Year 12 - Ext 1 - Trial and HSC Revision - Sheet 1

WORKED SOLUTIONS

Question 1 {Perms and Combs} - a group of 20 people are seated around a table. If a family of 5 have to sit next to each other, how many ways are there for this group to be seated?

Subtract 1 for the head of the table	19
Remove the family of 5	14
Add the family back in as one 'unit'	15
Arrange the 15 then allow for the internal arrangements of the family	15!5!

Question 2 {Inequalities} solve $x^2 - 4x - 12 \geq 0$

a) Find roots

$$x^2 - 4x - 12 = 0$$
$$(x-6)(x+2) = 0$$
$$x = 6\, or\, -2$$

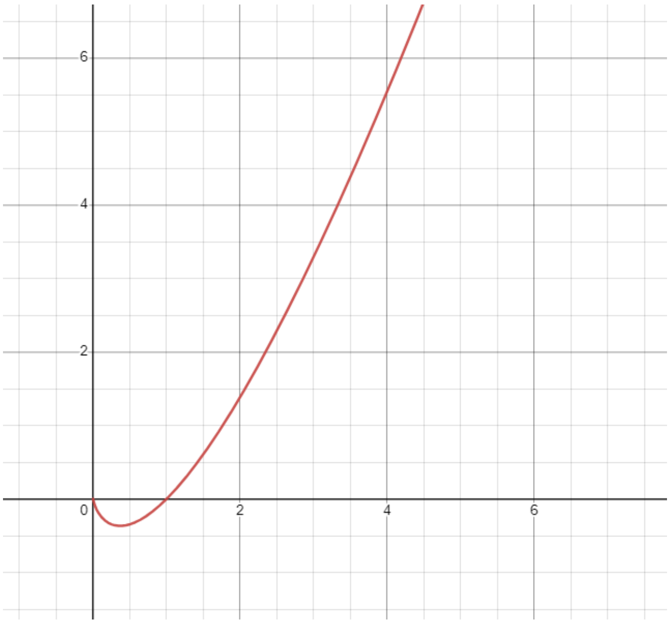
b) Test the function value outside and inside the roots

x	-3	-2	0	6	7
$f(x)$	9	0	-12	0	9

c) Answer the question

$$x^2 - 4x - 12 \geq 0$$
$$where\, x \leq -2\, or\, x \geq 6$$

Question 3 {inverse functions} - given the graph of $f(x)$ below, sketch the graph of $f^{-1}(x)$



Question 4 {related rates of change}

A cone-shaped candle whose height is 3 times its radius is melting at the constant rate of 1.4 $cm^3\ s^{-1}$. If the proportion of radius to height is preserved, find the rate at which the radius will be decreasing when it is 3.7 cm.

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

Assembling the pieces	**Putting the pieces together**
$\frac{dV}{dt} = 1.4$ $V = \frac{1}{3}\pi r^3 h$ $h = 3r$ $V = \frac{1}{3}\pi r^2 \cdot 3r$ $V = \pi r^3$ $\frac{dV}{dr} = 3\pi r^2$ $@r = 3.7$ $\frac{dV}{dr} = 3\pi(3.7)^2$ $\frac{dV}{dr} = 129.03$	$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$ $1.4 = 129.03 \times \frac{dr}{dt}$ $\therefore \frac{dr}{dt} = 0.01\, cms^{-1}\ (2dp)$

Question 5 {further logs and exponents}

2 a Show that $N = 45 + Ae^{0.14t}$ is a solution of $\frac{dN}{dt} = 0.14(N - 45)$.

b Given $N = 82$ when $t = 2$, find A to 2 decimal places.

c What is N when $t = 5$?

d Find t when $N = 120$.

e Sketch the graph of this function for values of t from 0 to 25.

Integrating

$$\frac{dN}{dt} = 0.14(N-45)$$

$$\therefore \frac{dt}{dN} = \frac{1}{0.14(N-45)}$$

$$\therefore t = \frac{1}{0.14}\int \frac{1}{N-45}dN$$

$$\therefore t = \frac{50}{7}\ln|N-45| + C$$

Rearranging

$$t - C = \frac{50}{7}\ln|N-45|$$
$$0.14(t-C) = \ln|N-45|$$
$$e^{0.1t-0.14C} = N - 45$$

$$Let\ A = e^{-0.14C}$$

$$Ae^{0.14t} = N - 45$$

$$\therefore N = Ae^{0.14t} + 45$$

Use initial value conditions

$$82 = Ae^{0.14\times 2} + 45$$

$$A = \frac{37}{e^{0.14\times 2}}$$

$$A \approx 27.96\ (2dp)$$

Find N when t = 5

$$\begin{aligned} N &= 27.96e^{0.14\times 5} + 45 \\ &= 101\ (0dp) \end{aligned}$$

Find t when N = 120

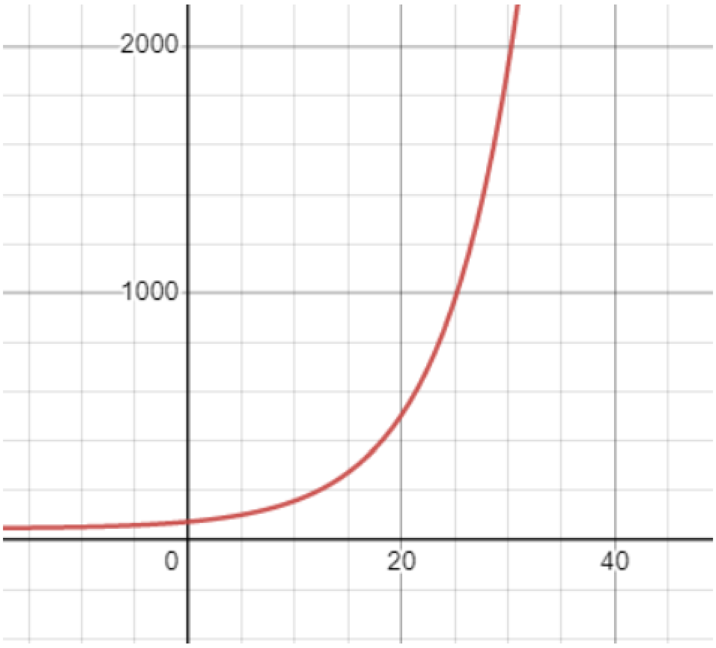
$$N = Ae^{0.14t} + 45$$

$$t = \frac{1}{0.14}\ln\left|\frac{N-45}{A}\right|$$

$$@N = 120$$

$$t = \frac{1}{0.14}\ln\left|\frac{120-45}{27.96}\right|$$

$$t \approx 7.05\ s\ (2dp)$$


2000
1000
0
20
40

Question 6 {vectors} for what value of a is the vector $\begin{pmatrix} a \\ 4 \end{pmatrix}$ perpendicular to $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$?

Vectors are perpendicular if the dot product = 0

$$\vec{u} \cdot \vec{v} = x_1 x_2 + y_1 y_2$$

$$-3a - 8 = 0$$
$$-3a = 8$$
$$a = -\frac{8}{3}$$

Question 7 {differentiation of inverse trig}

3 What is the derivative of $\tan^{-1}\frac{x}{2}$?

A. $\dfrac{1}{2\left(4 + x^2\right)}$

B. $\dfrac{1}{4 + x^2}$

C. $\dfrac{2}{4 + x^2}$

D. $\dfrac{4}{4 + x^2}$

From formula sheet $\dfrac{d}{dx}\tan^{-1} x = \dfrac{f'(x)}{1 + [f(x)]^2}$

$$f(x) = \frac{x}{2}, f'(x) = \frac{1}{2}$$

$$\therefore f'(x) = \frac{\frac{1}{2}}{1 + \frac{x^2}{4}}$$

$$= \frac{2}{4 + x^2}$$

Question 8 {mathematical induction}

(a) Prove by mathematical induction that, for all integers $n \geq 1$, **3**

$$1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1.$$

Step 1 - show for

$LHS\, is\, 1(1!) = 1$

$RHS = 2! - 1$
$= 1$
$= LHS$

Step 2 - assume for $n = k$

$1(1!) + 2(2!) + \ldots + k(k!) = (k+1)! - 1$

Step 3 - show that this implies it works for $n = k + 1$

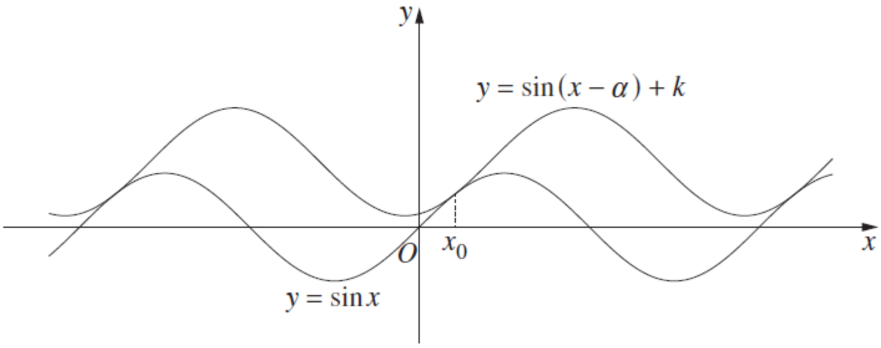
RTP $1(1!) + 2(2!) + \ldots + k(k!) + (k+1)(k+1)! = (k+2)! - 1$

LHS= $(k+1)! - 1 + (k+1)(k+1)!$	Using inductive step
$= (k+2)(k+1)! - 1$	Collecting like terms
$= (k+2)! - 1$	$(k+1)! = 1 \times 2 \times 3 \ldots \times (k+1)$ $(k+2)! = 1 \times 2 \times 3 \times \ldots \times (k+1) \times (k$ $\therefore (k+2)! = (k+1)! \times (k+2)$
= RHS	

Therefore, by induction, $P(n)$ is true for $n \geq 1$

Question 9 {trig equations}

The diagram shows the two curves $y = \sin x$ and $y = \sin(x - \alpha) + k$, where $0 < \alpha < \pi$ and $k > 0$. The two curves have a common tangent at x_0 where $0 < x_0 < \frac{\pi}{2}$.



(i) Explain why $\cos x_0 = \cos(x_0 - \alpha)$. **1**

(ii) Show that $\sin x_0 = -\sin(x_0 - \alpha)$. **2**

(iii) Hence, or otherwise, find k in terms of α. **2**

Part 1

The curves have a common tangent at x_0 and so have the same slope there.

$$y = \sin x$$
$$\frac{dy}{dx} = \cos x$$
$$y = \sin(x - \alpha) + k$$
$$\frac{dy}{dx} = \cos(x - \alpha)$$

and so $\cos x_0 = \cos(x_0 - \alpha)$.

Part 2

As $\cos x_0 = \cos(x_0 - \alpha)$

We have

$$\begin{aligned} \sin^2 x_0 &= 1 - \cos^2 x_0 \\ &= 1 - \cos^2(x_0 - \alpha) \qquad \text{(from (i))} \\ &= \sin^2(x_0 - \alpha) \\ \sin x_0 &= \pm \sin(x_0 - \alpha) \end{aligned}$$

But at x_0

$$\sin x_0 = \sin(x_0 - \alpha) + k$$

with $\quad k > 0$

so $\quad \sin x_0 \neq \sin(x_0 - \alpha)$

Hence $\quad \sin x_0 = -\sin(x_0 - \alpha)$

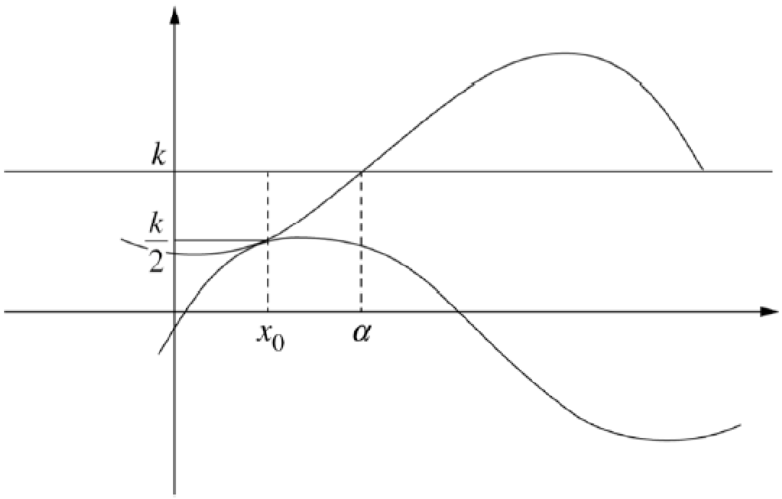
Part 3

At x_0 $\quad \sin x_0 = \sin(x_0 - \alpha) + k$

$\sin x_0 = -\sin x_0 + k$ $\qquad$ from (ii)

$2\sin x_0 = k$

$\sin x_0 = \dfrac{k}{2}$



Now $0 < \alpha < \pi$ and $0 < x_0 < \dfrac{\pi}{2}$

So $-\pi < x_0 - \alpha < \dfrac{\pi}{2}$

$\cos(x_0 - \alpha) = \cos x_0$

$\sin(x_0 - \alpha) = -\sin x_0$

Hence $\qquad x_0 - \alpha = 2n\pi - x_0$

and so $\qquad x_0 - \alpha = -x_0 \qquad as \quad -\pi < x_0 < \dfrac{\pi}{2}$

$2x_0 = \alpha$

$x_0 = \dfrac{\alpha}{2}$

Thus $\qquad k = 2\sin\dfrac{\alpha}{2}$