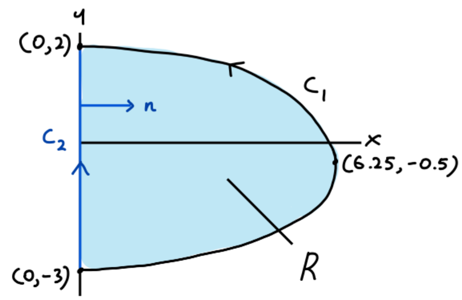
Problems

1. Let $F(x,y) = \left(xy^2 + 2y, e^{x^2} + y^4\right)$ be a vector field in $\mathbb{R}^2$. Let $C_1 \subseteq \mathbb{R}^2$ be the oriented curve parametrized by $\gamma_1(t) = (6 - t - t^2, t)$ for $-3 \le t \le 2$. Compute the normal flow of F across C_1 and include a well-labeled sketch illustrating your argument. *Hint:* Close the loop.

Define C_2 as the straight line from $(0,-3)$ to $(0,2)$. Parametrize it by $\phi : [-3,2] \to \mathbb{R}^2$ defined by $\phi(t) = (0,t)$ for $t \in [-3,2]$. The tangent to C_2 is $T_{C_2}(t) = (0,1)$ for $t \in [-3,2]$. Define $n : [-3,2] \to \mathbb{R}^2$ by $n(t) = (1,0)$ for $t \in [-3,2]$. Notice n is the normal to C_2 since for $t \in [-3,2]$, $n(t) \cdot T_{C_2} = 0$, $||n(t)|| = 1$, and $\{n(t), T_{C_2}\}$ is a positively oriented basis of $\mathbb{R}^2$ by the right hand rule.

Define the region $R = \{(x,y) \in \mathbb{R}^2 : -3 \le y \le 2, 0 \le x \le 6 - y - y^2\}$. A picture of C_1, C_2, and R with their orientations is shown below:



Notice $C_1 - C_2 = \{(x,y) \in \mathbb{R}^2 : x = 6 - y - y^2, y \in [-3,2]\} \cup \{(0,y) \in \mathbb{R}^2 : y \in [-3,2]\} = \partial R$ is a positively oriented piecewise curve. Also, F is C_1 on R, which is a regular region since $R = \overline{R^o}$. Thus, Green's theorem (divergence form) can be applied:

$$\oint_{\partial R} (F \cdot n) ds = \oint_{C_1 - C_2} (F \cdot n) ds = \int_{C_1} (F \cdot n) ds - \int_{C_2} (F \cdot n) ds = \iint_R \operatorname{div}(F) dA \tag{1}$$

by Lemma 11.3.16.

Then, we have the following:

$$\int_{C_2} (F \cdot n) ds = \int_{-3}^{2} F(\phi(t)) \cdot n(t) ||\phi'(t)|| dt = \int_{-3}^{2} (2t, 1 + t^4) \cdot (1,0) dt = \int_{-3}^{2} 2t dt = -5$$

$$\iint_R \operatorname{div}(F) dA = \iint_R (y^2 + 4y^3) dA = \int_{-3}^{2} \int_0^{6-y-y^2} (y^2 + 4y^3) dx dy = \int_{-3}^{2} (6y^2 + 23y^3 - 5y^4 - 4y^5) dy = -\frac{1625}{12}$$

Thus, by rearranging (1), we have

$$\int_{C_1} (F \cdot n) ds = -\frac{1625}{12} + (-5) = -\frac{1685}{12}$$

2. Multivariable calculus has shown how you can do calculus with all of your linear algebra. Now, near the end of your journey, it is time to do linear algebra with all of your calculus (in two dimensions).

   Let $U \subseteq \mathbb{R}^2$ be an open set. Let $C^\infty(U)$ be the set of real-valued functions $f : U \to \mathbb{R}$ with infinitely many partial derivatives; that is, $\partial^\alpha f$ exists and is continuous on U for all multi-indices $\alpha \in \mathbb{N}^2$. The space of C^∞ **scalar functions** $V = C^\infty(U)$ and space of C^∞ **vector fields** $V^2 = V \times V$ can each be thought of as a space of vectors. For example, the zero function belongs to V and acts as the zero vector. Moreover, any linear combination in V also belongs to V. Similar statements hold true for V^2.

   (2a) You can view the differential operators 'grad' and 'curl' as linear transformations on these spaces.

   - Gradient is a linear map of C^∞ scalar functions to C^∞ vector fields.
     That is, grad : $V \to V^2$ is a linear map. Hence, if $f \in V$ then $\mathrm{grad}(f) \in V^2$.
   - Two-dimensional curl is a linear map of C^∞ vector fields to C^∞ scalar functions.
     That is, curl : $V^2 \to V$ is a linear map. Hence, if $F \in V^2$ then $\mathrm{curl}(F) \in V$.

   Prove that curl is a linear map from V^2 to V. In other words, show $\mathrm{curl}(F + \lambda G) = \mathrm{curl}(F) + \lambda\,\mathrm{curl}(G)$ for any $F, G \in V^2$ and any $\lambda \in \mathbb{R}$. You may assume that the partial derivative operators $\partial_1 : V \to V$ and $\partial_2 : V \to V$ are linear maps.

Let $F, G \in V^2$ and $\lambda \in \mathbb{R}$. $\mathrm{curl}(F + \lambda G) = \partial_1((F + \lambda G)_2) - \partial_2((F + \lambda G)_1) = \partial_1(F_2 + \lambda G_2) - \partial_2(F_1 + \lambda G_1) = \partial_1 F_2 + \lambda \partial_1 G_2 - \partial_2 F_1 - \lambda \partial_2 G_1$ since ∂_1 and ∂_2 are linear maps. Rearranging terms, this equals $(\partial_1 F_2 - \partial_2 F_1) + \lambda(\partial_1 G_2 - \partial_2 G_1) = \mathrm{curl}(F) + \lambda\,\mathrm{curl}(G)$.

   (2b) The image of the gradient is contained in the kernel of curl. That is, $\mathrm{img(grad)} \subseteq \mathrm{ker(curl)}$. There are two ways to prove this fact: by boring calculation or by the "one true proof".

   Prove that $\mathrm{img(grad)} \subseteq \mathrm{ker(curl)}$ by a boring calculation with partial derivatives.

Let $f \in V$, so $\mathrm{grad}(f) = (\partial_1 f, \partial_2 f) \in V^2$. By Clairaut's theorem (Theorem 6.2.8), since U is open and f is C^1, $\partial_2 \partial_1 f = \partial_2 \partial_1 f$. Thus, $\mathrm{curl}(\mathrm{grad}(f)) = \partial_1 \partial_2 f - \partial_2 \partial_1 f = 0$, meaning $\mathrm{grad}(f) \in \mathrm{ker(curl)} = \{F \in V^2 : \mathrm{curl}(F) = 0\}$. Since f was arbitrary, $\mathrm{img(grad)} \subseteq \mathrm{ker(curl)}$.

(2c) The "one true proof" of img(grad) $\subseteq$ ker(curl) relies upon the fundamental theorem of line integrals and Green's theorem. Here is such an attempt to prove this containment.

> *1. Let* $F \in \operatorname{img}(\operatorname{grad})$ *and* $p \in U$. *Then* $\forall \varepsilon > 0, \oint_{\partial B_\varepsilon(p)} (F \cdot T)\, ds = 0.$
>
> *2.* $\implies \forall \varepsilon > 0, \iint_{\overline{B_\varepsilon(p)}} (\operatorname{curl} F)\, dA = 0 \implies \lim_{\varepsilon \to 0^+} \left[\frac{1}{\operatorname{area}(\overline{B_\varepsilon(p)})} \iint_{\overline{B_\varepsilon(p)}} (\operatorname{curl} F)\, dA \right] = 0$
>
> *3.* $\implies (\operatorname{curl} F)(p) = 0 \implies F \in \ker(\operatorname{curl})$

There are no serious errors but it is terribly written. Rewrite this into a well-written justified proof. Do **not** use Lemma 12.1.7. *Hint:* Use FTLI, Green's, and integral MVT.

Let $F \in \operatorname{img}(\operatorname{grad})$, so $F = \operatorname{grad}(f)$ for some $f \in V$. Let $\varepsilon > 0$, and let $p = (x, y) \in U$. Define $R = \overline{B_\varepsilon(p)}$ and $\partial R = \partial B_\varepsilon(p)$.

Parametrize ∂R by $\gamma : [0, 2\pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (x + \varepsilon\cos(t), y + \varepsilon\sin(t))$ for $t \in [0, 2\pi]$. Notice that $\partial R = \gamma([0, 2\pi])$, γ is continuous on $[0, 2\pi]$, γ is C^1 on $(0, 2\pi)$, $\gamma'(t) = (-\varepsilon\sin(t), \varepsilon\cos(t)) \neq 0$ for $t \in (0, 2\pi)$, and γ is injective on $(0, 2\pi)$. By definition 11.1.8, γ is a simple regular parametrization of ∂R, meaning ∂R is a curve. Furthermore, $\gamma(0) = \gamma(2\pi)$, so ∂R is closed.

By definition, the circulation of F along ∂R is $\oint_{\partial R} F \cdot T ds = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t) dt = \int_{\partial R} F \cdot d\gamma$, where $d\gamma = \gamma'(t)dt$. Since $F = \operatorname{grad}(f)$, this is equivalent to $\int_{\partial R} \operatorname{grad}(f) \cdot d\gamma$, which by the FTLI equals $f(\gamma(0)) - f(\gamma(2\pi)) = f(x+\varepsilon, y) - f(x+\varepsilon, y) = 0$. (The FTLI can be applied since f is C^1 on U, which contains ∂R.) Thus, $\oint_{\partial R} F \cdot T ds = 0$.

Note R is compact, Jordan measurable, and $\overline{R^\circ} = R$, so it is a regular region. By Problem Set 7 5a), the unit normal of ∂R is $n(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$, which points away from R, so ∂R is positively oriented. Thus, Green's theorem (curl form) can be applied: $\oint_{\partial R} F \cdot T ds = \iint_R \operatorname{curl}(F) dA = 0$. (*)

Note (*) holds true for all $\varepsilon > 0$. Thus,

$$\lim_{\varepsilon \to 0^+} \left[\frac{1}{\operatorname{area}(R)} \iint_R (\operatorname{curl} F)\, dA \right] = \lim_{\varepsilon \to 0^+} \frac{0}{\operatorname{area}(R)} = 0 \tag{2}$$

since $\operatorname{area}(R) = \operatorname{area}(\overline{B_\varepsilon(p)}) > 0$. By Corollary 8.2.8 (as derived from the integral MVT), since $R = \overline{B_\varepsilon(p)}$ and $\operatorname{curl} F$ is continuous, (2) is equal to $(\operatorname{curl} F)(p)$. Hence, $(\operatorname{curl} F)(p) = 0$, and since p was arbitrary in U, $F \in \ker(\operatorname{curl}) = \{F \in V^2 : \operatorname{curl}(F) = 0\}$. Since F was arbitrary in img(grad)), img(grad) $\subseteq$ ker(curl)).

(2d) The image may or may not equal the kernel in (2b). It depends on the topology of $U \subseteq \mathbb{R}^2$.

- Give an example of a set $U = U_1$ where img(grad) = ker(curl).
- Give an example of a set $U = U_2$ where img(grad) $\subsetneq$ ker(curl).

Briefly justify each of your examples. *Hint:* Check out the J5 readings and/or worksheet.

Define $U_1 = R^2$, which is a simply connected domain since it is open, path-connected, and the inside of every simple closed curve lying in U_1 is a subset of U_1. By Theorem 11.5.14, since U_1 is simply connected, every irrotational function $F \in V^2$ is conservative on U_1. Thus, $\forall F \in V^2$, $F \in \ker(\text{curl}) \implies F$ is irrotational $\implies \exists f : U_1 \to \mathbb{R}$ such that $F = \text{grad}(f)$ by Theorem 11.5.14. This implies $\ker(\text{curl}) \subseteq \text{img}(\text{grad})$ for U_1, so img(grad) = ker(curl).

Define $U_2 : R^2 \setminus \{(0,0,0)\}$ and $F : U_2 \to \mathbb{R}^2$ by $F(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ for $(x,y) \in U_2$. Notice that $\partial_1 F_2 = \frac{y^2-x^2}{(y^2+x^2)^2} = \partial_2 F_1$, so F is irrotational and $F \in \ker(\text{curl})$. However, since F is not a gradient vector field (hence not conservative) by Problem Set 7 Q4, there is no function $f \in V$ such that $F = \text{grad}(f)$. Thus, $\ker(\text{curl}) \nsubseteq \text{img}(\text{grad})$, so img(grad) $\subsetneq$ ker(curl).

(2e) These observations about grad and curl can be beautifully encapsulated in this elegant diagram.

$$V \xrightarrow{\text{grad}} V^2 \xrightarrow{\text{curl}} V$$

At first glance, this appears to just be a composition of maps but if you dig a bit deeper, you will notice it actually captures a lot of vector calculus in $\mathbb{R}^2$. Take an arbitrary element at the leftmost V in the diagram. Map it once to V^2 and then map it again to the next V. The element has moved two stages to the right. What happened to this element? How do the two core theorems of vector calculus in $\mathbb{R}^2$ relate to this phenomenon? Explain in two to three full sentences using the previous parts of this question.

For every element $f \in V$, its image under grad, $\text{grad}(f) \in V^2$, is conservative, and the image of grad(f) under curl is the zero function in V. This follows from FTLI and Green's by 2c, thus showing that conservative vector fields are irrotational (encapsulated in the $V^2 \to V$ transformation) and creating this phenomenon.

3. Fix $R, H > 0$. Let $S \subseteq \mathbb{R}^3$ be the part of the cone $H^2(x^2+y^2) = R^2z^2$ with $0 \leq z \leq H$.

(3a) Prove that S is a surface.

Define $D = [0,R] \times [0,2\pi]$ and $\phi : D \to \mathbb{R}^2$ by $\phi(r,t) = (r\cos t, r\sin t, \frac{H}{R}r)$. Note D is regular since it is compact, Jordan measurable, and $D = \overline{D^\circ}$. Also, D is path-connected, ϕ is continuous, and $\phi(D) = S$. Thus, by definition 13.1.1, ϕ is a (2-variable) parametrization of S.

For $p \in D^\circ$, ϕ is C^1 at p and $\{\partial_1\phi(p), \partial_2\phi(p)\} = \{(\cos t, \sin t, \frac{H}{R}), (-r\sin t, r\cos t, 0)\}$ is a linearly independent set. Thus, by definition 13.1.7, ϕ is regular.

To show ϕ is injective on D except along ∂D, suppose that $\exists x, y \in D$ such that $x, y \notin \partial D$, $\phi(x) = \phi(y)$ and $x \neq y$. This yields the system of equations

$$x_1\cos(x_2) = y_1\cos(y_2)$$
$$x_1\sin(x_2) = y_1\sin(y_2)$$
$$\frac{H}{R}x_1 = \frac{H}{R}y_1$$

By the third equation, $x_1 = y_1$. Since $x \neq y$, this means $x_2 \neq y_2$. By the first and second equations, $\cos(x_2) = \cos(y_2)$ and $\sin(x_2) = \sin(y_2)$, meaning either $x_2 = 0, y_2 = 2\pi$ or $x_2 = 2\pi, y_2 = 0$. This yields a contradiction since $(x, y) \in (\{(x_1, 0) \in \mathbb{R}^2 : x_1 \in [0,2\pi]\} \cup \{(x_1, 2\pi) \in \mathbb{R}^2 : x_1 \in [0,2\pi]\}) \subseteq \partial D$, or else $x_2 = y_2$ which implies $x = y$. Thus, $\forall x, y \in D$, either $\phi(x) = \phi(y) \implies x = y$ or $x, y \in \partial D$, meaning ϕ is simple by definition 13.1.12.

By definition 13.1.18, S is a surface since ϕ is a simple regular 2-variable parametrizaton of S.

(3b) Calculate the surface area of S by definition.

- Let $(r,t) \in D^\circ$. $\partial_1\phi = (\cos t, \sin t, \frac{H}{R})$ and $\partial_2\phi = (-r\sin t, r\cos t, 0)$, so

$$(\partial_1\phi \times \partial_2\phi)(r,t) = \begin{bmatrix} e_1 & e_2 & e_3 \\ \cos t & \sin t & H/R \\ -r\sin t & r\cos t & 0 \end{bmatrix}$$

$$= e_1\left(-\frac{H}{R}r\cos t\right) - e_2\left(\frac{H}{R}r\sin t\right) + e_3\left(r\cos^2 t + r\sin^2 t\right)$$

$$= \left(-\frac{H}{R}r\cos t, -\frac{H}{R}r\sin t, r\right)$$

meaning that

$$\|(\partial_1\phi \times \partial_2\phi)(r,t)\| = \sqrt{\frac{H^2}{R^2}r^2\cos^2 t + \frac{H^2}{R^2}r^2\sin^2 t + r^2} = r\sqrt{\frac{H^2}{R^2}+1}$$

- The surface area is

$$A(S) = \iint_D \|\partial_1\phi \times \partial_2\phi\| \, dA = \int_0^{2\pi}\int_0^R r\sqrt{\frac{H^2}{R^2}+1}\, dr\,dt = \int_0^{2\pi}\left(\frac{r^2}{2}\sqrt{\frac{H^2}{R^2}+1}\,\Big|_{r=0}^{r=R}\right)dt$$

$$= \int_0^{2\pi} \frac{R^2}{2}\sqrt{\frac{H^2}{R^2}+1}\, dt = t\,\frac{R^2}{2}\sqrt{\frac{H^2}{R^2}+1}\,\Big|_{t=0}^{t=2\pi} = \pi R^2\sqrt{\frac{H^2}{R^2}+1} = \pi R\sqrt{H^2+R^2}$$

by definition 13.2.1.

(3c) Orient S with upward unit normal. Find the flux of $F(x,y,z)=(x,y,z)$ across S by directly calculating the surface integral $\iint_S F\cdot n\,dS$. Do not use Stokes' theorem or the divergence theorem.

By definition 13.4.1 and using the cross product calculation from part b,

$$\begin{aligned}
\iint_S F\cdot n\,dS &= \iint_D (F\circ\phi)\cdot(\partial_1\phi\times\partial_2\phi)\,dA \\
&= \int_0^{2\pi}\int_0^R F(\phi(r,t))\cdot\left(-\frac{H}{R}r\cos(t),-\frac{H}{R}r\sin(t),r\right)drdt \\
&= \int_0^{2\pi}\int_0^R \left(r\cos(t),r\sin(t),\frac{H}{R}r\right)\cdot\left(-\frac{H}{R}r\cos(t),-\frac{H}{R}r\sin(t),r\right)drdt \\
&= \int_0^{2\pi}\int_0^R -\frac{H}{R}r^2\cos^2(t)-\frac{H}{R}r^2\sin^2(t)+\frac{H}{R}r^2\,drdt \\
&= \frac{H}{R}\int_0^{2\pi}\int_0^R r^2-r^2\,drdt \\
&= \frac{H}{R}\int_0^{2\pi}\int_0^R 0\,drdt \\
&= 0
\end{aligned}$$

Note that in the first to second lines of the above, $(\partial_1\phi\times\partial_2\phi)$ was chosen to be $(+1)(-\frac{H}{R}r\cos(t),-\frac{H}{R}r\sin(t),r)$. This is since the unit normal points upwards, which is consistent with the direction of $(\partial_1\phi\times\partial_2\phi)$ (which has a positive z component), meaning its sign is positive.

4. Let $G : A \to \mathbb{R}^3$ and $H : B \to \mathbb{R}^3$ be parametrizations of an oriented surface $S \subseteq \mathbb{R}^3$. Assume there exists a diffeomorphism $\varphi : U \to V$ such that $\det D\varphi > 0$, $A \subseteq U$, $B \subseteq V$, $B = \varphi(A)$, and $G(u) = (H \circ \varphi)(u)$ for $u \in A$.

(4a) Let F be a vector field in $\mathbb{R}^3$ that is continuous on S. Assuming

$$\forall u \in A^o, \quad (\partial_1 G \times \partial_2 G)(u) = (\partial_1 H \times \partial_2 H)(\varphi(u)) \det D\varphi(u), \tag{$\star$}$$

prove that

$$\iint_A (F \circ G) \cdot (\partial_1 G \times \partial_2 G)\, dA = \iint_B (F \circ H) \cdot (\partial_1 H \times \partial_2 H)\, dA$$

provided both integrals exist. In other words, the surface integral is invariant under reparametrization.

Assume both integrals exist. Define $\gamma : B^o \to \mathbb{R}$ by $\gamma(v) = (F \circ H)(v) \cdot (\partial_1 H \times \partial_2 H)(v)$ for $v \in B^o$. Thus, for $u \in A^o$,

$$\gamma(\varphi(u)) = (F \circ H)(\varphi(u)) \cdot (\partial_1 H \times \partial_2 H)(\varphi(u)) \tag{3}$$

For $u \in A^o$, $(F \circ G)(u) \cdot (\partial_1 G \times \partial_2 G)(u) = (F \circ (H \circ \varphi))(u) \cdot (\partial_1 H \times \partial_2 H)(\varphi(u)) \det D\varphi(u)$ by (*). Notice that by (3), this becomes $\gamma(\varphi(u)) \det D(\varphi(u)) = \gamma(\varphi(u))|\det D(\varphi(u))|$ since $\det D(\varphi(u)) > 0$.

Since G is a simple regular parametrization, its domain A is a regular region and hence a compact Jordan measurable set. Notice γ is integrable on $\varphi(A) = B$ since $\iint_B (F \circ H) \cdot (\partial_1 H \times \partial_2 H)\, dA$ exists, so by the change of variables theorem, $(\gamma \circ \varphi)|\det D\varphi|$ is integrable on A and the following holds:

$$\begin{aligned}
\iint_A (F \circ G) \cdot (\partial_1 G \times \partial_2 G)\, dA &= \iint_A (\gamma \circ \varphi)|\det D\varphi|\, dA \\
&= \iint_{\varphi(A)} \gamma\, dA \\
&= \iint_B (F \circ H) \cdot (\partial_1 H \times \partial_2 H) dA
\end{aligned}$$

by definition of γ and $\varphi(A) = B$.

(4b) By applying and citing properties of the cross product, prove that ($\star$) holds.

- Let $u \in A^\circ$. Since G, H, φ are C^1, by the chain rule,

$$DG(u) = D(H\circ\varphi)(u) = DH(\varphi(u))D\varphi(u)$$

$$= \begin{bmatrix} \partial_1 G_1(u) & \partial_2 G_1(u) \\ \partial_1 G_2(u) & \partial_2 G_2(u) \\ \partial_1 G_3(u) & \partial_2 G_3(u) \end{bmatrix} = \begin{bmatrix} \partial_1 (H\circ\varphi)_1(u) & \partial_2 (H\circ\varphi)_1(u) \\ \partial_1 (H\circ\varphi)_2(u) & \partial_2 (H\circ\varphi)_2(u) \\ \partial_1 (H\circ\varphi)_3(u) & \partial_2 (H\circ\varphi)_3(u) \end{bmatrix} = \begin{bmatrix} \partial_1 H_1(\varphi(u)) & \partial_2 H_1(\varphi(u)) \\ \partial_1 H_2(\varphi(u)) & \partial_2 H_2(\varphi(u)) \\ \partial_1 H_3(\varphi(u)) & \partial_2 H_3(\varphi(u)) \end{bmatrix} \begin{bmatrix} \partial_1 \varphi_1(u) & \partial_2 \varphi_1(u) \\ \partial_1 \varphi_2(u) & \partial_2 \varphi_2(u) \end{bmatrix}$$

$$= \begin{bmatrix} \partial_1 H_1(\varphi(u))\partial_1\varphi_1(u) + \partial_2 H_1(\varphi(u))\partial_1\varphi_2(u) & \partial_1 H_1(\varphi(u))\partial_2\varphi_1(u) + \partial_2 H_1(\varphi(u))\partial_2\varphi_2(u) \\ \partial_1 H_2(\varphi(u))\partial_1\varphi_1(u) + \partial_2 H_2(\varphi(u))\partial_1\varphi_2(u) & \partial_1 H_2(\varphi(u))\partial_2\varphi_1(u) + \partial_2 H_2(\varphi(u))\partial_2\varphi_2(u) \\ \partial_1 H_3(\varphi(u))\partial_1\varphi_1(u) + \partial_2 H_3(\varphi(u))\partial_1\varphi_2(u) & \partial_1 H_3(\varphi(u))\partial_2\varphi_1(u) + \partial_2 H_3(\varphi(u))\partial_2\varphi_2(u) \end{bmatrix} \quad (*)$$

- Then, $(\partial_1 G \times \partial_2 G)(u) = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ \partial_1 G_1(u) & \partial_1 G_2(u) & \partial_1 G_3(u) \\ \partial_2 G_1(u) & \partial_2 G_2(u) & \partial_2 G_3(u) \end{bmatrix} = \begin{bmatrix} (\partial_1 G_2(u))(\partial_2 G_3(u)) - (\partial_1 G_3(u))(\partial_2 G_2(u)) \\ (\partial_1 G_3(u))(\partial_2 G_1(u)) - (\partial_1 G_1(u))(\partial_2 G_3(u)) \\ (\partial_1 G_1(u))(\partial_2 G_2(u)) - (\partial_2 G_1(u))(\partial_1 G_2(u)) \end{bmatrix}$ by direct calculation of the cross product

$$= \begin{bmatrix} (\partial_1 H_2(\varphi(u))\partial_1\varphi_1(u) + \partial_2 H_2(\varphi(u))\partial_1\varphi_2(u))(\partial_1 H_3(\varphi(u))\partial_2\varphi_1(u) + \partial_2 H_3(\varphi(u))\partial_2\varphi_2(u)) \\ -(\partial_1 H_3(\varphi(u))\partial_1\varphi_1(u) + \partial_2 H_3(\varphi(u))\partial_1\varphi_2(u))(\partial_1 H_2(\varphi(u))\partial_2\varphi_1(u) + \partial_2 H_2(\varphi(u))\partial_2\varphi_2(u)) \\ (\partial_1 H_3(\varphi(u))\partial_1\varphi_1(u) + \partial_2 H_3(\varphi(u))\partial_1\varphi_2(u))(\partial_1 H_1(\varphi(u))\partial_2\varphi_1(u) + \partial_2 H_1(\varphi(u))\partial_2\varphi_2(u)) \\ -(\partial_1 H_1(\varphi(u))\partial_1\varphi_1(u) + \partial_2 H_1(\varphi(u))\partial_1\varphi_2(u))(\partial_1 H_3(\varphi(u))\partial_2\varphi_1(u) + \partial_2 H_3(\varphi(u))\partial_2\varphi_2(u)) \\ (\partial_1 H_1(\varphi(u))\partial_1\varphi_1(u) + \partial_2 H_1(\varphi(u))\partial_1\varphi_2(u))(\partial_1 H_2(\varphi(u))\partial_2\varphi_1(u) + \partial_2 H_2(\varphi(u))\partial_2\varphi_2(u)) \\ -(\partial_1 H_1(\varphi(u))\partial_2\varphi_1(u) + \partial_2 H_1(\varphi(u))\partial_2\varphi_2(u))(\partial_1 H_2(\varphi(u))\partial_1\varphi_1(u) + \partial_2 H_2(\varphi(u))\partial_1\varphi_2(u)) \end{bmatrix}$$

by the equivalence between $DG(u)$ and $DH(\varphi(u))D\varphi(u)$ at $(*)$

$$= \begin{bmatrix} \partial_1 H_2(\varphi(u))\partial_1\varphi_1(u)\partial_1 H_3(\varphi(u))\partial_2\varphi_1(u) + \partial_1 H_2(\varphi(u))\partial_1\varphi_1(u)\partial_2 H_3(\varphi(u))\partial_2\varphi_2(u) + \partial_2 H_2(\varphi(u))\partial_1\varphi_2(u)\partial_1 H_3(\varphi(u))\partial_2\varphi_1(u) \\ + \partial_2 H_2(\varphi(u))\partial_1\varphi_2(u)\partial_2 H_3(\varphi(u))\partial_2\varphi_2(u) - \partial_1 H_3(\varphi(u))\partial_1\varphi_1(u)\partial_1 H_2(\varphi(u))\partial_2\varphi_1(u) - \partial_1 H_3(\varphi(u))\partial_1\varphi_1(u)\partial_2 H_2(\varphi(u))\partial_2\varphi_2(u) \\ - \partial_2 H_3(\varphi(u))\partial_1\varphi_2(u)\partial_1 H_2(\varphi(u))\partial_2\varphi_1(u) - \partial_2 H_3(\varphi(u))\partial_1\varphi_2(u)\partial_2 H_2(\varphi(u))\partial_2\varphi_2(u) \\ \\ \partial_1 H_3(\varphi(u))\partial_1\varphi_1(u)\partial_1 H_1(\varphi(u))\partial_2\varphi_1(u) + \partial_1 H_3(\varphi(u))\partial_1\varphi_1(u)\partial_2 H_1(\varphi(u))\partial_2\varphi_2(u) + \partial_2 H_3(\varphi(u))\partial_1\varphi_2(u)\partial_1 H_1(\varphi(u))\partial_2\varphi_1(u) \\ + \partial_2 H_3(\varphi(u))\partial_1\varphi_2(u)\partial_2 H_1(\varphi(u))\partial_2\varphi_2(u) - \partial_1 H_1(\varphi(u))\partial_1\varphi_1(u)\partial_1 H_3(\varphi(u))\partial_2\varphi_1(u) - \partial_1 H_1(\varphi(u))\partial_1\varphi_1(u)\partial_2 H_3(\varphi(u))\partial_2\varphi_2(u) \\ - \partial_2 H_1(\varphi(u))\partial_1\varphi_2(u)\partial_1 H_3(\varphi(u))\partial_2\varphi_1(u) - \partial_2 H_1(\varphi(u))\partial_1\varphi_2(u)\partial_2 H_3(\varphi(u))\partial_2\varphi_2(u) \\ \\ \partial_1 H_1(\varphi(u))\partial_1\varphi_1(u)\partial_1 H_2(\varphi(u))\partial_2\varphi_1(u) + \partial_1 H_1(\varphi(u))\partial_1\varphi_1(u)\partial_2 H_2(\varphi(u))\partial_2\varphi_2(u) + \partial_2 H_1(\varphi(u))\partial_1\varphi_2(u)\partial_1 H_2(\varphi(u))\partial_2\varphi_1(u) \\ + \partial_2 H_1(\varphi(u))\partial_1\varphi_2(u)\partial_2 H_2(\varphi(u))\partial_2\varphi_2(u) - \partial_1 H_1(\varphi(u))\partial_2\varphi_1(u)\partial_1 H_2(\varphi(u))\partial_1\varphi_1(u) - \partial_1 H_1(\varphi(u))\partial_2\varphi_1(u)\partial_2 H_2(\varphi(u))\partial_1\varphi_2(u) \\ - \partial_2 H_1(\varphi(u))\partial_2\varphi_2(u)\partial_1 H_2(\varphi(u))\partial_1\varphi_1(u) - \partial_2 H_1(\varphi(u))\partial_2\varphi_2(u)\partial_2 H_2(\varphi(u))\partial_1\varphi_2(u) \end{bmatrix}$$

after expanding the terms

$$= \begin{bmatrix} \partial_1 H_2(\varphi(u))\partial_2 H_3(\varphi(u))\partial_1\varphi_1(u)\partial_2\varphi_2(u) - \partial_1 H_3(\varphi(u))\partial_2 H_2(\varphi(u))\partial_1\varphi_1(u)\partial_2\varphi_2(u) - \partial_1 H_2(\varphi(u))\partial_2 H_3(\varphi(u))\partial_1\varphi_2(u)\partial_2\varphi_1(u) + \partial_1 H_3(\varphi(u))\partial_2 H_2(\varphi(u))\partial_1\varphi_2(u)\partial_2\varphi_1(u) \\ \partial_1 H_3(\varphi(u))\partial_2 H_1(\varphi(u))\partial_1\varphi_1(u)\partial_2\varphi_2(u) - \partial_1 H_1(\varphi(u))\partial_2 H_3(\varphi(u))\partial_1\varphi_1(u)\partial_2\varphi_2(u) - \partial_1 H_3(\varphi(u))\partial_2 H_1(\varphi(u))\partial_1\varphi_2(u)\partial_2\varphi_1(u) + \partial_1 H_1(\varphi(u))\partial_2 H_3(\varphi(u))\partial_1\varphi_2(u)\partial_2\varphi_1(u) \\ \partial_1 H_1(\varphi(u))\partial_2 H_2(\varphi(u))\partial_1\varphi_1(u)\partial_2\varphi_2(u) - \partial_1 H_2(\varphi(u))\partial_2 H_1(\varphi(u))\partial_1\varphi_1(u)\partial_2\varphi_2(u) - \partial_1 H_1(\varphi(u))\partial_2 H_2(\varphi(u))\partial_1\varphi_2(u)\partial_2\varphi_1(u) + \partial_1 H_2(\varphi(u))\partial_2 H_1(\varphi(u))\partial_1\varphi_2(u)\partial_2\varphi_1(u) \end{bmatrix}$$

after cancelling some terms

$$= (\partial_1\varphi_1(u)\partial_2\varphi_2(u) - \partial_1\varphi_2(u)\partial_2\varphi_1(u)) \begin{bmatrix} \partial_1 H_2(\varphi(u))\partial_2 H_3(\varphi(u)) - \partial_1 H_3(\varphi(u))\partial_2 H_2(\varphi(u)) \\ \partial_1 H_3(\varphi(u))\partial_2 H_1(\varphi(u)) - \partial_1 H_1(\varphi(u))\partial_2 H_3(\varphi(u)) \\ \partial_1 H_1(\varphi(u))\partial_2 H_2(\varphi(u)) - \partial_1 H_2(\varphi(u))\partial_2 H_1(\varphi(u)) \end{bmatrix} = \det D\varphi(u) \det \begin{bmatrix} e_1 & e_2 & e_3 \\ \partial_1 H_1(\varphi(u)) & \partial_1 H_2(\varphi(u)) & \partial_1 H_3(\varphi(u)) \\ \partial_2 H_1(\varphi(u)) & \partial_2 H_2(\varphi(u)) & \partial_2 H_3(\varphi(u)) \end{bmatrix}$$

$= (\partial_1 H \times \partial_2 H)(\varphi(u)) \det D\varphi(u)$ as needed.