CSC373H1 - Ethics Module Assignment

Construct the network with the set of nodes $\{s, t, v_1, \ldots, v_n, w_1, \ldots, w_m\}$, where each v_i represents a country and each w_r represents a warehouse, with the following edges:

- (s, w_r) with capacity s_r for each w_r
- (v_i, t) with capacity c_i for each v_i
- (w_r, v_i) with capacity s_r if (i, r) is in the list of pairs and 0 otherwise

Then, the Ford-Fulkerson algorithm computes an integral maximal flow. We prove that there is a one-to-one correspondence between the flow of the network (not necessarily maximal) and total number of vaccines that can be allocated:

- (Flow of f in the network $\implies$ f vaccines can be allocated) Assume there is a flow f in the network. This means that f moves from s into the (w_r, v_i) edges with positive capacities and then into t. Since each w_r can receive at most s_r flow from s and each v_i can send at most c_i flow, each (w_r, v_i) edge cannot have more than $\min(s_r, c_i)$ flow. Equivalently, since each warehouse can supply up to s_j vaccines and each country can receive up to c_i vaccines, it must be that each warehouse j cannot send more than $\min(s_j, c_i)$ vaccines to a given country i. Additionally, if $(i, r) \notin$ the list of pairs, (w_r, s_i) cannot send flow, which means warehouse r does not send any vaccines to country i. Since the flow f meets all constraints in the original problem, f vaccines can therefore be allocated in total.
- (f vaccines can be allocated $\implies$ flow of f in the network) Assume that f vaccines can be allocated in total. Now, since each warehouse r can supply up to s_r doses in total and each country i is willing to purchase up to c_i doses, each warehouse r cannot send more than $\min(s_r, c_i)$ vaccines to a given country i. In addition, each warehouse r can only send vaccines to a country i if $(i, r) \in$ the list of pairs. In the constructed network, this is exactly equivalent to how each w_r can receive at most s_r flow from s and each v_i can send at most c_i flow to t, meaning each (w_r, v_i) edge cannot carry a flow of more than $\min(s_r, c_i)$. Moreover, (w_r, v_i) has positive flow only if $(i, r) \in$ the list of pairs. Thus, a flow of f is feasible in the network.

This proves the one-to-one correspondence, meaning that maximizing the flow of the network is equivalent to maximizing the total number of vaccines that can be allocated, as desired.