X1 - 2 Year 12 - Ext 1 - Trial and HSC Revision - Sheet 2

Name:

Question 1 {Polynomials}

a) Show that $(x-4)$ is one of the factors of $x^3 - 2x^2 - 11x + 12$. HINT: Show that there is no remainder when you divide by $x - 4$.

b) Hence or otherwise, express $x^3 - 2x^2 - 11x + 12$ in fully factored form.

a) Use the Remainder Theorem

$$\begin{aligned} f(4) &= 4^3 - 2(4)^2 - 11 \cdot 4 + 12 \\ &= 64 - 32 - 44 + 12 \\ &= 0 \end{aligned}$$

Therefore $(x-4)$ is a factor

b) Divide the polynomial by $(x-4)$

$x - 4\overline{)x^3 - 2x^2 - 11x + 12} = x^2 + 2x - 3$ (working left to the reader)

$$x^2 + 2x - 3 = (x+3)(x-1)$$

Therefore

$$x^3 - 2x^2 - 11x + 12 = (x-4)(x+3)(x-1)$$

Question 2 {Inverse functions} Find and sketch the inverse of $f(x) = x^2 - 4$ and state the domain and range of the inverse function.

$y = x^2 - 4$
$x = y^2 - 4$
$x + 4 = y^2$
$y = \pm\sqrt{x + 4}$

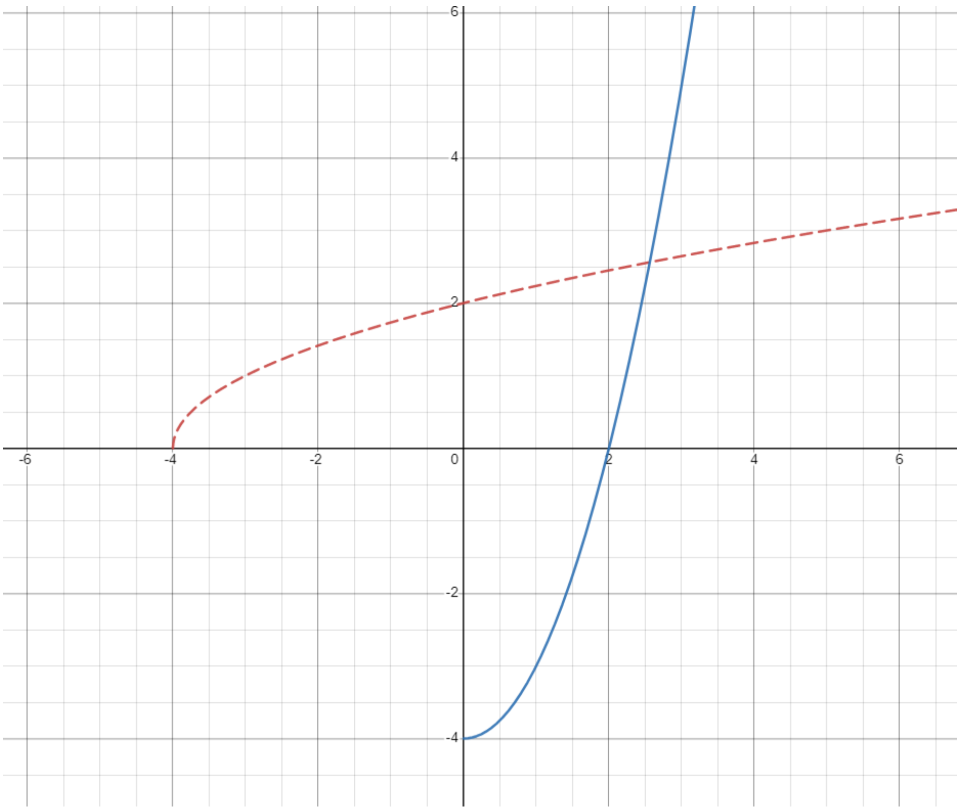
In order for the inverse to be a function, a domain restriction must be applied to the original function $x \geq 0$

Therefore, for the original function, the domain is $x \geq 0$ and the range is $y \geq 4$

When the inverse is found, the domain and range swap so, for the inverse function

Domain: $x \geq 4$
Range: $y \geq 0$

Original as solid line, inverse as dashed line



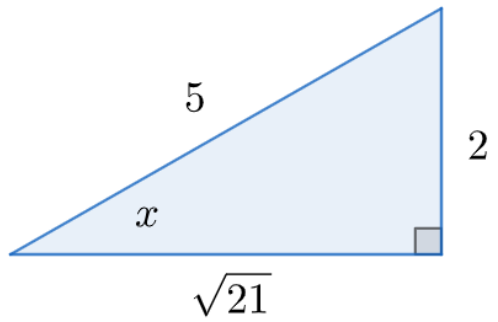
Question 3 {trig} - given $\sin x = \frac{2}{5}$, find the exact value of $\sin 2x$.

Hint: check page 610 of the Year 11 textbook (on Team).

Use a double angle formula

$\sin 2x = 2\sin x \cos x$

Draw a diagram to find $\cos x$



$\sin x = \frac{2}{5}$ as shown

Third side filled in using Pythagoras

$\therefore \cos x = \frac{\sqrt{21}}{5}$

Therefore

$$\begin{aligned}\sin 2x &= 2 \cdot \frac{2}{5} \cdot \frac{\sqrt{21}}{5} \\ &= \frac{4\sqrt{21}}{25}\end{aligned}$$

Question 4 {related rates of change}

A stone drops into a pond, creating a circular ripple. The radius of the ripple increases from 0 cm, at a constant rate of 5 cm s^{-1}.

At what rate is the area enclosed within the ripple increasing when the radius is 15 cm?

A. 25π $\text{cm}^2\,\text{s}^{-1}$

B. 30π $\text{cm}^2\,\text{s}^{-1}$

C. 150π $\text{cm}^2\,\text{s}^{-1}$

D. 225π $\text{cm}^2\,\text{s}^{-1}$

Assembling the pieces	**Putting it all together**
$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ $\frac{dr}{dt} = 5$- (given in question) $A = \pi r^2$ $\frac{dA}{dr} = 2\pi r$ $@r = 15$ $\frac{dA}{dr} = 30\pi$	$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ $\frac{dA}{dt} = 30\pi \cdot 5$ $= 150\pi\, cm^2 s^{-1}$

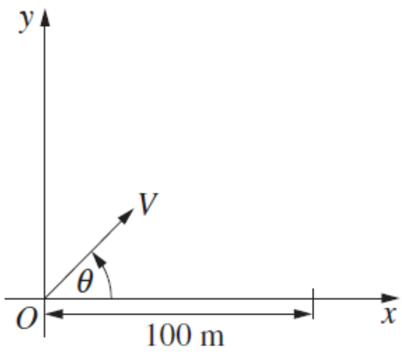
Question 5 {further logs and exponents}

At time t the displacement, x, of a particle satisfies $t = 4 - e^{-2x}$. **3**

Find the acceleration of the particle as a function of x.

Question 6 {vectors}

A golfer hits a golf ball with initial speed V m s^{-1} at an angle θ to the horizontal. The golf ball is hit from one side of a lake and must have a horizontal range of 100 m or more to avoid landing in the lake.



Neglecting the effects of air resistance, the equations describing the motion of the ball are

$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

where t is the time in seconds after the ball is hit and g is the acceleration due to gravity in m s^{-2}. Do NOT prove these equations.

(i) Show that the horizontal range of the golf ball is $\dfrac{V^2 \sin 2\theta}{g}$ metres. **2**

(ii) Show that if $V^2 < 100g$ then the horizontal range of the ball is less than 100 m. **1**

Part 1

The range is the distance from where the ball launches to where it lands (i.e. when the vertical displacement is zero).

$$\begin{aligned} Vt\sin\theta - \frac{1}{2}gt^2 &= 0 \\ Vt\sin\theta &= \frac{1}{2}gt^2 \\ V\sin\theta &= \frac{1}{2}gt \\ \frac{2V\sin\theta}{g} &= t \end{aligned}$$

Which gives us an expression for the time it takes for the ball to land.

Sub that into the formula for the horizontal displacement

$$\begin{aligned} x &= Vt\cos\theta \\ &= V\left(\frac{2V\sin\theta}{g}\right)\cos\theta \\ &= \frac{2V^2\sin\theta\cos\theta}{g} \end{aligned}$$

Looking at the numerator, it includes $2\sin\theta\cos\theta$ which we know, from double angle formulas, is the same as $\sin 2\theta$

So we can rearrange the formula to $x = \dfrac{V^2\sin 2\theta}{g}$ which is what we were asked to prove.

Part 2

Basically, the statement says that, if we don't hit the ball hard enough, it won't make the required length. Put another way, if it doesn't make the required length, we didn't hit the ball hard enough. So let's assume it doesn't make the required 100m

$$\frac{V^2\sin 2\theta}{g} < 100$$

$$V^2\sin 2\theta < 100g$$

Which is very close to the required solution.

Consider $\sin 2\theta$. The sine of anything, 2θ or otherwise, can't be higher than 1. This means that the highest value of the LHS is $V^2 \times 1 = V^2$

Which means that $V^2\sin 2\theta < 100g$ QED

Question 7 {induction}

Prove by mathematical induction that $8^{2n+1} + 6^{2n-1}$ is divisible by 7, for any integer $n \geq 1$. **3**

There's an important trick about half-way down this proof. This fact will be important

$$6^{2n+1} = 6^{2n-1+2}$$

Step 1 - check for n=1

$$\begin{aligned} 8^3 + 6^1 &= 518 \\ &= 7 \times 74 \end{aligned}$$

All good, step 1

Step 2 - assume for some arbitrary integer $n = k$

$$8^{2k+1} + 6^{2k-1} = 7p$$

Which we pre-emptively rearrange to

$$8^{2k+1} = 7p - 6^{2k-1}$$

Step 3 - RTP that this implies the proposition is true for $n = k + 1$

I.e. $8^{2k+3} + 6^{2k+1} = 7q$

Re-arranging the LHS so that we an use the inductive step

$$\begin{aligned} &= 8^{2k+1+2} + 6^{2k-1+2} \\ &= 8^2 \cdot 8^{2k+1} + 6^2 \cdot 6^{2k-1} \end{aligned}$$

Bring in the inductive step

$$= 8^2\left(7p - 6^{2k-1}\right) + 6^2 \cdot 6^{2k-1}$$

Then expand and collect like terms

$$\begin{aligned} &= 64 \cdot 7p - 28 \cdot 6^{2k-1} \\ &= 7\left(64p - 4 \cdot 6^{2k-1}\right) \end{aligned}$$

And because p and k are integers, this equals the RHS

Therefore, by induction, $P(n)$ is true for $n \geq 1$

Question 8 {binomial distribution}

8 In Havana, Cuba, around 85% of the cars were built before 1960. A sample of 112 cars was taken. Find the probability that in this sample the percentage of cars built before 1960 is:

a less than 80%

b at least 90%

c at least 75%

d between 75% and 80%

Start with sample probability

$$\mu = p = 0.85$$

$$\begin{aligned}\sigma^2 &= \frac{p(1-p)}{n} \\ &= \frac{0.85 \times 0.15}{112} \\ &= 0.00113\end{aligned}$$

$$\therefore \sigma = 0.034\,(2dp)$$

To find probabilities, use z-scores

a) 80%=0.8

$$\begin{aligned}z &= \frac{0.8-0.85}{0.034} \\ &= -1.48\end{aligned}$$

Consult the table to find $P(z < -1.48) = 0.0694$

b) 90%=0.9

$$\begin{aligned}z &= \frac{0.9-0.85}{0.034} \\ &= 1.48\end{aligned}$$

$$P(z = 1.48) = 0.9306$$

Therefore probability of at least 90% is 0.0694

c) 75% = 0.75

$$z = \frac{0.75 - 0.85}{0.034}$$
$$= -2.94$$

$$P(z = -2.94) = 0.0016$$

Therefore probability of at least 75 = 0.9984

d) 0.0678

Question 9 {differential equations}

2 The number of cattle N on a property is growing over t years according to the equation $\frac{dN}{dt} = 0.18N$.

a Solve this equation, given that the initial number of cattle is 600.

b Find the number of cattle after:

i 5 years **ii** 10 years

c Find how long it will take for the number of cattle to reach 2000.

d Find the rate at which the number of cattle is growing after:

i 5 years **ii** 10 years

Step 1 $\frac{dN}{dt} = 0.18N$ $\therefore \frac{dt}{dN} = \frac{1}{0.18N}$ $\therefore t = \frac{1}{0.18}\int \frac{1}{N}dN$ $\therefore t = \frac{1}{0.18}\ln\|N\| + C$ Rearranging $0.18(t-C) = \ln\|N\|$ $e^{0.18t-0.18C} = N$ $Ae^{0.18t} = N$ Use initial value conditions At $t = 0,\ N = 600$ $\therefore A = 600$ $N = 600e^{0.18t}$	b and c are left for the reader Find the rate at which …. We need to find $\frac{dN}{dt}$ $N = 600e^{0.18t}$ $\therefore \frac{dN}{dt} = 108e^{0.18t}$ Then apply the values of t.