

In PHYS 211 (or 405) every solution to the Schrodinger equation that you learned was a *single* particle solution. Meaning you only ever considered systems where you had to figure out the energy eigenvalues of a single particle in a box or a single electron and so on. Even when solving the He atom, where you have two electrons, the system is “simple enough” to do regular quantum mechanics. However, most real life systems, like solids, are systems of $N \sim 10^{23}$ particles all being shoved into *discrete* quantum states. For this, solving the Schrodinger equation is not feasible, and we have to rely on *Quantum statistical mechanics*.

In statistical mechanics, we focus instead on approximate behavior of ensembles of particles by computing averages based on 1-particle states. If you’re familiar with probability theory, you may know that to compute averages of an observable, we require a probability distribution function (PDF). This PDF should tell us how likely a specific energy state is, and from there we can obtain most of what we want.

Indistinguishable Particles

This may seem not too bad, however, the story gets more complicated when we remember that quantum particles are weird and they sometimes can be **indistinguishable**. In fact, most simple systems are indistinguishable – elementary particles like electrons, atoms, molecules, and even quasiparticles.

The next complication comes from the fact that even though particles can be indistinguishable, switching the position of two particles can sometimes have an effect on the system. This separates indistinguishable particles into two categories:

- **Bosons** - particles that can be exchanged with no consequences.
- **Fermions** - particles that can’t be exchanged without the addition of a $-$ sign (i.e. their position wavefunctions anticommute).

This is their definitions but for most purposes, it is better to identify Bosons and Fermions based on their **spin**:

- **Bosons are integer spin particles** ($s = 0, 1, 2, \dots$).
 - e.g. photons, phonons, mesons, etc.
- **Fermions are half-integer spin particles** ($s = 1/2, 3/2, 5/2, \dots$).
 - e.g. electrons, protons, neutrons, some nuclei etc.

In addition to their spin, and probably the most important fact that we need to remember for this class is:

- **More than 1 Boson can occupy the same quantum state**

- No more than 1 Fermion can occupy the same quantum state

The fact that Fermions cannot occupy the same state is probably better known as the *Pauli exclusion principle*, which you might have learned in chemistry. Hopefully, you can see how when dealing with 10^{23} particles, their behavior can be *drastically* different if they can bunch up in a single state, or if they are forced to spread out.

Quantum Distributions

Now that we know about Fermions vs. Bosons, we can write down their probability distributions and most importantly the average number of particles in a system, better known as the **occupation number**.

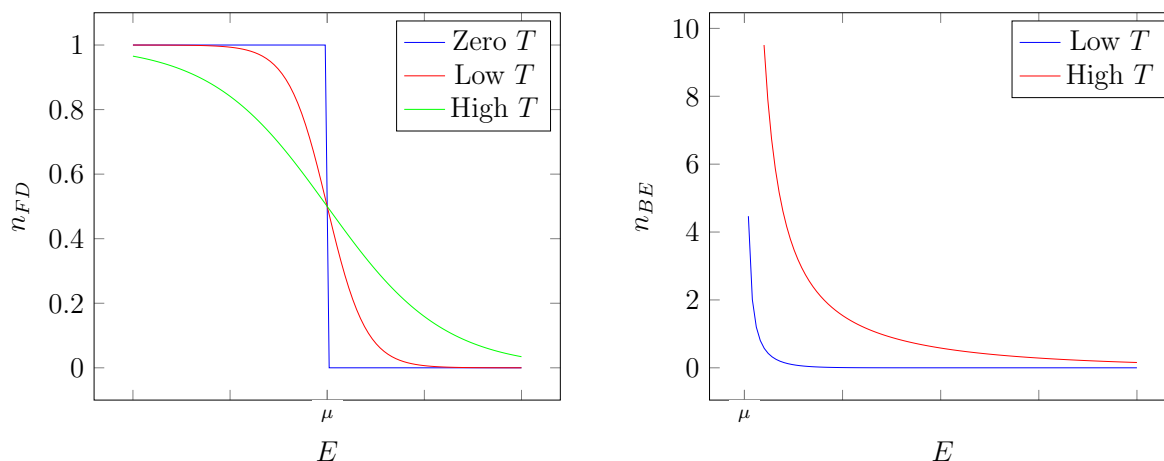
One can derive the probability distributions from something known as the *grand partition function* (which you'll probably do if you take a statistical mechanics class) but the important result is that we can determine the **occupation number** at some given energy state E and temperature $k_B T$ from this probability to be

$$n_{FD} = \frac{1}{e^{(E-\mu)/k_B T} + 1} \quad (\text{Fermions})$$

$$n_{BE} = \frac{1}{e^{(E-\mu)/k_B T} - 1} \quad (\text{Bosons})$$

where μ can be thought of as the energy associated with adding or removing a particle from the system. The occupation number is simply the probability that a particle occupies a specific energy state.

The occupation number for Fermions is known as the Fermi-Dirac distribution and caps the occupation number at 1, since only up to 1 fermion can occupy an energy state. At zero temperature, the F-D distribution is simply the Heavyside step function centered at μ and it softens at higher temperatures. Below, you can see three different temperature ranges of the F-D distribution.



The occupation number for Bosons is known as the Bose-Einstein distribution and, unlike the F-D distribution, it blows up as it approaches μ , reflecting the fact that all particles can occupy a single state. As temperature increases, the probability of getting more particles to spread out over higher energy states increases as seen in the right plot above.

More Statistics

Again, it is practically impossible to deterministically solve for the observables of a system with $\sim 10^{23}$ particles, so we must rely on computing averages based on the distributions above.

We already know the probability of a certain state with energy E being occupied, but what about the states that are available at all? For this we need the **density of states**, which tells us the allowed number of states within some energy range. That is, the density of states, $g(E)$, is defined as

$$\# \text{ of states between } E \text{ and } dE = g(E)dE.$$

We will see that the density of states will depend on both the geometry of the problem and the actual energies of the system. This may seem like an abstract concept, but think of it as just a way to keep track of the energies that are allowed in a system and we will worry about how to calculate it later when we need it. If you're more math inclined, it may help to think of the density of states as the piece that allows us to go from sums of discrete energy states, to integrals of "practically continuous" energy variables.

Now, what if we instead need to know not just the allowed states but the occupied states? For that we define the **density of occupied states**,

$$\text{density of occupied states} = n(E)g(E)$$

which is simply the probability of a state being occupied times how dense or likely that state is.

From these we can now just do regular statistics. Namely, we can count the number of particles in a system: just sum over all energies

$$N = \int n(E)g(E) dE$$

and if we want the mean energy, then we look at the how many particles are per energy and divide by the total:

$$\langle E \rangle = \frac{\int E n(E)g(E) dE}{\int n(E)g(E) dE}$$

Notice how I've been referring to adding but I am using integrals instead of sums. This is because we will be able to replace sums with integrals due to the vast size of our systems leading to a negligible spacing between energy states, making the energy spectrum practically continuous.