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Modeling Behavior of Networked Agents with Dynamic Opinions

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1 Introduction

This research uses a game theoretic approach to investigate the relationships between opinions, information sharing, and structural dynamics on social networks.

1.1 Motivation

The recent expansion of social networks has transformed how most people receive information and form opinions. The scale of these networks is enormous: as of 2017, more than 100 million active daily Twitter users sent on average 500 million tweets per day [2]. Consequently, in the same year, two-thirds of Americans consumed at least some of their news from social media sites [9].

This represents a fundamental change from the way people received information as recently as a few decades ago. In particular, information sources are distributed, and people have dramatically increased ability to control the information they receive. Since in modern networks, people both choose their connections and form opinions based on those connections, opinions and network topologies are intrinsically co-determined. Modeling this process to understanding the ways users interact with each other and with information on such networks enables users to engage knowledgeably, network engineers to design intentionally, and publicists to engage effectively.

The analysis of behavior on social networks is a critical aspect of the study of economics and computation and constitutes an apt application of the techniques studied in our class. Network science is a core topic in the economics and computation literature, as evidenced by multiple chapters on networks (omitted from our course) in our text, [8]. In particular, the importance of social networks to commerce and marketing make them of particular intrinsic interest in understanding the modern economy. Further, game theoretic models are widely applied to networked agents, and tools from class such as repeated games, optimality, and Nash equilibrium provide a fitting framework for analyzing networked agent behavior.

1.2 Related Work

Three relevant and well-researched aspects of social network dynamics are user opinion formation, information sharing, and graph structure.

The majority of research in networked opinion formation focuses on the problem of influence maximization [1, 5]. The present work instead considers opinion dynamics in the absence of external "manipulation." Bindel et al. investigate natural mechanisms by which a diversity of opinions is maintained in equilibrium, which is frequently observed in real-world networks but uncommon in theoretical analyses [3]. While most models consider separately opinion and network structure dynamics, Gu et al. describe a co-evolution model that both describes opinion convergence and divergence ("herding"), as well as performs well in predicting sponsorship of legislative bills in Congress [6].

Zinoviev simulates the diffusion of a single piece information under a multi-faceted agent motivation function to show that time and existence of convergence of knowledge varies by network size [12]. Sun et al. instead investigate diffusion of two dissenting pieces of information under a game theoretic user model [10]. Jiang et al. more explicitly incorporate agents' preferences in a game theoretic framework optimized by an evolutionary process to characterize stable network states with respect to information sharing [7].

Relatively little theoretical work exists on network connection dynamics. Most models describe network formation [4, 11], while little focuses on dynamics of mature networks. Connection dynamics in light of self-motivated information transmission has not been previously addressed.

1.3 Problems and Approach

Our understanding of network dynamics and equilibrium properties are particularly lacking in regard to two sets of endogenous variables. First, there are limited results for systems modeling both opinion dynamics and user sharing behavior. Second, there are no published results concerning co-dynamics of opinions and network connections. Further, relatively few models consider users modeled as agents endowed with game theoretic decision-making capabilities with regard to their interactions on the network.

We study the following questions.

1. How do users initial stances and the topology of a social network influence optimal and equilibrium information sharing?

2. Assuming truthful information sharing, how do user stances influence optimal and equilibrium "following" behavior–the primary determinant of network structure?

We attempt to answer these questions analytically, drawing on techniques from network theory and game theory.

1.4 Outline

Section 2 defines our operating network model and the games played on this network. This model differs from previous analyses by using a continuous representation of stances, as well as by providing a framework for games in which agents can alternately decide both sharing *and* following behavior. Section 3 contains an analytical study of optimal strategies and equilibria in the games presented. We show that for some network structures, the opinions of agents optimizing their sharing behavior will converge to an average of their initial opinions, weighted by agent's levels of impressionability. Additionally, we show that in equilibrium, a network of close-minded agents deciding whom to follow will display herding behavior into multiple groups if and only if the ratio of agents' hunger for information over their level of closed-mindedness is greater than the ratio of the disparity between the groups' stances over the size of the groups. The final section outlines applications and directions for future work.

2 Model Definitions

The Spectrum Model is a framework for analyzing continuous dynamics of opinions, information transmission, and structure on a social network. The model describes a single "topic" that is the subject of the model. Stances on the topic vary along a single dimension. For example, the model may be used to describe U.S. politics, in which case stances vary continuously from very conservative to moderate to very liberal.

This section formally defines the Spectrum framework. It then specifies two games that can be played on the network: the Sharing Game and the Following Game.

2.1 The Spectrum Model

Formally, a spectrum model is a tuple $(N, \mathbf{G}, \pi)$, representing a collection of agents, a dynamic following graph on the agents, and the agents' personalities, respectively.

2.1.1 Information

Atoms of information are each described by a stance towards the topic of the model such that $s_{i,t} \in [-1, 1]$ indicates the stance of the information sent by agent i at time t. The vector of the stances of information sent by each agent at time t is $s_t = (s_{i,t})_{t \in 1,2,...}$. We call s_t the *action profile* at time t. Stance quantifies the viewpoint expressed by a piece of information along a linear spectrum. In particular, $s_h > s_i$ if and only if atom h expresses a more favorable opinion of the topic than atom i. We interpret values of 1 and -1 as opposite extremes, and 0 as neutrality or moderateness.

2.1.2 Following Network

Following relationships between agents over time are described by a dynamic undirected graph $\mathbf{G} = \{G_t\}_{t \in \{1,2,...\}} = \{(V, E_t)\}_{t \in \{1,2,...\}}$. The set of vertices does not vary over time and corresponds to a constant population of agents. The set of edges is dynamic; $(i, j) \in E_t \Leftrightarrow i$ follows j at time t. For the converse relationship, we write $(j, i) \in E_t \Leftrightarrow$ i is a "followee" of j at time t. To accord with real-world networks and intuition, we restrict agents from following themselves: $(i, i) \notin E_t$ for any i and for all t.

2.1.3 Agents

The set of agents is denoted $N = \{1, 2, \ldots, n\}$. Agent i is represented by graph node v_i.

Each agent i also has a stance $p_{i,t} \in [-1, 1]$ which varies with time t. The interpretation of an agent's stance is analogous to the stance of an atom of information.

Further, each agent has a time-invariant personality vector $\pi_i = (\alpha_i, \beta_i, \gamma_i(\cdot))$ such that $\alpha_i \in [-1, 1]$, $\beta_i \in [0, 1]$, and non-decreasing $\gamma_i : \mathbb{N} \mapsto \mathbb{R}$. The matrix of all personality parameters is $\pi = (\pi_i)_{i \in N}$. Personality parameters are defined precisely through agents' utility functions (for α_i and γ_i) and the stance update rule (for β_i). Loosely, α_i is agent i's level of open-mindedness towards information of a different stance, β_i is its impressionability, and $\gamma_i(\cdot)$ describes the component of utility determined by the quantity of information i receives (its thirst for information). Note that negative values of α_i describe aversion to receiving opposing information. Thirst for information function $\gamma_i(\cdot)$ non-decreasing implies that ceteris paribus agents weakly prefer more information.

Throughout, let $R^{i,t} = \{j : (i,j) \in E_t\}$ be the set of agents i follows at time t. The stance update rule is to add the difference between the average stance of information received and previous stance, scaled by impressionability.

$$\begin{aligned} p_{i,t} &= \beta_i \left(\frac{\sum_{j \in R^{i,t-1}} s_{j,t-1}}{|R^{i,t-1}|} \right) + (1-\beta_i) p_{i,t-1} \\ &= p_{i,t-1} + \beta_i \left(\frac{\sum_{j \in R^{i,t-1}} s_{j,t-1}}{|R^{i,t-1}|} - p_{i,t-1} \right) \end{aligned} \tag{1}$$

We can also define an agent's utility function in terms of its personality. Agents' instantaneous utilities share a common form u. The utility of agent i at time t for action profile s_t is $\gamma_i(|R^{i,t}|)$ if $|R^{i,t}| = 0$, and otherwise

$$u_{i,t}(s_t) = \gamma_i(|R^{i,t}|) + \frac{1}{|R^{i,t}|} \sum_{j \in R^{i,t}} \alpha_i * |p_{i,t} - s_{j,t}|. \tag{2}$$

2.2 The Games

We model the flow of information and the dynamics of the network structure separately via two games, the Sharing Game and the Following Game. In both, agents are accorded game-theoretic decision making capabilities in pursuit of maximizing their utility with regard to the information they receive. For the sake of minimizing complexity, the games describe separately the two major, complimentary facets of user behavior on networks: information sharing (or "posting", or "tweeting", or "talking") and following (or "listening").

2.2.1 The Sharing Game

The Sharing Game is an infinitely repeated game that proceeds in rounds $T = \{1, 2, \ldots\}$ in the context of the Spectrum Model. The Sharing Game is played on a static network such that $G_t = G = (V, E)\ \forall t$.

Each round of the Sharing Game has the same flow: (1) agents' stances are updated, (2) agents make their sharing decision, and (3) agents' utilities are calculated. In each round, the players are all agents N. In round $t = 1$, agents' stances are initialized. In subsequent rounds $t > 1$, agents stances are updated according to Equation (1), based on the information they received in round $t - 1$. Then, for all t, each agent i makes a single decision of the stance of a single piece of information it shares with all of its followers, $s_{i,t} \in [-1, 1]$. Agents knowledge is universal: they are aware of the information shared by all agents in all previous rounds. Note, however, that agents do not know others' true stances. Finally, the instantaneous utility of agent i at time t, $u_{i,t}(s_t)$, is calculated as defined in Equation (2), where we let $\gamma_i(|R^{i,t}|) = 0\ \forall t \in N$, since agents' actions have no impact on the amount of information they receive. Agents discount utility from future rounds at rate $\delta \in [0, 1)$.

Formally, the stage game of the Sharing Game is the tuple $(N, (A_i)_{i\in N}, (K_i)_{i\in N}, (U_{i,t})_{i\in N})$, as described in Game 1.

Game 1: Stage Game at Time t of the Sharing Game

Players: N = {1,2,. . . ,n}
Action Set: $(A_{i,t})_{i\in N} = (s_{i,t} \in [-1, 1])_{i\in N}$
Knowledge: $(K_{i,t})_{i\in N} = \left(\bigcup_{\forall j\in V\ \forall u<t} s_{j,u}\right)_{i\in N}$
Utilities: $(U_{i,t})_{i\in N} = \left(\frac{1}{|R^{i,u}|} \sum_{j\in R^{i,u}} \alpha_i * |p_{i,u} - s_{j,u}|\right)_{i\in N}$

2.2.2 The Following Game

The Following Game also occurs in the context of the Spectrum Model and repeats infinitely over rounds $T = \{1, 2, \ldots\}$. In contrast to the Sharing Game, the Following Game occurs on a dynamic network, but with simplistic sharing behavior. The stage game is as follows.

Each round of the Following Game has the same flow: (1) agents' stances are updated, (2) agents make their following decision and then share truthfully, and (3) agents' utilities are calculated. In each round, the players are all agents N. First, agents' stances are updated according to Equation (1), or initialized if $t = 1$. At time t, each agent i then makes a single decision to add an agent j such that $(i, j) \notin E_{t-1} \setminus \{i\}$, remove an agent k such that $(i, k) \in E_{t-1}$, or do nothing, $\emptyset$. The result of their choices dictates the topology of the network E_t. Agents "truthfully" share information representing their actual stances $s_{i,t} = p_{i,t}$. Like in the Sharing Game, agents' knowledge is universal, which implies that agents know exactly the full history of the true stances of all agents. Finally, utilities are calculated based on E_t. Discounting occurs analogously to the Sharing Game.

Formally, the stage game of the Following Game is the tuple $(N, (A_i)_{i\in N}, \mathcal{D}, (K_i)_{i\in N}, (U_{i,t})_{i\in N})$, described in Game 2.

Game 2: Stage Game at Time t of the Following Game

Players: N = {1,2,. . . ,n}
Action Set: $(A_{i,t})_{i\in N} = (N \setminus i \cup \{\emptyset\})_{i\in N}$
Network Dynamics:
$E_t := \mathcal{D}((a_{i,t})_{i\in N}) = E_{t-1} \cup \{(i, a_{i,t})\forall i \in N\ :(i, a_{i,t}) \notin E_{t-1}\} \setminus \{(i, a_{i,t})\forall i \in N\ :(i, a_{i,t}) \in E_{t-1}\}$
Knowledge: $(K_{i,t})_{i\in N} = \left(\bigcup_{\forall j\in V\ \forall u<t} s_{j,u}\right)_{i\in N}$
Utilities: $(U_{i,t})_{i\in N} = \left(\gamma_i(|R^{i,t}|) + \frac{1}{|R^{i,u}|}\sum_{j\in R^{i,u}} \alpha_i * |p_{i,u} - s_{j,u}|\right)_{i\in N}$

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3 Strategic Analysis

3.1 Sharing Game Strategies and Equilibria

First we will show basic equilibrium properties on general graphs before considering specific graph structures.

Claim 1. *Any action profile $a = (a_1 = s_{1,t}, a_2 = s_{2,t}, \ldots, a_n = s_{n,t})$ is a Nash Equilibrium of the Sharing stage game in round t.*

Proof. We follow the convention of using $u_{i,t}(s_{i,t}, s_{-i,t})$ to denote the instantaneous utility of agent i at time t given agent i shares information of stance $s_{i,t}$ and agents $N\backslash i$ share information with stances $s_{-i,t} = (s_{1,t}, ..., s_{i-1,t}, s_{i+1,t}, ..., s_{n,t})$. From the definition of agent utility,

$$u_{i,t}(s_{i,t}, s_{-i,t}) = \frac{1}{|R^{i,t}|}\sum_{j\in R^{i,t}} \alpha_i * |p_{i,t} - s_{j,t}| \tag{3}$$

By definition, $(i, i) \notin E$, so $i \notin R^{i,t}$. Hence $u_{i,t}(s_{i,t}, s_{-i,t})$ can be written as $u_{i,t}(s_{-i,t})$, which does not depend on $s_{i,t}$. Therefore, $u_{i,t}(s_{i,t}, s_{-i,t}) = u_{i,t}(j_{i,t}, s_{-i,t})$ for all actions $j_{i,t} \in A_{i,t}$. Since this holds for all agents i, we have the condition for a Nash Equilibrium:

$$u_{i,t}(s_{i,t}, s_{-i,t}) = u_{i,t}(j_{i,t}, s_{-i,t})\ \ \forall i \in N\ \ \forall j \in A_{i,t}. \tag{4}$$

□

The property in Claim 1 may either seem trivial or otherwise indicate that the Sharing Game is somehow degenerate. This is not the case in general, as we show in Claim 2. Moreover, this property provides an interesting insight into the structure of this model. An agents' sharing choice at time t does not impact its own utility instantaneously in the same round t. Agents with particularly short foresight may fail to make wise sharing decisions.

We use the notation $u_{i,t}(s_{i,v}, s_{i,t})$ to explicitly denote the possible dependence of $u_{i,t}$ on the action $s_{i,v}$ of agent i in round v.

Claim 2. *There exists a Sharing Game on some graph G with some strategy profile for which there exist two distinct actions of i at time $v < t$, $s_{i,v} \neq j_{i,v}$, such that $u_{i,t}(s_{i,v}, s_{i,t}) \neq u_{i,t}(j_{i,v}, s_{i,t})$.*

Proof. Consider the cycle on two nodes $G = (\{1, 2\}, \{(1, 2), (2, 1)\})$ with initial agent stances $p_{1,1} = 0$ and $p_{2,1} = 1$ and personalities $\pi_i = (-1, \frac{1}{2})$ for $i = 1, 2$. Suppose agent 2 shares information "truthfully" with stance equal to its stance $p_{2,t}$ in each round t. In round 1, if agent 1 shares information with stance 1, then in round 2, agent 2 has stance $p_{2,2} = p_{2,1} + \beta(1 - 1) = 1$. Agent 1's resulting utility in round 2 is $\alpha_1 * |p_{1,2} - s_{2,2}| = \alpha_1 * |\frac{0+1}{2} - 1| = -\frac{1}{2}$. If, instead agent 1 shares information with stance 0 in round 1, then agent 2 shares information with stance $1 + \beta_2(0 - 1) = \frac{1}{2}$. Agent 1's utility in round 2 is then $\alpha_1 * |\frac{0+1}{2} - \frac{1}{2}| = 0$. □

We observe that when two mutually following agents share information with a stance that depends on their own stances, then the sharing behavior of both agents in round t affects their own utility in subsequent rounds. Intuitively, this results from the impact that of an agent's sharing on the stances of the community.

We now derive results for interesting graph structures.

3.1.1 On a Tree

One possible assumption is that agent i's sharing strategy does not depend on the sharing actions of nodes i does not follow.

Definition 3.1. (Forward Dependence). Strategies in a Sharing Game are *forward dependent* if for all agents i and for all rounds, the strategy of i is independent of the actions of all j such that $(i,j) \notin E$.

If the social network is a sufficiently inclusive representation of the relationships in a community, this assumption is sensible since it is difficult to conceive of how an agent i could alter its behavior on the basis of information it does not receive by any means. The tree network structure is special case for which under forward strategy dependence, agents' sharing behavior does not affect their own utilities at any time horizon.

Claim 3. *For all directed trees G, if strategies are forward dependent, then the utilities at time t in the Sharing Game $u_{i,t}(s_{i,v}) = u_{i,t}(j_{i,v})\ \forall i \in N\ \forall j_{i,v} \in A_{i,v}\ \forall v \leq t$.*

Proof. G is a directed tree implies that for all pairs of nodes $i \neq j$ if there is a directed path from j to i, there does not exist a directed path from i to j. By forward dependence, if there is no directed path from i to j, agent j's actions are independent of the actions of i. Since this holds for all j influencable by i, the actions of i are independent of the utility of i. So for all rounds t and agents i, we can write $u_{i,t}(s_{i,v}) = u_{i,t}(j_{i,v})$ for any action of i, $j_{i,v}$. □

3.1.2 One Friend Pair

First, we consider a directed cycle of length 2, $G = (\{1,2\}, \{(1,2),(2,1)\})$. This structure represents two agents who mutually follow each other. For example, this structure is typical of most friendships on Facebook, in which following through friendship is mutual. While real-world social networks are typically much larger, the 2-cycle allows us to investigate pressures on sharing exerted by a single other agent.

Specifically, what is the optimal sharing strategy of agent 1 given that it assumes its friend simply shares information corresponding to its stance?

Lemma 1. *If a close-minded ($\alpha_i < 0$) agent i mutually interacts with a single truthful agent, agent i's optimal sharing strategy in round t with respect to utility in rounds up to $t+1$ is $s_{1,t}^{opt} = \frac{(1-\beta_1)p_{1,t} + (\beta_1+\beta_2-1)p_{2,t}}{\beta_2}$, if this has absolute value less than 1, and the boundary condition -1 or 1, otherwise.*

Proof. Without loss of generality, consider $i = 1$. By Claim 1, agent 1's action $s_{1,t}$ does not affect its contemporaneous utility $u_{1,t}$. To determine the affect of $s_{1,t}$ on $u_{1,t+1}$, manipulate the utility function.

$$u_{1,t+1}(s_{1,t}) = \frac{1}{|R^{1,t+1}|} \sum_{j \in R^{1,t+1}} \alpha_1 * |p_{1,t+1} - s_{j,t+1}| \tag{5}$$

$$= \alpha_1 * |p_{1,t+1} - s_{2,t+1}| \tag{6}$$

$$= \alpha_1 * |(p_{1,t} + \beta_1(s_{2,t} - p_{1,t})) - s_{2,t+1}|. \tag{7}$$

In the final equation above, only $s_{2,t+1}$ depends on $s_{1,t}$. Specifically, if agent 2 shares truthfully,

$$s_{2,t+1} = p_{2,t+1} = p_{2,t} + \beta_2(s_{1,t} - p_{2,t}). \tag{8}$$

Hence, agent 1's utility in $t+1$ is

$$u_{1,t+1}(s_{1,t}) = \alpha_1 * |(p_{1,t} + \beta_1(s_{2,t} - p_{1,t})) - (p_{2,t} + \beta_2(s_{1,t} - p_{2,t}))| \tag{9}$$

$$= \alpha_1 * |(p_{1,t} + \beta_1(p_{2,t} - p_{1,t})) - (p_{2,t} + \beta_2(s_{1,t} - p_{2,t}))| \tag{10}$$

$$= \alpha_1 * |(1-\beta_1)p_{1,t} + (\beta_1 + \beta_2 - 1)p_{2,t} - \beta_2 s_{1,t}| \tag{11}$$

Using $\alpha_1 < 0$ and taking first order conditions with respect to $s_{1,t}$, we find

$$\max_{s_{1,t}} (u_{1,t+1}(s_{1,t})) = \min_{s_{1,t}} (|(1-\beta_1)p_{1,t} + (\beta_1+\beta_2-1)p_{2,t} - \beta_2 s_{1,t}|) \tag{12}$$

$$\Rightarrow s_{1,t}^* = \frac{(1-\beta_1)p_{1,t} + (\beta_1+\beta_2-1)p_{2,t}}{\beta_2} \tag{13}$$

Since agents are restricted $-1 < s_{1,t} < 1$, we have the optimal sharing strategy is $s_{1,t}^{opt} = \max(\min(s_{1,t}^*, 1), -1)$. □

We note that this strategy is implementable by agent 1 if and only if agent 1 knows the personality parameter β_2 of agent 2. Intuitively, this strategy says agent 1 should share information that is an average of the agents' stances, weighted by their impressionability, which determines how stances will change in the next round.

We can analyze the dependence of the optimal strategy on personality parameters α_i and β_i. If agent 1 has perfect impressionability ($\beta_1 = 1$), the optimal strategy reduces to

$$s^*_{1,t} = \frac{(1-\beta_1)p_{1,t} + (\beta_1+\beta_2-1)p_{2,t}}{\beta_2} = \frac{(1-1)p_{1,t} + (1+\beta_2-1)p_{2,t}}{\beta_2} = p_{2,t} \tag{14}$$

This accords with our intuitive understanding: if agent 1 is perfectly impressionable, it becomes a parrot, simply echoing the stance of its friend.

As for the open-mindedness parameter α_i, we see that $s^{opt}_{1,t}$ does not depend on α_1 beyond its sign. This is also reasonable: agent 1 wants agent 2 to share the single most agreeable stance of information possible regardless of the magnitude of its utility.

We now investigate possible sharing strategy equilibria under these assumptions.

Corollary 1.1. *If two close-minded ($\alpha_i < 0$) agent i mutually interact, both believing the other to share truthfully, the strategy profile*

$$\begin{aligned} s^*_{1,t} &= \frac{(1-\beta_1)p_{1,t} + (\beta_1+\beta_2-1)p_{2,t}}{\beta_2}, \\ s^*_{2,t} &= \frac{(1-\beta_2)p_{2,t} + (\beta_1+\beta_2-1)p_{1,t}}{\beta_1} \end{aligned} \tag{15}$$

is a Nash Equilibrium.

Proof. By Lemma 1, the strategy $s^{opt}_{i,t}$ is a dominant strategy for agent i. It follows immediately that both agents playing $s^{opt}_{i,t}$ consists of a Nash Equilibrium. Note, however, that this strategy profile being a Nash Equilibrium rests critically on both agents assuming the other to share truthfully. □

The next step in the analysis of the friend pair structure is to investigate agents' stances in equilibrium. We use the notation $-i = 1$ if $i = 2$ and $-i = 2$ if $i = 1$.

Theorem 2. *If two close minded ($\alpha_i < 0$) mutual friends both play Nash Equilibrium strategies from Corollary 1.1, the equilibrium stance $\lim_{t\to\infty} p_{i,t}$ of agent i equals*

1. *$p_{i,0}$ if $\beta_1 = \beta_2 = 0$,*
2. *$\frac{(\beta_i-1)p_{i,0}+(\beta_{-i}-1)p_{-i,0}}{\beta_1+\beta_2-2}$ if $0 < \beta_1 + \beta_2 < 2$, and*
3. *does not exist, i.e. alternates, if $\beta_1 = \beta_2 = 1$.*

Proof. For this analysis, for tractability we relax the $[-1, 1]$ constraint on stances of shared values. We can re-impose the constraint as needed in interpreting the final solution. Consider the values of $s^*_{i,t}$, the potentially unachievable optimal sharing values. First, take $i = 1$.

$$\begin{aligned} p_{1,t+1} &= p_{1,t} + \beta_1\left(s^*_{2,t} - p_{1,t}\right) && (16)\\ &= p_{1,t} + \beta_1\left(\frac{(1-\beta_2)p_{2,t} + (\beta_1+\beta_2-1)p_{1,t}}{\beta_1} - p_{1,t}\right) && (17)\\ &= (1-\beta_1)p_{1,t} + (1-\beta_2)p_{2,t} + (\beta_1+\beta_2-1)p_{1,t} && (18)\\ &= \beta_2(p_{1,t} - p_{2,t}) + p_{2,t} && (19)\end{aligned}$$

By symmetry, dynamics are analogous for $p_{2,t}$, so the system of recurrence equations describing the dynamics of stances is

$$\begin{aligned} p_{1,t+1} &= \beta_2(p_{1,t} - p_{2,t}) + p_{2,t} \\ p_{2,t+1} &= \beta_1(p_{2,t} - p_{1,t}) + p_{1,t}. \end{aligned} \tag{20}$$

The solution of this general system is

$$\begin{aligned} p_{1,t} &= \frac{1}{\beta_1+\beta_2-2}\left(p_{1,0}\left((\beta_2-1)(\beta_1+\beta_2-1)^t + \beta_1 - 1\right) - p_{2,0}\left((\beta_2-1)((\beta_1+\beta_2-1)^t - 1)\right)\right) \\ p_{2,t} &= \frac{1}{\beta_1+\beta_2-2}\left(p_{2,0}\left((\beta_1-1)(\beta_1+\beta_2-1)^t + \beta_2 - 1\right) - p_{1,0}\left((\beta_1-1)((\beta_1+\beta_2-1)^t - 1)\right)\right) \end{aligned} \tag{21}$$

The limiting behavior of this system depends on the value $c = \beta_1 + \beta_2 - 1$. The three possible cases are (1) $c = 1 \Rightarrow \beta_1 + \beta_2 = 0$; (2) $|c| < 1 \Rightarrow 0 < \beta_1 + \beta_2 < 2$; and (3) $|c| > 1, c = 1 \Rightarrow \beta_1 + \beta_2 < 0,\ \beta_1 + \beta_2 \geq 2$. These correspond to the cases of the limit of $p_{1,t}$ below.

$$\lim_{t\to\infty} p_{1,t} = \begin{cases} p_{1,0} & \beta_1 + \beta_2 = 0 \\ \frac{(\beta_1-1)p_{1,0}+(\beta_2-1)p_{2,0}}{\beta_1+\beta_2-2} & 0 < \beta_1 + \beta_2 < 2 \\ \infty & \beta_1 + \beta_2 < 0,\ \beta_1 + \beta_2 \geq 2. \end{cases} \tag{22}$$

Recall that the model restricts $0 \leq \beta_i \leq 1$. Hence, these conditions are a superset of those in the Theorem statement. □

Theorem 2 says that (1) if both agents have zero impressionability, their stances will remain unchanged in the long run, (2) if at least one agent has some, non-perfect impressionability, their stances converge to an average of their initial stances, weighted by their relative levels of impressionability, and (3) the limiting behavior alternates if both agents have perfect impressionability.

Our final remark in the friend pair structure analysis is a special case of the above theorem.

Corollary 2.1. *The stances of two close minded ($\alpha_i < 0$) mutual friends with equal impressionability $\beta_1 = \beta_2 < 1$ both playing Nash Equilibrium strategies from Corollary 1.1 will converge to the simple average of their initial stances, $p_{1,t} = p_{2,t} = \frac{1}{2}(p_{1,0} + p_{2,0})$ as $t \to \infty$.*

Proof. While this result follows directly from the statement of Theorem 2, we will take a different approach, in part as a sanity check of the Theorem.

The system of recurrence equations in the special case is $\beta := \beta_1 = \beta_2$

$$\begin{aligned} p_{1,t+1} &= \beta(p_{1,t} - p_{2,t}) + p_{2,t} \\ p_{2,t+1} &= \beta(p_{2,t} - p_{1,t}) + p_{1,t}. \end{aligned} \tag{23}$$

Solving the system we obtain,

$$\begin{aligned} p_{1,t} &= \frac{1}{2}\left(p_{1,0}(1 + (2\beta - 1)^t) + p_{2,0}(1 - (2b - 1)^t)\right) \\ p_{2,t} &= \frac{1}{2}\left(p_{1,0}(1 - (2\beta - 1)^t) + p_{2,0}(1 + (2b - 1)^t)\right), \end{aligned} \tag{24}$$

where $p_{1,0}$ and $p_{2,0}$ are interpreted as the initial stances. To characterize the equilibrium we examine the behavior in the limit $t \to \infty$.

$$\lim_{t\to\infty} p_{1,t} = \lim_{t\to\infty}\left(\frac{1}{2}\left(p_{1,0}(1 + (2\beta - 1)^t) + p_{2,0}(1 - (2\beta - 1)^t)\right)\right) \tag{25}$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\left((2\beta - 1)^t(p_{1,0} - p_{2,0}) + p_{1,0} + p_{2,0}\right)\right) \tag{26}$$

$$= \begin{cases} \frac{1}{2}(p_{1,0} + p_{2,0}) & \beta < 1 \\ p_{1,0} & \beta = 1 \\ \infty & \beta > 1. \end{cases} \tag{27}$$

By symmetry of the solutions for $p_{1,t}$ and $p_{1,t}$, the limiting behavior is analogous for $p_{2,t}$. Now, by assumption of the model, $0 \leq \beta \leq 1$. Hence, up to sharing constraints, the stances of any mutual friends with non-perfect identical impressionabilities both converge to $\frac{1}{2}(p_{1,0} + p_{2,0})$. □

While the results from in Theorem 2 and Corollary 2.1 may seem to be fairly specific, they can be helpful in cases where we believe a single friend pair's influence on each other dominates the effect of other friends in the network. In that case, then these results apply to any open-minded friend pair who (reasonably) believe the other to play truthfully. We have shown in most such cases, both individuals converge to the same opinion (basic herding behavior) which is a weighted average of their initial opinions.

3.2 Following Game Strategies and Equilibria

The Following Game describes the complementary action set to the actions in the Sharing Game. As formally specified in the Model section, agents in the Following Game decide whom to follow, but do not decide what stance of information to share–all agents share truthfully. Because agents determine their connections on the network, in the Following Game we use the general utility function in Equation (2), which depends explicitly on the quantity of information received.

In our analysis of the Following Game, first we derive an approximately optimal following strategy for networks of high degree. Then we investigate graph structures that result from agents playing this approximately optimal strategy.

3.2.1 Approximately Optimal Strategies

Agent i has three choices in round t: (1) do nothing, (2) add one followee (if one exists), or (3) remove one followee (if one exists). Hence the utility in time t of agent i for making action $a \in N \setminus \{i\}$ defined in terms of the network structure in time $t-1$ is

$$u_{i,t}(a) = \begin{cases} \gamma_i(|R^{i,t-1}|) + \frac{1}{|R^{i,t-1}|}\sum_{j\in R^{i,t-1}} \alpha_i * |p_{i,t} - s_{j,t}| & a = \emptyset \\ \gamma_i(|R^{i,t-1}|-1) + \frac{1}{|R^{i,t-1}|-1}\sum_{j\in R^{i,t-1}\setminus a} \alpha_i * |p_{i,t} - s_{j,t}| & a \in E_{i,t-1} \\ \gamma_i(|R^{i,t-1}|+1) + \frac{1}{|R^{i,t-1}|+1}\sum_{j\in R^{i,t-1}\cup\{a\}} \alpha_i * |p_{i,t} - s_{j,t}| & a \notin E_{i,t-1} \end{cases}$$

Consider the case of adding a followee. The difference to i's utility from adding l versus making no change, $u_{i,t}(l) - u_{i,t}(\emptyset)$, is

$$\gamma_i(|R^{i,t-1}|+1) - \gamma_i(|R^{i,t-1}|) + \frac{\alpha_i}{|R^{i,t-1}|+1}\sum_{j\in R^{i,t-1}\cup\{k\}} |p_{i,t} - s_{j,t}| - \frac{\alpha_i}{|R^{i,t-1}|}\sum_{j\in R^{i,t-1}} |p_{i,t} - s_{j,t}| \quad (28)$$

$$= \gamma_i(|R^{i,t-1}|+1) - \gamma_i(|R^{i,t-1}|) + \alpha_i\left(\frac{1}{|R^{i,t-1}|+1}|p_{i,t} - s_{l,t}| - \frac{1}{(|R^{i,t-1}|+1)|R^{i,t-1}|}\sum_{j\in R^{i,t-1}} |p_{i,t} - s_{j,t}|\right). \quad (29)$$

Since social networks typically have hundreds of edges per node, we consider the take the limit as $|R^{i,t}| \to \infty$. This is a reasonable approximation to the utilities over which an agent would maximize, and the error is certainly less than the impact of noise or limited rationality in real networks. In this case, the relative impact of the second re-weighting term in parentheses disappears.

$$\lim_{|R|\to\infty} (u_{i,t}(l) - u_{i,t}(\emptyset)) = \gamma_i(|R^{i,t-1}|+1) - \gamma_i(|R^{i,t-1}|) + \frac{\alpha_i}{|R^{i,t-1}|+1}|p_{i,t} - s_{l,t}|. \quad (30)$$

The case of removing a followee can be approximated analogously. This approximation motives the following definition.

Definition 3.2. (Approximately Optimal Following). Agent i follows an *approximately optimal following strategy* if in all rounds t, agent i takes action

$$a^{approx}_{i,t} = \arg\max_{a\in N\setminus\{i\}} d^{approx}_{i,t}(a) \quad (31)$$

where $d^{approx}_{i,t}(a)$ gives the approximate marginal utility of adding or removing agent a versus the null action,

$$d^{approx}_{i,t}(a) := \begin{cases} 0 & a = \emptyset \\ \gamma_i(|R^{i,t-1}|-1) - \gamma_i(|R^{i,t-1}|) - \frac{\alpha_i}{|R^{i,t-1}|-1}|p_{i,t} - s_{a,t}| & a \in E_{i,t-1} \\ \gamma_i(|R^{i,t-1}|+1) - \gamma_i(|R^{i,t-1}|) + \frac{\alpha_i}{|R^{i,t-1}|+1}|p_{i,t} - s_{a,t}| & a \notin E_{i,t-1} \setminus \{i\}. \end{cases} \quad (32)$$

Note that $d^{approx}_{i,t}(a^{approx}_{i,t}) \approx u_{i,t}(a^{approx}_{i,t}) + u_{i,t}(\emptyset)$, as $u^{approx}_{i,t}(a)$ describes the *difference* in approximate utility between the strategy of adding or removing agent a versus the baseline null strategy. The remainder of this section explicitly assumes that agents optimize these approximate utilities to play an approximately optimal following strategy, as opposed to a strictly optimal strategy.

3.2.2 Agent-Level Equilibrium Conditions

We investigate conditions on the agent level under the network structure is stable over time. Such conditions describe equilibrium graphical structures that we might expect to see in real world social networks. To do so, we first show two lemmas that result from approximately optimal sharing.

Lemma 3. *The network structure of a Following Game at time t is an equilibrium structure (stable), i.e., $G_u = G_t \ \forall s \geq t$, if all agents play an approximately optimal following strategy, and the following two conditions hold:*

$$a_{i,t}^{approx} = \emptyset \ \forall i \in N \tag{33}$$

$$s_{i,t+1} = s_{i,t} \ \forall i \in N. \tag{34}$$

Proof. Condition (33) implies that $G_t = G_{t-1}$. Since all agents follow an approximately optimal sharing strategy, condition (34) combined with the fact that the graph structure G is unchanged implies all agents solve the identical optimization in round $t+1$ as in round t. Hence, $a_{i,t+1}^{approx} = \emptyset \ \forall i \in N$. By induction, $a_{i,u}^{approx} = \emptyset \ \forall u \geq t$. Hence, no agent either adds or removes a followee in any period $u \geq t$, so network G_t is an equilibrium. □

Lemma 4. *If agent i plays an approximately optimal following strategy, then at time t, agent i does* not *add agent $l \notin E_{i,t-1} \setminus \{i\}$ if*

$$\begin{cases} \frac{\gamma_i(|R^{i,t-1}|+1)-\gamma_i(|R^{i,t-1}|)}{-\alpha_i} \geq \frac{|p_{i,t}-s_{l,t}|}{|R^{i,t-1}|+1} & \alpha_i > 0 \\ \frac{\gamma_i(|R^{i,t-1}|+1)-\gamma_i(|R^{i,t-1}|)}{-\alpha_i} \leq \frac{|p_{i,t}-s_{l,t}|}{|R^{i,t-1}|+1} & \alpha_i < 0, \end{cases} \tag{35}$$

and agent i does not *remove agent $k \in E_{i,t-1}$ if*

$$\begin{cases} \frac{\gamma_i(|R^{i,t-1}|-1)-\gamma_i(|R^{i,t-1}|)}{\alpha_i} \leq \frac{|p_{i,t}-s_{k,t}|}{|R^{i,t-1}|-1} & \alpha_i > 0 \\ \frac{\gamma_i(|R^{i,t-1}|-1)-\gamma_i(|R^{i,t-1}|)}{\alpha_i} \geq \frac{|p_{i,t}-s_{k,t}|}{|R^{i,t-1}|-1} & \alpha_i < 0. \end{cases} \tag{36}$$

Proof. By contrapositive. Assume that approximately optimal agent i adds agent l at time t. Then $l = a_{i,t}^{approx}$, so by Definition 3.2,

$$0 < \gamma_i(|R^{i,t-1}|+1) - \gamma_i(|R^{i,t-1}|) + \frac{\alpha_i}{|R^{i,t-1}|+1}|p_{i,t} - s_{a,t}| \tag{37}$$

$$\gamma_i(|R^{i,t-1}|+1) - \gamma_i(|R^{i,t-1}|) > -\alpha_i \frac{|p_{i,t} - s_{a,t}|}{|R^{i,t-1}|+1} \tag{38}$$

$$\Rightarrow \begin{cases} \frac{\gamma_i(|R^{i,t-1}|+1)-\gamma_i(|R^{i,t-1}|)}{-\alpha_i} < \frac{|p_{i,t}-s_{a,t}|}{|R^{i,t-1}|+1} & \alpha_i > 0 \\ \frac{\gamma_i(|R^{i,t-1}|+1)-\gamma_i(|R^{i,t-1}|)}{-\alpha_i} > \frac{|p_{i,t}-s_{a,t}|}{|R^{i,t-1}|+1} & \alpha_i < 0. \end{cases} \tag{39}$$

By contrapositive, this implies line (35) of the lemma. The second part in line (36) follows analogously. □

Lemma 4 has a nice interpretation. If agent i is close minded ($\alpha_i < 0$), it will not follow a new agent l unless the ratio of the marginal value of information received from j to i's level of close-mindedness is greater than the scaled magnitude of the disagreeableness of j's stance to i. This implies that, all else equal, increasing the marginal value of information from j, decreasing the close-mindedness of i, or bringing j's opinion closer to that of i's makes i more amenable to following j.

Theorem 5. *The network G_t consisting of all agents playing approximately optimal following strategies is an equilibrium structure if $s_{i,t+1} = s_{i,t} \ \forall i \in N$, and Equation (35) holds for all $l \notin E_{i,t-1} \setminus \{i\}$, and Equation (36) holds for all $k \in E_{i,t-1}$.*

Proof. This follows from the previous two lemmas. By Lemma 3, G_t is an equilibrium structure if $s_{i,t+1} = s_{i,t} \ \forall i \in N$, and $a_{i,t}^{approx} = \emptyset \ \forall i \in N$. The first condition is assumed explicitly by the theorem. For the second condition, if $\exists i \in N$ such that $a_{i,t}^{approx} \neq \emptyset$, agent i either adds or removes a followee. By Lemma 4, the conditions in Equations (35) and (36) together imply agent i neither adds or removes a followee. Hence, both conditions of Lemma 3 hold, so G_t is stable. □

3.2.3 Equilibrium Structures

With the agent-level requirements for network structure equilibrium from Theorem 5, we can characterize the global structure of some equilibrium network topologies.

Definition 3.3. (Party System). The state at time t of a Following Game is a *party system* if, at time t,

1. All agents are close-minded ($\alpha_i < 0$) and play approximately optimal following strategies,
2. The graph G can be partitioned into a set of m disjoint, completely connected subgraphs (groups) indexed by the set $M = \{1, 2, \ldots, m\}$. For all agents $i \in N$, let n_i denote the size of the group G_v such that $i \in G_v$, and
3. All agents within a group v share stance $p^v = p_{i,t}\ \forall i \in G_v$.

This definition is motivated by network structures frequently observed in political communities in which a collection of partisan individuals associate tightly with like-minded peers and distance themselves from opponents.

Corollary 5.1. *If a Following Game is a party system at time t, the network structure G_t is an equilibrium network structure if and only if for all i in N,*

$$\frac{\gamma_i(|R^{i,t-1}| + 1) - \gamma_i(|R^{i,t-1}|)}{-\alpha_i} < \min_{v \in M} \frac{|p_{i,t} - p^v|}{n_i + 1}. \tag{40}$$

Proof. By Theorem 5, G_t is an equilibrium structure if Equation (35) holds for all $l \notin E_{i,t-1} \setminus \{i\}$ and Equation (36) holds for all $k \in E_{i,t-1}$, and $s_{i,t+1} = s_{i,t}$ for all $i \in N$.

By assumption (1) of the party system, $\alpha_i < 0$, so part 2 of Equation (35) applies. By assumption (2) of the party system, $n_i = |R^{i,t}|$ so it follows immediately from the assumption in line (40) that Equation (35.2) holds for all $l \notin E_{i,t-1} \setminus \{i\}$.

Now considering Equation (36.2) for all $k \in E_{i,t-1}$, if $k \in E_{i,t-1}$ then, $p_{i,t} = p_{k,t} = s_{k,t} \Rightarrow |p_{i,t} - s_{k,t}| = 0$. Now, since $\gamma_i(\cdot)$ is assumed to be non-decreasing, $\gamma_i(|R^{i,t-1}| - 1) - \gamma_i(|R^{i,t-1}|) < 0$, so

$$\frac{\gamma_i(|R^{i,t-1}| - 1) - \gamma_i(|R^{i,t-1}|)}{\alpha_i} > 0. \tag{41}$$

Hence, the left-hand side of Equation (36.2) is positive, and the right-hand side is 0, so the equation holds.

Finally, all agents opinions are unchanged in round $t + 1$. By assumption (3) of the party system for all agents i, $p_{i,t} = p_{j,t} = s_{j,t}$ for all $j \in R^{i,t}$, so

$$\frac{1}{|R^{i,t}|} \sum_{j \in R^{i,t}} s_{j,t} = p_{i,t} \tag{42}$$

$$\Rightarrow s_{i,t+1} = p_{i,t+1} = p_{i,t} + \beta_i(p_{i,t} - p_{i,t}) = p_{i,t} = s_{i,t}. \tag{43}$$

□

Intuitively, since $p_{i,t} = p_{i,t+1}$ for all $i \in N$ and $G_t = G_{t+1}$, the stage game at round $t + 1$ is identical to that at time t. By induction, $G_{t+\tau} = G_t$ for all $\tau > 0$, and G_t is an equilibrium network structure.

Example 3.1. (Linear Thirst for Information Two-Party System). Consider a party system with two parties $M = \{1, 2\}$, and agents with linear thirst for information $\gamma_i(|R|) = c|R|\ \forall i \in N$. By Corollary 5.1, such a state is an equilibrium of the Following Game if and only if for all i in N,

$$\frac{c(|R^{i,t-1}| + 1) - c(|R^{i,t-1}|)}{-\alpha_i} < \min_{v \in M} \frac{|p_{i,t} - p^v|}{n_i + 1} \tag{44}$$

$$\frac{c}{-\alpha_i} < \frac{|p^1 - p^2|}{n_i + 1}. \tag{45}$$

Such a party system contains two herds of individuals who are connected and share a common stance, but who are not connected to the other herd and, hence, maintain their opinion in equilibrium. To disrupt such a system, comparative statics suggest (1) increasing agents' thirst for information, (2) increasing the size of the groups, (3) decreasing agents' close-mindedness, or (4) decreasing the disparity between the groups' stances. This example is a formal model that provides a stylized understanding of the existence of factions in self-determined networks such as online social networks, friendship relationships in a college, or party governments.

4 Conclusion

In this paper we defined a new model for the determination of opinions on a single topic on social networks. Agent behavior is modeled game theoretically, and alternately agents are able to chose either their sharing behavior or their following behavior. Potentially heterogeneous agents are parameterized by three personality variables which control how agents' opinions change when receiving information and how agents' utilities respond to the stances of information they receive.

In the case that agents chose their sharing behavior on a static network, we first showed that under realistic assumptions, tree network structures result in agents being ambivalent over all sharing choices. In the context of a real-world network, this finding suggests that only agents who largely follow their own followers feel pressure to communicate stances favorable to their peers. Next, we investigated the simple network topology of two mutual friends. Under the reasonable real-world assumption that both agents believe the other shares truthfully, we showed that the strategy profile in which agents share information of stance equal to a an average of their opinions weighted by their relative impressionability levels is a Nash Equilibrium of the sharing stage game. Finally, for a friend pair with non-boundary condition impressionability levels, agents' stances in equilibrium are uniquely determined to be an average of original agent stances, (differently) weighted by their relative impressionabilities. This finding accords with our observation that in small networks agents' opinions converge to the same stance in equilibrium, and also simply defines the point of this convergence in terms of how easily agents of each stance are influenced.

In the case that agents chose their following behavior while sharing truthfully, we first developed conditions for "approximately-optimal" following by linearizing the utility function. The approximately-optimal following strategy is a reasonable model for real-world optimizing behavior because it is simply calculable and within a small margin of error compared to the completely-optimal strategy for large networks. The main result for this game specifies the conditions under which factions of shared opinions exist in equilibrium.

4.1 Future Work

There remain many questions to investigate in the topic of co-determined opinions and network structure, even just in the context of the model proposed in this paper. In particular, for the Sharing Game, can we determine optimal behavior and equilibrium sharing and agent stances for more complex network topologies such as the complete graph or the Barabasi-Albert random graph? Can these equilibria be determined analytically or is simulation an appropriate method?

For the Following Game, can we generalize the conditions for an equilibrium party system to include slight discrepancies of the initial stances within groups? Further, while we have provided a characterization of equilibrium conditions for the party system, we have not detailed the initial network conditions that could lead to such an equilibrium. Investigating if realistic network creation processes, such as the preferential attachment model, can lead to this equilibrium may offer insight into how the party system structure can be reached or avoided.

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