CSC373H1 - Assignment 4

Question 1

a) For any $c \geq 1$, fix $B > c + c^2$. Suppose there are only proposals 1 and 2, where $x_1 = c, y_1 = 1 + c$ and $x_2, y_2 = B$. Only one proposal can be picked by any algorithm since $\sum_{i \in \{1,2\}} x_i = B + c > B$. ALG picks proposal 1 since $\frac{y_1}{x_1} = \frac{1+c}{c} > \frac{y_2}{x_2} = \frac{B}{B} = 1$. OPT picks proposal 2 since $y_2 = B > c + c^2 \geq y_1 = 1 + c$. Profit(ALG) $= 1 + c < \frac{B}{c} = \frac{1}{c}$ Profit(OPT) since $B > c + c^2$, as desired.

b) Assume that $k < n$, so project $k+1$ exists. Define $\epsilon = \frac{B - \sum_{i=1}^{k} x_i}{x_{k+1}}$, which $\in [0, 1)$ since $\sum_{i=1}^{k} x_i + x_{k+1} > B$. Define OPT as the optimal solution with only an integral number of projects. Define OPT' as the solution which extends OPT by spending its remaining budget to pick a fraction of a project.

Consider a solution S that sorts the projects as in part (a) and picks $\{1, \ldots, k\}$ and an ϵ-fraction of $k+1$, s.t. $\sum_{i=1}^{k} x_i + \epsilon x_{k+1} = B$. By Lemmas 1 and 2, this is the optimal solution if B is to be spent entirely. Then, Profit(S) $\geq$ Profit(OPT') $\geq$ Profit(OPT). From this, $2\max(\sum_{i=1}^{k} y_i, y_{k+1}) \geq \sum_{i=1}^{k} y_i + y_{k+1} > \sum_{i=1}^{k} y_i + \epsilon y_{k+1} =$ Profit(S) $\geq$ Profit(OPT). Since the modified algorithm picks $\max(\sum_{i=1}^{k} y_i,\ y_{k+1})$, it achieves a 2-approximation.

Lemma 1: Replacing any subset of the projects chosen by S with later projects cannot increase the total profit. Assume $\{1, \ldots, n\}$ are sorted as in part (a). Define $A \subseteq \{1, \ldots, k+1\}$ as the set of projects to be replaced, $C \subseteq \{k+2, \ldots, n\}$ as their replacement, and $\alpha_j \in (0, 1]$ as the fraction of $j \in C$ to include. Note that

- $\forall i \in A, \dfrac{y_i}{x_i} \geq \dfrac{y_{k+1}}{x_{k+1}}$, so $\sum_{i \in A} y_i \geq \dfrac{y_{k+1}}{x_{k+1}} \sum_{i \in A} x_i$, and
- $\forall j \in C, \alpha_j \dfrac{y_j}{x_j} \leq \alpha_j \dfrac{y_{k+1}}{x_{k+1}}$, so $\sum_{j \in C} \alpha_j y_j \leq \dfrac{y_{k+1}}{x_{k+1}} \sum_{j \in C} \alpha_j x_j$.

If $k + 1 \notin A$:

- $\sum_{j \in C} \alpha_j x_j \leq \sum_{i \in A} x_i$, or else the projects in C would not fit in the spaces in the budget left by removing A.
- Thus, $\sum_{j \in C} \alpha_j y_j \leq \dfrac{y_{k+1}}{x_{k+1}} \sum_{j \in C} \alpha_j x_j \leq \dfrac{y_{k+1}}{x_{k+1}} \sum_{i \in A} x_i \leq \sum_{i \in A} y_i$, which is the total profit of A.

If $k + 1 \in A$:

- $\sum_{j \in C} \alpha_j x_j \leq \sum_{i \in A \setminus \{k+1\}} x_i + \epsilon x_{k+1}$, or else the projects in C would not fit in the spaces in the budget left by removing A.
- Thus, $\sum_{j \in C} \alpha_j y_j \leq \dfrac{y_{k+1}}{x_{k+1}} \sum_{j \in C} \alpha_j x_j \leq \dfrac{y_{k+1}}{x_{k+1}} (\sum_{i \in A \setminus \{k+1\}} x_i + \epsilon x_{k+1}) \leq \sum_{i \in A \setminus \{k+1\}} y_i + \dfrac{y_{k+1}}{x_{k+1}} (\epsilon x_{k+1}) = \sum_{i \in A \setminus \{k+1\}} y_i + \epsilon y_{k+1}$, which is the total profit of A.

Lemma 2: Define $\alpha = \dfrac{B - \sum_{i \in \{1, \ldots, k+1\} \setminus \{j\}} x_i}{x_j}$ for $j < k+1$. If S instead picks the entire project

$k + 1$ and only picks an α-fraction of project j, the total profit cannot increase. This is since

$$\begin{aligned}
\frac{y_j}{x_j} \geq \frac{y_{k+1}}{x_{k+1}} &\iff \frac{\sum_{i=1}^{k+1} x_i - B}{x_j} y_j \geq \frac{\sum_{i=1}^{k+1} x_i - B}{x_{k+1}} y_{k+1} \\
&\iff (1 - \frac{B - \sum_{i \in \{1,\ldots,k+1\} \setminus \{j\}} x_i}{x_j}) y_j \geq (1 - \frac{B - \sum_{i=1}^{k} x_i}{x_{k+1}}) y_{k+1} \\
&\iff y_j + \epsilon y_{k+1} \geq \alpha y_j + y_{k+1}.
\end{aligned}$$