Exercises in Fundamental Physics

(Undergraduate L3 – Graduate M1 Level)

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Contents

Abstract

This document gathers a selection of original exercises in fundamental physics, designed with a transversal and progressive perspective, from the third year of undergraduate studies to the first year of a Master's degree. Each exercise is accompanied by a detailed solution (when available), and is embedded in a rigorous historical, theoretical, or practical context. Topics covered include special relativity, quantum mechanics, statistical physics, electrodynamics, and incursions into mathematical physics. A classification by level is proposed to guide the reader's progression.

Chapter 1

Introduction

This document is a compilation of exercises in Fundamental Physics that I designed with passion, in the spirit of an end-of-L3 / M1 course, and beyond. The aim is twofold: to provide rigorous, inspiring problems that highlight the formal and conceptual beauty of physics, and to offer a solid foundation for students wishing to deepen their understanding of major classical and modern theories. I hope to share my enthusiasm for physics that goes beyond what is typically covered in class, drawing on concepts that span multiple areas of physics.

Each exercise involves specific concepts (indicated in parentheses, such as **(SR)** for Special Relativity, **(QM)** for Quantum Mechanics, etc.) and is gradually supplemented with a detailed correction, accessible by clicking on the "(Correction)" link. Exercises are rated with stars (see 2.1), and you are free to start with the one that intrigues you the most.

As a first-year Master's student in Fundamental Physics at Sorbonne University (Pierre and Marie Curie campus), I want this collection to remain dynamic: solutions will be added regularly. Lastly, in the correction section, by clicking on the exercise titles (either in the heading or at the beginning of the solution), you can return to the corresponding exercise.

I hope that by reading and working through these exercises, you will find as much enjoyment as I had in writing them.

Chapter 2

Information

2.1 Notations

1. Vector quantities are written in bold, except for the operator ∇, which is never in bold. Four-vector quantities (in relativity) are written with a Greek letter, in superscript for covariant components, and in subscript for contravariant ones.
   Example: $\mathbf{v}$ for velocity, ∇p for the pressure gradient (which is a vector!), and x^μ for spacetime position in covariant form. Conversely, in Quantum Mechanics, vectors are denoted using kets, and operators in bold.
   Example: $|\psi\rangle$ for a state vector ψ and $\mathbf{H}$ for the Hamiltonian.

2. The notation d denotes the differential operator.

3. The notation ∂_u implicitly means $\frac{\partial}{\partial u}$ if u is a variable, and $\partial_\mu = \frac{\partial}{\partial x^\mu}, \partial^\mu = \frac{\partial}{\partial x_\mu}$ in relativity.

4. $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$ in Cartesian coordinates, is an operator that properly defines the gradient, divergence, and curl. Indeed, ∇f is the gradient of f, $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$, and $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$. The operator $\partial_\mu \partial^\mu = \Box$ is the d'Alembertian, invariant under Lorentz transformations.

5. The notation $\dot{x}$ denotes a time derivative: $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$. In a relativity exercise, the preferred notation will be $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$, where τ is the proper time, and $\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$.

6. The notation f' denotes a derivative with respect to the variable x, i.e., $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$.

7. The notation $[A]$ indicates the physical unit of the quantity A.

8. The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}$ denote the sets of real, complex, and natural numbers, respectively.

9. The metric used in special relativity is $g_{\mu\nu} = (-,+,+,+)$. We also recall that $a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu = g^{\mu\nu} a_\mu b_\nu$.

10. Stars indicate the difficulty level of the exercises, ranging from 1: ★ to 5 stars: ★★★★★.
    The difficulty assessment is based on the length, technical and mathematical complexity, and the academic level (L3, M1, M2) needed to be comfortable with the concepts involved.

11. A dagger † means the exercise is either taken from or inspired by an existing one. A footnote will also appear in such cases.

12. The symbol $\triangle$ indicates that the solution is still being written.

2.2 Fundamental Constants

Constant	Exact value	Units
Planck constant	$h = 6.62607015 \times 10^{-34}$	$\mathrm{J\,s}$
Dirac constant	$\hbar = \frac{h}{2\pi} = 1.054571817 \times 10^{-34}$	$\mathrm{J\,s}$
Speed of light	$c = 299792458$	$\mathrm{m\,s^{-1}}$
Elementary charge	$e = 1.602176634 \times 10^{-19}$	C
Electron mass	$m_e = 9.1093837015 \times 10^{-31}$	kg
Proton mass	$m_p = 1.67262192369 \times 10^{-27}$	kg
Neutron mass	$m_n = 1.675 \times 10^{-27}$	kg
Vacuum permittivity	$\varepsilon_0 = 8.854187817 \times 10^{-12}$	$\mathrm{F\,m^{-1}}$
Vacuum permeability	$\mu_0 = 4\pi \times 10^{-7}$	$\mathrm{N\,A^{-2}}$
Gravitational constant	$G = 6.67430 \times 10^{-11}$	$\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Boltzmann constant	$k_B = 1.380649 \times 10^{-23}$	$\mathrm{J\,K^{-1}}$
Avogadro number	$N_A = 6.02214076 \times 10^{23}$	$\mathrm{mol^{-1}}$
Ideal gas constant	$R = 8.314462618$	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Reference temperature (0°C)	$T_0 = 273.15$	K

Table 2.1: Fundamental physical constants with their exact values.

2.3 Formulary

2.3.1 Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss's Law)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(No magnetic monopoles)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's Law)}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampère-Maxwell Law)}$$

$$\nabla \times \mathbf{A} = \mathbf{B}, \quad -\partial_t \mathbf{A} - \nabla \varphi = \mathbf{E} \qquad \text{(Relation between potentials and EM field)}$$

$$P = \frac{q^2 a^2}{6\pi \varepsilon_0 c^3} \qquad \text{(Larmor Power)}$$

2.3.2 Special Relativity

$$E = \gamma m c^2 = \sqrt{p^2 c^2 + m^2 c^4} \qquad \text{(Relativistic energy)}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{(Lorentz factor)}$$

$$x' = \gamma (x - vt) \qquad \text{(Lorentz transformation)}$$

$$t' = \gamma \left(t - \frac{vx}{c^2} \right) \qquad \text{(Time transformation)}$$

$$\beta = \frac{v}{c}$$

$$\mathbf{p} = \gamma m \mathbf{v} \qquad \text{(Relativistic momentum vector)}$$

$$\mathbf{p} = \hbar \mathbf{k} \qquad \text{(Photon momentum vector)}$$

2.3.3 Quantum Mechanics

$$\mathbf{P} = -i\hbar\nabla \qquad \text{(Momentum operator)}$$

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \mathbf{H}\left|\psi\right\rangle \qquad \text{(Schrödinger equation)}$$

$$[\mathbf{X}_i, \mathbf{P}_j] = i\hbar\delta_{ij} \qquad \text{(Canonical commutation)}$$

$$\mathbf{X} = \sqrt{\frac{\hbar}{2m\omega}}(\mathbf{a} + \mathbf{a}^\dagger), \quad \mathbf{P} = i\sqrt{\frac{\hbar m\omega}{2}}(\mathbf{a}^\dagger - \mathbf{a}) \qquad \text{(Annihilation and creation operators)}$$

$$[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{1} = \mathbf{Id} \qquad \text{(Commutator)}$$

$$\mathbf{H} = \hbar\omega\left(\mathbf{N} + \frac{1}{2}\right) \qquad \text{(Harmonic oscillator Hamiltonian)}$$

$$\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}, \quad \mathbf{N}\left|n\right\rangle = n\left|n\right\rangle,\ n \in \mathbb{N} \qquad \text{(Number operator)}$$

$$\mathbf{L}_i = \varepsilon_{ijk}\mathbf{X}_j\mathbf{P}_k \qquad \text{(Angular momentum in Einstein notation)}$$

$$[\mathbf{J}_i, \mathbf{J}_j] = i\varepsilon_{ijk}\mathbf{J}_k \qquad \text{(Angular momentum algebra)}$$

2.3.4 Statistical Physics

In the canonical ensemble,

$$\beta = \frac{1}{k_B T} \qquad \text{(Thermal energy)}$$

$$Z = \sum_n e^{-\beta E_n} \qquad \text{(Partition function)}$$

$$\langle E\rangle = \frac{\sum_n E_n e^{-\beta E_n}}{Z} = -\partial_\beta \ln Z \qquad \text{(Average energy)}$$

2.3.5 Analytical Mechanics

$$\mathcal{L} = T - V \qquad \text{(Lagrangian)}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = 0 \qquad \text{(Lagrange equations)}$$

2.3.6 Subatomic Physics

$$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \qquad \text{(Infinitesimal solid angle)}$$

Where $\Omega \in [0, 4\pi]$ because by definition $\Omega = \frac{S}{R^2}$ with $S = 4\pi R^2$, the surface area of a sphere of radius R.

2.3.7 Operators in Curvilinear Coordinates

Cylindrical coordinates:

$$
\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z \\
\nabla \cdot \mathbf{A} &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\mathbf{e}_r \\
&\quad + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{e}_\theta \\
&\quad + \left(\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\mathbf{e}_z
\end{aligned}
$$

Spherical coordinates:

$$
\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \\
\nabla \times \mathbf{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right)\mathbf{e}_r \\
&\quad + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\mathbf{e}_\theta \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\mathbf{e}_\phi
\end{aligned}
$$

2.3.8 Trigonometric identities

$$
\sin^2(\theta) + \cos^2(\theta) = 1, \quad 1 + \tan^2(\theta) = \frac{1}{\cos^2(\theta)}.
$$

Addition formulas

$$
\begin{aligned}
\sin(a \pm b) &= \sin(a)\cos(b) \pm \cos(a)\sin(b), \\
\cos(a \pm b) &= \cos(a)\cos(b) \mp \sin(a)\sin(b).
\end{aligned}
$$

Double-angle formulas

$$
\begin{gathered}
\sin(2\theta) = 2\sin(\theta)\cos(\theta), \\
\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta).
\end{gathered}
$$

These formulas are very useful for variable changes in integration.

Expressions of $\sin(x)$, $\cos(x)$, and $\tan(x)$ in terms of $t = \tan\left(\frac{x}{2}\right)$

$$
\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad \tan(x) = \frac{2t}{1-t^2}.
$$

Variable substitution $t = \tan\left(\frac{x}{2}\right)$

This change of variable is often used to simplify trigonometric integrals. We also have:

$$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t.$$

2.4 Legend of thematic notations

- **(SR)**: Special Relativity
- **(QM)**: Quantum Mechanics
- **(EM)**: Electromagnetism
- **(AM)**: Analytical Mechanics
- **(SM)**: Statistical Mechanics
- **(SP)**: Subatomic Physics

2.5 Suggested paths depending on your level

To help readers navigate this dense collection of exercises, here are a few suggested paths based on your level and goals. Of course, every student is free to explore the problems that inspire them.

Level	**Recommended exercises**
Early Bachelor Year 3	3.1 – Two-body problem 3.2 – Rutherford cross section 3.4 – Pulsed magnetic field machine 3.13 – Electrodynamic instability of the classical atom
End of Bachelor / Beginning of Master 1	3.3 – Cherenkov effect 3.5 – Metric on a sphere 3.6 – Blackbody radiation 3.10 – Hydrogen atom and radial equation 3.12 – Pöschl–Teller potential
Advanced Master 1	3.7 – Minimization of gravitational potential 3.8 – Relativistic charged particle 3.9 – Relativistic hydrodynamics 3.11 – Towards a relativistic formalism

Chapter 3

Exercises

This collection of exercises was designed with the ambition to go beyond mere mechanical practice of methods. Each problem aims to highlight a certain form of mathematical elegance or physical depth — a careful eye will discover, behind the equations and techniques, a subtle coherence, sometimes even a formal beauty. Some exercises are demanding, both in their length and structure: they are sometimes inspired by competitive exams or realistic physical situations, and may require several hours of reflection. Their goal is not only to reinforce technical skills, but to make one feel, through progressive resolution, the deep unity between mathematical rigor and the physical reality it describes. This chapter is dynamic: new problems will be regularly added in the same spirit of elegance, clarity, and depth.

3.1 Two-body problem and quantization of the Bohr atom† (MA) ★★★

[1] (Solution)

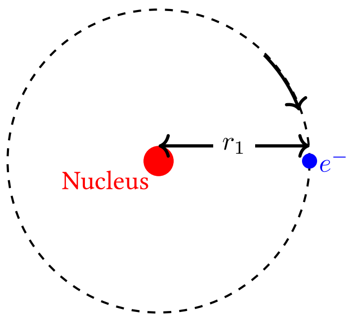


Figure 3.1: Diagram of the Bohr atom.

Consider a system of two particles with masses m_1 and m_2 interacting via a central potential $V(r) = -\frac{C}{r} = -\frac{\vartheta^2}{r}$[2], where r is the distance between the two particles and C is a real constant.
Here we use the Coulomb potential, but one could just as well use a gravitational potential. We will study in detail the bound states of the hydrogen atom according to the old quantum theory and, in particular, derive the energy associated with a given trajectory of the electron of mass m_1 around the nucleus of mass m_2.

3.1.1 Center of mass

Let $\mathbf{r}_1, \mathbf{r}_2$ be the position vectors of the electron and the nucleus relative to an arbitrary reference frame, and $\mathbf{v}_1, \mathbf{v}_2$ their respective velocities.

1. Write the Lagrangian $\mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2)$.

[1]† Inspired by Claude Aslangul, *Mécanique Quantique 1*, Chapter 7.
[2]We define $\vartheta^2 = \frac{e^2}{4\pi\varepsilon_0}$.

2. Let **R** be the center-of-mass position vector and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Show that the Lagrangian can be written as:

$$\mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2) = \mathcal{L}_G(\mathbf{V}) + \mathcal{L}_r(\mathbf{r}, \mathbf{v})$$

3. Explain why the total angular momentum about the center of mass G, denoted **J**, is a conserved quantity. Deduce a conclusion about the trajectory.

From here on, we examine only the internal motion through $\mathcal{L}_r$ in polar coordinates (r, θ) in the plane perpendicular to **J**.

3.1.2 Integration of the equations of motion

1. Write the Hamiltonian $\mathcal{H}$ for the internal motion and derive Hamilton's equations. Recover the conservation of angular momentum and interpret the equation involving only $\mathbf{r}, \dot{\mathbf{r}}$.

2. Determine the relationship between r and θ, i.e., the trajectory. To do so, eliminate time from the previous equations by setting $u = \frac{1}{r}$, and show that:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = K, \quad K = \frac{\mu\vartheta^2}{J^2}$$

3. Finally, deduce that the trajectory is a conic, whose equation can always be written in the form:

$$r(\theta) = \frac{p}{1 + \varepsilon\cos\theta}$$

   Give the expression of p, the conic parameter, and ε, the eccentricity. Check how the value of ε relative to 1 determines the nature of the corresponding state (bound or unbound).

3.1.3 Bohr quantization

In this part, we consider only bound states ($E < 0$) and apply Bohr's rules to select among all classically possible trajectories. These rules involve the action variables J_θ, J_r and are written:

$$J_\theta := \oint p_\theta \,\mathrm{d}\theta = n_\theta h$$
$$J_r := \oint p_r \,\mathrm{d}r = n_r h$$
$$n_\theta, n_r \in \mathbb{Z}$$

1. Determine the possible values of the angular momentum J as a consequence of the quantization of J_θ. Specify the possible values of the integer n_θ.

2. Quantize J_r and deduce the relation between ε and the integers n_r, n_θ[3]. Given:

$$\int_0^{2\pi} \frac{1}{1 + \varepsilon\cos\theta}\,\mathrm{d}\theta = \frac{2\pi}{\sqrt{1 - \varepsilon^2}}$$

3. Deduce that the energy E is quantized, with $n \in \mathbb{N}^*$ depending on n_θ, n_r, and that:

$$E_n = -\frac{\mu\vartheta^4}{2n^2\hbar^2}$$

[3] At first glance, one might say that $J_r = 0$; an integration by parts is necessary.

3.2 Rutherford Scattering Cross Section† (FS) ★★

[4] (Solution)

We consider the same situation as in the previous exercise: two particles, one of which is fixed, interacting through a potential of the form $V(r) = \frac{C}{r}$. Here, $C = \frac{Qq}{4\pi\varepsilon_0}, Q = Ze, q = 2e$. We will use some results from the previous exercise, so it is recommended to complete that one first.

3.2.1 Deflection of a charged particle by an atomic nucleus

We work in the polar coordinate system (r, φ), perpendicular to the angular momentum, since the motion is planar. The alpha particle arrives with initial velocity $\mathbf{v}_0$. We assume $\lim_{t\to-\infty} \varphi(t) = \pi$.

1. Determine the non-zero component of **J** as a function of r, φ. Also determine it in terms of b, v_0, where b is the impact parameter.

2. Write the equation of motion. Decompose $\mathbf{v} = \dot{\mathbf{r}}$ into a vector parallel and one perpendicular to the polar axis. Deduce that:
$$m\dot{v}_\perp = \frac{C}{r^2}\sin\varphi$$

3. We want to introduce the deflection angle θ. By integrating the equation, show that:
$$v_0 \sin\theta = \frac{C}{mbv_0}(\cos\theta + 1)$$

4. Using some trigonometric identities, deduce that:
$$\tan\frac{\theta}{2} = \frac{C}{2E_0 b}$$
where $E_0 = \frac{1}{2}mv_0^2$.

3.2.2 Rutherford Scattering Cross Section

1. Recall the formula for the differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$.

2. Deduce that:
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{C^2}{16E_0^2\sin^4\frac{\theta}{2}}$$

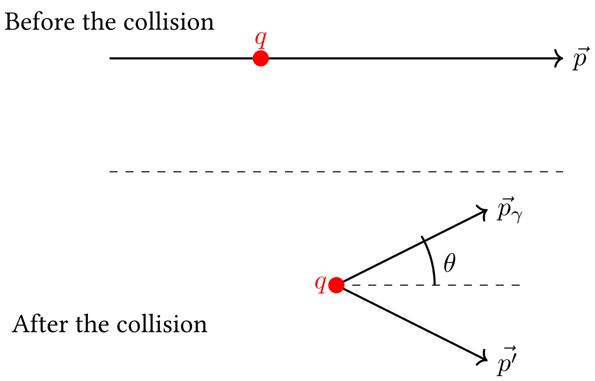
3. Deduce that this model is invalid for small deflection angles.

4. Explain why this experiment demonstrates the existence of atomic nucleus.

[4†] Inspired by Claude Aslangul, *Mécanique Quantique 1*, Chapter 3.

3.3 Cherenkov Effect† (SR, NP) ★★★

[5] (Solution)
The Cherenkov effect occurs when a charged particle travels through a dielectric medium at a speed v greater than the speed of light in that medium c/n, where n is the refractive index of the medium.



The momentum of the charged particle is $\mathbf{p}$ before the collision, $\mathbf{p}_\gamma$ is the photon momentum after the collision, and $\mathbf{p}'$ is the particle's momentum after the collision. The angle θ is the angle between $\mathbf{p}$ and $\mathbf{p}_\gamma$. Recall that $\lambda = \frac{c}{n\nu}$.

1. Express p_γ in terms of h, ν, c, n. Deduce the relation between p_γ and E_γ in a medium with refractive index n.
2. Write the momentum conservation equation for the elementary process.
3. Using the previous question, express $\mathbf{p}'^2$ in terms of the magnitudes of the momenta p, p_γ, and the angle θ.
4. Write the energy conservation equation.
5. Deduce that:
$$p'^2 = p^2 - 2\frac{E}{c^2}h\nu + \frac{p_\gamma^2}{n^2}$$
where E is the initial energy of the electron.
6. Express $\cos\theta$ in terms of $p, p_\gamma, E, h, n, c, \nu$.
7. Show that:
$$\cos\theta = \frac{c}{nv}\left[1 + \frac{h\nu}{2E}(n^2 - 1)\right]$$
8. What is the condition for the Cherenkov effect to occur?
9. In which frequency range are the photons emitted?
10. In which direction are the highest-energy photons emitted?
11. All photons are emitted within a cone; what is the half-apex angle ϕ of this cone? Estimate ϕ for $n = \frac{4}{3}$ and $v = \frac{4}{5}c$.
12. Compare the minimum kinetic energy required for the particle to produce Cherenkov radiation in the cases of an electron and a proton, for $n = \frac{4}{3}$.

[5] † Inspired by Claude Aslangul, *Mécanique Quantique 1*, Chapter 5.

3.4 Pulsed Magnetic Field Machine (EM) ★★

(Solution)

The magnetic stimulation machine is a non-invasive technology used in physiotherapy and rehabilitation. It works by generating pulsed magnetic fields using a circular coil. In practice, the machine sends current pulses through the coil, which creates a time-varying magnetic field. According to Faraday's law, this variation automatically induces an electric field in the surrounding tissues.

This induced electric field acts directly on the cellular membranes of muscles by activating ion channels. As a result, an action potential is triggered, leading to muscle contraction. This mechanism allows not only for the stimulation of weakened or atrophied muscles, but also improves blood circulation and reduces pain. Moreover, the absence of direct skin contact makes the treatment comfortable and safe for the patient.

The magnetic stimulation machine is especially used to:

- Support muscle rehabilitation after injury or surgery. - Relieve chronic pain linked to musculoskeletal disorders. - Improve muscle tone and prevent atrophy. - Stimulate blood and lymph circulation to speed up recovery.

In summary, thanks to an approach based on fundamental principles of electromagnetic induction, this technology provides an effective treatment for various muscular and nervous conditions, serving as a complementary solution to conventional rehabilitation therapies.

We consider a circular coil of radius R carrying a time-varying current:

$$I(t) = I_0 e^{-t/\tau} \sin(\omega t),$$

where I_0 is the current amplitude, τ is the damping time constant, and ω is the oscillation frequency. The coil's axis is assumed to coincide with the z-axis. The coil is considered thin and modeled as a single loop.

1. **Magnetic field of the coil**

   (a) Assuming the coil behaves like a magnetic dipole, express the magnetic field $\mathbf{B}$ along the central axis (at a distance z from the center) in terms of $I(t)$, R, z, and physical constants.

   (b) Show that for $z \gg R$, the field approximates that of a magnetic dipole and give its asymptotic expression.

2. **Induced electric field in biological tissue** We model the tissue as a thin conducting disk of radius a, placed under the coil.

   (a) Starting from the local Faraday law:
   $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
   express the induced electric field $\mathbf{E}$ in terms of $\frac{\mathrm{d}B}{\mathrm{d}t}$.

   (b) Assuming cylindrical symmetry (purely azimuthal field), derive the expression for the induced electric field $E_\theta(r,t)$ in the plane of the disk, distinguishing the cases $r < R$ and $r > R$.

3. **Effect on motor neurons** A motor neuron is assumed to be activated when the induced voltage exceeds a threshold V_{thresh}.

   (a) Express V in terms of the parameters of the problem.

   (b) Determine a condition on I_0, τ, ω, and the geometric parameters to ensure neuron activation.

   (c) Using realistic numerical values ($R = 5\,\text{cm}$, $a = 2\,\text{cm}$, $I_0 = 100\,\text{A}$, $\tau = 1\,\text{ms}$, $\omega = 10^4$ rad/s, and $V_{\text{thresh}} = 10\,\text{mV}$), check whether neuron activation is possible.

4. **Effect of pulsed magnetic field on muscles** Explain why a pulsed magnetic field, by inducing an electric field in tissues, can provoke muscle contraction. Briefly describe the physiological mechanism (activation of ion channels, generation of an action potential, muscle contraction).

3.5 Metric on a Sphere (MA) ★★★

(Solution)

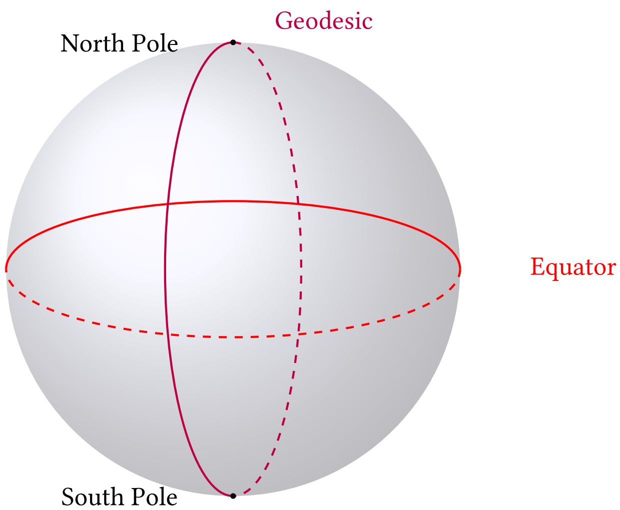


Figure 3.2: Diagram of a sphere and its geodesics.

Our goal is to determine the metric on a sphere and its geodesics. This will help us understand the optimal flight paths for an airplane. Recall that in spherical coordinates, for a fixed radius R,

$$\begin{aligned} x &= R\cos\varphi\sin\theta \\ y &= R\sin\varphi\sin\theta \\ z &= R\cos\theta \end{aligned}$$

1. Calculate the line element $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$ as a function of R, θ, and φ.
2. Using the action $S = \int \mathrm{d}s = \int \mathcal{L}\mathrm{d}\lambda$, where λ is a suitably chosen parameter, and the variational principle, determine the geodesic equations.
3. Solve the equations by using symmetries. One may use that

$$\begin{aligned} &\int \frac{\mathrm{d}\alpha}{\sin^2\alpha\sqrt{1-\frac{\lambda^2}{\sin^2\alpha}}} \quad \text{set } u = \cot\alpha, \\ &\int -\frac{\mathrm{d}t}{\sqrt{1-t^2}} = \arccos t + C. \end{aligned}$$

   Show that the geodesics have the following form:

$$(x, y, z) \in S^2, \quad ax + by + cz = 0,$$

   that is, the geodesics are intersections between planes passing through the origin and the sphere, or in other words, arcs of great circles.

3.6 Blackbody Radiation (PS) ★★★★

(Solution)

We seek to obtain the spectral energy density, that is the function

$$u(\nu, T) = \frac{\mathrm{d}^2 W}{\mathrm{d}\nu \mathrm{d}\mathcal{V}} = \frac{\mathrm{d}N}{\mathrm{d}\nu} \frac{\langle W \rangle}{\mathcal{V}}, \tag{2.6.1}$$

with W the energy and $\langle W \rangle$ the mean energy. We will also work in a historical framework, without using quantum mechanics, which was partly discovered thanks to the results we are about to demonstrate.

3.6.1 Number of Modes Excited per Frequency Unit

1. Consider a blackbody represented by a cubic cavity of side length L and volume $\mathcal{V}$. Write down the wave equation for the electric field $\mathbf{E}$ inside the cavity.

2. Solve the wave equation. Explain why the field $\mathbf{E}$ depends on three modes $n_x, n_y, n_z \in \mathbb{N}^*$.

3. Show that
$$n_x^2 + n_y^2 + n_z^2 = r^2 = \left(\frac{2L}{\lambda}\right)^2.$$

4. By counting unit cubes stacked along the axes n_x, n_y, n_z, we can enumerate the total number N of excited modes.

   Each cube can be represented as $\mathbf{r} = n_\mu \mathbf{e}^\mu$. When the cubes are very numerous, that is, when L is much larger than λ, it suffices to calculate the volume of a sphere of radius r.

   However, since the integers are strictly positive, only $1/8$ of the total sphere volume is taken. Also, a factor of 2 must be considered due to the two possible polarization planes of the electric field $\mathbf{E}$.

   Using these data, deduce that
$$\frac{\mathrm{d}N}{\mathrm{d}\nu} = \frac{8\pi\nu^2}{c^3}\mathcal{V}.$$

3.6.2 Ultraviolet Catastrophe

1. Explain why the ensemble associated with this problem — the calculation of u — corresponds to the canonical ensemble.

2. Calculate the Hamiltonian of a harmonic oscillator.

3. Give the probability of being in an energy state W. Deduce the partition function Z of a gas of harmonic oscillators.

4. Show that
$$\langle W \rangle = k_B T.$$

5. Deduce that
$$u(\nu, T) = 8\pi \frac{\nu^2}{c^3} k_B T,$$
   and explain the title of this subsection.

3.6.3 Planck's Law

The revolutionary idea is to estimate that the photon energy is quantized. Thus, we move from the idea of a continuous energy distribution to a discrete one. This idea arose from the fact that the average energy of an oscillator did not depend on the frequency ν. Planck suspected a simple proportionality relation between W and ν:

$$W_n = nh\nu.$$

Then came the idea of quanta, that energy is not continuous but distributed in indivisible packets called **quanta**[6].

1. Recalculate the partition function Z.
2. Deduce that
$$u(\nu, T) = 8\pi \frac{\nu^2}{c^3} \frac{h\nu}{e^{\beta h\nu} - 1}$$
with $\beta = \frac{1}{k_B T}$.

Thus, the ultraviolet catastrophe was resolved, and this result agreed perfectly with experiments. This function became integrable, which later led to Stefan's law.

3.6.4 Energy Flux Emitted by a Blackbody

Consider a cavity in thermal equilibrium filled with a photon gas at temperature T. The radiation is **isotropic** and characterized by a volumetric spectral energy density $u(\nu)$, such that

$$u(\nu)\,\mathrm{d}\nu = \text{electromagnetic energy per unit volume between frequencies } \nu \text{ and } \nu + \mathrm{d}\nu.$$

Let I be the total intensity (energy flux per unit surface perpendicular to it, integrated over all directions) emitted by the blackbody.

1. Recall the expression of the monochromatic energy flux emitted in a direction making an angle θ with respect to the surface normal, in terms of the directional spectral intensity I_ν and the solid angle $\mathrm{d}\Omega$.
2. Show that the total energy flux emitted at frequency ν per unit surface is given by
$$I(\nu) = \int_{\Omega_+} I_\nu \cos\theta \,\mathrm{d}\Omega,$$
where Ω_+ denotes the outgoing hemisphere ($0 \leq \theta \leq \pi/2$).
3. Assuming the radiation is isotropic, i.e., I_ν is independent of direction, show that
$$I(\nu) = \pi I_\nu.$$
4. By integrating over all frequencies, deduce that the total emitted intensity is
$$I = \int_0^\infty \pi I_\nu \,\mathrm{d}\nu.$$
5. Show that the volumetric spectral energy density $u(\nu)$ is given by
$$u(\nu) = \frac{1}{c} \int_{S^2} I_\nu(\mathbf{n}) \,\mathrm{d}\Omega.$$
Assuming isotropic radiation, deduce that
$$u(\nu) = \frac{4\pi}{c} I_\nu.$$
6. Deduce that
$$I = \frac{c}{4} \int_0^\infty u(\nu) \,\mathrm{d}\nu.$$

[6] Albert Einstein used Planck's idea in his annus mirabilis of 1905 to explain the photoelectric effect, which earned him the Nobel Prize in 1921.

3.6.5 Stefan's Law

Stefan's law states that for a blackbody,

$$I(T) = \sigma T^4,$$

where σ is the Stefan–Boltzmann constant. We will prove it.

1. Using the previous parts, show that
$$I = \frac{2\pi k_B^4}{h^3c^2}T^4 \int_0^\infty \frac{x^3}{e^x - 1}\mathrm{d}x.$$
2. Verify the convergence of the integral and express it as a series.
3. Finally, prove that
$$I(T) = \frac{2\pi^5 k_B^4}{15h^3c^2}T^4.$$
This is recognized as Stefan's law[7]
$$I = \sigma T^4.$$

3.6.6 Application: Solar Mass Loss by Electromagnetic Radiation

Assuming the Sun is a blackbody, determine $\dot{m}$, the mass loss per unit time. What is this mass loss rate in $\mathrm{kg} \cdot \mathrm{s}^{-1}$? Knowing that our Sun is approximately 4.6×10^9 years old, estimate how many Earth masses the Sun has lost so far.

Data: $R = 6.96 \times 10^8\,\mathrm{m}$, $T = 5775\,\mathrm{K}$, $m = 1.98 \times 10^{30}\,\mathrm{kg}$, $m_\oplus = 6 \times 10^{24}\,\mathrm{kg}$.

[7]Hence, $\sigma = \frac{2\pi^5 k_B^4}{15h^3c^2}$, which is rather unexpected.

3.7 Minimization of the Gravitational Potential by a Ball (MA) ★★★★★

(Solution)
This exercise involves notions of differential calculus.

We consider the following variational problem: among bounded open domains $\Omega \subset \mathbb{R}^3$ of fixed volume, find the one minimizing the **internal gravitational interaction** defined by the functional:

$$\mathcal{F}[\Omega] = \iint_{\Omega\times\Omega} \frac{1}{|x-x'|}\,\mathrm{d}^3x\,\mathrm{d}^3x'$$

Note that this expression is proportional to the gravitational self-interaction potential of a body with uniform density. Indeed, for $x \in \mathbb{R}^3$,

$$U(x) = -G\int_\Omega \frac{\rho}{|x-x'|}\,\mathrm{d}^3x'$$

The total gravitational potential energy of the system is then:

$$E[\Omega] = \frac{1}{2}\int_\Omega \rho U(x)\,\mathrm{d}^3x = -\frac{G}{2}\rho^2\int_\Omega\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x\,\mathrm{d}^3x'.$$

- We consider a domain $\Omega \subset \mathbb{R}^3$, i.e., a bounded open set of class $\mathcal{C}^2$, with boundary $\partial\Omega$.
- The volume of Ω is defined by:
$$V := \int_\Omega \mathrm{d}^3x$$
- We consider an infinitesimal normal deformation of the boundary of Ω, parametrized by $\varepsilon \in \mathbb{R}$, given by:
$$x \mapsto x + \varepsilon f(x)n(x), \quad \text{for } x \in \partial\Omega$$
where $f \in C^\infty(\partial\Omega)$ is a smooth function and $n(x)$ is the outward unit normal vector to $\partial\Omega$.
- The deformed domain is denoted Ω_ε, the bounded open set obtained by this deformation:
$$\Omega_\varepsilon := \{x + \varepsilon f(x)n(x) \mid x \in \Omega\} + o(\varepsilon)$$
(The deformation is assumed to be smoothly extended inside Ω to rigorously define Ω_ε.)

3.7.1 Hadamard's Formula

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^1$ function, and Ω_ε a smooth deformation of Ω such that for $x \in \partial\Omega$,

$$x \mapsto x + \varepsilon f(x)n(x)$$

and assume this deformation extends smoothly to all of Ω.

We want to prove that:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\int_{\Omega_\varepsilon} F(x)\,\mathrm{d}^3x = \int_{\partial\Omega} F(x)f(x)\,\mathrm{d}S(x)$$

where $\mathrm{d}S$ is the surface element associated to $\partial\Omega$.

1. We will study the function $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$.
   (a) Justify that $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$, $M \mapsto \det M$ is differentiable.
   (b) Prove that for all $M \in \mathcal{M}_n(\mathbb{R})$,
$$\det(I + \varepsilon M) = 1 + \varepsilon\,\mathrm{Tr}(M) + o(\varepsilon)$$

Deduce that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\det(I+\varepsilon M)\big|_{\varepsilon=0} = \mathrm{Tr}(M)$.

(c) Let $X \in \mathrm{GL}_n(\mathbb{R})$, $H \in \mathcal{M}_n(\mathbb{R})$. Prove that

$$\mathrm{d}(\det)(X)(H) = \mathrm{Tr}({}^t\,\mathrm{Com}(X)\,H)$$

2. Set the change of variables $x(u) = u + \varepsilon f(u)n(u)$, and compute the Jacobian $\det\left(\frac{\partial x}{\partial u}\right)$ at first order in ε, i.e., up to $o(\varepsilon)$.

3. Let $F : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$, $v : \mathbb{R}^n \to \mathbb{R}^n$, and $\varepsilon \in U$ a neighborhood of 0. By considering a well-chosen function, prove that for all $x \in \mathbb{R}^n$,

$$F(x+\varepsilon v(x)) = F(x) + \varepsilon v(x)\cdot\nabla F(x) + o(\varepsilon)$$

4. Deduce the desired result using the Divergence Theorem.

3.7.2 Connection with the Gravitational Potential

1. Show that $E[\Omega] < 0$, and that minimizing the energy is equivalent to maximizing the following quantity:

$$\mathcal{I}[\Omega] := \int_\Omega\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x\,\mathrm{d}^3x'.$$

2. Suppose $\Omega = B(0,R)$ is a ball centered at the origin of radius R such that $\mathrm{Vol}(\Omega) = \frac{4}{3}\pi R^3 = V$. Show that the gravitational potential at the center is given by:

$$U(0) = -G\rho\int_\Omega \frac{1}{|x'|}\,\mathrm{d}^3x'.$$

Calculate this integral explicitly.

3.7.3 The Sphere?

1. Prove that

$$\delta\mathcal{F} = 2\int_{\partial\Omega}\left(\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x'\right) f(x)\,\mathrm{d}S(x)$$

You may use or prove (for the more courageous) that for all $\Omega\subset\mathbb{R}^n$, for all $\varphi:\Omega\to\mathbb{R}^n$,

$$\int_{\partial(\Omega^2)}\varphi(x)\,\mathrm{d}\mu(x) = 2\int_{\Omega\times\partial\Omega}\varphi(x)\,\mathrm{d}\mu(x)$$

2. We want to minimize $\mathcal{F}$ under fixed volume constraint V. To do this, consider the Lagrangian:

$$\mathcal{L}(\lambda) = \mathcal{F} - \lambda V, \quad \lambda\in\mathbb{R}.$$

Deduce that the first variation of $\mathcal{F}$ writes:

$$\delta\mathcal{L} = \int_{\partial\Omega}\left(2\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x' - \lambda\right) f(x)\,\mathrm{d}S(x).$$

3. Using spherical symmetry, show that if Ω is a ball of radius R, then for all $x\in\partial\Omega$, the quantity

$$\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x'$$

is constant. Deduce that the ball satisfies the **stationary condition** $\delta\mathcal{L} = 0$ for all f.

4. (*Bonus*) Show that the ball is indeed a *local minimum* for $\mathcal{F}$ under volume constraint by studying the second variation.

5. Conclude and explain why large objects in the Universe are spherical.

3.8 Relativistic Motion of a Charged Particle (SR, AM, EM, PS) ★★★★★

(Solution)

3.8.1 Relativistic Lagrangian of a Charged Particle in an Electromagnetic Field

1. Show that using the principle of least action and Lorentz invariance, the action of a free particle of mass m can be written as $S = -mc \int \mathrm{d}s$ where $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2$. Deduce that the Lagrangian of the system is
$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\,,$$
where $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$.

2. By introducing the electromagnetic four-potential $A^\mu = (\phi/c, \mathbf{A})$, propose an interaction term L_{int} corresponding to a particle of charge q in this field. Show that it can be written as
$$\mathcal{L}_{\text{int}} = q\,\mathbf{A}\cdot\mathbf{v} - q\phi\,,$$
and deduce the total Lagrangian $\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_{\text{int}}$.

3. Starting from the total Lagrangian, calculate the generalized momentum $P_i = \partial\mathcal{L}_{\text{tot}}/\partial v^i$. Show that it can be expressed as
$$\mathbf{p} = \gamma m\mathbf{v} + q\mathbf{A}\,,$$
where $\gamma = (1 - v^2/c^2)^{-1/2}$.

4. Write the Euler–Lagrange equations associated with L_{tot} and show that they lead to the Lorentz equation in 3 dimensions,
$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = q\big(\mathbf{E} + \mathbf{v}\times\mathbf{B}\big)\,,$$
with $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla\times\mathbf{A}$.

5. Express the Lagrangian by parameterizing with proper time τ and deduce that,
$$\mathcal{L} = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + qg_{\mu\nu}A^\mu\dot{x}^\nu$$

6. Give the covariant form of this equation of motion: show that we obtain
$$m\ddot{x}_\mu = qF_{\mu\nu}\dot{x}^\nu$$
Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor.

7. Explicitly write the components of the tensor $F_{\mu\nu}$ and show that $F_{0i} = E_i/c$ and $F_{ij} = -\varepsilon_{ijk}B_k$. Interpret the physical meaning of these components.

8. Calculate the two invariants of the electromagnetic field,
$$I_1 = F_{\mu\nu}F^{\mu\nu}, \quad I_2 = \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},$$
and express them in terms of $\mathbf{E}$ and $\mathbf{B}$. What are the physical cases corresponding to $I_1 = 0$ and $I_2 = 0$?

9. Verify that under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$, the equations of motion remain unchanged. What is the associated symmetry?

3.8.2 Equations of Motion of a Charged Particle in a Plane Electromagnetic Wave

Consider a particle of mass m and charge q subjected to an electromagnetic field described by the tensor $F^{\mu\nu}$. Its motion is governed by the equation:

$$m\ddot{x}^\mu = qF^{\mu\nu}\dot{x}_\nu$$

where the dots denote derivatives with respect to the particle's proper time τ. We use natural units where $c = 1$.

A plane electromagnetic wave is modeled by a four-potential of the form:

$$A^\mu(x) = a^\mu f(k_\nu x^\nu)$$

where f is a $\mathcal{C}^1$ function, k^μ is a lightlike four-vector, hence $k^\mu k_\mu = 0$, and a^μ is a constant four-vector representing the polarization.

1. Show that
$$F^{\mu\nu}(x) = (k^\mu a^\nu - k^\nu a^\mu)\, f'(k_\rho x^\rho)$$

2. (a) Calculate $\partial_\mu A^\mu$ for the potential $A^\mu(x) = a^\mu f(k_\rho x^\rho)$.
   (b) Deduce that the Lorenz gauge condition $\partial_\mu A^\mu = 0$ implies:
$$a^\mu k_\mu = 0$$

3. Now consider the motion of a particle in this electromagnetic wave.
   (a) Using the expression for the tensor $F^{\mu\nu}$ found in question 1, show that:
$$F^{\mu\nu}\dot{x}_\nu = [k^\mu(a_\rho \dot{x}^\rho) - a^\mu(k_\rho \dot{x}^\rho)]\, f'(k_\rho x^\rho)$$
   (b) Deduce the equation of motion in the form:
$$m\ddot{x}^\mu = q\, [k^\mu(a_\rho \dot{x}^\rho) - a^\mu(k_\rho \dot{x}^\rho)]\, f'(k_\rho x^\rho)$$

4. Now we seek to integrate this equation.
   (a) Show that the scalar $k_\rho \dot{x}^\rho$ is constant during the motion.
   (b) Deduce that $\phi = k_\rho x^\rho(\tau)$ is an affine function of τ, which can be used as a new parameter.
   (c) Using the previous relations, integrate the equation of motion and determine the complete expression for the trajectory $\tau \mapsto x^\mu(\tau)$[8]

3.8.3 Field Theory

We define the action,
$$S = \int_\Omega -\frac{1}{4\mu_0} F^{\mu\nu}F_{\mu\nu} + A^\mu j_\mu \mathrm{d}^4x, \quad \Omega \subset \mathbb{R}^{1,3}$$
We can thus easily define a Lagrangian **density**.

1. For an action depending on a field φ (scalar, tensor, etc.):
$$S = \int_\Omega \mathcal{L}(\varphi, \partial_\mu\varphi, x^\mu)\mathrm{d}^4x$$
   Prove that the Euler-Lagrange equations remain valid for a field φ.
   To do this, we will postulate the principle of least action, meaning that for an infinitesimal transformation $\varphi \mapsto \varphi + \varepsilon\eta$[9], we have,
$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon}[\varphi + \varepsilon\eta, \partial_\mu(\varphi + \varepsilon\eta), x^\mu](0) = 0$$

[8] This exercise allows us to analytically determine the trajectory of a charged particle in a plane electromagnetic wave. You can then plot it in Python using the obtained functions.

[9] Where η is a $\mathcal{C}^1(\Omega)$ function, and $\forall x \in \partial\Omega, \eta(x) = 0$, i.e., the function vanishes at the boundaries.

2. Derive Maxwell's equations in tensor form,

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0\,,$$

where $j^\mu = (c\rho, \mathbf{j})$ is the four-current (four-current density).

3.8.4 Trajectory of a Charged Particle in a Constant Magnetic Field

Consider a particle of mass m and charge q moving relativistically in an electromagnetic field. In this section, we gradually introduce the effects of a constant magnetic field $\mathbf{B} = B\,\mathbf{e}_z$ (curved sector of a synchrotron) and an average braking force due to synchrotron radiation.

A. Synchrotron Radiation Neglected

1. Calculate $F^{\mu\nu}$.
2. Deduce that the motion is in the Oxy plane. Show that the energy is constant if radiation losses are neglected.
3. Show that, in the absence of energy loss, for $u^\mu = (\gamma c, 0, u_0 = \gamma v, 0)$[10],

$$x(t) = R\cos\left(\frac{\omega}{\gamma}t\right), \qquad y(t) = R\sin\left(\frac{\omega}{\gamma}t\right)$$

With (synchrotron law)[11]:

$$R = \frac{\gamma v}{\omega} = \frac{\gamma m v}{qB}$$

B. Study of the Real Motion

1. Synchrotron radiation leads to an average energy loss. Recall the formula for the average radiated power (relativistic Larmor) for a centripetal acceleration $a = v^2/R$,

$$P = -\frac{\mathrm{d}}{\mathrm{d}t}E \;=\; \frac{q^2}{6\pi\varepsilon_0 c^3}\,\gamma^4 a^2$$

Using $E = \gamma m c^2$, show that by expanding, we obtain the differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma = -C\,\gamma^2 \times v$$

Give the expression for the coefficient C in terms of $q, B, m, c, \varepsilon_0$.

2. Solve the differential equation for v[12].
3. Deduce the new trajectory of the charged particle. Study the limit as $t \to \infty$.
4. Plot the parametric curve $x(t), y(t)$ in Python. What problem does this generate?

3.8.5 Physics of Relativistic Colliders

Here, we will use natural units where the speed of light $c = 1$.

1. Define the square of the total energy-momentum invariant $s = (p_1 + p_2)^2$ for the collision of two particles with four-momenta p_1 and p_2. Express the total energy available in the center-of-mass frame (CMS) in terms of s.

[10] It would also be necessary to show that γ and v are constant and that $\tau(t) = \gamma t$.
[11] For this, we will need to switch to the laboratory frame.
[12] It is indeed much simpler to solve the equation for v than for γ, since here v depends on time.

2. For a head-on collision of two identical particles of mass m and energy E (each) in the laboratory frame, show that the CMS energy is $\sqrt{s} = 2E$ (assuming $E \gg m$).

3. For the case of a collision with a fixed target of mass m, derive the formula

$$s = m^2 + m^2 + 2mE_{\text{lab}},$$

and deduce the threshold energy for the production of two particles of mass m (extreme elastic collision).

4. Calculate the energy required in a fixed-target experiment to produce a new particle of mass M at threshold, and compare it to the energy required in a symmetric collider ($E_{\text{CM}} = M + M$). Why are colliders with counter-propagating beams more efficient for reaching high energies?

3.9 Relativistic Hydrodynamics and Heavy-Ion Collisions (SR,PS) ★★★★★

(Solution)

3.9.1 Classical Hydrodynamics

1. Write the mass conservation equation (continuity) for a classical fluid:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

Show that for an incompressible fluid ($\rho =$ constant), this reduces to $\nabla \cdot \mathbf{v} = 0$.

2. Write Euler's equation for a perfect (inviscid) fluid subjected to a gravitational field $\mathbf{g}$:

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\Big) = -\nabla p + \rho \mathbf{g}.$$

Briefly describe the physical meaning of each term in this equation.

3. Show how adding viscous effects leads to the Navier–Stokes equation:

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\Big) = -\nabla p + \eta \nabla^2 \mathbf{v} + \Big(\zeta + \frac{\eta}{3}\Big)\nabla(\nabla \cdot \mathbf{v}) + \rho \mathbf{g},$$

where η is the dynamic (shear) viscosity and ζ the bulk viscosity. Explain the role of these terms.

4. Explain the difference between the Lagrangian description (tracking fluid particle trajectories) and the Eulerian description (velocity field at a fixed point in space). In particular, show that the total derivative for a fluid is $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ in the Eulerian formalism.

5. Define streamlines in a fluid and show that these curves are tangent to the velocity field vector $\mathbf{v}$ at every point. Interpret the physical meaning of these lines.

6. Prove Bernoulli's theorem for a stationary, incompressible, and inviscid fluid. Show that along a streamline,

$$\frac{1}{2}\rho v^2 + p + \rho\Phi = \text{constant},$$

where Φ is a force potential (e.g., $\Phi = gz$ for a constant gravitational field $\mathbf{g}$).

3.9.2 Introduction to Relativistic Hydrodynamics

Relativistic hydrodynamics describes the evolution of continuous systems with high energy density (such as quark-gluon plasma) by incorporating the principles of special relativity. Here, we focus on perfect fluids (no viscosity or thermal conduction) and their covariant description.

1. **Energy-momentum tensor.** The energetic and dynamic content of a perfect fluid is encoded in the energy-momentum tensor:

$$T^{\mu\nu} = (\varepsilon + p)\,\frac{u^\mu u^\nu}{c^2} - p\, g^{\mu\nu},$$

where:

- ε is the energy density (in the fluid's rest frame),
- p is the pressure (same units as ε, i.e., J/m^3),
- u^μ is the fluid's four-velocity,
- $\eta^{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric.

(a) Verify that $T^{\mu\nu}$ is symmetric.

(b) Calculate $T^{\mu\nu}$ in the fluid's rest frame ($u^\mu = (c, 0, 0, 0)$).

(c) Interpret the physical components T^{00}, T^{0i}, and T^{ij}.

(d) Show that the trace $T^\mu_{\ \mu} = \varepsilon - 3p$.

2. **Energy and momentum conservation.** In any isolated system, the energy-momentum tensor is locally conserved:
$$\partial_\mu T^{\mu\nu} = 0.$$
This tensor equation (4 scalar equations) expresses energy conservation ($\nu = 0$) and momentum conservation ($\nu = 1, 2, 3$). It is the fundamental equation of relativistic hydrodynamics.

   (a) What are the dynamical unknowns of the problem?

   (b) Why is it necessary to supplement this system with an equation of state relating ε, p, and possibly T?

3. **Relativistic thermodynamics.** In the fluid's rest frame, we locally define:
$$T : \text{temperature}, s : \text{entropy density}, \mu : \text{chemical potential}, n : \text{particle number density}.$$
The first law of thermodynamics, expressed in local densities (i.e., in a volume element $\mathrm{d}V$), takes the form:
$$\mathrm{d}\varepsilon = T\,\mathrm{d}s + \mu\,\mathrm{d}n.$$

   (a) Assuming $\mu = 0$, show that $\mathrm{d}p = s\,\mathrm{d}T$.

   (b) Deduce the identity $\varepsilon + p = Ts$, called the Euler relation.

4. **Relativistic speed of sound.** The speed of sound c_s is defined by:
$$c_s^2 = \left(\frac{\partial p}{\partial \varepsilon}\right)_s.$$

   (a) Calculate c_s for an ultra-relativistic fluid where $p = \varepsilon/3$.

   (b) Compare it to the speed of light c and comment.

3.9.3 Relativistic Hydrodynamics

Consider a perfect fluid in special relativity. The total number of particles is given by
$$N = \int_\Sigma J^\mu\,\mathrm{d}\Sigma_\mu,$$
across a spacelike hypersurface Σ oriented toward the future (e.g., $t = \text{const}$). We assume N is conserved.

1. Show that particle number conservation is locally expressed as
$$\partial_\mu(nu^\mu) = 0,$$
where n is the particle density in the comoving frame, and u^μ is the fluid's four-velocity.

2. Show that for $u^\mu = (c, 0, 0, 0)$, we have $T^{00} = \varepsilon$ and $T^{ii} = p$. Interpret.

3. Using $\partial_\mu T^{\mu\nu} = 0$, derive the equation of motion (or *relativistic Euler equation*) for a perfect fluid without sources:
$$(\varepsilon + p)u^\mu \partial_\mu u^\nu + \left(g^{\mu\nu} + \frac{u^\mu u^\nu}{c^2}\right)\partial_\mu p = 0.$$

3.9.4 Application to Heavy-Ion Collisions

1. Describe the scenario of a central heavy-ion collision at RHIC or LHC: formation of a quark-gluon plasma (QGP), thermalization, hydrodynamic expansion, freeze-out[13].

2. Introduce Bjorken coordinates: $\tau = \sqrt{t^2 - z^2}$, $\eta = \frac{1}{2} \ln \frac{t+z}{t-z}$. Assuming boost-invariant flow, show that $\partial_\mu T^{\mu\nu} = 0$ leads to:
$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} + \frac{\varepsilon + p}{\tau} = 0.$$

3. For $p = \varepsilon/3$, solve the above equation and deduce:
$$\varepsilon(\tau) \propto \tau^{-4/3}, \quad T(\tau) \propto \tau^{-1/3}.$$

4. During the QGP $\to$ hadrons transition, the equation of state can be written:
$$p = \frac{\varepsilon - 4B}{3}.$$
Show that $p = 0$ at the transition implies $\varepsilon = 4B$ and deduce the critical temperature T_c.

5. Modeling a nucleus as a sphere of radius R, define the geometric cross-section $\sigma \approx \pi(2R)^2$. Relate this quantity to the distinction between central and peripheral collisions.

6. Show that the initial energy density ε_0 is larger for a central collision. Assuming $\varepsilon = aT^4$, estimate the initial temperature T_0 reached at RHIC ($\varepsilon_0 \sim 10$ GeV/fm^3).

7. Define the viscosity-to-entropy ratio η/s. Why do values close to $1/4\pi$ indicate a nearly perfect fluid? What is the effect of low viscosity on the elliptic flow v_2?

8. Explain the concepts of chemical freeze-out (inelastic reactions frozen) and kinetic freeze-out (elastic reactions frozen). Why does hydrodynamics cease to be valid at this stage?

9. How does relativistic hydrodynamics connect measured observables (transverse momentum spectrum, anisotropies, etc.) to the initial state of the QGP?

[13]The term "relativistic fluid" refers to any fluid whose constituents have kinetic energies comparable to their mass: $k_B T \gtrsim mc^2$. This can be a plasma (charged), but also a photon or neutrino gas. Thus, relativistic hydrodynamics is more general than plasma physics.

3.10 Hydrogen Atom and Radial Equation (QM) ★★★

(Solution)

In this problem, we study the hydrogen atom (an electron of mass m_e in the Coulomb potential $V(r) = -e^2/r$ of a fixed proton) in non-relativistic quantum mechanics. We use spherical coordinates (r, θ, ϕ) and the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m_e}\left[\frac{1}{r^2}\partial_r\left(r^2\partial_r\right) - \frac{\mathbf{L}^2}{\hbar^2 r^2}\right]\psi(r,\theta,\phi) - \frac{e^2}{r}\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi),$$

where $\mathbf{L}^2$ is the orbital angular momentum operator.

3.10.1 Separation of Variables and Radial Equation

1. Show that the wavefunction can be separated as $\psi(r,\theta,\phi) = R(r)\,Y_{\ell m}(\theta,\phi)$, where $Y_{\ell m}$ is a spherical harmonic eigenfunction of $\mathbf{L}^2$ and $\mathbf{L}_z$, with:

$$\mathbf{L}^2 Y_{\ell m} = \hbar^2\ell(\ell+1)\,Y_{\ell m}, \qquad \mathbf{L}_z Y_{\ell m} = \hbar m\,Y_{\ell m}.$$

   Deduce that the radial Schrödinger equation for $R(r)$ is:

$$-\frac{\hbar^2}{2m_e}\left[\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}R\right] - \frac{e^2}{r}R = ER.$$

2. Let $u(r) = rR(r)$. Show that the equation becomes:

$$-\frac{\hbar^2}{2m_e}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2m_e r^2} - \frac{e^2}{r}\right]u(r) = Eu(r).$$

   Define the parameter κ as:

$$\kappa = \sqrt{\frac{2m_e|E|}{\hbar^2}}.$$

   Show that introducing the dimensionless variable $\rho = \kappa r$, the equation takes the form:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2} = \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{\rho_0}{\rho} + 1\right]u(\rho),$$

   where $\rho_0 = \dfrac{m_e e^2}{\hbar^2\kappa}$.

3. Propose the ansatz:

$$u(\rho) = \rho^{\ell+1}e^{-\rho/2}v(\rho),$$

   and show that $v(\rho)$ satisfies the differential equation[14]:

$$\rho\frac{\mathrm{d}^2 v}{\mathrm{d}\rho^2} + (2\ell + 2 - \rho)\frac{\mathrm{d}v}{\mathrm{d}\rho} + (\rho_0 - 2\ell - 2)v = 0.$$

4. Expanding $v(\rho) = \sum_{k=0}^{\infty} c_k\rho^k$, show that the series generally diverges at infinity unless it terminates at a finite order. Deduce that the termination condition is:

$$\rho_0 = 2n, \qquad \text{where } n = \hat{k} + \ell + 1 \in \mathbb{N}^*.$$

5. Derive the expression for the bound energy levels of the hydrogen atom:

$$\kappa_n = \frac{m_e e^2}{\hbar^2}\cdot\frac{1}{2n} \quad\Rightarrow\quad E_n = -\frac{\hbar^2\kappa_n^2}{2m_e} = -\frac{m_e e^4}{2\hbar^2}\cdot\frac{1}{n^2}.$$

[14] This is a confluent hypergeometric equation.

6. What is the degeneracy of each energy level E_n? Show that it is n^2 by considering the possible values of ℓ (from 0 to $n-1$) and m (from $-\ell$ to $+\ell$). Explain why, in this non-relativistic model, the energy depends only on n and not on ℓ.

3.10.2 Ground State ($n = 1$) and Radial Properties

7. For the ground state ($n = 1$, $\ell = 0$), show that the normalized radial wavefunction is:

$$R_{1,0}(r) = \frac{2}{a_0^{3/2}}\, e^{-r/a_0}.$$

Deduce the full expression for $\psi_{1,0,0}(r, \theta, \phi)$ and verify its normalization $\int |\psi_{1,0,0}|^2 \mathrm{d}^3x = 1$ (note that $Y_0^0 = 1/\sqrt{4\pi}$).

8. Calculate the radial probability density $P(r) = 4\pi |R_{1,0}(r)|^2 r^2$ and sketch its qualitative profile as a function of r. Interpret the physical meaning of this density (most probable location of the electron).

9. Show that the expectation value of the distance $\langle r \rangle$ between the electron and the nucleus, as well as the variance $(\Delta r)^2$, are given by:

$$\langle r \rangle = \frac{3}{2} a_0, \qquad (\Delta r)^2 = \langle r^2 \rangle - \langle r \rangle^2 = \frac{3}{2} a_0^2 - \Big(\frac{3}{2} a_0\Big)^2.$$

(Hint: Use the integral $\int_0^\infty r^n e^{-2r/a_0} dr = n!(a_0/2)^{n+1}$ and verify the results.)

10. (Optional) Introduce the momentum representation. Compute the Fourier transform $\tilde{\psi}_{1,0,0}(\mathbf{p})$ of the ground state and interpret the associated momentum distribution (square modulus). What are the expectation values of the momentum $\langle \mathbf{p} \rangle$ and its square $\langle p^2 \rangle$?

11. *Interpretation*: Briefly discuss how the $1/n^2$ dependence of the energy levels E_n explains the fine structure of hydrogen spectral lines and the concept of the principal quantum number.

3.11 Toward a Relativistic Formalism (QM, SR) ★★★★★

(Solution)
We use the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and natural units $c = \hbar = 1$.

1. Consider a real scalar field $\phi(x)$ of mass m. Define the relativistic Lagrangian density:
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$
Show that $\mathcal{L}$ is invariant under Lorentz transformations.
2. Apply the Euler–Lagrange equations to the scalar field and derive the Klein–Gordon equation $(\Box + m^2)\phi = 0$.
3. Discuss the Klein–Gordon equation, recalling why a second-order equation poses interpretational challenges for a relativistic field (e.g., probability density issues).
4. Motivated by this difficulty, we seek a first-order relativistic wave equation (symmetric in time and space) for a multicomponent object $\psi(x)$ (a spinor). State the general form of such a linear equation in ∂_μ (e.g., $(iA^\mu\partial_\mu - m)\psi = 0$ with matrices A^μ).
5. We propose the Dirac Lagrangian for a 4-component spinor field $\psi_\alpha(x)$ and its Dirac conjugate $\bar{\psi} = \psi^\dagger\gamma^0$:
$$\mathcal{L}_D = \bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi.$$
   (a) Show that $\mathcal{L}_D$ is Hermitian (up to a total derivative).
   (b) Verify that $\mathcal{L}_D$ is invariant under Lorentz transformations (spinorial transformations of ψ).
6. Apply the Euler–Lagrange equations for multicomponent fields (varying with respect to $\bar{\psi}$) to derive the Dirac equation:
$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$
Using the definition of $\bar{\psi}$, write the adjoint equation satisfied by $\bar{\psi}$.
7. (a) Multiply the Dirac equation from the left by $(i\gamma^\nu\partial_\nu + m)$ and show that ψ satisfies the Klein–Gordon equation. Explain why the γ^μ matrices must satisfy anticommutation relations.
   (b) Explicitly derive the Dirac matrix anticommutation relations:
$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}.$$
8. (a) Give explicit representations of the γ^μ matrices in the Dirac basis (e.g., $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$).
   (b) Verify the Clifford algebra relations derived above for $\mu, \nu = 0, 1, 2, 3$.
9. We seek plane-wave solutions to the Dirac equation of the form:
$$\psi(x) = u(p)\,e^{-ip\cdot x},$$
where $p^\mu = (E, \mathbf{p})$, $p \cdot x = p_\mu x^\mu = -Et + \mathbf{p}\cdot\mathbf{x}$.
   (a) Show that $u(p)$ satisfies the algebraic equation:
$$(\gamma^\mu p_\mu - m)\,u(p) = 0.$$
   (b) Deduce the allowed values of E and interpret the solutions as positive/negative energy states.
   (c) How many linearly independent solutions (spinors u and v) exist for a given momentum? Relate this to the spin degrees of freedom of the particle and its antiparticle.

10. Define the Dirac current:
$$j^\mu = \bar{\psi}\,\gamma^\mu \psi.$$
    (a) Using the Dirac equation (and its adjoint), show that $\partial_\mu j^\mu = 0$ (current conservation).
    (b) Verify that the density $j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi$ is positive definite. Interpret j^0 as a probability density and compare with the Klein–Gordon case.

11. We now quantize the Dirac field.
    (a) State the canonical quantization conditions for the Dirac field ψ and $\psi^\dagger$ (or ψ and $\bar{\psi}$). Justify why anticommutators $\{\,,\,\}$ must be used instead of commutators.
    (b) Explain how quantization leads to the introduction of creation/annihilation operators for particles and antiparticles (Dirac's interpretation of negative-energy states as antiparticles).

12. Show that the Dirac Lagrangian $\mathcal{L}_D$ is invariant under the global phase transformation $\psi \to e^{i\alpha}\psi$. Using Noether's theorem, deduce the conservation of the current j^μ.

13. Introduce minimal coupling of the Dirac field to an electromagnetic potential A_μ by replacing ∂_μ with $D_\mu = \partial_\mu + ieA_\mu$ in $\mathcal{L}_D$. Show that this prescription makes the Lagrangian invariant under local gauge transformations $\psi \to e^{ie\Lambda(x)}\psi$, $A_\mu \to A_\mu - \partial_\mu\Lambda$. Briefly discuss the physical implications (introduction of electromagnetic interactions and charge conservation).

14. Summarize the physical implications of the Dirac equation: existence of a free spin-$1/2$ particle (and its antiparticle), prediction of the magnetic moment $\mu = g\frac{q}{2m}\mathbf{S}, \mu$ vector, with $g = 2$, etc. Conclude by explaining how this construction respects relativistic covariance and the role of Clifford algebra in describing relativistic fermions.

3.12 Pöschl–Teller Potential $V(x) = -\dfrac{V_0}{\cosh^2(\alpha x)}$ (QM) ★★★

(Solution)
We study a quantum system subject only to the potential $V(x) = -\dfrac{V_0}{\cosh^2(\alpha x)}$.
The system's Hamiltonian is written as

$$\mathbf{H} = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X}) = \frac{\mathbf{P}^2}{2m} - \frac{V_0}{\cosh^2(\alpha \mathbf{X})}$$

1. Write the time-independent Schrödinger equation for a wavefunction $\psi(x)$:

$$-\frac{\hbar^2}{2m}\psi''(x) - \frac{V_0}{\cosh^2(\alpha x)}\,\psi(x) = E\,\psi(x).$$

2. Show that the substitution $u = \tanh(\alpha x)$ leads to

$$\psi'(x) = \alpha\,(1-u^2)\,\frac{\mathrm{d}\phi}{\mathrm{d}u}, \quad \psi''(x) = \alpha^2\Big((1-u^2)\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} - 2u\frac{\mathrm{d}\phi}{\mathrm{d}u}\Big),$$

   with $\phi(u) = \psi(x(u))$.

3. Deduce that the equation for $u \in (-1, 1)$ becomes

$$(1-u^2)\,\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} - 2u\,\frac{\mathrm{d}\phi}{\mathrm{d}u} + \Big[\lambda(\lambda+1) - \frac{\mu^2}{1-u^2}\Big]\phi = 0,$$

   and express λ, μ in terms of $V_0, \alpha, m, \hbar, E$.

4. Identify λ, μ. Seek a solution of the form $\phi(u) = (1-u^2)^{\frac{\mu}{2}}\,P(u)$. Show that P must be a polynomial.

5. Deduce the energy quantization E_n in the form

$$E_n = -\frac{\hbar^2\alpha^2}{2m}\,(\lambda - n)^2, \quad n = 0, 1, \ldots, \lfloor\lambda\rfloor,$$

   where $\lambda(\lambda+1) = \frac{2mV_0}{\hbar^2\alpha^2}$.

6. Show that the number of bound states is finite: $N = \lfloor\lambda\rfloor + 1$.

7. Provide a physical interpretation of why only these N levels can exist.

3.13 Larmor Power and Electrodynamic Instability of the Classical Atom† (EM) ★★

[15] (Solution)
A confined (and thus accelerated) charge emits electromagnetic radiation. We now examine in more detail some consequences of classical Electromagnetic laws combined with those of dynamics (hence: *Electrodynamics*), and in particular, show that the classical atom is fundamentally unstable: the electron localized in the atom emits radiation and, as a result, gradually loses energy.

The description below relies on the assumption that the radiation effect is a minor phenomenon, although it ultimately leads to dramatic conclusions. We will therefore start with an ordinary dynamical description, to which we will add the perturbative effects of the source's (the confined electron's) radiation on its own motion.

3.13.1 Calculation of the radiation reaction force $\mathbf{F}_{\text{rad}}$.

The Larmor power is the power lost by an accelerated charge. We will deduce a radiation reaction force $\mathbf{F}_{\text{rad}}$ from it, which leads to dramatic consequences.

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{2q^2 a^2}{3c^3} \tag{3.13.1}$$

1. Write the work $\mathrm{d}E_{\text{at}} = \mathrm{d}W$, equal to the change in atomic energy over a time $\mathrm{d}t$ due to the radiation force $\mathbf{F}_{\text{rad}}$.
2. Write the energy variation of the atom over a time $\mathrm{d}t$ due to the radiated power of the electron.
3. By integrating by parts and assuming periodic motion, show that[16],

$$\mathbf{F}_{\text{rad}} = \frac{2\vartheta^2}{3c^3}\ddot{\mathbf{v}} \tag{3.13.2}$$

4. Apply Newton's second law with the previously calculated $\mathbf{F}_{\text{rad}}$[17] and a restoring force $\mathbf{F} = -m\omega_0^2\mathbf{r}$. Seeking a solution of the form $\mathbf{r}(t) = \mathrm{Re}\{\mathbf{r}_0 e^{i\omega t}\}$, and letting

$$\omega = \omega_0(1 + \alpha(\omega_0\tau) + o(\omega_0\tau)), \alpha \in \mathbb{R} \tag{3.13.3}$$

show that the solution is a damped oscillator.
N.B. Given $\tau = \frac{2e^2}{3mc^3} \simeq 6.4 \times 10^{-24}$ s, $\omega_0 = 3 \times 10^{15}$ rad.s^{-1}. Comment.

3.13.2 Conceptual issues raised by the radiation reaction force $\mathbf{F}_{\text{rad}}$.

The radiation reaction force $\mathbf{F}_{\text{rad}}$ written above is conceptually pathological, as the following analysis shows. Using the notations of Section 1.5, Volume I, the Abraham-Lorentz equation for a particle of charge e and mass m subjected to a force $\mathbf{F}$ (with $\mathbf{v} = \dot{\mathbf{r}}$) is:

$$m\ddot{\mathbf{r}} = m\tau\,\dddot{\mathbf{r}} + \mathbf{F}, \tag{3.13.4}$$

where $\tau = \frac{2e^2}{3mc^3} \simeq 6.4 \times 10^{-24}$ s is a characteristic time. One oddity of this equation is the appearance of a third derivative of the particle's position (defined by the radius vector $\mathbf{r}$), which is meant to represent the radiation damping effect.

Moreover, the perturbation of motion caused by this effect is fundamentally *singular*, in the sense that it alters the order of the motion equation, which changes from second to third order as soon as the charge is nonzero. In fact, it is precisely because the small parameter τ multiplies the highest derivative that the perturbation is called *singular*, by definition[18].

[15]† Inspired by Claude Aslangul, *Quantum Mechanics 1*, Chapter 1.
[16]Where $\vartheta^2 = \frac{e^2}{4\pi\varepsilon_0}$
[17]Note the appearance of a force depending on the derivative of the acceleration. We will study in the next part the issues caused by this force.
[18]The same phenomenon occurs in the Schrödinger eigenvalue equation, where it is Planck's constant that multiplies the highest derivative. A specific perturbation technique is used for such problems, known as the WKB (or BKW) method in the quantum context.

With these warnings in mind, we now examine the consequences of equation (3.13.4) as it stands, to highlight the deep conceptual issues it poses.

1. Using the standard method for solving a differential equation like (3.13.4), write the general expression for the acceleration $\ddot{\mathbf{r}}(t)$, assuming the acceleration at some instant t_0, $\ddot{\mathbf{r}}(t_0)$, is known.
2. Examine the particular case $\mathbf{F} = 0$, and show that the solution is physically aberrant.
3. Returning to the general solution obtained in 1 for $\mathbf{F} \neq 0$, show that the divergent solutions can formally be eliminated by a suitable choice of t_0. Comment on this choice — which, from a technical standpoint, expresses a boundary condition rather than an initial condition.
4. Deduce the regularized expression of the solution obtained in 1. Stepping back, analyze the integral kernel in this expression and verify that, in the limit of zero charge, the motion equation reduces to the standard dynamical equation.
5. To clearly exhibit the violation of a major physical principle, make a simple change of integration variable to obtain:
$$\dot{\mathbf{v}}(t) = \frac{1}{m} \int_0^{+\infty} e^{-s} \times \mathbf{F}(t + \tau s)\, \mathrm{d}s. \tag{3.13.5}$$
Comment on this equation and show that it violates a physical principle.
6. To highlight this violation even more spectacularly, treat the case of a particle with zero velocity at $t = -\infty$ and subjected to a step force:
$$\mathbf{F}(t) = \begin{cases} 0 & \text{if } t < 0, \\ \mathbf{F}_0 & \text{if } t > 0. \end{cases} \tag{3.13.6}$$
Summarize these results by plotting the time evolution of the acceleration and velocity. Note that the particle starts moving... **before the force is applied**!

Chapter 4

Exercise Solutions

As you may notice, not **all** exercises have been solved yet. Unsolved exercises are marked with the symbol $\triangle$. The remaining solutions will be added progressively. If you would like to submit a solution, please send it to the following email address in LaTeX format:
ryanartero2005@gmail.com.

Moreover, you can return to the exercise you were working on by clicking on its title, either at the top of the page or at the beginning of the exercise.

4.1 Two-Body Problem

4.1.1 Center of Mass

We denote by $\mathbf{r}_1, \mathbf{r}_2$ the position vectors of the electron and the nucleus with respect to an arbitrary reference frame, and by $\mathbf{v}_1, \mathbf{v}_2$ the corresponding velocities.

1. $\mathcal{L} = \frac{1}{2}(m_1\mathbf{v}_1^2 + m_2\mathbf{v}_2^2) - \frac{\vartheta^2}{\|\mathbf{r}_1 - \mathbf{r}_2\|}$.
2. $$\begin{aligned} \mathbf{R} &= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \implies \mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \\ \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \implies \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \\ \mu &= \frac{m_1 m_2}{m_1 + m_2} \\ \implies \mathcal{L} &= \frac{1}{2}(m_1 + m_2)\mathbf{V}^2 + \frac{1}{2}\mu\mathbf{v}^2 - \frac{\vartheta^2}{r} = \mathcal{L}_G(\mathbf{V}) + \mathcal{L}_r(\mathbf{r}, \mathbf{v}) \end{aligned}$$
3. The potential is central for the center of mass. This implies that **J** is a conserved quantity.

In the following, we focus exclusively on the internal motion described by $\mathcal{L}_r$ in polar coordinates (r, θ) in the plane perpendicular to **J**.

4.1.2 Integration of the Equations of Motion

1. The expression for the kinetic energy in polar coordinates in $\mathbb{R}^2$ is:

$$\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2),$$

which gives the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}, \quad \text{with } k = \vartheta^2.$$

The Euler-Lagrange equations are:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu\dot{r}) - \mu r\dot{\theta}^2 + \frac{k}{r^2} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu r^2\dot{\theta}\right) = 0.$$

Conjugate momenta are given by:

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = \mu\dot{r}, \qquad p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \mu r^2\dot{\theta}.$$

The Hamiltonian reads:

$$H = p_r\dot{r} + p_\theta\dot{\theta} - \mathcal{L} = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} - \frac{k}{r}.$$

Hamilton's equations are then:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu}, \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{\mu r^3} - \frac{k}{r^2},$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2}, \qquad \dot{p}_\theta = -\frac{\partial H}{\partial\theta} = 0.$$

p_θ is a conserved quantity (since θ is a cyclic variable); thus, $p_\theta = \mu r^2\dot{\theta}$ is constant — the angular momentum J, fixed by the initial conditions.

Indeed, $\mathbf{J} = \mu\mathbf{r}\times\dot{\mathbf{r}} = \mu r\mathbf{u}_r \times (\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta) = \mu r^2\dot{\theta} = p_\theta$.

The first integral of energy is:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} - \frac{k}{r}.$$

By differentiating $p_r = \mu\dot{r}$ and substituting:

$$\dot{p}_r = \mu\ddot{r} = \frac{J^2}{\mu r^3} - \frac{k}{r^2},$$

we recover the radial equation of motion:

$$\mu\ddot{r} = \frac{J^2}{\mu r^3} - \frac{k}{r^2}. \tag{7.25}$$

The first term on the right-hand side is the centrifugal force, the second is the attractive Coulomb force.

2. To eliminate time, we differentiate the composite function $r(\theta(t))$:
Let $r'(\theta) = \frac{\mathrm{d}r}{\mathrm{d}\theta}$ and $r''(\theta) = \frac{\mathrm{d}^2r}{\mathrm{d}\theta^2}$. Using $p_\theta = \mu r^2\dot{\theta} = J$, we get:

$$\dot{\theta} = \frac{J}{\mu r^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{J}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}.$$

Thus:

$$\dot{r} = r'\frac{J}{\mu r^2}, \qquad \ddot{r} = \frac{J}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(r'\frac{J}{\mu r^2}\right).$$

Setting $u = \frac{1}{r}$, we obtain:

$$\dot{r} = -\frac{J}{\mu}u', \qquad \ddot{r} = -\frac{J^2}{\mu^2}(u'' + u),$$

and substitution into (7.25) gives:

$$-\frac{J^2}{\mu^2}(u''+u) = \frac{J^2}{\mu}u^3 - \frac{k}{\mu}u^2.$$

Multiplying both sides by $-\frac{\mu^2}{J^2}$ yields:

$$u''+u = \frac{\mu k}{J^2}.$$

3. The differential equation in $u(\theta)$:

$$u''+u = \frac{\mu k}{J^2}$$

has the general solution:

$$u(\theta) = A\cos(\theta+\varphi) + \frac{\mu k}{J^2},$$

hence:

$$r(\theta) = \frac{1}{A\cos(\theta+\varphi)+\frac{\mu k}{J^2}}.$$

One can always choose the polar axis so that $r(\theta)$ is extremal at $\theta = 0$ (or π), which gives $\varphi = 0$:

$$r(\theta) = \frac{1}{\frac{\mu k}{J^2}(1+\varepsilon\cos\theta)}, \tag{7.26}$$

where $\varepsilon = \dfrac{AJ^2}{\mu k}$ is the eccentricity.

The constant A (or ε) is determined by the initial conditions or by the energy:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} - \frac{k}{r}.$$

Using $r(\theta)$ and $J = \mu r^2\dot{\theta}$, one can express E as a function of ε:

$$\varepsilon^2 = 1 + \frac{2EJ^2}{\mu k^2}. \tag{7.27}$$

4. Equation (7.26) defines a family of curves called **conic sections** (intersections of a cone with a plane). Three subfamilies are distinguished according to the value of ε:

- If $\varepsilon < 1$, the trajectory is an **ellipse**, closed, corresponding to energy $E < 0$: a bound and periodic motion (in particular, $\varepsilon = 0$ gives a circle).
- If $\varepsilon = 1$, the trajectory is a **parabola**: a limiting case $E = 0$ separating bound and unbound motions.
- If $\varepsilon > 1$, the denominator in (7.26) can vanish for some angle $\theta_\infty = \arccos\left(-\frac{1}{\varepsilon}\right)$: the trajectory is a (open) **hyperbola** with asymptotes; $E > 0$ corresponds to a particle arriving from infinity with nonzero initial velocity.

In all cases, the origin (center of force) is one of the foci of the conic.

4.1.3 Bohr Quantization

In this section, we consider only bound states with $E < 0$.

1. The quantization condition on the angle θ is immediate since $p_\theta = J$ is a conserved quantity:

$$J_\theta = \int_0^{2\pi} p_\theta \,\mathrm{d}\theta = 2\pi J, \quad \text{which implies} \quad J = n_\theta \hbar \quad \text{with } n_\theta \in \mathbb{N}^*.$$

n_θ cannot be zero, as this would correspond to a straight-line trajectory periodically crossing the nucleus. Ultimately:

$$J = n_\theta \hbar, \quad n_\theta \in \mathbb{N}^*.$$

2. We have:

$$\int p_r \,\mathrm{d}r = \int \mu \dot{r} \,\mathrm{d}r = \mu \int r'(\theta)\dot{\theta} \,\mathrm{d}r = \mu \int r'(\theta) \frac{J}{\mu r^2} \,\mathrm{d}r = \int \frac{J r'(\theta)}{r^2} \,\mathrm{d}r = \int \frac{J}{r^2} \frac{\mathrm{d}r}{\mathrm{d}\theta} \,\mathrm{d}\theta.$$

The quantization condition, taking into account equation (7.26), becomes:

$$\int_0^{2\pi} \frac{J\varepsilon \sin\theta}{(1+\varepsilon\cos\theta)^2} \,\mathrm{d}\theta = n_r h.$$

The integral evaluates to:

$$2\pi J \left(\int_0^{2\pi} \frac{\varepsilon \sin\theta}{(1+\varepsilon\cos\theta)^2} \,\mathrm{d}\theta \right) = -2\pi J \left(\int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{1}{1+\varepsilon\cos\theta} \right) \mathrm{d}\theta \right).$$

Integration by parts yields:

$$\int_0^{2\pi} \left(\frac{1}{1+\varepsilon\cos\theta} - 1 \right) \mathrm{d}\theta = \int_0^{2\pi} \left(\frac{1-(1+\varepsilon\cos\theta)}{1+\varepsilon\cos\theta} \right) \mathrm{d}\theta = \int_0^{2\pi} \left(\frac{-\varepsilon\cos\theta}{1+\varepsilon\cos\theta} \right) \mathrm{d}\theta.$$

The quantization condition thus becomes:

$$2\pi J \left(\frac{1}{\sqrt{1-\varepsilon^2}} - 1 \right) = n_r h.$$

Since $2\pi J = n_\theta h$, this also implies:

$$n_\theta \left(\frac{1}{\sqrt{1-\varepsilon^2}} - 1 \right) = n_r,$$

which can be rewritten as:

$$\frac{1}{\sqrt{1-\varepsilon^2}} = \frac{n}{n_\theta}, \quad \text{where } n = n_r + n_\theta. \tag{7.28}$$

3. From equation (7.27):

$$1 - \varepsilon^2 = -\frac{2EJ^2}{\mu\vartheta^4}, \quad \text{so} \quad E = -\frac{\mu\vartheta^4}{2J^2}(1-\varepsilon^2).$$

Since $J = n_\theta \hbar$ and $1 - \varepsilon^2 = \left(\frac{n_\theta}{n}\right)^2$, we finally obtain:

$$E_n = -\frac{\mu\vartheta^4}{2\hbar^2 n^2}. \tag{7.29}$$

4.2 Rutherford Scattering Cross-Section

4.2.1 Deflection of a Charged Particle by an Atomic Nucleus

We work in a polar coordinate system (r, φ) in the plane of motion.

1. **Angular Momentum:** The angular momentum in polar coordinates is:

$$J = mr^2\dot{\varphi}.$$

At past infinity, the particle has speed v_0 and an impact parameter b. The angular momentum is then:

$$J = -mbv_0.$$

The negative sign comes from the fact that φ decreases with time.

2. **Equation of Motion:** The central repulsive force is given by:

$$\mathbf{F} = \frac{C}{r^2}\hat{\mathbf{r}}, \quad \text{where } C = \frac{qQ}{4\pi\varepsilon_0}.$$

We decompose $\mathbf{v} = \dot{\mathbf{r}}$ into two components. Projecting onto the direction perpendicular to the polar axis, we find:

$$m\dot{v}_\perp = \frac{C}{r^2}\sin\varphi.$$

3. **Deflection Angle θ:** Multiplying the equation by $\mathrm{d}t$ and changing variables, we use:

$$r^2\dot{\varphi} = \frac{J}{m} \Rightarrow \mathrm{d}t = \frac{mr^2}{J}\mathrm{d}\varphi.$$

Integrating from $t = -\infty$ to $t = +\infty$:

$$v_0 \sin\theta = \int \dot{v}_\perp \mathrm{d}t = \frac{C}{J}(\cos\theta + 1).$$

4. **Relation to Kinetic Energy:** The initial energy is $E_0 = \frac{1}{2}mv_0^2$, so:

$$\tan\left(\frac{\theta}{2}\right) = \frac{C}{2E_0 b}.$$

4.2.2 Rutherford Scattering Cross-Section

1. **Expression for the Differential Cross-Section:** The general definition is $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$.

2. **Using $\tan(\theta/2)$:** With:

$$b = \frac{C}{2E_0}\cot\left(\frac{\theta}{2}\right), \quad \frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{C}{4E_0}\frac{1}{\sin^2(\theta/2)}, \implies \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{C}{4E_0}\right)^2 \frac{1}{\sin^4(\theta/2)}.$$

3. **Limit of the Model:** For $\theta \to 0$, we have $\sin(\theta/2) \to 0$ so $\mathrm{d}\sigma/\mathrm{d}\Omega \to \infty$. The integral over $\theta \in [0, \pi]$ diverges: the total cross-section is infinite. This reflects the infinite range of the Coulomb interaction.

4. **Experimental Interpretation:** This model explains Rutherford's experimental results: alpha particles can be strongly deflected. This implies the existence of a highly concentrated atomic nucleus, as such deflection requires a very intense field in a very localized region[1].

[1] By introducing the minimum approach distance $a_{\min}$ for a head-on collision ($b = 0$), we have:

$$a_{\min} = \frac{C}{E_0}.$$

4.3 Cherenkov Effect

1. $$p = \frac{E}{c} = \frac{nh\nu}{c}, \quad p_z = \frac{c}{c}(nh\nu) = \frac{nh\nu}{c}, \quad n = \frac{p_z c}{h\nu}.$$

2. The components of momentum are:
$$p = p' \cos\varphi + p_z \cos\theta, \quad 0 = -p' \sin\varphi + p_z \sin\theta.$$

3. We have:
$$p_z^2 = p^2 - 2pp_z \cos\varphi + p_z^2.$$

4. Energy conservation reads:
$$\sqrt{p^2c^2 + m^2c^4} = \sqrt{p_z^2c^2 + m^2c^4} + h\nu,$$
or equivalently:
$$\frac{1}{\sqrt{1-\beta^2}} mc^2 = \frac{1}{\sqrt{1-\beta_f^2}} mc^2 + h\nu.$$

5. Squaring both sides, we obtain:
$$p_z^2 = p^2 - \frac{2h\nu E}{c^2} + \frac{(h\nu)^2}{c^2}, \quad \text{where } E \text{ denotes the initial energy of the electron.}$$

6. Comparing the two expressions for p_z^2, we can write:
$$p^2 - 2pp_z \cos\varphi + p_z^2 = p^2 - \frac{2h\nu E}{c^2} + \frac{(h\nu)^2}{c^2},$$
from which, after simplification, we get:
$$\cos\varphi = \frac{h\nu}{pc}\left(1 - \frac{E}{pc}\right) + \frac{h\nu}{2pc},$$
with $E = \gamma mc^2$, $p = \gamma mv$, $p_z = \frac{nh\nu}{c}$, so that:
$$\cos\theta = \frac{1}{n\beta}\left(1 - \frac{1}{2}\frac{1}{\gamma^2}\right).$$

7. Finally:
$$\cos\theta = \frac{1}{n\beta}\left[1 + (n^2-1)\frac{1}{2\gamma^2}\right].$$
Since $E = \gamma mc^2$, this can also be written as:
$$\cos\theta = \frac{1}{n\beta}\left[1 + \frac{n^2-1}{2}\left(1-\beta^2\right)\right].$$

8. We must have:
$$\frac{1}{n\beta}\left[1 + (n^2-1)\frac{1}{2\gamma^2}\right] \leq 1.$$
Since the bracketed term is clearly greater than 1, it is necessary (though not sufficient) that:
$$\beta > \frac{1}{n}.$$

The differential cross-section can then be rewritten as:
$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{a_{\min}^2}{16} \cdot \frac{1}{\sin^4(\theta/2)}}.$$

9. Photons are emitted between $v = 0$ and a frequency ν_{max} such that $\cos\theta = 1$, i.e.:

$$0 \leq \nu \leq \frac{E}{h}\left(1 - \frac{1}{n\beta}\right), \quad \text{with } E = \nu_{\text{max}} h.$$

10. The most energetic photons are emitted in the direction $\theta = 0$.

11. All photons are emitted within a cone of half-angle φ, corresponding to the angle θ for a photon of zero frequency:

$$\varphi = \arccos\left(\frac{1}{n\beta}\right) = \arccos\left(\frac{1}{n}\right) \simeq 20°.$$

12. For the effect to occur, one needs $\nu > \frac{1}{n}$, i.e. $\beta > \frac{1}{n}$, hence:

$$E > \frac{1}{\sqrt{1 - \frac{1}{n^2}}} mc^2.$$

For an electron, this means $E > 0.77$ MeV, and for a proton, $E > 1.4$ GeV.

4.4 Pulsed Magnetic Field Machine

4.4.1 Magnetic Field of the Coil

(a) For a circular loop of radius R, the Biot–Savart law gives the magnetic field along the z-axis:

$$B_z(z,t) = \frac{\mu_0 I(t) R^2}{2\left(z^2+R^2\right)^{3/2}}.$$

This is obtained by integrating over the loop, exploiting the circular symmetry.

(b) For $z \gg R$, we can approximate $\left(z^2+R^2\right)^{3/2} \simeq z^3$. Thus,

$$B_z(z,t) \sim \frac{\mu_0 I(t) R^2}{2z^3},$$

which is the expression for the field of a magnetic dipole of moment $m = I(t)R^2$.

4.4.2 Induced Electric Field in Biological Tissue

The local form of Faraday's law in cylindrical coordinates (assuming the induced electric field is purely azimuthal) is:

$$\left(\nabla \times \mathbf{E}\right)_z = \frac{1}{r}\frac{\partial(rE_\theta)}{\partial r} = -\frac{\partial B_z}{\partial t}.$$

Differentiate B_z with respect to time:

$$\frac{\partial B_z}{\partial t} = \frac{\mu_0 R^2}{2\left(z^2+R^2\right)^{3/2}}\,\dot{I}(t).$$

So the local equation becomes:

$$\frac{1}{r}\frac{\partial(rE_\theta)}{\partial r} = -\frac{\mu_0 R^2\,\dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}.$$

Integration for $r < R$: Integrate from 0 to r, imposing $E_\theta(0,t) = 0$ (to avoid a singularity):

$$\int_0^r \frac{\partial(r'E_\theta(r',t))}{\partial r'}\,\frac{\mathrm{d}r'}{r'} = -\frac{\mu_0 R^2\,\dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}\int_0^r \mathrm{d}r'.$$

This yields:

$$rE_\theta(r,t) = -\frac{\mu_0 R^2\,\dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}}\cdot\frac{r^2}{2},$$

and therefore:

$$E_\theta(r,t) = -\frac{\mu_0 R^2\,\dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}}\,r \quad \text{for } r \le R.$$

Integration for $r > R$: For $r > R$, since the magnetic flux remains confined within the coil region, it is more appropriate to use the integral form of Faraday's law. Consider a circular path of radius $r > R$. The integral form gives:

$$\oint \mathbf{E}\cdot d\boldsymbol{\ell} = 2\pi r\,E_\theta = -\frac{\mathrm{d}\Phi}{\mathrm{d}t},$$

where the flux Φ is that through the area of the coil:

$$\Phi = \pi R^2\,B_z(z,t) = \pi R^2\frac{\mu_0 I(t)R^2}{2\left(z^2+R^2\right)^{3/2}}.$$

The time derivative of Φ is:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \pi R^2\frac{\mu_0 R^2}{2\left(z^2+R^2\right)^{3/2}}\,\dot{I}(t).$$

Thus,

$$2\pi r\, E_\theta = -\pi \frac{\mu_0 R^4\, \dot{I}(t)}{2\left(z^2+R^2\right)^{3/2}},$$

and so for $r > R$:

$$E_\theta(r,t) = -\frac{\mu_0 R^4\, \dot{I}(t)}{4r\left(z^2+R^2\right)^{3/2}}.$$

Summary:

$$E_\theta(r,t) = \begin{cases} -\dfrac{\mu_0 R^2\, \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}}\, r, & r \le R, \\ -\dfrac{\mu_0 R^4\, \dot{I}(t)}{4r\left(z^2+R^2\right)^{3/2}}, & r \ge R. \end{cases}$$

Continuity check: At $r = R$, the inner solution gives

$$E_\theta(R,t) = -\frac{\mu_0 R^3\, \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}},$$

and the outer solution gives the exact same result. Continuity is thus ensured.

4.4.3 Effect on Motor Neurons

The induced voltage over a disk of radius a is given by:

$$V = \int_0^a E(r,t)\, dr.$$

Using the expression of $E_\theta(r,t)$ for $r \le R$ (assuming $a \le R$ for simplicity), we get:

$$V = -\frac{\mu_0 R^2\, \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}} \int_0^a r\, dr = -\frac{\mu_0 R^2\, \dot{I}(t)}{4\left(z^2+R^2\right)^{3/2}} \cdot \frac{a^2}{2}.$$

Therefore,

$$V = -\frac{\mu_0 R^2\, a^2\, \dot{I}(t)}{8\left(z^2+R^2\right)^{3/2}}.$$

To activate the neuron, the condition is:

$$|V| \ge V_{\text{threshold}}.$$

That is,

$$\left| -\frac{\mu_0 R^2 a^2\, \dot{I}(t)}{8\left(z^2+R^2\right)^{3/2}} \right| \ge V_{\text{threshold}}.$$

Taking the absolute value, this gives the condition:

$$\frac{\mu_0 R^2 a^2\, |\dot{I}(t)|}{8\left(z^2+R^2\right)^{3/2}} \ge V_{\text{threshold}}.$$

Or equivalently:

$$|\dot{I}(t)| \ge \frac{8\left(z^2+R^2\right)^{3/2} V_{\text{threshold}}}{\mu_0 R^2 a^2}.$$

This gives a **lower bound** on the current's time derivative necessary to trigger a **motor neuron activation** via the induced electric field.

4.4.4 Effect of Oscillating Current

Assuming that

$$I(t) = I_0 e^{i\omega t},$$

then $\dot{I}(t) = i\omega I_0 e^{i\omega t}$ and the induced electric field becomes oscillatory:

$$E_\theta(r,t) = E_\theta(r)\, e^{i\omega t}.$$

This behavior reflects the presence of electromagnetic waves in the system, with phases and amplitudes modulated by the frequency ω.

4.4.5 Effect of Pulsed Magnetic Field on Muscles

When the magnetic stimulation device delivers rapid pulses, the time variation of the magnetic field induces an electric field in the surrounding tissues. In muscles, this electric field can cause depolarization of cell membranes by activating ion channels, which generates an action potential. This excitation leads to involuntary muscle contraction, exploited in physiotherapy to improve muscle rehabilitation, increase blood circulation, and reduce pain.

4.5 Metric of a Sphere

1. Using that $\mathrm{d}(\cos u) = -\sin u\,\mathrm{d}u$ and $\mathrm{d}(\sin u) = \cos u\,\mathrm{d}u$, we get

$$\begin{aligned}
\frac{\mathrm{d}x^2}{R^2} &= [-\sin\theta\sin\varphi\mathrm{d}\varphi + \cos\theta\cos\varphi\mathrm{d}\theta]^2\\
&= (\sin\theta\sin\varphi\mathrm{d}\varphi)^2 - 2\sin\theta\sin\varphi\mathrm{d}\varphi\cos\theta\cos\varphi\mathrm{d}\theta + (\cos\theta\cos\varphi\mathrm{d}\theta)^2,\\
\frac{\mathrm{d}y^2}{R^2} &= [\sin\theta\cos\varphi\mathrm{d}\varphi + \cos\theta\sin\varphi\mathrm{d}\theta]^2\\
&= (\sin\theta\cos\varphi\mathrm{d}\varphi)^2 + 2\sin\theta\sin\varphi\mathrm{d}\varphi\cos\theta\cos\varphi\mathrm{d}\theta + (\cos\theta\sin\varphi\mathrm{d}\theta)^2,\\
\frac{\mathrm{d}z^2}{R^2} &= \sin^2\theta\mathrm{d}\theta^2.
\end{aligned}$$

Adding these terms and using $\cos^2 + \sin^2 = 1$, we obtain

$$\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{3.5.1}$$

2. From equation (3.5.1), factoring by $\mathrm{d}\theta^2$ inside the square root, we have

$$\begin{aligned}
\mathrm{d}s &= R\sqrt{\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2}\\
&= R\sqrt{1+\sin^2\theta\,\varphi'^2}\mathrm{d}\theta, \quad \varphi' = \frac{\mathrm{d}\varphi}{\mathrm{d}\theta}\\
&= R\mathcal{L}\mathrm{d}\theta.
\end{aligned}$$

We notice that $\partial_\varphi\mathcal{L} = 0$, so φ is a cyclic variable. Thus,

$$\partial_{\varphi'}\mathcal{L} = \lambda \in \mathbb{R}$$

where λ is a constant.

3.

$$\begin{aligned}
&\partial_{\varphi'}\mathcal{L} = \lambda \in \mathbb{R}\\
\implies &\frac{\varphi'\sin^2\theta}{\sqrt{1+\sin^2\theta\varphi'^2}} = \lambda\\
\implies &\varphi'^2(\sin^4\theta - \lambda^2\sin^2\theta) = \lambda^2\\
\implies &\mathrm{d}\varphi = \lambda\frac{\mathrm{d}\theta}{\sin^2\theta\sqrt{1-\frac{\lambda^2}{\sin^2\theta}}}.
\end{aligned}$$

Integrating,

$$\begin{aligned}
\varphi - \varphi_0 &= \lambda\int^\theta \frac{\mathrm{d}\alpha}{\sin^2\alpha\sqrt{1-\frac{\lambda^2}{\sin^2\alpha}}}\\
&=_{u=\cot\alpha} -\lambda\int^{\cot\theta}\frac{\mathrm{d}u}{\sqrt{1-\lambda^2(1+u^2)}}\\
&=_{t=\frac{u}{\beta}} -\frac{\lambda}{\beta}\int^{\frac{\cot\theta}{\beta}}\frac{\mathrm{d}t}{\sqrt{1-t^2}}, \quad \beta^2 = 1-\lambda^2\\
&= \arccos\left(\frac{\cot\theta}{\beta}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\beta\cos(\varphi-\varphi_0) &= \cot\theta,\\
\beta\sin\theta\cos(\varphi-\varphi_0) &= \cos\theta.
\end{aligned}$$

Using some trigonometric formulas, we obtain

$$\begin{aligned} R \times \left(\beta \cos\varphi_0 \cos\varphi \sin\theta + \beta \cos\varphi_0 \sin\varphi \sin\theta\right) = \cos\theta \\ \implies ax + by - z = 0, \end{aligned}$$

where we substituted using spherical coordinates, with $a = \beta \cos\varphi_0 = b$.

4.6 Blackbody Radiation

4.6.1 Number of Modes Excited per Frequency Unit

1. This is the D'Alembert equation in vacuum,

$$\Box \mathbf{E} = 0.$$

2. The cavity enforces a stationary solution, thus

$$\mathbf{E} = \cos \omega t \sum_{\mu=1}^{3} E^{\mu} \sin(k_\mu x^\mu) \mathbf{e}_\mu.$$

For each μ, the boundary condition is $\mathbf{E}(x^\mu = L) = \mathbf{0}$. Hence,

$$\begin{aligned} \sin(k_\mu L) &= 0, \\ k_\mu L &= n_\mu \pi, \\ k_\mu &= \frac{n_\mu \pi}{L}. \end{aligned}$$

3. We know that the norm of $\mathbf{k}$ equals the sum over each component,

$$\begin{aligned} \|\mathbf{k}\|^2 &= \sum_\mu \left(\frac{n_\mu \pi}{L}\right)^2, \\ \left(\frac{2\pi}{\lambda}\right)^2 &= \frac{\pi^2}{L^2} \sum_\mu n_\mu^2, \\ r^2 = \left(\frac{2L}{\lambda}\right)^2 &= \sum_\mu n_\mu^2. \end{aligned}$$

4. The volume of modes up to frequency $\|\mathbf{k}\|$ is

$$V(\|\mathbf{k}\|) = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \left(\frac{2L}{\lambda}\right)^3 = \frac{4}{3}\pi \|\mathbf{k}\|^3.$$

The number of modes is the mode volume divided by the volume of a single mode, with some factors. Since $k_\mu = \frac{\pi}{L} n_\mu$ and $n_\mu \in \mathbb{N}^*$ (factor $\times \frac{1}{8}$), and polarization (factor $\times 2$), we have

$$\begin{aligned} \implies N &= \frac{V(\|\mathbf{k}\|)}{\left(\frac{\pi}{L}\right)^3} \\ &= \frac{1}{8} \times 2 \times \frac{\frac{4}{3}\pi \|\mathbf{k}\|^3}{\pi^3} L^3 \\ &= \frac{1}{8} \times 2 \times \frac{4}{3}\pi \frac{\left(\frac{2\pi}{\lambda}\right)^3}{\pi^3} L^3 \\ &= \pi \frac{8L^3}{\lambda^3} \\ &= \frac{8\pi\nu^3}{3c^3} L^3, \\ \implies \frac{\mathrm{d}N}{\mathrm{d}\nu} &= \frac{8\pi\nu^2}{c^3} \mathcal{V} \end{aligned} \tag{4.6.1}$$

4.6.2 Ultraviolet Catastrophe

1. The system is in contact with a thermostat at temperature T, and the system is closed.

2. In 1D,
$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}\omega^2 q^2$$

3.
$$p(W = \varepsilon) = \frac{1}{Z}\exp(-\beta\varepsilon)$$
We also have in 1D,
$$Z = \frac{1}{h}\int_{\mathbb{R}^2} e^{-\beta\mathcal{H}} \mathrm{d}q\mathrm{d}p$$
Hence,
$$\int_{\mathbb{R}} e^{-\beta\frac{p^2}{2m}} dp = \sqrt{\frac{2m\pi}{\beta}}$$
And,
$$\int_{\mathbb{R}} e^{-\beta\frac{m\omega^2 q^2}{2}} dq = \sqrt{\frac{2\pi}{m\omega^2\beta}}$$
Hence,
$$Z = \frac{1}{h}\frac{2\pi}{\omega\beta} = \frac{1}{h}\frac{T}{\beta}$$

4. We use the formula for the average energy,
$$\langle W\rangle = -\partial_\beta \ln Z = \partial_\beta \ln\beta = \frac{1}{\beta} = k_B T$$

5. It is then obvious to say that thanks to eq. 4.6.2 and the previous question,
$$u(\nu, T) = 8\pi\frac{\nu^2}{c^3}k_B T$$
Hence $u \propto \nu^2$, which implies, $\int_{\mathbb{R}+} u\mathrm{d}\nu \propto \int_{\mathbb{R}+} \nu^2\mathrm{d}\nu$, which diverges.

4.6.3 Planck's Law

1. The energy levels are discrete, so we sum:
$$Z = \sum_n e^{-\beta W_n} = \frac{1}{1 - e^{-\beta W_1}}$$
Thus, the average energy becomes by the same calculation,
$$-\partial_\beta \ln Z = \frac{h\nu}{e^{\beta h\nu} - 1}$$
Using, $W_1 = h\nu$.

2. It is then obvious that,
$$u(\nu, T) = 8\pi\frac{\nu^2}{c^3}\frac{h\nu}{e^{\beta h\nu} - 1} \tag{4.6.2}$$

4.6.4 Energy flux emitted by a black body

1. Monochromatic energy flux in a given direction.

   The directional spectral intensity $I_\nu(\theta, \varphi)$ is defined as the energy transported per unit area, time, frequency, and steradian, in direction (θ, φ).

   The monochromatic energy flux emitted in direction (θ, φ) relative to the surface normal is:
$$\mathrm{d}\Phi_\nu = I_\nu(\theta, \varphi)\cos\theta\,\mathrm{d}\Omega,$$

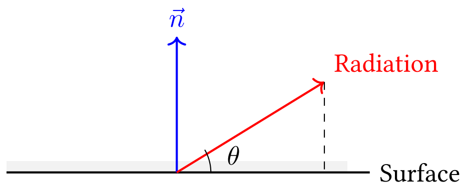


Figure 4.1: The radiation is emitted with an angle θ relative to the normal: only $\cos\theta$ contributes to the flux through the surface. Indeed, it goes out in all directions, so we integrate over $[0, \frac{\pi}{2}]$, and only the contribution of $\cos\theta$ (the projection) matters.

where $\mathrm{d}\Omega$ is the solid angle element around this direction, and $\cos\theta$ comes from the projection of the flux on the normal to the surface (cf. fig 4.1).

2. Total energy flux emitted at frequency ν.

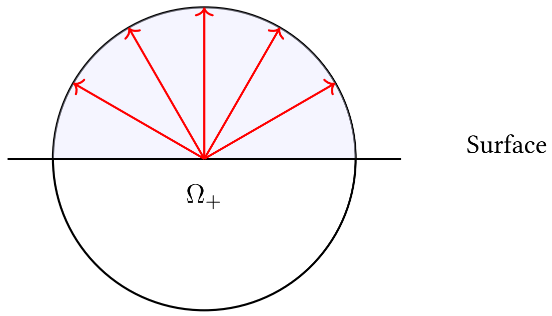


Figure 4.2: The radiation goes out in all directions of the hemisphere Ω_+: we integrate only for $\theta \in [0, \pi/2]$.

The total energy flux $I(\nu)$ emitted at frequency ν per unit surface is obtained by integrating the elementary flux over the entire outgoing hemisphere (i.e. directions such that $0 \le \theta \le \pi/2$, cf. fig 4.2):

$$I(\nu) = \int_{\Omega_+} I_\nu(\theta, \varphi) \cos\theta \,\mathrm{d}\Omega.$$

3. Case of isotropic radiation

If the radiation is isotropic, we have $I_\nu(\theta, \varphi) = I_\nu =$ constant (independent of direction). We can then take I_ν out of the integral:

$$I(\nu) = I_\nu \int_{\Omega_+} \cos\theta \,\mathrm{d}\Omega.$$

But:

$$\int_{\Omega_+} \cos\theta \,\mathrm{d}\Omega = \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi.$$

Calculating,

$$\int_0^{\pi/2} \cos\theta \sin\theta \,\mathrm{d}\theta = \frac{1}{2}, \quad \text{and} \quad \int_0^{2\pi} \mathrm{d}\varphi = 2\pi.$$

Hence,

$$I(\nu) = I_\nu \cdot 2\pi \cdot \frac{1}{2} = \pi I_\nu.$$

4. Total emitted intensity (all frequencies combined)

We want to show that the spectral volumetric energy density $u(\nu)$ can be expressed as a function of the directional intensity $I_\nu(\mathbf{n})$ by:

$$u(\nu) = \frac{1}{c} \int_{S^2} I_\nu(\mathbf{n}) \,\mathrm{d}\Omega.$$

- $u(\nu)\,\mathrm{d}\nu$ represents the electromagnetic energy contained in a unit volume, for waves whose frequency is between ν and $\nu + \mathrm{d}\nu$.

- $I_\nu(\mathbf{n})$ is the spectral intensity in the direction $\mathbf{n}$, that is, the energy transported per unit time, per unit perpendicular surface, per unit frequency, per unit solid angle.

Consider an elementary surface $\mathrm{d}s$ and a radiation beam incident along a direction $\mathbf{n}$ making an angle θ with the normal to $\mathrm{d}s$.

The volume V swept by the rays in the direction $\mathbf{n}$ during a short time interval $\mathrm{d}t$ is given by:

$$\mathrm{d}V = c\,\mathrm{d}t \cdot \mathrm{d}s \cdot \cos\theta.$$

The energy transported through the surface $\mathrm{d}s$ by these rays during this time is:

$$\mathrm{d}E = I_\nu(\mathbf{n}) \cdot \cos\theta \cdot \mathrm{d}s \cdot \mathrm{d}t \cdot \mathrm{d}\Omega.$$

We deduce that the energy per unit volume associated with the direction $\mathbf{n}$ is:

$$\frac{\mathrm{d}E}{\mathrm{d}V} = \frac{I_\nu(\mathbf{n}) \cdot \cos\theta \cdot \mathrm{d}s \cdot \mathrm{d}t \cdot \mathrm{d}\Omega}{c\,\mathrm{d}t \cdot \mathrm{d}s \cdot \cos\theta} = \frac{I_\nu(\mathbf{n})}{c}\mathrm{d}\Omega.$$

To obtain the total energy density, we sum over all propagation directions on the unit sphere:

$$u(\nu) = \frac{1}{c}\int_{S^2} I_\nu(\mathbf{n})\,\mathrm{d}\Omega.$$

If the radiation is isotropic, then $I_\nu(\mathbf{n}) = I_\nu$ is independent of direction. The integral becomes:

$$u(\nu) = \frac{I_\nu}{c}\int_{S^2}\mathrm{d}\Omega = \frac{I_\nu}{c}\cdot 4\pi.$$

Hence,

$$\boxed{u(\nu) = \frac{4\pi}{c} I_\nu}$$

5. Relation between total intensity and $u(\nu)$

   We take again the previous expression:

$$I = \int_0^\infty \pi I_\nu\,\mathrm{d}\nu,$$

and substitute $I_\nu = \frac{c}{4\pi}u(\nu)$:

$$I = \int_0^\infty \pi \cdot \frac{c}{4\pi}u(\nu)\,\mathrm{d}\nu = \frac{c}{4}\int_0^\infty u(\nu)\,\mathrm{d}\nu.$$

4.6.5 Stefan's Law

1. We previously demonstrated that,

$$I(T) = \frac{c}{4}\int_{\mathbb{R}+} u(\nu, T)\mathrm{d}\nu$$

   Replacing with what was obtained in eq. 4.6.2,

$$\begin{aligned} I &= \frac{c}{4}\frac{8\pi}{c^3}\int_{\mathbb{R}+}\frac{h\nu^3}{e^{\beta h\nu}-1}\mathrm{d}\nu \\ &\underset{x=\beta h\nu}{=} \frac{2\pi k_B^4}{h^3c^2}T^4\int_0^\infty \frac{x^3}{e^x-1}\mathrm{d}x \end{aligned}$$

   Recall that $\beta = \frac{1}{k_B T}$.

2. By performing a series expansion, one easily eliminates division by zero. Indeed, near zero,

$$e^x - 1 \underset{0}{=} x + o(x) \implies \frac{x^3}{e^x-1} \underset{0}{=} x^2 + o(x^2)$$

which converges well at zero. At infinity, the exponential ensures convergence of the integral.

$$\begin{aligned}\int_{\mathbb{R}+} \frac{x^3}{e^x - 1}\mathrm{d}x &= \int_{\mathbb{R}+} \mathrm{d}x\, x^3 e^{-x} \frac{1}{1 - e^{-x}} \\ &=_{\text{DSE}} \int_{\mathbb{R}+} \mathrm{d}x\, x^3 \sum_{n\in\mathbb{N}^*} e^{-nx} \\ &\underset{u=nx}{=} \sum_{n\in\mathbb{N}^*} \frac{1}{n^4} \int_{\mathbb{R}+} \mathrm{d}u\, u^3 e^{-u} \\ &= \zeta(4)\Gamma(4) \\ &= 6\zeta(4)\end{aligned}$$

3. Thanks to Fourier theory, one can show that $\zeta(4) = \frac{\pi^4}{90}$. We then have,

$$I(T) = \frac{2\pi^5 k_B^4}{15h^3c^2} T^4, \tag{4.6.3}$$

4.6.6 Application: Solar mass loss due to electromagnetic radiation

We consider the Sun as a black body at temperature $T = 5775\,\mathrm{K}$. The total power radiated by the Sun is given by Stefan-Boltzmann law:

$$P = I \cdot S = \sigma T^4 \cdot 4\pi R^2,$$

where

$$\sigma = 5{,}67 \times 10^{-8}\ \mathrm{W.m^{-2}K^{-4}}, \quad R = 6{,}96 \times 10^8\ \mathrm{m}$$

is the radius of the Sun.

Let's calculate P:

$$P = 5{,}67 \times 10^{-8} \times (5775)^4 \times 4\pi(6{,}96 \times 10^8)^2.$$

We estimate:

$$(5775)^4 \simeq 1{,}11 \times 10^{15},$$

$$4\pi(6{,}96 \times 10^8)^2 = 4\pi \times 4{,}84 \times 10^{17} \simeq 6{,}08 \times 10^{18}.$$

Thus,

$$P \simeq 5{,}67 \times 10^{-8} \times 1{,}11 \times 10^{15} \times 6{,}08 \times 10^{18} \simeq 3{,}83 \times 10^{26}\ \mathrm{W}.$$

According to Einstein's mass-energy equivalence relation,

$$E = mc^2,$$

the mass loss rate $\dot{m}$ per unit time related to this radiated power is

$$\dot{m} = \frac{P}{c^2},$$

with $c = 3{,}00 \times 10^8$ m/s.

Hence,

$$\dot{m} = \frac{3{,}83 \times 10^{26}}{(3{,}00 \times 10^8)^2} = \frac{3{,}83 \times 10^{26}}{9 \times 10^{16}} \simeq 4{,}26 \times 10^9\ \mathrm{kg/s}.$$

Knowing that the age of the Sun is about $t = 4{,}6 \times 10^9$ years, i.e.

$$t = 4{,}6 \times 10^9 \times 3{,}15 \times 10^7 \simeq 1{,}45 \times 10^{17}\ \mathrm{s},$$

the total lost mass is

$$\Delta m = \dot{m} \times t = 4{,}26 \times 10^9 \times 1{,}45 \times 10^{17} \simeq 6{,}18 \times 10^{26}\ \mathrm{kg}.$$

In number of Earth masses, with $m_T = 6 \times 10^{24}$ kg,

$$\frac{\Delta m}{m_T} = \frac{6{,}18 \times 10^{26}}{6 \times 10^{24}} \simeq 103.$$

Thus, the Sun loses about $4{,}3 \times 10^9$ kg/s by radiation. Since its formation, it has lost about 100 times the mass of the Earth.

4.7 Minimization of the gravitational potential by a ball

4.7.1 Hadamard's formula

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^1$ function, and let Ω_ε be a smooth deformation of Ω such that, for $x \in \partial\Omega$,

$$x \;\mapsto\; x + \varepsilon\, f(x)\, n(x),$$

extended on all Ω. We want to prove:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x \;=\; \int_{\partial\Omega} F(x)\, f(x)\, \mathrm{d}S(x), \tag{4.7.1}$$

where $\mathrm{d}S$ is the surface element on $\partial\Omega$.

1. **Study of the function** $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$**.**

   (a) *Differentiability of* $\det$.
   Recall that for $M = (m_{ij}) \in \mathcal{M}_n(\mathbb{R})$,

   $$\det(M) \;=\; \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)}.$$

   It is thus a polynomial in the n^2 variables m_{ij}. Any polynomial function $\mathbb{R}^{n^2} \to \mathbb{R}$ is of class $\mathcal{C}^\infty$. In particular, $\det$ is differentiable at every point of $\mathcal{M}_n(\mathbb{R})$, notably near the identity I.

   (b) *Expansion of* $\det(I + \varepsilon M)$.
   We want to show:

   $$\forall M \in \mathcal{M}_n(\mathbb{R}), \quad \det\big(I + \varepsilon M\big) \underset{\varepsilon \to 0}{=} 1 \;+\; \varepsilon\; \mathrm{Tr}(M) \;+\; o(\varepsilon),$$

   which implies $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} \det(I + \varepsilon M) = \mathrm{Tr}(M)$.
   It suffices to write M in upper triangular form, then the determinant is the product of the eigenvalues! Thus,

   $$\det(I + \varepsilon M) = \prod_{i=1}^{n}(1 + \varepsilon\lambda_i) = 1 + \varepsilon \sum_{i=1}^{n} \lambda_i + O(\varepsilon^2) = 1 + \varepsilon\, \mathrm{Tr}\, M + o(\varepsilon)$$

   which concludes the proof.

   (c) We reduce to the previous case by factoring out X.

   $$\begin{aligned} \det(X + H) &= \det X \det\big(I + X^{-1}H\big) \\ &= \det X \Big(1 + \mathrm{tr}\big(X^{-1}H\big) + o(\|H\|)\Big) \\ &= \det X + \mathrm{tr}\big({}^t\mathrm{Com}(X) H\big) + o(\|H\|) \end{aligned}$$

   Thus we have,

   $$\mathrm{d}(\det(H))\,(X) = \mathrm{Tr}\big({}^t\mathrm{Com}(X) H\big)$$

2. **Change of variables and calculation of the Jacobian.**
   We perform the change of variable

   $$x = x(u) = u + \varepsilon\, f(u)\, n(u), \qquad u \in \Omega.$$

   To compute $\det\Big(\frac{\partial x}{\partial u}\Big)$ at first order in ε, we write

   $$x_i(u) = u_i + \varepsilon\, f(u)\, n_i(u), \quad i = 1, \ldots, n.$$

   Then

   $$\frac{\partial x_i}{\partial u_j} = \delta_{ij} + \varepsilon\, \Big(\partial_j f(u)\Big)\, n_i(u) + \varepsilon\, f(u)\, \partial_j n_i(u).$$

Let the matrix $A(u) = \big(\partial_j f\, n_i + f\, \partial_j n_i\big)_{i,j}$. We have $\dfrac{\partial x}{\partial u} = I + \varepsilon\, A(u)$. By the previous expansion,

$$\det\Big(\frac{\partial x}{\partial u}\Big) = \det\big(I + \varepsilon\, A(u)\big) = 1 + \varepsilon\ \mathrm{Tr}\big(A(u)\big) + o(\varepsilon).$$

Noticing that $\mathrm{Tr}(A(u)) = \nabla\cdot\big(f\, n\big)$, we obtain

$$\det\Big(\frac{\partial x}{\partial u}\Big) = 1 + \varepsilon\, \nabla\cdot\big(f\, n\big)(u) + o(\varepsilon).$$

3. **Expansion of $F(x + \varepsilon\, v(x))$.**
Let $F : \mathbb{R}^n \to \mathbb{R} \in \mathcal{C}^1$, $v : \mathbb{R}^n \to \mathbb{R}^n$. Fixing x, define $\varphi(\varepsilon) = F\big(x + \varepsilon\, v(x)\big)$. By the chain rule in dimension 1,
$$\varphi'(\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(x + \varepsilon v(x)) = v(x)\ \cdot \nabla F\big(x + \varepsilon\, v(x)\big).$$
In particular, for $\varepsilon \to 0$,
$$\varphi(\varepsilon) = \varphi(0) + \varepsilon\, \varphi'(0) + o(\varepsilon) = F(x) + \varepsilon\, v(x) \cdot \nabla F(x) + o(\varepsilon).$$
Hence
$$\forall x \in \mathbb{R}^n, \quad F\big(x + \varepsilon\, v(x)\big) = F(x) + \varepsilon\, v(x) \cdot \nabla F(x) + o(\varepsilon).$$

4. **Derivation of Hadamard's formula.**
We perform the change $x(u)$ in $\displaystyle\int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x$. Then
$$\int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x = \int_{\Omega} F\big(x(u)\big)\ \det\Big(\frac{\partial x}{\partial u}\Big)\, \mathrm{d}^3 u.$$
From the two previous points,
$$F\big(x(u)\big) = F(u) + \varepsilon\, f(u)\, n(u) \cdot \nabla F(u) + o(\varepsilon), \quad \det\Big(\tfrac{\partial x}{\partial u}\Big) = 1 + \varepsilon\, \nabla\cdot\big(f\, n\big)(u) + o(\varepsilon).$$
Multiplying,
$$F\big(x(u)\big) \det\Big(\frac{\partial x}{\partial u}\Big) = F(u) + \varepsilon\, \Big[f\, n \cdot \nabla F + F\, \nabla\cdot(f\, n)\Big](u) + o(\varepsilon).$$
Therefore,
$$\int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x = \int_{\Omega} F(u)\, \mathrm{d}^3 u + \varepsilon \int_{\Omega} \Big[f\, n \cdot \nabla F + F\, \nabla\cdot(f\, n)\Big](u)\, \mathrm{d}^3 u + o(\varepsilon).$$
Then,
$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{\Omega_\varepsilon} F(x)\, \mathrm{d}^3 x = \int_{\Omega} \nabla\cdot\, \big(F\, f\, n\big)(u)\, \mathrm{d}^3 u,$$
using the product rule. Finally, by the divergence theorem,
$$\int_{\Omega} \nabla\cdot\, \big(F\, f\, n\big)\, \mathrm{d}^3 u = \int_{\partial\Omega} F\, f\, n \cdot n\, \mathrm{d}S(u) = \int_{\partial\Omega} F\, f\, \mathrm{d}S.$$
This concludes the proof of Hadamard's formula (4.7.1).

4.7.2 Connection with the gravitational potential

1. **Sign of $E[\Omega]$ and definition of $\mathcal{I}[\Omega]$.**
We have
$$E[\Omega] = -\frac{G}{2}\, \rho^2 \iint_{\Omega\times\Omega} \frac{1}{|x - x'|}\, \mathrm{d}^3 x\, \mathrm{d}^3 x'.$$

Since $G > 0$ and $\rho > 0$, it immediately follows that $E[\Omega] < 0$. Minimizing $E[\Omega]$ is therefore equivalent to *maximizing*

$$\mathcal{I}[\Omega] := \iint_{\Omega\times\Omega} \frac{1}{|x-x'|}\,\mathrm{d}^3x\,\mathrm{d}^3x'.$$

2. **Calculation of the potential at the center of a ball.**
Suppose $\Omega = B(0,R)$ with fixed volume $\frac{4}{3}\pi R^3 = V$. The density is ρ. For $x = 0$,

$$U(0) = -G\,\rho \int_\Omega \frac{1}{|x'|}\,\mathrm{d}^3x' = -G\,\rho \int_0^R \int_{S^2} \frac{1}{r}\,r^2\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r.$$

In spherical coordinates,

$$\int_{S^2} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 4\pi, \quad \text{and} \quad \int_0^R \frac{r^2}{r}\,\mathrm{d}r = \int_0^R r\,\mathrm{d}r = \frac{R^2}{2}.$$

Thus

$$U(0) = -G\,\rho\cdot 4\pi\cdot\frac{R^2}{2} = -2\pi\,G\,\rho\,R^2.$$

Hence the explicit expression of the potential at the center.

4.7.3 The sphere?

1. **First variation of $\mathcal{F}$.**
We write $\mathcal{F}[\Omega_\varepsilon]$ and apply Hadamard's formula with $F(x) = \int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x'$. Then

$$\delta\mathcal{F} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{1}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

Thus, using Hadamard's formula for Ω^2,

$$\delta\mathcal{F} = 2\int_{\partial\Omega}\left(\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x'\right) f(x)\,\mathrm{d}S(x).$$

2. **Introduction of the Lagrange multiplier λ.**
We want to minimize $\mathcal{F}$ under the constraint $V[\Omega] = V$. We define the Lagrangian functional

$$\mathcal{L}[\Omega] := \mathcal{F}[\Omega] - \lambda\,V[\Omega], \quad \lambda\in\mathbb{R}.$$

Its first variation writes

$$\delta\mathcal{L} = \delta\mathcal{F} - \lambda\,\delta V = 2\int_{\partial\Omega}\left(\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x'\right) f(x)\,\mathrm{d}S(x) \;-\; \lambda\int_{\partial\Omega} f(x)\,\mathrm{d}S(x).$$

By linearity,

$$\delta\mathcal{L} = \int_{\partial\Omega}\left(2\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x' - \lambda\right) f(x)\,\mathrm{d}S(x).$$

3. **Stationary condition for the ball.**
For $\delta\mathcal{L} = 0$ for *all* perturbations f, it is necessary and sufficient that

$$2\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x' - \lambda \;=\; 0, \quad \text{for all } x\in\partial\Omega.$$

In other words, the function $x \mapsto \displaystyle\int_\Omega \frac{1}{|x-x'|}\,\mathrm{d}^3x'$ is constant on $\partial\Omega$.

If $\Omega = B(0,R)$ is a ball, then by spherical symmetry, for every $x\in\partial B(0,R)$ (i.e. $|x| = R$), the integral $\displaystyle\int_{B(0,R)} \frac{1}{|x-x'|}\,\mathrm{d}^3x'$ depends only on $|x| = R$.
Thus it is *constant* on ∂B. We deduce that the ball satisfies the stationary condition $\delta\mathcal{L} = 0$ for all f.

4. **(*Bonus*) Second variation and local minimum.**
To show that the ball is a *local minimum* of $\mathcal{F}$ under volume constraint V, one must verify that the second variation $\delta^2\mathcal{L}[f]$ is strictly positive for any perturbation $f \neq 0$ satisfying $\int_{\partial\Omega} f \, \mathrm{d}S = 0$.

   Without full details here, the second variation can be written as a bilinear form:

$$\delta^2\mathcal{F}[f] = \int_{(\partial\Omega)^2} K(x,x')\, f(x)\, f(x')\, \mathrm{d}S(x)\, \mathrm{d}S(x') \; + \; \int_{\partial\Omega} f(x)^2 \kappa(x) \mathrm{d}S(x),$$

   with kernel $K(x,x') = \frac{1}{|x-x'|}$ and $\kappa(x)$ the mean curvature at x.
For the ball, thanks to the spherical harmonics expansion, one shows this form is strictly positive on $\left\{ f \mid \int_{\partial\Omega} f \, \mathrm{d}S = 0 \right\}$. This proves the ball is a local minimum.

5. **Physical conclusion.**
The ball minimizes the internal gravitational energy for a fixed volume. In physics, this explains that in the approximation of a massive self-gravitating body at rest, the stationary configuration of least energy is spherical. This is why large objects in the Universe (stars, planets in the absence of tidal forces or rapid rotation) tend to a spherical shape.

4.8 Relativistic motion of a charged particle △

4.9 Relativistic hydrodynamics and heavy ion collisions △

4.10 Hydrogen atom and radial equation

4.10.1 Separation of variables and radial equation

1. **Separation of variables**

   The Hamiltonian of the hydrogen atom, in the spherical basis, is written as:

$$\mathbf{H} = -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{e^2}{r}.$$

   In spherical coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\mathbf{L}^2}{\hbar^2 r^2},$$

   where $\mathbf{L}^2$ is the square of the orbital angular momentum.

   We look for a solution of the form

$$\psi(r,\theta,\phi) = R(r)Y_{\ell m}(\theta,\phi),$$

   where $Y_{\ell m}$ are the spherical harmonics simultaneous eigenfunctions of $\mathbf{L}^2$ and $\mathbf{L}_z$, satisfying

$$\mathbf{L}^2 Y_{\ell m} = \hbar^2\ell(\ell+1)Y_{\ell m}, \quad \mathbf{L}_z Y_{\ell m} = \hbar m Y_{\ell m}.$$

   Injecting into the stationary Schrödinger equation $\mathbf{H}\psi = E\psi$, we get the following radial equation:

$$-\frac{\hbar^2}{2m_e}\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{\ell(\ell+1)}{r^2}R\right] - \frac{e^2}{r}R = ER.$$

   Expanding the radial derivative,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r},$$

   which gives the announced equation:

$$-\frac{\hbar^2}{2m_e}\left(\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}R\right) - \frac{e^2}{r}R = ER.$$

2. **Change of function:** $u(r) = rR(r)$

   Putting $u(r) = rR(r)$, we calculate:

$$\frac{\mathrm{d}R}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2},$$

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2} = \frac{1}{r}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} - \frac{2}{r^2}\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{2u}{r^3}.$$

   Replacing in the radial equation, the terms in u/r^3 cancel and we get:

$$-\frac{\hbar^2}{2m_e}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2m_e r^2} - \frac{e^2}{r}\right]u = Eu.$$

3. **Dimensionless change of variable**

   We define

$$\kappa = \sqrt{\frac{2m_e|E|}{\hbar^2}}, \quad \rho = \kappa r.$$

   The equation becomes

$$-\frac{\hbar^2}{2m_e}\kappa^2\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2m_e r^2} - \frac{e^2}{r}\right]u = Eu.$$

Since $E = -|E|$, dividing the whole equation by $-\frac{\hbar^2\kappa^2}{2m_e}$:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} = \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{2m_e e^2}{\hbar^2\kappa}\frac{1}{\rho} + 1\right]u.$$

We then set

$$\rho_0 = \frac{m_e e^2}{\hbar^2\kappa}.$$

This gives the announced equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} = \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{\rho_0}{\rho} + 1\right]u.$$

4. **Ansatz on the form of $u(\rho)$**

   We set

   $$u(\rho) = \rho^{\ell+1}e^{-\rho/2}v(\rho).$$

   By calculating the second derivative of $u(\rho)$ and replacing into the differential equation, one finds that $v(\rho)$ satisfies:

   $$\rho\frac{\mathrm{d}^2v}{\mathrm{d}\rho^2} + (2\ell+2-\rho)\frac{\mathrm{d}v}{\mathrm{d}\rho} + (\rho_0 - 2\ell - 2)v = 0.$$

   This equation is that of the confluent hypergeometric function.

5. **Power series and termination condition**

   We develop

   $$v(\rho) = \sum_{k=0}^{\infty} c_k\rho^k.$$

   The equation gives a recurrence relation between coefficients c_k. Generally, this series diverges as $\rho \to \infty$ unless the series is a polynomial, i.e. it stops at some finite order $\hat{k}$. The termination condition is

   $$\rho_0 = 2n,$$

   where

   $$n = \hat{k} + \ell + 1 \in \mathbb{N}^*.$$

6. **Expression of bound energy levels**

   Reinjecting the definition of ρ_0,

   $$\rho_0 = \frac{m_e e^2}{\hbar^2\kappa} = 2n \quad \Rightarrow \quad \kappa = \frac{m_e e^2}{2\hbar^2 n}.$$

   Now

   $$E = -\frac{\hbar^2\kappa^2}{2m_e} = -\frac{m_e e^4}{2\hbar^2}\cdot\frac{1}{n^2}.$$

   These are the quantized energy levels of the hydrogen atom.

7. **Degree of degeneracy**

   For a level n, possible values of ℓ are

   $$\ell = 0, 1, 2, \ldots, n-1,$$

   and for each ℓ, the values of m go from

   $$m = -\ell, -\ell+1, \ldots, \ell-1, \ell,$$

   i.e. $(2\ell+1)$ values.

The degree of degeneracy is therefore

$$g_n = \sum_{\ell=0}^{n-1}(2\ell+1) = 2\sum_{\ell=0}^{n-1}\ell + \sum_{\ell=0}^{n-1} 1 = 2\frac{(n-1)n}{2} + n = n^2.$$

Interpretation: In this non-relativistic model without spin-orbit interactions or relativistic effects, the energy depends only on the principal quantum number n. This reflects the larger symmetry of the problem (rotational invariance and Runge-Lenz-type symmetry), which leads to this high degeneracy.

4.10.2 Ground state ($n = 1$) and radial properties

7. For $n = 1, \ell = 0, n_r = 0$:

$$u(r) = Are^{-r/a_0}, \quad \Rightarrow R(r) = \frac{u(r)}{r} = Ae^{-r/a_0}.$$

Normalization requires:

$$\int_0^\infty |R(r)|^2 r^2 \mathrm{d}r = |A|^2 \int_0^\infty e^{-2r/a_0} r^2 \mathrm{d}r = 1.$$

The integral yields $2!(a_0/2)^3 = a_0^3/4 \Rightarrow |A|^2 = 4/a_0^3$. Therefore:

$$R_{1,0}(r) = \frac{2}{a_0^{3/2}} e^{-r/a_0}.$$

The spherical harmonic is $Y_{00} = 1/\sqrt{4\pi}$, so:

$$\psi_{1,0,0}(r,\theta,\phi) = \frac{2}{a_0^{3/2}} e^{-r/a_0} \cdot \frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}.$$

The normalization is indeed satisfied:

$$\int |\psi|^2 \mathrm{d}^3x = \int_0^\infty |R|^2 r^2 \mathrm{d}r \int |Y|^2 \mathrm{d}\Omega = 1.$$

8. The radial probability density is:

$$P(r) = 4\pi |R(r)|^2 r^2 = 4\pi \left(\frac{2}{a_0^{3/2}}\right)^2 e^{-2r/a_0} r^2 = \frac{16\pi}{a_0^3} r^2 e^{-2r/a_0}.$$

It vanishes at $r = 0$ and as $r \to \infty$; the maximum is found at $r = a_0$. *Interpretation:* the most probable location to find the electron is at the Bohr radius.

9. We use:

$$\int_0^\infty r^n e^{-2r/a_0} \mathrm{d}r = n! \left(\frac{a_0}{2}\right)^{n+1}.$$

For $\langle r \rangle$:

$$\langle r \rangle = \int_0^\infty r|R(r)|^2 r^2 \mathrm{d}r = \frac{4}{a_0^3}\int_0^\infty r^3 e^{-2r/a_0} \mathrm{d}r = \frac{4}{a_0^3} \cdot 3! \left(\frac{a_0}{2}\right)^4 = \frac{3}{2}a_0.$$

For $\langle r^2 \rangle$:

$$\int r^4 e^{-2r/a_0} \mathrm{d}r = 4!(a_0/2)^5 = 24(a_0/2)^5 \Rightarrow \langle r^2 \rangle = 3a_0^2.$$

Thus:

$$(\Delta r)^2 = 3a_0^2 - (3a_0/2)^2 = 3a_0^2 - \frac{9}{4}a_0^2 = \frac{3}{4}a_0^2.$$

10. The Fourier transform of the ground state yields an isotropic distribution centered at $p = 0$. The expectation

value is $\langle \mathbf{p} \rangle = 0$ (even function), and:

$$\langle p^2 \rangle = \int \tilde{\psi}^*(\mathbf{p}) p^2 \tilde{\psi}(\mathbf{p}) \mathrm{d}^3 p.$$

It can be related to the average kinetic energy:

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = -E_1 = \frac{1}{2} E_0. \Rightarrow \langle p^2 \rangle = m_e E_0.$$

11. The $1/n^2$ dependence explains the structure of the spectral lines described by the Rydberg formula:

$$\frac{1}{\lambda} = \mathcal{R}_H \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right).$$

The principal quantum number n orders the energy levels. In non-relativistic QM, ℓ does not affect E_n, unlike the relativistic case (Lamb shift, spin-orbit coupling).

4.11 Towards a relativistic formalism △

4.12 Pöschl–Teller potential $V(x) = -\frac{V_0}{\cosh^2(\alpha x)}$ △

4.13 Electrodynamic Instability of the Classical Atom

4.13.1 Calculation of the braking and radiation force $\mathbf{F}_{\text{rad}}$.

1.
$$\mathrm{d}E_{\text{at}} = \mathrm{d}W = \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\mathrm{d}t \implies \Delta E_{\text{at}} = \int_{t_1}^{t_2} \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\mathrm{d}t \tag{4.13.1}$$

2. Be careful, this energy variation is the opposite of the energy radiated during the same interval:

$$\mathrm{d}E_{\text{at}} = -P\,\mathrm{d}t = -\frac{2e^2a^2}{3c^3}\mathrm{d}t = -\frac{2e^2\dot{\mathbf{v}}^2}{3c^3}\mathrm{d}t \tag{4.13.2}$$

We obtain:

$$\Delta E_{\text{at}} = -\frac{2e^2}{3c^3}\int_{t_1}^{t_2} \dot{\mathbf{v}}^2\mathrm{d}t \tag{4.13.3}$$

3. Moreover, by integrating by parts and assuming quasi-periodicity:

$$\Delta E_{\text{at}} = \frac{2e^2}{3c^3}\int_{t_1}^{t_2} \ddot{\mathbf{v}} \cdot \mathbf{v}\mathrm{d}t \tag{4.13.4}$$

By comparison with 4.13.1, a candidate force is the **Abraham-Lorentz radiation braking force**:

$$\mathbf{F}_{\text{rad}} = \frac{2e^2}{3c^3}\ddot{\mathbf{v}} \tag{4.13.5}$$

4. Now consider the **Thomson model**, in which the electron is bound to the origin by a harmonic restoring force. The equation of motion becomes:

$$m\ddot{\mathbf{r}} = -m\omega_0^2\mathbf{r} + \frac{2e^2}{3c^3}\dddot{\mathbf{r}} \tag{4.13.6}$$

We look for a solution of the form $r(t) = \mathrm{Re}\left[r(0)e^{i\omega t}\right]$. The perturbative expansion:

$$\omega = \omega_0\left[1 + a(\omega_0\tau) + \mathcal{O}((\omega_0\tau)^2)\right] \tag{4.13.7}$$

gives $a = \frac{1}{2}$, hence finally:

$$\mathbf{r}(t) = \mathbf{r}(0)e^{-\omega_0^2\tau t}\cos(\omega_0 t) \tag{4.13.8}$$

The motion is therefore a damped oscillator. The characteristic damping time, or typical lifetime of the atom in this model, is:

$$T_{\text{nat}} = \frac{1}{\omega_0^2\tau} \sim 10^{-8}\ \text{s} \tag{4.13.9}$$

The classical atom is thus fundamentally unstable: the electron spirals toward the nucleus, very slowly on the atomic (pseudo-period) scale, but very rapidly on the macroscopic scale.

4.13.2 Conceptual problems generated by the braking force $\mathbf{F}_{\text{rad}}$.

1. The equation to solve is:

$$\dot{\mathbf{v}} - \tau\ddot{\mathbf{v}} = \frac{1}{m}\mathbf{F}(t)$$

whose general solution is:

$$\dot{\mathbf{v}}(t) = v(t_0)e^{(t-t_0)/\tau} - \frac{1}{m\tau}\int_{t_0}^{t} e^{(t-t')/\tau}\mathbf{F}(t')\,\mathrm{d}t' \tag{4.13.10}$$

2. An unacceptable phenomenon, sometimes called *preacceleration of a charged particle*, appears: if $F = 0$, the above expression clearly shows that the acceleration diverges exponentially at large times.

3. One can formally eliminate the divergent solutions by taking $t_0 = +\infty$. This is a boundary condition that effectively eliminates the so-called "initial condition".

4. Taking $t_0 = +\infty$, we obtain:

$$\begin{aligned}\dot{\mathbf{v}}(t) &= -\frac{1}{m\tau}\int_t^{+\infty} e^{(t-t')/\tau}\mathbf{F}(t')\,\mathrm{d}t' \\ &= -\frac{1}{m}\int_t^{+\infty} K(t-t')\mathbf{F}(t')\,\mathrm{d}t' \end{aligned} \tag{4.13.11}$$

   with $K(t-t') = \frac{1}{\tau}e^{(t-t')/\tau}$.

   This is the *regularized form*, all the more so since the limit of zero charge correctly reproduces the Lorentz Force Electrodynamics (LFE).

   Indeed, in the limit $e \to 0$, we have $\tau \to 0$ and the kernel $K(t-t')$ tends to a Dirac delta function $\delta(t-t')$, yielding:

$$\dot{\mathbf{v}}(t) = \frac{1}{m}\mathbf{F}(t)$$

5. It is already apparent that the acceleration at instant t depends on future values of the force. This equation therefore violates the **principle of causality**. A change of variable makes this clear. Let $s = \frac{t'-t}{\tau}$, we get:

$$\dot{\mathbf{v}}(t) = -\frac{1}{m}\int_0^{+\infty} e^{-s}\mathbf{F}(t+\tau s)\,\mathrm{d}s \tag{4.13.12}$$

6. With a step force:

$$\mathbf{F}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mathbf{F}_0 & \text{if } t \geq 0 \end{cases}$$

   we obtain:

$$\begin{aligned} t < 0: \quad & \dot{\mathbf{v}}(t) = -\frac{1}{m\tau}\int_0^{+\infty} e^{(t-t')/\tau}\cdot 0\,\mathrm{d}t' = -\frac{\mathbf{F}_0}{m}e^{t/\tau} \\ t > 0: \quad & \dot{\mathbf{v}}(t) = -\frac{\mathbf{F}_0}{m\tau}\int_t^{+\infty} e^{(t-t')/\tau}\,\mathrm{d}t' = -\frac{\mathbf{F}_0}{m} \end{aligned}$$