X1 - 3 Year 12 - Ext 1 - Trial and HSC Revision - Sheet 3

Name:

Question 1 {Perms and Combs}

To complete a course, a student must choose and pass exactly three topics. **2**

There are eight topics from which to choose.

Last year 400 students completed the course.

Explain, using the pigeonhole principle, why at least eight students passed exactly the same three topics.

Hint: Given you have to pass 3 out of 8 topics, how many ways are there to pass the course?

Start by assessing how many ways there are to pass the course, which will be ${}_8\mathrm{C}_3 = 56$

Out of 400 kids, there are 56 ways to pass the course.

$400\, MOD\, 56 \;=\; 8$

Therefore, at least 8 students had to have done the same three topics.

Question 2 {inequalities} $f(x) = (x+5)\left(x^2+6x+5\right)$. By finding the roots or otherwise, solve the inequality $f(x) \geq 0$

$f(x) = (x+5)(x+5)(x+1)$

Therefore solve $(x+5)(x+5)(x+1) \geq 0$

The roots are therefore at $x = -5\, and\, x = -1$

Test around the roots

x	-6	-5	-4	-1	0			
$f(x)$	-5	0	-3	0	25			

Therefore $f(x) \geq 0\, at\, x = -5\, or\, x \geq -1$

Question 3 {further trig} - solve $4\sin\theta\cos\theta = -2$ for $0 \leq \theta \leq 2\pi$. HINT: divide by 2 first.

$2\sin\theta\cos\theta = -1$
$\sin 2\theta = -1$

Adjust domain to $0 \leq 2\theta \leq 4\pi$

Angle family $2\theta = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}$

ASTC doesn't apply here because of the -1

Therefore $\theta = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

Question 4 {further logs and exponents}

4 A chemical reaction causes the amount of chlorine to be reduced at a rate proportional to the amount of chlorine present at any one time. If the amount of chlorine is given by the formula $A = A_0e^{-kt}$ and 100 L reduces to 65 L after 5 minutes, find:

a the amount of chlorine after 12 minutes

b how long it will take for the chlorine to reduce to 10 L.

Set up	Use
$A = A_0e^{-kt}$ $@t = 0,\ A = 100$ $100 = A_0e^0$ $\therefore A_0 = 100$ $@t = 5,\ A = 65$ $65 = 100e^{-5k}$ $\dfrac{65}{100} = e^{-5k}$ $\ln\left\|\dfrac{65}{100}\right\| = -5k$ $k \approx 0.086\ (3dp)$	$@t = 12$ $A = 100e^{-0.086\times 12}$ $A \approx 35.63\ L$ $@A = 10$ $10 = 100e^{-0.086t}$ $0.1 = e^{-0.086t}$ $\ln 0.1 = -0.086t$ $t = \ln 0.1 \div -0.086$ $t \approx\ 26.77\, m\ (2dp)$

Question 5 {further logs and exponents}

At time t the displacement, x, of a particle satisfies $t = 4 - e^{-2x}$. **3**

Find the acceleration of the particle as a function of x.

This is an Ext 2 question

Question 6 {vectors}

A projectile is launched from a height of 20m, at an angle of 45^o and with an initial velocity of $30\,ms^{-1}$. Find the horizontal range of the projectile.

Firstly, derive the necessary equations

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Assume $a = -10$

Find velocity

$$v_j = \int a\,dt$$
$$= -10t + C$$

Use initial value conditions

$@t = 0,\ v_j = 30\sin 45$

$\therefore C = 30\sin 45 = \dfrac{30}{\sqrt{2}}$

Full statement of velocity

$$\begin{aligned} v &= [30\cos 45]_i + \left[-10t + \frac{30}{\sqrt{2}}\right]_j \\ &= \left[\frac{30}{\sqrt{2}}\right]_i + \left[-10t + \frac{30}{\sqrt{2}}\right]_j \end{aligned}$$

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Find displacement

$$s = \int v\,dt$$

$$s = \left[\frac{30t}{\sqrt{2}}\right]_i + \left[-5t^2 + \frac{30t}{\sqrt{2}} + D\right]_j$$

Use initial value conditions

$@t = 0,\ s_j = 20$

$\therefore\ D = 20$

$$\therefore s = \left[\frac{30t}{\sqrt{2}}\right]_i + \left[-5t^2 + \frac{30t}{\sqrt{2}} + 20\right]_j$$

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To answer the question

The range is the horizontal distance from take off to landing. At landing, the vertical displacement is zero.

$-5t^2 + 15\sqrt{2}t + 20 = 0$

Solve as a quadratic.

$$t = \frac{-15\sqrt{2} \pm \sqrt{\left(15\sqrt{2}\right)^2 - 4 \cdot -5 \cdot 20}}{-10}$$

$t \approx 5.04\, s\; (2dp)$

This gives a horizontal range of

$$\begin{aligned} s_i &= \frac{30 \cdot 5.04}{\sqrt{2}} \\ &= 106.91\, m\; (2\, dp) \end{aligned}$$

Question 7 {further vectors}

Given $\vec{u} = 3_i + 7_j$ and $\vec{v} = 2_i + 3_j$, find $proj_{\vec{u}}\vec{v}$

Using $proj_u v = \left(\frac{v \cdot u}{|u|^2}\right)\vec{u}$

$$v \cdot u = 3 \times 2 + 7 \times 3$$
$$= 27$$

$$|u|^2 = 3^2 + 7^2$$
$$= 58$$

$$\left(\frac{v \cdot u}{|u|^2}\right) = \frac{27}{58}$$

$$\left(\frac{v \cdot u}{|u|^2}\right)\vec{u} = \frac{81}{58}_i + \frac{189}{58}_j$$

Question 8 {trig equations}

Write $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x+\alpha)$ where $R > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$. **2**

$$
\begin{aligned}
R &= \sqrt{a^2+b^2} \\
&= \sqrt{3+1} \\
&= \sqrt{4} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \tan^{-1}\left(\frac{b}{a}\right) \\
&= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)
\end{aligned}
$$

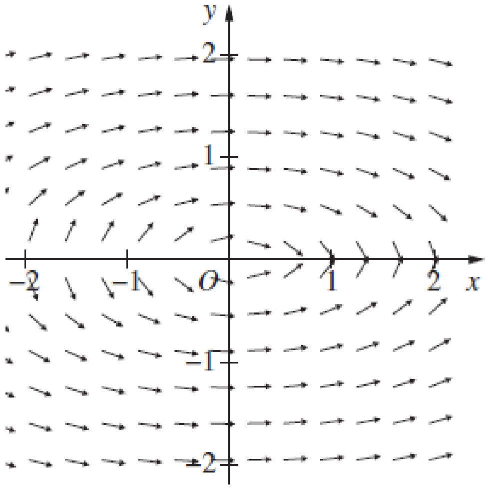
$$\therefore \alpha = \frac{\pi}{6}$$

$$\therefore \sqrt{3}\sin x + \cos x = 2\sin\left(x + \frac{\pi}{6}\right)$$

Question 9 {differential equations}

7 Which of the following best represents the direction field for the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{4y}$?

A.



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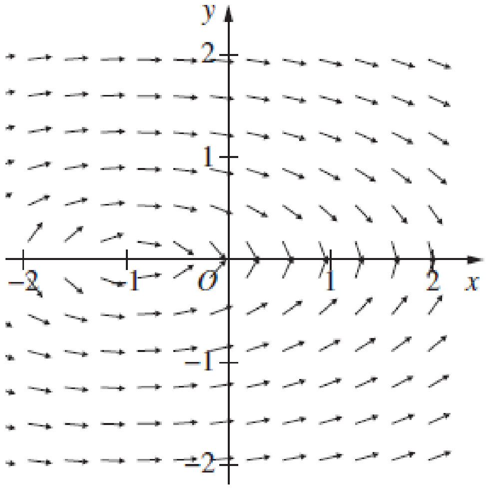

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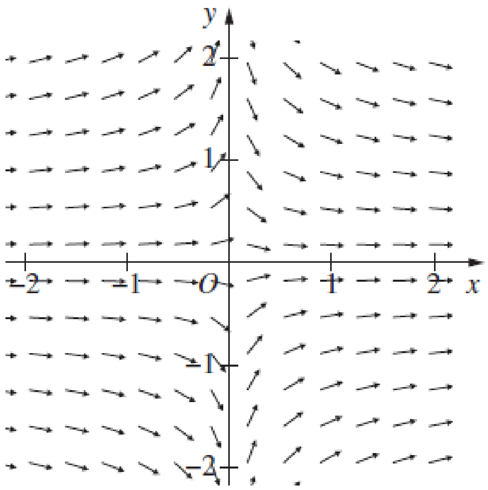

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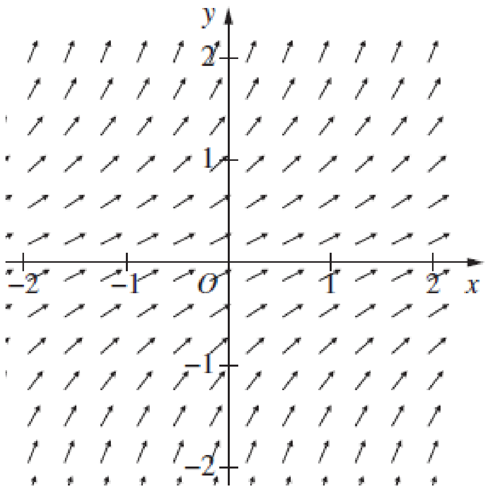

Answer = A