X2 - 4 Year 12 - Ext 2 - Trial and HSC Revision - Sheet 4

Name:

Question 1 {Integration} Evaluate the following intergrals

a) (by substitution) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)^2} dx$

$u = 1 + \sin x$
$\frac{du}{dx} = \cos x$
$du = \cos x \, dx$

$@\, x = 0,\ u = 1 + \sin(0) = 1$
$@x = \frac{\pi}{2},\ u = 1 + \sin\frac{\pi}{2} = 2$

Integral becomes

$$\int_1^2 \frac{1}{u^2} du$$

$$= \left[\frac{-1}{u}\right]_1^2$$

$$= \left[\frac{-1}{2} - \frac{-1}{1}\right]$$

$$= \frac{1}{2}$$

b) (by completing the square) $\int \frac{1}{x^2 - 2x + 5} dx$

Denominator

$x^2 - 2x + 5$

$= (x-1)^2 + 4$

Integral becomes

$$\int \frac{1}{4 + (x-1)^2} dx$$

Which, from our table of standard integrals (formula sheet) works out to be

$$= \frac{1}{2} \tan^{-1} \left(\frac{x-1}{2} \right) + C$$

c) (by partial fractions) $\displaystyle\int \frac{4+7x}{(x+1)^2(2+3x)}dx$

By re-writing in the form $\displaystyle\frac{A}{x+a}+\frac{Bx+C}{(x+a)^2}+\frac{D}{x+b}$

The partial fractions work out to be

$$\frac{A}{x+1}+\frac{Bx+C}{(x+1)^2}+\frac{D}{(2+3x)}$$

Question 2 {proof} -

Show that $1+r+r^2+r^3+\ldots+r^{n-1}=\dfrac{1-r^n}{1-r}$ for $r\neq 1$.

Step 1 - $n=1$

$$1=\frac{1-r}{1=r}$$

Step 2 - assume for $n=k$

$$1+r+r^2+r^3+\ldots+r^{k-1}=\frac{1-r^k}{1-r}$$

Step 3 - show that this implies truth for $n=k+1$

$$1+r+r^2+r^3+\ldots+r^{k-1}+r^k=\frac{1-r^{k+1}}{1-r}$$

LHS

$$=\frac{1-r^k}{1-r}+r^k$$

$$=\frac{1-r^k}{1-r}+\frac{r^k(1-r)}{1-r}$$

$$=\frac{1-r^k}{1-r}+\frac{r^k-r^{k+1}}{1-r}$$

$$=\frac{1-r^{k+1}}{1-r}$$

$$=RHS$$

Question 3 {proof}

Use proof by contradiction to prove that if a, b are integers, then $a^2 - 4b - 3 \neq 0$. **3**
You may wish to consider the case when a is even and the case when a is odd.

Contradiction

Assume $a^2 - 4b - 3 = 0$

$a^2 - 4b = 3$

$a^2 = 3 + 4b$

RHS is an odd number because

$4b + 2 + 1 = 2(2b + 1) + 1$ for any integer b

This means that, if a is even, it cannot equal the RHS because, if $a = 2p,\ a^2 = 4p^2 = 2\left(2p^2\right)$

Consider the case when $a = 2k + 1$ (i.e. when a is odd)

$a^2 = 4k^2 + 4k + 1$

Which leads us to the question, can the following be true?

$4k^2 + 4k + 1 = 4b + 3$

Or

$4k^2 + 4k = 4b + 2$

$4\left(k^2 + k\right) = 4b + 2$

So the LHS is divisible by 4 but the RHS can't be.

So, whether a is odd or even, it can't work out

Therefore, by contradiction, the premise is proven.

Question 4 {complex numbers}

(i) Use De Moivre's theorem to show that $(1+i\tan\theta)^5 = \dfrac{\cos 5\theta + i\sin 5\theta}{\cos^5\theta}$. **1**

(ii) Hence find expressions for $\cos 5\theta$ and $\sin 5\theta$ in terms of $\tan\theta$ and $\cos\theta$. **2**

(iii) Show that $\tan 5\theta = \dfrac{5t-10t^3+t^5}{1-10t^2+5t^4}$ where $t=\tan\theta$. **1**

(iv) Use the result of (iii) and an appropriate substitution **3**

to show that $\tan\dfrac{\pi}{5} = \sqrt{5-2\sqrt{5}}$.

Part 1

$$(1+i\tan\theta) = \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)$$

Be careful when you apply the power here. The $n\theta$ trick only applies to the numerator

$$(1+i\tan\theta)^5 = \frac{\cos 5\theta + i\ \sin 5\theta}{\cos^5\theta}$$

Part 2

By the binomial theorem,

$$(1+i\tan\theta)^5 = 1\cdot 1^5 + 5\cdot 1^4\cdot i\tan\theta + 10\cdot 1^3\cdot(i\tan\theta)^2 + 10\cdot 1^2(i\tan\theta)^3 + 5\cdot 1\cdot(i\tan\theta)^4 + 1\cdot(i\tan\theta)^5$$

Tidying up that mess

$$= 1 + 5i\tan\theta - 10\tan^2\ \theta - 10i\tan^3\theta + 5\tan^4\theta + i\tan^5\theta$$

Then arranging it in real and imaginary parts

$$= \left(1 - 10\tan^2\theta + 5\tan^4\theta\right) + i\left(5\tan\theta - 10\tan^3\theta + \tan^5\theta\right)$$

Which gives us two versions of $(1+i\tan\theta)^5$. Equate the real and imaginary parts of these versions

$$\frac{\cos 5\theta}{\cos^2 \theta} = 1 - 10\tan^2\theta + 5\tan^4\theta$$

$$\therefore \cos 5\theta = \cos^2\theta\left(1 - 10\tan^2\theta + 5\tan^4\theta\right)$$

$$\frac{\sin 5\theta}{\cos^2 \theta} = 5\tan\theta - 10\tan^3\theta + \tan^5\theta$$

$$\therefore \sin 5\theta = \cos^2\theta\left(5\tan\theta - 10\tan^3\theta + \tan^5\theta\right)$$

Part 3

Using $t = \tan\theta$ and $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$ and the results from above

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

> Wait, what? Where did the $\cos^2\theta$ go? Well, they were top and bottom of a fraction - they canceled each other out.

Part 4

Show that $\tan\dfrac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$

Say we make $\theta = \dfrac{\pi}{5}$ then $5\theta = \pi$

We can then look at $t = \tan\dfrac{\pi}{5}$ and $\tan 5\theta = \tan\pi = 0$

So we have

$$\frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4} = 0$$

Which, rather conveniently, makes the numerator equal to zero

$$t^5 - 10t^3 + 5t = 0$$

$$t\left(t^4 - 10t^2 + 5\right) = 0$$

So either $t = 0$ (which is clearly not true) or

$$t^4 - 10t^2 + 5 = 0$$

> Wait, what? Why is it clearly not true? Well, if $t = 0$ then $\tan\theta = 0$. The only times that's true is where $\theta = 0\, or\, \theta = \pi$ which our θ most certainly is not.

Using the quadratic formula (because it's just a quadratic in disguise)

$$\begin{aligned} t^2 &= \frac{10 \pm \sqrt{100 - 20}}{2} \\ &= \frac{10 \pm \sqrt{80}}{2} \\ &= 5 \pm 2\sqrt{5} \end{aligned}$$

Which gives us

$$t = \pm\sqrt{5 \pm 2\sqrt{5}}$$

How do we eliminate the plus / minus bits?

For a start, we know that $\frac{\pi}{5}$ is in the first quadrant so the tangent has to be positive

$$\therefore t = \sqrt{5 \pm 2\sqrt{5}}$$

From here, we need to box a bit clever. Let's look at the two options

$$5 + 2\sqrt{5} > 9$$

$$5 - 2\sqrt{5} < 1$$

We know that $\tan\frac{\pi}{4} = 1$ and, because $\frac{\pi}{5} < \frac{\pi}{4}$, $\tan\frac{\pi}{5} < \tan\frac{\pi}{4}$

Therefore, $\tan\frac{\pi}{5} < 1$

So the only answer that could be right is

$$t = \sqrt{5 - 2\sqrt{5}}$$

QED

Question 5 {vectors}

Relative to a fixed origin O, the points A, B and C have position vectors

$$\underset{\sim}{a} = \begin{pmatrix} -1 \\ \frac{4}{3} \\ 7 \end{pmatrix},\ \underset{\sim}{b} = \begin{pmatrix} 4 \\ \frac{4}{3} \\ 2 \end{pmatrix} \text{ and } \underset{\sim}{c} = \begin{pmatrix} 6 \\ \frac{16}{3} \\ 2 \end{pmatrix} \text{ respectively.}$$

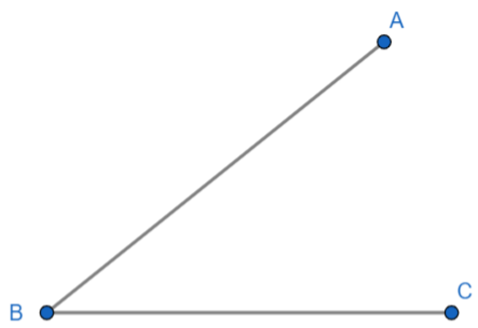
(i) Find the cosine of $\angle ABC$. **1**

(ii) Hence find the area of the triangle ABC. **1**

(iii) Use a vector method to find the shortest distance between the point A and the line passing through the points B and C. **2**

Part 1

Before we get too excited, remember that an angle only exists in two dimensions. $\angle ABC$ looks like this



So we're going to need the vectors $\vec{BA}$ and $\vec{BC}$

$$\vec{BA} = \begin{pmatrix} (-1) - 4 \\ \frac{4}{3} - \frac{4}{3} \\ 7 - 2 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}$$

$$\vec{BC} = \begin{pmatrix} 6-4 \\ \frac{16}{3} - \frac{4}{3} \\ 2-2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

Now we can use the two definitions of the dot product to find $\cos\theta$

$$x_1x_2 + y_1y_2 + z_1z_2 = |u||v|\cos\theta$$

$$\therefore \cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|u||v|}$$

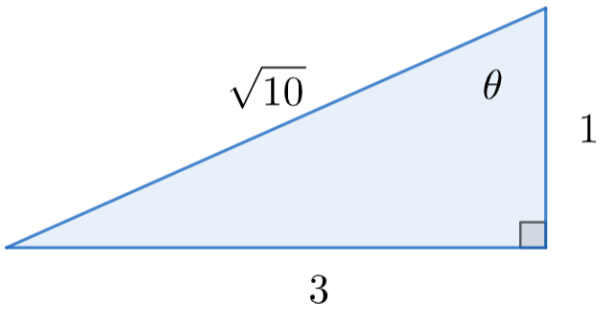
$$\therefore \cos\theta = \frac{-10+0+0}{\sqrt{50}\cdot\sqrt{20}}$$
$$= \frac{-10}{\sqrt{1000}}$$
$$= \frac{-1}{\sqrt{10}}$$

Part 2

$$A = \frac{1}{2}ab\sin C$$

We have $a\, and\, b$ just fine. We don't have $\sin C$ though.

We do know that $\cos\theta = \dfrac{-1}{\sqrt{10}}$. The negative part just tells us that the angle is in the second quadrant. The triangle below shows an angle that has $\cos\theta = \dfrac{1}{\sqrt{10}}$



Which means that $\sin\theta = \dfrac{3}{\sqrt{10}}$

And the angle is in the second quadrant (see above) so $\sin\theta$ is positive.

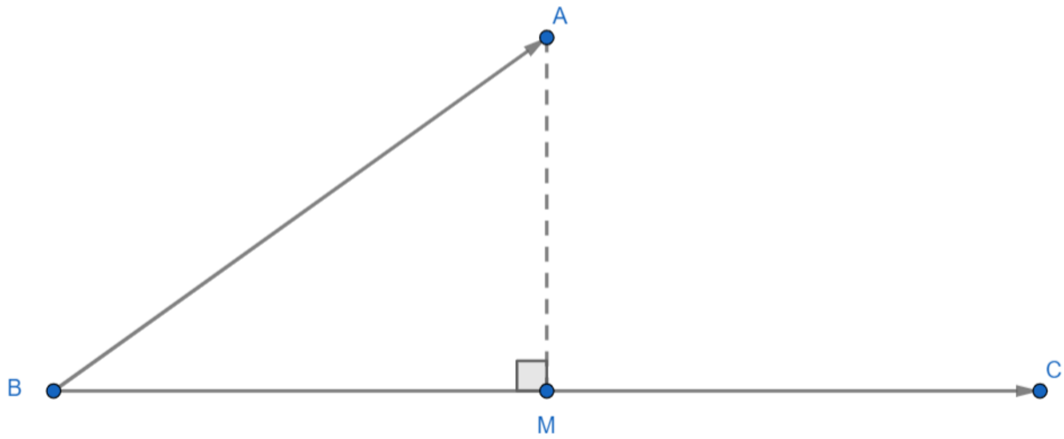
Coming back to the area question

$$\begin{aligned}
A &= \frac{1}{2}\cdot\sqrt{50}\cdot\sqrt{20}\cdot\frac{3}{\sqrt{10}} \\
&= \frac{3\sqrt{1000}}{2\sqrt{10}} \\
&= \frac{3\cdot\sqrt{100}\cdot\sqrt{10}}{2\sqrt{10}} \\
&= 15\, u^2
\end{aligned}$$

Part 3

Shortest distance from a to the line joining b and c

The shortest distance from a point to a line meets the line at a right angle, as shown here



In essence, we need to find the vector $\vec{BM}$ which is the projection of $\vec{BA}$ onto $\vec{BC}$

Using $proj_u v = \frac{u \cdot v}{|v|^2} v$

We already have most of these pieces

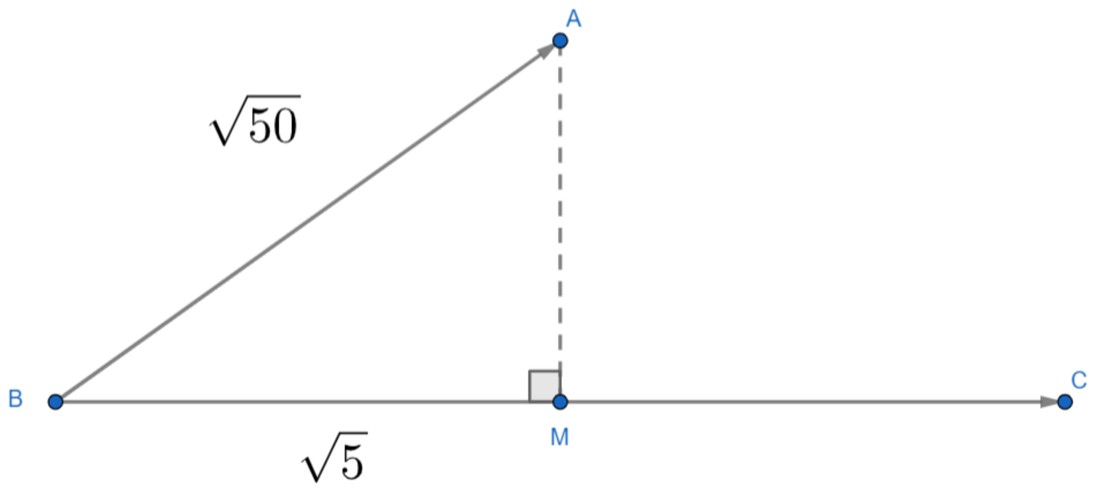
$$\vec{BA} \cdot \vec{BC} = -10$$

$$\left|\vec{BC}\right| = \sqrt{20}$$

$$\begin{aligned} \therefore\ \vec{BM} = proj_{BC} BA &= \frac{-10}{20}\begin{pmatrix}2\\4\\0\end{pmatrix} \\ &= -\frac{1}{2}\begin{pmatrix}2\\4\\0\end{pmatrix} \\ &= \begin{pmatrix}-1\\-2\\0\end{pmatrix} \end{aligned}$$

$$\left|\vec{BM}\right| = \sqrt{(-1)^2 + (-2)^2}$$
$$= \sqrt{5}$$

Which means our diagram now looks like this



Simple Pythagoras will now give us the length of AM, which is the shortest distance from the point to the line

$$\left|\vec{AM}\right| = \sqrt{\left(\sqrt{50}\right)^2 - \left(\sqrt{5}\right)^2}$$
$$= 3\sqrt{5}$$