Exercise 5

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In solving the problems, you may freely use and refer to results in the slides. Please hand in for marking problems 1–2. Problem 3 is optional and only to be covered in the classes in case you have time.

1. (An example of non-existence of a UMVUE in the presence of a nuisance parameter) For $\mu \in \mathbb{R}$ and $\sigma^2 \in (0, \infty)$ let $P_{\mu,\sigma^2} = \mathsf{N}(\mu, 1) \otimes \mathsf{N}(\mu, \sigma^2)$ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ (That is, we have two independent normal variables with the same mean but different variance.). The problem is to estimate μ in the presence of the *nuisance parameter* σ^2, based on a sample of size n. Following the below steps, we shall show that no UMVUE for μ exists.

   (a) Show that P_{μ,σ^2} has density

   $$\frac{1}{2\pi\sigma} \exp\left[-\frac{1}{2\sigma^2}\left(\sigma^2 x^2 + y^2 - 2\mu(\sigma^2 x + y) + \mu^2(\sigma^2 + 1)\right)\right]$$

   with respect to the Lebesgue measure λ_2 on $\mathcal{B}(\mathbb{R}^2)$.

   (b) Show that for each fixed $\sigma_0^2 \in (0, \infty)$ it holds that $\sum_{i=1}^n (\sigma_0^2 x_i + y_i)$ is a complete sufficient statistic for μ under $\{P^n_{\mu,\sigma_0^2} : \mu \in \mathbb{R}\}$.
   Hint: Think exponential families and use the results for sufficiency and completeness within these.

   (c) Show that

   $$\hat{\mu}_n\left((x_1, y_1), \ldots, (x_n, y_n); \sigma_0^2\right) := \frac{1}{(1 + \sigma_0^2)n} \sum_{i=1}^n (\sigma_0^2 x_i + y_i)$$

is the unique UMVUE for μ under $\{P^n_{\mu,\sigma_0^2} : \mu \in \mathbb{R}\}$.

(d) Conclude that no UMVUE exists for estimating μ in $\{P^n_{\mu,\sigma^2} : \mu \in \mathbb{R}, \sigma^2 \in (0,\infty)\}$.

Hint: As in (c), show that for $\sigma_1^2 \neq \sigma_0^2$

$$\hat{\mu}_n\left((x_1,y_1),\dots,(x_n,y_n);\sigma_1^2\right) := \frac{1}{(1+\sigma_1^2)n}\sum_{i=1}^n(\sigma_1^2 x_i + y_i)$$

is the unique UMVUE for μ under $\{P^n_{\mu,\sigma_1^2} : \mu \in \mathbb{R}\}$ and argue that

$$\hat{\mu}_n\left((x_1,y_1),\dots,(x_n,y_n);\sigma_1^2\right) \neq \hat{\mu}_n\left((x_1,y_1),\dots,(x_n,y_n);\sigma_0^2\right)$$

with P_{μ,σ^2}-probability one for all $\mu \in \mathbb{R}$ and $\sigma^2 \in (0,\infty)$. Thus, you have shown that the best unbiased estimator depends on the unknown nuisance parameter σ^2. Hence, no uniform minimum variance unbiased estimator exists for μ in $\{P^n_{\mu,\sigma^2} : \mu \in \mathbb{R}, \sigma^2 \in (0,\infty)\}$.

2. (Bounded loss functions and non-existence of unbiased estimators uniformly minimizing the risk) Let $\{P_\theta : \theta \in \mathbb{R}^k\}$ be a family of mutually absolutely continuous probability measures on a measurable space $(\mathcal{X},\mathcal{A})$. Assume, furthermore, that there exists an unbiased estimator $\delta : \mathcal{X} \to \mathbb{R}^k$ for θ and that the loss function $L : \mathbb{R}^k \times \mathbb{R}^k \to [0,\infty)$ satisfies that $\sup_{x,y\in\mathbb{R}^k} L(x,y) =: M < \infty$ and $L(x,y) = 0$ if and only if $x = y$. Fix some $\theta_0 \in \mathbb{R}^k$ and define, for $\pi \in (0,1)$,

$$\delta_\pi(x) = \begin{cases} \theta_0 & \text{with probability } 1-\pi \\ \frac{1}{\pi}[\delta(x)-\theta_0]+\theta_0 & \text{with probability } \pi. \end{cases}$$

(a) Show that δ_π is unbiased for θ for all $\pi \in (0,1)$.

(b) Show that the risk of δ_π at θ_0 is no greater than πM, i.e. show that $E_{\theta_0} L(\theta_0,\delta_\pi) \leq \pi M$.

(c) Show that no unbiased estimator exists that has a risk of 0 at θ_0.

Hint: Use the mutual absolute continuity of the P_θ.

(d) Conclude that no unbiased estimator exists that uniformly minimizes the risk.

Hint: What happens to the risk of δ_π as $\pi \to 0$?

Some context/interpretation: Recall that for convex loss functions the Rao-Blackwell-Lehmann-Scheffé Theorem guarantees the existence of unbiased estimators uniformly minimizing the risk in the presence of a complete sufficient statistic. The above problem applies, in particular, to the case of $P_\theta = \mathsf{N}(\theta, 1)$ (use $k = 1$), in which case a complete sufficient statistic and an unbiased estimator of the mean exists. Thus, the problem shows that even in this favorable setting no unbiased uniform minimum risk estimator exists when the loss function is bounded. Note that boundedness rules out convexity of the loss function (as we rule out constant loss functions). Thus, you have shown that for many non-convex loss functions the conclusion of the Rao-Blackwell-Lehmann-Scheffé Theorem ceases to hold. That is, convexity is an important assumption in that theorem.

3. (Uniformly most powerful tests) Consider the experiment $\big(\mathcal{X}, \mathcal{A}, \{P_\theta : \theta \in \Theta\}\big)$ in which we wish to test

$$\mathsf{H}_0 : \theta \in \Theta_0 \quad \text{vs.} \quad \mathsf{H}_1 : \theta \in \Theta_1,$$

where Θ_0 and Θ_1 are non-empty disjoint subsets of Θ.

(a) Assume that ψ is a uniformly most powerful test at level $\alpha \in (0,1)$. Show that if $E_\theta \psi < 1$ for some $\theta \in \Theta_1$, then $\alpha(\psi) := \sup_{\theta \in \Theta_0} E_\theta \psi = \alpha$.

Hint: Assume that $\alpha(\psi) < \alpha$ and show that a more powerful level α test than ψ exists that is a convex combination of ψ and the test 1 that always rejects.

(b) Consider the setting in (a) but assume now that the uniformly most powerful test ψ of level α satisfies that $E_\theta \psi = 1$ for all $\theta \in \Theta_1$. Then $\alpha(\psi) < \alpha$ may occur, but show that there still exists a test φ with $\alpha(\varphi) = \alpha$ and $E_\theta \varphi = 1$ for all $\theta \in \Theta_1$.

(c) Let $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$ and assume that $P_{\theta_0} \ll P_{\theta_1}$. Let ϕ be a level $\alpha \in (0,1)$ test. Show that $E_{\theta_1} \phi < 1$.

Hint: Assume, for contradiction, that $E_{\theta_1} \phi = 1$.