Statistics 153 (Introduction to Time Series) Homework Six

Due on 5 Dec, 2016 in the stat front office (3rd floor in Evans Hall)

November 21, 2016

1. **DFT and Convolution**: For two datasets $x_0, x_1, \ldots, x_{n-1}$ and $y_0, y_1, \ldots, y_{n-1}$, their *convolution* is the dataset $z_0, z_1, \ldots, z_{n-1}$ defined by

$$z_i = \sum_{j=0}^{n-1} x_{i-j} y_j \qquad \text{for } i = 0, \ldots, n-1$$

   where $x_{-k} = x_{-k+n}$. Find the DFT of $z_0, \ldots, z_{n-1}$ in terms of the DFTs of $x_0, \ldots, x_{n-1}$ and $y_0, \ldots, y_{n-1}$.

2. **Chirps**:

   (a) Consider the data $x_t = \sin(\pi t^2/256)$ for $t = 0, 1, \ldots, 127$. Plot the data. Also plot the magnitude (absolute) of the DFT coefficients $b_1, \ldots, b_{64}$. Comment on the two plots.

   (b) Repeat the previous exercise for $y_t = \sin(\pi t^2/512)$ and $z_t = \sin(\pi t^2/1024)$. Comment on the differences between the plots.

3. **DFT of a periodic series**: Suppose $x_0, x_1, \ldots, x_{n-1}$ is periodic with period h i.e., $x_{t+hu} = x_t$ for all integers t and u. Let n be an integer multiple of h i.e., $n = kh$ for an integer k. Suppose that the DFT the data $x_0, \ldots, x_{n-1}$ is $b_0, b_1, \ldots, b_{n-1}$. Also suppose that the DFT of the data in the first cycle (i.e., $x_0, x_1, \ldots, x_{h-1}$) is $\beta_0, \ldots, \beta_{h-1}$. Show that $b_0 = k\beta_0, b_k = k\beta_1, b_{2k} = k\beta_2, \ldots, b_{(h-1)k} = k\beta_{h-1}$ and that all other b_js are zero.

4. Consider the following seasonal AR model:

$$(1 - \phi B)(1 - \Phi B^s) X_t = Z_t,$$

   where $\{Z_t\}$ is white noise and $|\phi| < 1, |\Phi| < 1$.

   (a) Calculate the spectral density of $\{X_t\}$.

   (b) Plot the spectral density for $\phi = 0.5$, $\Phi = 0.9$, $\sigma_Z^2 = 1$ and $s = 12$.

   (c) Also plot the spectral density for the AR(1) process $(1 - 0.5B)X_t = Z_t$ and the seasonal AR(1) process $(1 - 0.9B^{12})X_t = Z_t$.

   (d) Compare and comment on the different plots.

5. The spectral density of a stationary time series $\{X_t\}$ is defined on $[-1/2, 1/2]$ by $f(\lambda) = 5$ for $1/6 \leq |\lambda| \leq 1/3$ and zero otherwise.

   (a) Evaluate the autocovariance function of $\{X_t\}$ at lags 0 and 1.

   (b) Find the spectral density of the process $\{Y_t\}$ defined by $Y_t = X_t - X_{t-12}$.

6. Consider the stationary Autoregressive process:

$$X_t - 0.99X_{t-3} = Z_t$$

   where $\{Z_t\}$ is white noise.

   (a) Compute and plot the spectral density of $\{X_t\}$.

   (b) Does the spectral density suggest that the sample paths of $\{X_t\}$ will exhibit approximately oscillatory behaviour? If yes, then with what period?

   (c) Simulate a sample of size 100 from this model. Plot the simulated data. Does this plot support the conclusion of part (b)?

   (d) Compute the spectral density of the filtered process:

$$Y_t = \frac{X_{t-1} + X_t + X_{t+1}}{3}. \tag{1}$$

   How does the spectral density of $\{Y_t\}$ compare to that of $\{X_t\}$?

   (e) From the simulated sample from $\{X_t\}$ in part (c), perform the averaging as in (1) to obtain a simulated sample from $\{Y_t\}$. Plot this sample. Does this plot support the spectral density plot in part (d)?

7. Without using the *arima.sim()* function in R, simulate $n = 400$ observations from the multiplicative seasonal ARMA model given by the difference equation:

$$(1 - 0.5B)(1 - 0.7B^{12})X_t = Z_t$$

   where $\{Z_t\}$ is white noise. Plot the sample autocorrelation function of the simulated observations and compare it with the true acf of the process.

8. Consider the first dataset, `q1train.csv`, which is uploaded to piazza. Remove the trend and seasonality by differencing first with order 52 and then a usual differencing. Call the resulting dataset $x_t, t = 1, \ldots, n$ to which a stationary model can be fit.

   (a) Estimate the spectral density of $\{X_t\}$ nonparametrically from the data $\{x_t\}$.

   (b) Fit a reasonable stationary model to $\{x_t\}$ and estimate the spectral density of $\{X_t\}$ by the spectral density of the fitted model.

   (c) Plot the two estimates of the spectral density on the same plot. Commment on the two plots.