Iverson computing competition
2016 may 31 solutions appended

name	______________________
school	______________________
city	___________
grade	______
cs teacher	______________________

are you taking AP computer science? (yes/no) ____
are you taking IB computer science? (yes/no) ____
have you taken advanced level courses? (3000 level, e.g. CSE3110 iterative algorithms I.) (yes/no/currently taking) ____________

illegible answers will not be marked

question	- - - -	marks	your score
1	tiling	11	
2	ascii maze	12	
3	blorks	10	
4	hex	7	
total	- - - -	40	

Exam by Zachary Friggstad and Ryan Hayward

Exam Format

This is a two-hour paper and pencil exam. There are four questions, each with multiple parts. Some part(s) might be easy. Solve as many parts of as many questions as you can.

Programming Language

Questions that require programming can be answered using any language (e.g. C/C++, Java, Python, ...) or pseudo-code. **Pseudo-code should be detailed enough to allow for a near direct translation into a programming language.** Clarify your code with appropriate comments. For full marks, an answer must be correct, well-explained, and as simple as possible.

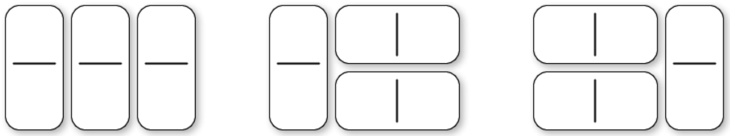
Our primary interest is in thinking skill rather than coding wizardry, so logical thinking and systematic problem solving count for more than programming language knowledge.

Suggestions

1. You can assume that the user enters only valid input in the coding questions.
2. In somes cases, sample executions of the desired program are shown. Review the samples carefully to make sure you understand the specifications. The samples may give hints.
3. Design your algorithm before writing any code. Use any format (pseudo-code, diagrams, tables) or aid to assist your design plan. We may give part marks for legible rough work, especially if your final answer is lacking. We are looking for key computing ideas, not specific coding details, so you can invent your own "built-in" functions for simple subtasks such as reading the next number, or the next character in a string, or loading an array. Make sure to specify such functions by giving a relationship between their inputs and outputs.
4. Read all questions before deciding which ones to attempt, and in which order. Start with the easiest parts of each question.

question 1: tiling

We want to tile $2 \times n$ grids using dominoes. A domino is a 2×1 or 1×2 tile. Here are all ways to tile a 2×3 grid:



(a) [1 mark] Draw all ways to tile a 2×4 grid.

(b) [2 marks] Let $f(n)$ denote the number of ways to tile a $2 \times n$ grid. So $f(0) = 1, f(1) = 1, f(2) = 2$, and $f(3) = 3$. Find $f(5)$ and $f(6)$ (you do not have to draw any tilings, but you can if it helps you).

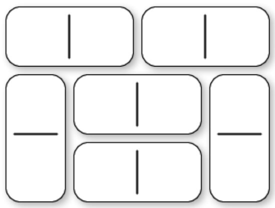
$f(5) =$

$f(6) =$

(c) [2 marks] Let $n \geq 2$. Assume that we know $f(k)$ for all values $0 \leq k \leq n-1$. Give a simple expression that calculates $f(n)$ using these known $f(k)$ values. Carefully justify your answer using at most **two sentences**.

(d) [2 marks] Write a function `tiling(n)` that returns the value $f(n)$ for a given integer $n \geq 0$. For full marks, it should compute $f(9999)$ in less than one second.

(e) [2 mark] Let $g(n)$ denote the number of ways to tile a $3 \times n$ grid using dominoes. Here is a tiling of a 3×4 grid:



So, $g(0) = 1, g(2) = 3$, $g(4) = 11$, and $g(6) = 41$.

What is $g(n)$ when n is odd? Justify your answer.

(f) [2 marks] Let $n \geq 2$ be an even integer. Assume that we know $g(k)$ for all $0 \leq k \leq n-1$. Give an expression that easily calculates $g(n)$ using these known $g(k)$ values. Justify your answer.

question 2: ascii mazes

One way to represent a maze using ASCII characters is `|`-*format*, which uses vertical bars `|`, underscores `_`, and spaces. A rectangular grid using only these three characters is a *valid maze* if

- there are at least 2 rows and at least 3 columns, and
- the top row has only `_` characters, and
- the first character in the second row is a space, indicating the entrance, and
- the last character in the last row is `_`, indicating the exit, and
- except for the entrance and exit, the first and last character on each row is `|`, and
- the bottom row has no spaces.

Example i

```
_________
  | ___ |
| | | | |
|_____|__
```

In `|`-format, a grid cell is *vacant* if it is a space, or if it is not on the top row and is `_`. Two vacant cells *join each other* if one is beside the other (up, down, left, or right). Neighbouring vacant cells join each other if and only if no wall separates them.

(a) [1 marks] For this maze, put a dot in each vacant cell, and draw a line between each pair of vacant cells that join each other.

```
____
  _|
| _|
|___
```

Another way to represent a maze is with `X`-*format*. `X`-format uses a space for vacant cells and an `X` otherwise. The first character of the second row is a space, for the entrance; the last character of the second-last row is a space, for the exit; every other character on the rectangular boundary of this grid is `X`.

Example ii

```
XXXXXXXXX
  X     X
X X XXX X
X X X X X
X X X X X
X     X
XXXXXXXXX
```

We can convert from `|`-format to `X`-format by adding extra rows, and placing a space or `X` between two cells that are on top of each other indicating whether they join each other. The `X`-format example above is what we get by converting from the `|`-format example. Notice that the number of vacant cells can change during this conversion. The maze in (b) is what you would get by converting from the maze in (a).

(b) [1 marks] For this maze, put a dot in each vacant cell, and draw a line between each pair of vacant cells that join each other.

```
XXXX
   X
X XX
X  X
X XX
X
XXXX
```

(c) [4 marks] Write a function `convert(maze)` that takes an array or list of strings representing a valid maze in `|`-format and prints the corresponding maze in `X`-format. For example, `maze` is from (a) then `convert(maze)` prints the maze from (b).

(d) [6 marks] Now write a function `search(maze)` that takes an array of strings representing a maze in `X`-format. It should print a path from the entrance to the exit using `*` characters. You may assume that there is exactly one way to travel from the entrance to the exit using a path that does not visit a cell more than once.

Example: calling `search(maze)` with `X`-format example ii gives this output:

```
XXXXXXXXX
**X*****X
X*X*XXX*X
X*X*X X*X
X*X*X X*X
X***  X**
XXXXXXXXX
```

question 3: blorks

A *binary string* contains only `0`s and `1`. A *blork* is a string containing only characters `0`, `1`, and `*`. A binary string `str` *matches* a blork `blrk` if each `*` character in `blrk` can be replaced with a binary string (possibly empty, and the binary strings do not all have to be the same) so that the resulting string equals `str`. We show such a matching by starting with the blork and then replacing each `*` with `(b)` where `b` is the needed binary string for that `*`.

Example

- `str = "01001"`, `blrk = "01*01"`, **match** `"01(0)01"`
- `str = "101"`, `blrk = "10**1"`, **match** `"10()()1"`
- `str = "10101101"`, `blrk = "10*10*"`, **match** `"10(101)10(1)" and "10()10(1101)"`
- `str = ""`, `blrk = ""` (both strings are empty), **match** `""`
- `str = "11101"`, `blrk = "101*"`, **no match possible**

(a) [2 marks] For each of the following, indicate whether it matches. If yes, give one replacement, in the same form as the example.

- `str = "1001"`, `blrk = "*1*0*1*"`

- `str = "110110001010100"`, `blrk = "1101*100*10*01*"`

- `str = "10001101001"`, `blrk = "100*11*101"`

- `str = ""`, `blrk = "*"`

- `str = "100101010100001010001011101101"`, `blrk = "*1001*001*11*101"`

- `str = "00011010110110001010101010101101"`, `blrk = "*1101*101010*110"`

(b) [3 marks] Write a function `extract(blrk)` that takes a blork `blrk` as parameter and returns an array or list of strings with the binary subtring *pieces* of `blrk`, namely the nonempty binary substrings left over if each $*$ is replaced with a space.

Example: for `blrk = "*10**1110*110"` the three pieces are `"10", "1110", "110"`.

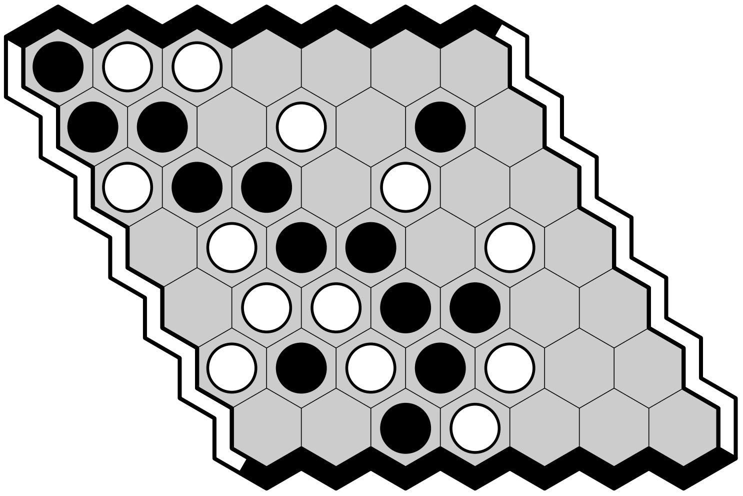
(c) [5 marks] Write a function `match(str, blrk)` that takes a binary string `str` and blork `blrk` as parameters and returns `true` if `str` matches `blrk` and `false` otherwise. If it is helpful, you may use the function described in part (b) even if you did not answer that question.

question 4: hex

Warning: this question can take time. Budget your time wisely.

The two-player game of Hex is played on an $n \times n$ board with hexagonal cells. Below is a 7×7 board. Players alternate turns. On a turn, a player puts a stone on an empty cell.

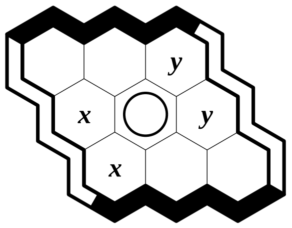
The first player is called *white* and places white stones. The second player is called *black* and places black stones. Two opposing sides of the board are white and the other two opposing sides are black. The winner is whoever connects their two sides with a connected path of their stones. In the example below, black has won.



An amazing property of Hex is that if the board is completely covered with stones then exactly one player has joined their two sides. So draws are not possible.

This question is about playing perfectly in Hex. We say that a player plays *perfectly* if, on each move, if there is some move that is part of a winning strategy for that player, then the player makes such a move. So, whoever can win always makes a winning move; whoever cannot win can play anywhere.

(a) [3 marks] On this 3×3 board white played her first move in the middle. (The letters are for the analysis below.) Now it is black's turn. We claim that white can win.



White can guarantee she gets at least one of the cells labelled x by placing a stone on one of these two cells if black ever places a stone on the other. Similarly, white can guarantee she gets at least one of the cells labelled y. But each x cell touches the left side and the middle, and each y cell touches the middle and the right side. So white will join her two sides if she follows this strategy.

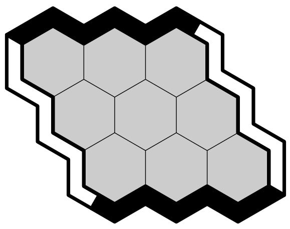
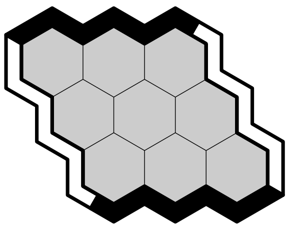
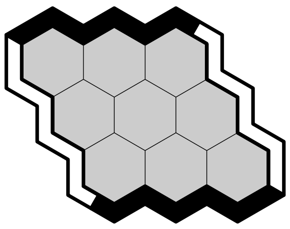
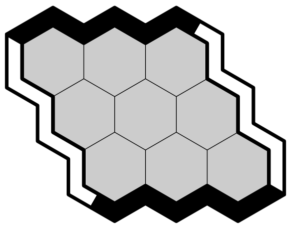
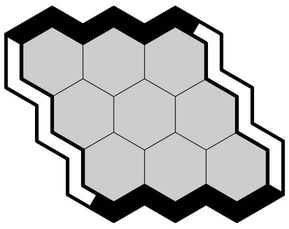
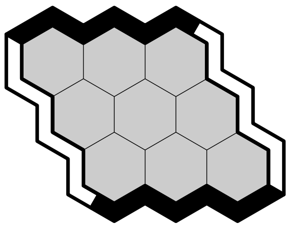
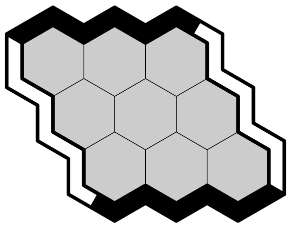
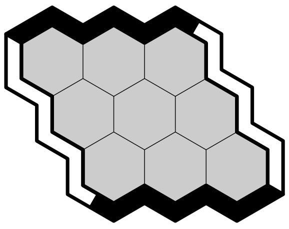
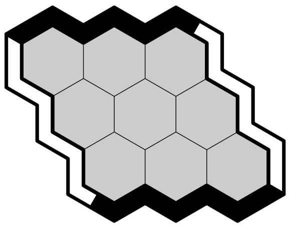
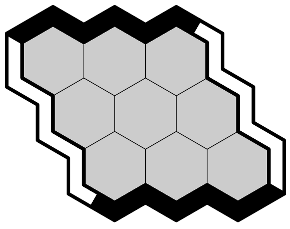
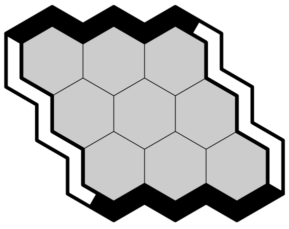
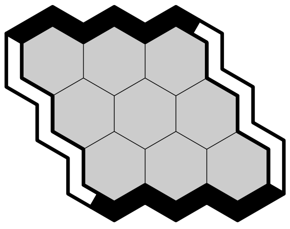
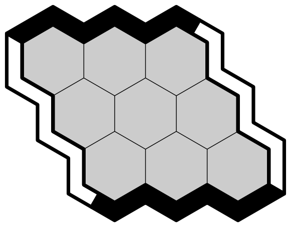
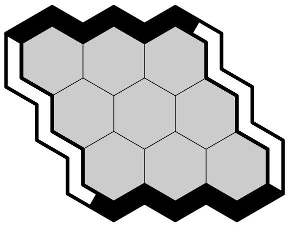
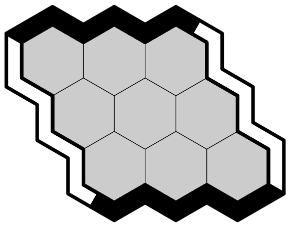
Now assume that white starts a game by by placing her first stone as shown below. It is black's turn. Who will win the game if both players play perfectly from now on? Justify your answer by giving the winning player's strategy **as concisely as possible**.
(We have included a page with blank 3×3 hex grids on the second last page of this exam. You can use these to help describe the strategy, just make sure that we can follow your reasoning.)

(b) [4 marks] For **each** cell on the 4×4 board below, show who wins (white or black) if that cell is white's first move, and both players play perfectly from then on: if white wins, then on that cell draw an empty circle (or the letter `W`); if black wins, then on that cell draw a filled circle (or the letter `B`).

You do not need to justify your answer: your score will be calculated as follows:

$$\max\left\{\frac{(\#\text{ of correct answers}) - (\#\text{ of incorrect answers})}{4}, \quad 0\right\}.$$

Cells left blank will be counted neither correct nor incorrect. Illegible entries will be counted incorrect. (Some empty 4×4 grids can be found at the end of this exam. These are for your scratch notes only and will not be looked at when we grade.)

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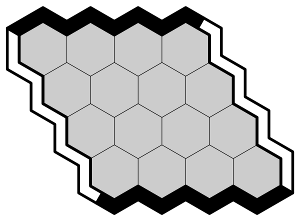
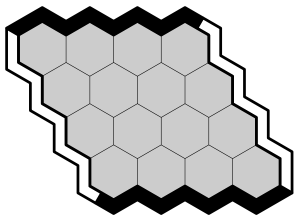
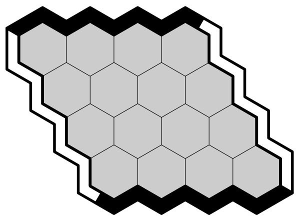
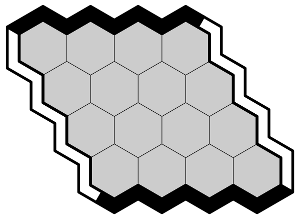
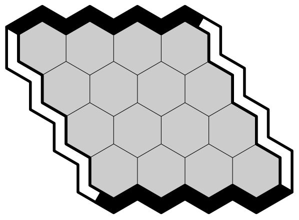
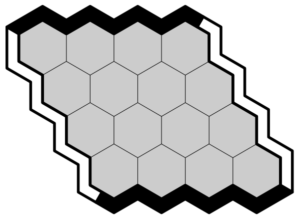
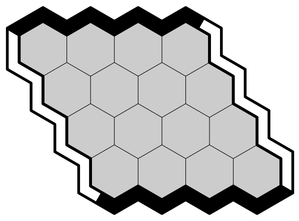
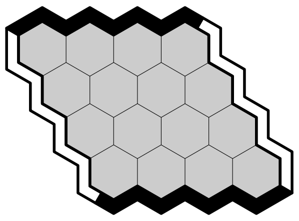
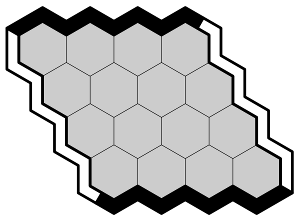
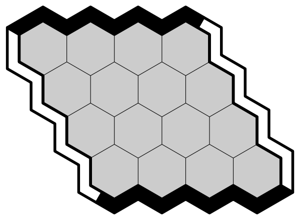
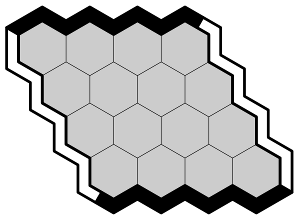
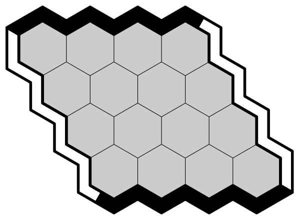
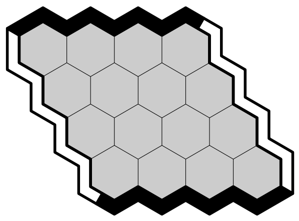
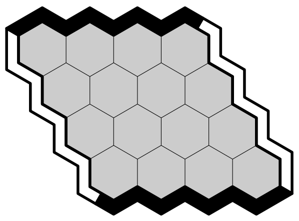
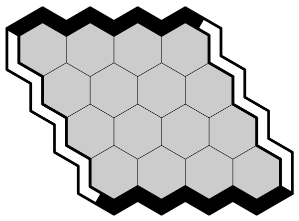
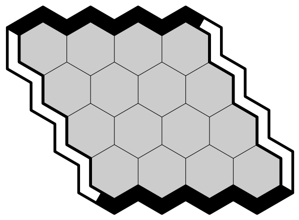
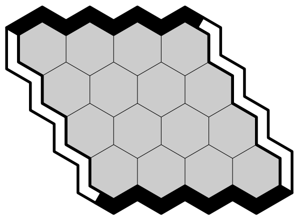
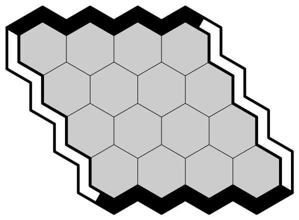
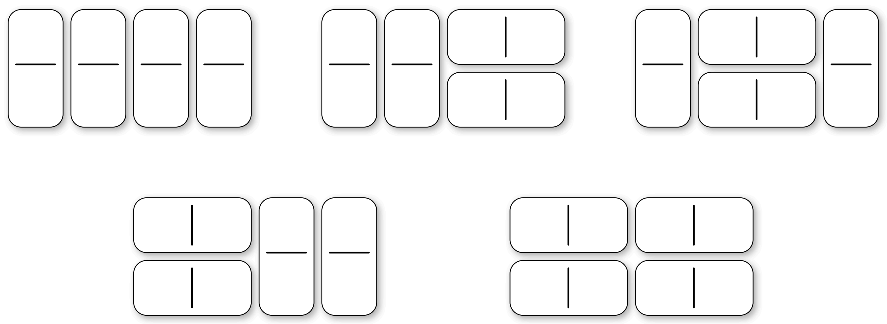



















Solutions

1(a)



1(b) $f(5) = 8, f(6) = 13$

1(c) $f(n) = f(n-1) + f(n-2)$

If the first column is covered by a vertical domino, then there are $f(n-1)$ ways to tile the remaining $2 \times (n-1)$ grid. Otherwise, the first two columns are covered by two horizontal dominoes, and the remaining $2 \times (n-2)$ grid can be tiled $f(n-2)$ ways.

1(d) Here are two solutions:

```
def tiling_sol1(n):
   a, b = 1, 1

   # invariant: just before iteration i
   # we have a = f(i), b = f(i+1)
   for i in range(n):
      a, b = b, a+b

   # postcondition: a = f(n), b = f(n+1)
   return a

def tiling_sol2(n):
   f = [1, 1]
   for i in range(2, n+1):
      f.append(f[i-1] + f[i-2])
   return f[n]
```

Some tried the following approach, but it is very slow. It cannot even compute $f(50)$ in any reasonable time.

```
def tiling_slow(n):
   if n <= 1:
      return 1
   else:
      return f(n-1) + f(n-2)
```

1(e) Each domino covers exactly 2 squares so any tiling by dominos can only cover an even number of squares. When n is odd, a $3 \times n$ grid contains an odd number of squares. So it cannot be tiled by dominos.

1(f) Here are two solutions:

1. $g(n) = 3 \cdot g(n-2) + 2 \cdot g(n-4) + 2 \cdot g(n-6) + \ldots + 2 \cdot g(2) + 2 \cdot g(0)$.

   To see this, any tiling either begins with three horizontal dominos which is followed by one of $g(n-2)$ possibilities. Or it begins with a "wall" (see Figure 1). There are two such walls of length k for every even $2 \leq k \leq n$, and each possible wall is followed by one of $g(n-k)$ possibilities.

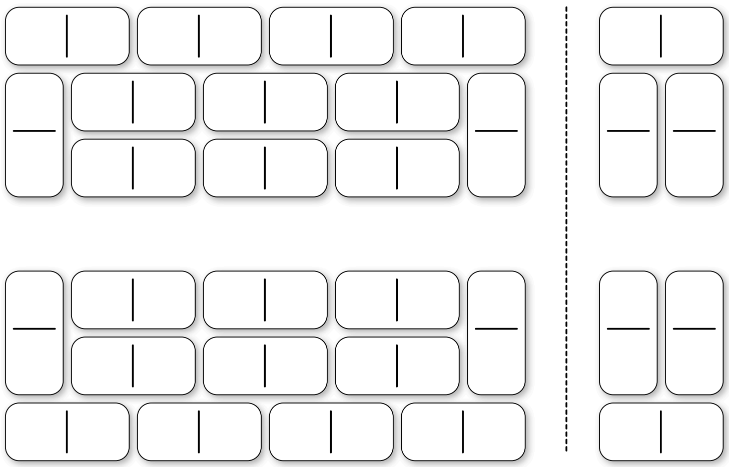


Figure 1: Left: the two walls of length 8. Right: the two walls of length 2.

2. $g(2) = 3$ and $g(n) = 4 \cdot g(n-2) - g(n-4)$ for $n \geq 4$.

   To see this, the number of tilings beginning with three horizontal dominos or a length 2 wall is $3 \cdot g(n-2)$. Every other tiling begins with a wall of length at least 4. Such tilings corresponds to tilings of the $3 \times (n-2)$ grid that starts with a wall (see Figure 2). The number of tilings of the $3 \times (n-2)$ grid that begin with a wall is exactly $g(n-2) - g(n-4)$.

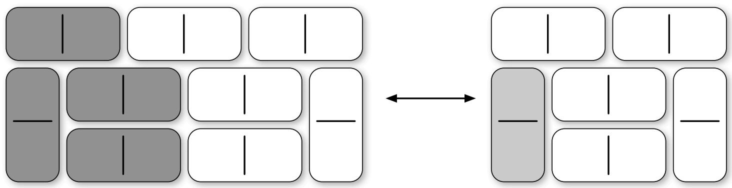
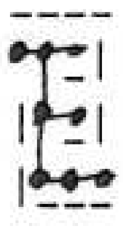
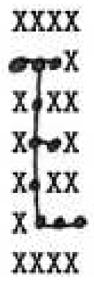


Figure 2: Swapping the dark grey dominos on the left with the single light grey domino shows how to convert between a length k wall and a length $k-2$ wall.

2(a) 

2(b) 

2(c) Here is a solution in python3:

```
def translate(maze):
  # this code would be cleaner if the input
  # format insisted that the left entrance
  # be _ instead of blank on top of |
  # ... either answer was ok
  newMaze = ['X'*len(maze[0])] #top border
  for j in range(1,len(maze)): #rest of maze
    newLine, newLine2  = '',''
    for k in range(len(maze[0])):
      if maze[j][k] == '|':
        newLine  += 'X'
        newLine2 += 'X'
      elif maze[j][k] == '_':
        newLine  += ' '
        newLine2 += 'X'
      else: # x must be ' '
        # this case would be simpler if input
        # format was changed as mentioned above
        newLine  += ' '
        if maze[j+1][k] == '|':
          newLine2 += 'X'
        else:
          newLine2 += ' '
    newMaze.append(newLine)
    newMaze.append(newLine2)
  return newMaze

test_maze = [
  "_________",
  "  |     |",
  "| | | | |",
  "|_____|__"
]

for line in translate(test_maze):
  print(line)
```

2(d)

One way to find a minimal path is to find a shortest path. One algorithm for unit-distance shortest path is breadth-first search. Each node in the algorithm is a cell in the maze. Each node has 4 neighbour locations: above, below, left, right

```
init()            # queue will hold cells we have seen but not processed
                  # parent[] pointers will show way back from dest'n
                  # seen[]  boolean: have we encountered this cell?
                  # wall(x): is there a wall at location x?
Q.add(start)
seen[start] = true
parent[start] = start
done = (start == dest) # done once we reach the destination
while (not done) and (not Q.isempty()):
  x = Q.removefirst()        # remove earliest cell in the queue
  for each neighbour y of x: # 4 neighbours: above, below, left, right
    if (not wall(y) and (not done) and (not seen[y]):
      Q.add(y)
      seen[y] = true
      parent[y] = x
      done = (y == dest)
```

You are told that there exists a path. It can be printed following parent pointers:

```
x = dest
while parent[x] <> x:
  print x   # so on our output grid, change the cell from space to *
  x = parent[x]
```

So now print the grid and you are done.

2(d) another solution. Traverse the maze via depth-first search, which can be implemented recursively. Be careful that the path that you find is minimal, i.e. has no shortcuts. Here is a working python3 solution:

```
def search(maze):
    # change the strings to lists of characters
    # (Python strings are immutable)
    maze = [list(line) for line in maze]

    # ensures the search won’t try to go out of the entrance
    maze[1][0] = ’*’

    def recursive_search(curr_r, curr_c):
        # if we found the exit, return True
        if curr_c == len(maze[0])-1:
            maze[curr_r][curr_c] = ’*’
            return True
        # if this cell is a wall or we have already visited it
        if maze[curr_r][curr_c] != ’ ’:
            return False
        # leave a bread crumb, we have visited this space before
        maze[curr_r][curr_c] = ’.’
        # try searching in each of the four directions from this spot
        for dr, dc in [(-1, 0), (1, 0), (0, -1), (0, 1)]:
            # if we found the exit, mark the path and return True
            if recursive_search(curr_r + dr, curr_c + dc):
                maze[curr_r][curr_c] = ’*’
                return True
        # no exit found
        return False

    # start the search from the right of the entrance
    recursive_search(1, 1)
    # all bread crumbs should be set to ’ ’ for printing
    for line in maze:
        for i in range(len(line)):
            if line[i] == ’.’: line[i] = ’ ’
    # print it out
    for line in maze:
        print(*line, sep = ’’)

test_maze = [
    "XXXXXXXXX",
    "  X     X",
    "X X XXX X",
    "X X X X X",
    "X X X X X",
    "X     X  ",
    "XXXXXXXXX"
]

search(test_maze)
```

3(a)

- **match**: ”()1()0(0)1()”
- **match**: ”1101()100(0)10(1)01(00)”
- **no match**
- **match**: ”()”
- **match**: ”()1001(01010100)001(0100010)11(101)101”
- **no match**

3(b) You could use a `split` function that is built in to most programming languages. However, you must take care to remove empty strings from the list of substrings.

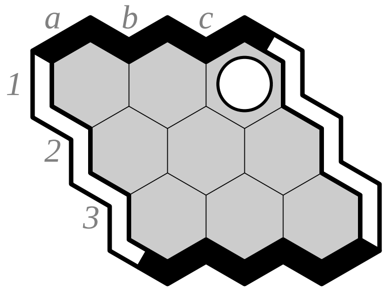
```
def extract(blrk):
    # split into substrings delimited by ’*’ character
    # the list comprehension will only use those that are nonempty
    substrings = blrk.split(’*’)
    return [str for str in substrings if str != ’’]
```

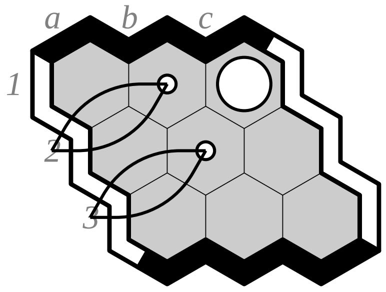
Without the split function, you could add the characters of `blrk` to a buffer, one character at a time. If you encounter the `*` character then the buffer string could be added to the output list (provided the buffer string is not empty).

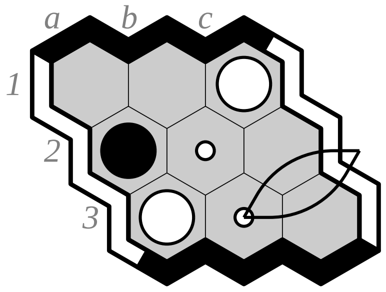
```
def extract_v2(blrk):
    buf = ’’
    substrings = []

    # ensures the last substring of blrk will be processed
    blrk += ’*’

    for c in blrk:
        if c == ’*’:
            if buf != ’’:
                substrings.append(str)
            str = ’’
        else:
            buf += c

    return substrings
```

3(c) The idea is to find the earliest occurence of each binary substring of `blrk` that appeared after the occurence we recorded for the previous binary substring. If we do not find all binary substrings this way, there is no match. We also have to make sure the first substring of `blrk` occurs as a prefix of `str` if `blrk` does not begin with `*` (and similarly for the last substring). Care must be taken if `blrk` contains no `*`.

```
def match(str, blrk):
    if '*' not in blrk:
        return str == blrk

    # here blrk has at least one *
    substrings = extract(blrk)
    if substrings == []:
        return True # anything matches an all-* blork

    # if requires, ensure the start and/or end of str matches the
    # appropriate substring
    if blrk[0] != '*' and str[:len(substrings[0])]:
return False

    for sub in substrings:
        # find the least index of str where sub appears as a substring
        loc = str.find(sub)
        if loc == -1:
            return False
        else:
            # remove all characters of str up to the end of the occurence of sub
            str = str[loc + len(sub):]

    return True
```



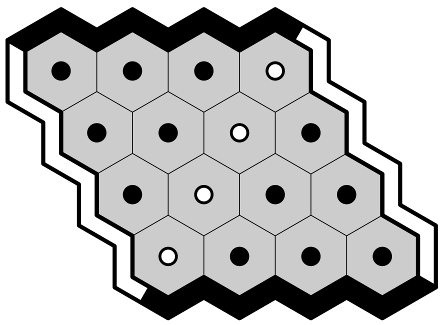




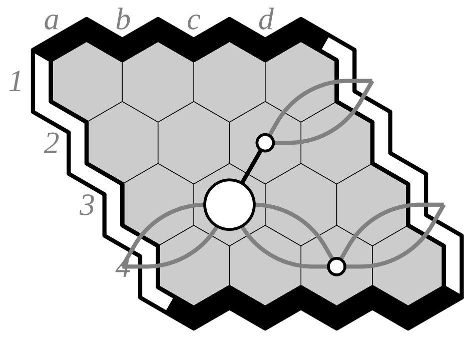
4(a) solution. (above middle) If black's 1st move is not in the set $\{a1, a2, b1\}$ then white wins by playing b1, and then on the next move one of $\{a1, a2\}$. Similarly, if black's 1st move is not in $\{a2, a3, b2\}$ then white wins by playing b2 and then on the next move one of $\{a2, a3\}$. So white wins unless black's 1st move is in *both* sets, so a2.

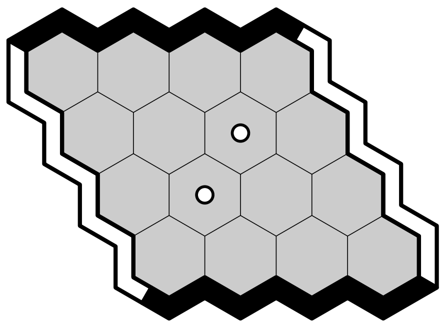
(above right) Now white wins by playing a3, and then on the next move either b2 or (if black is already there) b3, and then on the next move either c2 or c3.

4(b)

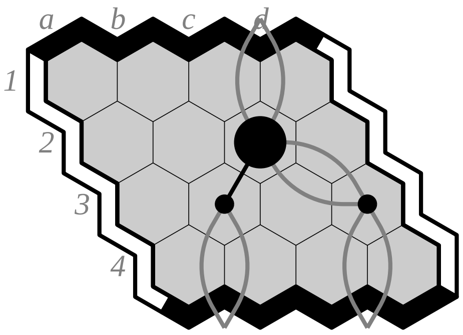


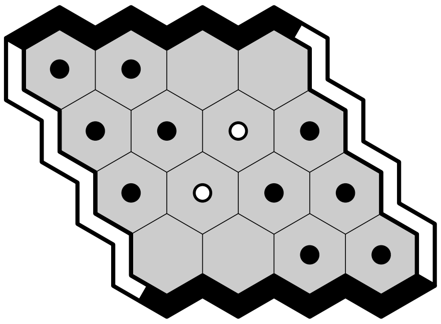
White wins by playing b3, as shown (below left). By a symmetric strategy (reflect the board through the centermost point) white also wins by playing c2. So far we have two winning openings.





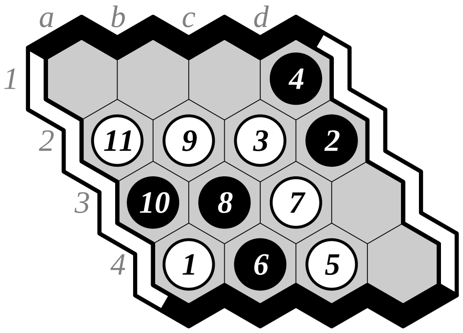
(below left) If white plays first in any of the unmarked cells, black can win. Considering also the symmetric black strategy yields 10 losing openings (below right).

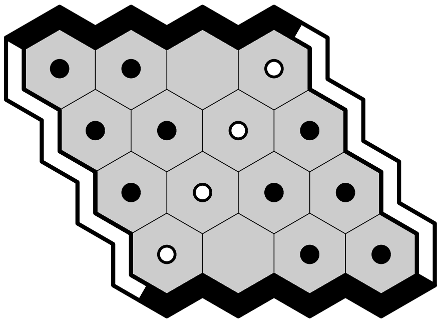




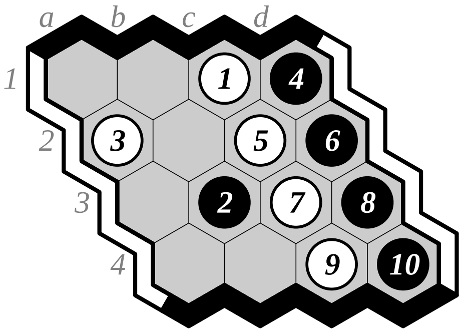
White can win by playing first at a4. If black's reply is not in {b3,b4,c3,d2,d3} then white wins by next playing c3. If black's reply is in one of these 5 cells, then white wins by playing c2. This is easy to see if black played in {b3,c3,d3}, and not too hard to see if black played in {b4,d2}. One such line of play is shown below.

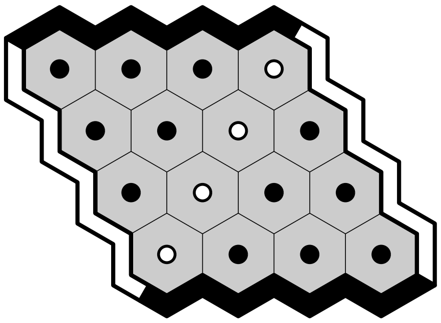
By symmetry, White also wins by playing first at d1.





There are only two opening moves left, and they are symmetric. If white plays first at c1 (below left), black can win: the main variation is shown, this is not a complete proof. By symmetry, if white plays first at b4, black wins. So we are done.





PS. Ryan Hayward is writing a book on Hex. If you would like to be notified when it is published, send him email: `hayward@ualberta.ca` .