Exam II

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1. Problem I

(a) Begin by writing the boundary conditions:
(i) $V = V_0 cos(\theta)$ at $r = R_1$
(ii) $V = V_0$ at $r = R_2$
(iii) $V = 0$ at $r \to \infty$
Now the problem has azimuthal symmetry (independent of ϕ), so Laplace's equation reads:

$$\frac{\partial}{\partial r}\Big(r^2\frac{\partial V}{\partial r}\Big) + \frac{1}{sin\theta}\frac{\partial}{\partial \theta}\Big(sin\theta\frac{\partial V}{\partial \theta}\Big) = 0$$

Using separation of variables as done in class yields solutions of the form:

$$V(r,\theta) = \sum_{n=0}^{\infty}\Big(A_n r^n + \frac{B_n}{r^{n+1}}\Big)P_n(cos\theta)$$

where the P_n's are the Legendre polynomials.
Consider the middle region, $R_1 < r < R_2$. Solutions take the complete form because r tends neither to ∞ nor 0. Write:

$$V_{R_1<r<R_2}(r,\theta) = \sum_{n=0}^{\infty}\Big(A_n r^n + \frac{B_n}{r^{n+1}}\Big)P_n(cos\theta)$$

Consider now the outer region, $r > R_2$. We must let $A_n = 0 \ \forall \ n$ because otherwise the potential goes to a nonphysical ∞ at large r. Write:

$$V_{r>R_2}(r,\theta) = \sum_{n=0}^{\infty}\Big(\frac{C_n}{r^{n+1}}P_n(cos\theta)\Big)$$

I replaced B_n with C_n because we need a different set of coefficients in the outer-region.
Consider now the innermost region, $r < R_1$. We must let $B_n = 0 \ \forall \ n$ because otherwise the potential goes to a nonphysical ∞ at small r. Write:

$$V_{r<R_1}(r,\theta) = \sum_{n=0}^{\infty}\Big(D_n r^n P_n(cos\theta)\Big)$$

Because of the spherical symmetry of the system, we can use Fourier's trick to write equations at the boundaries that only consider the case of $n = 0$ because other values of $P_n(cos\theta)$ produce orthogonal functions that when integrated by Fourier's trick go to 0. The general solution for any region becomes,

$$V(r,\theta) = A_0 + \frac{B_0}{r}$$

Write,

$$A_0 + \frac{B_0}{R_2} = \frac{C_0}{R_2} = V_0$$

$$A_0 + \frac{B_0}{R_1} = D_0 = V_0 cos\theta$$

The boundary cases can equate solutions in different areas by the condition of continuity at the boundary.

Consider the region of $R_1 < r < R_2$. Consider $A_0 + \frac{B_0}{R_2} = V_0$ and $A_0 + \frac{B_0}{R_1} = V_0 cos\theta$. Subtract one from the other:

$$B_0 = \frac{V_0 cos\theta - V_0}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

This implies by $A_0 = V_0 cos\theta - \frac{B_0}{R_1}$:

$$A_0 = V_0 cos\theta - \frac{1}{R_1}\frac{V_0 cos\theta - V_0}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

Plugging these value in to the general solution for $R_1 < r < R_2$,

$$V_{R_1<r<R_2}(r,\theta) = V_0 + \left(\frac{1}{r} - \frac{1}{R_1}\right)\frac{V_0 cos\theta - V_0}{\frac{1}{R_1} - \frac{1}{R_2}}$$

Consider the region of $r < R_1$. We have:

$$V_{r<R_1}(r,\theta) = D_0$$

but at R_1, $D_0 = V_0 cos\theta$. Thus,

$$V_{r<R_1}(r,\theta) = V_0 cos\theta$$

Consider the region of $r > R_2$. We have:

$$V_{r>R_2}(r,\theta) = \frac{C_0}{r}$$

but at R_2, $\frac{C_0}{R_2} = V_0 \implies C_0 = V_0 R_2$. Thus,

$$V_{r>R_2}(r,\theta) = \frac{V_0 R_2}{r}$$

(b) We know that $E(r,\theta) = -\nabla V$.
We will compute by region:
(1) $r < R_1$:

$$-\nabla V_0 cos\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}V_0 cos\theta = \frac{V_0 sin\theta}{r}\hat{\theta}$$

(2)$R_1 < r < R_2$:

$$-\nabla\Big(V_0 + \left(\frac{1}{r} - \frac{1}{R_1}\right)\frac{V_0 cos\theta - V_0}{\frac{1}{R_1} - \frac{1}{R_2}}\Big) = -\Big(\frac{-1}{r^2}\frac{V_0 cos\theta - V_0}{\frac{1}{R_1} - \frac{1}{R_2}}\hat{r} + \frac{1}{r}\Big(\frac{-V_0 sin\theta}{r(\frac{1}{R_1} - \frac{1}{R_2})} + \frac{V_0 cos\theta}{R_1(\frac{1}{R_1} - \frac{1}{R_2})}\Big)\hat{\theta}\Big)$$

$$= \frac{1}{r^2}\frac{V_0 cos\theta - V_0}{\frac{1}{R_1} - \frac{1}{R_2}}\hat{r} + \frac{1}{r}\Big(\frac{V_0 sin\theta}{r(\frac{1}{R_1} - \frac{1}{R_2})} - \frac{V_0 sin\theta}{R_1(\frac{1}{R_1} - \frac{1}{R_2})}\Big)\hat{\theta}$$

(3) $r > R_2$:

$$-\nabla\frac{V_0 R_2}{r} = -\frac{\partial}{\partial r}\frac{V_0 R_2}{r} = \frac{V_0 R_2}{r^2}\hat{r}$$

Now we will find the surface charge density. In general, $\sigma = \epsilon_0(E^{\perp}_{above} - E^{\perp}_{below})$. Note that only the $\hat{r}$ components are perpendicular to the surface.
(1) σ_1:

$$\sigma_1 = \frac{1}{R_1^2}\frac{V_0 cos\theta - V_0}{\frac{1}{R_1} - \frac{1}{R_2}}$$

(2) σ_2:

$$\sigma_2 = \frac{V_0 R_2}{R_2^2} - \frac{1}{R_2^2}\frac{V_0 cos\theta - V_0}{\frac{1}{R_1} - \frac{1}{R_2}}$$

2. Problem II

(a) Let $\lambda = \lambda_0$. We have that $\oint \bar{D} \cdot d\bar{a} = Q_{f_{enc}}$. Then:

$$D(2\pi r L) = \lambda L \implies \bar{D} = \frac{-\lambda}{2\pi r}\hat{r}$$

We have that this is a linear dielectric.
In general for linear media, $\bar{D} = \epsilon_0 \bar{E} + \bar{P} = \epsilon_0 \chi_e \bar{E} = \epsilon_0(1 + \chi_e)\bar{E}$.
This implies that $\bar{D} = \epsilon \bar{E}$ where $\epsilon = \epsilon_r \epsilon_0$. In our case, $\epsilon = \epsilon_0 \frac{(R^2+r^2)}{2rR}$.
We can now explicitly write $\bar{E}$.

$$\bar{E} = \frac{\bar{D}}{\epsilon} = \frac{-\lambda}{2\pi r}\frac{2rR}{\epsilon_0(R^2 + r^2)}\hat{r} = \frac{-\lambda R}{\pi\epsilon_0(R^2 + r^2)}\hat{r}$$

Now calculate ΔV.
We have that $C = C_{vac}\epsilon_r$. Find capacitance in vacuum. For a cylindrical capacitor of infinite length, we consider an analogy to a point outside a long line of charge a distance r from the axis. This has potential

$$V = \frac{\lambda}{2\pi\epsilon_0} ln(\frac{r_0}{r})$$

This holds because the charge on the outer cylinder doesn't contribute to the field between the cylinders. So,

$$\Delta V = V_{\sqrt{3}R} - V_R = \frac{\lambda}{2\pi\epsilon_0} ln(\frac{R}{\sqrt{3}R})$$

The total charge in length L is $Q = \lambda L$, so

$$C_{vac} = \frac{Q}{|\Delta V|} = \frac{2\pi\epsilon_0 L}{ln(\frac{1}{\sqrt{3}})}$$

Thus, we can conclude that,

$$C = C_{vac}\epsilon_r = \frac{2\pi\epsilon_0 L}{ln(\frac{1}{\sqrt{3}})}\epsilon_r$$

Now find the energy density u. From class, the energy density in a capacitor with linear dielectric is $u_E = \frac{1}{2}\bar{E} \cdot \bar{D}$. But $\bar{E}$ and $\bar{D}$ are parallel, so we can write $u_E = \frac{1}{2}|\bar{E}||\bar{D}|$.

$$u_E = \frac{1}{2}|\bar{E}||\bar{D}| = \frac{1}{2}|\frac{-\lambda R}{\pi\epsilon_0(R^2 + r^2)}\hat{r}||\frac{-\lambda}{2\pi r}\hat{r}| = \frac{1}{2}|\frac{\lambda R}{\pi\epsilon_0(R^2 + r^2)}||\frac{\lambda}{2\pi r}| = \frac{\lambda^2 R}{4\pi^2\epsilon_0 r(R^2 + r^2)}$$

(b) First find the polarization. We have $\bar{P} = \epsilon_0\chi_e\bar{E} = \epsilon_0(\epsilon_r - 1)\bar{E}$.

$$\bar{P} = \epsilon_0\Big(\frac{R^2 + r^2}{2rR} - 1\Big)\frac{-\lambda R}{\pi\epsilon_0(R^2 + r^2)}\hat{r} = \Big(-\frac{(R^2 + r^2)\lambda R}{\pi(R^2 + r^2)2rR} + \frac{\lambda R}{\pi(R^2 + r^2)}\Big)\hat{r}$$

$$= \Big(-\frac{\lambda}{2\pi r} + \frac{\lambda R}{\pi(R^2 + r^2)}\Big)\hat{r}$$

Now find the volumetric bound charge density. We have $\rho_b = -\nabla \cdot \bar{P}$.

$$\rho_b = -\frac{1}{r}\frac{\partial}{\partial r}\Big(-\frac{r\lambda}{2\pi r} + \frac{r\lambda R}{\pi(R^2 + r^2)}\Big) = -\frac{1}{r}\frac{\partial}{\partial r}\Big(-\frac{\lambda}{2\pi} + \frac{r\lambda R}{\pi(R^2 + r^2)}\Big)$$

$$= \frac{1}{r}\Big[r\Big(\frac{-\lambda R}{\pi(R^2 + r^2)^2}2r\Big) + \frac{r\lambda R}{\pi(R^2 + r^2)}\Big] = \frac{1}{r}\Big[\frac{-2r^2\lambda R}{\pi(R^2 + r^2)^2} + \frac{r\lambda R}{\pi(R^2 + r^2)}\Big]$$

$$= \frac{1}{r}\Big[\frac{\lambda R}{\pi}\Big(\frac{1}{R^2 + r^2} - \frac{2r^2}{(R^2 + r^2)^2}\Big)\Big] = \frac{1}{r}\frac{-\lambda R}{\pi}\frac{R^2 - r^2}{(R^2 + r^2)^2}$$

Now find the surface bound charge densities. We have that $\sigma_b = \bar{P}\cdot\hat{n}$.

$$\sigma_{1b} = \Big(-\frac{\lambda}{2\pi R} + \frac{\lambda R}{\pi(R^2+R^2)}\Big)\hat{r}\cdot\hat{n}$$

But because of cylindrical symmetry, $\hat{r}=\hat{n}$.

$$\sigma_{1b} = \Big(-\frac{\lambda}{2\pi R} + \frac{\lambda}{2\pi R}\Big) = 0$$

$$\sigma_{2b} = \Big(-\frac{\lambda}{2\pi\sqrt{3}R} + \frac{\lambda R}{\pi(R^2+(\sqrt{3}R)^2)}\Big) = \Big(-\frac{\lambda}{2\sqrt{3}\pi R} + \frac{\lambda R}{\pi(R^2+3R^2)}\Big)$$

$$\sigma_{2b} = \Big(-\frac{\lambda}{2\sqrt{3}\pi R} + \frac{\lambda}{4\pi R}\Big) = \frac{\lambda}{\pi R}\Big(\frac{-1}{2\sqrt{3}} + \frac{1}{4}\Big)$$

To satisfy that bound charges cancel, we want $\int_{\partial V}\sigma_b\cdot da + \int_V \rho_b\cdot d\tau = 0$.
First compute $\int_{\partial V}\sigma_b\cdot da$.

$$\int_0^L \frac{\lambda}{\pi R}\Big(\frac{-1}{2\sqrt{3}}+\frac{1}{4}\Big)2\pi r dz = \frac{\lambda}{\pi R}\Big(\frac{-1}{2\sqrt{3}}+\frac{1}{4}\Big)2\pi r\int_0^L dz$$

$$= 2\sqrt{3}L\lambda\Big(\frac{-1}{2\sqrt{3}}+\frac{1}{4}\Big)$$

Next compute $\int_V \rho_b\cdot d\tau$.

$$\int_V \rho_b\cdot d\tau = 2\pi L\int_R^{\sqrt{3}R}\frac{-\lambda_0 R}{\pi r}\frac{R^2-r^2}{(R^2+r^2)^2}r dr$$

$$= -2\lambda RL\int_R^{\sqrt{3}R}\frac{R^2-r^2}{(R^2+r^2)^2}dr = -2\lambda RL\Big[\frac{r}{R^2+r^2}\Big]\Big|_R^{\sqrt{3}R}$$

$$= -2\lambda RL\Big[\frac{R\sqrt{3}}{4R^2} - \frac{R}{2R^2}\Big]$$

Now compute the sum.

$$\int_{\partial V}\sigma_b\cdot da + \int_V\rho_b\cdot d\tau = 0 \implies \frac{\lambda}{\pi R}\Big(\frac{-1}{2\sqrt{3}}+\frac{1}{4}\Big) - 2\lambda RL\Big[\frac{R\sqrt{3}}{4R^2}-\frac{R}{2R^2}\Big] = 0$$

$$\implies \frac{\lambda}{\pi R}\Big(\frac{-1}{2\sqrt{3}}+\frac{1}{4}\Big) = 2\lambda RL\Big[\frac{R\sqrt{3}}{4R^2}-\frac{R}{2R^2}\Big]$$

$$\frac{\sqrt{3}}{4}-\frac{1}{2} = \frac{\sqrt{3}}{4}-\frac{1}{2}$$

This proves the desired cancellation.

3. Problem III

(a) First, find the surface current $\bar{K}$.

$$\int_{-\frac{d}{2}}^{\frac{d}{2}}\bar{J}dz = 2\Big(\frac{-J_0 z^2}{d}\Big)\Big|_0^{\frac{d}{2}}\hat{z} = \frac{-J_0 d}{2}\hat{z}$$

This is the current element carried in an infinitesimally thin strip with respect to y which runs from $x=-\infty\to\infty$ Thus, $\bar{K} = \frac{-J_0 d}{2}y\hat{x}$.
To calculate the magnetic field, use $\hat{J}$ and Ampere's Law.
By symmetry, the field is in the $\hat{y}$ direction. The field in the region above the xy plane will be $(-1)*$(the field in the region below the xy plane. Construct an amperian "pillbox" and

vary the z value to find $\hat{B}$ in and out of the slab:
At $0 < z < \frac{d}{2}$,

$$\bar{B} = \frac{\mu_0}{L}\int_S \bar{J}d\bar{a} = \frac{-2\mu_0 J_0 z^2}{d}\hat{y}$$

At $z > \frac{d}{2}$,

$$\bar{B} = \frac{\mu_0}{L}\int_S \bar{J}d\bar{a} = \frac{-2\mu_0 J_0 (\frac{d}{2})^2}{d}\hat{y} = \frac{-\mu_0 J_0 d}{2}\hat{y}$$

So we write,

$$\bar{B} = \frac{-\mu_0 J_0 d}{2}\hat{y},\ \frac{d}{2} < z$$
$$\bar{B} = \frac{\mu_0 J_0 d}{2}\hat{y},\ \frac{-d}{2} > z$$
$$\bar{B} = \frac{-2\mu_0 J_0 z^2}{d}\hat{y},\ \frac{-d}{2} < z < \frac{d}{2}$$

(b) We know that the direction of $\bar{A}$ is the same as the direction of the current. Thus, $\bar{A}$ is in the $\hat{x}$ direction.
Now we apply Stokes's theorem,

$$\oint \bar{A}\cdot d\bar{l} = \int_A \bar{B}\cdot d\bar{a}$$

For area A, define a rectangle above the xy plane and parallel to the xz plane. Let the sides of the rectangle parallel to the xy plane be length 1. Let the lower side be distance a and upper side be distance b from the plane. Now for $\frac{d}{2} < z$,

$$\int_A \bar{B}\cdot d\bar{a} = \frac{-\mu_0 J_0 d}{2}(b-a)$$

Now consider

$$\bar{A} = \frac{-\mu_0 J_0 d}{2}z\hat{x}$$

Check the divergence and curl. Trivially, the divergence is 0 because there is x element in the $\hat{x}$ direction. The curl is

$$\nabla \times \bar{A} = -\frac{\partial}{\partial z}\Big(\frac{-\mu_0 J_0 d}{2}z\Big)\hat{y} = \frac{-\mu_0 J_0 d}{2}\hat{y} = \bar{B}$$

This meets both conditions. Now for $z < \frac{d}{2}$, by symmetry we can write

$$\bar{A} = \frac{\mu_0 J_0 d}{2}z\hat{x}$$

This also satisfies the divergence and curl conditions.
Now for $\frac{-d}{2} < z < \frac{d}{2}$,

$$\int_A \bar{B}\cdot d\bar{a} = \frac{-2\mu_0 J_0 d}{2}(b-a)$$

Now consider

$$\bar{A} = \frac{-2\mu_0 J_0 d}{2}z\hat{x}$$

Check the divergence and curl. Trivially, the divergence is 0 as above. The curl is

$$\nabla \times \bar{A} = -\frac{\partial}{\partial z}\Big(\frac{-2\mu_0 J_0 d}{2}z\Big)\hat{y} = \frac{2\mu_0 J_0 d}{2}\hat{y} = \bar{B}$$

This satisfies all of the required conditions.