X2 - 1 Year 12 - Ext 2 - Trial and HSC Revision - Sheet 1

WORKED SOLUTIONS

Question 1 {Proofs}

Prove by contradiction that $\sqrt{2}$ is irrational.

Assume that $\sqrt{2}$ is rational and see if that leads to a contradiction

1	$\sqrt{2} = \frac{m}{n}$	Where $\frac{m}{n}$ is a fraction in its simplest possible form (i.e. m and n have no common factors)
2	$2 = \frac{m^2}{n^2}$	Square both sides
3	$2n^2 = m^2$	rearrange
4		Which means that m^2 is even. Which means that m is even
5	$m = 2k$ $m^2 = 4k^2$	Because m is even
6	$2n^2 = 4k^2$	Rewriting from line 3
7	$n^2 = 2k^2$	By the same logic as line 4, this means that n is also even
8		BUT the assumption was that m and n did not have a common factor. If they're both even, they have a common factor of 2.

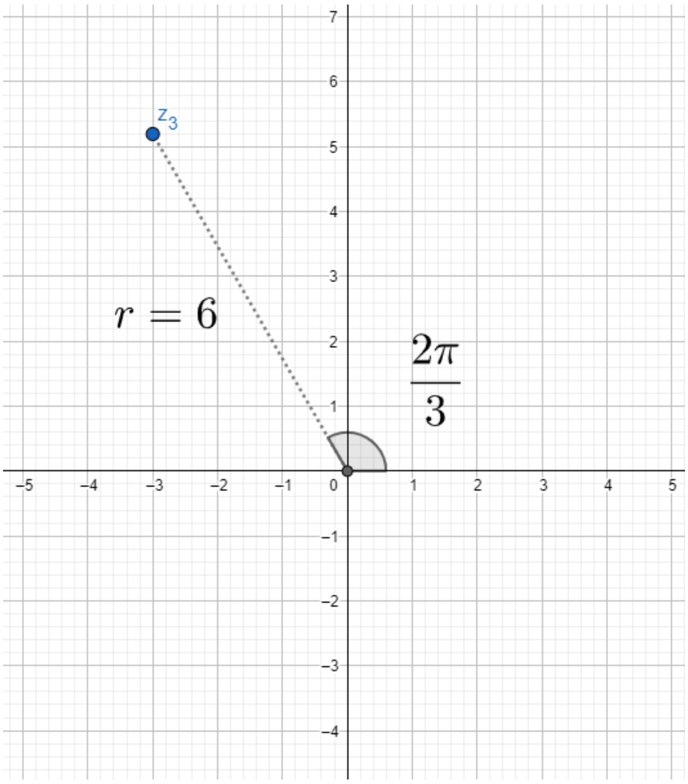
Therefore, by contradiction, $\sqrt{2}$ must be irrational.

Question 2 {Complex numbers}

Given $z_1 = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$ and $z_2 = 3\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$, sketch $z_1 z_2$ on the complex plane.

Using $z_1 z_2 = r_1 r_1 \text{cis}(\theta_1 + \theta_2)$

$z_3 = 6\text{cis}\left(\dfrac{2\pi}{3}\right)$



Question 3 {further induction} Prove by mathematical induction that $3^n + 2^n$ is divisible by 5 for all positive integers n such that n is odd.

Hint: the first step involves proving that it holds for $n = 1$

Step 1 - demonstrate that it works for $n = 1$

$3^1 + 2^1 = 5$

Which is clearly divisible by 5

Step 2 - assume that it works for some arbitrary odd integer $n = k$

$$3^k + 2^k = 5p$$

Which rearranges to $3^k = 5p - 2^k$

Step 3 - demonstrate that is mean it works for $n = k + 2$

Note: it has to be $k + 2$ because, if k is an odd number, the next odd number is $k + 2$

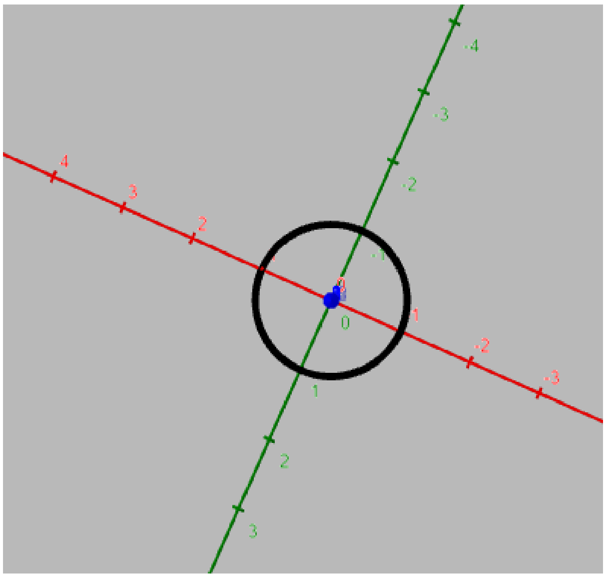
RTP: $3^{k+2} + 2^{k+2} = 5q$

$3^2 3^k + 2^2 2^k = 5q$	Index laws
LHS = $3^2\left(5p - 2^k\right) + 2^2 2^k$	Use inductive step
$= 9 \cdot 5p - 9 \cdot 2^k + 4 \cdot 2^k$	Expand brackets
$= 9 \cdot 5p - 5 \cdot 2^k$	Collect like terms
$= 5\left(9p - 2^k\right)$	Factorise
$= \ RHS$	Because p and k are integers

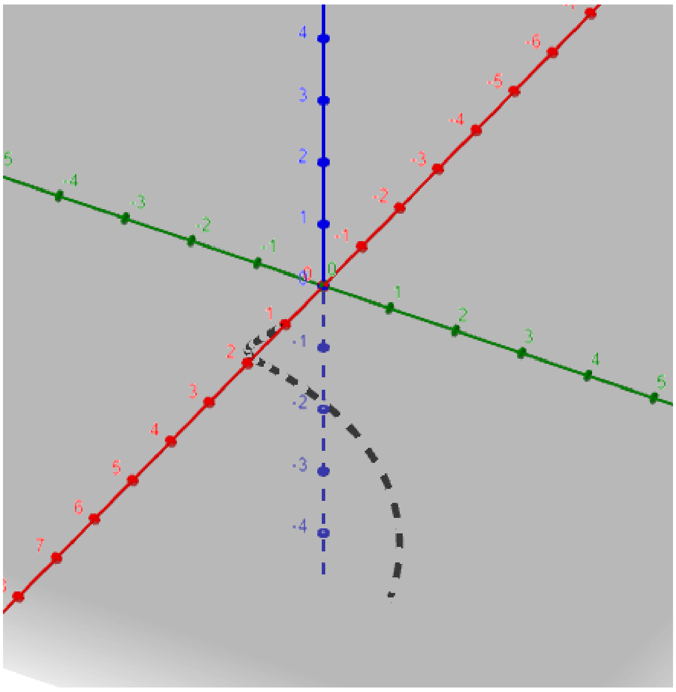
Therefore, by induction, $P(n)$ is true for all positive odd integers n

Question 4 {vectors} Describe / sketch the graph of the vector function $\begin{pmatrix} \cos t \\ \sin t \\ -t \end{pmatrix}$ for $t \geq 0$

Ignoring the z-axis, it looks like this



The value on the z-axis just descends as t gets bigger. So the net effect is a spiral that descends from the x-y plane, radius 1.



Question 5 {integration} Integrate the following

a) $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}dx$

Complete the square

$$
\begin{aligned}
&3-2x-x^2 \\
&=3-\left(2x+x^2\right) \\
&=3-\left[(x+1)^2-1\right] \\
&=4-(x+1)^2
\end{aligned}
$$

Therefore the integral becomes

$$\int \frac{1}{\sqrt{4-(x+1)^2}}dx$$

Which we recognise from the formula sheet as matching the pattern

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a}+c$$

Where $a=2,\ f(x)=x+1,\ f'(x)=1$

Therefore the integral evaluates to $\sin^{-1}\dfrac{x+1}{2}+C$

b) $\int \frac{3x+1}{(x-3)(x+2)} dx$

Use partial fractions and equate coefficients

$$\frac{3x+1}{(x-3)(x+2)} = \frac{a}{x-3} + \frac{b}{x+2}$$

$$= \frac{a(x+2)+b(x-3)}{(x-3)(x+2)}$$

$$= \frac{(a+b)x+(2a-3b)}{(x-3)(x+2)}$$

Therefore

$a+b=3$
$2a-3b=1$

$a=3-b$	Rearrange (1)
$2(3-b)-3b=1$	Sub into (2)
$6-5b=1$	Expand and collect like terms
$b=1$	Solve
$a=2$	Sub back into (1)

So the integral becomes

$$\int \frac{2}{x-3} dx + \int \frac{1}{x+2} dx$$

$$= 2\ln|x-3| + \ln|x+2| + C$$