THE UNIVERSITY OF MELBOURNE
SCHOOL OF COMPUTING AND INFORMATION SYSTEMS
COMP30026 MODELS OF COMPUTATION

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Challenge 1 Answer Part A

$((\neg P \wedge Q) \Rightarrow \neg R) \Rightarrow R$ (start here)

$\neg((\neg P \wedge Q) \Rightarrow \neg R) \vee R$ (remove outside implication)

$\neg(\neg(\neg P \wedge Q) \vee \neg R) \vee R$ (remove inside implication)

$\neg((P \vee \neg Q) \vee \neg R) \vee R$ (push inside negation in with demorgan's)

$(\neg(P \vee \neg Q) \wedge R) \vee R$ (push inside negation in with demorgan's)

$((\neg P \wedge Q) \wedge R) \vee R$ (push inside negation in with demorgan's)

$(\neg P \wedge Q \wedge R) \vee R$ (associativity)

Where $(\neg P \wedge Q \wedge R) \vee R$ is equivalent to R via truth table:

$(\neg$	P	$\wedge$	Q	$\wedge$	$R)$	$\vee$	R — R
T	F	F	F	F	F	F	F — F
T	F	F	F	F	T	T	T — T
T	F	T	T	F	F	F	F — F
T	F	T	T	T	T	T	T — T
F	T	F	F	F	F	F	F — F
F	T	F	F	F	T	T	T — T
F	T	F	T	F	F	F	F — F
F	T	F	T	F	T	T	T — T

Therefore, R is the shortest formula equivalent to $((\neg P \wedge Q) \Rightarrow \neg R) \Rightarrow R$

Challenge 2

From the information we are given, we can derive the following:

a. P is telling the truth, and Q is telling the truth, i.e.
$Fa = ((P \wedge P' \wedge \neg Q \wedge Q') \vee (\neg P \wedge \neg P' \wedge \neg Q \wedge Q') \vee (Q \wedge Q' \wedge P \wedge \neg P') \vee (\neg Q \wedge \neg Q' \wedge P \wedge \neg P'))$

b. P is lying (either knave/sick knight), and Q is telling the truth, i.e.
$Fb = ((P \wedge P' \wedge \neg Q \wedge Q') \vee (P \wedge \neg P' \wedge Q \wedge \neg Q') \vee (Q \wedge Q' \wedge P \wedge \neg P') \vee (\neg Q \wedge \neg Q' \wedge P \wedge \neg P'))$

c. P is telling the truth, and Q is lying, i.e.
$Fc = ((P \wedge P' \wedge \neg Q \wedge Q') \vee (P \wedge \neg P' \wedge \neg Q \wedge Q') \vee (\neg Q \wedge Q' \wedge \neg P \wedge P') \vee (\neg Q \wedge \neg Q' \wedge \neg P \wedge \neg P'))$

d. P is lying, and Q is lying, i.e.
$Fd = ((\neg P \wedge P' \wedge \neg Q \wedge \neg Q') \vee (P \wedge \neg P' \wedge Q \wedge \neg Q') \vee (\neg Q \wedge Q' \wedge \neg P \wedge P') \vee (Q \wedge \neg Q' \wedge \neg P \wedge P'))$

We can express the above as a truth table:

P	P'	Q	Q'	$Functions Fa, Fb, Fc, Fd$
F	F	F	F	F
F	F	F	T	F
F	F	T	F	F
F	F	T	T	F
F	T	F	F	F
F	T	F	T	F
F	T	T	F	T**
F	T	T	T	F
T	F	F	F	F
T	F	F	T	F
T	F	T	F	F
T	F	T	T	F
T	T	F	F	F
T	T	F	T	F
T	T	T	F	F
T	T	T	T	F

As shown above, from these statements we can derive that **P is a healthy knave** and **Q is a sick knight**.

Challenge 1 Answer Part B

$(P \Rightarrow (Q \vee R)) \wedge (P \Leftrightarrow Q)$ (start here)

$(\neg P \vee (Q \vee R)) \wedge ((P \wedge Q) \vee (\neg P \wedge \neg Q))$ (remove implication and biimplication)

$(\neg P \vee Q \vee R) \wedge ((P \wedge Q) \vee (\neg P \wedge \neg Q))$ (remove inner bracket)

$((\neg P \vee Q \vee R) \wedge (P \wedge Q)) \vee ((\neg P \vee Q \vee R) \wedge (\neg P \wedge \neg Q))$ (expand out via distributivity)

$(\neg P \wedge P \wedge Q) \vee (Q \wedge P \wedge Q) \vee (R \wedge P \wedge Q) \vee (\neg P \wedge \neg P \wedge \neg Q) \vee (Q \wedge \neg P \wedge \neg Q) \vee (R \wedge \neg P \wedge \neg Q)$
(expand out again via distributivity)

In the above, we can see that $(\neg P \wedge P \wedge Q)$ and $(Q \wedge \neg P \wedge \neg Q)$ evaluate to false, and therefore can be removed as they do not effect the formula. $(Q \wedge P \wedge Q)$ and $(\neg P \wedge \neg P \wedge \neg Q)$ can also be shortened via the absorption law.

$(Q \wedge P) \vee (R \wedge P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (R \wedge \neg P \wedge \neg Q)$

In the above, the terms $(R \wedge P \wedge Q)$ and $(R \wedge \neg P \wedge \neg Q)$ can be removed as they are weaker statements and therefore R has no effect on the final result.

$(P \Leftrightarrow Q)$ (converting into biimplication from form $(Q \wedge P) \vee (\neg P \wedge \neg Q)$)

Challenge 1 Answer Part C

$(P \Rightarrow (P \Leftrightarrow Q))$ (start here)

$P \Rightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$ (remove biimplication)

$\neg P \vee (P \wedge Q) \vee (\neg P \wedge \neg Q)$ (remove implication)

$((\neg P \vee P) \wedge (\neg P \vee Q)) \vee (\neg P \wedge \neg Q)$ (expand via distributivity)

In the above, $(\neg P \vee P)$ always evaluates to True, therefore it can be removed.

$(\neg P \vee Q) \vee (\neg P \wedge \neg Q)$

$(\neg P \vee Q \vee \neg P) \wedge (\neg P \vee Q \vee \neg Q)$ (expand via distributivity)

In the above, the second part $(\neg P \vee Q \vee \neg Q)$ always evaluates to True, so therefore it can be removed from the formula.

$(\neg P \vee Q)$ (absorption)

$(P \Rightarrow Q)$ (convert to implication so that only one type of connective is used)

Challenge 1 Answer Part D

$P \vee (R \Rightarrow Q)$ (start here)

$P \vee (\neg R \vee Q)$ (remove implication)

$P \vee \neg R \vee Q)$ (associativity)

$P \vee \neg R \vee Q \vee (Q \wedge \neg Q)$

In the above, we add $(Q \wedge \neg Q)$ which always evaluates to false, which, when added to a statement in disjunct form, does not change the statement.

$Q \vee \neg R \vee ((P \wedge Q) \vee (P \wedge \neg Q))$ (expand via distributivity)

$Q \vee \neg R \vee (P \wedge Q) \vee (P \wedge \neg Q)$ (associativity)

$\neg R \vee ((P \wedge Q) \vee Q) \vee (P \wedge \neg Q)$ (associativity)

Where $((P \wedge Q) \vee Q)$ is equivalent to Q via truth table:

$((P$	$\wedge$	$Q)$	$\vee$	$Q)$ — Q
F	F	F	F	F — F
F	F	T	T	T — T
T	F	F	F	F — F
T	T	T	T	T — T

$(\neg R \vee Q) \vee (P \wedge \neg Q)$ (after applying the above)

$(R \Rightarrow Q) \vee \neg(P \Rightarrow Q)$ (convert to implication)

$(P \Rightarrow Q) \Rightarrow (R \Rightarrow Q)$ (convert outer to implication)

Challenge 3 Answer Part A

$\forall x(P(x)) \Rightarrow Q$ is *not* logically equivalent to $\forall x(P(x) \Rightarrow Q)$
This can be seen in the counter proof below:
Let Q always evaluate to False
Let domain $D : \{1, 2\}$
Let $P(x)$ mean "x is even"

$\forall x(P(x)) \Rightarrow Q$
evaluates domain D to:
$False \Rightarrow False$
$True$

Whereas in the case of $x : 2$
$\forall x(P(x) \Rightarrow Q)$ evaluates to:
$False$

$True \not\equiv False$

Therefore, $\forall x(P(x)) \Rightarrow Q$ is *not* logically equivalent to $\forall x(P(x) \Rightarrow Q)$

Challenge 3 Answer Part B

$\exists x(Q \Rightarrow R(x))$ is logically equivalent to $Q \Rightarrow \exists x(R(x))$

This can be done by translating one claim to another:
$\exists x(\neg Q \vee R(x))$ (start here)

$\neg Q \vee R(f(x))$ (skolemization)

$\neg Q \vee \exists x(R(x))$

$Q \Rightarrow \exists x(R(x))$

Therefore, $\exists x(Q \Rightarrow R(x))$ is logically equivalent to $Q \Rightarrow \exists x(R(x))$

Challenge 4 Answer

First Order Predicate Logic

Here are the following statements in First-Order Predicate Logic:

a. $\forall x(\exists y(M(x, y)) \Rightarrow F(x))$

b. $\forall x(\exists y(M(x, y) \wedge U(y)) \Rightarrow U(x))$

c. $\forall x((F(x) \wedge U(x) \wedge B(x)) \Rightarrow D(x))$

d. $\forall x \forall y((U(x) \wedge M(x, y) \wedge D(x)) \Rightarrow D(y))$

e. $\forall x((S(x) \wedge U(x)) \Rightarrow B(x))$

Horn Clause Form

Converting to Horn Clauses:

a. $\forall x(\exists y(M(x, y)) \Rightarrow F(x))$
$\forall x(\neg \exists y(M(x, y)) \vee F(x))$
$\forall x(\forall y(\neg M(x, y)) \vee F(x))$
$\forall x((\neg M(x, y)) \vee F(x))$
$(\neg M(x, y)) \vee F(x)$
$\{\neg M(x, y), F(x)\}$

b. $\forall x(\exists y(M(x, y) \wedge U(y)) \Rightarrow U(x))$
$\forall x(\neg(\exists y(M(x, y) \wedge U(y)) \vee U(x))$
$\forall x(\forall y(\neg M(x, y) \vee \neg U(y)) \vee U(x))$
$\forall x((\neg M(x, y) \vee \neg U(y)) \vee U(x))$
$\{\neg M(x, y), \neg U(y), U(x)\}$

c. $\forall x((F(x) \land U(x) \land B(x)) \Rightarrow D(x))$
$\forall x(\neg(F(x) \land U(x) \land B(x)) \lor D(x))$
$(\neg F(x) \lor \neg U(x) \lor \neg B(x) \lor D(x))$
$\{\neg F(x), \neg U(x), \neg B(x), D(x)\}$

d. $\forall x \forall y((U(x) \land M(x,y) \land D(x)) \Rightarrow D(y))$
$\forall x \forall y(\neg(U(x) \land M(x,y) \land D(x)) \lor D(y))$
$\forall x \forall y(\neg U(x) \lor \neg M(x,y) \lor \neg D(x) \lor D(y))$
$\forall x(\neg U(x) \lor \neg M(x,y) \lor \neg D(x) \lor D(y))$
$\{\neg U(x), \neg M(x,y), \neg D(x), D(y)\}$

e. $\forall x((S(x) \land U(x)) \Rightarrow B(x))$
$\forall x(\neg(S(x) \land U(x)) \lor B(x))$
$\forall x(\neg S(x) \lor \neg U(x) \lor B(x))$
$\{\neg S(x), \neg U(x), B(x)\}$

Resolution

Here, the statement "Any unicorn whose mother is Syldavian suffers from dermal asthenia" can be translated to: $\forall x(U(x) \land \exists y(M(y,x) \land S(y)) \Rightarrow D(x))$.
To check for logical consequence, we negate the statement and then check for unsatisfiability.

$\neg \forall x((U(x) \land \exists y(M(y,x) \land S(y))) \Rightarrow D(x))$

$\neg \forall x(\neg(U(x) \land \exists y(M(y,x) \land S(y))) \lor D(x))$

$\neg \forall x(\neg U(x) \lor \neg \exists y(M(y,x) \land S(y)) \lor D(x))$

$\neg \forall x(\neg U(x) \lor \forall y(\neg M(y,x) \lor \neg S(y)) \lor D(x))$

$\exists x(\neg\neg U(x) \land \neg \forall y(\neg M(y,x) \lor \neg S(y)) \land \neg D(x))$

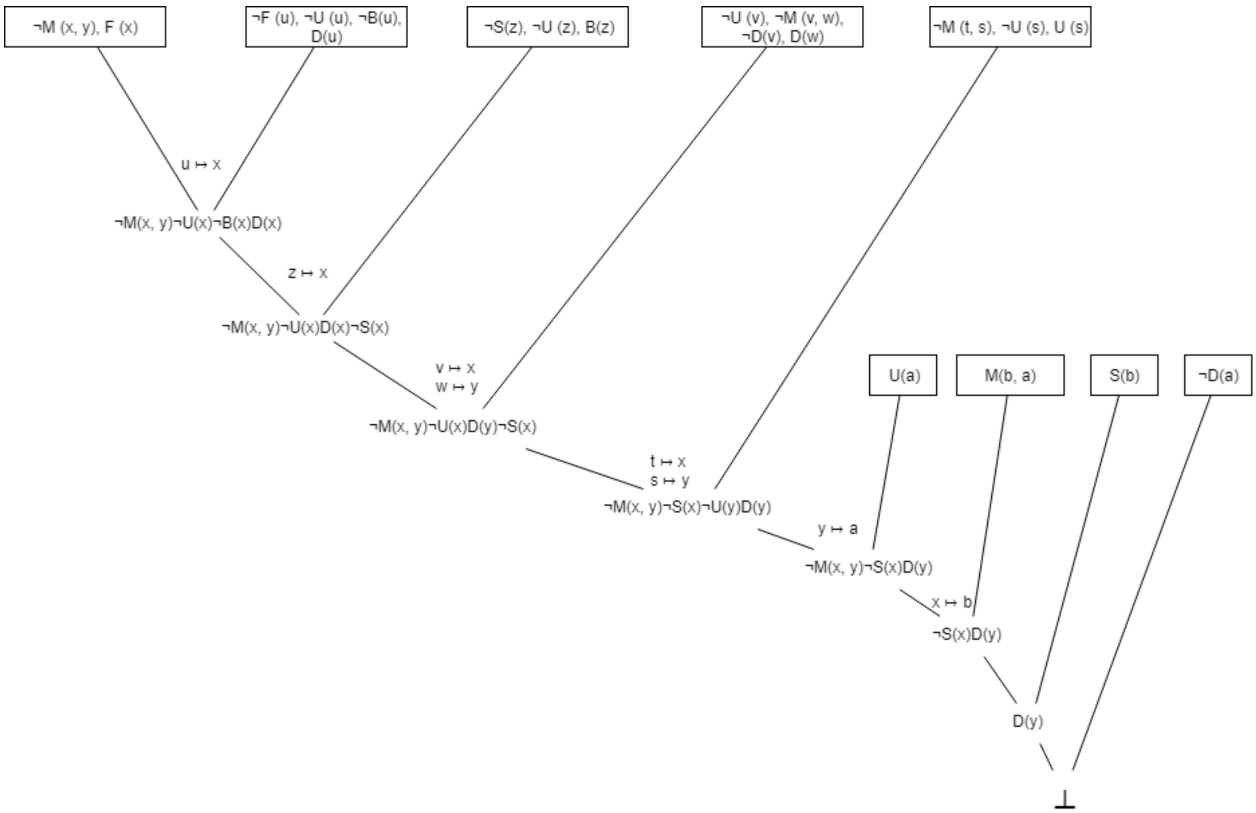
$\exists x(U(x) \land \exists y(M(y,x) \land S(y)) \land \neg D(x))$ (pushing negation in)

Now we perform skolemization:

$(U(a) \land (M(b,a) \land S(b)) \land \neg D(a))$

$\{\{U(a)\}, \{M(b,a)\}, \{S(b)\}, \{\neg D(a)\}\}$

Now for unification:



Challenge 5 Answer Part A

a. $\forall x \forall y (N(x, y) \Rightarrow N(y, x))$

b. $\forall x \forall y (\exists u (M(u, x) \land \neg M(u, y)) \Rightarrow N(x, y))$

c. $\forall x (\exists u (\neg M(u, x)) \Rightarrow F(x))$

d. $\forall x \forall y \forall u ((D(x, y) \land M(u, x)) \Rightarrow \neg M(u, y))$

Challenge 5 Answer Part B

a. $\forall x \forall y (N(x, y) \Rightarrow N(y, x))$
$\forall x \forall y (\neg N(x, y) \lor N(y, x))$
$\{\neg N(x, y), N(y, x)\}$

b. $\forall x \forall y (\exists u (M(u, x) \land \neg M(u, y)) \Rightarrow N(x, y))$
$\forall x \forall y (\forall u (\neg M(u, x) \lor M(u, y)) \lor N(x, y))$
$\{\neg M(u, x), M(u, y), N(x, y)\}$

c. $\forall x (\neg \exists u (M(u, x)) \Rightarrow E(x))$
$\forall x (\neg\neg \exists u (M(u, x)) \lor E(x))$
$\forall x (\exists u (M(u, x)) \lor E(x))$
$\forall x ((M(f(x), x)) \lor E(x))$ (skolemize)

$\{M(f(x), x), E(x)\}$

d. $\forall x \forall y \forall u((D(x,y) \wedge M(u,x)) \Rightarrow \neg M(u,y))$
$\forall x \forall y \forall u(\neg(D(x,y) \wedge M(u,x)) \vee \neg M(u,y))$
$\forall x \forall y \forall u((\neg D(x,y) \vee \neg M(u,x)) \vee \neg M(u,y))$
$\{\neg D(x,y), \neg M(u,x), \neg M(u,y)\}$

Challenge 5 Answer Part C

Our fifth statement can be described by the following:
$\forall x \forall y(D(x,y) \Rightarrow (N(x,y) \vee (E(x) \wedge E(y))))$

To show this statement is a logical consequence of the statements (a1-a4), we can negate it and resolve for unsatisfiability:
$\neg \forall x \forall y(D(x,y) \Rightarrow (N(x,y) \vee (E(x) \wedge E(y))))$

$\neg \forall x \forall y(\neg D(x,y) \vee (N(x,y) \vee (E(x) \wedge E(y))))$

$\exists x \exists y \neg(\neg D(x,y) \vee (N(x,y) \vee (E(x) \wedge E(y))))$

$\exists x \exists y(\neg\neg D(x,y) \wedge \neg(N(x,y) \vee (E(x) \wedge E(y))))$

$\exists x \exists y(D(x,y) \wedge (\neg N(x,y) \wedge \neg(E(x) \wedge E(y))))$

$\exists x \exists y(D(x,y) \wedge (\neg N(x,y) \wedge (\neg E(x) \vee \neg E(y))))$

After Skolemization:
$(D(a,b) \wedge (\neg N(a,b) \wedge (\neg E(a) \vee \neg E(b))))$

$\{\{D(a,b)\}, \{\neg N(a,b)\}\{\neg E(a), \neg E(b)\}\}$

Challenge 5 Answer Part D

Finally, unification: