X2 - 2 Year 12 - Ext 2 - Trial and HSC Revision - Sheet 2

Name:

Question 1 {Proofs}

Consider the proposition:

'If $2^n - 1$ is not prime, then n is not prime'.

Given that each of the following statements is true, which statement disproves the proposition?

A. $2^5 - 1$ is prime

B. $2^6 - 1$ is divisible by 9

C. $2^7 - 1$ is prime

D. $2^{11} - 1$ is divisible by 23

Consider the contrapositive $\neg Q \to \neg P$

Which would read "If n is prime, $2^n - 1$ is prime"

To show that this is not true, we will disprove by counter-example.

Look at D. If $2^{11} - 1$ is divisible by 23, it's not prime. Yet $n = 11$ is prime.

Therefore the contrapositive is disproven.

Therefore the original proposition is disproven by D.

Question 2 {Complex numbers}

Consider the complex numbers $w = -1 + 4i$ and $z = 2 - i$.

(i) Evaluate $|w|$. **1**

(ii) Evaluate $w\overline{z}$. **2**

$$
\begin{aligned}
|w| &= \sqrt{(-1)^2 + 4^2} \\
&= \sqrt{17}
\end{aligned}
$$

$$\overline{z} = 2 + i$$

$$
\begin{aligned}
\therefore w\overline{z} &= (-1 + 4i)(2 + i) \\
&= -2 - i + 8i - 4 \\
&= -6 + 7i
\end{aligned}
$$

Question 3 {vectors}

Consider the two lines in three dimensions given by **3**

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}.$$

By equating components, find the point of intersection of the two lines.

X-components

$$3 + \lambda_1 = 3 - 2\lambda_2$$
$$\lambda_1 = -2\lambda_2$$

Y-components (including incorporating the results from the x-components)

$$-1 + 2\lambda_1 = -6 + \lambda_2$$
$$-1 + 2(-2\lambda_2) = -6 + \lambda_2$$
$$-1 - 4\lambda_2 = -6 + \lambda_2$$
$$5 = 5\lambda_2$$
$$\lambda_2 = 1$$

Using the x-component result again

$$\lambda_1 = -2$$

Confirm the results using the z-components

$$7 + \lambda_1 = 2 + 3\lambda_2$$
$$7 + (-2) = 2 + 3(1)$$
$$5 = 5$$

Therefore finding the point

$$x = 3 + \lambda_1 = 1$$
$$y = -1 + 2\lambda_1 = -5$$
$$z = 7 + \lambda_1 = 5$$

So the point of intersection is $(1, -5, 5)$

Question 4 {induction}

Prove by mathematical induction that, for $n \geq 2$, **4**

$$\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}<\frac{n-1}{n}.$$

Step 1 - check for $n = 2$

$$\frac{1}{2^2}+\frac{1}{3^2}<\frac{2-1}{2}$$
$$\frac{1}{4}+\frac{1}{9}<\frac{1}{2}$$
$$\frac{9}{36}+\frac{4}{36}<\frac{18}{36}$$
$$\frac{13}{36}<\frac{18}{36}$$

Which is true so Step 1 is proven

Step 2 - assume for some integer $k \geq 2$

$$\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{k^2}<\frac{k-1}{k}$$

Step 3 - check if this implies it works for $n = k+1$

RTP $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{k^2}+\frac{1}{(k+1)^2}<\frac{k}{k+1}$

$\frac{k-1}{k}+\frac{1}{(k+1)^2}<\frac{(k+1)-1}{k+1}$	Use inductive step
LHS = $\frac{(k-1)(k+1)^2+k}{k(k+1)^2}$	Common denominator
$=\frac{k^3+k^2-1}{k(k+1)^2}$	Expanding and collecting like terms

RHS = $\dfrac{k^2(k+1)}{k(k+1)^2}$	Over the same denominator as the LHS to enable comparison
$= \dfrac{k^3+k^2}{k(k+1)^2}$	Expanding the top

Given that the numerator of the LHS is one less than the numerator of the RHS,

$LHS < RHS$

Therefore, by induction, $P(n)$ is proven for $n \geq 2$

Question 5 {integration}

Use integration by parts to evaluate $\displaystyle\int_1^e x \ln x \, dx$. **3**

Using $\displaystyle\int uv' dx = uv - \int vu' dx$

$u = \ln x,\ u' = \dfrac{1}{x}$

$v' = x,\ v = \dfrac{x^2}{2}$

Therefore

$$\begin{aligned}\int x \ln x \, dx &= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x^2 \cdot \frac{1}{x} dx \\ &= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x \, dx \\ &= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C\end{aligned}$$

Applying the bounds of integration (which means we can lose the C)

$$\begin{aligned}\int_1^e x \ln x \, dx &= \left[\frac{x^2 \ln x}{2} - \frac{x^2}{4}\right]_1^e \\ &= \left[\left(\frac{e^2}{2} - \frac{e^2}{4}\right) - \left(\frac{1^2 \cdot 0}{2} - \frac{1}{4}\right)\right] \\ &= \frac{e^2}{4} + \frac{1}{4} \\ &= \frac{1}{4}\left(e^2 + 1\right)\end{aligned}$$