X1 - 4 Year 12 - Ext 1 - Trial and HSC Revision - Sheet 4

WORKED SOLUTIONS

Question 1 {Polynomials}

Given that $\alpha, \beta\, and\, \gamma$ are roots of $2x^3 + 5x^2 - x - 3$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

Hint 1: When adding fractions you need a common denominator
Hint 2: Year 11 textbook, page305

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

From the formula sheet

$$\alpha\beta\gamma = \frac{-d}{a} = \frac{3}{2}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{-1}{2}$$

Therefore

$$\frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{-1}{2} \times \frac{2}{3} = \frac{-1}{3}$$

Question 2 {inequalities} Solve $\frac{3}{x-2} \geq 4$

Hint: multiply both sides by the square of the denominator (Yr 11 Textbook, p 78)

$$\frac{3}{x-2} \geq 4$$

$$3(x-2) \geq 4(x-2)^2$$

$$3x - 6 \geq 4\left(x^2 - 4x + 4\right)$$

$$3x - 6 \geq 4x^2 - 16x + 16$$

$$4x^2 - 19x + 22 \leq 0$$

Consider critical points and roots

Critical points at $x = 2$ because it would make the denominator zero.

Roots

$$x = \frac{19 \pm \sqrt{(-19)^2 - 4 \cdot 4 \cdot 22}}{8}$$

$$x = 2.75 \, or \, 2$$

Therefore test around critical points and roots

x	0	2	2.5	2.75	3
$f(x)$	22	Not defined	-13	0	1

So $2 < x \leq 2.75$

Question 3 {trig} - Show that $1 + \tan^2\theta = \sec^2\theta$

LHS

$1 + \tan^2\theta$

$= \dfrac{\cos^2}{\cos^2} + \dfrac{\sin^2}{\cos^2}$

$= \dfrac{1}{\cos^2}$

$= \sec^2$

$= RHS$

QED

Question 4 {related rates of change}

A spherical meteor enters the Earth's atmosphere and burns up (loses volume) at a rate that is proportional to its surface area. Assuming the meteor stays spherical, show that the rate of change of the radius is a constant.

Hint: mathematically, you start with two ideas

1. $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$ (standard chain rule)
2. $\frac{dV}{dt} = kS$ (the information from the question)

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

The surface area of a sphere is

$S = 4\pi r^2$

So $\frac{dV}{dt} = \frac{dr}{dt} S = kS$

Therefore $\frac{dr}{dt} = k$

Which is a constant

Question 5 {mathematical induction}

Prove that $n^3 + 2n$ is divisible by 3 for all integers n.

Step 1 - $n = 1$

LHS = $1^3 + 2(1) = 3$ which is divisible by 3

Step 2 - assume for some arbitrary $n = k$

$k^3 + 2k = 3p$

$k^3 = 3p - 2k$

Step 3 - show that this implies the truth of $P(k+1)$

RTP $(k+1)^3 + 2(k+1) = 3q$

Consider LHS

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

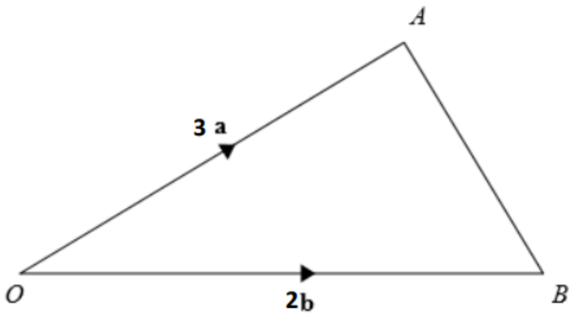
Rearranging to use the inductive step

$= k^3 + 2k + 3k^2 + 3k + 3$
$= 3p + 3(k^2 + k + 1)$

Because both p and k are integers, this is divisible by 3

Therefore, by induction, $P(n)$ is true for $n \geq 1$

Question 6 {vectors}



OAB is a triangle
$\overrightarrow{OA}$ = **3a**
$\overrightarrow{OB}$ = **2b**

P is a point on *AB* so that *AP : PB* is 1 : 3

Given that $\overrightarrow{OP}$ = k (9**a** + 2**b)**

Find the value of k

$$\vec{A}B = \vec{A}O + \vec{O}B$$
$$= -3a + 2b$$

$$\vec{A}P = \frac{1}{4}\vec{A}B$$
$$= \frac{1}{4}[-3a + 2b]$$
$$= \frac{-3}{4}a + \frac{2}{4}b$$

$$\vec{O}P = \vec{O}A + \vec{A}P$$
$$= 3a + \frac{-3}{4}a + \frac{2}{4}b$$
$$= \frac{9}{4}a + \frac{2}{4}b$$
$$= \frac{1}{4}[9a + 2b]$$

$$\therefore k = \frac{1}{4}$$

Question 7 {Binomial distribution}

In externally marked exam papers, an average of 7.5% of students miss doing the questions on the back page. A random sample of 100 students' exam papers were checked for this student error.

a How many students in the sample would be expected to make this error?

b If the sample proportion is approximately normally distributed, find its mean and standard deviation.

c Find the z-score for each percentage of students making this error:

i 4% **ii** 5% **iii** 8% **iv** 10%

d Find the probability that the percentage of students making this error is:

i less than 5% **ii** less than 10% **iii** more than 8%

iv more than 4% **v** between 4% and 10%

Hint: you will need the table on pages 633 and 634 of the textbook.

a. $0.075 \times 100 = 8$ (rounded to the nearest whole student.

b. $\mu = p = 0.075$

$$\begin{aligned} \sigma^2 &= \frac{p(1-p)}{n} \\ &= \frac{0.075(0.925)}{100} \\ &= 0.00069375 \end{aligned}$$

$$\therefore \sigma = 0.0263\ (3dp)$$

c. $z = \dfrac{x - \mu}{\sigma}$ etc

Question 8 {further calculus}

Find the derivative of the inverse function of $f(x) = x^2e^x$

Firstly, swap x and y to find the inverse function

$$y = x^2e^x$$
$$x = y^2e^y$$

Which is way too hard to rearrange to make y the subject

So we'll use this trick

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Using the product rule

$$\begin{aligned}\frac{dx}{dy} &= 2ye^y + y^2e^y \\ &= ye^y(2+y)\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1}{ye^y(2+y)}$$

Question 9 {differential equations}ln

An element of mass M is decaying over t years according to the formula

$$\frac{dM}{dt} = -0.045M.$$

The initial mass is 100 g.

a Solve the differential equation to find the equation for the mass of the element.

b Find the mass after 20 years.

c What is the rate at which the mass is decaying after 20 years?

d Find the half-life of the element (the time it takes to halve its mass).

$$\frac{dt}{dM} = \frac{-1}{0.045} \cdot \frac{1}{M}$$
$$\therefore t = \frac{-1}{0.045} \int \frac{1}{M} dM$$
$$\therefore t = \frac{-1}{0.045} \ln|M| + C$$

Rearranging

$$-0.045(t - C) = \ln|M|$$

$$e^{-0.045t+0.045C} = M$$

Setting $A = e^{0.045C}$

$$M = Ae^{-0.045t}$$

Use initial value conditions

$$100 = Ae^0$$
$$\therefore A = 100$$

$$@t = 20,\ M = 100e^{-0.045\times 20}$$
b) $M \approx 40.66\, g\, (2dp)$

c) Rate of decay is $\frac{dM}{dt}$

$M = 100e^{-0.045t}$

$\dfrac{dM}{dt} = -4.5e^{-0.045t}$

$@\, t \; = \; 20$

$\dfrac{dM}{dt} = -1.83\, gy^{-1}$

d)

$50 = 100e^{-0.045t}$

$\dfrac{1}{2} = e^{-0.045t}$

$\ln\dfrac{1}{2} = -0.045t$

$t = \; \ln\dfrac{1}{2} \div -0.045$

$t \approx 15.4\, years\, (1\, dp)$