Numerical investigation of f-plane geostrophic adjustment

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Abstract.
Numerical models of the ocean and atmosphere constructed based on finite-difference methods frequently employ staggered grids when approximating spatial derivatives in the equations governing geophysical fluid motion. Here the utility of grid staggering is explored within the simplified context of the geostrophic adjustment problem, to which analytical transient solutions are often difficult or impossible to obtain. Solutions are compared on staggered and unstaggered spatial grids for leapfrog and forward-backward time stepping methods and presented for a variety of initial states.

1. Introduction and motivation

First introduced by *Arakawa and Lamb* [1977], the concept of computational grid staggering remains widely used in numerical models of oceanic and atmospheric circulation. For the computation of prognostic variables, the atmospheric component of the GFDL CM2.1 coupled climate model uses the so-called Arakawa B-grid, for example.

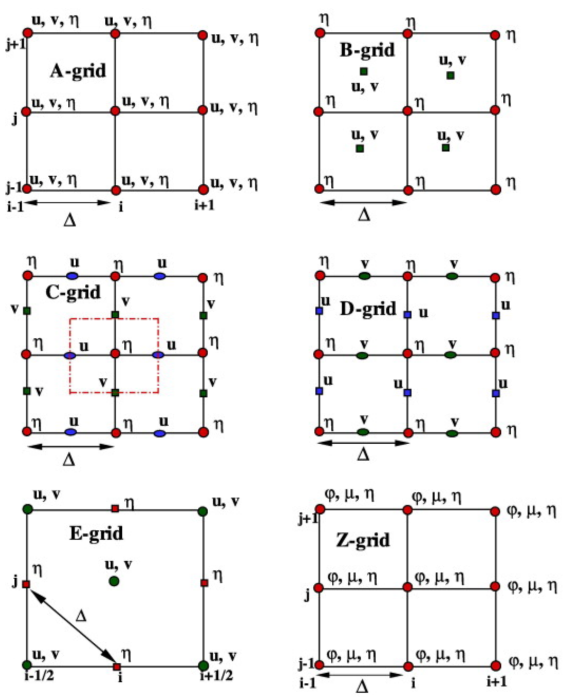

Figure 1. Arakawa grids $(A), (B), ..., (E)$ commonly used in numerical models of the ocean and atmosphere. Also shown is the Z-grid due to *Randall* [1994]. From *Rajpoot et al.* [2012].

The configuration of this grid is shown in Figure 1 within the context of the two-dimensional shallow water equations along with other grid layouts used in numerical models of the ocean and atmosphere. To gain preliminary insight into the use of such grids, I explored the use of spatial grid staggering in the simplest of possible ways: in one spatial dimension, to solve a basic problem relevant to large-scale ocean-atmosphere circulation. Here I briefly present solutions to the linear rotating shallow water equations in an examination of the geostrophic adjustment problem, and discuss the results obtained on both staggered and unstaggered grid configurations.

2. The basic problem

Let us consider the problem of geostrophic adjustment originally considered by *Rossby* [1938]. We are thus concerned with the following question: given a prescribed perturbation to the free surface or distribution of momentum to a shallow single layer of rotating fluid, how will the system respond in time? The answer, in marked contrast to the case of a non-rotating fluid, is that gravity waves modified by rotation, or Poincaré waves, will radiate away from the initial disturbance carrying (a fraction of the total) energy away and leaving behind a steady state current in geostrophic balance. Following *Gill* [1982], we model the motion of the ocean or atmosphere as a single-layer of homogeneous fluid rotating about a vertical axis at constant angular speed $f/2$ governed by the linear rotating shallow water equations. For simplicity, we examine cases in which perturbations quantities are a function of x only. In non-dimensional form, the equations to be solved are

$$\frac{\partial u}{\partial t} - v = -\frac{\partial \eta}{\partial x}, \tag{1a}$$

$$\frac{\partial v}{\partial t} + u = 0, \tag{1b}$$

$$\frac{\partial \eta}{\partial t} + \left(\frac{\lambda_R}{L}\right)^2 \frac{\partial u}{\partial x} = 0, \tag{1c}$$

where $\lambda = c/f$ is the barotropic radius of deformation, and $c = \sqrt{gH}$ is the phase speed of long gravity waves in a non-rotating system. The only non-dimensional parameter in these equations left to characterize the solution's behavior is in the equation for mass conservation (1c). For $L \ll \lambda_R$, the divergence term can not be neglected, and horizontal convergence is predominantly balanced by changes in the location of the free surface (as in the case of long surface gravity waves). On the other hand, for $L \gg \lambda_R$, the free surface remains approximately motionless and inertial oscillations dominate the response. The behavior of the transient solution is also evident in the Poincaré wave dispersion relation,

$$(\omega/f)^2 = 1 + k^2\lambda_R^2. \tag{2}$$

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which may be found by elimination of u and v in favor of η (*i.e.*, by forming the Klein-Gordon equation), and assuming a plane wave solution. Because frequency ω depends nonlinearly on wavenumber k, these waves are dispersive and it is straightforward to see that waves of relatively short wavelength travel fastest.

Lastly, it is worth noting that this system conserves a linearized form of shallow water potential vorticity (PV), such that

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{L^2}{\lambda_R^2}\eta\right) = 0. \tag{3}$$

The above form of PV in parentheses is thus constant in time, although spatial variations are permitted as determined by the distribution of relative vorticity and/or free surface displacement of the initial state. This simple fact can be exploited to obtain analytical solutions for the steady geostrophic mode (c.f., *Gill*, 7.2, 7.3; *Pedlosky*, Lec. 12). However, the full solution is also comprised of a transient part that in general is much more difficult to obtain.

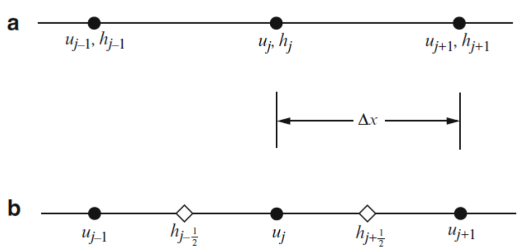

Figure 2. 1D computational grids for the solution of equation (1). Note that $\eta = h$. From *Durran* [2010].

3. Numerical approach

In solving equation (1), leapfrog and forward-backward time stepping were each applied on regular and staggered grids using boundary conditions periodic in x. This amounts to four solution methods that were applied and compared. The two grids on which solutions were analyzed include (a) a regular grid on which u, v, and η are collocated, and (b) a staggered grid where η points are defined $\Delta x/2$ to the right of points where u, v are both defined, as seen in Figure 2. As brief examples, on an unstaggered grid the solution algorithm with leapfrog time stepping is

$$u_j^{n+1} = u_j^{n-1} + 2\Delta t\left[v_j^n - \frac{\eta_{j+1}^n - \eta_{j-1}^n}{2\Delta x}\right] \tag{4a}$$

$$v_j^{n+1} = v_j^{n-1} + 2\Delta t u_j^n \tag{4b}$$

$$\eta_j^{n+1} = \eta_j^{n-1} + \frac{\Delta t}{\Delta x}\left[u_{j+1}^n - u_{j-1}^n\right] \tag{4c}$$

In contrast, on a staggered grid, the forward-backward scheme gives rise to the following partial difference equations

$$u_j^{n+1} = u_j^n + \Delta t\left[v_j^n - \frac{\eta_{j+1/2}^n - \eta_{j-1/2}^n}{\Delta x}\right] \tag{5a}$$

$$v_j^{n+1} = v_j^n + \Delta t u_j^{n+1} \tag{5b}$$

$$\eta_{j+1/2}^{n+1} = \eta_{j-1/2}^n + \frac{\Delta t}{\Delta x}\left[u_j^{n+1} - u_j^{n+1}\right] \tag{5c}$$

It is worth noting that, despite the appearance of u^{n+1} on the right hand side of equations (5b) and (5c), this scheme is fully explicit as u^{n+1} has for each n been previously calculated in (5a).

4. Comparison of methods

To serve as a case for comparison, consider a motionless initial state with free surface displacement $\eta(x,0)$ =sech$^2(x)$, and $L = \lambda_R$ so that buoyancy and rotation are of equal importance. First I computed solutions on a regular grid for both of the time stepping methods discussed above.

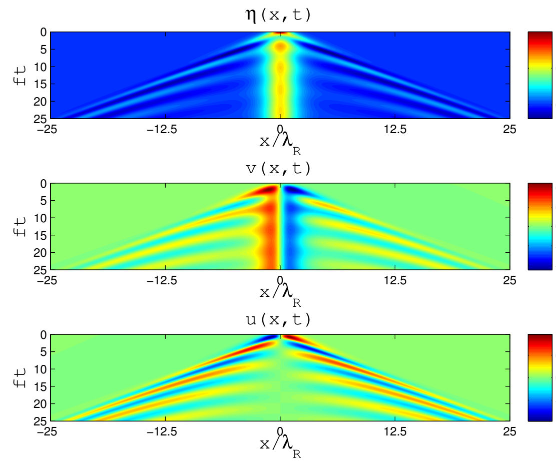

Figure 3. Adjustment to geostrophic balance of a free surface displacement $\eta(x,0)$ =sech$^2(x)$ with $v(x,0) = u(x,0) = 0$.

The solution for forward-backward time differencing on a regular grid is unconditionally unstable. In contrast, the leapfrog scheme (equation 4) with a Robert filter (coefficient $R = 0.03$) remains stable for $\sigma < 1$ but gives rise to a spurious computational oscillation between even and odd time steps, such that the solution jumps back and forth for even and odd n. Fortunately, using a staggered grid appears to be beneficial to both time marching schemes under consideration. The leapfrog method no longer suffers from the spurious oscillatory behavior found on a regular grid, although the stability transition occurs at a more stringent value of $\sigma = 1/10$. The solution based on forward-backward differencing on a staggered grid (equation 5) is shown in Figure 3. Of all methods considered here, this scheme performs best and remains stable for any Δt and can be used for long integrations in time.

In Figure 3 shorter waves are seen to radiate faster outward from the initial disturbance consistent with the dispersion relation for inertia-gravity waves. The free surface perturbation oscillates toward a new equilibrium position, retaining potential energy in the form of an elevated free surface. Geostrophic currents come into steady-state with the equilibrium free surface displacement flowing in opposite directions on each side.

5. Other solutions

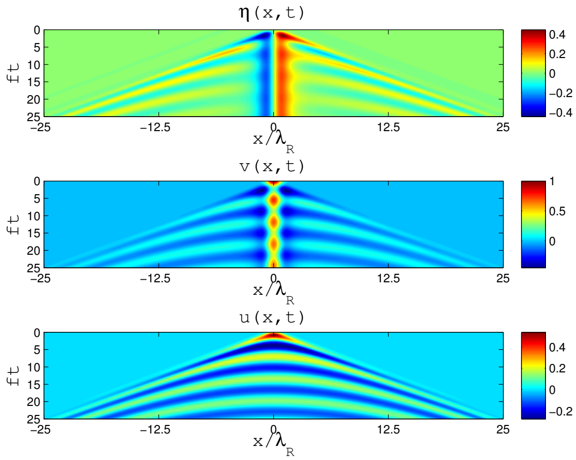

Figure 4. Adjustment to geostrophic balance of a jet $v(x, 0) = e^{-x^2} \cos(x)$ with $\eta(x, 0) = u(x, 0) = 0$.

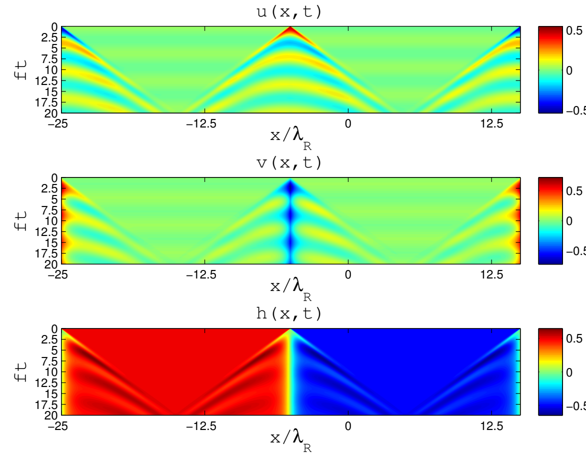

Figure 5. Adjustment to geostrophic balance of a unit displacement of the free surface as considered analytically by *Gill* (1982).

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