PHYS7721 Homework 6

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Exercise 1. Derive, for all three statistics, the relevant expressions for the quantity $\langle n_\epsilon^2\rangle - \langle n_\epsilon\rangle^2$ from the respective probabilities $p_\epsilon(n)$. Show that, quite generally,

$$\langle n_\epsilon^2\rangle - \langle n_\epsilon\rangle^2 = kT\Big(\frac{\partial\langle n_\epsilon\rangle}{\partial\mu}\Big)_T;$$

compare with the corresponding result, (4.5.3), for a system embedded in a grand canonical ensemble.

Solution. Begin with Bose-Einstein,

$$p_\epsilon(n) = (1-r)r^n$$

where $r = \frac{\langle n_\epsilon\rangle}{\langle n_\epsilon\rangle+1}$ and $n \in \mathbb{N}$. Then

$$\langle n_\epsilon\rangle = (1-r)\sum_{n=0}^{\infty} nr^n = \frac{r}{1-r}$$

$$\langle n_\epsilon^2\rangle = (1-r)\sum_{n=0}^{\infty} n^2r^n = \frac{r(1+r)}{(1-r)^2}$$

and altogether,

$$\langle n_\epsilon^2\rangle - \langle n_\epsilon\rangle^2 = \frac{r}{(1-r)^2}$$

Now Fermi-Dirac,

$$\langle n_\epsilon^2\rangle = \sum_{n=0}^{1} n^2 p_\epsilon(n) = p_\epsilon(1) = \langle n_\epsilon\rangle$$

$$\langle n_\epsilon^2\rangle - \langle n_\epsilon\rangle^2 = \langle n_\epsilon\rangle - \langle n_\epsilon\rangle^2$$

Now Maxwell-Boltzmann,

$$\langle n_\epsilon(n_\epsilon-1)\rangle = \sum_n n(n-1)\frac{\langle n_\epsilon\rangle^n}{n!}e^{-\langle n_\epsilon\rangle} = \langle n_\epsilon\rangle^2$$

$$\langle n_\epsilon^2\rangle - \langle n_\epsilon\rangle^2 = \langle n_\epsilon\rangle$$

Using $\langle n_\epsilon\rangle^{-1} = e^{(\epsilon-\mu)/kT} + a$ and differentiative with respect to μ we find

$$\frac{-1}{kT}\Big[\langle n_\epsilon\rangle^{-1} - a\Big]$$

And altogether,

$$kT\Big[\frac{\partial\langle n_\epsilon\rangle}{\partial\mu}\Big]_T = \langle n_\epsilon\rangle - a\langle n_\epsilon\rangle^2$$

So the expression requested holds in all cases up to a constant. This expression is similar to (4.5.3) with the different types of averages switched. □

Exercise 2. Refer to Section 6.2 and show that, if the occupation number n_ϵ of an energy level ϵ is restricted to the values $0, 1, ..., l$, then the mean occupation number of that level is given by

$$\langle n_\epsilon \rangle = \frac{1}{z^{-1}e^{\beta\epsilon} - 1} - \frac{l+1}{(z^{-1}e^{\beta\epsilon})^{l+1} - 1}.$$

Check that while $l = 1$ leads to $\langle n_\epsilon \rangle_{F.D}$, $l \to \infty$ leads to $\langle n_\epsilon \rangle_{B.E}$.

Solution. With the hint to start from equation (6.2.15),

$$\mathrm{L}(z, V, T) = \prod_\epsilon \Big[\sum_{n_\epsilon=0}^{l} (ze^{-\beta\epsilon})^{n_\epsilon} \Big] = \prod_\epsilon \Big[\frac{1 - (ze^{-\beta\epsilon})^{l+1}}{1 - ze^{-\beta\epsilon}} \Big]$$

and converting to natural log form, prods to sums,

$$q(z, V, T) = \sum_\epsilon \Big(\ln(1 - (ze^{-\beta\epsilon})^{l+1}) - \ln(1 - ze^{-\beta\epsilon}) \Big)$$

Now we use the definition of $\langle n_\epsilon \rangle$,

$$\langle n_\epsilon \rangle = \frac{-1}{\beta} \Big(\frac{\partial q}{\partial \epsilon} \Big)_{z,T,\epsilon} = \frac{1}{z^{-1}e^{\beta\epsilon} - 1} - \frac{l+1}{(z^{-1}e^{\beta\epsilon})^{l+1} - 1}$$

which in the case $l = 1$ is the given result for Fermi-Dirac and if we take $l \to \infty$ and consider that $z^{-1}e^{\beta\epsilon} > 1$ gives the Bose-Einstein Result. □

Exercise 3. An ideal classical gas composed of N particles, each of mass m, is enclosed in a vertical cylinder of height L placed in a uniform gravitational field (of acceleration g) and is in thermal equilibrium; ultimately, both N and $L \to \infty$. Evaluate the partition function of the gas and derive expressions for its major thermodynamic properties. Explain why the specific heat of this system is larger than that of a corresponding system in free space.

Solution. This is just an application of the new statistics of the partition function.,

$$Q_N(\beta) = \frac{1}{N!} Q_1^N(\beta)$$

$$Q_1(\beta) = \frac{1}{h^3} \int e^{-\beta(\frac{p^2}{2m} + mgz)} dp_x dp_y dp_z dx dy dz = \Big(\frac{2\pi m}{\beta h^2} \Big)^{\frac{3}{2}} \frac{A(1 - e^{-\beta mgL})}{\beta mg}$$

where A is the cross sectional area. Taking $L \to \infty$ we get

$$Q_1(\beta) = \Big(\frac{2\pi m}{\beta h^2} \Big)^{\frac{3}{2}} \frac{A}{\beta mg}$$

which is proportional to $T^{5/2}$. Taking

$$E = -\frac{\ln Q}{\partial \beta} = \frac{5}{2} NkT \implies C_V = \frac{5}{2} Nk$$

The specific heat is larger because we have potential energy in the system.

□

Exercise 4. Consider the effusion of molecules of a Maxwellian gas through an opening of area a in the walls of a vessel of volume V.

(a) Show that, while the molecules inside the vessel have a mean kinetic energy $\frac{3}{2}kT$, the effused ones have a mean kinetic energy $2kT$, T being the quasistatic equilibrium temperature of the gas.

(b) Assuming that the effusion is so slow that the gas inside is always in a state of quasistatic equilibrium, determine the manner in which the density, the temperature, and the pressure of the gas vary with time.

Solution.

(a) This is like section 6.4. By analogy to 6.4.11,

$$\left\langle \frac{1}{2}mu_z^2 \right\rangle = \frac{1}{2}m\langle u^2\cos^2\theta\rangle = \frac{1}{2}m\frac{\int_0^{\pi/2}\int_0^{\infty}(u^3\cos^3\theta)f(\mathbf{u})u^2\sin\theta du d\theta}{\int_0^{\pi/2}\int_0^{\infty}(u\theta)f(\mathbf{u})u^2\sin\theta du d\theta}$$

I did this integral in maple and found $= \frac{1}{4}m\frac{\langle u^2\rangle}{\langle u\rangle}$. Now the expression $\frac{\langle u^2\rangle}{\langle u\rangle} = \frac{4}{\beta m}$ which gives kT for the potential contribution. Now the kinetic energy adds $frac12kT$ per degree of freedom so the total mean energy of an effused molecule is $2kT$.

(b) Quasistatic conditions give the ideal gas E and P. But then,

$$\frac{dE}{dt} = \frac{d}{dt}(\frac{3}{2}NkT) = 2kT\frac{dN}{dt} \implies \frac{dT}{T} = \frac{1}{3}\frac{dN}{N}$$

so T goes like $N^{1/3}$ and P goes like $N^{4/3}$.

$$\frac{dN}{dt} = \frac{-1}{4}an\langle u\rangle = \frac{-1}{4}\frac{aN}{V}\left(\frac{8kT}{\pi m}\right)^{1/2}$$

which altogether gives,

$$\frac{dT}{T} = \frac{-1}{3}\frac{a}{V}\left(\frac{kT}{2\pi m}\right)^{1/2}dt$$

which gives how N, V, T vary with time.

□