} k^\mu \left[\alpha_0 (\sin\phi - \sin\phi_0) - \frac{qa^2}{2m\omega} \left(\frac{\phi - \phi_0}{2} - \frac{\sin 2\phi - \sin 2\phi_0}{4}\right)\right] - \frac{q}{m} a^\mu (\sin\phi - \sin\phi_0). \tag{4.258}$$

The integrated trajectory components are:

$$ct(\tau) = ct_0 + \frac{u^0(\tau_0)}{\omega} \Delta\phi + \frac{q}{m\omega^2} k^0 \left[\alpha_0 \Delta(-\cos\phi) - \frac{qa^2}{4m\omega} \left(\Delta\phi - \frac{\Delta(\sin 2\phi)}{2}\right)\right] - \frac{q}{m\omega} a^0 \Delta(-\cos\phi), \tag{4.259}$$

$$x^i(\tau) = x^i(\tau_0) + \frac{u^i(\tau_0)}{\omega} \Delta\phi + \frac{q}{m\omega^2} k^i \left[\alpha_0 \Delta(-\cos\phi) - \frac{qa^2}{4m\omega} \left(\Delta\phi - \frac{\Delta(\sin 2\phi)}{2}\right)\right] - \frac{q}{m\omega} a^i \Delta(-\cos\phi), \tag{4.260}$$

where

$$\phi = \omega\tau + \phi_0, \quad \alpha_0 = a_\mu u^\mu(\tau_0), \quad a^2 = a_\mu a^\mu. \tag{4.261}$$

This expression gives the full analytic trajectory of a charged particle in a monochromatic sinusoidal plane electromagnetic wave.

4.8.3 Field Theory

1. For an action depending on a field φ (scalar, tensor, etc.):

$$S = \int_\Omega \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu)\,\mathrm{d}^4x \tag{4.262}$$

We consider a variation $\varphi \mapsto \varphi + \varepsilon\eta$ with $\eta \in \mathcal{C}^1_c(\Omega)$ (compactly supported and continuously differentiable). Then:

$$\delta S = \frac{\mathrm{d}S}{\mathrm{d}\varepsilon}[\varphi + \varepsilon\eta]\bigg|_{\varepsilon=0} \tag{4.263}$$

$$= \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \varphi}\eta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\mu \eta\right)\mathrm{d}^4x \tag{4.264}$$

We integrate by parts the term $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\mu \eta$:

$$u = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Rightarrow \mathrm{d}u = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}, \quad \mathrm{d}v = \partial_\mu \eta \Rightarrow v = \eta \tag{4.265}$$

We obtain:

$$\delta S = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right)\right)\eta\,\mathrm{d}^4x \tag{4.266}$$

Since η vanishes on the boundary $\partial\Omega$, we obtain the Euler-Lagrange equation for fields:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right) = 0} \tag{4.267}$$

2. We consider the electromagnetic action:

$$\mathcal{L} = -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} - A^\mu j_\mu \tag{4.268}$$

We vary the potential: $A^\mu \to A^\mu + \varepsilon\eta^\mu$:

$$\delta S = \frac{\mathrm{d}S}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} = \int_\Omega \left(-\frac{1}{2\mu_0}F^{\mu\nu}\delta F_{\mu\nu} - j_\mu \eta^\mu\right)\mathrm{d}^4x \tag{4.269}$$

Now,

$$\delta F_{\mu\nu} = \partial_\mu \eta_\nu - \partial_\nu \eta_\mu \tag{4.270}$$

So:

$$\delta S = \int_\Omega \left(-\frac{1}{\mu_0}F^{\mu\nu}\partial_\mu \eta_\nu - j_\mu \eta^\mu\right)\mathrm{d}^4x \tag{4.271}$$

$$= \int_\Omega \left(\frac{1}{\mu_0}\partial_\mu F^{\mu\nu} - j^\nu\right)\eta_\nu \mathrm{d}^4x \tag{4.272}$$

Since this must hold for all η_ν, we obtain:

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 j^\nu} \tag{4.273}$$

The homogeneous Maxwell equations follow from the definition of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which implies:

$$\boxed{\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0} \tag{4.274}$$

4.8.4 Trajectory of a Charged Particle in a Constant Magnetic Field

1. The field tensor $F^{\mu\nu}$: in the reference frame where $\mathbf{E} = 0$ and $\mathbf{B} = B\mathbf{e}_z$, we have:

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.275}$$

2. The equation of motion $\mathrm{d}u^\mu/\mathrm{d}\tau = (q/m)F^{\mu\nu}u_\nu$ implies $u^3 = \text{constant} = 0$, hence planar motion. The energy $E = \gamma mc^2$ is conserved since $F^{0\nu} = 0$.

3. The equation $\mathrm{d}u^\mu/\mathrm{d}\tau = (q/m)F^{\mu\nu}u_\nu$ gives:

$$\frac{\mathrm{d}u^1}{\mathrm{d}\tau} = (q/m)F^{12}u^2 = -(qB/m)u^2, \tag{4.276}$$
$$\frac{\mathrm{d}u^2}{\mathrm{d}\tau} = (q/m)F^{21}u^1 = (qB/m)u^1. \tag{4.277}$$

This describes uniform circular motion, thus:

$$\omega = \frac{qB}{m}, \qquad x(t) = R\cos\left(\frac{\omega}{\gamma}t\right), \quad y(t) = R\sin\left(\frac{\omega}{\gamma}t\right), \tag{4.278}$$

with γ constant, implying $\tau = t/\gamma$, and:

$$R = \frac{\gamma mv}{qB}. \tag{4.279}$$

4. Starting from:

$$P = -\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{q^2}{6\pi\varepsilon_0 c^3}\gamma^4 a^2, \qquad E = \gamma mc^2, \quad a = \frac{v^2}{R}, \tag{4.280}$$

we obtain:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\frac{q^2}{6\pi\varepsilon_0 c^5 m}\gamma^3 a^2 = -C(\gamma^2 - 1), \tag{4.281}$$

with:

$$C = \frac{q^4B^2}{6\pi\varepsilon_0 c^5 m^3}. \tag{4.282}$$

5. Solving:

$$\frac{\mathrm{d}\gamma}{\gamma^2 - 1} = -C\mathrm{d}t \Rightarrow \frac{1}{2}\ln\left|\frac{\gamma - 1}{\gamma + 1}\right| = -Ct + C_0, \tag{4.283}$$

we find:

$$\boxed{\gamma(t) = \coth(Ct + C_0)}. \tag{4.284}$$

6. Using $\omega = \frac{qB}{m}$ and $v(t) = c\sqrt{1 - 1/\gamma(t)^2}$, we obtain:

$$R(t) = \frac{\gamma(t)mv(t)}{qB}, \quad \theta(t) = \int_0^t \frac{\omega}{\gamma(s)}\,\mathrm{d}s, \tag{4.285}$$

and then:

$$x(t) = R(t)\cos\theta(t), \tag{4.286}$$
$$y(t) = R(t)\sin\theta(t). \tag{4.287}$$

7. The trajectory spirals toward the origin since $R(t) \to 0$ and $\omega/\gamma(t) \to 0$, although oscillations persist. Numerically, this leads to error accumulation, requiring adaptive time steps to resolve the fast oscillations at early times.

4.8.5 Relativistic Collider Physics

We work in natural units, with $c = 1$.

1. The square of the total energy-momentum invariant is defined as:

$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2. \tag{4.288}$$

In the center-of-mass frame (CMS), the total available energy is:

$$E_{\text{tot}}^{\text{(CMS)}} = \sqrt{s}. \tag{4.289}$$

2. For a head-on collision of two identical particles of mass m and energy E each (in the laboratory frame), the four-momenta are:

$$p_1 = (E, \mathbf{p}), \tag{4.290}$$
$$p_2 = (E, -\mathbf{p}), \tag{4.291}$$

which gives:

$$s = (p_1 + p_2)^2 = 2m^2 + 2(E^2 - \mathbf{p}^2) = 4E^2, \tag{4.292}$$

using $E^2 - \mathbf{p}^2 = m^2$ and $E \gg m$. Therefore:

$$\boxed{\sqrt{s} = 2E}. \tag{4.293}$$

3. For a fixed-target collision:

$$p_1 = (E_{\text{lab}}, \mathbf{p}), \tag{4.294}$$
$$p_2 = (m, 0), \tag{4.295}$$

then:

$$s = (p_1 + p_2)^2 = m^2 + m^2 + 2mE_{\text{lab}} = 2m^2 + 2mE_{\text{lab}}, \tag{4.296}$$

hence:

$$\boxed{s = 2m^2 + 2mE_{\text{lab}}}. \tag{4.297}$$

At the threshold for producing two new particles of mass m, we set $\sqrt{s} = 2m$, yielding:

$$2m = \sqrt{2m^2 + 2mE_{\text{lab}}} \Rightarrow E_{\text{lab}} = 2m. \tag{4.298}$$

4. To produce a single particle of mass M at threshold in a fixed-target experiment:

$$s = M^2 = m^2 + m^2 + 2mE_{\text{lab}} \Rightarrow E_{\text{lab}} = \frac{M^2 - 2m^2}{2m}. \tag{4.299}$$

In contrast, in a symmetric collider:

$$E_{\text{CM}} = \sqrt{s} = 2E = 2M \Rightarrow E = M. \tag{4.300}$$

Thus, for the same center-of-mass energy, the required lab-frame energy is much larger in a fixed-target setup than in a symmetric collider. This is why head-on colliders are more efficient for producing heavy particles at high energy.

4.9 Relativistic hydrodynamics △

4.10 Hydrogen atom and radial equation

4.10.1 Separation of variables and radial equation

1. **Separation of variables**

   The Hamiltonian of the hydrogen atom, in the spherical basis, is written as:

$$\mathbf{H} = -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{e^2}{r}. \tag{4.301}$$

   In spherical coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\mathbf{L}^2}{\hbar^2 r^2}, \tag{4.302}$$

   where $\mathbf{L}^2$ is the square of the orbital angular momentum.

   We look for a solution of the form

$$\psi(r,\theta,\phi) = R(r)Y_{\ell m}(\theta,\phi), \tag{4.303}$$

   where $Y_{\ell m}$ are the spherical harmonics simultaneous eigenfunctions of $\mathbf{L}^2$ and $\mathbf{L}_z$, satisfying

$$\mathbf{L}^2 Y_{\ell m} = \hbar^2\ell(\ell+1)Y_{\ell m}, \quad \mathbf{L}_z Y_{\ell m} = \hbar m Y_{\ell m}. \tag{4.304}$$

   Injecting into the stationary Schrödinger equation $\mathbf{H}\psi = E\psi$, we get the following radial equation:

$$-\frac{\hbar^2}{2m_e}\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{\ell(\ell+1)}{r^2}R\right] - \frac{e^2}{r}R = ER. \tag{4.305}$$

   Expanding the radial derivative,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r}, \tag{4.306}$$

   which gives the announced equation:

$$-\frac{\hbar^2}{2m_e}\left(\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}R\right) - \frac{e^2}{r}R = ER. \tag{4.307}$$

2. **Change of function:** $u(r) = rR(r)$

   Putting $u(r) = rR(r)$, we calculate:

$$\frac{\mathrm{d}R}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2}, \tag{4.308}$$

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2} = \frac{1}{r}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} - \frac{2}{r^2}\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{2u}{r^3}. \tag{4.309}$$

   Replacing in the radial equation, the terms in u/r^3 cancel and we get:

$$-\frac{\hbar^2}{2m_e}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2m_e r^2} - \frac{e^2}{r}\right]u = Eu. \tag{4.310}$$

3. **Dimensionless change of variable**

   We define
$$\kappa = \sqrt{\frac{2m_e|E|}{\hbar^2}}, \quad \rho = \kappa r. \tag{4.311}$$
   The equation becomes
$$-\frac{\hbar^2}{2m_e}\kappa^2\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2m_e r^2} - \frac{e^2}{r}\right]u = Eu. \tag{4.312}$$
   Since $E = -|E|$, dividing the whole equation by $-\frac{\hbar^2\kappa^2}{2m_e}$:
$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} = \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{2m_e e^2}{\hbar^2\kappa}\frac{1}{\rho} + 1\right]u. \tag{4.313}$$
   We then set
$$\rho_0 = \frac{m_e e^2}{\hbar^2\kappa}. \tag{4.314}$$
   This gives the announced equation:
$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} = \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{\rho_0}{\rho} + 1\right]u. \tag{4.315}$$

4. **Ansatz on the form of $u(\rho)$**

   We set
$$u(\rho) = \rho^{\ell+1}e^{-\rho/2}v(\rho). \tag{4.316}$$
   By calculating the second derivative of $u(\rho)$ and replacing into the differential equation, one finds that $v(\rho)$ satisfies:
$$\rho\frac{\mathrm{d}^2v}{\mathrm{d}\rho^2} + (2\ell + 2 - \rho)\frac{\mathrm{d}v}{\mathrm{d}\rho} + (\rho_0 - 2\ell - 2)v = 0. \tag{4.317}$$
   This equation is that of the confluent hypergeometric function.

5. **Power series and termination condition**

   We develop
$$v(\rho) = \sum_{k=0}^{\infty} c_k\rho^k. \tag{4.318}$$
   The equation gives a recurrence relation between coefficients c_k. Generally, this series diverges as $\rho \to \infty$ unless the series is a polynomial, i.e. it stops at some finite order $\hat{k}$. The termination condition is
$$\rho_0 = 2n, \tag{4.319}$$
   where
$$n = \hat{k} + \ell + 1 \in \mathbb{N}^*. \tag{4.320}$$

6. **Expression of bound energy levels**

   Reinjecting the definition of ρ_0,
$$\rho_0 = \frac{m_e e^2}{\hbar^2\kappa} = 2n \quad \Rightarrow \quad \kappa = \frac{m_e e^2}{2\hbar^2 n}. \tag{4.321}$$

Now

$$E = -\frac{\hbar^2\kappa^2}{2m_e} = -\frac{m_e e^4}{2\hbar^2}\cdot\frac{1}{n^2}. \tag{4.322}$$

These are the quantized energy levels of the hydrogen atom.

7. **Degree of degeneracy**

   For a level n, possible values of ℓ are

$$\ell = 0, 1, 2, \ldots, n-1, \tag{4.323}$$

   and for each ℓ, the values of m go from

$$m = -\ell, -\ell+1, \ldots, \ell-1, \ell, \tag{4.324}$$

   i.e. $(2\ell+1)$ values.

   The degree of degeneracy is therefore

$$g_n = \sum_{\ell=0}^{n-1}(2\ell+1) = 2\sum_{\ell=0}^{n-1}\ell + \sum_{\ell=0}^{n-1}1 = 2\frac{(n-1)n}{2} + n = n^2. \tag{4.325}$$

   Interpretation: In this non-relativistic model without spin-orbit interactions or relativistic effects, the energy depends only on the principal quantum number n. This reflects the larger symmetry of the problem (rotational invariance and Runge-Lenz-type symmetry), which leads to this high degeneracy.

4.10.2 Ground state ($n = 1$) and radial properties

7. For $n = 1$, $\ell = 0$, $n_r = 0$:

$$u(r) = Are^{-r/a_0}, \quad \Rightarrow R(r) = \frac{u(r)}{r} = Ae^{-r/a_0}. \tag{4.326}$$

   Normalization requires:

$$\int_0^\infty |R(r)|^2 r^2 \mathrm{d}r = |A|^2 \int_0^\infty e^{-2r/a_0} r^2 \mathrm{d}r = 1. \tag{4.327}$$

   The integral yields $2!(a_0/2)^3 = a_0^3/4 \Rightarrow |A|^2 = 4/a_0^3$. Therefore:

$$R_{1,0}(r) = \frac{2}{a_0^{3/2}}e^{-r/a_0}. \tag{4.328}$$

   The spherical harmonic is $Y_{00} = 1/\sqrt{4\pi}$, so:

$$\psi_{1,0,0}(r,\theta,\phi) = \frac{2}{a_0^{3/2}}e^{-r/a_0}\cdot\frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}. \tag{4.329}$$

   The normalization is indeed satisfied:

$$\int |\psi|^2 \mathrm{d}^3x = \int_0^\infty |R|^2 r^2 \mathrm{d}r \int |Y|^2 \mathrm{d}\Omega = 1. \tag{4.330}$$

8. The radial probability density is:

$$P(r) = 4\pi|R(r)|^2r^2 = 4\pi\left(\frac{2}{a_0^{3/2}}\right)^2 e^{-2r/a_0}r^2 = \frac{16\pi}{a_0^3}r^2e^{-2r/a_0}. \tag{4.331}$$

It vanishes at $r = 0$ and as $r \to \infty$; the maximum is found at $r = a_0$. *Interpretation:* the most probable location to find the electron is at the Bohr radius.

9. We use:

$$\int_0^\infty r^n e^{-2r/a_0}\mathrm{d}r = n!\left(\frac{a_0}{2}\right)^{n+1}. \tag{4.332}$$

For $\langle r\rangle$:

$$\langle r\rangle = \int_0^\infty r|R(r)|^2r^2\mathrm{d}r = \frac{4}{a_0^3}\int_0^\infty r^3e^{-2r/a_0}\mathrm{d}r = \frac{4}{a_0^3}\cdot 3!\left(\frac{a_0}{2}\right)^4 = \frac{3}{2}a_0. \tag{4.333}$$

For $\langle r^2\rangle$:

$$\int r^4e^{-2r/a_0}\mathrm{d}r = 4!(a_0/2)^5 = 24(a_0/2)^5 \Rightarrow \langle r^2\rangle = 3a_0^2. \tag{4.334}$$

Thus:

$$(\Delta r)^2 = 3a_0^2 - (3a_0/2)^2 = 3a_0^2 - \frac{9}{4}a_0^2 = \frac{3}{4}a_0^2. \tag{4.335}$$

10. The Fourier transform of the ground state yields an isotropic distribution centered at $p = 0$. The expectation value is $\langle \mathbf{p}\rangle = 0$ (even function), and:

$$\langle p^2\rangle = \int \tilde{\psi}^*(\mathbf{p})p^2\tilde{\psi}(\mathbf{p})\mathrm{d}^3p. \tag{4.336}$$

It can be related to the average kinetic energy:

$$\langle T\rangle = \frac{\langle p^2\rangle}{2m} = -E_1 = \frac{1}{2}E_0. \Rightarrow \langle p^2\rangle = m_eE_0. \tag{4.337}$$

11. The $1/n^2$ dependence explains the structure of the spectral lines described by the Rydberg formula:

$$\frac{1}{\lambda} = \mathcal{R}_H\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right). \tag{4.338}$$

The principal quantum number n orders the energy levels. In non-relativistic QM, ℓ does not affect E_n, unlike the relativistic case (Lamb shift, spin-orbit coupling).

4.11 Towards a relativistic formalism △

4.12 Pöschl–Teller potential $V(x) = -\dfrac{V_0}{\cosh^2(\alpha x)}$

1. The time-independent Schrödinger equation reads:

$$-\frac{\hbar^2}{2m}\psi''(x) - \frac{V_0}{\cosh^2(\alpha x)}\psi(x) = E\psi(x). \tag{4.339}$$

2. Let $u = \tanh(\alpha x)$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = \alpha(1-u^2)$.

$$\begin{aligned}
\psi'(x) &= \frac{\mathrm{d}\phi}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x} = \alpha(1-u^2)\frac{\mathrm{d}\phi}{\mathrm{d}u},\\
\psi''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha(1-u^2)\frac{\mathrm{d}\phi}{\mathrm{d}u}\right)\\
&= \alpha\left(-2u\alpha(1-u^2)\frac{\mathrm{d}\phi}{\mathrm{d}u} + (1-u^2)\alpha(1-u^2)\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2}\right)\\
&= \alpha^2\left((1-u^2)^2\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} - 2u(1-u^2)\frac{\mathrm{d}\phi}{\mathrm{d}u}\right).
\end{aligned}$$

3. The equation becomes:

$$-\frac{\hbar^2}{2m}\alpha^2\left((1-u^2)^2\phi'' - 2u(1-u^2)\phi'\right) - \frac{V_0}{1-u^2}\phi = E\phi. \tag{4.340}$$

Divide by $(1-u^2)$ and set:

$$\lambda(\lambda+1) = \frac{2mV_0}{\hbar^2\alpha^2}, \qquad \mu^2 = -\frac{2mE}{\hbar^2\alpha^2}, \tag{4.341}$$

which gives:

$$(1-u^2)\phi'' - 2u\phi' + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-u^2}\right]\phi = 0. \tag{4.342}$$

4. We look for a solution of the form

$$\phi(u) = (1-u^2)^{\mu/2}P(u), \tag{4.343}$$

where $P(u)$ is a smooth function on $]-1,1[$, denoted simply as $Q(u) := (1-u^2)^{\mu/2}$ for brevity.

We then compute the derivatives of ϕ using the product rule:

$$\phi'(u) = Q'(u)P(u) + Q(u)P'(u), \tag{4.344}$$

$$\phi''(u) = Q''(u)P(u) + 2Q'(u)P'(u) + Q(u)P''(u). \tag{4.345}$$

Substitute into the differential equation from the previous question:

$$(1-u^2)\phi''(u) - 2u\,\phi'(u) + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-u^2}\right]\phi(u) = 0. \tag{4.346}$$

Compute explicitly $Q'(u)$ and $Q''(u)$. We have:

$$Q(u) = (1-u^2)^{\mu/2}, \quad \Rightarrow \quad Q'(u) = -\mu u(1-u^2)^{\frac{\mu}{2}-1}, \tag{4.347}$$

$$Q''(u) = -\mu(1-u^2)^{\frac{\mu}{2}-1} + \mu(\mu-2)u^2(1-u^2)^{\frac{\mu}{2}-2}. \tag{4.348}$$

Substituting into the full equation:

$$(1-u^2)\left[Q''(u)P(u) + 2Q'(u)P'(u) + Q(u)P''(u)\right] - 2u\left[Q'(u)P(u) + Q(u)P'(u)\right] + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-u^2}\right]Q(u)P(u) = 0.$$

All terms contain a common factor $Q(u)$, which can be factored out since $Q(u) \neq 0$ on $]-1,1[$. This yields a differential equation for $P(u)$ only:

$$(1-u^2)P''(u) - 2(\mu+1)uP'(u) + [\lambda(\lambda+1) - \mu(\mu+1)]P(u) = 0. \tag{4.349}$$

This is a differential equation of the type associated with Jacobi (or generalized Legendre) polynomials, which have polynomial solutions under certain quantization conditions (see next question).

5. (a) We have:

$$\int_{\mathbb{R}} |\psi(x)|^2 \mathrm{d}x = \int_{-1}^{1} |\phi(u)|^2 \frac{\mathrm{d}u}{1-u^2} = \int_{-1}^{1} |P(u)|^2(1-u^2)^{\mu-1}\mathrm{d}u < \infty. \tag{4.350}$$

This is the normalization condition.

(b) Set $P(u) = \sum_{p=0}^{\infty} a_p u^p$. Substituting into the ODE gives:

$$a_{p+2} = \frac{p(p+2\mu+1) - C}{(p+2)(p+1)} a_p \underset{\infty}{\sim} \frac{p^2}{p^2} a_p, \tag{4.351}$$

with $C = \lambda(\lambda+1) - \mu(\mu+1)$. So $a_{p+2} \sim a_p$ as $p \to \infty$.

(c) Thus,

$$\exists c \in \mathbb{R}^*, \quad P(u) \underset{u\to 1,\, p\to\infty}{\sim} \sum c u^p = \frac{c}{1-u}, \tag{4.352}$$

which implies:

$$|\phi(u)|^2 \sim \frac{1}{(1-u)^{2-\mu}}. \tag{4.353}$$

This is integrable only if $\mu > 1$. For bound states we have $\mu > 0$, hence divergence occurs if $\mu \leq 1$. Conclusion: for the integral to converge, the series must terminate $\Rightarrow P$ is a polynomial.

6. As shown earlier, $P(u)$ is a polynomial of degree $n \in \mathbb{N}$, and the condition for series termination is

$$a_{n+2} = 0. \tag{4.354}$$

Using the recurrence relation:

$$a_{p+2} = \frac{p(p+2\mu+1) - \lambda(\lambda+1) + \mu(\mu+1)}{(p+2)(p+1)} a_p, \tag{4.355}$$

and applying it to $p = n$, gives:

$$n(n+2\mu+1) = \lambda(\lambda+1) - \mu(\mu+1). \tag{4.356}$$

Developing both sides:

$$\begin{aligned} &\text{LHS: } n(n+2\mu+1) = n^2 + 2n\mu + n, \\ &\text{RHS: } \lambda(\lambda+1) - \mu(\mu+1) = \lambda^2 + \lambda - \mu^2 - \mu. \end{aligned}$$

Gathering terms:

$$\mu^2 + (2n+1)\mu + (n^2 + n - \lambda^2 - \lambda) = 0. \tag{4.357}$$

We get a quadratic equation in μ:

$$\mu^2 + (2n+1)\mu + A = 0, \quad \text{where } A = n(n+1) - \lambda(\lambda+1). \tag{4.358}$$

Its discriminant is:

$$\Delta = (2n+1)^2 - 4A = (2n+1)^2 - 4(n(n+1) - \lambda(\lambda+1)). \tag{4.359}$$

Expand:

$$\begin{aligned} \Delta &= 4n^2 + 4n + 1 - 4n(n+1) + 4\lambda(\lambda+1) \\ &= (4n^2 + 4n + 1 - 4n^2 - 4n) + 4\lambda(\lambda+1) \\ &= 1 + 4\lambda(\lambda+1). \end{aligned}$$

Thus,

$$\Delta = (2\lambda+1)^2. \tag{4.360}$$

So the discriminant is a perfect square, and the equation admits two real roots:

$$\mu_\pm = \frac{-(2n+1) \pm (2\lambda+1)}{2}. \tag{4.361}$$

We compute both:

$$\mu_1 = \frac{-(2n+1) + (2\lambda+1)}{2} = \lambda - n, \quad \mu_2 = \frac{-(2n+1) - (2\lambda+1)}{2} = -(\lambda+n+1). \tag{4.362}$$

The only physically admissible solution is the first, since for a bound state $\mu > 0$ (because $E < 0 \Rightarrow \mu^2 > 0$).

Therefore, the termination condition gives:

$$\mu = \lambda - n, \quad \text{with } n \in \mathbb{N},\ n < \lambda. \tag{4.363}$$

7. From Eq. (4.341), we deduce:

$$E_n = -\frac{\hbar^2\alpha^2}{2m}(\lambda - n)^2, \quad n = 0, 1, \ldots, \lfloor\lambda\rfloor. \tag{4.364}$$

8. The number of bound states is $N = \lfloor\lambda\rfloor + 1$, a finite number.

9. Physically, the potential $V(x) = -V_0/\cosh^2(\alpha x)$ decays exponentially at infinity ($V(x) \sim -4V_0 e^{-2\alpha|x|}$), too rapidly to allow for an infinite number of bound states. This is a finite-width potential well: the particle can only be confined to a finite number of energy levels.

4.13 Electrodynamic Instability of the Classical Atom

4.13.1 Calculation of the braking and radiation force $\mathbf{F}_{\text{rad}}$.

1. $$\mathrm{d}E_{\text{at}} = \mathrm{d}W = \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\mathrm{d}t \implies \Delta E_{\text{at}} = \int_{t_1}^{t_2} \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\mathrm{d}t \tag{4.365}$$

2. Be careful, this energy variation is the opposite of the energy radiated during the same interval:

$$\mathrm{d}E_{\text{at}} = -P\,\mathrm{d}t = -\frac{2e^2a^2}{3c^3}\mathrm{d}t = -\frac{2e^2\dot{\mathbf{v}}^2}{3c^3}\mathrm{d}t \tag{4.366}$$

We obtain:

$$\Delta E_{\text{at}} = -\frac{2e^2}{3c^3}\int_{t_1}^{t_2} \dot{\mathbf{v}}^2\mathrm{d}t \tag{4.367}$$

3. Moreover, by integrating by parts and assuming quasi-periodicity:

$$\Delta E_{\text{at}} = \frac{2e^2}{3c^3}\int_{t_1}^{t_2} \ddot{\mathbf{v}} \cdot \mathbf{v}\mathrm{d}t \tag{4.368}$$

By comparison with 4.365, a candidate force is the **Abraham-Lorentz radiation braking force**:

$$\mathbf{F}_{\text{rad}} = \frac{2e^2}{3c^3}\ddot{\mathbf{v}} \tag{4.369}$$

4. Now consider the **Thomson model**, in which the electron is bound to the origin by a harmonic restoring force. The equation of motion becomes:

$$m\ddot{\mathbf{r}} = -m\omega_0^2\mathbf{r} + \frac{2e^2}{3c^3}\dddot{\mathbf{r}} \tag{4.370}$$

We look for a solution of the form $r(t) = \text{Re}\left[r(0)e^{i\omega t}\right]$. The perturbative expansion:

$$\omega = \omega_0\left[1 + a(\omega_0\tau) + \mathcal{O}((\omega_0\tau)^2)\right] \tag{4.371}$$

gives $a = \frac{1}{2}$, hence finally:

$$\mathbf{r}(t) = \mathbf{r}(0)e^{-\omega_0^2\tau t}\cos(\omega_0 t) \tag{4.372}$$

The motion is therefore a damped oscillator. The characteristic damping time, or typical lifetime of the atom in this model, is:

$$T_{\text{nat}} = \frac{1}{\omega_0^2\tau} \sim 10^{-8}\text{ s} \tag{4.373}$$

The classical atom is thus fundamentally unstable: the electron spirals toward the nucleus, very slowly on the atomic (pseudo-period) scale, but very rapidly on the macroscopic scale.

4.13.2 Conceptual problems generated by the braking force $\mathbf{F}_{\text{rad}}$.

1. The equation to solve is:

$$\dot{\mathbf{v}} - \tau\ddot{\mathbf{v}} = \frac{1}{m}\mathbf{F}(t) \tag{4.374}$$

whose general solution is:

$$\dot{\mathbf{v}}(t) = v(t_0)e^{(t-t_0)/\tau} - \frac{1}{m\tau}\int_{t_0}^{t} e^{(t-t')/\tau}\mathbf{F}(t')\,\mathrm{d}t' \tag{4.375}$$

2. An unacceptable phenomenon, sometimes called *preacceleration of a charged particle*, appears: if $F = 0$, the above expression clearly shows that the acceleration diverges exponentially at large times.

3. One can formally eliminate the divergent solutions by taking $t_0 = +\infty$. This is a boundary condition that effectively eliminates the so-called "initial condition".

4. Taking $t_0 = +\infty$, we obtain:

$$\begin{aligned}\dot{\mathbf{v}}(t) &= -\frac{1}{m\tau}\int_{t}^{+\infty} e^{(t-t')/\tau}\mathbf{F}(t')\,\mathrm{d}t' \\ &= -\frac{1}{m}\int_{t}^{+\infty} K(t-t')\mathbf{F}(t')\,\mathrm{d}t'\end{aligned} \tag{4.376}$$

with $K(t-t') = \frac{1}{\tau}e^{(t-t')/\tau}$.

This is the *regularized form*, all the more so since the limit of zero charge correctly reproduces the Lorentz Force Electrodynamics (LFE).

Indeed, in the limit $e \to 0$, we have $\tau \to 0$ and the kernel $K(t-t')$ tends to a Dirac delta function $\delta(t-t')$, yielding:

$$\dot{\mathbf{v}}(t) = \frac{1}{m}\mathbf{F}(t) \tag{4.377}$$

5. It is already apparent that the acceleration at instant t depends on future values of the force. This equation therefore violates the **principle of causality**. A change of variable makes this clear. Let $s = \frac{t'-t}{\tau}$, we get:

$$\dot{\mathbf{v}}(t) = -\frac{1}{m}\int_{0}^{+\infty} e^{-s}\mathbf{F}(t+\tau s)\,\mathrm{d}s \tag{4.378}$$

6. With a step force:

$$\mathbf{F}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mathbf{F}_0 & \text{if } t \geq 0 \end{cases} \tag{4.379}$$

we obtain:

$$t < 0: \quad \dot{\mathbf{v}}(t) = -\frac{1}{m\tau}\int_{0}^{+\infty} e^{(t-t')/\tau}\cdot 0\,\mathrm{d}t' = -\frac{\mathbf{F}_0}{m}e^{t/\tau} \tag{4.380}$$

$$t > 0: \quad \dot{\mathbf{v}}(t) = -\frac{\mathbf{F}_0}{m\tau}\int_{t}^{+\infty} e^{(t-t')/\tau}\,\mathrm{d}t' = -\frac{\mathbf{F}_0}{m} \tag{4.381}$$

4.14 Geodesics in an Optical Medium △

4.15 Bose-Einstein Condensation

1. (a) **Expression for the average occupation number $\langle n_\varepsilon \rangle$ according to Bose-Einstein statistics**

For a bosonic system at thermal equilibrium in the grand-canonical ensemble, the average number of particles in an energy state ε is given by the Bose-Einstein distribution. The partition function for this state is:

$$Z_\varepsilon = \sum_{n_\varepsilon=0}^{\infty} e^{-\beta n_\varepsilon(\varepsilon-\mu)} = \frac{1}{1-e^{-\beta(\varepsilon-\mu)}}, \tag{4.382}$$

where $\beta = \frac{1}{k_B T}$ and μ is the chemical potential (which must satisfy $\mu < \varepsilon$ for convergence).

The average number of particles in this state is:

$$\langle n_\varepsilon \rangle = \frac{1}{Z_\varepsilon} \sum_{n_\varepsilon=0}^{\infty} n_\varepsilon e^{-\beta n_\varepsilon(\varepsilon-\mu)}. \tag{4.383}$$

By differentiating Z_ε with respect to $\beta(\varepsilon-\mu)$, we find:

$$\langle n_\varepsilon \rangle = -\frac{1}{Z_\varepsilon}\frac{\partial Z_\varepsilon}{\partial[\beta(\varepsilon-\mu)]} = -\frac{\partial \ln Z_\varepsilon}{\partial[\beta(\varepsilon-\mu)]} = \frac{1}{e^{\beta(\varepsilon-\mu)}-1}. \tag{4.384}$$

(b) **Expression of the density of states $g(\varepsilon)$ for a non-relativistic free gas in a cubic box**

Consider a free particle gas in a volume V with periodic boundary conditions. The non-relativistic kinetic energy is:

$$\varepsilon = \frac{\hbar^2 k^2}{2m}, \tag{4.385}$$

where $k = |\mathbf{k}|$ is the wave vector magnitude.

The number of states with wave vectors inside a sphere of radius k is given by the quantization in k-space:

$$\mathcal{N}(k) = \frac{V}{(2\pi)^3} \times \frac{4\pi k^3}{3}. \tag{4.386}$$

Differentiating with respect to k gives the density of states in k:

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}k} = \frac{V}{2\pi^2}k^2. \tag{4.387}$$

Using the change of variables $\varepsilon = \frac{\hbar^2 k^2}{2m}$, we have:

$$k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}, \quad \mathrm{d}k = \frac{m}{\hbar^2 k}\mathrm{d}\varepsilon. \tag{4.388}$$

Thus the density of states per unit energy is:

$$g(\varepsilon) = \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}k}\frac{\mathrm{d}k}{\mathrm{d}\varepsilon} = \frac{V}{2\pi^2}k^2 \times \frac{m}{\hbar^2 k} = \frac{V}{2\pi^2}\frac{m}{\hbar^2}k = \frac{V}{2\pi^2}\frac{m}{\hbar^2}\sqrt{\frac{2m\varepsilon}{\hbar^2}}. \tag{4.389}$$

Simplifying yields:

$$g(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}. \tag{4.390}$$

2. **Formula for the total average number of particles**

   The total average number of particles is the integral over all energy levels of the average occupation weighted by the density of states:

$$\langle N\rangle = \int_0^{+\infty} \langle n_\varepsilon\rangle g(\varepsilon)\mathrm{d}\varepsilon = \int_0^{+\infty} \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1}\mathrm{d}\varepsilon, \tag{4.391}$$

   which matches the given expression.

3. **Chemical potential as a function of temperature and density**

   In a closed system with N particles and density $\rho = N/V$, the canonical and grand-canonical ensembles are equivalent, so we set $\langle N\rangle = N$.

   Using the substitution $x = \beta\varepsilon$ and defining $\varphi(T) = e^{\mu(T)/(k_BT)}$, the integral becomes:

$$\rho = \frac{N}{V} = \left(\frac{2mk_BT}{4\pi^2\hbar^2}\right)^{3/2} \int_0^{+\infty} \frac{x^{1/2}}{e^x/\varphi(T)-1}\mathrm{d}x = \left(\frac{2mk_BT}{4\pi^2\hbar^2}\right)^{3/2} \int_0^{+\infty} \frac{x^{1/2}}{e^{x/\varphi(T)}-1}\mathrm{d}x, \tag{4.392}$$

   which corresponds to equation (3.159).

4. **Behavior of the chemical potential $\mu(T)$**

   Since $\varphi(T) = e^{\mu(T)/(k_BT)}$ and $\mu(T) < 0$, we have $0 < \varphi(T) < 1$. The function

$$f(\varphi) = \int_0^{+\infty} \frac{x^{1/2}}{e^x/\varphi - 1}\mathrm{d}x \tag{4.393}$$

   is increasing in φ on $(0, 1)$. When T decreases, the prefactor $\left(\frac{2mk_BT}{4\pi^2\hbar^2}\right)^{3/2}$ decreases as well. To keep the equality, $\varphi(T)$ must increase, so $\mu(T)$ increases.

5. **Constraint on chemical potential and critical temperature**

   The chemical potential μ must be strictly less than the lowest energy level (taken here as zero, the ground state), so $\mu < 0$. At the limit $\mu \to 0^-$, we define the critical temperature T_{BE} by:

$$\rho = \left(\frac{2mk_BT_{\mathrm{BE}}}{4\pi^2\hbar^2}\right)^{3/2} \int_0^{+\infty} \frac{x^{1/2}}{e^x-1}\mathrm{d}x. \tag{4.394}$$

   Using the approximation (3.160), we get:

$$T_{\mathrm{BE}} = \frac{2\pi\hbar^2}{mk_B}\left(\frac{\rho}{\zeta(3/2)}\right)^{2/3}, \tag{4.395}$$

   where $\zeta(3/2) \simeq 2.612$ is the Riemann zeta function.

6. **Breakdown of equation (3.159) for $T \le T_{\mathrm{BE}}$**

   For $T \le T_{\mathrm{BE}}$, setting $\mu = 0$ no longer satisfies equation (3.159) because the integral saturates and cannot increase further. The issue is that the population of the ground state, which can contain a macroscopic fraction of particles (the condensate), has not been taken into account.

7. **Population of the ground state N_0**

   Isolating the population of the ground state, N_0, we write:

$$N = N_0 + \int_0^{+\infty} \frac{g(\varepsilon)}{e^{\beta\varepsilon}-1}\mathrm{d}\varepsilon = N_0 + \left(\frac{2m}{4\pi^2\hbar^2}\right)^{3/2} V \int_0^{+\infty} \frac{\varepsilon^{1/2}}{e^{\beta\varepsilon}-1}\mathrm{d}\varepsilon, \tag{4.396}$$

with $\mu = 0$. Therefore, the condensate fraction is:

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{BE}}}\right)^{3/2}. \tag{4.397}$$

The condensate fraction exists only for $T < T_{\text{BE}}$ and grows as temperature decreases.

8. **Grand potential $\mathcal{J}$ and pressure for $T \leq T_{\text{BE}}$**

   For $T \leq T_{\text{BE}}$, the grand potential is:

$$\frac{\mathcal{J}}{k_B T} = -\ln(1 + N_0) + \left(\frac{2m}{4\pi^2\hbar^2}\right)^{3/2} V \int_0^{+\infty} \varepsilon^{1/2} \ln\left(1 - e^{-\beta\varepsilon}\right) \mathrm{d}\varepsilon. \tag{4.398}$$

   In the thermodynamic limit, N_0 is very large so $\ln(1 + N_0) \simeq \ln N_0$, which becomes negligible at the intensive scale. The pressure $P = -\mathcal{J}/V$ is then:

$$P = \frac{k_B T}{\lambda_{\text{th}}^3} \int_0^{+\infty} \frac{x^{3/2}}{e^x - 1} \mathrm{d}x, \tag{4.399}$$

   where $\lambda_{\text{th}} = \sqrt{\frac{2\pi\hbar^2}{mk_B T}}$ is the thermal de Broglie wavelength. Using (3.163), we see the pressure is independent of N_0, depends only on T, and decreases with temperature.

4.16 Decay Chain

4.16.1 Physical modeling of the decay chain

1. The first nucleus N_1 decays spontaneously with decay constant λ_1, so:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\lambda_1 N_1. \tag{4.400}$$

Each nucleus N_k (for $k \in [\![2, n-1]\!]$) is created from the decay of N_{k-1} and disappears by its own decay. Thus:

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = -\lambda_k N_k + \lambda_{k-1} N_{k-1}, \quad \text{for } k \in [\![2, n]\!]. \tag{4.401}$$

2. For $n = 2$, the system is:

$$\begin{cases} \frac{\mathrm{d}N_1}{\mathrm{d}t} = -\lambda_1 N_1, \\ \frac{\mathrm{d}N_2}{\mathrm{d}t} = -\lambda_2 N_2 + \lambda_1 N_1. \end{cases} \tag{4.402}$$

Solving:

$$N_1(t) = N_0 e^{-\lambda_1 t}, \tag{4.403}$$

Then:

$$N_2(t) = \int_0^t \lambda_1 N_1(s) e^{-\lambda_2 (t-s)} \mathrm{d}s = \lambda_1 N_0 \int_0^t e^{-\lambda_1 s} e^{-\lambda_2 (t-s)} \mathrm{d}s. \tag{4.404}$$

Explicitly:

$$N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) \quad (\text{for } \lambda_1 \neq \lambda_2). \tag{4.405}$$

3. We compute:

$$N_1(t) + N_2(t) = N_0 e^{-\lambda_1 t} + \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) = N_0 \left(e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \right). \tag{4.406}$$

Simplifying:

$$N_1(t) + N_2(t) = N_0 \left(\frac{\lambda_2 - \lambda_1 + \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t} \right) = N_0. \tag{4.407}$$

This expresses conservation of the total number of nuclei.

4. $N_2(t)$ reaches a maximum when $\frac{\mathrm{d}N_2}{\mathrm{d}t} = 0$:

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -\lambda_2 N_2 + \lambda_1 N_1 = 0 \quad \Rightarrow \quad N_2 = \frac{\lambda_1}{\lambda_2} N_1(t). \tag{4.408}$$

Inserting $N_1(t) = N_0 e^{-\lambda_1 t}$ into the expression of $N_2(t)$, and solving for t yields:

$$t_{\max} = \frac{1}{\lambda_2 - \lambda_1} \ln \left(\frac{\lambda_2}{\lambda_1} \right). \tag{4.409}$$

4.16.2 Mathematical analysis

1. If the λ_k are pairwise distinct, A is triangular with distinct eigenvalues $-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-1}, 0$, so it is diagonalisable over $\mathbb{R}$.

2. Since A is a constant matrix, the Cauchy-Lipschitz theorem guarantees the existence and uniqueness of a global solution for all $t \geq 0$.

3. Let $E(t) = \|\mathbf{N}(t)\|^2 = \sum_{k=1}^{n} N_k(t)^2$. Since $\mathbf{N}(t)$ is differentiable (solution of a linear ODE), $E(t)$ is differentiable as well.

   (a) We have

   $$\langle x|Ax\rangle = x^\top Ax = (x^\top Ax)^\top = \frac{1}{2}x^\top(A + A^\top)x = x^\top Sx = \langle x|Sx\rangle\,,$$

   where $S = \frac{A+A^\top}{2}$ is the symmetric part of A. This identity holds for all $x \in \mathbb{R}^n$.

   (b) The matrix S is symmetric and tridiagonal, with:

   $$S_{kk} = -\lambda_k, \quad S_{k,k+1} = S_{k+1,k} = \frac{\lambda_k}{2}, \quad \text{for } k = 1, \ldots, n-1.$$

   Then for any vector $x = (x_1, \ldots, x_n)$, we compute:

   $$\begin{aligned} x^\top Sx &= \sum_{k=1}^{n} S_{kk}x_k^2 + 2\sum_{k=1}^{n-1} S_{k,k+1}x_k x_{k+1} \\ &= \sum_{k=1}^{n}(-\lambda_k)x_k^2 + \sum_{k=1}^{n-1}\lambda_k x_k x_{k+1} \\ &= -\frac{1}{2}\sum_{k=1}^{n-1}\lambda_k(x_k - x_{k+1})^2 - \frac{\lambda_1}{2}x_1^2. \end{aligned}$$

   Since all $\lambda_k \geq 0$, this scalar product is always less than or equal to zero.

   (c) Applying this to $\mathbf{N}(t)$, we obtain:

   $$E'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{N}(t)\|^2 = 2\left\langle \mathbf{N}(t)\middle|\dot{\mathbf{N}}(t)\right\rangle = 2\left\langle \mathbf{N}(t)|A\mathbf{N}(t)\right\rangle = 2\left\langle \mathbf{N}(t)|S\mathbf{N}(t)\right\rangle \leq 0.$$

   Therefore, $E(t)$ is a non-increasing function, and the system is stable.

4. The dynamical system is given by:

   $$\dot{\mathbf{N}}(t) = A\mathbf{N}(t),$$

   where $A \in \mathcal{M}_n(\mathbb{R})$ is a chain-decay matrix with $\lambda_k \geq \alpha > 0$ for all $k \in [\![1, n-1]\!]$, and $\lambda_n = 0$.

   The matrix A is triangular or diagonalizable over $\mathbb{R}$, with one eigenvalue equal to 0 (corresponding to the stable isotope) and all other eigenvalues satisfying $\mathrm{Re}(\lambda) \leq -\alpha$.

   We consider an operator norm (equivalent to the usual norm), adapted to the spectrum of A, such that:

   $$\forall x \in \mathbb{R}^n, \quad \|e^{At}x\| \leq Ce^{\mu t}\|x\|, \quad \text{with } \mu = \max\{\mathrm{Re}(\lambda) \mid \lambda \in \mathrm{Sp}(A), \lambda \neq 0\}.$$

   By hypothesis, $\mu \leq -\alpha < 0$.

Note that $\mathbf{N}_\infty \in \ker A$, hence $A\mathbf{N}_\infty = 0$, and we have:

$$\mathbf{N}(t) - \mathbf{N}_\infty = e^{At}(\mathbf{N}_0 - \mathbf{N}_\infty).$$

Taking norms:

$$\|\mathbf{N}(t) - \mathbf{N}_\infty\| \le Ce^{-\alpha t}\|\mathbf{N}_0 - \mathbf{N}_\infty\| \le Ce^{-\alpha t}\|\mathbf{N}_0\|.$$

This proves exponential convergence toward the equilibrium.

5. Summing the differential system:

$$\sum_{k=1}^{n} \frac{\mathrm{d}N_k}{\mathrm{d}t} = -\sum_{k=1}^{n-1} \lambda_k N_k + \sum_{k=2}^{n} \lambda_{k-1} N_{k-1} = 0, \tag{4.410}$$

which implies:

$$\sum_{k=1}^{n} N_k(t) = \sum_{k=1}^{n} N_k(0), \quad \forall t \ge 0. \tag{4.411}$$

Thus, the total number of nuclei is conserved.

4.16.3 Nonlinear model with saturating decay rates

1. Each decay rate is given by:

$$\lambda_k(N_k) = \frac{\lambda_k^0}{1 + a_k N_k}, \quad \text{with } \lambda_k^0 > 0,\ a_k > 0. \tag{4.412}$$

This function is $\mathcal{C}^\infty$ on $\mathbb{R}_+^n$ and locally Lipschitz. So the system

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = -\frac{\lambda_k^0}{1 + a_k N_k} N_k + \frac{\lambda_{k-1}^0}{1 + a_{k-1} N_{k-1}} N_{k-1} \tag{4.413}$$

admits a unique local solution by the Cauchy-Lipschitz theorem.

2. Using Grönwall's lemma or by observing the positivity of flows, one proves that $N_k(t) \ge 0$ for all $t \ge 0$, and solutions extend globally on $\mathbb{R}_+$.

3. At equilibrium, we impose $\frac{\mathrm{d}N_k}{\mathrm{d}t} = 0$. The equilibrium equations are:

$$0 = -\frac{\lambda_1^0}{1 + a_1 N_1^*} N_1^*, \quad 0 = -\frac{\lambda_k^0}{1 + a_k N_k^*} N_k^* + \frac{\lambda_{k-1}^0}{1 + a_{k-1} N_{k-1}^*} N_{k-1}^*, \quad k \in [\![2, n]\!]. \tag{4.414}$$

4. The first equation imposes $N_1^* = 0$. By recurrence, all $N_k^* = 0$ for $k \in [\![1, n-1]\!]$. Then necessarily $N_n^* = N_\infty$ (total mass), so:

$$\mathbf{N}^* = (0, \ldots, 0, N_\infty). \tag{4.415}$$

5. The Jacobian matrix $J(\mathbf{N})$ is lower triangular. At $\mathbf{N} = \mathbf{0}$, we compute:

$$J(\mathbf{0}) = A = \begin{pmatrix} -\lambda_1^0 & 0 & \cdots & 0 \\ \lambda_1^0 & -\lambda_2^0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-1}^0 & 0 \end{pmatrix}. \tag{4.416}$$

6. The eigenvalues of $J(\mathbf{0})$ are the same as in the linear case: $-\lambda_1^0, \ldots, -\lambda_{n-1}^0, 0$. Thus, the fixed point $\mathbf{N}_\infty$ is locally asymptotically stable.

7. Increasing the parameters a_k decreases the decay rate $\lambda_k(N_k)$, which slows down the dynamics. The system takes longer to relax, and transient states may persist longer compared to the linear model.