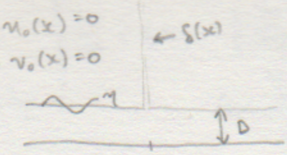
Geostrophic adjustment, Fourier Transform methods

$u_t - fv = -g\eta_x \qquad u(x,0) = u_0(x) = 0$

$v_t + fu = -g\cancel{\eta_y}^0 \qquad v(x,0) = v_0(x) = 0$



$\eta_t + D(u_x + \cancel{v_y}^0) = 0 \qquad \eta(x,0) = \eta_0(x) = \delta(x)\ x=0$

Eliminate u, v to reduce the problem to an ODE in η

$\eta_{ttt} + D u_{xtt} = 0\,, \quad u_{xtt} = -g\eta_{xxt} + f v_{xt}$

$\eta_{ttt} - gD\eta_{xxt} + fDv_{xt} = 0\,, \quad v_{xt} = -fu_x = \frac{f}{D}\eta_t$

$\Rightarrow \boxed{\eta_{ttt} - gD\eta_{xxt} + f^2\eta_t = 0}$

$\eta(x,0) = \eta_0(x) = \delta(x)$

$\eta_t(x,0) = -Du_{0x}(x,0) = 0$

$\eta_{tt}(x,0) = -Du_{0xt}(x,0) = gD\eta_{xx} - fD\cancel{v_x}^0$

Fourier transform equation and ICs

$\mathcal{F}(\eta_{ttt}) = \int_{-\infty}^{\infty} e^{ikx}\eta_{ttt}\,dx = \frac{\partial^3}{\partial t^3}\int_{-\infty}^{\infty} e^{ikx}\eta\,dx = \frac{\partial^3}{\partial t^3}\bar{\eta}$

$\mathcal{F}(gD\eta_{xxt}) = \left[gD\int_{-\infty}^{\infty} e^{ikx}\frac{\partial^2\eta}{\partial x^2}dx\right]_t = gD\frac{\partial}{\partial t}\left[e^{ikx}\frac{\partial\eta}{\partial x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} ik\,e^{ikx}\frac{\partial\eta}{\partial x}dx\right]$

$= gD\frac{\partial}{\partial t}\left[-ik\left(e^{ikx}\eta\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} ik\,e^{ikx}\eta\right)\right] = -gDk^2\bar{\eta}_t$

$\mathcal{F}(f^2\eta_t) = f^2\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{ikx}\eta\,dx = f^2\bar{\eta}_t$

$\Rightarrow \boxed{\bar{\eta}_{ttt} + (gDk^2 + f^2)\bar{\eta}_t = 0}$

$\mathcal{F}[\eta_0(x)] = \int_{-\infty}^{\infty} e^{ikx}\delta(x)\,dx = 1 = \bar{\eta}(k,t)$

$\mathcal{F}(\eta_{0t}(x,0)) = 0 = \bar{\eta}_t(k,t)$

$\mathcal{F}[\eta_{0tt}(x,0)] = gD\int_{-\infty}^{\infty} e^{ikx}\eta_{xx}\,dx = -gDk^2\,\cancel{\bar{\eta}(k,0)}^1$

$\bar{\eta}_{tt} + \sigma^2\bar{\eta} = C$

$\bar{\eta}(k,t) = C + Ae^{-i\sigma t} + Be^{i\sigma t}$

$\bar{\eta}(k,0) = C + A + B = 1$

$\bar{\eta}_t(k,0) = -i\sigma A + i\sigma B = 0 \Rightarrow A = B$

$\bar{\eta}_{tt}(k,0) = -\sigma^2 A - \sigma^2 B = -gDk^2$

$\Rightarrow 2A = \frac{gDk^2}{\sigma^2}$

$C + \frac{gDk^2}{\sigma^2} = 1$

$C = 1 - \frac{gDk^2}{\sigma^2}$

$= \frac{f^2}{\sigma^2}$

$\bar{\eta}(k,t) = \frac{f^2}{\sigma^2} + \frac{gDk^2}{2\sigma^2}\left(e^{-i\sigma t} + e^{i\sigma t}\right)$

FT pair

$\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\bar{\eta}(k,t)\,dk = \mathcal{F}^{-1}(\bar{\eta})$

$\bar{\eta}(k,t) = \int_{-\infty}^{\infty} e^{ikx}\eta(x,t)\,dx = \mathcal{F}(\eta)$

$\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\frac{f^2}{gDk^2 + f^2}\,dk$

$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{gDk^2}{2\sigma^2}\left[e^{-ikx - i\sigma t} + e^{-ikx + i\sigma t}\right]dk$

= steady geostrophic state + Poincare waves

$\eta_s(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\left(\frac{f^2/gD}{k^2 + f^2/gD}\right)dk$

$= \frac{1}{2\pi}\frac{f}{\sqrt{gD}}\int_{-\infty}^{\infty} e^{-ikx}\left[\frac{f/\sqrt{gD}}{k^2 + f^2/gD}\right]dk$

$= \frac{1}{2\pi}a^{-1}e^{-\frac{|x|}{a}}$

$a = \frac{c}{f} = \frac{\sqrt{gD}}{f} \equiv$ barotropic Rossby radius of deformation

$\eta(x,t) = \frac{1}{2\pi a}e^{-|x|/a} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{gDk^2}{2\sigma^2}\left[e^{-i(kx+\sigma t)} + e^{-i(kx-\sigma t)}\right]dk$

right + left travelling Poincare waves