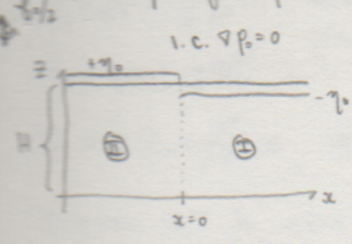
Barotropic geostrophic adjustment [Linear]



local depth $D = H + \eta$

$\eta(t=0) = -\eta_0 \,\mathrm{sgn}(x)$

$\mathrm{sgn}(x) = \begin{cases} +1 & x>0 \\ -1 & x<0 \end{cases}$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial p}{\partial z} = -\rho g \Rightarrow \int_z^{H+\eta} p_z\, dz = -\rho g \int_z^{H+\eta} dz$$

$\rho = \rho_0$

$$\underset{0}{p(H+\eta)} - p(z) = -\rho g (H+\eta-z)$$

$$p = \rho g \eta \quad \text{(dynamic)}$$

Small amplitude $\eta_0/H \ll 1$

linear

$$\Rightarrow \frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

local change in relative vorticity due to stretching of planetary vorticity filaments ONLY

$$\boxed{\frac{\partial}{\partial t}\zeta + f\delta = 0} \;*$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\zeta - \frac{f}{(H+\cancel{\eta})}\eta\right) = 0$$

$$\frac{\partial}{\partial t}\underbrace{\left(\zeta - \frac{f\eta}{H}\right)}_{} = 0$$

linear SW PV $= q(x,y) =$ constant

$$\int_0^{H+\eta}\left[\vec{\nabla}_3\cdot\vec{u} = 0\right]dz$$

$$(H+\eta)\delta + w(H+\eta) - \cancel{w(0)}^{\,0} = 0$$

$$\frac{D}{Dt}(\cancel{H}+\eta) + (H+\eta)\delta = 0$$

$$\frac{\partial \eta}{\partial t} + \nabla\cdot\left[(H+\eta)\vec{u}\right] = 0$$

linearized $\Rightarrow$

$$\frac{\partial \eta}{\partial t} + (H+\eta)\delta + u\cancel{\frac{\partial}{\partial x}\eta} + v\cancel{\frac{\partial}{\partial y}\eta} = 0$$

$$\Rightarrow \boxed{\delta = -\frac{1}{(H+\eta)}\frac{\partial \eta}{\partial t}}$$

$t\to\infty \sim t=0$

$$\frac{g}{f}\nabla^2\eta - \frac{f\eta}{H} = -\frac{f}{H}\eta(t=0)$$

$$= \frac{f\eta_0}{H}\,\mathrm{sgn}(x)$$

Assume $\partial_y = 0$ since I.C. only $f(x)$

$$\frac{d^2\eta}{dx^2} - \frac{f^2}{gH}\eta = \frac{f^2}{gH}\eta_0\,\mathrm{sgn}(x)$$

Must solve in two regions I + II

region

I $x>0$

$$\frac{d^2}{dx^2}(\eta+\eta_0) = \frac{f^2}{gH}(\eta+\eta_0) \Rightarrow \eta+\eta_0 = Ae^{-x/a} + \cancel{Be^{x/a}} \;;\; a = \frac{\sqrt{gH}}{|f|}$$

bounded solution

II $x<0$

$$\frac{d^2}{dx^2}(\eta-\eta_0) = \frac{1}{a^2}(\eta-\eta_0) \Rightarrow \eta-\eta_0 = \cancel{\bar{A}e^{-x/a}} + \bar{B}e^{x/a}$$

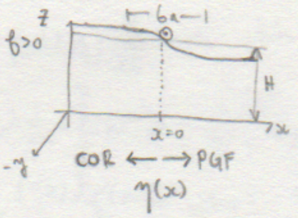
$\bar{A} = 0$

$$\left.\begin{array}{l} \eta(0+) = \eta(0-) \Rightarrow A-\eta_0 = \bar{B}+\eta_0 \\ \frac{\partial \eta}{\partial x}(0+) = \frac{\partial \eta}{\partial x}(0-) \Rightarrow -\frac{A}{\cancel{\alpha}} = \frac{\bar{B}}{\cancel{\alpha}} \end{array}\right\} \Rightarrow 2A = 2\eta_0$$

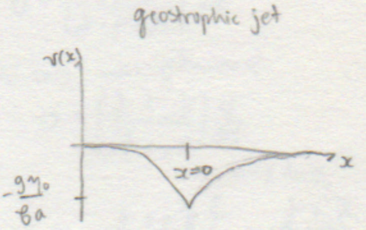
$$v_g = \frac{g}{f}\frac{\partial \eta}{\partial x} = \frac{g}{f}\begin{cases} -\frac{\eta_0}{a}e^{-x/a} & x>0 \\ -\frac{\eta_0}{a}e^{x/a} & x<0 \end{cases}$$

$$= -\frac{g}{f}\frac{\eta_0}{a}e^{-|x|/a}$$

$$\eta(x) = \begin{cases} \eta_0\left(e^{-x/a} - 1\right), & x>0 \\ \eta_0\left(1 - e^{x/a}\right), & x<0 \end{cases}$$



geostrophic jet



[Nonlinear]

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial}{\partial x}\zeta + v\frac{\partial}{\partial y}\zeta + (f+\zeta)\delta + \beta v = 0$$

$\Downarrow$

$$\frac{D}{Dt}\left[\zeta + f\right] + (f+\zeta)\delta = 0$$

$$\frac{\partial \eta}{\partial t} + \nabla\cdot\left[(H+\eta)\vec{u}\right] = 0$$

$\Downarrow$

$$\frac{D}{Dt}(H+\eta) + (H+\eta)\delta = 0$$

$$\Rightarrow \frac{D}{Dt}(\zeta+f) - \frac{(f+\zeta)}{(H+\eta)}\frac{D}{Dt}(H+\eta) = 0$$

$$\frac{1}{H+\eta}D_t(\zeta+f) - \frac{(\zeta+f)}{(H+\eta)^2}\frac{D}{Dt}(H+\eta) = 0$$

$$\frac{D}{Dt}\left[\frac{\zeta+f}{H+\eta}\right] = 0 \quad \frac{\text{absolute vorticity}}{\text{depth of fluid}} \quad \text{conserved following fluid motion}$$

$$Q = \frac{\zeta+f}{H+\eta} = \frac{\zeta+f}{H}\left(\frac{1}{1+\frac{\eta}{H}}\right) \simeq \frac{\zeta+f}{H}\left[1 - \frac{\eta}{H} + O\left(\frac{\eta}{H}\right)^2\right] = \frac{1}{H}\left[\zeta - \frac{\zeta\eta}{H} + f - \frac{f\eta}{H} + \dots\right.$$

$\frac{\eta}{H} \ll 1$ and

$$(1-x)(1+x+x^2+\dots+x^N) = 1 + \cancel{x} + \cancel{x^2} + \dots - \cancel{x} - \cancel{x^2} - x^{N+1}$$

$$\Rightarrow \sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$

if $|x|<1$ then

$$\lim_{N\to\infty}\sum_{n=0}^{N} x^n = \frac{1}{1-x}$$

therefore

$$\frac{1}{1+x} = \sum_{n=0}^{N}(-x)^n = 1 - x + x^2 - \dots$$

if $\frac{\zeta}{f} \ll 1$ (Ro $\ll 1$) then

$$Q = \frac{q}{H} + O\left(\frac{\zeta\eta}{H}, \dots\right)$$

H constant

$$\Rightarrow \underset{=0}{\frac{D}{Dt}\left[Q\right]} \rightarrow \frac{\partial}{\partial t}\left(\zeta - \frac{f\eta}{H}\right) = 0$$

linearization

$\frac{\partial Q}{\partial t} = -\vec{u}\cdot\nabla Q$ advection may change local value of Q, unlike for q (linear)