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SIO 250

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Advanced Atmospheric Dynamics

taught by:

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An Introduction to Dynamic Meteorology

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$U = Q + W$

$\delta U = \delta Q - p\,\delta V$

$\delta U + p\,\delta V = \delta Q$

$\frac{DU}{Dt} + p\frac{D\alpha}{Dt} = \frac{DQ}{Dt} = J$

$c_v\frac{DT}{Dt} + p\frac{D\alpha}{Dt} = J$

First law of thermodynamics (extensive form)

$$c_v\frac{DT}{Dt} + p\frac{D\alpha}{Dt} = J \leftarrow \text{diabatic heating rate per unit mass}$$

$c_v\frac{DT}{Dt}$: change in internal energy per unit mass following parcel

$p\frac{D\alpha}{Dt}$: p-V work rate / unit mass

$$\frac{D}{Dt}\left(\frac{p}{\rho}\right) = \frac{D}{Dt}(RT) \Rightarrow \alpha\frac{Dp}{Dt} + p\frac{D\alpha}{Dt} = R\frac{DT}{Dt}$$

$$c_v\frac{DT}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = J$$

$$c_v\frac{DT}{Dt} + R\frac{DT}{Dt} - \alpha\frac{Dp}{Dt} = J$$

$$\underbrace{(c_v+R)}_{c_p}\frac{DT}{Dt} - \alpha\underbrace{\frac{Dp}{Dt}}_{\omega} = J$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p} - \frac{\alpha}{c_p}\omega = \frac{J}{c_p}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \left(\frac{\partial T}{\partial p} - \frac{\alpha}{c_p}\right)\omega = \frac{J}{c_p}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)T - S_p\omega = J/c_p$$

$\theta = T\left(\frac{p_0}{p}\right)^{\kappa} \rightarrow \ln\theta = \ln T + \kappa(\ln p_0 - \ln p)$

$\frac{1}{\theta}\frac{\partial\theta}{\partial p} = \frac{1}{T}\frac{\partial T}{\partial p} - \frac{\kappa}{p}\frac{\partial p}{\partial p}$

$\frac{T}{\theta}\frac{\partial\theta}{\partial p} = \frac{\partial T}{\partial p} - \frac{RT}{c_p p} = -S_p$

Recall: $\frac{T}{\theta}\frac{\partial\theta}{\partial z} = \Gamma_d - \Gamma$ (derivation same as above)

$\frac{dp}{dz} = -\rho g \Rightarrow dz = \frac{dp}{-\rho g} \Rightarrow -\frac{T}{\theta}\frac{\partial\theta}{\partial p} = \frac{\Gamma - \Gamma_d}{\rho g}$

$\Gamma_d \approx 9.8\,K/km$

typical $\rightarrow \Gamma \approx 6.5\,K/km$

$\Gamma < \Gamma_d$ stable: parcel cools faster than environment

$$S_p = \frac{\alpha}{c_p} - \frac{\partial T}{\partial p} = \frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial\theta}{\partial p}$$

static stability parameter

θ decreases w/ increasing pressure for a statically stable environment, $S_p > 0$

- if $S_p = 0$, $\frac{\partial\theta}{\partial p} = 0$ there is no temperature change due to vertical motion.
- if $S_p > 0$, $-\underbrace{\frac{\partial\theta}{\partial p}}_{<0} > 0$ ← colder air near the surface, for example, and $\omega < 0$ upward motion induces adiabatic cooling; $\omega > 0$ downward motion induces adiabatic warming
- if $S_p < 0$, $-\underbrace{\partial\theta/\partial p}_{>0} < 0$ and $\omega > 0$ cools the atmosphere locally, etc., but this is statically unstable.

Quasi-geostrophy

$$\vec{\nabla}\cdot\vec{v}_g = 0 = \frac{\partial}{\partial x}u_g + \frac{\partial}{\partial y}v_g = \frac{\partial}{\partial x}\left(\frac{-1}{f_0}\frac{\partial\Phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f_0}\frac{\partial\Phi}{\partial x}\right)$$

Horizontal momentum: $\frac{D\vec{v}}{Dt} + f\hat{k}\times\vec{v} = -\vec{\nabla}\Phi$

$\vec{v} = \vec{v}_g + \vec{v}_a$, $\frac{|v_a|}{|v_g|} \sim O(Ro)^1$ $\quad \beta L \ll f_0$

Thermodynamic: $\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)T - S_p\omega = J/c_p$

$\zeta_g = \frac{1}{f_0}\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2\Phi$ $\quad f = f_0 + \beta y$

Vertical momentum: $-\partial\Phi/\partial p = RT/p = \alpha$ (hydrostatic)

mass conservation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial\omega}{\partial p} = 0 \longrightarrow \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial\omega}{\partial p} = 0$

$$f\hat{k}\times\vec{v} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & f\\ u & v & w\end{vmatrix} = \hat{i}(-fv) + \hat{j}(fu)$$

Horizontal momentum (Q-G)

$$\frac{D}{Dt}(\vec{v}_g + \cancel{\vec{v}_a}) + (f_0 + \beta y)\hat{k}\times(\vec{v}_g + \vec{v}_a) = -\nabla\Phi$$

$$\cdots + \underbrace{f_0\hat{k}\times\vec{v}_g}_{-\nabla\Phi} + f_0\hat{k}\times\vec{v}_a + \beta y\hat{k}\times\vec{v}_g + \underbrace{\cancel{\beta y\hat{k}\times\vec{v}_a}}_{\text{small}} = -\nabla\Phi$$

$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}$

$$\frac{D_g}{Dt}\vec{v}_g + f_0\hat{k}\times\vec{v}_a + \beta y\hat{k}\times\vec{v}_g = 0 \quad \begin{cases}\frac{\partial u_g}{\partial t} + u_g\frac{\partial u_g}{\partial x} + v_g\frac{\partial u_g}{\partial y} - f_0 v_a - \beta y v_g = 0\\ \frac{\partial v_g}{\partial t} + u_g\frac{\partial v_g}{\partial x} + v_g\frac{\partial v_g}{\partial y} + f_0 u_a + \beta y u_g = 0\end{cases}$$

Thermodynamic (Q-G)

$T_{total} = T_0(p) + T(x,y,p,t)$

$$\left(\frac{\partial}{\partial t} + \vec{v}_g\cdot\nabla\right)\frac{RT}{p} - \underbrace{S_p\frac{R}{p}}_{\sigma}\omega = \frac{JR}{c_p p} \rightarrow \left(\frac{\partial}{\partial t} + \vec{v}_g\cdot\nabla\right)\underbrace{\left(-\frac{\partial\Phi}{\partial p}\right)}_{=\alpha} - \sigma\omega = \frac{JR}{c_p p}$$

α: specific volume, or thickness between pressure levels

$\theta_{total} = \theta_0(p) + \theta(x,y,p,t)$

$\left|\frac{d\theta_0}{dp}\right| \gg \left|\frac{\partial\theta}{\partial p}\right| \Rightarrow S_p = -\frac{T}{\theta}\frac{\partial\theta}{\partial p} \approx -\frac{T}{\theta_0}\frac{d\theta_0}{dp}$

Take $\hat{k} \cdot \vec{\nabla} \times$ (Q-G momentum) and use continuity to eliminate ageostrophic cmpt.

$$\frac{\partial}{\partial t}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + \cancel{\frac{\partial u_g}{\partial x}\frac{\partial v_g}{\partial x}} + \cancel{u_g \frac{\partial}{\partial x}\left(\frac{\partial v_g}{\partial x}\right)} + \cancel{\frac{\partial v_g}{\partial x}\frac{\partial v_g}{\partial y}} + \cancel{v_g \frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial x}\right)} + f_0 \frac{\partial u_a}{\partial x} + \cancel{\beta y \frac{\partial u_g}{\partial x}}$$

$$- \cancel{\frac{\partial u_g}{\partial y}\frac{\partial u_g}{\partial x}} - \cancel{u_g \frac{\partial}{\partial x}\left(\frac{\partial u_g}{\partial y}\right)} - \cancel{\frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial y}} - \cancel{v_g \frac{\partial}{\partial y}\left(\frac{\partial u_g}{\partial y}\right)} + f_0 \frac{\partial v_a}{\partial y} + \cancel{\beta y \frac{\partial v_g}{\partial y}} + v_g \beta = 0$$

$$\frac{\partial}{\partial t}\zeta_g + u_g \frac{\partial}{\partial x}\zeta_g + v_g \frac{\partial}{\partial y}\zeta_g + \cancel{\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right)}\zeta_g + f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) + v_g \beta = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{f_0}\nabla^2 \Phi\right) + \vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) + f_0 \underbrace{\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right)}_{-\frac{\partial \omega}{\partial p}} = 0 \qquad \text{Q-G vorticity equation}$$

- vertical stretching forces a positive tendency in geostrophic relative vorticity and hence a decrease in geopotential
- this applies to both hemispheres, as f_0 accompanies both of these terms

but recall $\zeta_g > 0$ is cyclonic in the northern hemisphere,
$\zeta_g < 0$ is cyclonic in the southern hemisphere (ζ_g cyclonic has same sign as f_0)

so saying a positive tendency $\frac{\partial \zeta_g}{\partial t} > 0$ *only applies in the NH*.

Geopotential tendency equation, $\chi \equiv \frac{\partial \Phi}{\partial t}$ local rate of change of geopotential height of constant pressure sfc

Q-G vorticity may be re-written $\frac{1}{f_0}\nabla^2 \chi + \vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - f_0 \frac{\partial \omega}{\partial p} = 0$

This equation depends exclusively on Φ and ω. So does Q-G thermodynamic, so an equation in terms of Φ only may be derived by eliminating ω.

$$\frac{\partial}{\partial p}\frac{f_0}{\sigma}\left(\frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial p}\right)\right) + \frac{\partial}{\partial p}\frac{f_0}{\sigma} u_g \frac{\partial}{\partial x}\left(-\frac{\partial \Phi}{\partial p}\right) + \frac{\partial}{\partial p}\frac{f_0}{\sigma} v_g \frac{\partial}{\partial y}\left(-\frac{\partial \Phi}{\partial p}\right) - \frac{\partial}{\partial p}\frac{f_0}{\cancel{\sigma}}\cancel{\sigma}\,\omega - \frac{\partial}{\partial p}\frac{f_0}{\sigma}\left(\frac{J R}{c_p p}\right) = 0$$

Adding these equations gives, after multiplying the second by -1:

$$\frac{1}{f_0}\nabla^2\chi + \frac{\partial}{\partial p}\frac{f_0}{\sigma}\frac{\partial}{\partial p}\chi + \vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right) - \cancel{f_0 \frac{\partial \omega}{\partial p}} - \frac{\partial}{\partial p}\frac{f_0}{\sigma}\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) + \cancel{f_0\frac{\partial \omega}{\partial p}} + \frac{\partial}{\partial p}\left(\frac{J R f_0}{\sigma c_p p}\right) = 0$$

or

$$\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = \underbrace{-f_0 \vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right)}_{\text{absolute vorticity advection}} \underbrace{- \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma} \cdot -\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right)}_{\text{vertical derivative of thickness advection}} \underbrace{- \frac{R}{c_p} f_0 \frac{\partial}{\partial p}\left(\frac{J}{\sigma p}\right)}_{\text{vertical derivative of diabatic heating}}$$

if χ is assumed to be wave-like, the LHS ~~will be like~~
$-\left(k^2 + l^2 + \frac{f_0^2 \pi^2}{\sigma p^2}\right)\chi$
i.e. $\propto -\chi$

Consider NH case $f_0 > 0$

absolute vorticity advection:
> 0 cyclonic adv. $\rightarrow \Phi$ falls
< 0 anticyclonic adv. $\rightarrow \Phi$ rises

$-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) > 0$ warm advection
< 0 cold advection

if cold advection dec. w/ height this term is positive so $\chi < 0$ geopotential falls

vorticity contours parallel to isobars near surface

→ Vorticity advection typically largest in mid-troposphere

Temperature contours parallel to isobars in mid troposphere

→ ~~Temperature advection typically largest in lower troposphere~~

omega equation

∇^2(Q-G thermodynamic) $+ \frac{\partial}{\partial p}$(Q-G vorticity)

$$\nabla^2 \frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial p}\right) = -\nabla^2\left[\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] + \sigma \nabla^2 \omega + \frac{R}{c_p p}\nabla^2 J$$

$$\frac{\partial}{\partial p}\nabla^2 \chi = -\nabla^2\left[-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] - \sigma \nabla^2 \omega - \frac{R}{c_p p}\nabla^2 J$$

$$\frac{\partial}{\partial p}\left(\frac{1}{f_0}\nabla^2 \chi\right) = -\frac{\partial}{\partial p}\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) + f_0 \frac{\partial^2 \omega}{\partial p^2}$$

$$\frac{\partial}{\partial p}\nabla^2 \chi = -f_0 \frac{\partial}{\partial p}\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) + f_0^2 \frac{\partial^2 \omega}{\partial p^2}$$

$$-f_0 \frac{\partial}{\partial p}\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) + f_0^2 \frac{\partial^2 \omega}{\partial p^2} = -\nabla^2\left[-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] - \sigma \nabla^2 \omega - \frac{R}{c_p}\frac{1}{p}\nabla^2 J$$

$$\sigma \nabla^2 \omega + f_0^2 \frac{\partial^2 \omega}{\partial p^2} = -f_0 \frac{\partial}{\partial p}\left[-\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right)\right] - \nabla^2\left[-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] - \frac{R}{c_p p}\nabla^2 J$$

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right)\right]}_{\text{vertical derivative of absolute vorticity advection}} \underbrace{- \frac{1}{\sigma}\nabla^2\left[-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]}_{\text{non-uniform temperature advection}} \underbrace{- \frac{R}{c_p p \sigma}\nabla^2 J}_{\text{isolated diabatic processes}}$$

$$\sim -\left(k^2 + l^2 + \frac{f_0^2 \pi^2}{\sigma p^2}\right)\omega$$

$$\propto -\omega$$

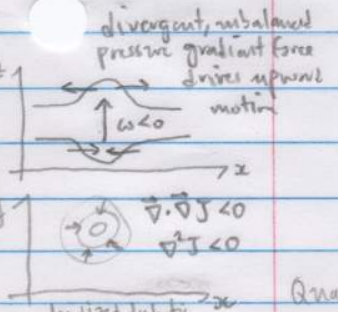
$-\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) > 0$ cyclonic

$-\frac{\partial}{\partial p}\left[-\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right)\right] > 0$ cyclonic vorticity advection increasing w/ altitude

$\Rightarrow \omega < 0$ upward motion

$-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) > 0$ warm advection

localized heating or cooling will induce expansion or contraction of otherwise flat pressure surfaces + create unbalanced ∇p forces



Quasi-geostrophic potential vorticity equation

→ obtain a powerful conservation law for the Q-G system

Starting from geopotential tendency, we may expand the second term and rearrange to

$$\left[\nabla^2 + \frac{\partial}{\partial p}\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right]\frac{\partial \Phi}{\partial t} = -f_0 \vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) - \frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] - f_0 \frac{R}{c_p}\frac{\partial}{\partial p}\left[\frac{J}{\sigma p}\right]$$

$$= -f_0 \vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) + \frac{\partial}{\partial p}\frac{f_0^2}{\sigma}\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) + \text{diabatic term}$$

The second term can be written

$$\frac{f_0^2}{\sigma}\cancelto{0}{\frac{\partial \vec{v}_g}{\partial p} \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)} + f_0^2 \vec{v}_g \cdot \vec{\nabla}\left(\frac{\partial}{\partial p}\frac{1}{\sigma}\frac{\partial \Phi}{\partial p}\right)$$

but $\vec{v}_g = \frac{1}{f_0}\hat{k} \times \vec{\nabla}\Phi \Rightarrow \frac{\partial \vec{v}_g}{\partial p} = \underbrace{\frac{1}{f_0}\hat{k} \times \vec{\nabla}\left(\frac{\partial \Phi}{\partial p}\right)}$

$\frac{\partial \vec{v}_g}{\partial p} \perp$ to $\vec{\nabla}\frac{\partial \Phi}{\partial p}$

$$\left[\nabla^2 + \frac{\partial}{\partial p}\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right]\frac{\partial \Phi}{\partial t} = -f_0 \vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) - f_0 \vec{v}_g \cdot \vec{\nabla}\left(\frac{\partial}{\partial p}\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right) - f_0 \frac{R}{c_p}\frac{\partial}{\partial p}\left[\frac{J}{\sigma p}\right]$$

$$\frac{\partial}{\partial t}\left[\frac{1}{f_0}\nabla^2 \Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right] = -\vec{v}_g \cdot \vec{\nabla}\left[\frac{1}{f_0}\nabla^2 \Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right] - \frac{R}{c_p}\frac{\partial}{\partial p}\left[\frac{J}{\sigma p}\right]$$

$$\frac{Dq}{Dt} = -\frac{\partial}{\partial p}\left[\frac{J R}{\sigma c_p p}\right]$$

$$q = \frac{1}{f_0}\nabla^2 \Phi + f + \frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right]$$

quasi-geostrophic potential vorticity

In the shallow water system

$$-\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\underbrace{\left(-\frac{\partial \Phi}{\partial p}\right)}\right) \sim -f_0 \frac{\eta}{H}$$

Rigid lid approx thus states that absolute vorticity conserved

specific volume $\alpha = \frac{1}{\rho}$

$$q_{sw} = \zeta_g + f - f_0 \frac{\eta}{H}$$

$\eta > 0 \Rightarrow \zeta_g + f > 0$

vertical stretching compensated by spin up omega equation

$\nabla^2(\text{Q-G thermodynamic}) + \frac{\partial}{\partial p}(\text{Q-G vorticity})$

$$\nabla^2 \frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial p}\right) = -\nabla^2\left[\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] + \sigma \nabla^2 \omega + \frac{R}{c_p p}\nabla^2 J$$

$$\frac{\partial}{\partial p}\nabla^2 \chi = -\nabla^2\left[-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] - \sigma \nabla^2 \omega - \frac{R}{c_p p}\nabla^2 J$$

$$\frac{\partial}{\partial p}\left(\frac{1}{f_0}\nabla^2 \chi\right) = -\frac{\partial}{\partial p}\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) + f_0 \frac{\partial^2 \omega}{\partial p^2}$$

$$\frac{\partial}{\partial p}\nabla^2 \chi = -f_0 \frac{\partial}{\partial p}\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) + f_0^2 \frac{\partial^2 \omega}{\partial p^2}$$

$$-f_0 \frac{\partial}{\partial p}\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) + f_0^2 \frac{\partial^2 \omega}{\partial p^2} = -\nabla^2\left[-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] - \sigma \nabla^2 \omega - \frac{R}{c_p}\frac{1}{p}\nabla^2 J$$

$$\sigma \nabla^2 \omega + f_0^2 \frac{\partial^2 \omega}{\partial p^2} = -f_0 \frac{\partial}{\partial p}\left[-\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right)\right] - \nabla^2\left[-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] - \frac{R}{c_p p}\nabla^2 J$$

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + \beta\right)\right]} - \underbrace{\frac{1}{\sigma}\nabla^2\left[-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right]} - \underbrace{\frac{R}{c_p p \sigma}\nabla^2 J}$$

$\sim -\left(k^2 + l^2 + \frac{f_0^2 \pi^2}{\sigma p^2}\right)\omega$

$\propto -\omega$

vertical derivative of absolute vorticity advection

$-\vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) > 0$ cyclonic

$-\frac{\partial}{\partial p}\left[-\vec{v}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2 \Phi + f\right)\right] > 0$ cyclonic vorticity advection increasing w/ altitude

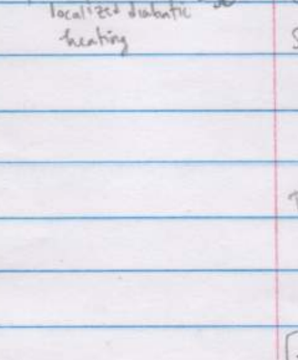
$\Rightarrow \omega < 0$ upward motion

non-uniform temperature advection

$-\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) > 0$ warm advection

isolated diabatic processes

localized heating or cooling will induce expansion or contraction of otherwise flat pressure surfaces + create unbalanced ∇p forces



Quasi-geostrophic potential vorticity equation

→ obtain a powerful conservation law

Starting from geopotential tendency, we may expand the second term and rearrange to for the Q-G system

$$\left[\nabla^2 + \frac{\partial}{\partial p}\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right]\frac{\partial \Phi}{\partial t} = -f_0 \vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + f\right) - \frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] - f_0 \frac{R}{c_p}\frac{\partial}{\partial p}\left[\frac{J}{\sigma p}\right]$$

$$= -f_0 \vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla \Phi + f\right) + \frac{\partial}{\partial p}\frac{f_0^2}{\sigma}\vec{v}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) + \text{diabatic term}$$

The second term can be written

$$\frac{f_0^2}{\sigma}\cancelto{0}{\frac{\partial \vec{v}_g}{\partial p} \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)} + f_0^2 \vec{v}_g \cdot \vec{\nabla}\left[\frac{\partial}{\partial p}\frac{1}{\sigma}\frac{\partial \Phi}{\partial p}\right]$$

but $\vec{v}_g = \frac{1}{f_0}\hat{k} \times \vec{\nabla}\Phi \Rightarrow \frac{\partial \vec{v}_g}{\partial p} = \underline{\frac{1}{f_0}\hat{k} \times \vec{\nabla}\left(\frac{\partial \Phi}{\partial p}\right)}$

$\frac{\partial v_g}{\partial p} \perp$ to $\vec{\nabla}\frac{\partial \Phi}{\partial p}$

$$\left[\nabla^2 + \frac{\partial}{\partial p}\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right]\frac{\partial \Phi}{\partial t} = -f_0 \vec{v}_g \cdot \vec{\nabla}\left(\frac{1}{f_0}\nabla^2 \Phi + \beta\right) - f_0 \vec{v}_g \cdot \vec{\nabla}\left(\frac{\partial}{\partial p}\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right) - f_0 \frac{R}{c_p}\frac{\partial}{\partial p}\left[\frac{J}{\sigma p}\right]$$

$$\frac{\partial}{\partial t}\left[\frac{1}{f_0}\nabla^2 \Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right] = -\vec{v}_g \cdot \vec{\nabla}\left[\frac{1}{f_0}\nabla^2 \Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right] - \frac{R}{c_p}\frac{\partial}{\partial p}\left[\frac{J}{\sigma p}\right]$$

$$\frac{D q}{D t} = -\frac{\partial}{\partial p}\left[\frac{J R}{\sigma c_p p}\right]$$

$$q = \frac{1}{f_0}\nabla^2 \Phi + f + \frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right]$$

quasi-geostrophic potential vorticity

In the shallow water system.

$$-\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\left(-\frac{\partial \Phi}{\partial p}\right)\right) \sim -f_0 \frac{\eta}{H}$$

specific volume $\alpha = \frac{1}{\rho}$

$q_{SW} = \zeta_g + f - f_0 \frac{\eta}{H}$

$\eta > 0 \Rightarrow \zeta_g + f > 0$

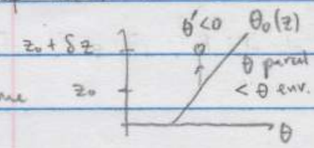
vertical stretching compensated by spin up

Rigid lid approx thus states that absolute vorticity conserved

Ryan Scott, SIO 250

Adiabatic oscillations in a stratified fluid

basic state characterized by $\frac{dp_0}{dz} = -\rho_0 g$, $p_T = p_0(z) + p$, $\rho_T = \rho_0 + \rho$, $\theta_T = \theta_0(z) + \theta$



equation of motion for fluid parcel → $\frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$ parcel: p, ρ environment: p_0, ρ_0

If we assume the pressure experienced by the parcel is the same as the ambient environmental pressure, then $p = p_0$ and we have

$$\frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{dp_0}{dz} = -g - \frac{(-\rho_0 g)}{\rho} = g\left(\frac{\rho_0}{\rho} - 1\right) = g\left(\frac{\rho_0 - \rho}{\rho}\right)$$

note: if $\rho < \rho_0$, i.e., parcel density less than environment, $\frac{\partial w}{\partial t} > 0$

$p = \rho R T$

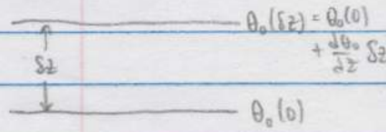
$$= g\left(\frac{\cancel{p_0}}{\cancel{R}T_0} - \frac{\cancel{p}}{\cancel{R}T}\right)\Big/ \frac{\cancel{p}}{\cancel{R}T} \underset{p=p_0}{=} g\left(\frac{1/T_0 - 1/T}{1/T}\right) \cdot \frac{T}{T} = g\left(\frac{T}{T_0} - 1\right)$$

$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p} = \frac{p}{\rho R}\left(\frac{p_0}{p}\right)^{R/c_p}$

parcel temp. $T_0 + T'$

$$= g\left(\frac{T - T_0}{T_0}\right) = g\frac{T'}{T_0} = g\frac{\theta'}{\theta_0} \to \text{drop primes} \to g\frac{\theta}{\theta_0}$$

$T > T_0 \Rightarrow \partial w/\partial t > 0$ basic state stratification



for simplicity, let $z_0 = 0$

$\theta_0(0)$ parcel potential temperature is conserved

$$\theta_0(\delta z) = \theta_0(0) + \frac{d\theta_0}{dz}\delta z + \dots$$

$$\theta_0(0) - \theta_0(\delta z) = \cancel{\theta_0(0)} - \left[\cancel{\theta_0(0)} + \frac{d\theta_0}{dz}\delta z + \dots\right] \approx -\frac{d\theta_0}{dz}\delta z$$

$$\Rightarrow \frac{Dw}{Dt} = \frac{\partial w}{\partial t} = \frac{\partial^2}{\partial t^2}(\delta z) = -\frac{g}{\theta_0}\frac{d\theta_0}{dz}\delta z$$

$$\frac{\partial^2}{\partial t^2}\delta z + N^2 \delta z = 0 \Rightarrow r^2 + N^2 = 0 \Rightarrow r = \pm\sqrt{-N^2}$$

subject to initial conditions $\delta z(t=0)$ and $w(t=0)$ specified

cases: $N^2 = -\frac{g}{\theta_0}\frac{d\theta_0}{dz}$

(i) $d\theta_0/dz < 0 \to N^2 < 0$ and $r = \pm N$, $\delta z = Ae^{Nt} + Be^{-Nt}$

static instability [cold above warm], $\Gamma > \Gamma_d$

$\ln\theta = \ln T + \frac{R}{c_p}[\ln p_0 - \ln p]$

$\frac{1}{\theta}\frac{d\theta}{dz} = \frac{1}{T}\frac{dT}{dz} - \frac{R}{c_p p}\frac{dp}{dz}$

$\frac{T}{\theta}\frac{d\theta}{dz} = \frac{dT}{dz} - \frac{RT}{c_p p}(-\rho g)$, $\frac{RT}{p} = 1/\rho$

$\frac{T}{\theta}\frac{d\theta}{dz} = \frac{g}{c_p} + \frac{dT}{dz}$, with $\frac{g}{c_p} = \Gamma_d$ and $\frac{dT}{dz} = -\Gamma$

→ (ii) $d\theta_0/dz = 0$ (neutral stability) $\to N^2 = 0$ and $r = 0$, δz = constant in time

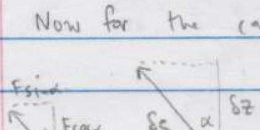
(iii) $d\theta_0/dz > 0 \to N^2 > 0$ and $r = \pm iN$, $\delta z = Ae^{iNt} + Be^{-iNt}$

statically stable $\Gamma < \Gamma_d$

$= \bar{A}\cos(Nt + \delta)$

$= A^* \sin(Nt) + B^* \cos(Nt)$

Now for the case of oscillations slanted at an angle α from the vertical,



$\delta z = \delta s \cos\alpha$

$\frac{\partial^2}{\partial t^2}\delta s = -N^2 \delta z \cos\alpha = \frac{\text{Force}}{\text{mass}}$, but only the component of force along δs

$$= -\underbrace{N^2\cos^2\alpha}_{\nu^2}\,\delta s$$

$\nu = \pm N\cos\alpha$ is the dispersion relationship for non-rotating gravity waves obtained in Ch. 7 by more rigorous mathematics

alternatively written $\nu = \pm\frac{Nk}{(k^2+m^2)^{1/2}}$ see gravity wave notes

$L_z << H = \frac{RT}{g}$

Atmospheric gravity waves

- no hydrostatic balance (except for the basic state) i.e., vertical accelerations are important
- $f = 0$ (consider this case first), so (x, y) isotropic directions
- adiabatic, Boussinesq $\rho = \rho_0$ except w/ buoyancy

no radiative heating, latent heating, etc. →

Zonal + vertical momentum, continuity, thermodynamic equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = 0, \quad \theta = T\left(\frac{p_s}{p}\right)^{\kappa} = \frac{p}{\rho R}\left(\frac{p_s}{p}\right)^{\kappa}, \; p_s = 10^3\,\text{hPa}$$

side

$$\ln\theta = \ln\left(\frac{p}{\rho R}\right) + \kappa[\ln p_s - \ln p]$$
$$= \ln\frac{p}{\rho R} - \kappa\ln p + \underbrace{\kappa \ln p_s}_{\text{const}}$$
$$= \left(1 - \frac{R}{c_p}\right)\ln p - \ln \rho R$$
$$= \left(1 - \frac{R}{c_v + R}\right)\ln p - [\ln\rho + \ln R]$$
$$= \left(\frac{c_v + R - R}{c_v + R}\right)\ln p - \ln\rho + C$$
$$= \frac{c_v}{c_p}\ln p - \ln\rho + C$$
$$= \gamma^{-1}\ln p - \ln\rho + C$$

$\frac{d\bar{p}}{dz} < 0$

$\frac{d\bar{\theta}}{dz} > 0$

Consider basic state

ρ_0, $\bar{u}$ constant

$\bar{w} = 0$, $\bar{p}(z)$, $\bar{\theta}(z)$

and allow perturbations

$$\rho = \rho_0 + \rho' \qquad p = \bar{p}(z) + p' \qquad \frac{\rho'}{\rho_0} \approx -\frac{\theta'}{\bar{\theta}}$$
$$u = \bar{u} + u' \qquad \theta = \bar{\theta}(z) + \theta'$$
$$w = \cancelto{0}{\bar{w}} + w'$$

see below

$$\bar{\theta}(z) = \frac{\bar{p}}{R\rho_0}\left(\frac{p_s}{\bar{p}}\right)^{\kappa}$$

$$\frac{d\bar{p}}{dz} = -\rho_0 g \qquad \ln\bar{\theta} = \gamma^{-1}\ln\bar{p} - \ln\rho_0 + C$$

---

perturbation equations

$$\cancelto{0}{\frac{\partial}{\partial t}\bar{u}} + \frac{\partial u'}{\partial t} + (\bar{u} + u')\left(\cancelto{0}{\frac{\partial \bar{u}}{\partial x}} + \frac{\partial u'}{\partial x}\right) + w'\left(\cancelto{0}{\frac{\partial \bar{u}}{\partial z}} + \frac{\partial u'}{\partial z}\right) = -\frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial x}\left(\cancelto{0}{\bar{p}(z)} + p'\right)$$

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial}{\partial x}u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$\frac{1}{1+x} \approx 1 - x$

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0 + \rho'}\left[\frac{d\bar{p}}{dz} + \frac{\partial p'}{\partial z}\right] - g = -\frac{1}{\rho_0}\left[\frac{1}{1 + \frac{\rho'}{\rho_0}}\right]\left[-\rho_0 g + \frac{\partial p'}{\partial z}\right] - g$$

$$\approx -\frac{1}{\rho_0}\left[1 - \frac{\rho'}{\rho_0}\right]\left(-\rho_0 g + \frac{\partial p'}{\partial z}\right) - g$$

$$= -\frac{1}{\rho_0}\Big[-\rho_0 g + \frac{\partial p'}{\partial z} + \rho' g - \underbrace{\cancel{\frac{\partial p'}{\partial z}\frac{\rho'}{\rho_0}}}_{\text{small}}\Big] - g$$

$$= \cancel{g} - \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0}g - \cancel{g} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0}g$$

$(1-x)(1 + x + \cdots + x^{N-1})$

$1 + x + x^2 + \cdots x^{N-1}$

$-x - x^2 - \cdots - x^{N-1} - x^N$

$= 1 - x^N$

so $\sum_{k=0}^{N-1} x^k = \frac{1 - x^N}{1 - x}$

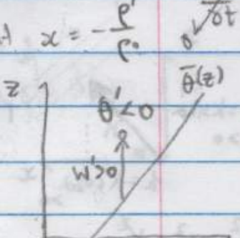
$N \to \infty$ considering we can $|x| < 1$

$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

at right $x = -\frac{\rho'}{\rho_0}$

$$\cancelto{0}{\frac{\partial}{\partial x}\bar{u}} + \frac{\partial u'}{\partial x} + \cancelto{0}{\frac{\partial \bar{w}}{\partial z}} + \frac{\partial w'}{\partial z} = 0 \Rightarrow u'_x + w'_z = 0$$

$$\cancelto{0}{\frac{\partial \bar{\theta}}{\partial t}} + \frac{\partial \theta'}{\partial t} + (\bar{u} + u')\left(\cancelto{0}{\frac{\partial \bar{\theta}}{\partial x}} + \frac{\partial \theta'}{\partial x}\right) + w'\left(\frac{d\bar{\theta}}{dz} + \frac{\partial \theta'}{\partial z}\right) = 0 \qquad \frac{\partial \theta'}{\partial t} + \bar{u}\frac{\partial \theta'}{\partial x} + w'\frac{d\bar{\theta}}{dz} = 0$$



$$\left.\begin{aligned}
&\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}\\
&\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)w' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0}g\\
&\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0\\
&\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\theta' + w'\frac{d\bar{\theta}}{dz} = 0
\end{aligned}\right\}$$

perturbation equations. It is useful too to express $\frac{\rho'}{\rho}$ in terms of θ

$$\ln(1 + \varepsilon) \approx \varepsilon$$

$$\ln\left[\bar{\theta}\left(1 + \frac{\theta'}{\bar{\theta}}\right)\right] = \frac{1}{\gamma}\ln\left[\bar{p}(z)\left(1 + \frac{p'}{\bar{p}}\right)\right] - \ln\left(\rho_0\left(1 + \frac{\rho'}{\rho_0}\right)\right) \Rightarrow \ln\bar{\theta} + \ln\left(1 + \frac{\theta'}{\bar{\theta}}\right) = \frac{1}{\gamma}\ln\bar{p}(z) + \frac{1}{\gamma}\ln\left(1 + \frac{p'}{\bar{p}}\right) - \ln\rho_0 - \ln\left(1 + \frac{\rho'}{\rho_0}\right)$$

$$\cancel{\frac{1}{\gamma}\ln\bar{p} - \ln\rho_0 + C} + \frac{\theta'}{\bar{\theta}} \approx \cancel{\frac{1}{\gamma}\ln\bar{p}} + \frac{1}{\gamma}\frac{p'}{\bar{p}} - \cancel{\ln\rho_0} - \frac{\rho'}{\rho_0} \Rightarrow \boxed{\rho' = \underbrace{\frac{\rho_0 p'}{\gamma\bar{p}}}_{\to c_s^2} - \frac{\theta'}{\bar{\theta}}\rho_0}$$

so $\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)w' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\theta'}{\bar{\theta}}g$

$\nabla\times\vec{u} = \xi\hat{i}+\eta\hat{j}+\zeta\hat{k}$

$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} = \hat{i}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+\hat{j}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$

$\frac{\partial}{\partial x}$(vertical momentum) $-\frac{\partial}{\partial z}$(horizontal momentum)

$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial w'}{\partial x} + \frac{1}{\rho_0}\frac{\partial}{\partial x}\left(\frac{\partial p'}{\partial z}\right) - \frac{g}{\bar{\theta}}\frac{\partial\theta'}{\partial x} = 0$

$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial u'}{\partial z} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial p'}{\partial x}\right) = 0$

① subtracting $\Rightarrow$ $\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial w'}{\partial x}-\frac{\partial u'}{\partial z}\right) - \frac{g}{\bar{\theta}}\frac{\partial\theta'}{\partial x} = 0$ $\quad -\eta$ component of vorticity equation

② $\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z} = 0$, $\quad \frac{\partial\theta'}{\partial t}+\bar{u}\frac{\partial\theta'}{\partial x}+w'\frac{d\bar{\theta}}{dz} = 0$ ③

$\partial_x(-\eta$ component vorticity equation$)$

$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 w'}{\partial x^2}-\frac{\partial^2 u'}{\partial z\partial x}\right) - \frac{g}{\bar{\theta}}\frac{\partial^2\theta'}{\partial x^2} = 0$ $\quad$ but $\frac{\partial^2 u'}{\partial z\partial x} = -\frac{\partial^2 w'}{\partial z^2}$

$\frac{\partial^2}{\partial t^2}\nabla^2 w' + \bar{u}\frac{\partial}{\partial t}\frac{\partial}{\partial x}\nabla^2 w' - \frac{g}{\bar{\theta}}\frac{\partial}{\partial t}\frac{\partial^2\theta'}{\partial x^2} = 0$

Now need to eliminate $-\frac{g}{\bar{\theta}}\frac{\partial}{\partial t}\frac{\partial^2\theta'}{\partial x^2}$

∂_x ① $\quad \left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\nabla^2 w' - \frac{g}{\bar{\theta}}\frac{\partial}{\partial x}\left[-\frac{1}{\bar{u}}\left(\frac{\partial\theta'}{\partial t}+w'\frac{d\bar{\theta}}{dz}\right)\right] = 0$

but $\frac{\partial\theta'}{\partial x} = -\frac{1}{\bar{u}}\left[\theta_t'+w'\bar{\theta}_z\right]$

$\frac{\partial}{\partial x}\frac{\partial}{\partial t}\nabla^2 w' + \bar{u}\frac{\partial^2}{\partial x^2}\nabla^2 w' + \frac{g}{\bar{\theta}\bar{u}}\frac{\partial^2}{\partial x^2}\frac{\partial\theta'}{\partial t} + \frac{g}{\bar{\theta}\bar{u}}\frac{\partial^2}{\partial x^2}w'\frac{d\bar{\theta}}{dz} = 0$

$\bar{u}\frac{\partial}{\partial x}\frac{\partial}{\partial t}\nabla^2 w' + \bar{u}^2\frac{\partial^2}{\partial x^2}\nabla^2 w' + \frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dz}\frac{\partial^2 w'}{\partial x^2} = -\frac{g}{\bar{\theta}}\frac{\partial}{\partial t}\frac{\partial^2\theta'}{\partial x^2}$

$\frac{\partial^2}{\partial t^2}\nabla^2 w' + 2\bar{u}\frac{\partial}{\partial x}\frac{\partial}{\partial t}\nabla^2 w' + \bar{u}^2\frac{\partial^2}{\partial x^2}\nabla^2 w' + \frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dz}\frac{\partial^2 w'}{\partial x^2} = 0$

$$\boxed{\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2\nabla^2 w' + N^2\frac{\partial^2 w'}{\partial x^2} = 0}$$

$\hat{w} = w_r + i w_i$

$\hat{w}e^{i\phi} = (w_r+iw_i)(\cos\phi+i\sin\phi)$
$= w_r\cos\phi + iw_r\sin\phi + iw_i\cos\phi - w_i\sin\phi$

~~Now consider case $\bar{u}=0$ (no mean flow)~~

Seek plane wave solution $w' = \mathrm{Re}\{\hat{w}e^{i\phi}\} = \mathrm{Re}\{\hat{w}e^{i(kx+mz-\nu t)}\} = w_r\cos\phi - w_i\sin\phi$

$\frac{\partial w'}{\partial t} = -i\nu w'$

$\frac{\partial^2 w'}{\partial t^2} = -\nu^2 w'$

$\frac{\partial w'}{\partial x} = ikw'$

$\frac{\partial^2 w'}{\partial x^2} = -k^2 w'$

$\frac{\partial^2 w'}{\partial z^2} = -m^2 w'$

$\frac{\partial^2}{\partial t^2}\nabla^2 w' = \frac{\partial^2}{\partial t^2}\left(-(k^2+m^2)w'\right)$
$= -(k^2+m^2)\times -\nu^2$

$-(k^2+m^2)km$

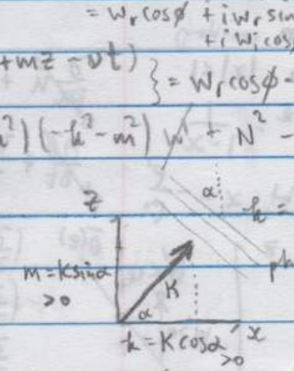
$\nu^2(k^2+m^2)\cancel{w'} + 2\bar{u}(-k^2-m^2)ik\cdot -i\nu\cancel{w'} + \bar{u}^2(-k^2)(-k^2-m^2)\cancel{w'} + N^2\cdot -k^2\cancel{w'} = 0$

$(\nu^2 - 2\bar{u}k + \bar{u}^2k)(k^2+m^2) - N^2k^2 = 0$

$(\nu-\bar{u}k)^2 = \frac{N^2k^2}{k^2+m^2}$

$\hat{\nu}$ is wave frequency moving with mean flow

$\hat{\nu}^2 = \frac{N^2k^2}{k^2+m^2}$



$\hat{\nu}^2 = \frac{N^2\cos^2\alpha\, K^2}{K^2(\cos^2\alpha+\sin^2\alpha)}$

$\hat{\nu} = \pm N\cos\alpha$

$m \in \mathbb{C}$ most generally so that waves can decay exponentially. But consider $m\to 0$, $L_z\to\infty$

$\hat{\nu} = N$

vertical phase lines

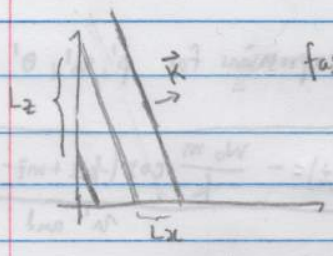
internal gravity wave dispersion relationship

wave frequency relative to mean flow $\rightarrow \hat{\nu} = \nu - \bar{u}k = \dfrac{\pm N k}{\|\vec{K}\|}$ $\qquad \|\vec{K}\| = (k^2+m^2)^{1/2}$

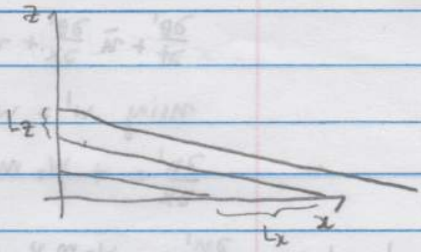
PHASE SPEEDS:

$c_x = \dfrac{\nu}{k} = \bar{u} \pm \dfrac{N}{\|\vec{K}\|}$ since $\|\vec{K}\| \propto \dfrac{1}{L}$ long waves propagate fastest zonally

$c_z = \dfrac{\nu}{m} = \bar{u}\dfrac{k}{m} \pm \dfrac{Nk/m}{\|\vec{K}\|}$ $\quad k/m \propto \dfrac{1/Lx}{1/Lz} = Lz/Lx \leftarrow$ waves with this ratio large travel fastest vertically



faster upward phase propagation



$\vec{c}_g$ = group velocity $= \nabla_{\vec{k}}\nu = \hat{i}\dfrac{\partial \nu}{\partial k} + \hat{k}\dfrac{\partial \nu}{\partial m}$

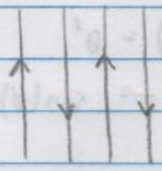
$c_{gx} = \dfrac{\partial \nu}{\partial k} = \bar{u} \pm \dfrac{N\sqrt{k^2+m^2} - Nk\frac{1}{2}(k^2+m^2)^{-1/2}\cdot 2k}{k^2+m^2} = \bar{u} \pm \dfrac{N(k^2+m^2) - Nk^2}{(k^2+m^2)^{3/2}} = \bar{u} \pm \dfrac{Nm^2}{(k^2+m^2)^{3/2}}$

$c_{gz} = \dfrac{\partial \nu}{\partial m} = \pm Nk \dfrac{\partial}{\partial m}\dfrac{1}{\sqrt{k^2+m^2}} = \pm Nk(-\tfrac{1}{2})(k^2+m^2)^{-3/2}\cdot 2m = \pm \dfrac{(-Nkm)}{(k^2+m^2)^{3/2}}$

$\vec{c}_g = \bar{u}\hat{i} \pm \dfrac{Nm}{(k^2+m^2)^{3/2}}(m\hat{i} - k\hat{k}) \Rightarrow \boxed{\vec{c}_g \cdot \vec{K} = 0}$ group and phase propagation are orthogonal

$L_z \to \infty$

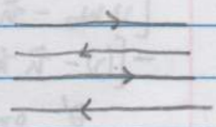
$m = 0$



- fluid parcels oscillate vertically at buoyancy frequency

$L_x \to \infty$

$k = 0$



$\hat{\nu} = \nu = 0$

- no buoyant restoring force
- Coriolis becomes important for low frequency, long period motions

$\hat{\nu} = \nu - \bar{u}k = \pm N$

$c_{px} = \bar{u} \pm \dfrac{N}{k}, \quad \vec{c}_g = \bar{u}$

Now, from the perturbation equations

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial}{\partial x}u' + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0$$

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\theta'}{\bar{\theta}}g = 0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$\frac{\partial \theta'}{\partial t} + \bar{u}\frac{\partial \theta'}{\partial x} + w'\frac{d\bar{\theta}}{dz} = 0$, we can derive expressions for p', u', θ'

using $w' = w_0\cos(kx+mz-\nu t)$

$\frac{\partial u'}{\partial x} = + w_0 m \sin(kx+mz-\nu t) \Rightarrow \boxed{u'(x,z,t) = -\frac{w_0 m}{k}\cos(kx+mz-\nu t)} + \cancel{f(z)}^{\,0}$

u' and w' in phase

$\boxed{w' = w_0\cos(\phi)}$

p' next: $\frac{\partial u'}{\partial t} = -\frac{w_0 m \nu}{k}\sin(\phi)$, $\bar{u}\frac{\partial u'}{\partial x} = \bar{u}\, w_0 m \sin(\phi)$

$$-\frac{\partial p'}{\partial x} = \rho_0 w_0 m(\bar{u} - c_x)\sin(kx+mz-\nu t)$$

$$-p' = \rho_0 w_0 m(\bar{u} - c_x)\frac{-1}{k}\cos(kx+mz-\nu t)$$

$\boxed{p' = \rho_0 w_0 \frac{m}{k}(\bar{u} - c_x)\cos(kx+mz-\nu t)}$ p' in phase with u', w'

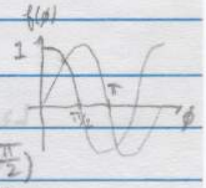
θ' next $+\nu w_0\sin(\phi) - \bar{u}w_0 k\sin(\phi) + w_0\frac{m}{k}(\bar{u}-c_x)(-m\sin(\phi)) = -\frac{\theta'}{\bar{\theta}}g$

$$\left[\nu w_0 - \bar{u}k w_0 - w_0\frac{m^2}{k}(\bar{u}-c_x)\right]\sin\phi$$

$K = \nabla\phi = (k, m)$

$$-\left[(\nu - \bar{u}k) + \frac{m^2}{k}(c_x - \bar{u})\right]\frac{\bar{\theta}}{g} \times \sin(kx+mz-\nu t) = \theta'$$

θ' out of phase with u', w', p' by $\frac{\pi}{2}$ $\longrightarrow$ $\sin(\phi) = \cos(\phi + \frac{\pi}{2})$



$\sin(0) = \cos(\frac{\pi}{2}) = 0$

$\sin(\pi) = \cos(\frac{3\pi}{2}) = 0$

$\sin(\frac{\pi}{2}) = \cos(\pi) = -1$

etc.

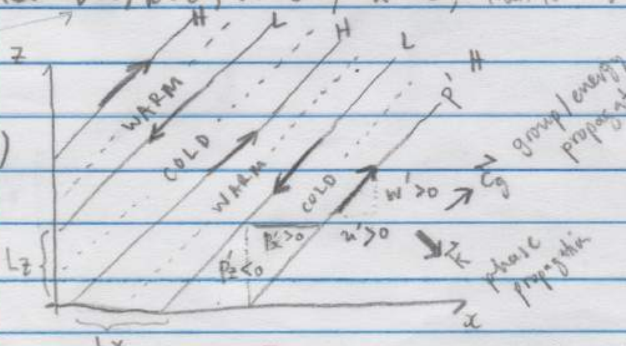
$\hat{\nu} = \nu - \cancel{\bar{u}k}^{\,0} = \pm\frac{Nk}{\|K\|}$ Consider $\nu > 0$, $k > 0$, $m < 0$, $\bar{u} = 0$, that is: $\nu = +\frac{Nk}{\|K\|}$

$\vec{c}_g = \cancel{\bar{u}\hat{i}}^{\,0} \pm \frac{Nm}{(k^2+m^2)^{3/2}}(m\hat{i} - k\hat{k})$

$= \frac{\partial \nu}{\partial k}\hat{i} + \frac{\partial \nu}{\partial m}\hat{k}$

previous page

choosing $k>0$ $m<0$ easy to see upward and rightward group propagation



$$\overline{u'w'} = -w_0^2\frac{m}{k}\int \cos^2(\vec{k}\cdot\vec{x} - \nu t)\,d\phi$$

$= -\frac{1}{2}\frac{m}{k}w_0^2$ $m<0$, $k>0$ means that $\overline{u'w'} > 0$

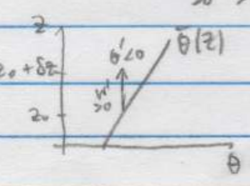
$$\frac{\partial \theta'}{\partial t} = -w'\frac{d\bar{\theta}}{dz} < 0$$

(with $w' > 0$, $\frac{d\bar{\theta}}{dz} > 0$)

$\frac{\partial w'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}$

$\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$

Downward phase propagation is easy to see: Upward motion causes adiabatic cooling and hence downward propagation of cold part of wave whereas downward motion causes adiabatic warming and downward propagation of warm anomaly



stable stratification $\partial\bar{\theta}/\partial z > 0$

$\theta' < 0$ because its colder than $\bar{\theta}$ at $z = z_0 + \delta z$

Stationary gravity waves, flow over infinitely long sinusoidal topography

$$\cancel{\frac{\partial^2}{\partial t^2}}\nabla^2 w' + 2\bar{u}\frac{\partial}{\partial x}\cancel{\frac{\partial}{\partial t}}\nabla^2 w' + \bar{u}^2\frac{\partial^2}{\partial x^2}\nabla^2 w' + N^2\frac{\partial^2 w'}{\partial x^2} = 0$$

$$\nabla^2 w' + \frac{N^2}{\bar{u}^2} w' = 0$$

$\phi = kx + mz$ $\qquad w' = \hat{w} e^{i\phi} \Rightarrow (-k^2 - m^2)\cancel{\hat{w}} + \frac{N^2}{\bar{u}^2}\cancel{\hat{w}} = 0$

$$m^2 = \frac{N^2}{\bar{u}^2} - k^2$$

OR: original dispersion relation $(\cancel{\hat{\nu}} - \bar{u}k)^2 = \frac{N^2 k^2}{k^2+m^2} \Rightarrow \bar{u}^2 \cancel{k^2} = \frac{N^2 \cancel{k^2}}{k^2+m^2}$ yields the same thing

Now if $\frac{N^2}{\bar{u}^2} > k^2 \Rightarrow \left|\frac{N}{\bar{u}}\right| > |k|$ or simply $|\bar{u}k| < N$, $m \in \mathbb{R}$

$\frac{N^2}{\bar{u}^2} < k^2 \Rightarrow \left|\frac{N}{\bar{u}}\right| < |k| \Rightarrow$ $\qquad |\bar{u}k| > N$, $m \in \mathbb{C}$, $m = m_r + i m_i$

$\bar{u} > 0$

frequency rel. mean flow → $\hat{\nu} = \cancel{\nu} - \bar{u}k = \pm\frac{Nk}{(k^2+m^2)^{1/2}}$, Choose $\hat{\nu} > 0$ so $k < 0$, take − root

eastward advection by zonal wind is balanced by westward propagation ($k<0$) so that the waves remain stationary relative to the surface ($\nu = 0$)

$\hat{c}_x = \frac{\hat{\nu}}{k} = -\bar{u}$ (zonal phase speed equal and opposite to mean flow)

$\hat{c}_{gx} = \frac{\partial\hat{\nu}}{\partial k} = -\bar{u}$ same for energy propagation

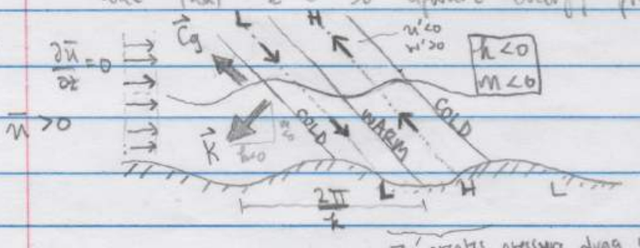
$\hat{c}_z = \frac{\hat{\nu}}{m} = -\bar{u}\frac{k}{m}$

$\hat{c}_{gz} = \frac{\partial\hat{\nu}}{\partial m} = \pm\frac{\partial}{\partial m}\frac{Nk}{(k^2+m^2)^{1/2}} = \pm Nk\left[-\frac{1}{\cancel{2}}(k^2+m^2)^{-3/2}\cdot\cancel{2}m\right] = \pm\frac{(-Nkm)}{(k^2+m^2)^{3/2}}$ $\left(\text{also} = \frac{\partial\nu}{\partial m}\right)$

but recall that the − root was chosen so here

and that $k<0$ so upward energy propagation requires $m<0$

$\overline{u'w'} < 0$ $\Rightarrow$



$-\nabla p'$ creates pressure drag on mean flow

for $m \in \mathbb{R}$

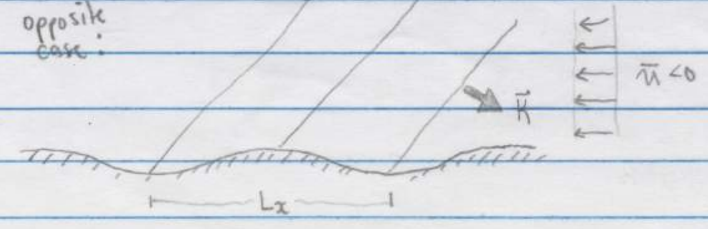
stationary phase lines

$w' = w_0\cos(kx+mz)$

$|\bar{u}k| < N$ occurs under

- strong stratification, N larger
- weak mean flow, $\bar{u}$ small
- large wavelength topography L_x large, k small

opposite case:



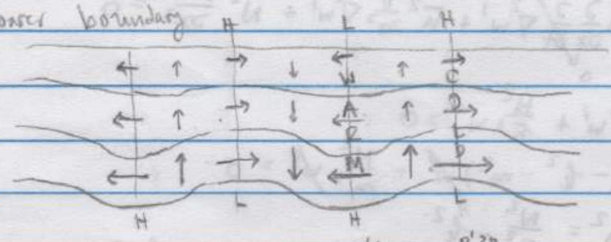
$k>0$, $m<0$

$(\hat{w} \in \mathbb{C})$

$\hat{w} = w_0 e^{i\delta}$ phase shift

trapped waves if $m \in \mathbb{C}$

$w' = \hat{w} e^{i(kx + (m_r + i m_i)z)} = \hat{w} e^{i(kx + m_r z)} e^{-m_i z}$, decays upward from the lower boundary



$u'w' = 0$ everywhere implies there cannot be upward propagation of momentum

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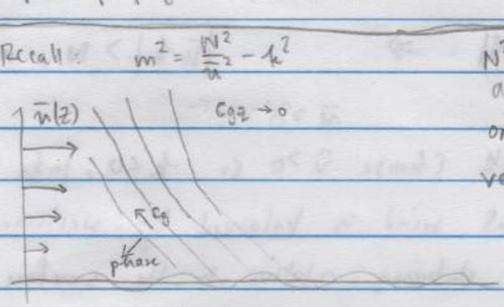
We chose $\bar{u}>0$ so $k<0$ so phase lines propagate westward against the mean flow

Recall $m^2 = \frac{N^2}{\bar{u}^2} - k^2$

N^2 constant (linear stratification, k constant) as $\bar{u}$ increases m decreases, or L_z decreases, and $m \to 0$ as $L_z \to \infty$

vertical trapping



$c_{gz} = \frac{Nkm}{(k^2+m^2)^{3/2}} \to 0$ as $m \to 0$, $L_z \to \infty$

k fixed

critical level

$c_{gx} \to \bar{u}$

Inertial oscillations and inertial instability

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= f_0 v \\ \frac{\partial v}{\partial t} &= -f_0 u\end{aligned}\right\}\quad \frac{\partial^2 u}{\partial t^2} = f_0\frac{\partial v}{\partial t} = -f_0^2 u \qquad \frac{\partial^2 u}{\partial t^2} + f_0^2 u = 0 \quad \text{simple harmonic oscillation}$$

$u(t) = A\cos(f_0 t) + B\sin(f_0 t)$

$u(0) = V \quad \partial u/\partial t\,(0) = 0 \Rightarrow A = V$ and $-Vf_0\sin(\cancel{f_0 t})^{0} + Bf_0\cos(\cancel{f_0 t})^{1} = 0, \quad B = 0$

$\boxed{u(t) = V\cos(f_0 t)}$ and $-V\cancel{f_0}\sin(f_0 t) = f_0 v(t)$, so

$\boxed{v(t) = -V\sin(f_0 t)}$

equation of circle $x^2+y^2=r^2$

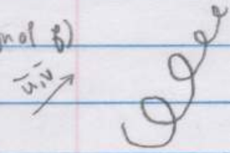
$x(t) = \int V\cos(f_0 t)\,dt = \frac{V}{f_0}\sin(f_0 t) \Longrightarrow \frac{V^2}{f_0^2}\left[\cos^2(f_0 t) + \sin^2(f_0 t)\right] = r^2$ (the bracket $= 1$)

$y(t) = -\int V\sin(f_0 t)\,dt = \frac{V}{f_0}\cos(f_0 t)$

parcel executes circular motion with radius $\frac{V}{|f_0|}$, speed V

inertial circles become smaller closer to the poles

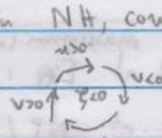
clockwise in NH, counterclockwise SH ; ANTICYCLONIC (opposite sign of f)



$\zeta > 0$ counter-clockwise

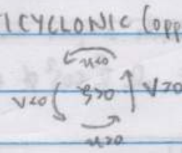
$\zeta < 0$ clockwise

NH case:



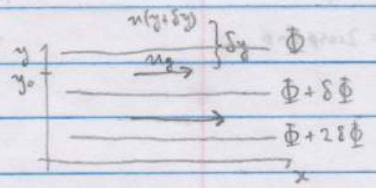
$f_0 > 0$: $\frac{\partial u}{\partial t} = f_0 v$, $\frac{\partial v}{\partial t} = -f_0 u$

SH case:



$f_0 < 0$: $\frac{\partial u}{\partial t} = f_0 v$, $\frac{\partial v}{\partial t} = -f_0 u$

Generalization to mean zonal flow $u_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial y}$



$\frac{Du}{Dt} = f_0 v = f_0\frac{Dy}{Dt} \Rightarrow u(y) = fy$

$\frac{Dv}{Dt} = -f_0 u - \frac{\partial \Phi}{\partial y} = -f_0 u + f_0 u_g = f_0(u_g - u)$

If a parcel in the mean zonal flow at $y = y_0$ is displaced by δy from its equilibrium latitude

$$u(y_0+\delta y) = u(y_0) + f_0\,\delta y = u_g(y_0) + f_0\,\delta y$$

whereas the geostrophic wind there is

$$u_g(y_0+\delta y) = u_g(y_0) + \frac{\partial u_g}{\partial y}\delta y$$

so that

$$\frac{\partial v}{\partial t} = \frac{\partial^2}{\partial t^2}\delta y = f_0\left[\cancel{u_g(y_0)} + f_0\,\delta y - \left(\cancel{u_g(y_0)} + \frac{\partial u_g}{\partial y}\delta y\right)\right]$$

$$\frac{\partial^2}{\partial t^2}\delta y = \underbrace{f_0\left(f_0 - \frac{\partial u_g}{\partial y}\right)}\delta y = f_0\frac{\partial M}{\partial y}\delta y, \quad \text{where } M = f_0 y - u_g$$

Holton defines M to be the absolute angular momentum. As Joel points out, this increases poleward in contrast to Earth's actual $\vec{L} = \vec{r}\times\vec{p}$. So, Holton's definition is poor

$\frac{\partial M}{\partial y} = f_0 - \frac{\partial u_g}{\partial y} = f_0 + \zeta_g = \eta$;

			NH ($f_0>0$)	SH ($f_0<0$)
$f_0\eta =$	$f_0\frac{\partial M}{\partial y} > 0$	stable	$\eta > 0$	$\eta < 0$
	$f_0\frac{\partial M}{\partial y} = 0$	neutral	$\eta = 0$	$\eta = 0$
	$f_0\frac{\partial M}{\partial y} < 0$	unstable	$\eta < 0$	$\eta > 0$

magnitude of geostrophic ζ smaller than planetary vorticity

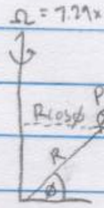
$\eta = \zeta + f < 0 \rightarrow f < \frac{\partial u}{\partial y}$ satisfied if $\frac{\partial u}{\partial y} = f + A$ for some $A > 0$ $\quad \frac{\partial u}{\partial y} > 0$

$\theta = 90° - \phi$

$0° \le \theta \le 180°$

$\theta = 0$ at NP
180 at SP

Earth's angular momentum



$r = R\cos\phi$

$\vec{l} = \vec{r}\times\vec{p} = |\vec{r}||\vec{p}|\sin(\vec{r},\vec{p}) = r\,m\,v\sin\left(\frac{\pi}{2}\right) = m v R\cos\phi$

$\frac{d\vec{l}}{dt} = \vec{r}\times\frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt}\times\vec{p} = \vec{\tau}$ = torque

ϕ: latitude, R: Earth's "radius"

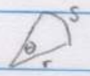
So the angular momentum per unit mass, or the specific angular momentum in this case is $\vec{l}/m$

$$L = \underbrace{\Omega(R\cos\phi)}_{V=\omega r}\ \underbrace{R\cos\phi}_{r} + u\,\underbrace{R\cos\phi}_{r} = (v+u)\,r$$

velocity due to Earth's rotation as seen in the inertial reference frame

distance to rotation axis

zonal velocity relative to Earth's surface



$s = r\theta$

$\frac{ds}{dt} = r\frac{d\theta}{dt}$, $\frac{ds}{dt} = v$, $\frac{d\theta}{dt} = \omega$

Absolute angular momentum

$$\boxed{L = \Omega R^2\cos^2\phi + uR\cos\phi = \Omega R^2 + uR \text{ at EQ (maximum } (u=0)) \quad L \ge 0\ \forall\phi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ = 0 \text{ at NP/SP}}$$

$y = R\phi$

$\frac{d}{dy} = \frac{1}{R}\frac{d}{d\phi}$

$\frac{d}{d\phi}\cos^2\phi = 2\cos\phi\sin\phi$

$$\frac{dL}{dy} = \frac{1}{R}\frac{dL}{d\phi} = \underbrace{-2\Omega\sin\phi}_{f}\ \underbrace{\cos\phi R}_{r} + \frac{1}{R}\frac{d}{d\phi}(uR\cos\phi)$$

$u\sin\phi$

$$= -fr + \frac{d}{dy}(ur) = -fr + r\underbrace{\frac{du}{dy}}_{f+A} + u\frac{dr}{dy}$$

$$= -fr + (f+A)r = Ar \qquad \text{So } \frac{dL}{dy} > 0 \text{ if } A>0$$
$$< 0 \text{ if } A<0$$

$L = \Omega R^2\cos^2\phi + uR\cos\phi \ne M = fy - ug$ in Holton

$\frac{dL}{dy} > 0$ unstable $\qquad \frac{dM}{dy} < 0$ unstable

Inertia gravity waves

Hydrostatic regime $\frac{\partial w}{\partial t} = 0$

$f = f_0 \neq 0$, $\bar{u} = 0$ for simplicity

$f < \nu << N$

$$\frac{\partial u'}{\partial t} - f v' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + f u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

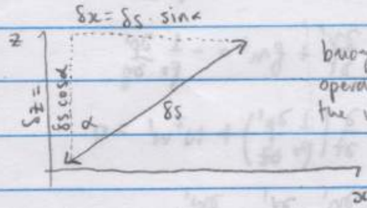
this has important ramifications $\longrightarrow$ $$\underbrace{\cancel{\frac{\partial w'}{\partial t}}}_{0} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\theta'}{\theta_0} g$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial \theta'}{\partial t} + w' \frac{d\theta_0}{dz} = 0$$

Heuristic, physical reasoning

$\delta x = \delta s \cdot \sin\alpha$



buoyancy operates in the vertical

Coriolis operates in the horizontal

$$\frac{\partial^2}{\partial t^2}\delta s = \frac{\text{Force}}{\text{mass}}\left(\begin{matrix}\text{component}\\ \text{along } \delta s\end{matrix}\right)$$

$$= -N^2 \delta z \cdot \cos\phi - f^2 \delta x \sin\phi$$

$$= -\left(N^2\cos^2\phi + f^2 \sin^2\phi\right)\delta s$$

$$\nu^2 = N^2\cos^2\phi + f^2\sin\phi$$

We are here interested in the large angle α limit

$$\nu^2 = N^2 \underbrace{\cancel{\cos^2\phi}}_{0} + f^2 \underbrace{\cancel{\sin^2\phi}}_{1}$$

Seek $(u', v', w', p', \theta') = (\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{\theta}) e^{i(kx + ly + mz - \nu t)}$

$$-i\nu \hat{u} e^{i(\cdots)} - f_0 \hat{v} e^{i(\cdots)} = -\frac{1}{\rho_0}\hat{p}\, i k\, e^{i(\cdots)}$$

$$-i\nu \hat{v} + f_0 \hat{u} e^{i(\cdots)} = -\frac{1}{\rho_0}\hat{p}\, i l\, e^{i(\cdots)}$$

$$0 = -\frac{1}{\rho_0}\hat{p}\, i m\, e^{i(\cdots)} + \frac{\hat{\theta}}{\theta_0} g\, e^{i(\cdots)}$$

$$(ik\hat{u} + il\hat{v} + im\hat{w}) e^{i(\cdots)} = 0$$

$$-i\nu\hat{\theta} e^{i(\cdots)} + \hat{w} e^{i(\cdots)}\frac{d\theta_0}{dz} = 0$$

$$-i\nu\hat{u} - f_0\hat{v} + \frac{1}{\rho_0} ik\hat{p} + 0\hat{\theta} + 0\hat{w} = 0$$

$$-i\nu\hat{v} + f_0\hat{u} + \frac{1}{\rho_0} il\hat{p} + 0\hat{\theta} + 0\hat{w} = 0$$

$$0\hat{u} + 0\hat{v} + \frac{1}{\rho_0} im\hat{p} + \frac{g}{\theta_0}\hat{\theta} + 0\hat{w} = 0$$

$$ik\hat{u} + il\hat{v} + im\hat{w} + 0\hat{p} + 0\hat{\theta} = 0$$

$$0\hat{u} + 0\hat{v} + \frac{d\theta_0}{dz}\hat{w} + 0\hat{p} - i\nu\hat{\theta} = 0$$

$$\underbrace{\begin{bmatrix} -i\nu & -f_0 & 0 & \frac{ik}{\rho_0} & 0 \\ f_0 & -i\nu & 0 & \frac{il}{\rho_0} & 0 \\ 0 & 0 & 0 & \frac{im}{\rho_0} & g/\theta_0 \\ ik & il & im & 0 & 0 \\ 0 & 0 & \frac{d\theta_0}{dz} & 0 & -i\nu \end{bmatrix}}_{\substack{\text{determinant of this matrix must be zero for}\\ \text{non-trivial solutions}}} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{p} \\ \hat{\theta} \end{bmatrix} = \vec{0}$$

$\rightarrow$ gives I-G wave dispersion relationship

OR

eliminate θ' by taking

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho_0}\frac{\partial p'}{\partial z}\right) = \frac{g}{\theta_0}\frac{\partial \theta'}{\partial t} = -\frac{g}{\theta_0}\frac{d\theta_0}{dz} w' = -N^2 w'$$ $\Rightarrow$ down to 4 equations, see next page

$$\frac{\partial u'}{\partial t} - fv' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + fu' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho_0}\frac{\partial p'}{\partial z}\right) + N^2 w' = 0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

and again (first two equations $\times i$)

$$-i\nu\hat{u} - f\hat{v} = -\frac{1}{\rho_0} ik\hat{p} \longrightarrow \nu\hat{u} - if\hat{v} = \frac{1}{\rho_0}k\hat{p}$$

$$-i\nu\hat{v} + f\hat{u} = -\frac{1}{\rho_0} il\hat{p} \longrightarrow \nu\hat{v} + if\hat{u} = \frac{1}{\rho_0}l\hat{p}$$

$$\frac{-i\nu(im)}{\rho_0}\hat{p} + N^2\hat{w} = 0 \longrightarrow \hat{w} = -\frac{\nu m}{\rho_0 N^2}$$

$$ik\hat{u} + il\hat{v} + im\hat{w} = 0$$

eliminate $\hat{u}$

$$\nu\hat{v} = \frac{1}{\rho_0}l\hat{p} - if\left(\frac{if}{\nu}\hat{v} + \frac{1}{\rho_0}\frac{k}{\nu}\hat{p}\right)$$

$$= \frac{1}{\rho_0}l\hat{p} + \left(\frac{+f^2}{\nu}\hat{v} - \frac{ifk}{\rho_0\nu}\hat{p}\right)$$

$$(\nu^2 - f^2)\hat{v} = \frac{\nu}{\rho_0}l\hat{p} - \frac{if}{\rho_0}k\hat{p}$$

$$\hat{v} = \frac{\nu l - ifk}{\rho_0(\nu^2 - f^2)}\hat{p}$$

eliminate $\hat{v}$

$$\nu\hat{u} = \frac{1}{\rho_0}k\hat{p} + if\left(\frac{1}{\rho_0}l\hat{p} - if\hat{u}\right)\frac{1}{\nu}$$

$$\nu^2\hat{u} = \frac{\nu}{\rho_0}k\hat{p} + if\frac{l}{\rho_0}\hat{p} + f^2\hat{u}$$

$$(\nu^2 - f^2)\hat{u} = \cdots$$

$$\hat{u} = \frac{\nu k + ifl}{\rho_0(\nu^2 - f^2)}\hat{p}$$

so now continuity gives

$$\frac{\cancel{\nu}\cancel{i}k^2 - \cancel{fkl}}{\cancel{\rho_0}(\nu^2 - f^2)} + \frac{\cancel{\nu i}l^2 + \cancel{fkl}}{\cancel{\rho_0}(\nu^2 - f^2)} - \frac{\cancel{i\nu}m^2}{\cancel{\rho_0}N^2} = 0$$

$$\frac{k^2 + l^2}{\nu^2 - f^2} = \frac{m^2}{N^2} \Rightarrow \boxed{\nu^2 = f^2 + N^2\left(\frac{k^2 + l^2}{m^2}\right) \quad \text{or} \quad m^2 = \frac{N^2(k^2 + l^2)}{\nu^2 - f^2}}$$

- for propagating waves we must have $m \in \mathbb{R}$ ($m^2 > 0$) $\nu > f$, hence the Coriolis frequency is the low ν cutoff
- for nearly hydrostatic waves: $\frac{k^2 + l^2}{m^2} \approx \frac{K_H^2}{m^2} = \frac{L_z^2}{L_x^2} \ll 1$; i.e. $L_{x,y} \gg L_z$ ($l = 0$ for simplicity)

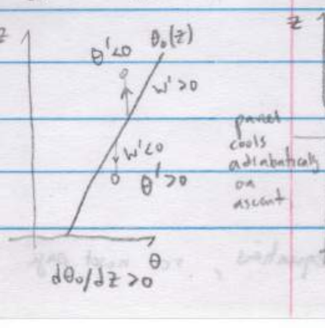
so that phase lines are nearly parallel to the ground, or α close to $\frac{\pi}{2}$ and Coriolis forces dominate over buoyancy forces

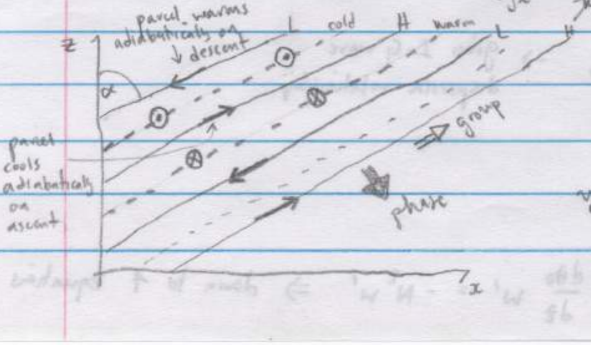
$$C_{gx} = \frac{\partial \nu}{\partial k} = \pm\frac{\partial}{\partial k}\left[f^2 + \frac{N^2}{m^2}(k^2 + l^2)\right]^{1/2} = \pm\frac{1}{\cancel{2}}\frac{N^2}{m^2}\cdot\cancel{2}k\left[f^2 + \frac{N^2}{m^2}(k^2 + l^2)\right]^{-1/2} = \pm\frac{k}{\sqrt{\nu}}\frac{N^2}{m^2}, \quad C_{gy} = \pm\frac{l}{\sqrt{\nu}}\frac{N^2}{m^2}$$

$$C_{gz} = \frac{\partial \nu}{\partial m} = \pm\frac{1}{2}\left[f^2 + \frac{N^2}{m^2}(k^2 + l^2)\right]^{-1/2}\cdot N^2(k^2 + l^2)(-\cancel{2}m^{-3}) = \pm\left(\frac{-(k^2 + l^2)}{\sqrt{\nu}}\frac{N^2}{m^3}\right)$$

letting $l = 0$ for simplicity $\quad \frac{C_{gz}}{C_{gx}} = \frac{-k^{\cancel{2}}\cancel{N^2}/\cancel{\sqrt{\nu}}\cdot m^{\cancel{3}}}{\cancel{k}\cancel{N^2}/\cancel{\sqrt{\nu}}\cdot \cancel{m^2}} = -k/m = \left[\frac{\nu^2 - f^2}{N^2}\right]^{1/2}$

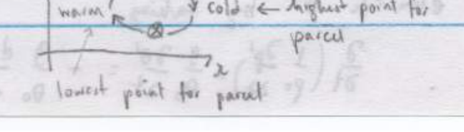
$$\frac{\partial \theta'}{\partial t} = -w'\frac{d\theta_0}{dz}$$





choosing $\nu > 0$ (positive root)

$$\vec{C_g}\cdot\vec{K} = C_{gx}k + \underset{\substack{\| \\ -k/m\, C_{gx}}}{C_{gz}}m = 0$$



Inertia - Gravity waves

Next, recall that for a homogeneous shallow fluid potential vorticity of the form $q = \zeta - f_0 \frac{\eta}{H}$ is conserved for linear motions (no advective terms).

perturbations from motionless basic state

$\bar{u}, \bar{v}, \bar{w} = 0$

$\rho = \rho_0$

$$\frac{\partial u'}{\partial t} - f v' = -g \frac{\partial \eta'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + f u' = -g \frac{\partial \eta'}{\partial y}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

$$\updownarrow$$

$$\frac{\partial \eta'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)$$

$$\int_0^{H+\eta} \frac{\partial w}{\partial z} dz = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \int_0^{H+\eta} dz$$

$$w(H) - \cancel{w(0)} = -\vec{\nabla}_H \cdot \vec{u}(H+\eta)$$

$$\uparrow$$

$$\frac{D}{Dt}(H+\eta) = \frac{D}{Dt}\eta$$

linearize to get $\rightarrow$

$$\frac{\partial}{\partial t}\zeta' + f\underbrace{\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)}_{-\frac{\partial w'}{\partial z} \text{ or } -\frac{1}{H}\frac{\partial \eta'}{\partial t}} = 0 \rightarrow \frac{\partial \zeta'}{\partial t} = f \frac{\partial w'}{\partial z}$$

after vertical integration of continuity

For the preceding case

$$\frac{\partial \theta'}{\partial t} + w' \frac{d\theta_0}{dz} = 0$$

or $w' = -\partial\theta'/\partial t \,/\, d\theta_0/dz$

and

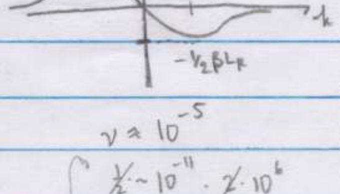
$$\frac{\partial \zeta'}{\partial t} = f \frac{\partial}{\partial z}\left(\frac{-\partial\theta'/\partial t}{d\theta_0/dz}\right)$$

I'll have to get the 5th edition of Holton

Wave frequencies

gravity: $f \le \nu \le N$, $\sim 10^{-4}\, s^{-1} \le \nu \le 10^{-2}\, s^{-1}$

Rossby: $\nu = -\dfrac{\beta k}{k^2 + 1/L_R^2}$



$l = 0$

For $l=0$

max. Rossby wave frequency

$$0 = \frac{\partial \nu}{\partial k} = \frac{-(k^2 + 1/L_R^2)\beta - (-\beta k) 2k}{(k^2 + 1/L_R^2)^2}$$

$$0 = -\beta k^2 - \beta 1/L_R^2 + 2\beta k^2$$

$$0 = \cancel{\beta} k^2 - \cancel{\beta} 1/L_R^2$$

$$k^2 = 1/L_R^2 \rightarrow k = 1/L_R$$

$$\nu = -\frac{\beta\, 1/L_R}{2/L_R^2} = -\frac{1}{2}\beta L_R$$

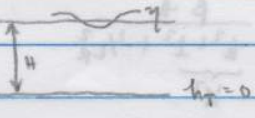
$\nu \approx 10^{-5}$

$\cancel{\frac{1}{2}} \sim 10^{-11} \cdot \cancel{2} \cdot 10^{6}$

highest Rossby wave frequency is an order of magnitude lower than that of gravity waves.

hence these are long period waves

Rossby waves



- shallow water equations

$$\partial u/\partial t + \vec{u}\cdot\vec{\nabla}u - fv = -g\,\partial\eta/\partial x$$

$$\partial v/\partial t + \vec{u}\cdot\vec{\nabla}v + fu = -g\,\partial\eta/\partial y$$

$$\frac{\partial\eta}{\partial t} + \vec{\nabla}\cdot((H+\eta)\vec{u}) = \frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}((H+\eta)u) + \frac{\partial}{\partial y}((H+\eta)v) = 0$$

consider basic state $\bar{u}, \bar{v} = 0$ f_0 = constant = f-plane

$$\cancel{\frac{\partial\bar{u}}{\partial t}}^{0} + \frac{\partial u'}{\partial t} + (\cancel{\bar{u}} + u')\frac{\partial}{\partial x}(\cancel{\bar{u}} + u') + (\cancel{\bar{v}} + v')\frac{\partial}{\partial y}(\cancel{\bar{u}} + u') - f(\cancel{\bar{v}} + v') = -g\frac{\partial\eta}{\partial x}$$

perturbation equations
$$\left\{\begin{aligned} &\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial\eta}{\partial x} \\ &\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial\eta}{\partial y} \\ &\frac{\partial\eta}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \end{aligned}\right. \Rightarrow \begin{aligned} &\eta_{ttt} + H(u_{xtt} + v_{ytt}) = 0 \\ &u_{xtt} = f_0 v_{xt} - g\eta_{xxt} \\ &v_{ytt} = -f_0 u'_{yt} - g\eta_{yyt} \end{aligned}$$

$$\eta_{ttt} - gH(\eta_{xx} + \eta_{yy})_t + f_0 H \zeta_t = 0$$

$$v_{xt} + f_0 u_{x} = -g\eta_{xy}$$

$$-u_{yt} + f_0 v_y = +g\eta_{xy}$$

$$\zeta_t + f_0\underbrace{(u_x + v_y)}_{-\frac{1}{H}\frac{\partial\eta}{\partial t}} = 0$$

$$\boxed{\frac{\partial}{\partial t}\left(\zeta - \frac{f_0\eta}{H}\right) = 0}$$

linear conservation of potential vorticity, q

$$\frac{\partial^3\eta}{\partial t^3} - gH\frac{\partial}{\partial t}\nabla^2\eta + f_0^2\frac{\partial\eta}{\partial t} = 0 \quad \text{3rd-order ODE in } \eta$$

$$\eta = \hat{\eta}e^{i\phi} = \hat{\eta}e^{i(kx+ly-\nu t)}$$

Dispersion relationship for SW inertia-gravity waves

$$\nu(\nu^2 - (f_0^2 + gH(k^2 + l^2))) = 0$$

if $\nu \neq 0$ then $\nu^2 = f_0^2 + gH(k^2 + l^2)$ $\min\{\nu\} = \pm f_0$ ($L_x, L_y \to \infty$ long waves)

if $\nu = 0$ then we have the steady geostrophic mode, which corresponds

$c_x = c_{gx} = 0$

to no divergence, geostrophic balance, non-propagating Rossby wave.

no $\beta = \frac{df}{dy}$ $\quad f_0^2 + gH(k^2 + l^2) = 0 \Rightarrow K^2 = -\frac{f_0^2}{gH} = -\frac{1}{L_R^2}$

conservation of SW potential vorticity: easy to derive from nonlinear SW equations, by taking $\hat{k}\cdot\nabla\times$ (horizontal momentum equations) combined with mass conservation

gravity waves do not possess potential vorticity q. The quasi-geostrophic formalism filters them out.

$$\frac{D}{Dt}\left(\frac{\zeta + f}{H+\eta}\right) = 0 \qquad \frac{\zeta + f}{H+\eta} = \frac{f}{H}\frac{\left(\frac{\zeta}{f}+1\right)}{\left(1 + \frac{\eta}{H}\right)} = \frac{1}{H}(\zeta + f)\left(1 - \frac{\eta}{H}\right) = \frac{1}{H}\left(\zeta - \underbrace{\frac{\zeta\eta}{H}}_{\text{small}} + f - \frac{f\eta}{H}\right)$$

assume $\zeta \ll f$, $\eta \ll H$

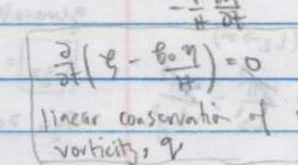
$$= \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$$

$$\zeta_g = \frac{g}{f_0}\nabla^2\eta = \nabla^2\psi', \quad \psi' = \frac{g}{f_0}\eta = \frac{\Phi}{f_0} \quad u' = -\frac{\partial\psi'}{\partial y} \quad v' = -\frac{\partial\psi'}{\partial x}$$

$$\frac{D}{Dt}\left(\zeta_g + f - \frac{f_0\eta}{H}\right) = 0$$

consider basic state zonal flow, $u = \bar{u}$, $\bar{v} = 0$ $\quad \psi = \bar{\psi} + \psi' = -\bar{u}y + \psi'$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\frac{g}{f_0}\nabla^2\eta - \frac{f_0}{H}\eta\right) + \beta v' = 0$$



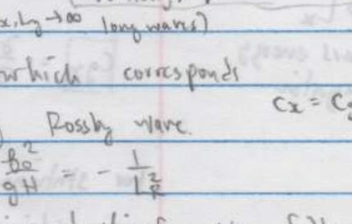
$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\nabla^2\psi' - \frac{f_0^2}{gH}\psi'\right) + \beta\frac{\partial\psi'}{\partial x} = 0; \text{ seek } \psi' = \hat{\psi}e^{i\phi} \text{ (plane wave solution)}$$

$\frac{\partial}{\partial t}\eta \sim -i\nu$

$$(-\cancel{i}\nu + \bar{u}\cancel{i}k)\left(-k^2 - l^2 - \frac{f_0^2}{gH}\right)\psi' + \beta\cancel{i}k\psi' = 0$$

$$(\nu - \bar{u}k)\left(k^2 + l^2 + \frac{1}{L_R^2}\right) + \beta k = 0$$

$$\nu = \bar{u}k - \frac{\beta k}{k^2 + l^2 + 1/L_R^2} \quad \text{Rossby wave dispersion relationship}$$



$\hat{\nu} = \nu - \bar{u}k$

same as taking $\vec{\nabla}\cdot\vec{u} = 0$ → Holton makes the rigid lid assumption, $L_R = \frac{\sqrt{gH}}{|f_0|} \to \infty$

$$c_x = \frac{\nu}{k} = \bar{u} - \beta/(k^2 + l^2 + 1/L_R^2) \quad \text{wave phase propagation westward relative to the mean zonal flow}$$

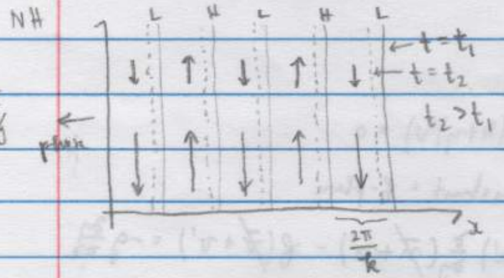
$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{1}{f}\frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial \Phi}{\partial x}\right)$$
$$= \cancel{-\frac{1}{f}\frac{\partial^2 \Phi}{\partial x \partial y}} + \cancel{\frac{1}{f}\frac{\partial^2 \Phi}{\partial x \partial y}} + \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y}\left(\frac{1}{f}\right)$$
$$= -\frac{1}{f^2}\frac{df}{dy}\frac{\partial \Phi}{\partial x} = -\frac{\beta v_g}{f_0}$$

recall: in β-plane approximation $f = f_0$ except where differentiated w.r.t. y

$$\nu = \bar{u}k - \frac{\beta k}{\underbrace{k^2 + l^2 + 1/L_R^2}_{K^2}}$$

$L_R = \frac{\sqrt{gH}}{|f|}$ barotropic Rossby radius of deformation

$l = 0 = \frac{2\pi}{L_y}$, $L_y \to \infty$



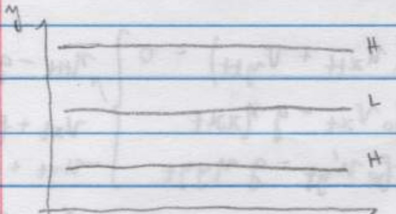
$$\vec{\nabla} \cdot \vec{v}_g = -\frac{\beta v_g}{f_0}$$

$v_g > 0$ convergence $\mathcal{C}$, ridging

$v_g < 0$ divergence $\mathcal{D}$, troughing

$$v_g = \frac{g}{f_0}\frac{\partial \eta}{\partial x} = \frac{1}{f_0}\frac{\partial \Phi}{\partial x}$$

$k = 0 = \frac{2\pi}{L_x}$, $L_x \to \infty$



wave doesn't propagate

$\nu = 0$, $C_x = 0$ $\quad (u', v') = (-\psi_y, 0)$

Steady flow

fluid parcels do not cross pv contours and hence experience no restoring force (pv gradient)

generally

$$\boxed{C_{gx}} = \frac{\partial \nu}{\partial k} = \bar{u} - \left[\frac{(K^2 + 1/L_R)\beta - \beta k(2k)}{(k^2 + l^2 + 1/L_R^2)^2}\right] = \bar{u} - \beta\left[\frac{-k^2 + l^2 + 1/L_R^2}{(K^2 + 1/L_R^2)^2}\right] = \boxed{\bar{u} + \beta\left[\frac{k^2 - l^2 - 1/L_R^2}{(K^2 + 1/L_R^2)^2}\right]}$$

$C_{gx} > 0$, if $k^2 > l^2 + 1/L_R^2$, likely when L_x small compared to L_y (see left)

rigid lid ($L_R \to \infty$)

$k^2 > l^2 \to \frac{4\pi^2}{L_x^2} > \frac{4\pi^2}{L_y^2}$

$L_y > L_x$

eastward energy propagation

$$\boxed{C_{gy}} = \frac{\partial \nu}{\partial l} = +\beta k\left[2l\right] / \left(K^2 + 1/L_R^2\right)^2 = \boxed{\frac{2\beta k l}{K^2 + 1/L_R^2}}$$

← north or southward depending on whether $l \gtrless 0$

For stationary Rossby waves, there is a balance between westward phase propagation and eastward advection by the mean flow, so the wave frequency relative to the ground $\nu = 0$.

$$\bar{u}\cancel{k} = \frac{\beta \cancel{k}}{k^2 + l^2 + 1/L_R^2} \Rightarrow \boxed{k^2 + l^2 = \frac{\beta}{\bar{u}} - \frac{1}{L_R^2}} \Rightarrow \|\vec{K}_s\| = \sqrt{\frac{\beta}{\bar{u}} - \frac{1}{L_R^2}}$$

Consider case of rigid lid: $1/L_R^2 = 0$, i.e., $L_R \to \infty$

recall L_R is the e-folding scale arising in the geostrophic adjustment problem, $L_R \to \infty$ is the case of NO rotation: PE fully → KE.

$l^2_{stationary} = \frac{\beta}{\bar{u}} - k^2$, or $K_s^2 = \frac{\beta}{\bar{u}}$

$$C_{gx\ stationary} = \bar{u} + \frac{\bar{u}(k^2 + l^2)(k^2 - l^2)}{(k^2 + l^2)^2} \Rightarrow \text{simplifying this gives } C_{gx\ stationary} = \frac{2\bar{u}k^2}{k^2 + l^2}$$

$$C_{gy\ stationary} = \frac{2\bar{u}(k^2 + l^2)kl}{(k^2 + l^2)^2} = \frac{2\bar{u}kl}{k^2 + l^2}$$

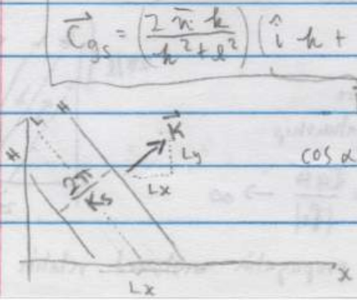
$$\boxed{\vec{C}_{g_s} = \left(\frac{2\bar{u}k}{k^2 + l^2}\right)\underbrace{(\hat{i}k + \hat{j}l)}_{\vec{K}}}$$, group/energy propagation parallel to $\vec{K}$ (perpendicular to wave crests)

$\vec{K} = \hat{i}k + \hat{j}l$

$K^2 = k^2 + l^2$

$k > 0$

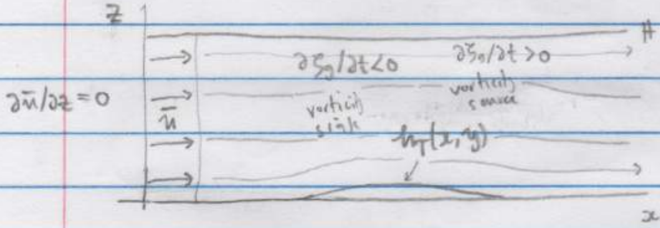
$l > 0$



$$\cos\alpha = \frac{L_x}{\sqrt{L_x^2 + L_y^2}} \to \frac{L_x/2\pi}{\frac{1}{2\pi}\sqrt{L_x^2 + L_y^2}} = \frac{1/k}{\sqrt{\left(\frac{1}{2\pi}\right)^2(L_x^2 + L_y^2)}} = \frac{1/k}{\sqrt{1/k^2 + 1/l^2}}$$

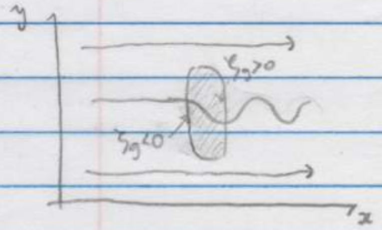
Topographic forcing of Rossby Waves

- the free barotropic Rossby wave section in Holton assumes terrain is flat, and, makes the rigid lid approximation. Continue the latter but now with $h_T(x,y) \neq 0$



$h_T \ll H$

$\zeta_g \ll f_0$



$$\frac{D}{Dt}\left[\frac{\zeta_g+f}{H-h_T}\right] = \frac{1}{H-h_T}\frac{D}{Dt}(\zeta_g+f) - \underbrace{(\zeta_g+f)}_{f_0}\frac{D}{Dt}(H-h_T)\frac{1}{(H-h_T)^2} = 0$$

$$\frac{1}{\cancel{H}}\left(\frac{\partial}{\partial t}+\vec{V}\cdot\vec{\nabla}\right)(\zeta_g+f) - \frac{1}{H\cancel{2}} f_0\left(-\frac{Dh_T}{Dt}\right) = 0$$

$u = \bar{u} + u' \qquad \zeta_g = \cancel{\bar{\zeta}}^{\,0} + \zeta'$

$v = \cancel{\bar{v}}^{\,0} + v'$

$$\left(\frac{\partial}{\partial t}\zeta_g + \vec{V}\cdot\vec{\nabla}(\zeta_g+f)\right) = -\frac{f_0}{H}\frac{Dh_T}{Dt}$$

$$\frac{\partial \zeta_g}{\partial t} + \underbrace{\vec{V}\cdot\vec{\nabla}\zeta_g}_{\bar{u}\frac{\partial}{\partial x}\zeta_g} + \beta v' = -\frac{f_0}{H}\frac{Dh_T}{Dt} = -\frac{f_0}{H}\left(\cancel{\frac{\partial h_T}{\partial t}}^{\,0} + \bar{u}\frac{\partial h_T}{\partial x}\right)$$

$$\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial}{\partial x}\zeta' + \beta v' = -\frac{f_0}{H}\bar{u}\frac{\partial h_T}{\partial x}$$

(β-plane and linearized about basic state $\bar{u}$)

consider $\bar{u} > 0$, $\partial h_T/\partial x > 0$, $f_0 > 0$ $\Rightarrow$ vorticity sink due to fluid column squashing; $\zeta_{gt} < 0$ vorticity becomes increasingly anticyclonic

Allow $h_T(x,y)$ to have the form $h_T(x,y) = \Re\{h_0 e^{ikx}\}\cos ly$ $\qquad \frac{\partial h_T}{\partial x} = ik h_0 e^{ikx}\cos ly$

and $\psi(x,y) = \Re\{\psi_0 e^{ikx}\}\cos ly$

$$\cancel{\frac{\partial \zeta_g}{\partial t}}^{\,0} + \bar{u}\cancel{i}k(-k^2-l^2)\psi + \beta\cancel{i}k\psi = -\frac{f_0}{H}\bar{u}\cancel{i}k h_0 \cancel{e^{ikx}}\cos ly$$

steady solution

$\psi = \psi_0 e^{ikx}\cos ly$

$$\left(-\bar{u}(k^2+l^2)+\beta\right)\cancel{k}\psi_0 = -\frac{f_0}{H}\bar{u}\cancel{k}h_0 \Rightarrow \psi_0 = \frac{-f_0 h_0 \cancel{\bar{u}}}{H\left(\frac{\beta}{\bar{u}} - \cancel{\bar{u}}K^2\right)} = \frac{f_0 h_0}{H(K^2-K_s^2)}$$

$\psi_0 \to \infty$ as $K \to K_s$ resonant singularity of barotropic system

ψ_0 in phase with h_0 if $K^2 > K_s^2$ $\quad \psi_0 \sim h_0$ $\quad \Rightarrow \quad L_s > L_x$

out of phase with h_0 if $K^2 < K_s^2$ $\quad \psi_0 \sim -h_0$ $\quad L_x > L_s$

This singular behavior may be removed by inserting a linear damping of ζ'

$$\frac{d\zeta'}{dt} = -r\zeta' \qquad \zeta'(0) = \zeta'(t=0)$$

$$\zeta' = \nabla^2\psi' = \zeta'(0)e^{-rt} = e^{-r(t-t_0)}$$

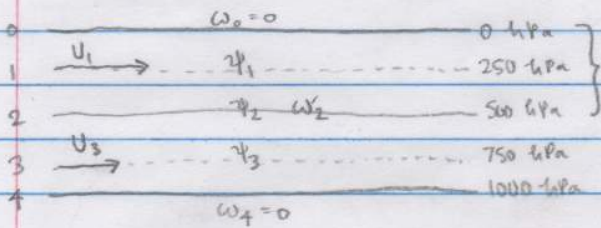
Also see Chapter 5.4

Two-layer model of baroclinic instability

$\psi = \frac{\Phi}{f_0}$ $\quad \left.\frac{\partial\omega}{\partial p}\right|_1 \approx \frac{\omega_2-\omega_0}{\delta p} = \frac{\omega_2}{\delta p} = \frac{\omega_2'}{\delta p}$

no basic state vertical motion

$\omega_2 = \cancel{\bar\omega_2}^{\,0} + \omega_2'$



$\vec v_i = \hat k \times \nabla\psi_i$ $\quad \left.\frac{\partial\omega}{\partial p}\right|_3 \approx \frac{\omega_4-\omega_2}{\delta p} = -\frac{\omega_2}{\delta p} = -\frac{\omega_2'}{\delta p}$

$\zeta_g = \nabla^2\psi = \frac{1}{f_0}\nabla^2\Phi$

$-\left.\frac{\partial\Phi}{\partial p}\right|_2 \approx \frac{\Phi_1-\Phi_3}{\delta p} = \frac{f_0(\psi_1-\psi_3)}{\delta p}$

Q-G vorticity at levels ① + ③

① $\frac{\partial}{\partial t}(\nabla^2\psi_1) + \vec v_1\cdot\vec\nabla(\nabla^2\psi_1) + \beta\frac{\partial\psi_1}{\partial x} = f_0\left.\frac{\partial\omega}{\partial p}\right|_1 = \frac{f_0}{\delta p}\omega_2$

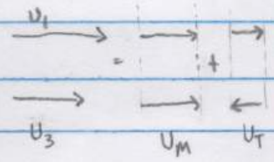
③ $\frac{\partial}{\partial t}(\nabla^2\psi_3) + \vec v_3\cdot\vec\nabla(\nabla^2\psi_3) + \beta\frac{\partial\psi_3}{\partial x} = -\frac{f_0}{\delta p}\omega_2$

downward vertical motion at 500 hPa (e.g. $\omega_2 > 0$) causes spin up of cyclonic vorticity at 250 hPa and spin down at 750 hPa

Q-G thermodynamic

② $\frac{\partial}{\partial t}\left(-\frac{\partial\Phi}{\partial p}\right) + \vec v_2\cdot\vec\nabla\left(-\frac{\partial\Phi}{\partial p}\right) - \sigma\omega = \cancel{\text{diabatic heating}}^{\,0}$

$\frac{\partial}{\partial t}(\psi_1-\psi_3) + \vec v_2\cdot\vec\nabla(\psi_1-\psi_3) - \frac{\sigma\,\delta p}{f_0}\omega_2 = 0$



basic state zonal flow + perturbation

$\psi_1 = \bar\psi_1 + \psi_1' = -U_1 y + \psi_1'$ (250 hPa)

$\psi_3 = \bar\psi_3 + \psi_3' = -U_3 y + \psi_3'$ (750 hPa)

$\psi_2 = \frac{\psi_1+\psi_3}{2} = \frac{1}{2}\left[-(U_1+U_3)y + \psi_1' + \psi_3'\right]$ (500 hPa)

$= -U_M y + \psi_2'$

$= -U_M y + \psi_M'$ is the barotropic streamfunction

another important streamfunction:

$\psi_T = \frac{\psi_1-\psi_3}{2} = \frac{1}{2}\left[-(U_1-U_3)y + (\psi_1'-\psi_3')\right]$

$= -U_T y + \psi_T'$

vertically avg'd mean zonal wind

$U_M = \frac{U_1+U_3}{2}$

$U_T = \frac{U_1-U_3}{2}$

mean thermal wind

With these expressions for ψ_1, ψ_2, ψ_3, the perturbation equations are

$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right) + U_1\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right) + \beta\frac{\partial\psi_1'}{\partial x} = \frac{f_0}{\delta p}\omega_2 \rightarrow$

$$\boxed{\begin{aligned}&\left(\frac{\partial}{\partial t} + \underbrace{U_1}_{U_M+U_T}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi_1'}{\partial x^2}\right) + \beta\frac{\partial\psi_1'}{\partial x} = \frac{f_0}{\delta p}\omega_2'\\ &\left(\frac{\partial}{\partial t} + \overbrace{U_3}^{U_M-U_T}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi_3'}{\partial x^2}\right) + \beta\frac{\partial\psi_3'}{\partial x} = -\frac{f_0}{\delta p}\omega_2'\end{aligned}}$$

similarly for the 750 hPa level:

$\vec v_2 = -\frac{\partial\psi_2}{\partial y}\hat i + \frac{\partial\psi_2}{\partial x}\hat j$

$-\frac{\partial\psi_2}{\partial y} = U_M - \frac{\partial\psi_2'}{\partial y}$

$\frac{\partial}{\partial t}\left(-\cancel{U_1 y} + \psi_1' - (-\cancel{U_3 y} + \psi_3')\right) - \frac{\partial\psi_2}{\partial y}\frac{\partial}{\partial x}(\psi_1-\psi_3) + \frac{\partial\psi_2}{\partial x}\frac{\partial}{\partial y}(\psi_1-\psi_3) - \frac{\sigma\,\delta p}{f_0}\omega_2' = 0$

$\frac{\partial}{\partial t}(\psi_1'-\psi_3') + U_M\frac{\partial}{\partial x}(\psi_1'-\psi_3') - \cancel{\frac{\partial\psi_2'}{\partial y}\frac{\partial}{\partial x}(\psi_1'-\psi_3')} + \frac{\partial\psi_M'}{\partial x}(-2U_T) - \frac{\sigma\,\delta p}{f_0}\omega_2' = 0$

thickness advection by perturbation zonal velocity small

$\left[\frac{\partial}{\partial t} + U_M\frac{\partial}{\partial x}\right](\psi_1'-\psi_3') - \frac{\partial}{\partial x}\left(\frac{\psi_1'+\psi_3'}{\cancel{2}}\right)\cancel{2}U_T - \frac{\sigma\,\delta p}{f_0}\omega_2' = 0$

$$\boxed{\left[\frac{\partial}{\partial t} + U_M\frac{\partial}{\partial x}\right](\psi_1'-\psi_3') - U_T\frac{\partial}{\partial x}(\psi_1'+\psi_3') = \frac{\sigma\,\delta p}{f_0}\omega_2'}$$

Terms in perturbation vorticity equations are easy to interpret; for perturbation thermodynamic $\psi_1'-\psi_3'$ is proportional to perturbation thickness (see above), which may change due to advection by the mean zonal wind (i.e. 500 hPa wind), vertical motion, and the other term

$\frac{\partial}{\partial t}(\psi_1'-\psi_3') = U_T\frac{\partial}{\partial x}(\psi_1'+\psi_3') = 2\underset{>0}{U_T}\,\underset{\text{meridional wind at 500 hPa (level 2)}}{\frac{\partial\psi_2'}{\partial x}}$

Since $\frac{\partial\bar T}{\partial y} < 0$ for $U_T > 0$ we see that thickness increases locally for northward v_2'

decreases locally for southward v_2'

Add vorticity equations and divide by 2

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right) + (U_m+U_T)\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right) + \beta\frac{\partial\psi_1'}{\partial x} = \cancel{\frac{f_0}{\delta p}}\,\omega_2'$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi_3'}{\partial x^2}\right) + (U_m-U_T)\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_3'}{\partial x^2}\right) + \beta\frac{\partial\psi_3'}{\partial x} = -\cancel{\frac{f_0}{\delta p}}\,\omega_2'$$

gives

$$\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\psi_1'+\psi_3'}{2}\right)\right] + U_m\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}\left(\frac{\psi_1'+\psi_3'}{2}\right)\right) + U_T\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}\left(\frac{\psi_1'-\psi_3'}{2}\right)\right) + \beta\left(\frac{\partial(\psi_1'+\psi_3')}{2\,\partial x}\right) = 0$$

governs barotropic vorticity (vertically averaged)

$$\boxed{\left[\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right]\left(\frac{\partial^2\psi_m'}{\partial x^2}\right) + U_T\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_T'}{\partial x^2}\right) + \beta\frac{\partial\psi_m'}{\partial x} = 0}$$

$\xrightarrow{U_T}$ $\zeta'>0$ $\zeta'<0$

$\longleftarrow$ $\zeta'<0$ $\zeta'>0$

$\zeta'>0$ become aligned in the vertical

Subtracting them gives

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}\left(\frac{\psi_1'-\psi_3'}{2}\right)\right) + U_m\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}\left(\frac{\psi_1'-\psi_3'}{2}\right)\right) + U_T\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}\left(\frac{\psi_1'+\psi_3'}{2}\right)\right) + \beta\frac{\partial}{\partial x}\left(\frac{\psi_1'-\psi_3'}{2}\right) = \frac{\cancel{2}f_0}{\cancel{2}\,\delta p}\,\omega_2'$$

but from the perturbation thermodynamic $\omega_2' = \frac{f_0}{\sigma\delta p}\left[\ \cdots\ \right]$

$$\left[\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right]\left(\frac{\partial^2\psi_T'}{\partial x^2}\right) + U_T\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_M'}{\partial x^2}\right) + \beta\frac{\partial\psi_T'}{\partial x} = \frac{f_0^2}{\sigma\,\delta p^2}\left[\left(\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right)2\psi_T' - U_T\frac{\partial}{\partial x}2\psi_m'\right]$$

$$\boxed{\left[\frac{\partial}{\partial t} + U_m\frac{\partial}{\partial x}\right]\left[\frac{\partial^2\psi_T'}{\partial x^2} - 2\lambda^2\psi_T'\right] + U_T\frac{\partial}{\partial x}\left[\frac{\partial^2\psi_m'}{\partial x^2} + 2\lambda^2\psi_m'\right] + \beta\frac{\partial\psi_T'}{\partial x} = 0}$$

governs baroclinic vorticity (thermal)

If the solutions are assumed to take the form of plane waves, then

$\psi_m' = Ae^{ik(x-ct)}$, $\psi_T' = Be^{ik(x-ct)}$, we obtain

first equation

$$-\cancel{ik}c\left[-k^2A\cancel{e^{i(\cdot\cdot)}}\right] + U_m\cancel{ik}\left[-k^2A\cancel{e^{i(\cdot\cdot)}}\right] + U_T\cancel{ik}\left(-k^2B\cancel{e^{i(\cdot\cdot)}}\right) + \beta\cancel{ik}A\cancel{e^{i(\cdot\cdot)}} = 0$$

$$\left[(c-U_m)k^2 + \beta\right]A + \left(-k^2U_T\right)B = 0$$

second equation

$$-\cancel{ik}c\left[-k^2B\cancel{e^{i(\cdot\cdot)}} - 2\lambda^2B\cancel{e^{i(\cdot\cdot)}}\right] + U_m\cancel{ik}\left[-k^2B\cancel{e^{i(\cdot\cdot)}} - 2\lambda^2B\cancel{e^{i(\cdot\cdot)}}\right] + U_T\cancel{ik}\left[-k^2A\cancel{e^{i(\cdot\cdot)}} + 2\lambda^2A\cancel{e^{i(\cdot\cdot)}}\right] + \beta\cancel{ik}B\cancel{e^{i(\cdot\cdot)}} = 0$$

$$\left[U_T(2\lambda^2-k^2)\right]A + \left[c(k^2+2\lambda^2) - U_m(k^2+2\lambda^2) + \beta\right]B = 0$$

$$\begin{bmatrix}(c-U_m)k^2+\beta & -k^2U_T\\ U_T(2\lambda^2-k^2) & (c-U_m)(k^2+2\lambda^2)+\beta\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

nontrivial solutions require $\det\underline{M} = 0$ where here $\underline{M}\vec{a} = \vec{0}$

in this case $x = c - U_m$

$ax^2+bx+c=0$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\left[(c-U_m)(k^2+2\lambda^2)+\beta\right]\left[(c-U_m)k^2+\beta\right] + k^2U_T^2(2\lambda^2-k^2) = 0$$

$$(c-U_m)^2k^2(k^2+2\lambda^2) + (c-U_m)(k^2+2\lambda^2)\beta + \beta(c-U_m)k^2 + \beta^2 + k^2U_T^2(2\lambda^2-k^2) = 0$$

$$\underbrace{(k^2+2\lambda^2)k^2}_{a}\left[c-U_m\right]^2 + \left[c-U_m\right]\underbrace{\beta(2k^2+2\lambda^2)}_{b} + \underbrace{\beta^2 + k^2U_T^2(2\lambda^2-k^2)}_{c} = 0$$

$$(c-U_m) = \frac{-\cancel{2}\beta(k^2+\lambda^2)}{\cancel{2}(k^2+2\lambda^2)k^2} \pm \underbrace{\sqrt{\frac{b^2-4ac}{4a^2}}}_{\delta} = \frac{-\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} \pm \left(\frac{\cancel{4}\beta^2(\cancel{k^4+2k^2\lambda^2}+\lambda^4) - \cancel{4}\,\cancel{k^2(k^2+2\lambda^2)}\left(\beta^2+k^2U_T^2(2\lambda^2-k^2)\right)}{\cancel{4}\,k^4(k^2+2\lambda^2)^2}\right)$$

$$c-U_m = \frac{-\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} \pm \underbrace{\left(\frac{\beta^2\lambda^4 - k^4U_T^2(k^2+2\lambda^2)(2\lambda^2-k^2)}{k^4(k^2+2\lambda^2)^2}\right)^{1/2}}_{\delta}$$

$c\in\mathbb{C}$ if $\delta<0$

and

$e^{ik(x-(c_r+ic_i)t)} = e^{i(kx-\underbrace{kc_r}_{\nu}t)}\,e^{\overbrace{kc_i}^{}t}$ growth rate $\alpha = kc_i$

$c = c_r + 0i$

$$\delta = \frac{\beta^2 \lambda^4}{k^4 (k^2+2\lambda^2)^2} - \frac{U_T^2 (2\lambda^2 - k^2)}{(k^2+2\lambda^2)}$$

> 0 $c \in \mathbb{R}$ propagating wave, stable

$= 0$ neutral stability

< 0 $c \in \mathbb{C}$ propagating wave if $c_r \neq 0$

$c = c_r + i c_i$ unstable

Consider case of $U_T = 0$ $(U_1 = U_3)$

Barotropic

$$c = U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} \pm \sqrt{\frac{\beta^2\lambda^4}{k^4(k^2+2\lambda^2)^2}} \quad > 0 \quad c \in \mathbb{R}$$

positive root

$$c_1 = U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} + \frac{\beta\lambda^2}{k^2(k^2+2\lambda^2)} = U_m - \frac{\beta}{k^2}\left(\frac{k^2+\cancel{2\lambda^2}}{\cancel{k^2+2\lambda^2}}\right)^{1} > 0$$

so these cases are baroclinically stable

negative root

$$c_2 = U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} - \frac{\beta\lambda^2}{k^2(k^2+2\lambda^2)} = U_m - \frac{\beta}{\cancel{k^2}}\left(\frac{\cancel{k^2}+\lambda^2-\lambda^2}{k^2+2\lambda^2}\right) = U_m - \frac{\beta}{(k^2+2\lambda^2)} > 0$$

c_1 is the barotropic Rossby wave dispersion relation for the case $\frac{D}{Dt}(\zeta + f) = 0$

$$\frac{\partial}{\partial t}\nabla^2\psi + \bar{U}\frac{\partial}{\partial x}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = 0 \rightarrow -\cancel{i}\nu(-k^2) + \bar{U}\cancel{i}k(-k^2) + \beta\cancel{i}k = 0$$

$$\nu - \bar{U}k = -\frac{\beta k}{k^2} \Rightarrow c_x = \frac{\nu}{k} = \bar{U} - \frac{\beta}{k^2}$$

fluid

c_2 is the barotropic R-wave dispersion relation without invoking a rigid-lid, or constant depth

$$\frac{D}{Dt}\left(\frac{\zeta+f}{H+\eta}\right) \rightarrow \frac{D}{Dt}\left(\zeta + f - f_0\frac{\eta}{H}\right) = 0 \Rightarrow \nu - \bar{U}k = -\frac{\beta k}{k^2 + 1/L_R^2}$$

$L_R = \frac{\sqrt{gH}}{|f_0|}$ barotropic Rossby radius of deformation

$$\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)\left(\nabla^2\psi - \frac{\psi}{L_R^2}\right) + \beta\frac{\partial\psi}{\partial x} = 0$$

except here $f_0^2/gH \Rightarrow 2\lambda^2$

this may also be considered the short wave limit

Consider case of $\beta = 0$ (f-plane dynamics), then

$$c = U_m \pm U_T\left(\frac{-(2\lambda^2 - k^2)}{(k^2+2\lambda^2)}\right)^{1/2} \quad c \in \mathbb{C} \text{ if } k^2 < 2\lambda^2 = \frac{2f_0^2}{\sigma(\delta p)^2}$$

waves longer than L_c are unstable; waves shorter than L_c stable [recall $\beta = 0$]

Therefore $k^2 = \frac{4\pi^2}{L_c^2} < 2\lambda^2$ provides the critical wavelength for instability

Recall

$$\sigma = \frac{p}{R}S_p = -\frac{p}{R}\frac{I}{\theta_0}\frac{d\theta_0}{dp} > 0$$

for stable stratification

$$L_c^2 > \frac{4\pi^2}{2\lambda^2} = \frac{2\pi^2}{\lambda^2} \Rightarrow L_c > \frac{\sqrt{2}\,\pi}{\lambda} \qquad \lambda = \frac{f_0}{\sqrt{\sigma}\,\delta p}$$

$\delta p = 500$ hPa, $f_0 = 10^{-4}\ s^{-1}$, $\sigma = 2.5 \times 10^{-6}\ m^2\ Pa^{-2}\ s^{-2}$ — typical values

$$= \frac{\sqrt{2\sigma}\,\pi\,\delta p}{f_0}$$

critical wavelength for baroclinic instability

increases w/ static stability

decreases with increasing f_0 (polar regions)

$$= \left[\sqrt{2(2.5\times10^{-6}\ \tfrac{m^2}{Pa^2 s^2})}\ (3.14159265\ldots)(50000\ Pa)\right] / 10^{-4}\ s^{-1}$$

for these values

$$\approx 3.5\times10^6\ m = 3500\ km$$

Recall $\psi \sim e^{ik(x-ct)} = e^{ik(x-(c_r+ic_i)t)} = e^{i(kx - kc_rt)}\,e^{kc_it}$

if $k^2 < 2\lambda^2$

$$c = U_m \pm i\,U_T\left(\frac{2\lambda^2 - k^2}{k^2+2\lambda^2}\right)^{1/2} = c_r \pm i\,c_i$$

$$\alpha = k\,U_T\left(\frac{2\lambda^2 - k^2}{k^2+2\lambda^2}\right)^{1/2}$$

growth rate increases linearly with mean thermal wind

greater slope implies more APE

U_T can be thought of as being proportional to meridional slope of 500 mb surface. Recall in two layer shallow water system

$$\Delta u = -\frac{g'}{f}\frac{\partial h}{\partial y}$$

For the more general case including β and U_T, the locus of conditions bearing marginal stability can be obtained by setting $\delta = 0$

$$\frac{\beta^2\lambda^4}{k^4(2\lambda^2+k^2)^{\cancel{2}}} = \frac{U_T^2(2\lambda^2-k^2)}{\cancel{(k^2+2\lambda^2)}}$$

solve for $k^4/2\lambda^4$

$$\frac{\beta^2 \cancel{2}\lambda^4}{k^4} = \cancel{2}U_T^2(2\lambda^2-k^2)(2\lambda^2+k^2)$$

$$= \cancel{2}U_T^2(\cancel{2\lambda^2k^2}+4\lambda^4-k^4-\cancel{2\lambda^2k^2})$$

$$= \cancel{2}U_T^2(4\lambda^4-k^4)$$

$$\beta^2 = U_T^2\frac{k^4}{\lambda^4}(4\lambda^4-k^4) = 4U_T^2k^4 - U_T^2\frac{(k^4)^2}{\lambda^4}$$

$$\frac{-1}{U_T^2 4\lambda^4}\beta^2 = -\frac{4}{2}\cancel{U_T^2}\left(\frac{k^4}{2\lambda^4}\right) + \cancel{U_T^2}\underbrace{\left(\frac{k^4}{2\lambda^4}\right)}_{x}{}^2$$

$$x = \frac{2 \pm \sqrt{4-4(1)\left(\frac{\beta^2}{4\lambda^4U_T^2}\right)}}{2} = 1 \pm \left(1-\frac{\beta^2}{4\lambda^4U_T^2}\right)^{1/2} = \frac{k^4}{2\lambda^4}$$

Holton plots in 8.3 the regions of stability and instability as r as a function of $k^2/2\lambda^2$, not x as he defines it... misleading

Let $r = \frac{2\lambda^2U_T}{\beta}$, $x = 1 \pm \left(1-\frac{1}{r^2}\right)^{1/2}$

$$x-1 = \pm\left(1-\frac{1}{r^2}\right)^{1/2}$$

$$(x-1)^2 = 1-\frac{1}{r^2} \Rightarrow r^2 = \frac{1}{1-(x-1)^2}$$

---

old notes

Joel (S10 217 B) says to let $x = \frac{k^2}{2\lambda^2}$, $y = \frac{2\lambda^2U_T}{\beta}$ as actually plotted in Fig. 8.3

From $\delta = 0$ → $\frac{\beta^2\lambda^4}{\cancel{k^4}} = U_T^2(4\lambda^4-k^4)\frac{k^4}{\lambda^4}$

$$\beta^2 = U_T^2 4k^4 - U_T^2\frac{(k^4)^2}{\lambda^4}$$

$4x^2\lambda^4 = k^2$

$x^4 16\lambda^8 = (k^2)^4 = (k^4)^2$

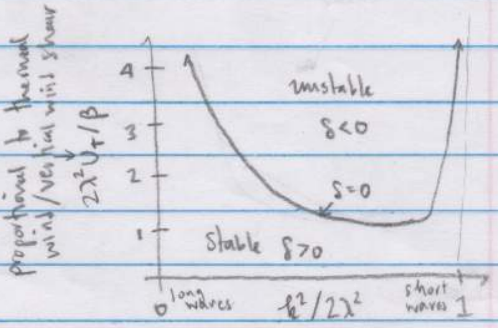
$$\beta^2 = 4U_T^2\left(\underbrace{x^2(2\lambda^2)^2}_{4\lambda^4} - \frac{(k^4)^2}{4\lambda^4}\right)$$

$$\frac{\beta^2}{4U_T^2} = \left(4x^2\lambda^4 - \frac{\cancel{4}x^4 16\lambda^{\cancel{8}4}}{\cancel{4}\cancel{\lambda^4}}\right)$$

$$\frac{\beta^2}{4U_T^2} = (4x^2\lambda^4 - 4x^4\lambda^4)$$

$$= 4\lambda^4(x^2-x^4)$$

$$\frac{\beta^2}{16U_T^2\lambda^4} = x^2(1-x^2)$$



why each region corresponds to unstable or stable perturbations is shown on the back of this page

proportional to horizontal wavenumber-squared

U_T necessary for instability depends strongly on k

$$\Rightarrow 4y^2 = [x^2(1-x^2)]^{-1}$$

or $\boxed{y^2 = \frac{1}{4x^2(1-x^2)}}$ ← gives $\delta = 0$ curve

plotted in matlab, looks correct

To see that the region above the curve is unstable

$\delta < 0 \Rightarrow$

$$\delta = \frac{\beta^2\lambda^4}{k^4(k^2+2\lambda^2)^2} - \frac{U_T^2(2\lambda^2-k^2)}{(k^2+2\lambda^2)} < 0$$

$$\frac{\beta^2\lambda^4}{k^4(k^2+2\lambda^2)^{\cancel{2}}} < \frac{U_T^2(2\lambda^2-k^2)}{\cancel{(k^2+2\lambda^2)}}$$

$$\frac{\beta^2\lambda^4}{k^4} < U_T^2(2\lambda^2-k^2)(k^2+2\lambda^2)$$

$x = \frac{k^2}{2\lambda^2}$

$$\beta^2 < U_T^2(4\lambda^4-k^4)\frac{k^4}{\lambda^4}$$

$y = \frac{2\lambda^2 U_T}{\beta}$

$$\frac{\beta^2}{U_T^2} < 4k^4 - \frac{1}{\lambda^4}(k^4)^2$$

$k^4 = x^2 4\lambda^4$

$4k^4 = x^2 16\lambda^4$

$(k^4)^2 = x^4 16\lambda^8$

$$\frac{\beta^2}{U_T^2} < 16x^2\lambda^4 - x^4 16\lambda^{\overbrace{8-4}^{4}}$$

$$\frac{\beta^2}{16\lambda^4U_T^2} < x^2 - x^4$$

$$\frac{16\lambda^4U_T^2}{\beta^2} > \frac{1}{x^2-x^4}$$

$$y^2 > \frac{1}{4x^2(1-x^2)} \quad \text{unstable}$$

Vertical Motion in Baroclinic waves

continuation of analysis of the two-layer model

$\omega_0 = 0$ — 0
— 1
ω_2 — 2
— 3
$\omega_4 = 0$ — 4

Apply Q-G omega equation at layer 2 (500-hPa)

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g\cdot\vec{\nabla}\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] - \frac{1}{\sigma}\nabla^2\left[-\vec{V}_g\cdot\vec{\nabla}\left(-\frac{\partial\Phi}{\partial p}\right)\right] - \cancel{\frac{\kappa}{p}J}$$

$$= -\frac{f_0}{\sigma}\left[-\frac{\partial\vec{V}_g}{\partial p}\cdot\vec{\nabla}(\zeta_g + f) - \vec{V}_g\cdot\vec{\nabla}\frac{\partial}{\partial p}(\zeta_g + f)\right] - \frac{1}{\sigma}\nabla^2[\cdots]$$

$\frac{\partial \zeta_2}{\partial p} = 0$

$\omega \rightarrow \omega_2'$ $\qquad$ $\nabla^2 \rightarrow \frac{\partial^2}{\partial x^2}$

$$\left.\frac{\partial^2\omega}{\partial p^2}\right|_2 = \frac{\left.\frac{\partial\omega}{\partial p}\right|_3 - \left.\frac{\partial\omega}{\partial p}\right|_1}{\delta p} = \frac{\frac{\cancel{\omega_4} - \omega_2}{\delta p} - \frac{\omega_2 - \cancel{\omega_0}}{\delta p}}{\delta p} = -\frac{2\omega_2}{\delta p^2}$$

$\Phi = f_0 \psi$

$$\frac{\Phi_1 - \Phi_3}{\delta p} = -\frac{\partial\Phi}{\partial p} = \frac{f_0}{\delta p}(\psi_1 - \psi_3)$$

$$\left.-\frac{\partial\vec{V}_g}{\partial p}\right|_2 = \frac{U_1 - U_3}{\delta p}\hat{i} + \frac{v_1 - v_3}{\delta p}\hat{j} = \frac{2}{\delta p}\left(-\frac{\partial\psi_T}{\partial y}\right)\hat{i} + \frac{2}{\delta p}\frac{\partial}{\partial x}\psi_T'\,\hat{j}$$

recall: $\psi_T = \overline{\psi}_T + \psi_T' = -\underbrace{\frac{(U_1 - U_3)}{2}}_{U_T}y + \frac{\psi_1' - \psi_3'}{2}$ and $v_i = \frac{\partial\psi_i'}{\partial x}$ $(i = 1,3)$

The Q-G omega equation discretizes to

$$\left(\frac{\partial^2}{\partial x^2} - \frac{f_0^2}{\sigma}\frac{2}{(\delta p)^2}\right)\omega_2' = -\frac{f_0}{\sigma}\left[\frac{2}{\delta p}U_T\frac{\partial\zeta_2}{\partial x} + \frac{2}{\delta p}\frac{\partial\psi_T'}{\partial x}\left(\frac{\partial\zeta_2}{\partial y} + \beta\right)\right] + \frac{1}{\sigma}\frac{\partial^2}{\partial x^2}\left[U_m\frac{\partial}{\partial x}\frac{f_0}{\delta p}\overbrace{(\psi_1 - \psi_3)}^{2\psi_T} + \frac{\partial\psi_m'}{\partial x}\frac{\partial}{\partial y}\frac{f_0}{\delta p}\overbrace{(\psi_1 - \psi_3)}^{2\psi_T}\right]$$

$\lambda^2 = \frac{f_0^2}{\sigma(\delta p)^2}$

$$= -\frac{2f_0}{\sigma\delta p}U_T\frac{\partial\zeta_2}{\partial x} \underbrace{- \frac{2f_0}{\sigma\delta p}\frac{\partial\psi_T'}{\partial x}\left(\frac{\partial\zeta_2}{\partial y} + \beta\right)}_{\text{small}} + \underbrace{\frac{2f_0}{\sigma\delta p}\frac{\partial^2}{\partial x^2}U_m\frac{\partial\psi_T}{\partial x}}_{\text{small}} + \frac{2f_0}{\sigma\delta p}\frac{\partial^2}{\partial x^2}\underbrace{\frac{\partial\psi_m'}{\partial x}}_{v_{500}'}\underbrace{\frac{\partial\psi_T}{\partial y}}_{-U_T}$$

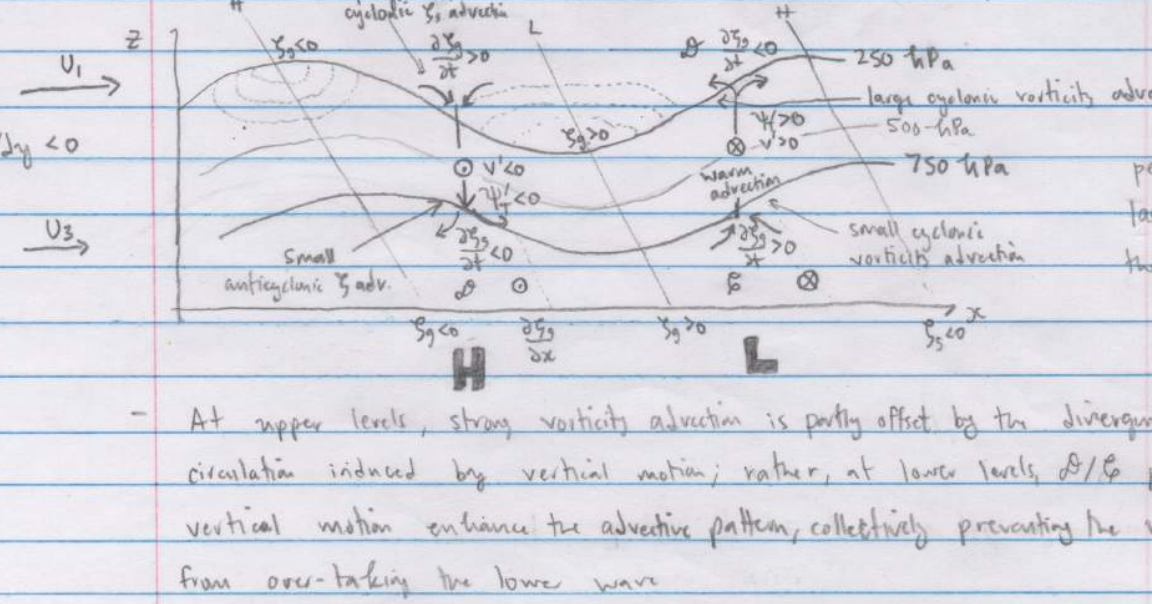
$$\frac{\partial\zeta_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial v'}{\partial x}\right)$$

$$\left[\frac{\partial^2}{\partial x^2} - 2\lambda^2\right]\omega_2' = -\frac{4f_0}{\sigma\delta p}\underbrace{U_T}_{>0}\frac{\partial\zeta_2}{\partial x} + \text{higher order terms}$$

$U_T \propto -\frac{d\overline{T}}{dy}$

$$-\omega_2' \propto -U_T\frac{\partial\zeta_2'}{\partial x} \propto \frac{d\overline{T}}{dy}\frac{\partial^2}{\partial x^2}v_2' \propto -v_2'\frac{d\overline{T}}{dy}$$

or simply $\omega_2' \propto v_2'\underbrace{\frac{d\overline{T}}{dy}}_{<0}$ $\Big\}$ upward motion where $v_2' > 0$, northward; downward motion when $v_2' < 0$, southward



perturbation thickness field lags $\frac{\pi}{2}$ in phase with the geopotential height field

- At upper levels, strong vorticity advection is partly offset by the divergent ageostrophic circulation induced by vertical motion; rather, at lower levels, $\mathcal{D}/\mathcal{C}$ pattern and vertical motion enhance the advective pattern, collectively preventing the upper level wave from over-taking the lower wave
- Stratification strengthened (weakened) in lower troposphere by $\omega' > 0$ ($\omega' < 0$) over the surface high (low)

Energy equations for the 2-layer model

• K' = perturbation kinetic energy of entire system

$= \frac{1}{2}(v_1'^2 + v_3'^2)$

multiply vorticity' equation by $-\psi_1'$ and average over a wave : $\overline{(\ \ )} = \frac{1}{L}\int_0^L (\ \ )\,dx$

250-hPa : $-\psi_1 \frac{\partial}{\partial t}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right) - \psi_1 U_1 \frac{\partial}{\partial x}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right) - \psi_1'\beta\frac{\partial\psi_1'}{\partial x} = -\psi_1\frac{f_0}{\delta p}\omega_2'$

first term $-\frac{1}{L}\int_0^L \psi_1'\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\frac{\partial\psi_1'}{\partial x}\right)\right)dx = -\frac{1}{L}\left[\overbrace{\cancel{\psi_1'\frac{\partial}{\partial t}\left(\frac{\partial\psi_1'}{\partial x}\right)\Big|_0^L}}^{0\text{ periodic}} - \int_0^L \frac{\partial\psi_1'}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial\psi_1'}{\partial x}\right)dx\right] = \frac{1}{L}\int_0^L\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial\psi_1'}{\partial x}\right)^2\right]dx$

Second term $-\frac{U_1}{L}\int_0^L \psi_1'\frac{\partial}{\partial x}\left(\frac{\partial^2\psi_1'}{\partial x^2}\right)dx = -\frac{U_1}{L}\left[\overbrace{\cancel{\psi_1'\frac{\partial^2\psi_1'}{\partial x^2}\Big|_0^L}}^{0\text{ periodic}} - \int_0^L \frac{\partial\psi_1'}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\psi_1'}{\partial x}\right)dx\right] = \frac{U_1}{L}\int_0^L\frac{\partial}{\partial x}\left[\frac{1}{2}\left(\frac{\partial\psi_1'}{\partial x}\right)^2\right]dx = 0$

third term $-\beta\int_0^L \psi_1'\frac{\partial\psi_1'}{\partial x}dx = -\beta\int_0^L\frac{\partial}{\partial x}\left(\frac{1}{2}\psi_1'^2\right)dx = 0$

$\rightarrow \frac{\partial}{\partial t}\overline{\left[\frac{1}{2}\left(\frac{\partial\psi_1'}{\partial x}\right)^2\right]} = -\frac{f_0}{\delta p}\overline{\omega_2'\psi_1'}$, The results for 750-hPa are straightforward to see from this.

Adding the results for both layers gives

$$\frac{\partial}{\partial t}\frac{1}{2}\left[\overline{\left(\frac{\partial\psi_1'}{\partial x}\right)^2} + \overline{\left(\frac{\partial\psi_3'}{\partial x}\right)^2}\right] = \frac{f_0}{\delta p}\overline{\omega_2'(\psi_3'-\psi_1')} = -\frac{2f_0}{\delta p}\overline{\omega_2'\psi_T'}$$

$$\boxed{\frac{\partial K'}{\partial t} = -\frac{2f_0}{\delta p}\overline{\omega_2'\psi_T'}}$$

K' produced by : upward motion where thickness large
: downward motion where thickness small
↓
P' used up

• $P' = \frac{\lambda^2}{2}(\psi_1'-\psi_3')^2$ multiply perturbation thermodynamic equation by $(\psi_1'-\psi_3')$ + avg. over wave

$$(\psi_1'-\psi_3')\frac{\partial}{\partial t}(\psi_1'-\psi_3') + U_m(\psi_1'-\psi_3')\frac{\partial}{\partial x}(\psi_1'-\psi_3') - U_T(\psi_1'-\psi_3')\frac{\partial}{\partial x}(\psi_1'+\psi_3') = \frac{\sigma\,\delta p}{f_0}\omega_2'\underbrace{(\psi_1'-\psi_3')}_{2\psi_T}$$

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{(\psi_1'-\psi_3')^2} + \cancel{\frac{U_m}{L}\int_0^L\frac{\partial}{\partial x}\left[\frac{1}{2}(\psi_1'-\psi_3')^2\right]dx} - U_T\overline{(2\psi_T')\frac{\partial\psi_m'}{\partial x}}\cdot 2 = \dots$$ next multiply through by λ^2

$$\boxed{\frac{\partial P'}{\partial t} = 4U_T\lambda^2\overline{\psi_T'\underbrace{\frac{\partial\psi_m'}{\partial x}}_{v_2'}} + \frac{2f_0}{\delta p}\overline{\omega_2'\psi_T'}}$$

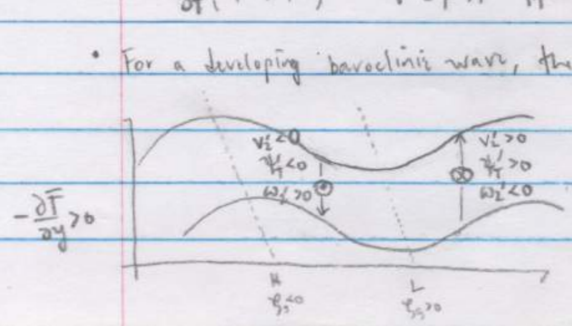
From these equations it's obvious that $\overline{\omega_2'\psi_T'} < 0$, corresponding to downward motion ($\omega_2' > 0$) where the perturbation thickness is low ($\psi_T' < 0$), and vice versa, represents a conversion of perturbation potential into kinetic energy.

$$\frac{\partial}{\partial t}(P'+K') = 4U_T\lambda^2\overline{\psi_T'\frac{\partial\psi_m'}{\partial x}} = \overbrace{4U_T\lambda^2\overline{\psi_T'v_2'}}^{>0}$$

ψ_T' ~ 500 hPa temp. OR 250–750 thickness

v_2' ~ 500 hPa meridional velocity

• For a developing baroclinic wave, the two terms that influence perturbation potential energy development are always opposite in sign.



• The perturbation total energy of the disturbance increases from poleward transport of warm air, $\overline{\underset{>0}{v_2'}\,\underset{>0}{\psi_T'}} > 0$ and equatorward transport of cold air $\overline{\underset{<0}{\psi_T'}\,\underset{<0}{v_2'}} > 0$

$\delta = kx_0$
phase shift to the left if $x_0 > 0$

Let $\psi_m' = A_m \cos(kx - kct)$, $\psi_T' = A_T \cos(kx + kx_0 - kct)$

$$\overline{\psi_T' \frac{\partial \psi_m'}{\partial x}} = \frac{1}{L} A_m A_T (-k) \int_0^L \sin(\underbrace{kx - kct}_{a}) \cos(\underbrace{kx - kct}_{a} + \underbrace{kx_0}_{b})\, dx$$

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$$-\frac{k A_m A_T}{L} \int_0^L [\sin(a)\cos(a)\cos(b) - \sin^2(a)\sin(b)]\, dx$$

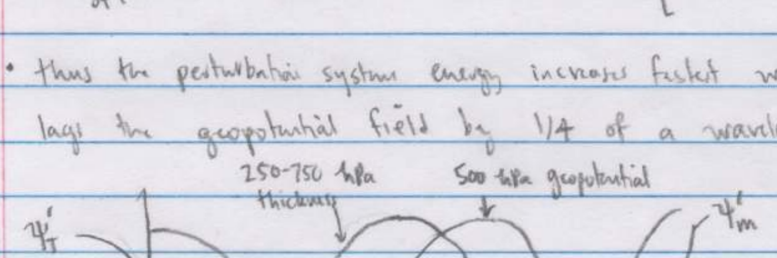
$$= \frac{k A_m A_T \sin(kx_0)}{L} \int_0^L \overbrace{\sin^2(kx - kct)\, dx}^{1/2} - \frac{k A_m A_T \cos(kx_0)}{L} \int_0^L \sin(kx - kct)\cos(kx - kct)\, dx$$

$$= \frac{k A_m A_T \sin(kx_0)}{2L} - \ldots$$

$$\Rightarrow \frac{\partial}{\partial t}[P' + K'] = 4 U_T \lambda^2 \overline{\psi_T' \frac{\partial \psi_m'}{\partial x}} = \frac{2 U_T \lambda^2 k A_m A_T \sin(kx_0)}{L}$$

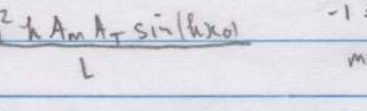
$-1 \le \sin\phi \le 1$
max[$\sin\phi$] occurs for $\phi = \frac{\pi}{2}$

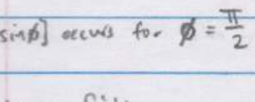
- thus the perturbation system energy increases fastest when the thickness field lags the geopotential field by 1/4 of a wavelength, $\pi/2$



- note that the disturbance energy growth is directly proportional to the basic state thermal wind, which itself is proportional to the slope of θ surfaces

$$\langle APE \rangle = \frac{1}{2} \int_0^{P_0} \frac{\overline{\theta'^2}}{\Theta_0^2} \frac{g}{N^2}\, dp$$, Here $\theta' \rightarrow$ is on a p-surface

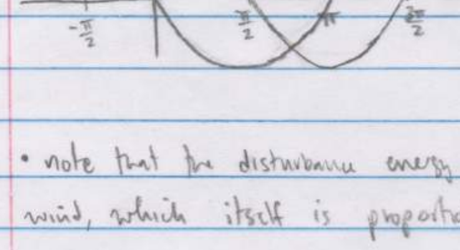


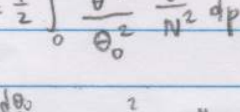


Recall $N^2 = \frac{g}{\theta_0}\frac{d\theta_0}{dz}$. N^2 smaller for dotted lines since $d\theta_0/dz$ is smaller; i.e. the stratification is weaker, and in this case the available potential energy is higher, as the system minimum possible potential energy is attained when cold air sits below warm and θ-surfaces are flat. Conversely, N^2 large implies more stratification and less less meridional heat transport needed to flatten isentropes.

$\frac{\partial \theta_0}{\partial z} > 0$

$\frac{\partial \theta_0}{\partial y} < 0$





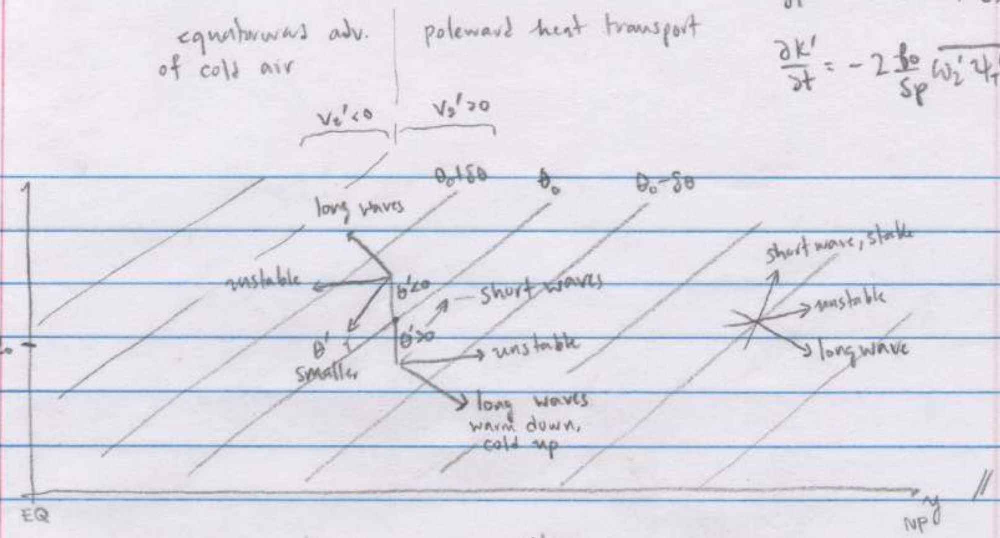
$\leftarrow$ No APE although it does possess gravitational potential energy

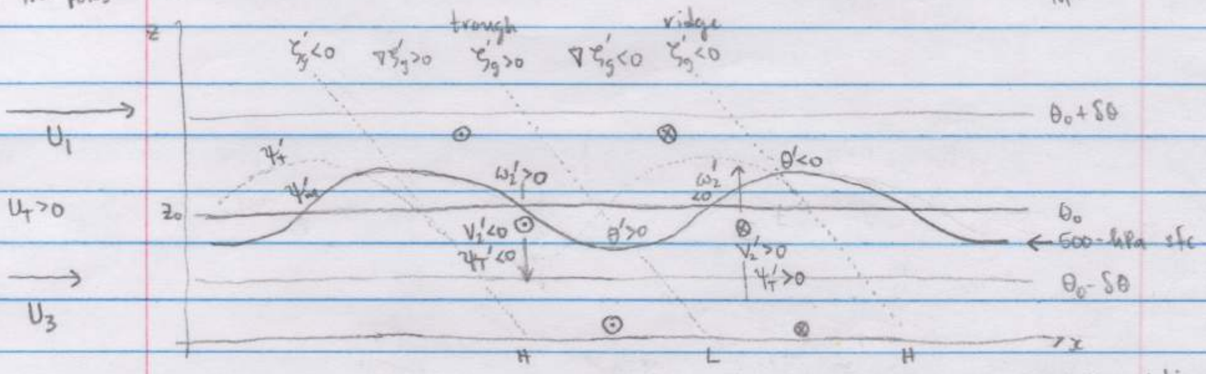
$$\frac{\partial}{\partial t} P' = 4 U_T \lambda^2 \overline{\psi_T' \frac{\partial \psi_m'}{\partial x}} + 2 \frac{f_0}{\delta p} \overline{\omega_2' \psi_T'}$$

$$\frac{\partial K'}{\partial t} = -2 \frac{f_0}{\delta p} \overline{\omega_2' \psi_T'}$$

$\nabla \theta_0 = \left(\cancel{\frac{\partial \theta_0}{\partial x}}, \frac{\partial \theta_0}{\partial y}, \frac{\partial \theta_0}{\partial z} \right)$ with $\frac{\partial \theta_0}{\partial x} = 0$, $\frac{\partial \theta_0}{\partial y} < 0$, $\frac{\partial \theta_0}{\partial z} > 0$

Basic state stratification ~~is statically stable~~ and is warm in the tropics and cold at the poles





ψ_T' field lags ψ_m' by $\pi/2$, $1/4\, L_x$

← perturbation relative vorticity advection

For short wave $\frac{\partial \zeta}{\partial x}$ is very large, + since $U_3 < U_1$, $-U \frac{\partial \zeta'}{\partial x}$ increases strongly with height which requires a stronger vertical circulation to maintain hydrostatic/geostrophic balance.

So, vertical parcel displacements will become resisted by static stability ($N^2 > 0$) for waves shorter than L_c.

In this case the poleward/upward warm air advection and equatorward/downward advection of cold air will be resisted by buoyancy. This case is similar to $\beta = 0$ since

$$\underbrace{\partial/\partial t\, \nabla^2 \psi + U \frac{\partial}{\partial x} \nabla^2 \psi}_{\text{these terms dominate}} + \underbrace{\beta \frac{\partial \psi}{\partial x}}_{\text{neglected}} = \cdots$$

Log-pressure coordinates

Recall that for an isothermal atmosphere $\frac{dp}{dz} = -\rho g = -\frac{pg}{RT} = -\frac{p}{H} \Rightarrow p = p_s e^{-z/H} \to \ln\left(\frac{p}{p_s}\right) = -\frac{z}{H}$.

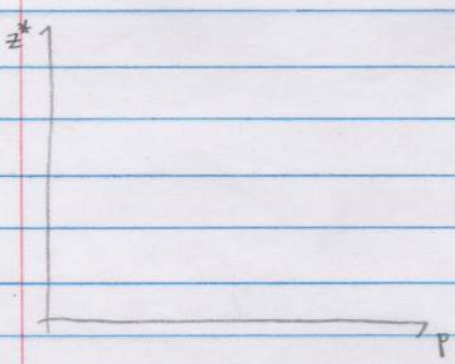
$\boxed{z^* = -H\ln\left(\frac{p}{p_s}\right)}$ and $\boxed{H = \frac{R\bar{T}}{g}}$ where $\bar{T}$ denotes a global avg,
defines a system of coordinates in which log-pressure and geopotential height coincide if the atmosphere has constant temperature throughout
but of course, in reality $-\partial T_0/\partial y > 0$ $-\partial T_0/\partial z > 0$ are typical

$\boxed{w^* = \frac{Dz^*}{Dt} = -\frac{H}{p}\frac{Dp}{Dt} = -\frac{H}{p}\omega}$

$\boxed{dz^* = -H\,d\ln p}$

$$\frac{\partial\omega}{\partial p} = \frac{\partial}{\partial p}\left(-\frac{p}{H}w^*\right) = -\frac{w^*}{H} + \frac{\partial w^*}{\partial p} = -\frac{w^*}{H} + \left(-\frac{p}{H}\frac{\partial w^*}{\partial p}\right) = -\frac{w^*}{H} + \frac{\partial w^*}{\partial z^*}$$

thus $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p}$ becomes $\boxed{\frac{\partial\omega}{\partial p} = \frac{\partial w^*}{\partial z^*} - \frac{w^*}{H}}$



$p(z) = \rho_0(z)RT$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w^*}{\partial z^*} - \frac{w^*}{H} = 0}$$ continuity, or conservation of mass

This can also be reached by examination of

$$\frac{1}{\rho_0}\frac{\partial}{\partial z^*}(\rho_0 w^*) = \frac{\partial w^*}{\partial z^*} + \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z^*}w^*, \text{ but } \frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z^*} = \frac{-1}{\rho_0 H}\frac{\partial \rho_0}{\partial \ln p}$$

$$= -\frac{p}{\rho_0 H}\frac{\partial}{\partial p}\left(\frac{p}{RT}\right) = -\frac{1}{H} \Rightarrow$$ same result as above

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z^*}(\rho_0 w^*) = 0}$$

$\frac{dp}{dz} = -\rho g \Rightarrow \frac{dp}{dz} = -\frac{p}{RT}g$

$\Rightarrow \frac{g\,dz}{dp} = -\frac{RT}{p}$

Hydrostatic: $\frac{\partial\Phi}{\partial z^*} = -\frac{\partial\Phi}{H\,\partial\ln p} = -\frac{p}{H}\frac{\partial\Phi}{\partial p} = \frac{\cancel{p}}{H}\left(\frac{RT}{\cancel{p}}\right) = \frac{RT}{H} \Rightarrow \boxed{\frac{\partial\Phi}{\partial z^*} = \frac{RT}{H}}$

Zonal Mean Atmospheric Circulation (Holton Ch. 10)

The governing equations in log-p coordinates are:

$$\frac{Du}{Dt} - fv = -\frac{\partial\Phi}{\partial x} + X, \quad \frac{Dv}{Dt} + fu = -\frac{\partial\Phi}{\partial y} + Y, \quad \frac{\partial\Phi}{\partial z} = \frac{RT}{H}$$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w) = 0$, and still need the log-pressure thermodynamics

The thermodynamic equation: $c_p\frac{DT}{Dt} - \alpha\overbrace{\frac{Dp}{Dt}}^{\omega} = J \Rightarrow c_p\frac{DT}{Dt} + \overset{\frac{RT}{p}}{\cancel{\alpha}}\frac{p}{H}w^* = J$

$$\frac{DT}{Dt} + \underbrace{\frac{R}{c_p}}_{\kappa}\frac{T}{H}w^* = \frac{J}{c_p} \to \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \underbrace{\cancel{\omega}}_{-\frac{p}{H}w^*}\frac{\partial T}{\partial p} + \frac{\kappa T}{H}w^* = \frac{J}{c_p}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \underbrace{\left(\frac{\kappa T}{H} + \frac{\partial T}{\partial z^*}\right)}_{\frac{H}{R}N^2}w^* = \frac{J}{c_p} \Rightarrow \boxed{\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \frac{H}{R}N^2 w^* = \frac{J}{c_p}}$$

where $N^2 = \frac{R}{H}\left(\frac{\kappa T}{H} + \frac{\partial T}{\partial z^*}\right)$

First Law of Thermodynamics in enthalpy form

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J$$

$$\cancel{c_p}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}\right) - \frac{\alpha\omega}{c_p} = \frac{J}{c_p}$$

$$-\frac{\partial \Phi}{\partial p} = \frac{RT}{p}$$

$$\left(\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T\right) - \underbrace{\left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p}\right)}_{S_p}\omega = \frac{J}{c_p}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(-\frac{p}{R}\frac{\partial \Phi}{\partial p}\right) - \left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p}\right)\omega = \frac{J}{c_p}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(-\frac{p}{H}\frac{\partial \Phi}{\partial p}\right) - \frac{R}{H}\left(\frac{RT}{pc_p} - \frac{\partial T}{\partial p}\right)\omega = \frac{J}{c_p}\frac{R}{H}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(\frac{\partial \Phi}{\partial z^*}\right) - \frac{R}{H}\left(\frac{RT}{pc_p} - \frac{\partial T}{\partial p}\right)\left(-\frac{p}{H}w^*\right) = \frac{JR}{c_p H}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(\frac{\partial \Phi}{\partial z^*}\right) + \frac{R}{H}\left(\frac{RT}{Hc_p} - \frac{p}{H}\frac{\partial T}{\partial p}\right)w^* = \frac{JR}{c_p H}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(\frac{\partial \Phi}{\partial z^*}\right) + \underbrace{\frac{R}{H}\left(\frac{\kappa T}{H} + \frac{\partial T}{\partial z^*}\right)}_{N^2}w^* = \frac{JR}{c_p H}$$

$$\boxed{\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(\frac{\partial \Phi}{\partial z^*}\right) + N^2 w^* = \frac{JR}{c_p H}}$$

or

$$\frac{DT}{Dt} + \frac{\kappa T}{H}w^* = \frac{J}{c_p}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}, \qquad \omega = -\frac{p}{H}w^*$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \underbrace{\frac{\kappa T}{H}w^* + \frac{\partial T}{\partial z^*}w^*}_{\left(\frac{\kappa T}{H} + \frac{\partial T}{\partial z^*}\right)w^*} = \frac{J}{c_p}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(\frac{RT}{H}\right) + N^2 w^* = \frac{JR}{c_p H}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)T + \frac{H}{R}N^2 w^* = \frac{J}{c_p}$$

Zonal average of zonal momentum equation:

$$\overline{(\ )} = \frac{1}{L}\int_0^L (\ )\,dx \qquad L = 2\pi r$$

$$\overline{ab} = \overline{(\bar{a}+a')(\bar{b}+b')} = \overline{\bar{a}\bar{b} + \bar{a}b' + a'\bar{b} + a'b'} = \bar{a}\bar{b} + \overline{a'b'} \quad \text{because } \overline{a'} = \overline{b'} = 0$$

flux form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w^*\frac{\partial u}{\partial z^*} + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w^*}{\partial z^*} - \frac{w^*}{H}\right) = +fv - \frac{\partial \Phi}{\partial x} + X$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z^*}(uw^*) - \frac{uw^*}{H} = +fv - \frac{\partial \Phi}{\partial x} + X,$$

drop *'s on w and z, write $\bar{a}+a'$, now take the zonal avg

$$\frac{\partial}{\partial t}\overline{(\bar{u}+\cancel{u'})} + \cancel{\frac{\partial}{\partial x}}\overline{(u^2)} + \frac{\partial}{\partial y}\overline{((\bar{u}+u')(\bar{v}+v'))} + \frac{\partial}{\partial z}\overline{((\bar{u}+u')(\bar{w}+w'))} - \frac{\overline{(\bar{u}+u')(\bar{w}+w')}}{H} = f\overline{(\bar{v}+\cancel{v'})} - \cancel{\frac{\overline{\partial\Phi}}{\partial x}} + \bar{X}$$

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial y}\left(\cancel{\bar{u}\bar{v}} + \overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\cancel{\bar{u}\bar{w}} + \cancel{\overline{u'w'}}\right) - \frac{\bar{u}\bar{w}}{H} - \cancel{\frac{\overline{u'w'}}{H}} = f\bar{v} + \bar{X}$$

small small small small small

$$\boxed{\frac{\partial \bar{u}}{\partial t} - f\bar{v} = -\frac{\partial}{\partial y}\left(\overline{u'v'}\right) + \bar{X}}$$

Zonal avg meridional momentum equation

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + v\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - \frac{w}{H}\right) = -fu - \frac{\partial \Phi}{\partial y} + Y$$

$$\frac{\partial}{\partial t}(\bar{v}+v') + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2) + \frac{\partial}{\partial z}(vw) - \frac{vw}{H} = -fu - \frac{\partial \Phi}{\partial y} + Y$$

$$\cancel{\frac{\partial}{\partial t}}\bar{v} + \frac{\partial}{\partial x}(\cancel{\bar{u}\bar{v}}) + \frac{\partial}{\partial x}(\cancel{\overline{u'v'}}) + \underbrace{\frac{\partial}{\partial y}\overline{(\bar{v}^2)} + \frac{\partial}{\partial y}\overline{(v'^2)} + \frac{\partial}{\partial z}(\bar{v}\bar{w}) + \frac{\partial}{\partial z}\overline{(v'w')} - \frac{\bar{v}\bar{w}}{H} - \frac{\overline{v'w'}}{H}}_{\text{small}} = -f\bar{u} - \frac{\partial \bar{\Phi}}{\partial y} + \cancel{\bar{Y}}$$

small 0 0 small?

$$\boxed{f_0\bar{u} = -\frac{\partial \bar{\Phi}}{\partial y}}$$

zonal average thermodynamic

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)T + \frac{H}{R}N^2 w^* = \frac{J}{c_p}$$

$$\frac{\partial}{\partial t}T + \vec{v}\cdot\vec{\nabla}T + T\left(\vec{\nabla}\cdot\vec{v} + \frac{\partial w^*}{\partial z^*} - \frac{w^*}{H}\right) + \frac{H}{R}N^2 w = \frac{J}{c_p}$$, drop stars

$$\overline{\frac{\partial}{\partial t}T} + \overline{\cancel{\frac{\partial}{\partial x}(uT)}} + \overline{\frac{\partial}{\partial y}(vT)} + \overline{T\frac{\partial w}{\partial z}} - \overline{\frac{Tw}{H}} + \frac{H}{R}\overline{N^2 w} = \overline{\frac{J}{c_p}}$$

$$\frac{\partial}{\partial t}\overline{(\bar{T}+\cancel{T'})} + \frac{\partial}{\partial y}(\bar{v}\bar{T}) + \frac{\partial}{\partial y}\overline{(v'T')} + \overline{(\bar{T}+T')\frac{\partial}{\partial z}(\bar{w}+w')} + \frac{\overline{(\bar{T}+T')(\bar{w}+w')}}{H} + \frac{H}{R}N^2\bar{w} = \frac{\bar{J}}{c_p}$$

$$\frac{\partial \bar{T}}{\partial t} + \underbrace{\bar{v}\frac{\partial \bar{T}}{\partial y}}_{\text{small}} + \underbrace{\bar{T}\frac{\partial \bar{v}}{\partial y}}_{\text{small}} + \frac{\partial}{\partial y}\overline{(v'T')} + \underbrace{\bar{T}\frac{\partial \bar{w}}{\partial z}}_{\text{small}} + \underbrace{\overline{T'\frac{\partial w'}{\partial z}}}_{\text{small}} + \underbrace{\cancel{\frac{\bar{T}\bar{w}}{H}}}_{\text{small}} + \underbrace{\cancel{\frac{\overline{T'w'}}{H}}}_{\text{small}} + \frac{H}{R}N^2\bar{w} = \frac{\bar{J}}{c_p}$$

$$\boxed{\frac{\partial \bar{T}}{\partial t} + \frac{H}{R}N^2\bar{w} = -\frac{\partial}{\partial y}\overline{(v'T')} + \bar{J}/c_p}$$

Recap: Zonal average of governing equations scaled quasi-geostrophically

zonal momentum: $\frac{\partial \bar{u}}{\partial t} - f_0\bar{v} = -\frac{\partial}{\partial y}\overline{(u'v')} + \bar{X}$

meridional momentum: $+f_0\bar{u} = -\partial\bar{\Phi}/\partial y$

hydrostatic: $\frac{\partial\bar{\Phi}}{\partial z} = \frac{R\bar{T}}{H}$

$$f_0\frac{\partial\bar{u}}{\partial z} = -\frac{\partial}{\partial y}\left(\frac{\partial\bar{\Phi}}{\partial z}\right) = -\frac{R}{H}\frac{\partial\bar{T}}{\partial y} \Rightarrow \boxed{f_0\frac{\partial\bar{u}}{\partial z} + \frac{R}{H}\frac{\partial\bar{T}}{\partial y} = 0}$$

mass conservation: $\frac{\partial}{\partial y}\bar{v} + \rho_0^{-1}\frac{\partial}{\partial z}(\rho_0\bar{w}) = 0$

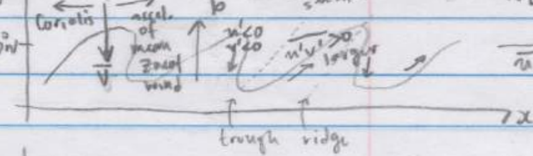
thermodynamic: $\frac{\partial\bar{T}}{\partial t} + \frac{H}{R}N^2\bar{w} = -\frac{\partial}{\partial y}\overline{(v'T')} + \frac{\bar{J}}{c_p}$

In steady state, we have

$$-f_0\bar{v} = -\frac{\partial}{\partial y}\overline{(u'v')} + \underbrace{\bar{X}}_{\text{drag term}}$$

drag is not large enough to balance eddy momentum convergence →

zonal eddy momentum flux convergence balanced by Coriolis torque associated with a mean meridional flow



NE – SW tilt necessary

$\vec{\bar{u}}$

$$0 = -f_0\bar{u} - \frac{\partial\bar{\Phi}}{\partial y} \quad : \text{geostrophy}$$

$$f_0\frac{\partial\bar{u}}{\partial z} = -\frac{R}{H}\frac{\partial\bar{T}}{\partial y} \quad : \text{thermal wind balance}$$

If $N^2 = 0$ then vertical motion does not cause adiabatic warming or cooling

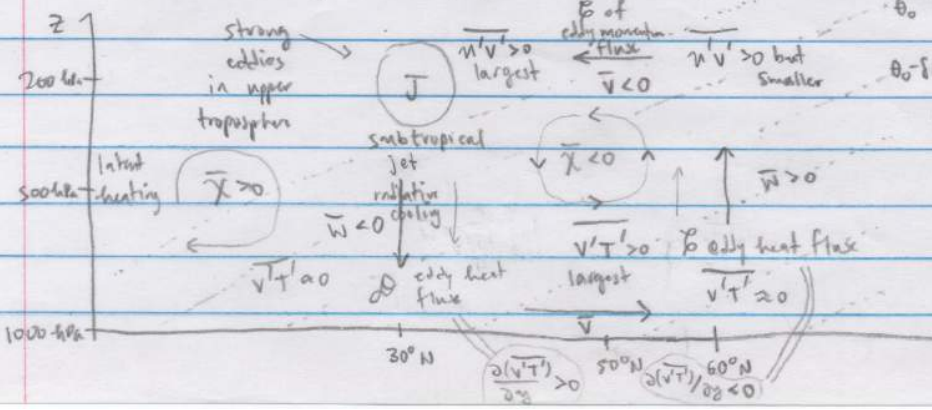
$\frac{\partial\theta_0}{\partial z} = 0 \Rightarrow \Gamma = \Gamma_d$

$$\frac{H}{R}N^2\bar{w} = -\frac{\partial}{\partial y}\overline{(v'T')} + \cancel{\frac{\bar{J}}{c_p}}$$

$\frac{\bar{J}}{c_p} \to 0$ if adiabatic flow, which is important to consider later

eddy heat flux convergence balanced by upward motion

T contours parallel to p contours in upper troposphere, hence $v'T' \approx 0$ there



This is the Ferrel Cell.

- thermally indirect because ~~cold air rises and warm~~ air sinks

$$\frac{\partial\bar{T}}{\partial y}<0 \Rightarrow \frac{\partial\bar{u}}{\partial z}>0$$ from thermal wind relation

Hadley cell thermally direct, driven by diabatic processes

$$\frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w) = 0$$

To proceed it is necessary that we assume $\partial \rho_0/\partial y < 0$

$$\frac{\partial v}{\partial y} + \frac{1}{\rho_0}\left(\frac{\partial}{\partial z}\rho_0\right) w + \frac{1}{\cancel{\rho_0}}\cancel{\rho_0}\frac{\partial w}{\partial z} = 0$$

$$\frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} w + \frac{\partial w}{\partial z} = 0$$

Recall that $\rho_0 = \rho_s e^{-z^*/H}$ so that this is indeed the case

$$v\frac{\partial \rho_0}{\partial y} + \rho_0\frac{\partial v}{\partial y} + \frac{\partial \rho_0}{\partial z} w + \rho_0\frac{\partial w}{\partial z} = 0$$

$$\frac{\partial}{\partial y}(\rho_0 v) + \frac{\partial}{\partial z}(\rho_0 w) = 0$$

We thus define a meridional mass streamfunction $\bar{\chi}$ such that

$$\rho_0 \bar{v} = -\frac{\partial \bar{\chi}}{\partial z} \quad \text{and} \quad \rho_0 \bar{w} = \frac{\partial \bar{\chi}}{\partial y}$$

$$\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{\chi}}{\partial z} \qquad \frac{\partial \bar{v}}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial \bar{\chi}}{\partial z}\right)$$

$$\bar{w} = \frac{1}{\rho_0}\frac{\partial \bar{\chi}}{\partial y}$$

$$\frac{\partial \bar{w}}{\partial y} = \frac{1}{\rho_0}\frac{\partial^2 \bar{\chi}}{\partial y^2}$$

Now take: $f_0\frac{\partial}{\partial z}\left(\frac{\partial \bar{u}}{\partial t} - f_0\bar{v} = -\frac{\partial}{\partial y}(\overline{u'v'}) + \bar{X}\right) + \frac{R}{H}\frac{\partial}{\partial y}\left(\frac{\partial \bar{T}}{\partial t} + N^2\frac{H}{R}\bar{w} = -\frac{\partial}{\partial y}(\overline{v'T'}) + \bar{J}/c_p\right)$

Note same form of equation as omega-eqn

the Rossby radius appears due to quasi-geostrophic scaling

$$\frac{\partial}{\partial t}\Big[\underbrace{f_0\frac{\partial \bar{u}}{\partial z} + \frac{R}{H}\frac{\partial}{\partial y}\bar{T}}_{=0}\Big] - \frac{\partial \bar{v}}{\partial z}f_0^2 + N^2\frac{\partial \bar{w}}{\partial y} = -f_0\frac{\partial}{\partial z}\frac{\partial}{\partial y}(\overline{u'v'}) + f_0\frac{\partial \bar{X}}{\partial z} - \frac{R}{H}\frac{\partial^2}{\partial y^2}(\overline{v'T'}) + \frac{R}{c_pH}\frac{\partial \bar{J}}{\partial y}$$

$$\Rightarrow \frac{N^2}{\rho_0}\frac{\partial^2 \bar{\chi}}{\partial y^2} + f_0^2\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial \bar{\chi}}{\partial z}\right) = f_0\frac{\partial \bar{X}}{\partial z} + \frac{R}{c_pH}\frac{\partial \bar{J}}{\partial y} - f_0\frac{\partial}{\partial z}\frac{\partial}{\partial y}(\overline{u'v'}) - \frac{R}{H}\frac{\partial^2}{\partial y^2}(\overline{v'T'})$$

$\bar{\chi}$ must be zero on boundaries since

$\frac{\partial \bar{\chi}}{\partial z} = 0 \Rightarrow \bar{v} = 0$ on N-S wall

$\frac{\partial \bar{\chi}}{\partial y} = 0 \Rightarrow \bar{w} = 0$ on top and bottom

~~$\bar{\chi}$ = constant~~ take it to be zero

$\bar{\chi}$ represented as double Fourier series

$$\underbrace{\frac{\partial^2 \bar{\chi}}{\partial y^2} + \frac{f_0^2}{\rho_0 N^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial \bar{\chi}}{\partial z}\right)}_{\bar{\chi}} = \frac{\rho_0}{N^2}\left[\frac{R}{H}\left(\frac{\partial \bar{J}}{c_p\partial y} - \frac{\partial^2}{\partial y^2}(\overline{v'T'})\right) + f_0\left[\frac{\partial \bar{X}}{\partial z} - \frac{\partial}{\partial z}\frac{\partial}{\partial y}(\overline{v'u'})\right]\right]$$

$\bar{\chi} \propto -\frac{\partial}{\partial y}$(diabatic heating)

- In the NH, diabatic heating decreases to the north for all y, so

Typically $\partial \bar{J}/\partial y < 0$, which drives $\bar{\chi} > 0$

$\frac{\partial^2}{\partial y^2}(\overline{v'T'})$

Typically $\frac{\partial^2}{\partial y^2}(\overline{v'T'}) < 0$, which drives $\bar{\chi} < 0$

eddy meridional heat flux $\overline{v'T'} > 0$ largest at $50°N$

$\frac{\partial}{\partial y}(\overline{v'T'}) > 0 \rightarrow$ between $30-50°N$, $\bar{w} < 0$

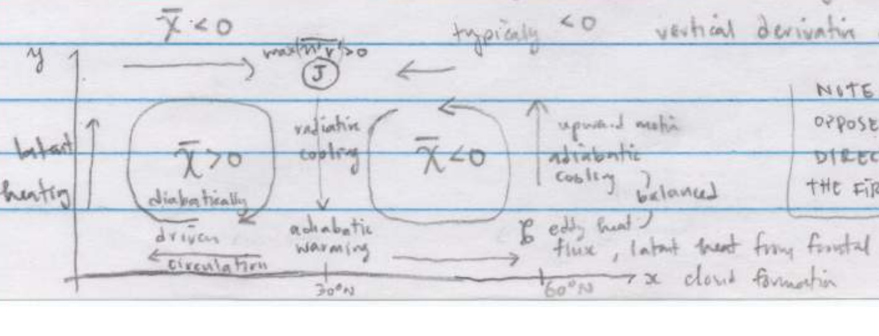
$\frac{\partial}{\partial y}(\overline{v'T'}) < 0 \rightarrow$ between $50-60°N$, $\bar{w} > 0$

$-\frac{\partial}{\partial z}$(zonal drag) $\quad \frac{\partial \bar{u}}{\partial t} = \bar{X}$

atmos. gain momentum from Earth for an easterly wind

$\frac{\partial}{\partial z}\frac{\partial}{\partial y}(\overline{u'v'})$ vertical derivative of large-scale eddy momentum flux divergence, OR the vertical derivative of the meridional (minus) vorticity flux

typically < 0



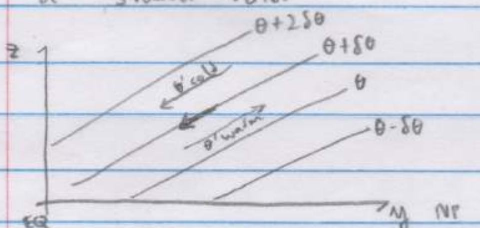
NOTE THAT THE EDDY FORCINGS OPPOSE THE TENDENCY TOWARD A DIABATIC, DIRECT CIRCULATION THAT IS FORCED BY THE FIRST TERM. THUS EDDY FORCING DOMINATES AT MID-LATITUDES AND FORCES ~ ADIABATIC CIRCULATION

In latent heating of tropical atmosphere / radiative cooling and subsidence in subtropics

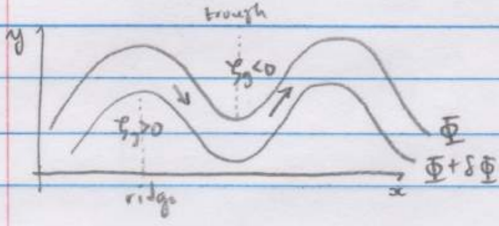
Zonal mean circulation

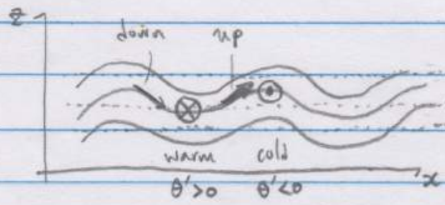
Tropical atmospheric circulation is driven by diabatic processes and is thus thermally direct. Eddy fluxes of momentum and heat play a small role.
Mid-latitude circulation is primarily driven by eddy fluxes of momentum and heat, and the circulation is mostly adiabatic; upward motion + adiabatic cooling of atmosphere where there is heat flux convergence, downward motion + compressional warming of the region of divergent eddy heat flux. Diabatic forcing plays a small role.

Ferrel Circulation : Adiabatic circulation



θ - conserved for adiabatic motion, thus the motion is along the isentropic surface
equatorward → downward
poleward → upward





$\theta = \theta_0(z) + \theta'$
$\theta(z)$ basic state stratification
warm air lowered gives higher temp at a given altitude where θ-surface is depressed, vice versa

In the conventional Eulerian mean considered above

NOTE THAT INDIVIDUAL PARCELS DO EXPERIENCE DIABATIC HEATING, BUT NOT FLUX $\mathcal{B}/\partial$

- there is cancelation between adiabatic expansion/compression and heat flux convergence/divergence
- to see the role of diabatic processes in the Ferrel Cell, we examine the

Transformed Eulerian Mean

recall the zonal average equations in log-pressure coordinates:

$z = -H \ln\left(\frac{p}{p_s}\right)$

$$\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v} = -\frac{\partial}{\partial y}\left(\overline{u'v'}\right) + \bar{X} \qquad \frac{\partial \bar{v}}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \bar{w}\right) = 0 \qquad \rho_0 = \rho_s e^{-z/H}$$

$$f_0 \bar{u} = -\frac{\partial \bar{\Phi}}{\partial y}$$

$$\frac{\partial \bar{T}}{\partial t} + \frac{H}{R} N^2 \bar{w} = -\frac{\partial}{\partial y}\left(\overline{v'T'}\right) + \frac{\bar{J}}{c_p} \qquad \frac{\partial \bar{\Phi}}{\partial z} = \frac{R\bar{T}}{H}$$

If $\frac{\partial \bar{T}}{\partial t} = 0$, upward vertical motion cools the atmosphere in regions of eddy heat flux convergence and warms it via subsidence where there's meridional eddy heat flux D. As seen in the conventional Eulerian mean, this cancelation constitutes adiabatic flow and masks the effect of diabatic processes. We define the residual motion driven by diabatic processes by subtracting that induced by eddies

$$\underbrace{\cancel{\frac{\partial \bar{T}}{\partial t}}}_{0} + \frac{H}{R} N^2 \bar{w} - -\frac{\partial}{\partial y}\left(\overline{v'T'}\right) = \frac{\bar{J}}{c_p}$$

$$\bar{w}^* = \frac{\bar{J} R}{H c_p N^2} = \bar{w} + \frac{R}{H N^2}\frac{\partial}{\partial y}\left(\overline{v'T'}\right) = \bar{w} + \frac{R}{H}\frac{\partial}{\partial y}\left(\frac{\overline{v'T'}}{N^2}\right)$$

The residual meridional circulation $\bar{v}^*$ is devised through continuity

$$\frac{\partial \bar{v}}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left[\rho_0 \bar{w}^* - \rho_0 \frac{R}{H}\frac{\partial}{\partial y}\left(\frac{\overline{v'T'}}{N^2}\right)\right] = 0$$

$$\frac{\partial}{\partial y}\underbrace{\left[\bar{v} - \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \frac{R}{H}\frac{\overline{v'T'}}{N^2}\right)\right]}_{\bar{v}^*} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \bar{w}^*\right) = 0$$

with these definitions the governing equations become

$$\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v} + f_0 \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \frac{R}{H}\frac{\overline{v'T'}}{N^2}\right) = -\frac{\partial}{\partial y}\left(\overline{u'v'}\right) + \frac{f_0}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \frac{R}{H}\frac{\overline{v'T'}}{N^2}\right) + \bar{X}$$

$$\frac{\partial \bar{u}}{\partial t} - f_0\left[\bar{v} - \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \frac{R}{H}\frac{\overline{v'T'}}{N^2}\right)\right] = \frac{-1}{\rho_0}\frac{\partial}{\partial y}\left(\rho_0 \overline{u'v'}\right) + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left[f_0 \rho_0 \frac{R}{H}\frac{\overline{v'T'}}{N^2}\right] + \bar{X}$$

$$\boxed{\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v}^* = \frac{1}{\rho_0}\vec{\nabla}\cdot\vec{F} + \bar{X}, \quad \vec{F} = -\rho_0 \overline{u'v'}\,\hat{j} + f_0 \rho_0 \frac{R}{H}\frac{\overline{v'T'}}{N^2}\,\hat{k}}$$

$\vec{F}$ is a vector in the y-z or latitude-height plane

$\uparrow$ log-p

$$\frac{\partial \bar{T}}{\partial t} + \frac{H}{R} N^2 \bar{w} = -\frac{\partial}{\partial y}\left(\overline{v'T'}\right) + \frac{\bar{J}}{c_p} \qquad f_0 \bar{u} = -\frac{\partial \bar{\Phi}}{\partial y}$$

$$\frac{\partial \bar{T}}{\partial t} + \frac{H}{R} N^2 \bar{w} + \frac{\partial}{\partial y}\left(\overline{v'T'}\right) = \bar{J}/c_p \qquad \frac{\partial \bar{v}^*}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \bar{w}^*\right) = 0$$

$$\boxed{\frac{\partial \bar{T}}{\partial t} + \frac{H}{R} N^2 \underbrace{\left(\bar{w} + \frac{R}{H}\frac{\partial}{\partial y}\left(\frac{\overline{v'T'}}{N^2}\right)\right)}_{\bar{w}^*} = \frac{\bar{J}}{c_p}} \qquad \frac{\partial \bar{\Phi}}{\partial z} = \frac{R\bar{T}}{H}$$

these remain unchanged

$$f_0 \frac{\partial \bar{u}}{\partial z} = -\frac{\partial}{\partial y}\left(\frac{\partial \bar{\Phi}}{\partial z}\right) = -\frac{R}{H}\frac{\partial \bar{T}}{\partial y}$$

$$\frac{\partial \bar{v}^*}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \bar{w}^*\right) = 0$$

$\rho_0 = \rho_0(z)$

$$\bar{v}^* \frac{\partial \rho_0}{\partial y} + \rho_0 \frac{\partial \bar{v}^*}{\partial y} + \frac{\partial}{\partial z}\left(\rho_0 \bar{w}^*\right) = 0$$

$$\frac{\partial}{\partial y}\left(\rho_0 \bar{v}^*\right) + \frac{\partial}{\partial z}\left(\rho_0 \bar{w}^*\right) = 0$$

residual stream function

We want to define $\bar{\chi}^*$ such that

$$\rho_0 \bar{v}^* = -\frac{\partial \bar{\chi}^*}{\partial z}, \quad \rho_0 \bar{w}^* = \frac{\partial \bar{\chi}^*}{\partial y}$$

$$\rho_0\left[\bar{v} - \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \frac{R}{H}\frac{\overline{v'T'}}{N^2}\right)\right] = -\frac{\partial \bar{\chi}^*}{\partial z}$$

$$\bar{v}^* = -\frac{1}{\rho_0}\frac{\partial \bar{\chi}^*}{\partial z} \Rightarrow \frac{\partial \bar{v}^*}{\partial z} = \frac{\partial}{\partial z}\left(-\frac{1}{\rho_0}\frac{\partial \bar{\chi}^*}{\partial z}\right)$$

$$\bar{w}^* = \frac{1}{\rho_0}\frac{\partial \bar{\chi}^*}{\partial y} \Rightarrow \frac{\partial \bar{w}^*}{\partial y} = \frac{1}{\rho_0}\frac{\partial^2 \bar{\chi}^*}{\partial y^2}$$

$$\underbrace{\int \rho_0 \bar{v}\, dz}_{-\bar{\chi}} - \rho_0 \frac{R}{H}\left(\frac{\overline{v'T'}}{N^2}\right) = -\bar{\chi}^*$$

$$-\bar{\chi} - \rho_0 \frac{R}{H}\left(\frac{\overline{v'T'}}{N^2}\right) = -\bar{\chi}^*$$

$$\Rightarrow \bar{\chi}^* = \bar{\chi} + \rho_0 \frac{R}{H}\left(\frac{\overline{v'T'}}{N^2}\right)$$

$$f_0 \frac{\partial}{\partial z}\left[\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v}^* = \frac{1}{\rho_0}\nabla\cdot\mathbf{F} + \bar{X} \equiv \bar{G}\right] + \frac{R}{H}\frac{\partial}{\partial y}\left[\frac{\partial \bar{T}}{\partial t} + \frac{H}{R}N^2 \bar{w}^* = \bar{J}/c_p\right]$$

$$\frac{\partial}{\partial t}\underbrace{\left(f_0 \frac{\partial \bar{u}}{\partial z} + \frac{R}{H}\frac{\partial}{\partial y}\bar{T}\right)}_{0 \text{ thermal wind}} - f_0^2 \frac{\partial \bar{v}^*}{\partial z} + N^2 \frac{\partial \bar{w}^*}{\partial y} = f_0 \frac{\partial \bar{G}}{\partial z} + \frac{\kappa}{H}\frac{\partial \bar{J}}{\partial y}$$

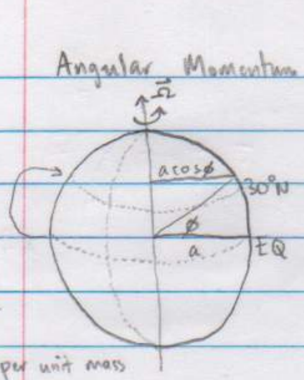
$$\frac{\rho_0}{N^2} f_0^2 \frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial \bar{\chi}^*}{\partial z}\right) + \cancel{\frac{N^2}{\rho_0}} \frac{\partial^2 \bar{\chi}^*}{\partial y^2} = \left[f_0 \frac{\partial \bar{G}}{\partial z} + \frac{\kappa}{H}\frac{\partial \bar{J}}{\partial y}\right]\frac{\rho_0}{N^2}$$

$$\frac{\partial^2 \bar{\chi}^*}{\partial y^2} + \rho_0 \frac{f_0^2}{N^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial \bar{\chi}^*}{\partial z}\right) = \frac{\rho_0}{N^2}\left[\frac{\kappa}{H}\frac{\partial \bar{J}}{\partial y} + f_0 \frac{\partial \bar{G}}{\partial z}\right]$$

$$-\bar{\chi}^* \propto \underbrace{\frac{\partial \bar{J}}{\partial y}}_{<0}, \quad \underbrace{\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\nabla\cdot\mathbf{F}\right)}_{<0 \text{ see Holton Fig 10.8}}, \quad \frac{\partial \bar{X}}{\partial z}$$

$\bar{\chi}^* > 0$ in tropics and mid-latitudes

Angular Momentum balance of the Earth-atmosphere system



As air parcel moves poleward in Hadley cell it gains an eastward zonal flow component to compensate for the decrease in distance to the axis of rotation

$\vec{L} = \vec{r} \times m\vec{v}$

$\vec{\tau} = \frac{D\vec{L}}{Dt} = \frac{D\vec{r}}{Dt} \times \vec{p} + \vec{r} \times \frac{D\vec{p}}{Dt} = \vec{r} \times \vec{F}$ if we assume that $\frac{D\vec{r}}{Dt} \times \vec{p} = 0$, where $\vec{F} = \frac{D\vec{p}}{Dt}$. The fractional change in radius for a parcel moving throughout the depth of the tropopause is $\frac{12\ \text{km}}{6371\ \text{km}} \times 100\% = 0.18\%$, so it is reasonable to let $r = a + z \rightarrow a$.

The specific (per unit mass) angular momentum about Earth's rotation axis can be written as the sum of the angular momentum associated with Earth's solid body rotation and that due to zonal motion relative to the Earth's surface

$$|\vec{L}| = M = (u_E + u)\, a\cos\phi$$

$$\vec{\Omega} \times \vec{r} = |\vec{\Omega}||\vec{r}| \sin(\vec{\Omega}, \vec{r}) = \Omega a\cos\phi \cdot \underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{1}$$

$$= (\Omega a\cos\phi + u)\, a\cos\phi$$

- For flow that conserves angular momentum, an approximate expression for an air parcel initially at rest at the EQ that then moves poleward is $u(\phi) = \Omega a \sin^2\phi / \cos\phi$

To see this: $M(t_1) = \Omega a^2 \underbrace{\cos^2(0)}_{1} = M(t_2) = \Omega a^2\cos^2\phi + u\, a\cos\phi$

$$\Omega a^2 (1 - \cos^2\phi) = u\, a\cos\phi$$ ~~This was part of HW 4.~~

$$u(\phi) = \Omega a \frac{\sin^2\phi}{\cos\phi}.$$

This expression gives zonal winds that quickly exceed observed wind speeds in the subtropical jet stream that would also require a very large (i.e. much larger than observed) meridional temperature gradient for thermal wind balance.

Thus, the specific angular momentum of an air parcel must decrease as it moves poleward in the thermally direct Hadley Cell. As noted above, $|\vec{L}|$ can change only via zonal torques acting on the air parcel of concern:

$\rho_0\, dz = -\frac{dp}{g}$

$$\frac{D}{Dt}|\vec{L}| = \frac{DM}{Dt} = a\cos\phi\left[-\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau_E^x}{\partial z}\right] = -a\cos\phi\left[\frac{\partial \Phi}{\partial x} + g\frac{\partial \tau_E^x}{\partial p}\right]$$

$\tau_E^x = \rho_0 \overline{u'w'}$

$$= a\cos\phi\left[-\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\overline{u'w'}\right)\right]$$

To examine the redistribution and exchange of angular momentum between the Earth and the atmosphere, it is useful to transform to σ-coordinates so that the Earth's surface is a coordinate surface. In p-coords, coordinate surfaces intersect terrain, and do so differently at different times.

Sigma - coordinates

$\sigma = p/p_s$ where $p(x,y,z,t)$ and $p_s(x,y,t)$

$\sigma = 0$ at "TOA" and $\sigma = 1$ at surface $\Rightarrow \dot\sigma = \frac{D\sigma}{Dt} = 0$ at $\sigma = 1$

$$\nabla_p(\ ) = \nabla_\sigma(\ ) - \sigma \nabla \ln p_s \frac{\partial}{\partial\sigma}(\ )$$
$$= \nabla_\sigma(\ ) - \frac{\sigma}{p_s}\nabla p_s \frac{\partial}{\partial\sigma}(\ )$$

- Horizontal momentum

$$\boxed{\frac{D\vec{V}}{Dt} + f\hat{k}\times\vec{V} = -\nabla_p\Phi = -\nabla_\sigma\Phi + \frac{\sigma}{p_s}\nabla p_s\frac{\partial\Phi}{\partial\sigma}}$$

- Vertical momentum (hydrostatic)

$d\sigma = \frac{dp}{p_s}$

$$-\frac{\partial\Phi}{\partial p} = \frac{RT}{p} \Rightarrow -p_s\frac{\partial\Phi}{\partial p} = \frac{RT}{p}p_s = \boxed{-\frac{\partial\Phi}{\partial\sigma} = \frac{RT}{\sigma}}$$

requires more work $\rightarrow$

- Continuity: $\vec{\nabla}_p\cdot\vec{V} + \frac{\partial\omega}{\partial p} = 0 \Rightarrow p_s(\vec{\nabla}_p\cdot\vec{V}) + \frac{\partial\omega}{\partial\sigma} = 0$

$$\vec{\nabla}_p\cdot\vec{V} = \vec{\nabla}_\sigma\cdot\vec{V} - \frac{\sigma}{p_s}\nabla p_s\cdot\frac{\partial\vec{V}}{\partial\sigma}$$

$$\dot\sigma = \frac{D\sigma}{Dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p\sigma + \omega\frac{\partial\sigma}{\partial p}$$
$$= \frac{p}{-p_s^2}\left(\frac{\partial p_s}{\partial t} + u\frac{\partial p_s}{\partial x} + v\frac{\partial p_s}{\partial y}\right) + \frac{\omega}{p_s}$$
$$= -\frac{\sigma}{p_s}\left(\frac{\partial p_s}{\partial t} + u\frac{\partial p_s}{\partial x} + v\frac{\partial p_s}{\partial y}\right) + \omega/p_s$$
$$\Rightarrow \frac{\partial\dot\sigma}{\partial\sigma} = -\frac{1}{p_s}\left(\frac{\partial p_s}{\partial t} + \vec{V}\cdot\nabla p_s\right) - \frac{\sigma}{p_s}\frac{\partial}{\partial\sigma}\cancel{\left(\frac{\partial p_s}{\partial t}\right)} - \frac{\sigma}{p_s}\frac{\partial u}{\partial\sigma}\frac{\partial p_s}{\partial x} - \frac{\sigma}{p_s}\frac{\partial v}{\partial\sigma}\frac{\partial p_s}{\partial y} + \frac{1}{p_s}\frac{\partial\omega}{\partial\sigma}$$
$$\Rightarrow p_s\frac{\partial\dot\sigma}{\partial\sigma} + \frac{\partial p_s}{\partial t} + \vec{V}\cdot\nabla p_s + \sigma\nabla p_s\cdot\frac{\partial\vec{V}}{\partial\sigma} = \frac{\partial\omega}{\partial\sigma}$$

we therefore have

$$p_s(\nabla_\sigma\cdot\vec{V}) - \cancel{\sigma\nabla p_s\cdot\frac{\partial\vec{V}}{\partial\sigma}} + p_s\frac{\partial\dot\sigma}{\partial\sigma} + \frac{\partial p_s}{\partial t} + \vec{V}\cdot\nabla p_s + \cancel{\sigma\nabla p_s\cdot\frac{\partial\vec{V}}{\partial\sigma}} = 0$$

$$\boxed{\frac{\partial p_s}{\partial t} + \vec{\nabla}\cdot(p_s\vec{V}) + p_s\frac{\partial\dot\sigma}{\partial\sigma} = 0}$$

- Thermodynamic: from 2.46

$$c_p\frac{D\ln\theta}{Dt} = \frac{J}{T} \Rightarrow \boxed{\frac{\partial\theta}{\partial t} + \vec{V}\cdot\vec{\nabla}\theta + \dot\sigma\frac{\partial\theta}{\partial\sigma} = \frac{J\theta}{c_pT}}$$

Another useful result: An element of mass in Cartesian coordinates is

$$\rho_0\,dx\,dy\,dz = -\cancel{\rho_0}\,dx\,dy\frac{dp}{\cancel{\rho_0}g} = -\frac{1}{g}p_s\,dx\,dy\,d\sigma$$

$d\sigma = \frac{dp}{p_s}$ because g is considered to be constant

ρ_0 in Cartesian coordinates is similar to p_s in sigma coordinates geopotential height $\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_z M + w\frac{\partial M}{\partial z} = -a\cos\phi \frac{1}{\rho_0}\left(\frac{\partial p}{\partial x}\right)_z + a\cos\phi \frac{1}{\rho_0}\frac{\partial \tau_E^x}{\partial z}$

pressure $\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p M + \omega\frac{\partial M}{\partial p} = -a\cos\phi\left[\left(\frac{\partial \Phi}{\partial x}\right)_p + g\frac{\partial \tau_E^x}{\partial p}\right]$

sigma $\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_\sigma M + \dot\sigma\frac{\partial M}{\partial \sigma} = -a\cos\phi\left[\left(\frac{\partial \Phi}{\partial x}\right)_\sigma - \frac{\sigma}{p_s}\frac{\partial p_s}{\partial x}\underbrace{\frac{\partial \Phi}{\partial \sigma}}_{-\frac{RT}{\sigma}} + g\frac{\partial \tau_E^x}{\partial \sigma}\overbrace{\cancel{\frac{\partial \sigma}{\partial p}}}^{\frac{1}{p_s}}\right]$

$$\left(\frac{\partial}{\partial t} + \vec v\cdot\nabla + \dot\sigma\frac{\partial}{\partial \sigma}\right)M = -a\cos\phi\left[\frac{\partial \Phi}{\partial x} + \frac{RT}{p_s}\frac{\partial p_s}{\partial x} + \frac{g}{p_s}\frac{\partial \tau_E^x}{\partial \sigma}\right]$$

To put this equation in flux form for zonal averaging, multiply it by p_s and add M times the continuity equation

$$p_s\frac{\partial M}{\partial t} + p_s\vec v\cdot\nabla M + p_s\dot\sigma\frac{\partial M}{\partial \sigma} = -p_s a\cos\phi[\cdots]$$

$$M\frac{\partial p_s}{\partial t} + M\vec\nabla\cdot(p_s\vec v) + M p_s\frac{\partial \dot\sigma}{\partial \sigma} = 0$$

$\downarrow$

$$\frac{\partial}{\partial t}(p_s M) + \vec\nabla\cdot(p_s\vec v\cdot M) + \frac{\partial}{\partial \sigma}(\dot\sigma p_s M) = -a\cos\phi\left[p_s\frac{\partial \Phi}{\partial x} + RT\frac{\partial p_s}{\partial x} + g\frac{\partial \tau_E^x}{\partial \sigma}\right]$$

$dx = a\cos\phi\, d\lambda$

$dy = a\, d\phi$

$$\vec\nabla\cdot(\ ) = \frac{1}{a\cos\phi}\left[\frac{\partial}{\partial \lambda}(\ \ ) + \frac{\partial}{\partial \phi}\left((\cdot)\cos\phi\right)\right]$$ in spherical coordinates

this $\cos\phi$ factor due to convergence of meridians

Note that

$$p_s\frac{\partial \Phi}{\partial x} + RT\frac{\partial p_s}{\partial x} = p_s\frac{\partial}{\partial x}(\Phi - RT) + \frac{\partial}{\partial x}(p_s RT)$$
$$= p_s\frac{\partial \Phi}{\partial x} - \cancel{p_s\frac{\partial}{\partial x}(RT)} + \cancel{p_s\frac{\partial}{\partial x}}RT + RT\frac{\partial p_s}{\partial x}$$

$\frac{\partial \Phi}{\partial \sigma} = -\frac{RT}{\sigma}$

and $\Phi - RT = \Phi + \sigma\frac{\partial \Phi}{\partial \sigma} = \frac{\partial}{\partial \sigma}(\sigma\Phi)$ and thus the first two terms on RHS can be written

$$p_s\frac{\partial \Phi}{\partial x} + RT\frac{\partial p_s}{\partial x} = p_s\frac{\partial}{\partial x}\left(\frac{\partial}{\partial \sigma}(\sigma\Phi)\right) + RT\frac{\partial p_s}{\partial x} = \frac{\partial}{\partial \sigma}\left(p_s\sigma\frac{\partial \Phi}{\partial x}\right) + RT\frac{\partial p_s}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial t}(p_s M) = -\frac{1}{a\cos\phi}\left[\frac{\partial}{\partial \lambda}(p_s M\vec v) + \frac{\partial}{\partial \phi}(p_s v M\cos\phi)\right] - \frac{\partial}{\partial \sigma}(\dot\sigma p_s M) - a\cos\phi\frac{\partial}{\partial \sigma}\left(p_s\sigma\frac{\partial \Phi}{\partial x}\right) - a\cos\phi\, RT\frac{\partial p_s}{\partial x} - a\cos\phi\, g\frac{\partial \tau_E^x}{\partial \sigma}$$

$\overline{(\ )} = \frac{1}{2\pi}\int_0^{2\pi}(\ )\,d\lambda\, a\cos\phi$ $\Rightarrow$ $\frac{\partial}{\partial t}\left(\overline{p_s M}\right) = -\frac{1}{a\cos\phi}\left[\frac{\partial}{\partial \phi}\left(\overline{p_s v M}\cos\phi\right)\right] - \frac{\partial}{\partial \sigma}\left[\overline{\dot\sigma p_s M} + a\cos\phi\,\overline{p_s\sigma\frac{\partial \Phi}{\partial x}} + a\cos\phi\, g\,\overline{\tau_E^x}\right]$

$$\frac{\partial}{\partial t}\left(\overline{p_s M}\right) = -\frac{1}{\cos\phi}\left[\frac{\partial}{\partial y}\left(\overline{p_s v M}\cos\phi\right)\right] - \frac{\partial}{\partial \sigma}\left[\overline{\dot\sigma p_s M} + a\cos\phi\,\overline{p_s\sigma\frac{\partial \Phi}{\partial x}} + a\cos\phi\, g\,\overline{\tau_E^x}\right]$$

rate of change of zonal-mean zonal angular momentum = horizontal convergence of horizontal flux of angular momentum and vertical convergence of vertical flux of angular momentum

M has units angular momentum per mass, but since

$p_s \sim \rho_0$

σ $\quad z$

$p_s M$ is like angular momentum / volume

and since we've (averaged) integrated in x, this equation represents the angular momentum of a ring of atmosphere per unit meridional distance dy and occupying a differential element in the vertical $d\sigma$

Next, we integrate the equation in the vertical, from the surface to the top of the atmosphere. First recall that

$d\sigma = \frac{dp}{p_s}$

$$\int_0^\infty (\ )\,dz = -\int_{p_s}^0 (\ )\frac{dp}{\rho_0 g} = -\frac{1}{g}\int_1^0 (\,)\frac{p_s\, d\sigma}{\rho}$$

$$\dot\sigma(1) = \dot\sigma(0) = 0$$

$$\frac{1}{g}\int_0^1 \frac{\partial}{\partial t}(\overline{p_s M})\, d\sigma = -\frac{1}{\cos\phi}\frac{1}{g}\frac{\partial}{\partial y}\int_0^1 \overline{p_s M v}\cos\phi\, d\sigma - \frac{1}{g}\int_0^1 \frac{\partial}{\partial\sigma}\left(\cancel{\dot\sigma\, p_s M}\right)^0 d\sigma$$

$$- \frac{1}{g}\int_0^1 \frac{\partial}{\partial\sigma}\left(a\cos\phi\ \overline{p_s\sigma\frac{\partial\Phi}{\partial x}}\right)d\sigma - \frac{1}{\cancel{g}}\int_0^1 a\cos\phi\,\cancel{g}\,\frac{\partial \tau_E^x}{\partial\sigma}\,d\sigma$$

$$\frac{\partial}{\partial t}\int_0^1 (\overline{p_s M})\frac{d\sigma}{g} = -\frac{1}{g\cos\phi}\frac{\partial}{\partial y}\int_0^1 \overline{p_s M v}\cos\phi\, d\sigma - \frac{a\cos\phi}{g}\ \overline{p_s\sigma\frac{\partial\Phi}{\partial x}}\Big|_{\sigma=1} - a\cos\phi\, \tau_E^x\Big|_{\sigma=1}$$

$\frac{d}{dy} = \frac{1}{a}\frac{d}{d\phi}$

$\Phi = gh$

$h(x,y) = \frac{1}{g}\Phi(x,y,\sigma=1)$

$$\frac{\partial}{\partial t}\int_0^1 \overline{p_s M}\,\frac{d\sigma}{g} = -\frac{1}{\cos\phi}\frac{\partial}{\partial y}\int_0^1 \overline{p_s v M}\cos\phi\,\frac{d\sigma}{g} - a\cos\phi\left(\overline{p_s\frac{\partial h}{\partial x}}\right) - a\cos\phi\, \tau_E^x\Big|_{\sigma=1}$$

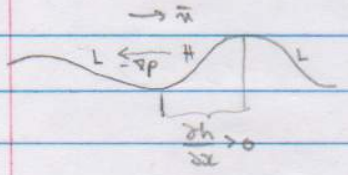
rate of change of angular momentum for zonal ring of air

↑ meridional convergence of the (zonal- and vertical-mean) meridional flux of zonal angular momentum

↑ surface pressure torque

↑ torque due to small-scale eddy friction effects

When $\overline{p_s\frac{\partial h}{\partial x}} > 0$ the atmosphere transfers angular momentum to the Earth and the pressure gradient slows down the mean zonal wind



In the NH, there are more mountains and thus the pressure torque plays a large role than in the SH.

It is instructive to decompose the flow into zonal-mean and eddy components,

$$M = \overline{M} + M' = (\Omega a\cos\phi + \bar u + u')\,a\cos\phi$$

$$= \underbrace{\Omega a^2\cos^2\phi + \bar u\, a\cos\phi}_{\overline{M}} + \underbrace{u' a\cos\phi}_{M'}$$

$$p_s v = \overline{p_s v} + (p_s v)'$$

$$\Rightarrow \overline{p_s v M} = \overline{p_s v}\,\overline{M} + \overline{(p_s v)' M'}$$

$$= \Omega a^2\cos^2\phi\,\overline{p_s v} + \bar u\, a\cos\phi\,\overline{p_s v} + \overline{u'(p_s v)'}\,a\cos\phi$$

$$= \left[\Omega a\cos\phi\,\overline{p_s v} + \bar u\,\overline{p_s v} + \overline{u'(p_s v)'}\right] a\cos\phi$$

↑ meridional Ω-momentum flux

↑ meridional drift ↑ important in tropics

↑ meridional eddy momentum flux ↑ important in mid-latitudes

Continuity equation :

$$\frac{\partial p_s}{\partial t} + \vec{\nabla}\cdot(p_s \vec{v}) + p_s \frac{\partial \dot{\sigma}}{\partial \sigma} = 0$$

average zonally : $$\frac{\partial \overline{p_s}}{\partial t} + \cancel{\frac{\partial}{\partial x}\overline{(p_s u)}} + \frac{\partial}{\partial y}\overline{(p_s v)} + \overline{p_s \frac{\partial \dot{\sigma}}{\partial \sigma}} = 0$$

$\dot{\sigma}(1) = \dot{\sigma}(0) = 0$

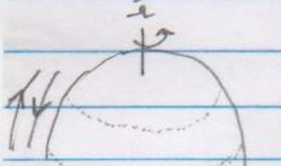
integrate vertically: $$\cancel{\frac{1}{g}}\int_0^1 \frac{\partial \overline{p_s}}{\partial t}\, d\sigma + \cancel{\frac{1}{g}}\frac{\partial}{\partial y}\int_0^1 \overline{p_s v}\, d\sigma + \cancel{\frac{1}{g}}\int_0^1 p_s \cancel{\frac{\partial \dot{\sigma}}{\partial \sigma}}\, d\sigma = 0$$

$\frac{\partial}{\partial t}\int_0^\infty \overline{\rho_0(z)}\, dz$ ← similar → $$\frac{\partial}{\partial t}\int_0^1 \overline{p_s}\, d\sigma = \frac{-1}{a\cos\phi}\frac{\partial}{\partial \phi}\int_0^1 \overline{p_s v}\cos\phi\, d\sigma$$

$$= -\frac{1}{\cos\phi}\frac{\partial}{\partial y}\int_0^1 \overline{p_s v}\cos\phi\, d\sigma$$

- For the zonal mean surface pressure (column mass) to decrease/increase there must be divergence/convergence of atmospheric mass from one latitudinal region to another.
  E.g. the annular modes
- Averaged over time, LHS = 0 and thus so is the net mass flow across latitudinal circles.

Now that we have expanded $\overline{p_s v M}$, it is instructive to re-examine the angular momentum budget for a zonal ring of air from the sfc to TOA.



zero net vertically integrated transport

$$\frac{\partial}{\partial t}\int_0^1 \overline{p_s M}\,\frac{d\sigma}{g} = -\frac{1}{\cancel{\cos\phi}}\frac{\partial}{\partial y}\int_0^1 \cancel{\cos\phi}\left[\Omega a\cos\phi\,\overline{p_s v} + \bar{u}\,\overline{p_s v} + \overline{u'(p_s v)'}\right] a\cos\phi\,\frac{d\sigma}{g}$$

First and Second RHS terms small in mid-latitudes

LORENZ ENERGY CYCLE

Holton 10.4

Governing equations for the zonal mean flow

$$\boxed{\begin{aligned} &\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v} = -\frac{\partial}{\partial y}(\overline{u'v'}) + \bar{X} \\ &+ f_0 \bar{u} = -\frac{\partial \bar{\Phi}}{\partial y} \\ &\frac{R\bar{T}}{H} = \frac{\partial \bar{\Phi}}{\partial z} \\ &\frac{\partial \bar{v}}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 \bar{w}) = 0 \\ &\frac{\partial \bar{T}}{\partial t} + \frac{H}{R}N^2 \bar{w} = -\frac{\partial}{\partial y}(\overline{v'T'}) + \frac{\bar{J}}{c_p} \rightarrow \frac{\partial}{\partial t}\left(\frac{\partial \bar{\Phi}}{\partial z}\right) + N^2 \bar{w} = -\frac{\partial}{\partial y}\left(\overline{v'\frac{\partial \Phi'}{\partial z}}\right) + \frac{\bar{J}R}{c_p H}\end{aligned}}$$

Governing equations for eddy motions: requires taking a step back to the full equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{\partial \Phi}{\partial x} + X \quad (10.1,\ u \text{ component})$$

expand each variable $u = \bar{u} + u'$, $v = \bar{v} + v'$, ... as an anomaly from the zonal mean

$$\frac{\partial}{\partial t}(\bar{u}+u') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') + (\cancel{\bar{v}}+v')\frac{\partial}{\partial y}(\bar{u}+u') + (\cancel{\bar{w}}+w')\frac{\partial}{\partial z}(\bar{u}+u') - f(\bar{v}+v') =$$

$$-\frac{\partial}{\partial x}(\bar{\Phi}+\Phi') + \bar{X} + X'$$

neglected because w' very small

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + v'\frac{\partial \bar{u}}{\partial y} + \cancel{w'\frac{\partial \bar{u}}{\partial z}} - f_0 v' = -\frac{\partial \Phi'}{\partial x} + X'$$

$$\boxed{\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)u' - v'\left(f_0 - \frac{\partial \bar{u}}{\partial y}\right) = -\frac{\partial \Phi'}{\partial x} + X'}$$

same as $\bar{u}$

$\frac{\partial}{\partial x}$(zonal avg. quantity) is zero

$$\frac{\partial}{\partial t}(\bar{v}+v') + (\bar{u}+u')\frac{\partial}{\partial x}(\cancel{\bar{v}}+v') + (\bar{v}+v')\frac{\partial}{\partial y}(\bar{v}+v') + (\bar{w}+w')\frac{\partial}{\partial z}(\bar{v}+v') + f_0(\bar{u}+u') = -\frac{\partial \bar{\Phi}}{\partial y} - \frac{\partial \Phi'}{\partial y} + \bar{Y} + Y'$$

$$\boxed{\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)v' + f_0 u' = -\frac{\partial \Phi'}{\partial y} + Y'}$$

$$\frac{\partial}{\partial t}(\bar{T}+T') + (\bar{u}+u')\frac{\partial}{\partial x}(\cancel{\bar{T}}+T') + (\bar{v}+v')\frac{\partial}{\partial y}(\bar{T}+T') + (\bar{w}+w')\frac{\partial}{\partial z}(\bar{T}+T') + \frac{\kappa(\bar{T}+T')}{H}(\bar{w}+w') = \frac{\bar{J}+J'}{c_p}$$

$$\frac{\partial}{\partial t}T' + \bar{u}\frac{\partial}{\partial x}T' + v'\frac{\partial \bar{T}}{\partial y} + w'\frac{\partial \bar{T}}{\partial z} + \frac{\kappa \bar{T}}{H}w' = \frac{J'}{c_p}$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)T' + v'\frac{\partial \bar{T}}{\partial y} + w'\underbrace{\left(\frac{\partial \bar{T}}{\partial z} + \frac{\kappa \bar{T}}{H}\right)}_{N^2 \frac{H}{R}} = J'/c_p$$

$$\boxed{\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial \Phi'}{\partial z}\right) + v'\frac{\partial}{\partial y}\left(\frac{\partial \bar{\Phi}}{\partial z}\right) + N^2 w' = \frac{J'R}{H c_p}}$$

$$\frac{\partial}{\partial x}(\bar{u}+u') + \frac{\partial}{\partial y}(\bar{v}+v') + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 \bar{w} + \rho_0 w') = 0$$

$$\boxed{\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w') = 0}$$

We next derive energy equations for the mean zonal flow

$$\rho_0 \bar{u} \frac{\partial \bar{u}}{\partial t} = \rho_0 \bar{u} \bar{v} f_0 - \rho_0 \bar{u} \frac{\partial}{\partial y}(\overline{u'v'}) + \bar{X} \rho_0 \bar{u}$$

$$\rho_0 \bar{v} \left(\frac{\partial \bar{v}}{\partial t}\right) = -\rho_0 \bar{v} \bar{u} f_0 - \rho_0 \bar{v} \frac{\partial \bar{\Phi}}{\partial y}$$

0

$$\frac{\partial}{\partial t}\left(\frac{\rho_0}{2}\bar{u}^2\right) = -\rho_0 \bar{u} \frac{\partial}{\partial y}(\overline{u'v'}) + \rho_0 \bar{u} \bar{X} - \rho_0 \bar{v} \frac{\partial \bar{\Phi}}{\partial y}$$

$$= -\frac{\partial}{\partial y}(\rho_0 \bar{v} \bar{\Phi}) + \rho_0 \bar{\Phi} \frac{\partial \bar{v}}{\partial y} - \frac{\partial}{\partial y}(\rho_0 \bar{u}\, \overline{u'v'}) + \rho_0 \overline{u'v'} \frac{\partial \bar{u}}{\partial y} + \rho_0 \bar{u} \bar{X}$$

$$= -\rho_0 \bar{v} \frac{\partial \bar{\Phi}}{\partial y} - \cancel{\bar{\Phi} \rho_0 \frac{\partial \bar{v}}{\partial y}} + \cancel{\rho_0 \bar{\Phi} \frac{\partial \bar{v}}{\partial y}} - \rho_0 \bar{u} \frac{\partial}{\partial y}(\overline{u'v'}) - \cancel{\overline{u'v'} \rho_0 \frac{\partial \bar{u}}{\partial y}} + \cancel{\rho_0 \overline{u'v'} \frac{\partial \bar{u}}{\partial y}} + \rho_0 \bar{u} \bar{X} \quad ✓$$

$$\frac{d}{dt} \int_0^\infty \int_{-D}^{D} \int_0^L \frac{\rho_0 \bar{u}^2}{2} dx\, dy\, dz \frac{1}{A} = \frac{-1}{A} \int_0^\infty \int_0^L \int_{-D}^{+D} \frac{\partial}{\partial y}(\rho_0 \bar{v} \bar{\Phi})\, dy\, dx\, dz + \frac{1}{A} \iiint \rho_0 \bar{\Phi} \frac{\partial \bar{v}}{\partial y} dV$$

$\bar{v} = 0$ at $\pm D$

$$- \frac{1}{A} \int_0^\infty \int_0^L \int_{-D}^{+D} \frac{\partial}{\partial y}(\rho_0 \bar{u}\, \overline{u'v'})\, dy\, dx\, dz + \frac{1}{A} \iiint \rho_0 \overline{u'v'} \frac{\partial \bar{u}}{\partial y} dV + \frac{1}{A} \iiint \rho_0 \bar{u} \bar{X} dV$$

$\bar{u} = 0$ at $\pm D$?

$\overline{u'v'} = 0$ at $\pm D$ ✓

$$\boxed{\frac{d}{dt}\left\langle \rho_0 \bar{u}^2/2 \right\rangle = \left\langle \rho_0 \bar{\Phi} \frac{\partial \bar{v}}{\partial y} \right\rangle + \left\langle \rho_0 \overline{u'v'} \frac{\partial \bar{u}}{\partial y} \right\rangle + \left\langle \rho_0 \bar{u} \bar{X} \right\rangle}$$

$\Phi = gh = \frac{p}{\rho_0} \rightarrow \rho_0 \Phi \sim p$

These RHS terms evidently represent

(1) work done by the zonal mean pressure force

(2) conversion of eddy kinetic to zonal-mean kinetic energy

(3) dissipation of zonal mean flow due to eddy stresses

$$\rho \frac{De}{Dt} = -p \vec{\nabla} \cdot \vec{v} + \rho J$$

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2} \vec{u} \cdot \vec{u} + \Phi\right) = -\vec{u} \cdot \nabla p$$

$$\rho_0 \int_{-D}^{+D} \bar{\Phi} \frac{\partial \bar{v}}{\partial y} dy = \cancel{\bar{\Phi} \bar{v} \Big|_{-D}^{+D}} - \int_{-D}^{+D} \bar{v} \frac{\partial \bar{\Phi}}{\partial y} dy \leftarrow$$

0

$$\left\langle \rho_0 \bar{\Phi} \frac{\partial \bar{v}}{\partial y} \right\rangle = -\left\langle \rho_0 \bar{v} \frac{\partial \bar{\Phi}}{\partial y} \right\rangle$$

$-\vec{v} \cdot \vec{\nabla} p$

Term (1) can be rewritten

$$\left\langle \rho_0 \bar{\Phi} \frac{\partial \bar{v}}{\partial y} \right\rangle = \frac{1}{A} \int_0^L \int_{-D}^{+D} \int_0^\infty \cancel{\rho_0} \bar{\Phi} \frac{\partial}{\partial z}(\rho_0 \bar{w}) \cancel{\rho_0^{-1}}\, dz\, dx\, dy = \frac{-1}{A} \int_0^L \int_{-D}^{+D} \left[\bar{\Phi} \cancel{\rho_0 \bar{w}} \Big|_0^\infty - \int_0^\infty \rho_0 \bar{w} \frac{\partial \bar{\Phi}}{\partial z} dz \right] dy\, dx$$

0

$$= \left\langle \rho_0 \bar{w} \frac{\partial \bar{\Phi}}{\partial z} \right\rangle = \frac{R}{H} \left\langle \rho_0 \bar{w} \bar{T} \right\rangle > 0$$ when cold air sinks or warm air rises on average (zonally averaged, that is)

The zonal-mean available potential energy is defined as

$$\bar{P} = \frac{1}{2} \frac{\rho_0}{N^2} \left(\frac{\partial \bar{\Phi}}{\partial z}\right)^2$$

$$\frac{\rho_0}{N^2}\left(\frac{\partial \bar{\Phi}}{\partial z}\right) \frac{\partial}{\partial t}\left(\frac{\partial \bar{\Phi}}{\partial z}\right) + \frac{\rho_0}{\cancel{N^2}} \cancel{N^2}\, \bar{w} \frac{\partial \bar{\Phi}}{\partial z} = -\frac{\rho_0}{N^2} \frac{\partial \bar{\Phi}}{\partial z} \frac{\partial}{\partial y}\left(\overline{v' \frac{\partial \Phi'}{\partial z}}\right) + \frac{\rho_0}{N^2} \frac{\partial \bar{\Phi}}{\partial z} \frac{\bar{J} R}{H c_p}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2} \frac{\rho_0}{N^2} \left(\frac{\partial \bar{\Phi}}{\partial z}\right)^2\right) = -\rho_0 \bar{w} \frac{\partial \bar{\Phi}}{\partial z} - \frac{\rho_0}{N^2} \frac{\partial \bar{\Phi}}{\partial z} \frac{\partial}{\partial y}\left(\overline{v' \frac{\partial \Phi'}{\partial z}}\right) + \frac{\rho_0}{N^2} \frac{\partial \bar{\Phi}}{\partial z} \frac{\bar{J} R}{H c_p}$$

$$\boxed{\frac{d}{dt}\left\langle \frac{1}{2} \frac{\rho_0}{N^2}\left(\frac{\partial \bar{\Phi}}{\partial z}\right) \right\rangle = \underbrace{-\left\langle \rho_0 \bar{w} \frac{\partial \bar{\Phi}}{\partial z} \right\rangle}_{} - \underbrace{\left\langle \frac{\rho_0}{N^2} \frac{\partial \bar{\Phi}}{\partial z} \frac{\partial}{\partial y}\left(\overline{v' \frac{\partial \Phi'}{\partial z}}\right) \right\rangle}_{} + \underbrace{\left\langle \frac{\rho_0}{N^2} \frac{\partial \bar{\Phi}}{\partial z} \frac{\bar{J} R}{c_p H} \right\rangle}_{}}$$

equal + opposite to first term in $\overline{KE}$ equation
- conversion between $\overline{APE}$ and $\overline{KE}$

this term is like $\bar{T}$ times the meridional eddy heat flux convergence
- represents conversion between zonal-mean APE + eddy PE'

correlation between temperature and diabatic heating
warming of regions w/ high T generates globally-avg'd APE

$$\rho_0 u' \frac{\partial u'}{\partial t} + \rho_0 \bar{u} u' \frac{\partial u'}{\partial x} - \rho_0 u' v' \left(f_0 - \frac{\partial \bar{u}}{\partial y}\right) = -\rho_0 u' \frac{\partial \Phi'}{\partial x} + \rho_0 u' X'$$

$$\rho_0 v' \frac{\partial v'}{\partial t} + \rho_0 \bar{u} v' \frac{\partial v'}{\partial x} + \rho_0 v' u' f_0 = -\rho_0 v' \frac{\partial \Phi'}{\partial y} + \rho_0 v' Y'$$

$$\frac{\partial}{\partial t}\left[\frac{\rho_0}{2}(u'^2+v'^2)\right] + \rho_0 \bar{u} \frac{\partial}{\partial x}\left(\frac{1}{2}u'^2 + \frac{1}{2}v'^2\right) + \rho_0 u'v' \frac{\partial \bar{u}}{\partial y} = -\rho_0(\vec{v}'\cdot\vec{\nabla}\Phi') + \rho_0(u'X'+v'Y')$$

$$\frac{\partial}{\partial t}\int_0^L \left[\frac{\rho_0(u'^2+v'^2)}{2}\right]\frac{dx}{L} + \frac{\rho_0\bar{u}}{L}\int_0^L \frac{\partial}{\partial x}\left(\frac{u'^2+v'^2}{2}\right)dx + \frac{\rho_0}{L}\int_0^L u'v'\frac{\partial\bar{u}}{\partial y}dx = -\frac{\rho_0}{L}\int_0^L \vec{v}'\cdot\vec{\nabla}\Phi'\, dx + \frac{\rho_0}{L}\int_0^L (u'X'+v'Y')dx$$

$$\frac{\partial}{\partial t}\left[\frac{\rho_0(\overline{u'^2}+\overline{v'^2})}{2}\right] = -\rho_0 \overline{u'v'\frac{\partial\bar{u}}{\partial y}} - \rho_0\overline{\vec{v}'\cdot\vec{\nabla}\Phi'} + \rho_0\overline{(u'X'+v'Y')}$$

$$\frac{\partial}{\partial t}\int_0^\infty\int_{-D}^{+D} \rho_0\frac{(\overline{u'^2}+\overline{v'^2})}{2}\,dy\,dz = -\left\langle \rho_0\overline{u'v'}\frac{\partial\bar{u}}{\partial y}\right\rangle - \underbrace{\frac{1}{A}\int_0^\infty\int_{-D}^{+D}\rho_0\int_0^L\left(u'\frac{\partial\Phi'}{\partial x}+v'\frac{\partial\Phi'}{\partial y}\right)dx\,dy\,dz} + \frac{1}{A}\int_0^\infty\int_{-D}^{+D}\rho_0\overline{(u'X'+v'Y')}\,dy\,dz$$

$\langle\ \rangle = \frac{1}{2D}\int_0^\infty\int_{-D}^{+D}(\ )\,dy\,dz$

because

$\langle\overline{(\cdot)}\rangle = \underbrace{\frac{1}{2DL}}_{A}\iiint(\cdot)\,dV$

periodic BCs

$$\frac{1}{A}\int_0^\infty\int_{-D}^{+D}\rho_0\left[\cancelto{0}{u'\Phi'\Big|_0^L} - \int_0^L \Phi'\frac{\partial u'}{\partial x}dx\right]dy\,dz + \frac{1}{A}\int_0^\infty\int_0^L\rho_0\left[\cancelto{0}{v'\Phi'\Big|_{-D}^{+D}} - \int_{-D}^{+D}\Phi'\frac{\partial v'}{\partial y}dy\right]dx\,dz$$

$$\frac{d}{dt}\left\langle\rho_0\left(\frac{\overline{u'^2}+\overline{v'^2}}{2}\right)\right\rangle = -\left\langle\rho_0\overline{u'v'}\frac{\partial\bar{u}}{\partial y}\right\rangle - \underbrace{\left\langle\rho_0\overline{\Phi'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)}\right\rangle} + \left\langle\rho_0\overline{(u'X'+v'Y')}\right\rangle$$

$$\left\langle\rho_0\overline{\Phi'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)}\right\rangle = \frac{1}{2D}\int_0^\infty\int_{-D}^{+D}\left[\frac{1}{L}\int_0^L\Phi'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)dx\right]dy\,dz$$

$$= \left\langle\cancel{\rho_0}\overline{\Phi'\left[-\frac{1}{\cancel{\rho_0}}\frac{\partial}{\partial z}(\rho_0 w')\right]}\right\rangle = -\left\langle\overline{\Phi'\frac{\partial}{\partial z}(\rho_0 w')}\right\rangle = -\frac{1}{A}\int_0^L\int_{-D}^{+D}\int_0^\infty \Phi'\frac{\partial}{\partial z}(\rho_0 w')\,dz\,dy\,dx$$

$$= -\frac{1}{A}\int_0^L\int_{-D}^{+D}\left[\cancelto{0}{\Phi'\rho_0 w'\Big|_0^\infty} - \int_0^\infty \rho_0 w'\frac{\partial\Phi'}{\partial z}dz\right]dy\,dx = \left\langle\rho_0\overline{w'\frac{\partial\Phi'}{\partial z}}\right\rangle$$

global Eddy KE equation

$$\boxed{\frac{d}{dt}\left\langle\frac{\rho_0}{2}\left[\overline{u'^2}+\overline{v'^2}\right]\right\rangle = -\left\langle\rho_0\overline{u'v'}\frac{\partial\bar{u}}{\partial y}\right\rangle + \left\langle\rho_0\overline{w'\frac{\partial\Phi'}{\partial z}}\right\rangle + \left\langle\rho_0\overline{(u'X'+v'Y')}\right\rangle}$$

Next, we still need an equation for the global eddy potential energy

$$\frac{\rho_0}{N^2}\frac{\partial\Phi'}{\partial z}\frac{\partial}{\partial t}\left(\frac{\partial\Phi'}{\partial z}\right) + \frac{\rho_0}{N^2}\frac{\partial\Phi'}{\partial z}\bar{u}\frac{\partial}{\partial x}\left(\frac{\partial\Phi'}{\partial z}\right) + \frac{\rho_0}{N^2}v'\frac{\partial}{\partial y}\left(\frac{\partial\bar{\Phi}}{\partial z}\right)\frac{\partial\Phi'}{\partial z} + \rho_0\frac{\partial\Phi'}{\partial z}w' = \frac{\rho_0}{N^2}\frac{\partial\Phi'}{\partial z}\frac{J'R}{Hc_p}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\frac{\rho_0}{N^2}\left(\frac{\partial\Phi'}{\partial z}\right)^2\right] + \frac{\rho_0}{N^2}\bar{u}\frac{\partial}{\partial x}\left(\frac{1}{2}\left(\frac{\partial\Phi'}{\partial z}\right)^2\right) + \frac{\rho_0}{N^2}\frac{\partial\Phi'}{\partial z}v'\frac{\partial^2\bar{\Phi}}{\partial y\partial z} + \rho_0\frac{\partial\Phi'}{\partial z}w' = \cdots$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\frac{\rho_0}{N^2}\overline{\left(\frac{\partial\Phi'}{\partial z}\right)^2}\right] + \frac{\rho_0}{N^2}\frac{\bar{u}}{L}\int_0^L\cancelto{0}{\frac{\partial}{\partial x}\left(\frac{1}{2}\left(\frac{\partial\Phi'}{\partial z}\right)^2\right)}dx + \frac{\rho_0}{N^2}\overline{v'\frac{\partial\Phi'}{\partial z}}\frac{\partial^2\bar{\Phi}}{\partial y\partial z} + \rho_0\overline{w'\frac{\partial\Phi'}{\partial z}} = \frac{\rho_0}{N^2}\frac{R}{Hc_p}\overline{J'\frac{\partial\Phi'}{\partial z}}$$

$$\boxed{\frac{d}{dt}\left\langle\frac{1}{2}\frac{\rho_0}{N^2}\overline{\left(\frac{\partial\Phi'}{\partial z}\right)^2}\right\rangle = -\left\langle\rho_0\overline{w'\frac{\partial\Phi'}{\partial z}}\right\rangle + \left\langle\frac{\rho_0 R}{N^2 Hc_p}\overline{J'\frac{\partial\Phi'}{\partial z}}\right\rangle - \left\langle\frac{\rho_0}{N^2}\overline{v'\frac{\partial\Phi'}{\partial z}}\frac{\partial^2\bar{\Phi}}{\partial y\partial z}\right\rangle}$$

The equations thus far derived are

Kinetic energy of the latitudinal and height averaged mean zonal flow →

$$\frac{d}{dt}\underbrace{\left\langle \frac{\rho_0}{2}\bar{u}^2 \right\rangle}_{\bar{K}} = \underbrace{\left\langle \rho_0 \bar{w} \frac{\partial \bar{\Phi}}{\partial z} \right\rangle}_{= [\bar{P}\cdot\bar{K}]} + \underbrace{\left\langle \rho_0 \overline{u'v'} \frac{\partial \bar{u}}{\partial y} \right\rangle}_{[K'\cdot\bar{K}]} + \underbrace{\left\langle \rho_0 \bar{u}\bar{X} \right\rangle}_{= \bar{\varepsilon}}$$

available potential energy of the mean global atmosphere

$$\frac{d}{dt}\underbrace{\left\langle \frac{1}{2}\frac{\rho_0}{N^2}\left(\frac{\partial \bar{\Phi}}{\partial z}\right)^2 \right\rangle}_{= \bar{P}} = -\underbrace{\left\langle \rho_0 \bar{w} \frac{\partial \bar{\Phi}}{\partial z} \right\rangle}_{[\bar{P}\cdot\bar{K}]} - \underbrace{\left\langle \frac{\rho_0}{N^2} \frac{\partial \bar{\Phi}}{\partial z} \frac{\partial}{\partial y}\left(\overline{v' \frac{\partial \Phi'}{\partial z}}\right) \right\rangle}_{= [P'\cdot\bar{P}] = -[\bar{P}\cdot P']} + \underbrace{\left\langle \frac{\rho_0}{N^2}\frac{R}{c_p H} \bar{J} \frac{\partial \bar{\Phi}}{\partial z} \right\rangle}_{= \bar{R}}$$

$\bar{R} > 0$ where diabatic heating occurs when it's warm thereby increasing the slope of θ surfaces

global mean eddy kinetic energy

$$\frac{d}{dt}\underbrace{\left\langle \frac{\rho_0}{2}\left(\overline{u'^2} + \overline{v'^2}\right) \right\rangle}_{= K'} = \underbrace{\left\langle \rho_0 \overline{w' \frac{\partial \Phi'}{\partial z}} \right\rangle}_{= [P'\cdot K']} - \underbrace{\left\langle \rho_0 \overline{u'v'} \frac{\partial \bar{u}}{\partial y} \right\rangle}_{[K'\cdot\bar{K}]} + \underbrace{\left\langle \rho_0 \left(\overline{u'X' + v'Y'}\right) \right\rangle}_{= \varepsilon'}$$

$$\frac{d}{dt}\underbrace{\left\langle \frac{1}{2}\frac{\rho_0}{N^2}\overline{\left(\frac{\partial \Phi'}{\partial z}\right)^2} \right\rangle}_{= P'} = -\underbrace{\left\langle \rho_0 \overline{w' \frac{\partial \Phi'}{\partial z}} \right\rangle}_{= [P'\cdot K']} - \underbrace{\left\langle \frac{\rho_0}{N^2} \overline{v' \frac{\partial \Phi'}{\partial z}} \frac{\partial^2 \bar{\Phi}}{\partial y \partial z} \right\rangle}_{= \,? \leftarrow [P'\cdot\bar{P}]} + \underbrace{\left\langle \frac{\rho_0}{N^2}\frac{R}{H c_p} \overline{J' \frac{\partial \Phi'}{\partial z}} \right\rangle}_{= R'}$$

see below ↓

$$\frac{1}{A}\iiint \frac{\rho_0}{N^2} \overline{v' \frac{\partial \Phi'}{\partial z}} \frac{\partial}{\partial y}\left(\frac{\partial \bar{\Phi}}{\partial z}\right) dy\,dx\,dz = \frac{1}{A}\iint \frac{\rho_0}{N^2}\left[\cancel{\overline{v' \frac{\partial \Phi'}{\partial z}} \frac{\partial \bar{\Phi}}{\partial z}\Bigg|_{-D}^{+D}}^{\,0} - \int_{-D}^{+D} \frac{\partial \bar{\Phi}}{\partial z} \frac{\partial}{\partial y}\left(\overline{v' \frac{\partial \Phi'}{\partial z}}\right) dy \right] dx\,dz$$

$$= -\left\langle \frac{\rho_0}{N^2} \frac{\partial \bar{\Phi}}{\partial z} \frac{\partial}{\partial y}\left(\overline{v' \frac{\partial \Phi'}{\partial z}}\right) \right\rangle$$

$$\Rightarrow \frac{d\bar{K}}{dt} = [\bar{P}\cdot\bar{K}] + [K'\cdot\bar{K}] + \bar{\varepsilon} = [\bar{P}\cdot\bar{K}] + [K'\cdot\bar{K}] + \bar{\varepsilon}$$

$$\frac{d\bar{P}}{dt} = -[\bar{P}\cdot\bar{K}] - [\bar{P}\cdot P'] + \bar{R} = -[\bar{P}\cdot\bar{K}] + [P'\cdot\bar{P}] + \bar{R}$$

$$\frac{dK'}{dt} = [P'\cdot K'] - [K'\cdot\bar{K}] + \varepsilon' = [P'\cdot K'] - [K'\cdot K'] + \varepsilon'$$

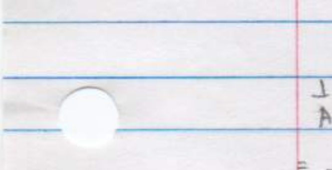
$$\frac{dP'}{dt} = -[P'\cdot K'] + [\bar{P}\cdot P'] + R' = -[P'\cdot K'] - [P'\cdot\bar{P}] + R'$$

$$\frac{d}{dt}\left[\bar{K} + \bar{P} + K' + P'\right] = \bar{R} + R' + \bar{\varepsilon} + \varepsilon'$$

For adiabatic, inviscid flow the total energy is conserved

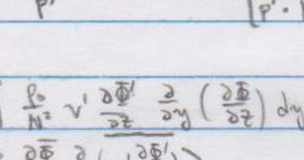
- the below diagram may be summarized as follows: radiative heating near 30°N cooling at higher latitudes creates zonal mean available potential energy. Baroclinic instability extracts APE converting P', R' into eddy kinetic energy in the form of vertical motion (and lateral), which is dissipated internally and at the surface. Eddy kinetic energy is sometimes converted to zonal mean KE $= \bar{K}$, itself which is converted to $\bar{P}$ associated with the Ferrell circulation, which cold air rises and warm air sinks

$$\begin{array}{ccccccc} \bar{R} & \rightarrow & \bar{P} & \rightarrow & P' & \leftarrow & R' \\ & & \uparrow{\scriptstyle \text{Ferrell, small}} & & \downarrow & & \\ \bar{\varepsilon} & \underset{\text{small}}{\swarrow} & \bar{K} & \leftarrow & K' & \rightarrow & \varepsilon' \end{array}$$

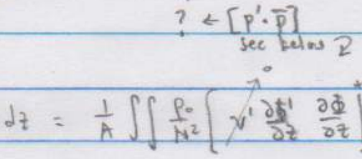


smaller meridional gradient in OLR

$\bar{R} > 0$ since troposphere is heated diabatically where warm



- greater thickness at lower latitudes where warm
- westerlies increasing w/ height; thermal wind
- sloping P surfaces → $\overline{APE} = \bar{P} > 0$ zonal mean available potential energy

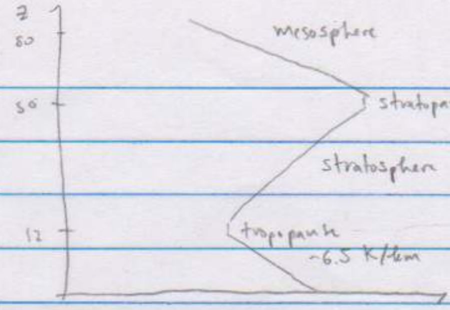


- baroclinic instability results in poleward heat transport, EQ-ward transport of cold air
- For dry atmos. $R' < 0$ since in warm location (σT_B^4) radiative emission greater than background in cold location (OLR less)

← That is, the atmosphere is cooled where warm + heated where cool (gains heat from new surroundings)

- for moist atmos. $R' > 0$ since latent heating when warm > radiative process

The middle atmosphere



O_3 absorption of solar UV
CO_2 radiative (IR) cooling

radiative cooling
convective upward transport
~~of latent and sensible heat~~
due to solar absorption
at surface

- stratosphere ~12 km - 50 km
- mesosphere ~50 - 80 km
- See Holton for zonal mean temperature + zonal wind structure near the solstices
- thermal wind equation: $\frac{\partial \bar{u}}{\partial z} = -\frac{R}{f_0 H}\frac{\partial \bar{T}}{\partial y}$
- the summer pole is constantly illuminated by sunlight and O_3 absorption in the stratosphere gives rise to warm temperatures

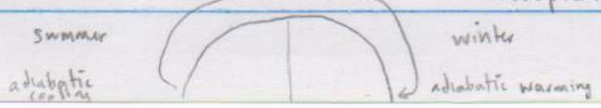
	NH ($f_0 > 0$)	SH ($f_0 < 0$)
January :	$\frac{\partial \bar{T}}{\partial y} < 0 \Rightarrow \frac{\partial \bar{u}}{\partial z} > 0$ westerlies	$\frac{\partial \bar{T}}{\partial y} < 0 \Rightarrow \frac{\partial \bar{u}}{\partial z} < 0$ easterlies
July :	$\frac{\partial \bar{T}}{\partial y} > 0 \Rightarrow \frac{\partial \bar{u}}{\partial z} < 0$ easterlies	$\frac{\partial \bar{T}}{\partial y} > 0 \Rightarrow \frac{\partial \bar{u}}{\partial z} > 0$ westerlies

- thus, the summer hemisphere middle atmosphere has easterlies, whereas the winter hemisphere has westerlies
- In contrast the tropospheric winds are always westerly, but the subtropical jets more poleward in summer, following the radiatively driven ∇T. NH: $\partial \bar{T}/\partial y < 0 \Rightarrow \frac{\partial \bar{u}}{\partial z} > 0$; SH: $\partial \bar{T}/\partial y > 0 \Rightarrow \frac{\partial \bar{u}}{\partial z} > 0$
- In the troposphere the atmosphere cools radiatively, while the surface heats up due to solar absorption, thereby destabilizing $\Gamma = -\frac{\partial T}{\partial z}$
- strong static stability of stratosphere generally inhibits troposphere - stratosphere exchange
- the middle atmosphere is closer to a state of radiative equilibrium i.e., the divergence of the radiative flux is zero, $\partial F_{net}/\partial z = 0$, so $(\partial T/\partial t)_{rad} = \frac{1}{\rho c_p}(\partial F_{net}/\partial z) = 0$, radiation in = radiation out.
- this radiatively determined state, however, is disturbed by dynamics: upward propagating waves (gravity, inertia-gravity, Kelvin, Rossby, Rossby-gravity)

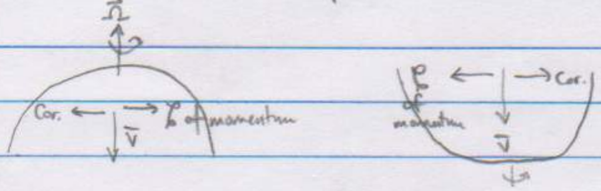
temperature difference between the two poles is smaller than observed

- the observed temperature structure is cooler than the radiative equilibrium state at the summer pole and warmer than the RE state at the winter pole
- vertically propagating gravity waves break in the mesosphere, depositing zonal momentum; upon reaching a critical level their amplitude increases with altitude as the atmosphere becomes more tenuous
- this deposition of momentum accelerates, or decelerates, the mean zonal flow such that Coriolis opposes it in steady state with a meridional cell with rising + adiabatic cooling at the summer pole, sinking and adiabatic warming at the winter pole

mesospheric circulation



Consider the NH summer: Easterlies in NH middle atmosphere, southward meridional circulation in the mesosphere



For this type of mesospheric circulation, gravity wave momentum convergence must be opposite in both hemispheres.

Gravity wave critical levels:

$$(c_x - \bar{u})^2 = \frac{N^2}{k^2 + m^2} \Rightarrow m^2 = \frac{N^2}{(c_x - \bar{u})^2} - k^2$$

$c_x \to \bar{u}$

LHS $= 0$

$c_x > 0$ eastward propagating gravity wave cannot penetrate westerly zonal wind $\bar{u} > 0$. $\Rightarrow m^2 \to \infty$, $L_z \to 0$

$c_x < 0$, westward propagating gravity wave can penetrate westerly wind $\bar{u} > 0$ LHS can't be zero

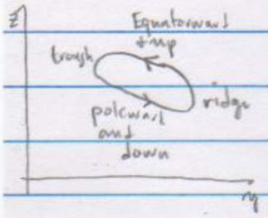
but $c_x < 0$, westward propagating gravity wave cannot penetrate easterly wind

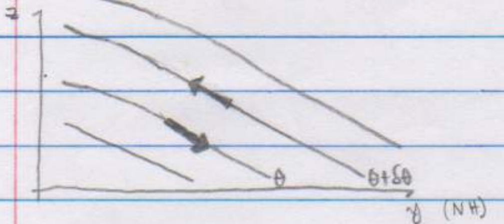
$$\overline{u'w'} = -\frac{1}{2}\frac{m}{k} W_0^2$$
> 0

$c_x > 0 \Rightarrow k > 0$, $m < 0$ downward phase propagation corresponds to upward energy propagation

Eulerian stratospheric circulation

adiabatic circulation must be along isentropes, or surfaces of constant θ

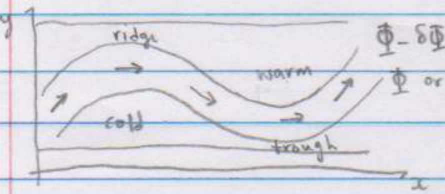


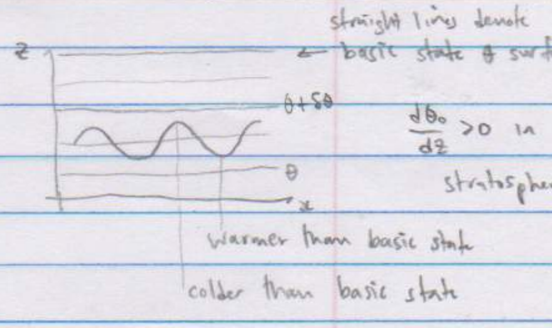


$$c_p \frac{D \ln\theta}{Dt} = \frac{J}{T} \quad (2.46)$$ or entropy

i.e. potential temperature can only change through diabatic heating

planetary wave





$\frac{d\theta_0}{dz} > 0$ in stratosphere

must account for diabatic processes

Transformed Eulerian Mean

$$\frac{\partial \bar u}{\partial t} - f_0 \bar v^* = \frac{1}{\rho_0}\vec\nabla\cdot\vec F + \bar X \equiv \bar G$$ total zonal force

$$\frac{\partial \bar T}{\partial t} + \frac{H}{R}N^2\bar w^* = -\alpha_r\left[\bar T - \bar T_r(y,z,t)\right]$$

radiative equilibrium state

$$\frac{\partial \bar v^*}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 \bar w^*) = 0$$

$$f_0\frac{\partial \bar u}{\partial z} + \frac{R}{H}\frac{\partial \bar T}{\partial y} = 0$$

Recall:

$$\vec F = -\rho_0\overline{u'v'}\,\hat j + \frac{\rho_0 f_0 R}{N^2 H}\overline{v'T'}\,\hat k$$

is the Eliassen-Palm flux vector

$[\alpha_r] = \frac{1}{\tau}$

τ is the radiative relaxation timescale, or ~~the time over which~~ the radiatively determined equilibrium state would be approached in the ~~absence of eddy motions~~

An idealized model in which $T_r(y,z,t) \to T_r(z)$ can be used to examine the influence of eddy/wave forcing of the stratospheric circulation. This would apply only for a small latitudinal strip. An equation only in $\bar w^*$ can be derived:

$$\frac{\partial}{\partial t}\left(\frac{\partial \bar u}{\partial y}\right) - f_0\frac{\partial \bar v^*}{\partial y} = \frac{\partial \bar G}{\partial y} \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial \bar u}{\partial y}\right) + \frac{f_0}{\rho_0}\frac{\partial}{\partial z}(\rho_0\bar w^*) = \frac{\partial \bar G}{\partial y} \Rightarrow \frac{\partial}{\partial t}\frac{\partial}{\partial y}\left(\frac{\partial \bar u}{\partial z}\right) + f_0\frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\bar w^*)\right] = \frac{\partial}{\partial z}\frac{\partial \bar G}{\partial y}$$

Figuring this entire derivation out took me at least 2 hrs →

$$\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial y}\left(-\frac{R}{f_0 H}\frac{\partial \bar T}{\partial y}\right) + f_0\frac{\partial}{\partial t}\frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\bar w^*)\right] = \frac{\partial}{\partial t}\frac{\partial}{\partial z}\frac{\partial \bar G}{\partial y}$$

$$\frac{\partial^2}{\partial y^2}\frac{\partial^2 \bar T}{\partial t^2} + \frac{H}{R}N^2\frac{\partial}{\partial t}\frac{\partial^2 \bar w^*}{\partial y^2} = -\alpha\frac{\partial^2}{\partial y^2}\frac{\partial \bar T}{\partial t} = -\alpha\frac{\partial}{\partial y}\frac{\partial}{\partial t}\left(\frac{\partial \bar T}{\partial y}\right) = -\alpha\frac{\partial}{\partial y}\frac{\partial}{\partial t}\left(-\frac{f_0 H}{R}\frac{\partial \bar u}{\partial z}\right) = \frac{\alpha f_0 H}{R}\overbrace{\frac{\partial}{\partial y}\left[\frac{\partial}{\partial t}\left(\frac{\partial \bar u}{\partial z}\right)\right]}$$

$$= \frac{\alpha f_0 H}{R}\left[\frac{\partial}{\partial z}\frac{\partial \bar G}{\partial y} - f_0\frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\bar w^*)\right]\right] \Rightarrow \frac{\partial^2}{\partial t^2}\frac{\partial^2 \bar T}{\partial y^2} = \frac{\alpha f_0 H}{R}[\dots] - \frac{H}{R}N^2\frac{\partial}{\partial t}\frac{\partial^2 \bar w^*}{\partial y^2}$$

$$-\frac{\cancel{R}}{f_0\cancel{H}}\left[\frac{\alpha f_0\cancel{H}}{\cancel{R}}\left(\frac{\partial}{\partial z}\frac{\partial \bar G}{\partial y} - f_0\frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\bar w^*)\right]\right) - \frac{\cancel{H}}{\cancel{R}}N^2\frac{\partial}{\partial t}\frac{\partial^2 \bar w^*}{\partial y^2}\right] + f_0\frac{\partial}{\partial t}\frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\bar w^*)\right] = \frac{\partial}{\partial t}\frac{\partial}{\partial z}\frac{\partial \bar G}{\partial y}$$

$$\left(\frac{\partial}{\partial t} + \alpha\right)f_0\frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\bar w^*)\right] + \frac{N^2}{f_0}\frac{\partial}{\partial t}\frac{\partial^2 \bar w^*}{\partial y^2} = \left(\frac{\partial}{\partial t} + \alpha\right)\frac{\partial}{\partial z}\frac{\partial \bar G}{\partial y}$$

Now if we take $\overline{w}^* = \hat{w}(y,z)\, e^{i\sigma t}$

$\overline{G} = \hat{G}(y,z)\, e^{i\sigma t}$ periodic forcing drives periodic vertical motion

$\frac{\partial}{\partial t}(\;) \to i\sigma(\;)$, so that we obtain

$$(i\sigma + \alpha)\frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 \hat{w})\right] + \frac{N^2}{f_0^2}\, i\sigma \frac{\partial^2 \hat{w}}{\partial y^2} = (i\sigma + \alpha)\frac{1}{f_0}\frac{\partial}{\partial z}\frac{\partial \hat{G}}{\partial y}$$

$$\boxed{\frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 \hat{w})\right] + \frac{i\sigma}{i\sigma + \alpha}\frac{N^2}{f_0^2}\frac{\partial^2 \hat{w}}{\partial y^2} = \frac{1}{f_0}\frac{\partial}{\partial z}\frac{\partial \hat{G}}{\partial y}}$$

Apparently this is an elliptic equation (like the Laplace, Poisson, + Helmholtz equations)

The term on the right represents forcing of $\overline{w}^*$, which depends on the vertical and meridional structure of $\overline{G}$. Localized forcing of the mean zonal wind will produce meridional/vertical motion so as to maintain zonal flow in thermal wind balance with the meridional ∇T.

We consider three cases of forcing:

High frequency

(1) $\sigma \gg \alpha$, in which case $\frac{i\sigma}{i\sigma+\alpha} \to 1$. Away from forcing, $\overline{G} \approx 0$ and scaling shows that

typical values
$f_0 = 10^{-4}$
$N = 10^{-2}$

$$\frac{\cancel{\rho_0}\hat{w}}{\cancel{\rho_0}(\delta z)^2} = -\frac{N^2}{f_0^2}\frac{\hat{w}}{(\delta y)^2} \to \delta z \sim \frac{f_0}{N}\,\delta y$$

$$\doteq \frac{10^2}{10^4}\,\delta y$$

$$= 10^{-2}\,\delta y$$

vertical motions much smaller than meridional motions.

Low frequency such as the annual cycle

(ii) $\sigma < \alpha$ vertical penetration scale increased from before

(iii) steady state, $\sigma/\alpha \to 0 \Rightarrow \overline{w}^* = \hat{w}(y,z)$ and $\overline{G} = \hat{G}(y,z)$

From the zonal momentum equation, $\frac{\partial \overline{u}}{\partial t} = 0 \Rightarrow -f_0\overline{v}^* = \overline{G} = \frac{1}{\rho_0}\vec{\nabla}\cdot\vec{F} + \overline{X}$

Thus, there is a balance between Coriolis and the zonal force

$$\frac{\partial \overline{v}^*}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\overline{w}^*) = 0 \Rightarrow -\frac{1}{f_0}\frac{\partial \overline{G}}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\overline{w}^*) = 0 \Rightarrow \frac{\partial}{\partial z}(\rho_0\overline{w}^*) = \frac{\rho_0}{f_0}\frac{\partial \overline{G}}{\partial y}$$

$$\int_z^\infty \frac{\partial}{\partial z}(\rho_0\overline{w}^*)\,dz = \frac{1}{f_0}\int_z^\infty \rho_0\frac{\partial \overline{G}}{\partial y}\,dz' \Rightarrow \underbrace{\cancel{\rho_0\overline{w}^*}(z\to\infty)}_{0} - (\rho_0\overline{w}^*)(z) = \frac{1}{f_0}\int_z^\infty \rho_0\frac{\partial \overline{G}}{\partial y}\,dz'$$

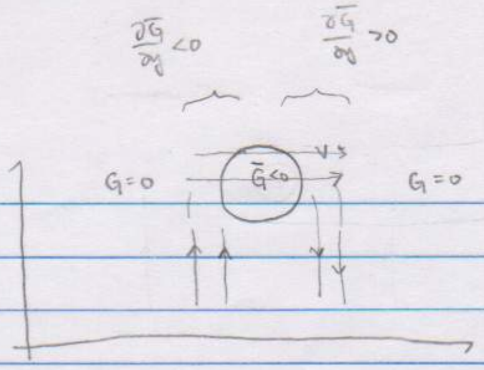
$$\boxed{\overline{w}^* = \frac{-1}{\rho_0 f_0}\frac{\partial}{\partial y}\int_z^\infty \rho_0\overline{G}\,dz'}$$

above the forcing region, $\overline{G} = 0$ and thus so is $\overline{w}^*$.

Consider a westward zonal force, $\bar{G}$ of easterly momentum

$$-f\bar{v}^* = \bar{G}$$

$$\bar{w}^* = -\frac{1}{\rho_0 f_0}\frac{\partial}{\partial y}\int_z^\infty \rho_0 \bar{G}\, dz'$$



integral, and hence $\bar{w}^*$, zero above this level

constant $\bar{w}^*$ below this level

Also, note that

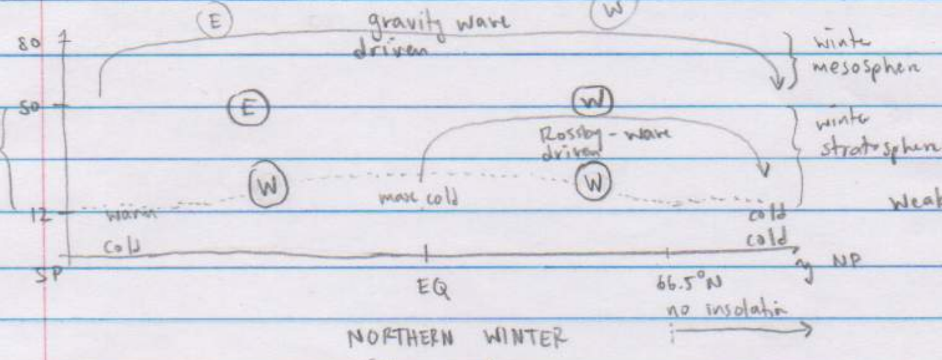
$$\cancel{\frac{\partial \bar{T}}{\partial t}}^{\,0} + \frac{H}{R}N^2\bar{w}^* = -\alpha\left[\bar{T} - T_r(z)\right]$$

$$\Rightarrow \frac{N^2 H}{R\rho_0 f_0}\frac{\partial}{\partial y}\int_z^\infty \rho_0 \bar{G}\, dz' = \alpha\left[T - T_r(z)\right] \text{, or}$$

$$\bar{T}(y,z) - \bar{T}_r(z) = \frac{N^2 H}{\alpha R \rho_0 f_0}\frac{\partial}{\partial y}\int_z^\infty \rho_0 \bar{G}\, dz$$

Thus, in steady state the departure of the temperature field from radiative equilibrium is proportional to the meridional gradient of the zonal force distribution

Looking at Holton figures 12.2 + 12.4, the largest deviation from radiative equilibrium occurs in the summer and winter mesosphere and the winter polar stratosphere
cooler, warmer than RE — warmer than RE ($\partial F_{net}/\partial z = 0$)



strong tropopause due to solar UV absorption by O_3 in stratosphere

weakly defined tropopause

NORTHERN WINTER
(JANUARY)

As we will see,
Rossby waves cannot vertically penetrate easterly zonal winds

Vertical Propagation of Rossby waves in a constant basic state zonal flow, $\bar{u}$

$$\frac{D_g q}{Dt} = \left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right) q = 0 \ , \quad q = \nabla^2\psi + f + \frac{f_0^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial\psi}{\partial z}\right) \ , \quad \rho_0 = \rho_s e^{-z/H}$$

$$\psi = \bar{\Psi} + \psi' = -\bar{u}y + \psi' \ , \quad \vec{v}_g = \hat{k}\times\nabla\psi \quad u = \bar{u}+u' \quad v = \bar{v}^{0} + v' \quad \zeta = \bar{\zeta}^{0} + \zeta'$$

in reality $\bar{u} = \bar{u}(y,z)$

$$q = \bar{q} + q' = \nabla^2(\bar{\Psi}+\psi') + f_0 + \beta y + \frac{1}{\rho_0}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\left(\rho_0\frac{\partial}{\partial z}(\bar{\Psi}+\psi')\right) \qquad N^2 \text{ const.}$$

$$\therefore \bar{q} = \beta y \ , \quad q' = \nabla^2\psi' + \frac{1}{\rho_0}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\left(\rho_0\frac{\partial\psi'}{\partial z}\right)$$

$\frac{\partial\bar{\Psi}}{\partial z} = 0$ here, so the ambient potential vorticity gradient is β

linearized equation

$$\frac{\partial}{\partial t}(\cancel{\bar{q}} + q') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{q}+q') + v'\frac{\partial\bar{q}}{\partial y} = 0 \Rightarrow \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)q' + v'\beta = 0$$

seek plane wave solutions in x, y allowing for vertical structure

$$\psi' = \Psi(z)e^{i(kx+ly-\nu t)+z/2H} \Rightarrow q' = -(k^2+l^2)\psi' + \cdots$$

$$\frac{\partial\psi'}{\partial z} = \frac{d\Psi}{dz}e^{i(\cdots)+z/2H} + \frac{\Psi(z)}{2H}e^{i(\cdots)+z/2H} \Rightarrow \rho_0\frac{\partial\psi'}{\partial z} = \rho_s\frac{d\Psi}{dz}e^{\overbrace{i(\cdots)+z/2H-z/H}^{-z/2H}} + \frac{\rho_s}{2H}\Psi(z)e^{i(\cdots)-z/2H}$$

$$\frac{\partial}{\partial z}\left(\rho_0\frac{\partial\psi'}{\partial z}\right) = \rho_s\frac{d^2\Psi}{dz^2}e^{i(\cdots)-z/2H} + \cancel{\rho_s\frac{d\Psi}{dz}\left(-\frac{1}{2H}\right)e^{i(\cdots)-z/2H}} + \cancel{\rho_s\left(\frac{1}{2H}\right)\frac{d\Psi}{dz}e^{i(\cdots)-z/2H}} - \frac{\rho_s}{4H^2}\Psi(z)e^{i(\cdots)-z/2H}$$

$$\Rightarrow \frac{1}{\rho_0}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\left(\rho_0\frac{\partial\psi'}{\partial z}\right) = \frac{e^{z/H}}{\cancel{\rho_s}}\frac{f_0^2}{N^2}\left[\cancel{\rho_s}\frac{d^2\Psi}{dz^2}e^{i(\cdots)-z/2H} - \frac{\cancel{\rho_s}}{4H^2}\Psi(z)e^{i(\cdots)-z/2H}\right] = \frac{f_0^2}{N^2}\left[\frac{d^2\Psi}{dz^2}e^{i(\cdots)+z/2H} - \frac{\Psi(z)}{4H^2}e^{i(\cdots)+z/2H}\right]$$

$$\text{Therefore, } (-\cancel{i}\nu + \cancel{i}k\bar{u})\left[-(k^2+l^2)\Psi(z)\cancel{e^{i(\cdots)+z/2H}} + \frac{f_0^2}{N^2}\frac{d^2\Psi}{dz^2}\cancel{e^{i(\cdots)+z/2H}} - \frac{f_0^2}{N^2}\frac{\Psi(z)}{4H^2}\cancel{e^{i(\cdots)+z/2H}}\right] + \beta\cancel{i}k\Psi(z)\cancel{e^{i(\cdots)+z/2H}} = 0$$

$$(\bar{u}k-\nu)\frac{f_0^2}{N^2}\frac{d^2\Psi}{dz^2} + \Psi(z)\left[(\bar{u}k-\nu)(-k^2-l^2) - \frac{f_0^2}{N^2}\frac{(\bar{u}k-\nu)}{4H^2} + \beta k\right] = 0$$

$$\frac{d^2\Psi}{dz^2} + \underbrace{\left(\frac{N^2}{f_0^2}\left[\frac{\beta k}{\bar{u}k-\nu} - (k^2+l^2)\right] - \frac{1}{4H^2}\right)}_{m^2}\Psi(z) = 0 \ , \quad m^2 = \frac{N^2}{f_0^2}\left[\frac{\beta}{\bar{u}-c_x} - (k^2+l^2)\right] - \frac{1}{4H^2}$$

if $m^2 > 0$ then $\Psi(z) = \hat{\Psi}e^{imz}$ and vertical propagation is possible

$m^2 > 0$ requires:

$$\frac{\beta}{\bar{u}-c_x} - \overbrace{(k^2+l^2)}^{K^2} - \frac{f_0^2}{4HN^2} > 0$$

- For short waves, $k^2+l^2 = K^2$ is large and m^2 becomes negative prohibiting vertical propagation.

$$\frac{\beta}{\bar{u}-c_x} - \left[K^2 + \frac{f_0^2}{4HN^2}\right] > 0$$

- If $\bar{u} < 0$, $m^2 < 0$ no matter what
  easterlies ← also cannot produce stationary waves

$$\bar{u} - c_x < \frac{\beta}{K^2 + f_0^2/4HN^2}$$

- westerlies less than U_c required for vertical propagation
- winter stratosphere!

$$0 < \bar{u} < \underset{\substack{0 \\ \text{for stationary}}}{\cancel{c_x}} + \frac{\beta}{\left(K^2 + \frac{f_0^2}{4HN^2}\right)} \equiv U_c$$

critical Rossby velocity, a function of k, l for fixed environmental conditions

as wavelength increases, K^2 decreases, and upper bound on $\bar{u}$ inc.

dispersion relation

$$m^2 = \frac{N^2}{f_0^2}\left[\frac{\beta}{\bar{u}-c_x} - (k^2+l^2)\right] - \frac{1}{4H^2} \rightarrow \frac{\beta}{\bar{u}-c_x} = \frac{f_0^2}{N^2}\left(m^2 + \frac{1}{4H^2}\right) + k^2 + l^2$$

$$\frac{f_0^2}{N^2}\left(m^2 + \frac{1}{4H^2}\right) = \frac{\beta}{\bar{u}-c_x} - (k^2+l^2)$$

$$\bar{u} - c_x = \frac{\beta}{k^2+l^2+\frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)} \Rightarrow c_x = \bar{u} - \frac{\beta}{K^2+\frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)}$$

reduces to barotropic R-wave if $m = 0$ ($\lambda_z \to \infty$) or $H \to \infty$ constant density fluid

$$c_{gx} = \frac{\partial \nu}{\partial k} = \bar{u} - \frac{[\cdots]\beta - \beta k(2k)}{[\cdots]^2} = \bar{u} - \frac{\beta\left[k^2+l^2+\frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)\right] - 2\beta k^2}{\left[k^2+l^2+\frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)\right]^2} = \bar{u} - \frac{-\beta k^2 + l^2 + \frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)}{[\cdots]^2}$$

$$c_{gx} = \bar{u} + \frac{\beta k^2 - \left[l^2 + \frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)\right]}{\left[k^2+l^2+\frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)\right]^2} \qquad c_{gy} = \frac{\partial \nu}{\partial l} = \frac{2\beta k l}{\left[k^2+l^2+\frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)\right]^2}$$

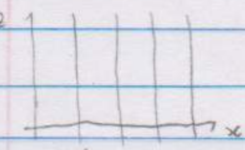
$$\nu = \bar{u}k - \frac{\beta k}{k^2+l^2+\frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)}$$

c_{gx}, c_{gy} similar to barotropic case

$$c_{gz} = \frac{\partial \nu}{\partial m} = \underbrace{\frac{\partial}{\partial m}(\cancel{\bar{u}k})}_{0} - \left[\frac{-\beta k\, 2m \frac{f_0^2}{N^2}}{\left[k^2+l^2+\frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)\right]^2}\right] = \frac{2\beta k m \frac{f_0^2}{N^2}}{\underbrace{\left[k^2+l^2+\frac{f_0^2}{N^2}\left(m^2+\frac{1}{4H^2}\right)\right]}_{K^2}{}^2} \to 0 \text{ as } m \to 0$$

$\lambda_z \to \infty$

i.e. when phase lines go vertical



So vertically trapped waves do not cause upward energy propagation

Also, $c_z = \frac{k}{m} c_x = \frac{2\pi}{L_x}\cdot\frac{L_z}{2\pi} c_x = \frac{L_z}{L_x} c_x$

So upward phase propagation is much slower than zonal, which itself is already small (low frequency waves)

The former analysis can be generalized to a zonal-mean zonal wind that varies with y, z; that is: $\bar{u}(y,z)$. Recall $q = \nabla^2(\bar{\Psi}+\psi') + f_0 + \beta y + \frac{1}{\rho_0}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\left(\rho_0 \frac{\partial \psi}{\partial z}\right)$

$\psi' = \Psi(y,z)\, e^{i(kx-\nu t) + z/2H}$ Thus $\bar{q} = \nabla^2\bar{\Psi} + f_0 + \beta y + \frac{1}{\rho_0}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\left(\rho_0\frac{\partial\bar{\Psi}}{\partial z}\right)$

$u' = -\frac{\partial \psi'}{\partial y} = -\frac{\partial \Psi}{\partial y} e^{i(\dots)+z/2H}$ $\qquad q' = \nabla^2\psi' + \frac{1}{\rho_0}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\left[\rho_0\frac{\partial\psi'}{\partial z}\right]$

$\nabla^2\psi' = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi' = -k^2\psi' + \frac{\partial^2\Psi}{\partial y^2} e^{i(\dots)+z/2H}$, $\frac{1}{\rho_0}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\left[\rho_0\frac{\partial\Psi}{\partial z}\right]$ is the same as before,

$\hookrightarrow = \frac{f_0^2}{N^2}\left[\frac{\partial^2\Psi}{\partial z^2} e^{i(\dots)+z/2H} - \frac{\Psi(y,z)}{4H^2} e^{i(\dots)+z/2H}\right]$

$$\frac{\partial}{\partial t}(\bar{q}+q') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{q}+q') + v'\frac{\partial\bar{q}}{\partial y} = 0$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)q' + v'\frac{\partial \bar{q}}{\partial y} = 0$$

$$0 = \left(-\cancel{ik}c_x + \bar{u}\,\cancel{ik}\right)\left[-k^2\Psi(y,z)\cancel{e^{i(\dots)+z/2H}} + \frac{\partial^2\Psi}{\partial y^2}\cancel{e^{i(\dots)+\frac{z}{2H}}} + \frac{f_0^2}{N^2}\frac{\partial^2\Psi}{\partial z^2}\cancel{e^{i(\dots)+\frac{z}{2H}}} - \frac{\Psi}{4H^2}\frac{f_0^2}{N^2}\cancel{e^{i(\dots)+z/2H}}\right] + \cancel{ik}\frac{\partial\bar{q}}{\partial y}\Psi\cancel{e^{i(\dots)}}$$

$$(\bar{u}-c_x)\left[\frac{\partial^2\Psi}{\partial y^2} + \frac{f_0^2}{N^2}\frac{\partial^2\Psi}{\partial z^2} - \left(k^2+\frac{f_0^2}{4N^2H^2}\right)\Psi\right] + \Psi\frac{\partial\bar{q}}{\partial y} = 0$$

Rossby wave refracts like E-M wave propagating in non-uniform material

$$\frac{\partial^2\Psi}{\partial y^2} + \frac{f_0^2}{N^2}\frac{\partial^2\Psi}{\partial z^2} + \underbrace{\left[\frac{\partial\bar{q}/\partial y}{(\bar{u}-c_x)} - \left(k^2+\frac{f_0^2}{4N^2H^2}\right)\right]}_{n_k^2}\Psi = 0$$

same equation as that governing electromagnetic wave propagation through media w/ variable index of refraction

$n = n_r + i n_i$

n_r: phase speed; n_i: absorption

$1 < n_r = \frac{c}{v}$ — c - speed of light in vacuum, v - speed in material

Generally $\frac{\partial\bar{q}}{\partial y} > 0$ since f_0 dominates and increases with latitude ($\beta > 0$ for all latitudes) and Q-G requires $\underbrace{\nabla^2\bar{\Psi}}_{=\zeta} \ll f_0$. $n_k^2 > 0$ associated with $\bar{u}$ not large (westerly)

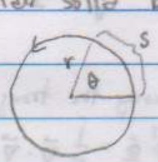
< 0 associated with $\bar{u}$ strong westerly

Circulation and Vorticity (Refresher)

$$\Gamma \equiv \oint \vec{u}\cdot d\mathbf{l} = \iint \vec{\nabla}\times\vec{u}\cdot\hat{n}\,dA \quad \text{by Stokes Theorem}$$

$$\vec{\omega} = \lim_{A\to 0}\frac{\Gamma}{A} = \vec{\nabla}\times\vec{u}$$

Consider solid body rotation



$$s = r\theta \Rightarrow \underset{\frac{ds}{dt}}{v} = r\underset{\frac{d\theta}{dt}}{\Omega} \qquad \Rightarrow \Gamma = \int_0^{2\pi}\Omega r^2\,d\theta = 2\Omega\pi r^2$$

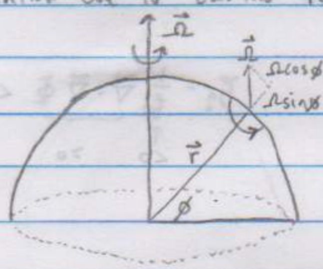
$$\frac{\Gamma}{\pi r^2} = 2\Omega$$ circulation per area in this case is twice the ambient rotation rate

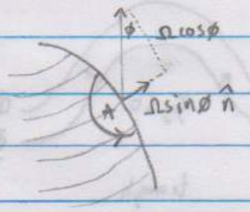
Kelvin's circulation Theorem:

$$\frac{D}{Dt}C_{abs} = \frac{D}{Dt}(C_e + C_{relative}) = 0$$

$$\frac{D}{Dt}(\zeta + f) = 0$$

Circulation due to Earth's rotation (and the origins of planetary vorticity)





The tangential velocity due to solid body rotation of the Earth is given by

$$|\vec{\Omega}||\vec{r}|\sin(\vec{\Omega},\vec{r}) = \vec{u}_e = \vec{\Omega}\times\vec{r} = \begin{vmatrix} i & j & k \\ \Omega_x & \Omega_y & \Omega_z \\ x & y & z \end{vmatrix} = \hat{i}\underbrace{(\Omega_y z - \Omega_z y)}_{u_x} + \hat{j}\underbrace{(\Omega_z x - \Omega_x z)}_{u_y} + \hat{k}\underbrace{(\Omega_x y - \Omega_y x)}_{u_z}$$

$$\sin\left(\tfrac{\pi}{2}-\phi\right) \quad \Omega(a+\underset{\text{small}}{z})\cos\phi$$

The circulation around A due to Earth's rotation is thus

$$C_e = \oint \vec{u}_e\cdot d\mathbf{l} = \iint \vec{\nabla}\times\vec{u}_e\cdot\hat{n}\,dA$$

$$\vec{\nabla}\times\vec{u}_e = \vec{\nabla}\times(\vec{\Omega}\times\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}u_z - \frac{\partial}{\partial z}u_y\right) + \hat{j}\left(\frac{\partial}{\partial z}u_x - \frac{\partial}{\partial x}u_z\right) + \hat{k}\left(\frac{\partial}{\partial x}u_y - \frac{\partial}{\partial y}u_x\right)$$
$$= \hat{i}(\Omega_x + \Omega_x) + \hat{j}(\Omega_y + \Omega_y) + \hat{k}(\Omega_z + \Omega_z) = 2\vec{\Omega}$$

Therefore $C_e = \iint 2\vec{\Omega}\cdot\hat{n}\,dA = 2\Omega\sin\phi\iint dA = 2\Omega\sin\phi\,A$

$$\frac{C_e}{A} = 2\Omega\sin\phi = f = \lim_{A\to 0}\frac{1}{A}\iint 2\vec{\Omega}\cdot\hat{n}\,dA \quad \text{is the planetary vorticity}$$

Absolute vorticity: $\vec{\omega}_{abs} = \vec{\nabla}\times\vec{u}_{abs} = \vec{\nabla}\times(\vec{\Omega}\times\vec{r} + \underset{\uparrow}{\vec{u}}) = f + \vec{\nabla}\times\vec{u} = f + \zeta$ if we only consider the vertical component

velocity relative to planet

Relative vorticity: $\vec{\omega} = \vec{\nabla}\times\vec{u}$

$$= \xi\hat{i} + \eta\hat{j} + \zeta\hat{k} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$\zeta = \hat{k}\cdot\vec{\nabla}\times\vec{u}$ is most important in large-scale atmosphere-ocean dynamics

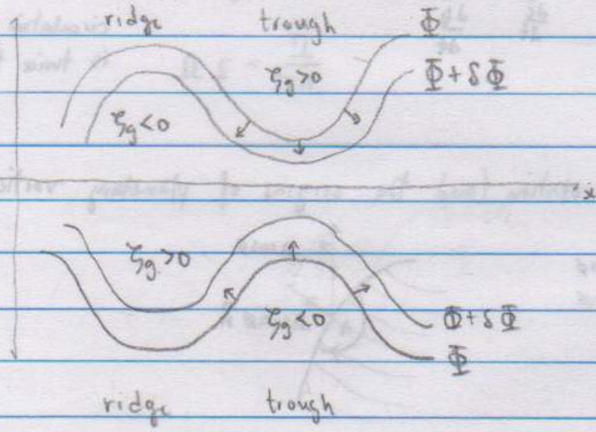
- cyclonic vorticity has the same sign as f

	NH ($f>0$)	SH ($f<0$)
cyclonic	$\zeta>0$	$\zeta<0$
anticyclonic	$\zeta<0$	$\zeta>0$

if (u,v) are geostrophic $\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2 \Phi$

NH

SH



considering the trough

$$\zeta_g = \frac{1}{f_0}\nabla^2\Phi = \underset{>0}{\frac{1}{f_0}}\underbrace{\vec{\nabla}\cdot\vec{\nabla}\Phi}_{>0} > 0$$

$$\zeta_g = \underset{<0}{\frac{1}{f_0}}\underbrace{\vec{\nabla}\cdot\vec{\nabla}\Phi}_{>0} < 0$$