X2 - 3 Year 12 - Ext 2 - Trial and HSC Revision - Sheet 3

Name:

Question 1 {Complex numbers}

(a) Let $z = \dfrac{2-3i}{1+i}$.

(i) Find $\overline{z}$ in the form $x + iy$. **2**

(ii) Evaluate $|z|$. **1**

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Part 1

Realise the denominator

$$z = \frac{2-3i}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{-1-5i}{2}$$
$$= \frac{-1}{2} - \frac{5}{2}i$$

Now find the complex conjugate

$$\bar{z} = -\frac{1}{2} + \frac{5}{2}i$$

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Part 2

$$|z| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{-5}{2}\right)^2}$$
$$= \sqrt{\frac{1}{4} + \frac{25}{4}}$$
$$= \sqrt{\frac{26}{4}}$$
$$= \frac{\sqrt{26}}{2}$$

</td>
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Question 2 {Complex numbers}

Consider $w = -\sqrt{3} + i$.

(i) Express w in modulus-argument form. **2**

(ii) Hence or otherwise show that $w^7 + 64w = 0$. **2**

Part 1

$$w = -\sqrt{3} + i$$

$$\begin{aligned} |w| &= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} \\ &= \sqrt{3+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\arg(w) = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$$

Using an exact value triangle, this means $\arg(w)$ is in the $\frac{\pi}{6}$ family of angles

We also know that w is in the second quadrant. (negative real part, positive imaginary part)

So $\arg(w) = \frac{5\pi}{6}$

Therefore

$$w = 2\text{cis}\left(\frac{5\pi}{6}\right)$$

Or

$$w = 2e^{i\frac{5\pi}{6}}$$

Part 2

We need to show that

$$w^7 = -64w$$

$$\begin{aligned} w^7 &= \left(2e^{i\frac{5\pi}{6}}\right)^7 \\ &= 2^7 e^{i\frac{35\pi}{6}} \\ &= 128e^{i\frac{35\pi}{6}} \end{aligned}$$

$$64w = 128e^{i\frac{5\pi}{6}}$$

Which seems like we're getting somewhere. Let's look at the arguments.

w^7 has an argument of $\frac{35\pi}{6}$ which is $\frac{\pi}{6}$ short of 6 full rotations (so in the 4th quadrant)

$64w$ has an argument of $\frac{5\pi}{6}$ which is $\frac{\pi}{6}$ short of half a rotation (so in the 2nd quadrant, diagonally opposite $\frac{35\pi}{6}$

Given that they have the same modulus (i.e. length) and their directions are opposite each other, we can say that

$$w^7 = -64w$$

Which is what we had to prove - QED

Question 3 {vectors}

Relative to a fixed origin O, the respective position vectors of three points A , B and C are:

$$\begin{pmatrix}3\\2\\9\end{pmatrix}, \begin{pmatrix}-5\\11\\6\end{pmatrix} \text{ and } \begin{pmatrix}4\\0\\-8\end{pmatrix}.$$

(i) Determine, in component form, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. **2**

(ii) Hence find, to the nearest degree, $\angle BAC$. **2**

(iii) Calculate the area of ΔBAC. **2**

Trying to do this on a graph would be very hard. We'll need to trust the algebra

Part 1

$\vec{AB}$ goes from A to B.

Therefore

$$\begin{aligned}\vec{AB} &= \begin{pmatrix}-5-3\\11-2\\6-9\end{pmatrix}\\ &= \begin{pmatrix}-8\\9\\-3\end{pmatrix}\end{aligned}$$

Similarly

$$\begin{aligned}\vec{BC} &= \begin{pmatrix}4-(-5)\\0-11\\-8-6\end{pmatrix}\\ &= \begin{pmatrix}9\\-11\\-14\end{pmatrix}\end{aligned}$$

Part 2

Use the two formulas for the dot product

$$x_1x_2 + y_1y_2 + z_1z_2 = |u||v|\cos\theta$$

Which rearranges to

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|u||v|}$$

$$\begin{aligned} |AB| &= \sqrt{(-8)^2 + 9^2 + (-3)^2} \\ &= \sqrt{154} \end{aligned}$$

$$\begin{aligned} |BC| &= \sqrt{9^2 + (-11)^2 + (-14)^2} \\ &= \sqrt{398} \end{aligned}$$

Therefore

$$\cos\theta = \frac{(-8)9 + 9(-11) + (-3)(-14)}{\sqrt{154}\cdot\sqrt{398}}$$

$$\begin{aligned} &\therefore \cos\theta \approx -0.52 \\ &\therefore \theta \approx 121^o \ (nd) \end{aligned}$$

Part 3

This is just a simple application of the formula for the area of a triangle

$$A = \frac{1}{2}ab\sin C$$

$$A = \frac{1}{2}\cdot\sqrt{154}\cdot\sqrt{398}\cdot\sin\left(121^o\right)$$

$$A \approx 105.65\, u^2\, (2\, dp)$$

Question 4 {integrals}

By completing the square find $\displaystyle\int \frac{1}{\sqrt{6 - x^2 - x}}\,dx.$ **2**

Consider

$$\begin{aligned}
&6 - x^2 - x \\
&= 6 - \left(x^2 + x\right) \\
&= 6 - \left[\left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4}\right] \\
&= \frac{25}{4} - \left(x + \frac{1}{2}\right)^2
\end{aligned}$$

Therefore, our integral becomes

$$\int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2}} dx$$

Which we recognise as being the pattern for $\sin^{-1}$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1}\frac{f(x)}{a} + c$$

Therefore our integral becomes

$$\sin^{-1}\frac{\left(x + \frac{1}{2}\right)}{\frac{5}{2}} + C$$

$$= \sin^{-1}\frac{2x + 1}{5} + C$$

Question 5 {proofs}

Prove if $x, y \in \mathbb{Z}$, then $x^2 - 4y \neq 2$. **3**

Let's try proving by contradiction

Assume that $x^2 - 4y = 2$ and see if that leads to a contradiction. Remember x and y need to be integers.

$x^2 - 4y = 2$	assumption
$x^2 = 2 + 4y$	rearrange
$x^2 = 2(1 + 2y)$	factorise
$x = \pm\sqrt{2(1 + 2y)}$	Square root both sides
$x = \pm\sqrt{2} \cdot \sqrt{1 + 2y}$	Separating the surds

Which makes us doubt that x could be an integer because it has a factor of $\sqrt{2}$

HOWEVER, is there any way that $\sqrt{2}$ could be a factor of an integer?

Yes, there is. For examples:

$$4 = 2\sqrt{2} \times \sqrt{2} = \sqrt{8} \times \sqrt{2} = \sqrt{16}$$

$$6 = 3\sqrt{2} \times \sqrt{2} = \sqrt{18} \times \sqrt{2} = \sqrt{36}$$

$$8 = 4\sqrt{2} \times \sqrt{2} = \sqrt{32} \times \sqrt{2} = \sqrt{64}$$

Which means that the only way an integer can have a factor of $\sqrt{2}$ is for the other factor to be the square root of an even number. If you're not convinced, consider that the only way to get rid of the $\sqrt{2}$ is to multiply it by itself.

So can $\pm\sqrt{2} \cdot \sqrt{1 + 2y}$ be an integer?

Asked another way, can the other factor, $\sqrt{1 + 2y}$, be the square root of an even number?

In short, no. Why? Because $2y + 1$ is, by definition, an odd number.

Therefore, our assumption has led us to a contradiction. Which means that the original proposition was true.

Question 6 {proofs}

Suppose that a_n $(n \geq 1)$ is a sequence defined by:

$$a_1 = 1, \quad a_2 = 3 \quad \text{and} \quad a_k = a_{k-1} + a_{k-2} \quad \forall\, k \geq 3.$$

Prove that $\forall\, n \geq 1$, we have $a_n \leq \left(\frac{7}{4}\right)^n$. **3**

This is a **very tough** induction question with a definite twist or two in it.

Step 1

First of all, we will need to establish that it's true for the first **two** terms which aren't covered by the definition involving k. This is different from the usual $n = 1$ requirement

$a_1 = 1$

$$\left(\frac{7}{4}\right)^1 = \frac{7}{4} \geq 1$$

$a_2 = 3$

$$\left(\frac{7}{4}\right)^3 = \frac{343}{64} \approx 5.36 \geq 3$$

So we're all good for the first two terms.

Step 2

Assume that it holds for some arbitrary $n = k$

I.e. that $a_k \leq \left(\frac{7}{4}\right)^k$

Because of the nature of the recursive relationship, that uses the two preceding terms, we are also going to need to assume that

$$a_{k-1} \leq \left(\frac{7}{4}\right)^{k-1}$$

Step 3

RTP that this infers that

$$a_k + a_{k-1} \leq \left(\frac{7}{4}\right)^{k+1}$$

<table>
<tr><td>$LHS \leq \left(\frac{7}{4}\right)^{k} + \left(\frac{7}{4}\right)^{k-1}$</td><td>Using the inductive step</td></tr>
<tr><td colspan="2">As some point I need to turn the LHS into $\left(\frac{7}{4}\right)^{k+1}$ to match the RHS.

Which means I’m going to have to multiply either $\left(\frac{7}{4}\right)^{k}$ or $\left(\frac{7}{4}\right)^{k-1}$ by some power of $\frac{7}{4}$ to add to the indices

e.g. $\left(\frac{7}{4}\right)^{k-1} \times \left(\frac{7}{4}\right)^{2} = \left(\frac{7}{4}\right)^{k+1}$</td></tr>
<tr><td>$LHS \leq \left(\frac{7}{4}\right)^{k-1} \left(\frac{7}{4} + 1\right)$</td><td>So I create a factorisation which gives me a number, independent of any k, as a factor which I could turn into some power of $\frac{7}{4}$</td></tr>
<tr><td>$LHS \leq \left(\frac{7}{4}\right)^{k-1} \left(\frac{11}{4}\right)$</td><td>Just tidying up the brackets</td></tr>
<tr><td colspan="2">At this point, I’m not sure how to get a power of $\frac{7}{4}$ out of that factor.

What I do know, though, is that $\left(\frac{7}{4}\right)^{2} = \frac{49}{16}$

And $\frac{44}{16} < \frac{49}{16}$

And $\frac{11}{4} = \frac{44}{16}$</td></tr>
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$LHS \leq \left(\frac{7}{4}\right)^{k-1}\left(\frac{44}{16}\right)$	Turn $\frac{11}{4}$ into its equivalent $\frac{44}{16}$
$LHS \leq \left(\frac{7}{4}\right)^{k-1}\left(\frac{44}{16}\right) \leq \left(\frac{7}{4}\right)^{k-1}\left(\frac{49}{16}\right)$	It would then be fair to say this about the LHS
$LHS \leq \left(\frac{7}{4}\right)^{k-1}\left(\frac{49}{16}\right)$	Or putting it simply
$LHS \leq \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}\right)^{2}$	There's that power of $\frac{7}{4}$ I needed
$LHS \leq \left(\frac{7}{4}\right)^{k+1}$	Applying index laws

Which is what we had to prove

Phew!