Please complete the following questions. You are allowed to work in groups but remember to show your own work.

Problem 1. (Monoatomic Chain of Atoms)

Let's do a few more things for the monoatomic chain that we didn't really talk about in detail during lecture.

(a) Write down the dispersion relation and explain how the Debye model can be recovered from this dispersion.

(b) Show that the oscillatory mode with wave vector k is equivalent to a mode with wave vector $k+\frac{2\pi}{a}$. In your own words, explain how this allows us to only focus on what happens only within the 1st Brillouin zone.

(c) Write down the displacement of two neighboring atoms, u_n and u_{n+1} and find the phase difference between the two atoms. What happens to this phase difference in the long wavelength limit? Sketch the corresponding motion of the atoms in the chain in the long wavelength limit.

(d) What is the phase difference at the smallest wavelength (the edge of the Brillouin zone)? Sketch the corresponding motion of the atoms.

(e) Find the expression for the density of states, $g(\omega)$, of the monoatomic chain. Plot your expression for $g(\omega)$ (you can set all constants to 1). Your plot should diverge at a specific value of ω. What is the significance of this value?

(f) Write down an integral expression for the heat capacity of the monoatomic lattice (This is a nasty integral that you can't solve). Show that this gives you the Dulong-Petit result in the high temperature limit. (Note: you can just show you get the proper temperature scaling. The resulting low T integral is improper due to the singularity in the density of states)

Problem 2. (Diatomic Chain of Atoms)

In class, we setup the diatomic scenario and arrived to the following coupled system of equations

$$
\begin{aligned}
m_1\ddot{u}_n &= \gamma_2(v_n - u_n) + \gamma_1(v_{n-1} - u_n)\\
m_2\ddot{v}_n &= \gamma_1(u_{n+1} - v_n) + \gamma_2(u_n - v_n)
\end{aligned}
$$

but I said I would leave the details of the normal mode calculations to you. For your calculations, you may assume that $m_1 = m_2 = m$ for simplicity.

(a) Use the ansatz

$$u_n = Ae^{-i(kna-\omega t)} \qquad v_n = Be^{-i(kna-\omega t)}$$

to find a system of equation for the amplitudes A and B.

(b) Show that your system has a solution only if

$$\omega_\pm = \sqrt{\frac{\gamma_1+\gamma_2}{m} \pm \frac{1}{m}\sqrt{(\gamma_1+\gamma_2)^2 - 4\gamma_1\gamma_2\sin^2\left(\frac{ka}{2}\right)}}$$

(*hint:* turn your system into a matrix equation)

(c) Plot the first Brillouin zone of the dispersion relation above for $\gamma_2 = 1.5\gamma_1$.

(d) In your own words, comment on the physical significance/origins of the name given to each branch of the dispersion relation you plotted above.

Problem 3. (Thermal Dependence in the Drude Model) We saw in lecture some of the predictions that the Drude model succeeds at making. We can also make some predictions about the temperature dependence of some electrical transport properties.

(a) In your own words explain the assumptions of the Drude model. You don't have to list all the assumptions, just give an overview of what the Drude model tries to do.

(b) Write down the resistivity predicted by the Drude model. Show that the Drude model then predicts that the resistivity has the following form:

$$\rho(T) = KT^n$$

where K is a temperature dependent constant and n is some exponent, both of which you should find.

(c) Experimental observations of the temperature dependence predict that the empirical form of the resistivity of a solid is of the form

$$\rho(T) = \rho_0(1 + \alpha(T - T_0))$$

where α is some constant. T_0 and ρ_0 are the room temperature and room temperature resistivity, respectively. Based on your results from part (a), do you think the Drude model succeeds at explaining the temperature dependence of electrical resistivity?

Problem 4. (AC Hall Effect)

The Hall effect is extremely useful in sensor physics (have you heard of Hall probes?) so let's review it in a slightly more sophisticated setting (this looks hard but this shouldn't be much harder than what we did in lecture!!).

Let's start with the same Hall effect setting but instead of applying some constant electric and magnetic fields, let's apply an AC field. We can write an AC electric $\vec{E}$ and $\vec{B} \perp \vec{E}$ on the z-axis as follows:

$$\vec{E} = E_0 e^{i\omega t}\hat{x}$$
$$\vec{B} = B_0\hat{z}$$

The electric field induces an AC current density

$$\vec{J} = J_0 e^{i\omega t}\hat{x}$$

For simplicity, let's assume $\tau \approx 0$ which means we don't have to worry about the drift velocity.

(a) Write down the Drude equation of motion (for $\tau \to 0$) and show that

$$E_0\hat{x} = \left(\frac{i\omega m_e}{ne^2}\right) J_0\hat{x} - \frac{B}{ne} J_0\hat{y}$$

(Think about why you can't find a steady state solution like we did in class)

(b) Write down the resistivity matrix from your expression above. (Note that the equations would look the same but with $\hat{y}$'s instead of $\hat{x}$'s for the y component of the field)

(c) In matrix form, conductivity and resistivity are still inverses of each other, however, inverse now refers to the matrix inverse. Find the conductivity matrix $\overleftrightarrow{\sigma} = \overleftrightarrow{\rho}^{-1}$. (Feel free to make a computer calculate it for you)

(d) The entries of your conductivity matrix should diverge to some value of ω. What is this value?

(e) In a real lab setting where $\tau > 0$, this divergence in conductivity is finite and shows up as a peak in conductance. Explain how you could use this resonance to measure the mass of the electron.

Problem 5. (Electron Gas Density of States)

In class, we saw that the Fermi/Free electron gas treatment has a parabolic dispersion relation given by

$$E(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m} \tag{1}$$

and we calculated that the density of states in 3D has the form $g_{3D}(E) \propto \sqrt{E}$.

(a) Show that in 1D the density of states has the form $g_{1D}(E) \propto 1/\sqrt{E}$.

(b) Show that in 2D the density of states has the form $g_{2D}(E) \sim$ const.

Problem 6. (Fermi Energy and Surface of atomic K)

In this problem we will look more closely at the concept of the Fermi energy and look at some real life examples.

(a) In your own words, explain what the Fermi energy represents.

(b) Explain what we mean by a Fermi surface and explain what the free electron model says about their shapes.

(c) Go to the Fermi surface database and take a look at what real Fermi surfaces look like for different metals. The free electron model is a very good approximation for alkali metals but not for others. Based on what you seen in the database, explain why this is the case.

(d) Potassium (K) has a Fermi energy of approximately $E_F = 2.12$ eV. Calculate its Fermi temperature, Fermi wave vector and free electron density.

(e) Focusing on your value of T_F, explain why we can say $\mu \approx E_F$ at room temperature.

(f) Calculate the *total* electron density of K. From this, estimate the percent of electrons that are available for conduction. You will need to look up some data about potassium (e.g. density, atomic mass and atomic number all of which is available on the wikipedia page for K).