X1 - 5 Year 12 - Ext 1 - Trial and HSC Revision - Sheet 5

WORKED SOLUTIONS

Question 1 {further calculus}

Which of the following is an equation of a curve that intersects at right angles every curve of the family $y = \frac{1}{x} + k$ (where k is a constant)?

(A) $y = x^2$

(B) $y = -x^2$

(C) $y = -\frac{1}{3}x^3$

(D) $y = \frac{1}{3}x^3$

Any equation that intersects the original at right angles has to be normal to the original at that point.

Which means $m_1 = \frac{-1}{m_2}$

Start with the original

$y = x^{-1} + k$

$\therefore y' = -x^{-2}$

$y' = \frac{-1}{x^2}$

So any curve normal to that has a gradient of

$\frac{-1}{\frac{-1}{x^2}} = x^2$

Which makes D the right answer

Question 2 {mathematical induction}

Emma Jane made an error proving that $3^{2n}-1$ is divisible by 8 (for $n \geq 1$), using mathematical induction. Part of her proof is shown below.

Step 2: Assume the result true for $n = k$

$3^{2k} - 1 = 8P$ where P is an integer.

Hence $3^{2k} = 8P + 1$

Step 3: To prove the result is true for $n = k + 1$

$RTP:\ 3^{2(k+1)} - 1 = 8Q$ where Q is an integer. Line 1

$LHS = 3^{2k+2} - 1$ Line 2

$= 3^{2k} \times 3^2 - 1$

$= (8P+1) \times 3^2 - 1$ (*using the assumption*) Line 3

$= 72P + 1 - 1$ Line 4

$= 72P$

$= 8(9P)$

$= 8Q$

$= RHS$

In which line did Emma Jane make an error?

(A) Line 1 (B) Line 2 (C) Line 3 (D) Line 4

Line 4 - the brackets weren't expanded properly.

Question 3 {trig equations} -

Solve $\sin 2\theta = \cos\theta$ for $0 \leq \theta \leq \pi$. 2

Hint: move the $\cos\theta$ to the LHS and factorise, don't just divide by $\cos\theta$. You'll need a double-angle formula

$$\sin 2\theta - \cos\theta = 0$$
$$2\sin\theta\cos\theta - \cos\ \theta = 0$$
$$\cos\theta(2\sin\theta - 1) = 0$$

By Null Factor

$$\cos\theta = 0\, or\ \sin\theta = \frac{1}{2}$$

Note: everything is in terms of θ now, not 2θ, so no need to adjust the domain

Going through the usual steps in the given domain

$$\theta = \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Question 4 {polynomials}

The cubic polynomial $P(x)=ax^3+bx^2-6x+8$ has a factor of $(x-1)$ and a remainder of -24 when divided by $(x+2)$.

(i) Show that $a=3$ and $b=-5$. **2**

(ii) Hence without using calculus, sketch $P(x)$, showing all axes intercepts. **3**

Hint: use the Remainder Theorem, it'll be much much easier.

Use the first piece of information $P(1)=0$ by the remainder theorem

$$a(1)^3+b(1)^2-6(1)+8=0$$
$$a+b=-2$$

Use the second piece of information

$$P(-2)=-24$$
$$a(-2)^3+b(-2)^2-6(-2)+8=-24$$
$$-8a+4b+12+8=-24$$
$$-2a+b=-11$$

Solving simultaneously

$a=-2-b$	From (1)
$-2(-2-b)+b=-11$	Into (2)
$b=-5$	Collect like terms and rearrange
$\therefore a=3$	Putting result back into 1

Which is what they asked us to prove

Part 2

We now know that the polynomial is $P(x)=3x^3-5x^2-6x+8$

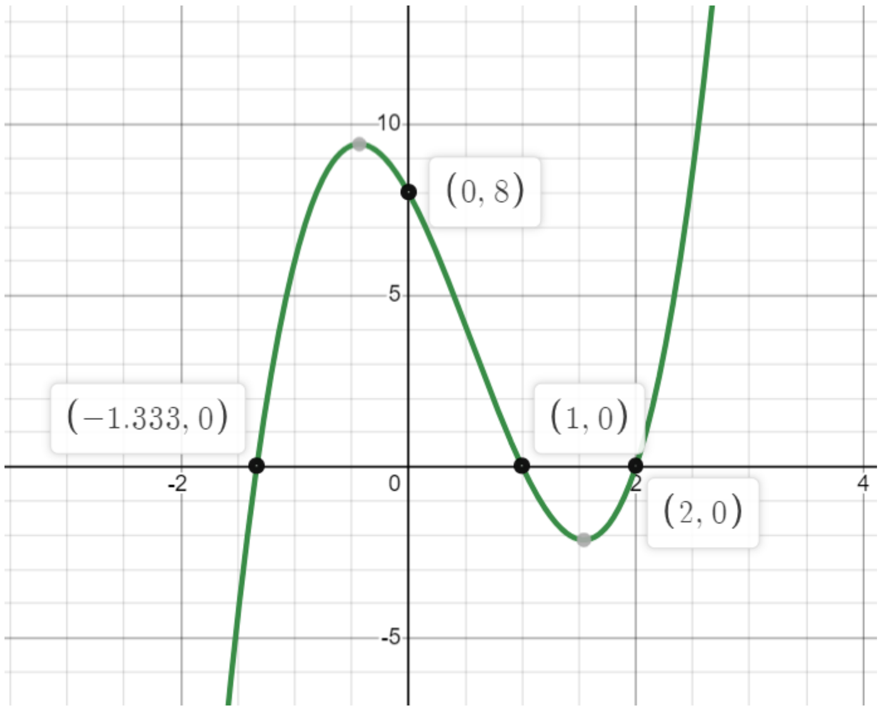
We know that $(x-1)$ is a factor

Do the long division to get $P(x)=(x-1)\left(3x^2-2x-8\right)$

Factorise the quadratic to get $P(x) = (3x+4)(x-2)(x-1)$

Which gives us roots at $\dfrac{-4}{3}, 1\, and\, 2$

It’s a positive cubic, so we know the basic shape



Question 5 {polynomials}

Find the term independent of x in the expansion of $\left(5x^2+\frac{1}{x}\right)^{12}$. **3**

Hint: “independent term” means the term that doesn’t have an x in it. AKA the constant term.

For there to be an independent term the power of x on the top will have to be the same as the power of x on the bottom.

Given that the whole thing is raised to the power of 12, we have powers 1 to 12 to choose from

List the first few terms and look for the pattern

$$ {}_{12}\mathrm{C}_0\left(5x^2\right)^{12}\left(\frac{1}{x}\right)^0 + {}_{12}\mathrm{C}_1\left(5x^2\right)^{11}\left(\frac{1}{x}\right)^1 + \ldots $$

Cleaning up the indices

$$ {}_{12}\mathrm{C}_0\left(5^{12}x^{24}\right)\left(\frac{1^0}{x^0}\right) + {}_{12}\mathrm{C}_1\left(5^{11}x^{22}\right)\left(\frac{1}{x^1}\right)+\ldots $$

If we look at the pattern of indices for the x value of the first term, they will be 24, 22, 20 ….

If we look at the pattern of indices for the x value of the second term, they will be 0, 1, 2 ….

We need to know when they will be the same (i.e. when they will cancel each other out).

Work it out manually or using AP and you get the 9th term (the common index will be 8)

So we just need the 9th term of the binomial expansion. Which will be

$$ {}_{12}\mathrm{C}_8\left(5x^2\right)^4\left(\frac{1}{x}\right)^8 $$

Question 6 {polynomials}

Let the cubic polynomial, $P(x) = x^3 - 3x^2 - 4x + 12$ have roots α, β and γ.

(i) Find the value of $\alpha + \beta + \gamma$. **1**

(ii) Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **2**

(iii) Given that two of its roots have a sum of zero, find the values of α, β and γ. **2**

Part 1

$$\begin{aligned} \alpha + \beta + \gamma &= \frac{-b}{a} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

Part 2

Given that we need a common denominator to add fractions

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$
$$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

$$\begin{aligned} \alpha\beta + \alpha\gamma + \beta\gamma &= \frac{c}{a} \\ &= \frac{-4}{1} \\ &= -4 \end{aligned}$$

$$\begin{aligned} \alpha\beta\gamma &= \frac{-d}{a} \\ &= \frac{-12}{1} \\ &= -12 \end{aligned}$$

$$\therefore \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} = \frac{-4}{-12} = \frac{1}{3}$$

Part 3

Assume

$$\alpha + \beta = 0$$
$$\therefore \alpha = -\beta$$

We already know that

$$\alpha + \beta + \gamma = 3$$
$$\therefore 0 + \gamma = 3$$
$$\therefore \gamma = 3$$

We also know that

$$\alpha\beta\gamma = -12$$

Using our first result

$$\alpha\beta = -4$$
$$\alpha = \frac{-4}{\beta}$$

Equate what we know about alpha

$$-\beta = \frac{-4}{\beta}$$
$$-\beta^2 = -4$$
$$\beta = 4$$
$$\beta = \pm 2$$

Which makes $\alpha = \mp 2$

We can’t make it any more certain than this list

$$\alpha = \mp 2,\ \beta = \pm 2,\ \gamma = 3$$

Question 7 {vectors}

Alysha hits a golf ball off the ground with velocity V at an angle of projection θ to the horizontal. The equations of motion are as follows and **do not need to be proven**:

$\ddot{x} = 0$	$\ddot{y} = -g$
$\dot{x} = V\cos\theta$	$\dot{y} = V\sin\theta - gt$
$x = Vt\cos\theta$	$y = Vt\sin\theta - \frac{1}{2}gt^2$

(i) Show that Alysha's golf ball reaches a maximum height of $\dfrac{V^2\sin^2\theta}{2g}$. **2**

(ii) Kristine hits a second golf ball projected from the same horizontal plane. **4**

It has velocity $V \times \sqrt{\dfrac{5}{2}}$ and is projected at an angle $\dfrac{\theta}{2}$ to the horizontal.

What angles should Alysha and Kristine project their golf balls if they are to reach the same maximum height?

Part 1

At the maximum height, the ball stops moving vertically for just an instant. Therefore, $\dot{y} = 0$

$$V\sin\theta - gt = 0$$
$$V\sin\theta = gt$$
$$t = \frac{V\sin\theta}{g}$$

This gives us **when** the max occurs. To find the vertical height at that point, sub that into the expression for y

$$y = Vt\sin\theta - \frac{1}{2}gt^2$$
$$y = V\left(\frac{V\sin\theta}{g}\right)\sin\theta - \frac{1}{2}gt^2$$
$$y = \frac{V^2\sin^2\,\theta}{g} - \frac{1}{2}g\left(\frac{V^2\sin^2\theta}{g^2}\right)$$
$$y = \frac{V^2\sin^2\,\theta}{g} - \frac{V^2\sin^2\theta}{2g}$$
$$y = \frac{V^2\sin^2\theta}{2g}$$

QED

The basic equations for the two golfers don't change, just the inputs

For Kristine, her max height is at

$$y = \frac{\left(\sqrt{\frac{5}{2}}V\right)^2\sin^2\frac{\theta}{2}}{2g}$$

And we need to solve for θ when the two max heights are the same

$$\frac{V^2\sin^2\theta}{2g} = \frac{\left(\sqrt{\frac{5}{2}}V\right)^2\sin^2\frac{\theta}{2}}{2g}$$

$$V^2\sin^2\theta = \frac{5}{2}V^2\sin^2\frac{\theta}{2}$$
$$2V^2\sin^2\theta = 5V^2\sin^2\frac{\theta}{2}$$
$$2\sin^2\theta = 5\sin^2\frac{\theta}{2}$$

Before we go too much further, there's a trivial case here - i.e. where $\theta = 0$. However, this would mean mully-grubbing the ball and the max height would also be 0. We will note this case but dismiss it; they're probably not that bad at golf.

$$\frac{2}{5}\sin^2\theta = \sin^2\frac{\theta}{2}$$

$$\frac{\sqrt{2}}{\sqrt{5}}\sin\theta = \sin\frac{\theta}{2}$$

$$\sqrt{\frac{2}{5}}\sin\theta - \sin\frac{\theta}{2} = 0$$

Which looks impossible until we make a u substitution

$$u = \frac{\theta}{2}$$
$$\therefore \theta = 2u$$

Question 8 {second derivatives}

Prove that the graph of $y = \log_e x$ is concave down for all $x > 0$. **2**

A graph is concave down if the second derivative is always negative

$$y = \ln x$$
$$\therefore \frac{dy}{dx} = \frac{1}{x}$$
$$\therefore \frac{d^2y}{dx^2} = \frac{-1}{x^2}$$

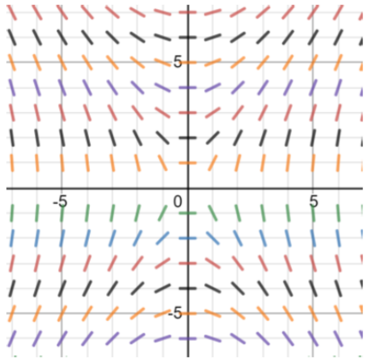
Given $x^2 \geq 0$ and the original function is not defined at $x = 0$

$$\frac{-1}{x^2} < 0$$

Therefore the curve is concave down for all $x > 0$

Question 9 {differential equations}

The slope field below could represent which of the following differential equations?



A. $\dfrac{dy}{dx} = \dfrac{2x}{y}$ B. $\dfrac{dy}{dx} = \dfrac{2y}{x}$ C. $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$ D. $\dfrac{dy}{dx} = \dfrac{y^2}{x^2}$

We can rule out C and D first because the fractions would always yield a positive number, regardless of the values of x and y. Therefore, the gradients of each little line would also have to be positive. Clearly they're not in quadrants 2 and 4

Pick a point - say (2,3) - and examine the remaining options

If A were true, the gradient at that point would be $\frac{4}{3}$ which is close-ish to 1

If B were true, the gradient at that point would be $\frac{6}{2}$ which is very steep

By inspection, it seems that A is more likely.