Option Pricing with Delayed Information

By Connor Schwarz and Ryan Bergner

In modern-day trading, information moves faster than ever before. When timing becomes the difference between a large profit and a disastrous loss in volatile markets, beating someone to the punch is a high priority. Unfortunately, the speed of the market exceeds that of the trader. Delays in information have been a continuous factor in judgments since the creation of the stock market. Two critical factors must be accounted for: Order execution time and delay in receiving information. Here, we examine a paper that introduces a novel method of addressing these issues.

Market incompleteness is a result of delay in information processing. An incomplete prevents perfect replication, often leading to the trader falling out of the money on large trades. Market volatility makes fighting the delay of information an arduous task. The authors of "Option Pricing with Delayed Information" propose a methodology to counter these obstacles by accounting for delay.

The Problem

In a perfect world, a trader would have access to all the information needed always to complete a trade on hand. However, this perspective fails in real-world situations. All traders are limited by the speed of information. Consider the following trade: A put option is placed in the early morning but only executed in the late afternoon. This delay is what comprises most of the struggle that comes with option pricing. By the time a trade is executed, it is possible the natural movement of the market in the time of the delay varies wildly from the predicted movement. In a perfect world, these trades would be completed instantaneously. As a result, losses are incurred or the associated risks increase.

The authors define their problem by addressing the overarching problem of option pricing with delayed information: A market with delayed information is incomplete; perfect replication is impossible in these conditions. This increases the complexity of models and creates inaccurate replications in the portfolio. The authors propose a different method for creating pricing measures to combat this issue.

Why is it important?

Computational measurements often aid trading strategies. In this field, precision is key. When working in markets with delayed information, the ability to accurately predict and create realistic models is severely inhibited. This inhibition makes traders act on impulse and prior knowledge, exaggerating the "volatility smile." However, the author's model suggests a new understanding of how delayed information affects the market. "Our model…suggests that the smile observed in the market might not all be by the market itself, and it could have been exaggerated because of how we interact with delayed information". The model mentioned uses modified price measures that converge to a Black-Scholes price process but with enlarged volatility. This approach creates a 'complete' market that allows for perfect replication. The enlarged volatility is accounting for both transaction costs and delayed information. This change is highly advantageous due to its ability to create more risk-averse trades in the long run.

Solution

To approach this problem, we must distinguish between the discrete case and the continuous case:

The Discrete Case

Using an N-period binomial tree model, the following stochastic process is developed $\Omega = \{0,1\}^N$, $\omega = (\omega_1,..,\omega_N) \in \Omega, S_k\colon \Omega \to \Re$

This case is developed with a filtration, $\mathfrak{I} = \{\mathfrak{I}_k, k = 0,\ldots,N\}$. Two equations are generated from this stochastic process: one is a risky asset price process, and the other is a discounted price process.

How do we introduce the delay into our binomial model? The authors consider the case when a trader places and order at time t, but is not executed until time $t + H$ with $H \in \{0,\ldots N-1\}$ $delay\ periods$. Giving us a refined filtration: $G_k = F_k - H, k = H,\ldots,N.$ This filtration is the information set of the princess process until time $min(k - H, 0)$. This time adjustment allows for a representation of this delay between ordering and execution. The portfolio process develops as follows:

$$(2.3)\quad V_H(x_0,\Delta)(\omega) := x_0 \cdot e^{rH} + \Delta_H \cdot S_H(\omega), \quad V_0(x_0,\Delta)(\omega) := e^{-rH} \cdot V_H(x_0,\Delta)(\omega) = x_0 + \Delta_H \cdot \widetilde{S}_H(\omega),$$

and in general

$$(2.4)\quad V_k(x_0,\Delta)(\omega) := \begin{cases} e^{-r(H-k)} \cdot V_H(x_0,\Delta)(\omega), & k = 0,\ldots,H-1, \\ e^{rk}x_0 + \sum_{l=H}^{k-1} S_l(\omega) \cdot \left(\Delta_{(l-1)\vee H} - \Delta_l\right) + S_k(\omega) \cdot \Delta_{(k-1)\vee H}, & k = H,\ldots,N. \end{cases}$$

The authors go on to prove the absence of arbitrage in the models. It is to be noted that portfolio value, $V_0(x_0, \Delta)$, is a random variable due to the existence of the delay. This difference creates an adjustment in the classical notion of arbitrage within the domain of the process. Using an updated probability measure, P that the optional projection P of the discounted stock price on the delayed filtration, is a P martingale. The authors go on to prove this point in relation to the models produced. They arrive at the fact that the process is a martingale, showing no arbitrage opportunity from time H to N.

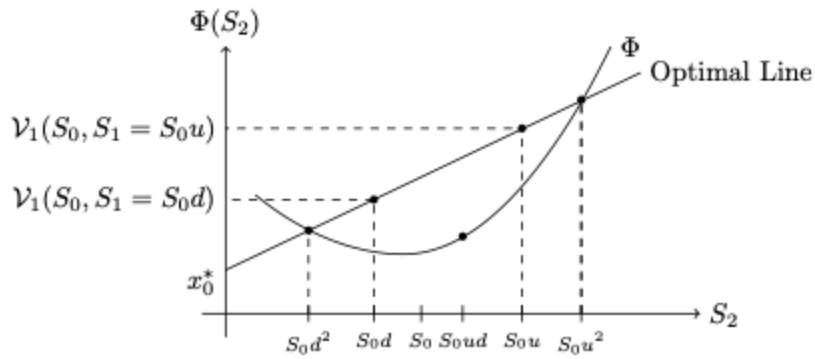
It is worth noting that the results are determined under European options with convex payoff curves. Given the discussed models, the super-replicating price $\pi(\varphi)$ is defined as:

$$\text{(2.7)} \qquad \bar{\pi}(\varphi) := \inf_{(x_0,\Delta)\in\Gamma} \max_{\omega\in\Omega} \left\{ V_0(x_0,\Delta)(\omega) = x_0 + \Delta_H \widetilde{S}_H(\omega) \right\},$$

where

$$\text{(2.8)} \qquad \Gamma := \{(x_0,\Delta) \in \mathbb{R} \times \mathcal{A}_{\mathfrak{G}} : V_N(x_0,\Delta) \geq \varphi \ \ \mathbb{P} - a.s.\} .$$

The authors proceed with various examples of the properties that follow from these models. The author's note two special cases: An N-period binomial model with H = N -1 periods of delay and an N-period binomial model with H periods of delay. Elaboration on approaches to these systems are divided into a dynamic programming approach and a direct approach. The direct approach utilizes markov chains and developed transition matrices to assert that a downward trend holds a higher probability compared to an upward movement preceded by an upward move. This distribution increases the variance of the risky asset price from the initial *P* measure. For example, a 2 period binomial model with a 1-period delay:



The optimal line denotes the super-replicating strategy, with a price defined to be: $\pi(\varphi) = max\{v_1(S_0, S_1 = S_0d), v_1(S_0, S_1 = S_0u)\}$.

$$\text{(2.1)}\, S_k(\omega) := S_0\, u^{I_k(\omega)}\, d^{k-I_k(\omega)}, \quad I_k(\omega) := \sum_{l=1}^{k} Z_l(\omega), \quad \widetilde{S}_k(\omega) := e^{-rk}\, S_k(\omega), \quad k = 1,\ldots,N,$$

Both the direct and dynamic programming approach are discussed in further detail in the continuous case.

The Continuous Case

In simulated computation, the Black-Scholes Model, determining the payoff of a certain option with delayed information, can be shown to produce a normal distribution with the same expected value as it would have with an identically parameterized model (i.e. perfectly-timed information). However, to account for increased uncertainty of the option's value caused by lagging information, the model with delayed information has a higher variance due to enlarged volatility. To show this, we will consider the continuous case of the Black-Scholes Model by taking an asymptotic approximation to establish probability bounds, which will simultaneously show us the level of asymptotic effect delayed information has on the model:

As shown in the discrete case in 2.7, :

Definition 2.2 (Super-replication price and the value process of super-replicating portfolio). *For any contingent claim with payoff function $\varphi : \Omega \to \mathbb{R}$ and expiration time N, its super-replication price $\bar{\pi}(\varphi)$ is defined as the minimal initial value of portfolio which exceeds the value φ at time N, i.e.,*

$$\bar{\pi}(\varphi) := \inf_{(x_0,\Delta)\in\Gamma} \max_{\omega\in\Omega} \left\{ V_0(x_0,\Delta)(\omega) = x_0 + \Delta_H \widetilde{S}_H(\omega) \right\}, \tag{2.7}$$

The definition is used to first establish the no-arbitrage price, shown on the discrete model, fits the stated definition for all $\omega \in \Omega$:

$$\inf_{(x_0,\Delta)\in\Gamma} \left\{ V_0(x_0,\Delta)(\omega) = x_0 + \Delta_H \widetilde{S}_H(\omega) \right\} = V_0(x_0^*,\Delta^*)(\omega) = x_0^* + \Delta_H^* \cdot \widetilde{S}_H(\omega). \tag{2.13}$$

Doing this will ensure $x_0 \in \mathbb{R}$ and $\boldsymbol{\Delta} \in \boldsymbol{\mathcal{A}}_{\mathfrak{G}}$ across the entire space of $\boldsymbol{\Gamma}$ to validate the option's overall delta sensitivity with respect to its change in price. (i.e. $V_N(x_0, \boldsymbol{\Delta}) \geq \boldsymbol{\rho}(S_N)$) But, from 2.10, note that in the case of H = N − 1 for all $x \in S_N$:

$$V_N(x_0,\Delta) = (e^{rN} x_0 + x \cdot \Delta_{N-1})|_{x=S_N}$$

Since the model is defined in terms of a function, we can expect every x-value to take a y-value in accordance with the above definition. Since the payoff function Φ is convex, we are able to establish:

$$\Gamma = \{(x_0, \Delta) \in \mathbb{R} \times \mathcal{A}_{\mathfrak{G}} : e^{rN} x_0 + S_0 u^N \cdot \Delta_{N-1} \geq \Phi(S_0 u^N), \quad e^{rN} x_0 + S_0 d^N \cdot \Delta_{N-1} \geq \Phi(S_0 d^N)\} . \tag{2.14}$$

This is an advantageous representation of **Γ** because not only have we the continuity for asymptotic calculations been created, but we have reduced the discrete binomial tree model (Theorem 2.2) into a linear programming problem, Ichiba and Mousavi already found the x and **Δ** parameters from **Γ** can be determined sufficiently by minimizing $x_0 + \boldsymbol{\Delta}_H \cdot S_H(\omega)$ with respect to **Γ**. Taking the Lagrangian as:

$$\mathcal{L} := x_0 + \Delta_H \widetilde{S}_H(\omega) + \lambda_1[\Phi(S_0 u^N) - (e^{rN} x_0 + S_0 u^N \Delta_H)] + \lambda_2[\Phi(S_0 d^N) - (e^{rN} x_0 + S_0 d^N \Delta_H)] ,$$

the delayed information model's parameters can be shown to be:

$$x_0^* = e^{-rN} \cdot \frac{u^N \Phi(S_0 d^N) - d^N \Phi(S_0 u^N)}{u^N - d^N} , \quad \Delta_H^* = \frac{\Phi(S_0 u^N) - \Phi(S_0 d^N)}{S_0 u^N - S_0 d^N} ,$$
$$\lambda_1^* = \frac{\widetilde{S}_H(\omega) - e^{-rN} S_0 d^N}{S_0 \cdot (u^N - d^N)} , \quad \lambda_2^* = \frac{e^{-rN} S_0 u^N - \widetilde{S}_H(\omega)}{S_0 u^N - S_0 d^N}$$

Where $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are lagrangian multipliers. Solving with Karush-Kuhn-Tucker conditions in mind, the super-replication price can be defined as:

$$\bar{\pi}(\varphi) = \max_{\omega \in \Omega} \inf_{(x_o, \Delta) \in \Gamma} V_0(x_0, \Delta)(\omega) = \max_{\omega \in \Omega} V_0(x_0^*, \Delta^*)(\omega) . \tag{2.16}$$

Which is in alignment with Ichiba and Mousavi's initial assertion that the maximum volatility over H, particularly at time 0 for $\omega \in \Omega$, is indeed 0. Realizing that this is the same result as the initial continuous equation (2.13), it is proven that:

$$\bar{\pi}(\varphi) = \max\left(x_0^* + e^{-rH} \Delta_H^* \cdot S_0 u^H , x_0^* + e^{-rH} \Delta_H^* \cdot S_0 d^H\right) , \tag{2.11}$$

where the corresponding strategy (x_0^*, Δ^*) *is given by* $\Delta_j^* \equiv 0 , \ j = 0, 1, \ldots, H-1$,

$$\Delta_H^* = \Delta_{N-1}^* = \frac{\Phi(S_0 u^N) - \Phi(S_0 d^N)}{S_0 \cdot (u^N - d^N)} \quad \text{and} \quad x_0^* = e^{-rN} \cdot \frac{u^N \Phi(S_0 d^N) - d^N \Phi(S_0 u^N)}{u^N - d^N} . \tag{2.12}$$

Therefore, the portfolio value with delayed information across N periods and experiencing H periods with insufficient information to be strikingly similar to the discrete case (2.4), as well as the original Black-Scholes model without delayed information present:

$$\mathcal{V}_k(S_{k-H}, S_{k-H}u^H) = e^{-r\widetilde{N}}\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0} = 1\right), \tag{2.31}$$

$$\mathcal{V}_k(S_{k-H}, S_{k-H}d^H) = e^{-r\widetilde{N}}\mathbb{E}^{\mathbb{Q}_k}\left(\Phi\left(S_N\right)|Z_{k,0} = 0\right). \tag{2.32}$$

The authors then go on to relate the options value at time k for k = H,…N - 2, N-1 to the structure of the space (Ω_k, F_k, Q_k). Since we know how to construct the (k + 1)'th iteration of these spaces from the k'th, the portfolio value at time k can be rewritten and reduced as follows:

$$\begin{aligned}\mathcal{V}_k(S_{k-H}, S_{k-H}u^H) &= e^{-r\widetilde{N}}\left[\mathfrak{p}_u\mathbb{E}^{\mathbb{Q}_{k+1}}\left(\varphi\left(S_N\right)|Z_{k+1,0} = 1\right) + \mathfrak{q}_u\mathbb{E}^{\mathbb{Q}_{k+1}}\left(\varphi\left(S_N\right)|Z_{k+1,0} = 0\right)\right], \\ &= e^{-r}\Big[\mathfrak{p}_u\mathcal{V}_{k+1}(S_{k-H+1} = S_{k-H}u, S_{k+1} = S_{k-H}u^{H+1}) \\ &\quad +\mathfrak{q}_u\mathcal{V}_{k+1}(S_{k-H+1} = S_{k-H}d, S_{k+1} = S_{k-H}d^{H+1})\Big],\end{aligned}$$

This shows our original claim that the super-replication price of:

$$\bar{\pi}(\varphi) = e^{-rH}\max\left(\mathcal{V}_H(S_0, S_0u^H), \mathcal{V}_H(S_0, S_0d^H)\right).$$

to be a correct adjustment to the Black-Scholes Model to handle delayed information, given the presence of H random periods information of a finite amount as the initial subset of N - 1 total periods where N = k - H and N + 1 = k - H + 1 (because H = N - 1 by definition and k can be thought of the length of a certain information delay).

Asymptotics of the Price and Return Processes

For investors and options traders, one of the more natural ways to utilize the information above would be to determine the probability an option's value will eventually reach the same value as the predefined strike price. To accomplish this, one must determine the asymptotic behavior of the parameters with delayed information at play.

One of the beneficial properties of the Black-Scholes-Merton-Model is that the probability measures of the stock moving up or down can be written as a markov chain transition matrix $\boldsymbol{Q}_n$ where:

$$Q_n = \begin{pmatrix} \mathfrak{q}_{n,d} & \mathfrak{p}_{n,d} \\ \mathfrak{q}_{n,u} & \mathfrak{p}_{n,u} \end{pmatrix} \quad \text{on } \{0,1\}. \tag{3.2}$$

Besides, for $m = n - H, \dots, n$,

$$\mathbb{Q}^n\left(Z_n^n = \cdots = Z_{n-H}^n = 1 | Z_{n-H-1}^n = 1\right) = \mathfrak{p}_{n,u}, \ \mathbb{Q}^n\left(Z_n^n = \cdots = Z_{n-H}^n = -1 | Z_{n-H-1}^n = 1\right) = \mathfrak{q}_{n,u},$$
$$\mathbb{Q}^n\left(Z_n^n = \cdots = Z_{n-H}^n = 1 | Z_{n-H-1}^n = 0\right) = \mathfrak{p}_{n,d}, \ \mathbb{Q}^n\left(Z_n^n = \cdots = Z_{n-H}^n = -1 | Z_{n-H-1}^n = 0\right) = \mathfrak{q}_{n,d}, \tag{3.3}$$

To define $\boldsymbol{Q}_n$ for j = 1, …, H in 2.19, Ichiba and Mousavi defined Z_k, k = 1, … , N to be independent Bernoulli random variables $P(Z_k = 1) = P(Z_k = 0) = ½$, k = 1, … , N. Since p and q are the parameters that must be tuned to fit this definition of independence, they showed p_j and q_j are given by:

$$\mathbb{Q}_j(I_N = N) := \mathfrak{p}_j = 1 - \mathbb{Q}_j(I_N = 0) = 1 - \mathfrak{q}_j\,, \quad \mathfrak{p}_j := \frac{u^j d^{H-j} e^r - d^{H+1}}{u^{H+1} - d^{H+1}}\,, \quad j = 0, \dots, H\,.$$

So, by applying the same logic to all $Z_l^n l$, l = 1, . . . , n such that n - H -1 timesteps in l , we can derive the adjusted transition probabilities as:

$$\mathfrak{p}_{n,d} := \frac{d_n^H e^{r_n} - d_n^{H+1}}{u_n^{H+1} - d_n^{H+1}} = 1 - \mathfrak{q}_{n,d}, \quad \mathfrak{p}_{n,u} := \frac{u_n^H e^{r_n} - d_n^{H+1}}{u_n^{H+1} - d_n^{H+1}} = 1 - \mathfrak{q}_{n,u}.$$

Thus, the authors were able to obtain:

$$S_\ell^n = S_0 \exp\left[\ell\mu_n + \sigma_n \sum_{i=1}^{\ell} X_i^n\right], \quad \ell = 0, \ldots, n,$$

where $X_{in} = 2Z_{ni} - 1$. By discretizing the functions for p and q and letting $t := T \cdot 1/n$ and evaluating the function in a constant manner for all $t \in T$ and because $S^{(n)}$ have trajectories that are right continuous and have left limits, we get:

$$S_t^{(n)} := S^n_{\lfloor nt \rfloor / T}, \quad 0 \le t \le T, \tag{3.8}$$

where $\lfloor . \rfloor$ is the floor function.

The process $S^{(n)}$ has trajectories which are right continuous with left limits. Note that in particular

$$S^{(n)}_{t^n_\ell} = S^n_\ell, \quad \ell = 0, \ldots, n.$$

Now, by interpolating over all S_l^n for $= 1, \ldots, n$, we can see that:

Lemma 3.1. *We have*

$$\mathfrak{p}_{n,u} = \frac{2H+1}{2(H+1)} - \left(\frac{\mu - r}{2\,(H+1)\,\sigma} + \frac{2H+1}{4(H+1)}\sigma\right)\sqrt{T}\delta_n + \mathcal{O}\left(\delta_n^2\right), \tag{3.6}$$

$$\mathfrak{p}_{n,d} = \frac{1}{2(H+1)} - \left(\frac{\mu - r}{2\,(H+1)\,\sigma} + \frac{2H+1}{4(H+1)}\sigma\right)\sqrt{T}\delta_n + \mathcal{O}\left(\delta_n^2\right). \tag{3.7}$$

Applying the Taylor Series to the parameters u_n, d_n, and r_n, and plugging into the probabilities for $\boldsymbol{Q}_n$ achieves the above definition. Ichiba and Mousavi went on to prove that for all $l = 1, \ldots,$ n - H - 1 that based on the definition of all other parameters, taking the limit of n to infinity in the form of Gruber and Schweizer:

$$\mathfrak{p}_n(\ell, x) = \frac{1}{2}\left[1 + \phi\delta_n + \lambda_n x\right] + \mathcal{O}(\delta_n^2),\ \phi = -2\left[\frac{\mu - r}{2\,(H+1)\,\sigma} + \frac{2H+1}{4\,(H+1)}\sigma\right]$$

The definition of $\boldsymbol{\lambda}$ is then shown to asymptotically approach:

$$\lambda_n = \frac{H}{H+1} + \mathcal{O}(\delta_n^2).$$

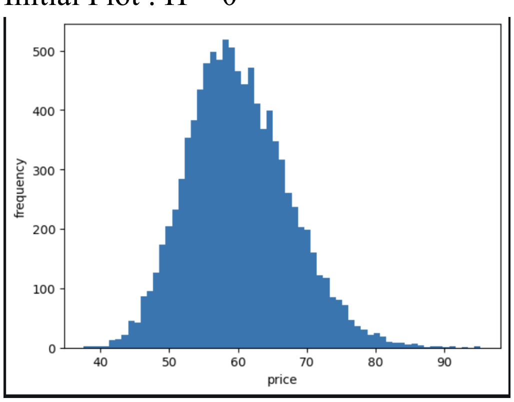
Finally taking the limit as n approaches infinity, the asymptotic approximation of the new volatility is given by:

$$\widetilde{\sigma} = \sqrt{\frac{1 + \lim_{n\to\infty} a_n(t, Y_t)}{1 - \lim_{n\to\infty} a_n(t, Y_t)}} \cdot \sigma = \sqrt{2H+1}\,\sigma,$$

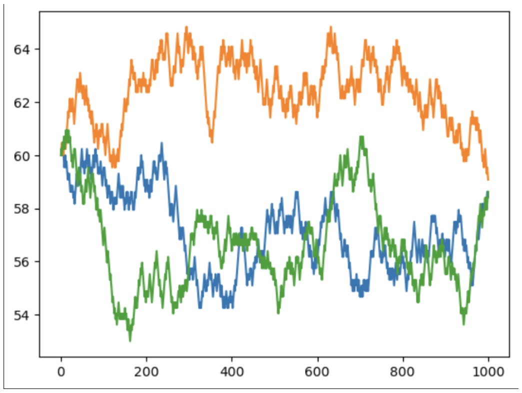
where $\boldsymbol{\sigma}$* is constant, but inflated to have a greater volatility than $\boldsymbol{\sigma}$. The new distribution of the price has essentially become log-normal as a result of this definition.

Simulation Results

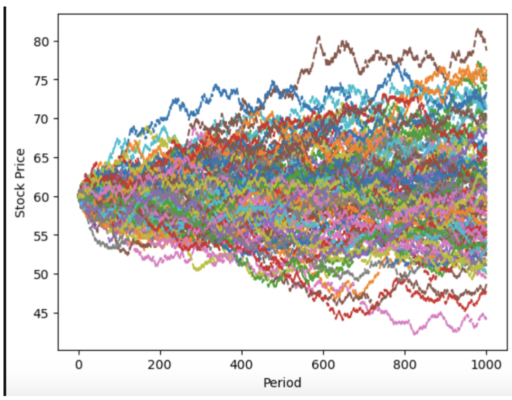
To demonstrate the findings, 10,000 random paths of a potential European call option are simulated with the number of delay periods H = (0, 10, 100) with all other parameters held equal (see attached jupyter notebook for values). In the case of H = 0, the Black-Sholes-Merton model experiences no delay, and is therefore normally distributed:

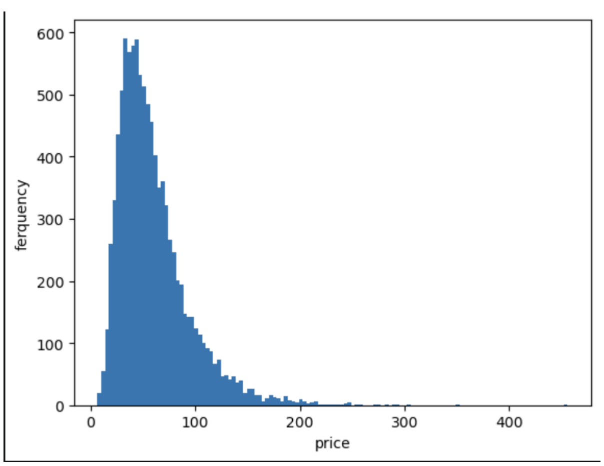
- Initial Plot : H = 0
- 

- N = 3

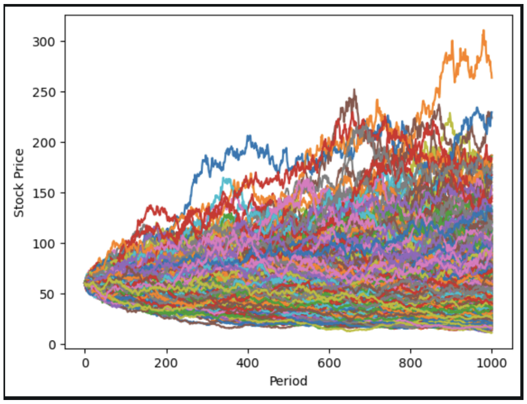
- 

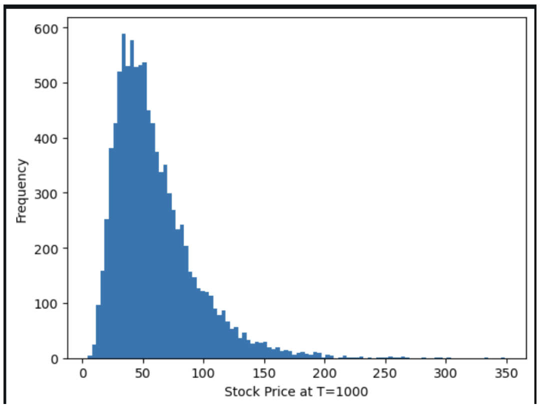
- N = 1000

- 

- Mean = 60.069, Variance = 27.166
- Initial Plot : H = 10
-
- 

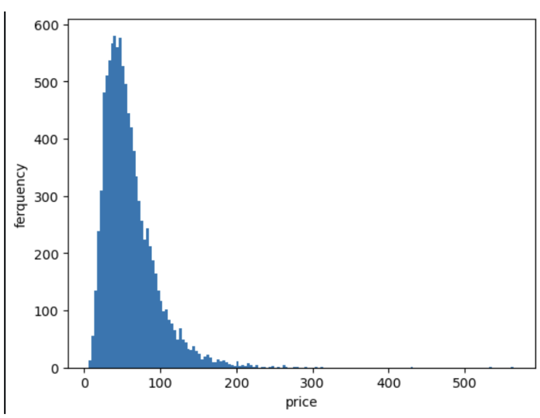
- N = 1000

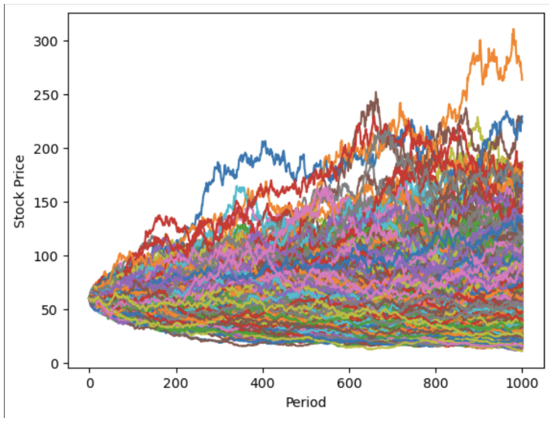
- 

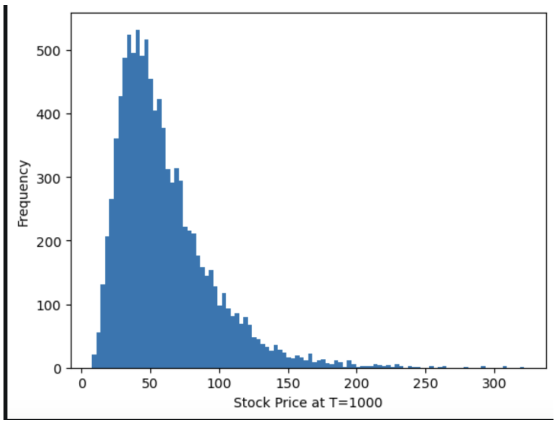
- 

- Mean = 60.018, Variance = 572.44

- Initial Plot : H = 100



- 

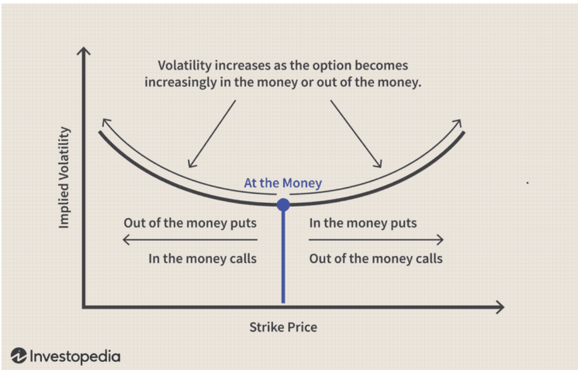
- 

- Mean = 60.164, Variance = 577.18

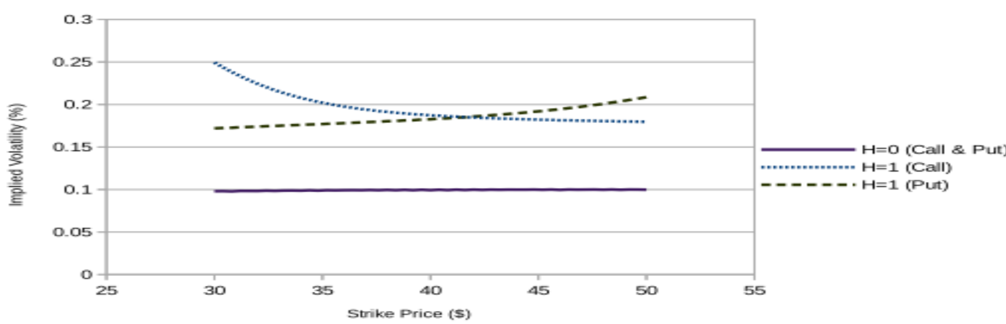
So, the author's initial assertions appear to be correct, and a great way to model the distribution of stock price and returns with delayed information.

Improvements

Delayed information is a formidable force within stock price movement. The introduced volatility from delayed information leads to a sequence known as a "volatility smile", a graph plotting implied volatility and strike price.



In a normal black-scholes model, a volatility smile is not predicted. The important idea introduced by this paper is that delayed information will exaggerate the volatility smile. This finding confirms a trader's belief that delay will increase volatility. The difference between a regular model and a delayed information model can be seen with the following graph:



Without the delay (H = 0), the implied volatility does not change over the strike prices. When delayed information is introduced, the tails of the graphs are curled, increasing the implied volatility.

Moreover, one of the main, novel improvements was using delayed information to satisfy complete market conditions. Due to delayed information, the markets are not considered complete, which prevents perfect replication of a portfolio. With this adjustment, the model is able to find perfect replication by utilizing the increased volatility as mentioned above. This improvement creates a more risk-averse portfolio.

Referenced Paper

- Ichiba, T., & Mousavi, S. M. (2017). Option Pricing with Delayed Information. *arXiv*.https://doi.org/10.48550/arXiv.1707.01600