Open: $\forall p \in A,\ \exists \varepsilon > 0,\ A^c \cap B_\varepsilon(p) = \emptyset$ (equivalently, $B_\varepsilon(p) \subseteq A$)

Closed: $\forall p \in \mathbb{R}^n,\ \forall \varepsilon > 0,\ A \cap B_\varepsilon(p) \neq \emptyset \Rightarrow p \in A$

Bounded: $\forall p \in A,\ \exists \varepsilon > 0,\ A \subseteq B_\varepsilon(p)$ (equivalently, $A \cap B_\varepsilon(p)^c = \emptyset$)

Problems

1. Let $A \subseteq \mathbb{R}^n$ be a set. Each statement is an equivalent definition for one of the given set properties. Identify the property by filling in EXACTLY ONE circle. No justification necessary. (unfilled ○ filled ●)

(1a) $\exists p \in A$ s.t. $\forall \varepsilon > 0, A^c \cap B_\varepsilon(p) \neq \varnothing$ $[-1,1] \cup \{2\}$ $[-\infty,1] \cup \{2\} \to$ closed

$(-1,1) \cup \{2\}$

○ A is open. ○ A is closed. ○ A is bounded. ○ A is compact.

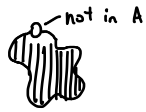
● A is not open. ○ A is not closed. ○ A is not bounded. ○ A is not compact.

(1b) $\exists p \in A$ s.t. $\forall \varepsilon > 0, A \cap B_\varepsilon(p)^c \neq \varnothing$

○ A is open. ○ A is closed. ○ A is bounded. ○ A is compact.

○ A is not open. ○ A is not closed. ● A is not bounded. ○ A is not compact.

(1c) $\exists p \notin A$ s.t. $\forall \varepsilon > 0, A \cap B_\varepsilon(p) \neq \varnothing$

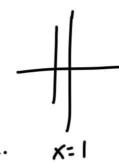


$A = \mathbb{R} \setminus \{0\}, p = 0$

○ A is open. ○ A is closed. ○ A is bounded. ○ A is compact.

○ A is not open. ● A is not closed. ○ A is not bounded. ○ A is not compact.

(1d) $\forall p \in \mathbb{R}^n, \left(\forall \varepsilon > 0, A \cap B_\varepsilon(p) \neq \varnothing \text{ AND } A^c \cap B_\varepsilon(p) \neq \varnothing\right) \implies p \in A$



○ A is open. ● A is closed. ○ A is bounded. ○ A is compact.

○ A is not open. ○ A is not closed. ○ A is not bounded. ○ A is not compact.

(1e) $\forall p \in \mathbb{R}^n, \forall$ sequences $\{x(k)\}_k \subseteq A, \left(\lim_{k\to\infty} x(k) = p \implies p \in A\right)$

○ A is open. ● A is closed. ○ A is bounded. ○ A is compact.

○ A is not open. ○ A is not closed. ○ A is not bounded. ○ A is not compact.

(1f) $\forall p \in A, \cancel{\forall \varepsilon > 0,} \forall$ sequences $\{x(k)\}_k \subseteq \mathbb{R}^n, \left(\lim_{k\to\infty} x(k) = p \implies \exists K \in \mathbb{N}^+ \text{ s.t. } \{x(k)\}_{k=K}^{\infty} \subseteq A\right)$

● A is open. ○ A is closed. ○ A is bounded. ○ A is compact.

○ A is not open. ○ A is not closed. ○ A is not bounded. ○ A is not compact.

(1g) $\exists$ sequence $\{x(k)\}_k \subseteq A$ s.t. $\left(\lim_{k\to\infty} ||x(k)|| = +\infty \text{ OR } \exists p \notin A \text{ s.t. } \lim_{k\to\infty} x(k) = p\right)$

○ A is open. ○ A is closed. ○ A is bounded. ○ A is compact.

○ A is not open. ○ A is not closed. ○ A is not bounded. ● A is not compact.

2. Lúcio is trying to prove the following lemma using the definition of compactness.

Lemma. *Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$. If A and B are compact, then $A \times B$ is compact.*

(2a) Lúcio writes an attempted argument, but it is incorrect.

> 1. *Let $\{(x(k), y(k))\}_{k=1}^{\infty}$ be a sequence in $A \times B$, so $x(k) \in A$ and $y(k) \in B$ for $k \geq 1$.*
> 2. *Since A is compact, there exists an increasing function $m_1 : \mathbb{N}^+ \to \mathbb{N}^+$ such that the subsequence $\{x(m_1(k))\}_{k=1}^{\infty}$ in A converges to a point $a \in A$.*
> 3. *Since B is compact, there exists an increasing function $m_2 : \mathbb{N}^+ \to \mathbb{N}^+$ such that the subsequence $\{y(m_2(k))\}_{k=1}^{\infty}$ in B converges to a point $b \in B$.*
> 4. *The subsequence $\{(x(m_1(k)), y(m_2(k)))\}_{k=1}^{\infty}$ in $A \times B$ therefore converges to $(a, b) \in A \times B$.*
> 5. *Since the sequence $\{(x(k), y(k))\}_{k=1}^{\infty}$ was arbitrary, this proves that $A \times B$ is compact.*

One line has a critical error. Identify the line and in a **single** sentence, explain what is the flaw.

○ Line 1 ○ Line 2 ○ Line 3 ● Line 4 ○ Line 5

Since $m_1(k)$ is not necessarily equal to $m_2(k)$ for some k, $\{(x(m_1(k)), y(m_2(k))\}_{k=1}^{\infty}$ might not belong to $\{x(k), y(k)\}_{k=1}^{\infty}$ (i.e., there can be mismatched terms of $x(k)$ and $y(k)$).

(2b) Rewrite a corrected proof using the definition of compactness.

Proof: Let $\{(x(k), y(k))\}_{k=1}^{\infty}$ be a sequence in $A \times B$, so $x(k) \in A$ and $y(k) \in B$ for $k \geq 1$.

- Since A is compact, there exists an increasing function $m_1 : \mathbb{N}^+ \to \mathbb{N}^+$ such that the subsequence $\{x(m_1(k))\}_{k=1}^{\infty}$ in A converges to a point $a \in A$.
- $\{y(k)\}_{k=1}^{\infty}$ also has the subsequence $\{y(m_1(k))\}_{k=1}^{\infty}$ in B. Since B is compact, $\{y(k)\}_{k=1}^{\infty}$ is bounded, so $\{y(m_1(k))\}_{k=1}^{\infty}$ is bounded too.
- Every bounded sequence has a convergent subsequence, so $\{y(m_1(k))\}_{k=1}^{\infty}$ has a convergent subsequence $\{y(m_2(k))\}_{k=1}^{\infty}$ where $m_2 : \mathbb{N}^+ \to \mathbb{N}^+$ is an increasing function whose range is inside the range of m_1. Thus, $\{y(m_2(k))\}_{k=1}^{\infty}$ converges to some point $b \in B$, so $y(m_2(k)) \to b$.
- Note that $\{x(m_2(k))\}_{k=1}^{\infty}$ is a subsequence of $\{x(m_1(k))\}_{k=1}^{\infty}$, which converges to a. Since every subsequence of a convergent sequence converges to the same point, $x(m_2(k)) \to a$.
- The subsequence $\{x(m_2(k)), y(m_2((k))\}_{k=1}^{\infty}$ in $A \times B$ thus converges to $(a, b) \in A \times B$.
- This shows $A \times B$ is compact (since $\{x(k), y(k)\}_{k=1}^{\infty}$ was arbitrary).

3. Limits can measure the rate at which functions tend to zero (or infinity) and polynomials are one of your favourite functions of all time. Let $j_1, \ldots, j_n \in \mathbb{N}$ and let $J \in \mathbb{N}^+$. You will compare the rate at which the monomial $x_1^{j_1} \cdots x_n^{j_n}$ and the norm $||x||^J$ tend to infinity as $||x|| = ||(x_1, \ldots, x_n)|| \to \infty$.

(3a) Prove that if $n \geq 2$ and $j_1 + \cdots + j_n \geq J$ then $\lim\limits_{||x|| \to \infty} \dfrac{x_1^{j_1} \cdots x_n^{j_n}}{||x||^J}$ does not exist.

Proof: Assume $n \geq 2$ and $j_1 + \cdots + j_n \geq J$.

- Define $f(x) = \dfrac{x_1^{j_1} \cdots x_n^{j_n}}{||x||^J}$.
- Define $\{x(k)\}_{k=1} = \{(\overbrace{k, \ldots, 0}^{n \text{ elements}})\}_k$, where every element is 0 except for the first,

$$\{y(k)\}_{k=1} = \{(\overbrace{k^{j_1}, \ldots, k^{j_n}}^{n \text{ elements}})\}_{k=1}.$$

- Notice that $\|x(k)\| = \sqrt{k^2} = k \to \infty$.
- $\|y(k)\| = \sqrt{\sum_{i=1}^{n} k^{2j_i}} \to \infty$ since $k \to \infty$, $j_1, \ldots, j_n \geq 0$.
- $f(x(k)) = \dfrac{k^{j_1} \cdots (0^{j_n})}{\|x(k)\|^J} = 0$ since $\|x(k)\| = \sqrt{k^2} = k \geq 1 > 0$,

  so $\lim\limits_{k \to \infty} f(x(k)) = 0$.
- $f(y(k)) = \dfrac{k^{j_1} \cdots k^{j_n}}{(k^{j_1} + \ldots + k^{j_n})^{J/2}} = \dfrac{k^{j_1 + \ldots + j_n}}{(k^{j_1} + \ldots + k^{j_n})^{J/2}} \geq \dfrac{k^J}{(k^{j_1} + \ldots + k^{j_n})^{J/2}} > 0$

  since $k \geq 1$, $j_1, \ldots, j_n \geq 0$, $J \geq 1$, so $\lim\limits_{k \to \infty} f(y(k)) > 0$.

- Since $\lim\limits_{k \to \infty} f(x(k)) \neq \lim\limits_{k \to \infty} f(y(k))$ where $\|x(k)\|$ and $\|y(k)\|$ both $\to \infty$, the limit DNE.

$\square$

(3b) Prove that if $n \geq 2$ and $j_1 + \cdots + j_n < J$, then $\lim_{||x|| \to \infty} \frac{x_1^{j_1} \cdots x_n^{j_n}}{||x||^J} = 0$.

Proof: Assume $n \geq 2$ and $j_1 + \cdots + j_n < J$.

- Let $\varepsilon > 0$. Take $M = \varepsilon^{1/(j_1 + \dots + j_n - J)}$. Let $x \in \mathbb{R}^n$. Assume $\|x\| > M$,

  which means that $\|x\|^{J - j_1 - \dots - j_n} > M^{J - j_1 - \dots - j_n}$ since $J - j_1 - \dots - j_n > 0$.

- Note that for $1 \leq i \leq n$, $|x_i| \leq \sqrt{x_1^2 + \dots + x_n^2} = \|x\|$.

- $$\left\| \frac{x_1^{j_1} \cdots x_n^{j_n}}{\|x\|^J} \right\| = \frac{|x_1^{j_1} \cdots x_n^{j_n}|}{\|x\|^J}$$

$$= \frac{|x_1^{j_1}| \cdots |x_n^{j_n}|}{\|x\|^J}$$

$$= \frac{|x_1|^{j_1} \cdots |x_n|^{j_n}}{\|x\|^J} \qquad (\text{since } j_1, \dots, j_n \in \mathbb{N} \geq 0)$$

$$\leq \frac{\|x\|^{j_1} \cdots \|x\|^{j_n}}{\|x\|^J} \qquad (\text{since } |x_i| \leq \|x\|,\ 1 \leq i \leq n)$$

$$= \frac{\|x\|^{j_1 + \dots + j_n}}{\|x\|^J}$$

$$= \|x\|^{j_1 + \dots + j_n - J} < M^{j_1 + \dots + j_n - J} \qquad (\text{since } j_1 + \dots + j_n - J < 0)$$

$$= \left(\varepsilon^{1/(j_1 + \dots + j_n - J)} \right)^{j_1 + \dots + j_n - J}$$

$$= \varepsilon.$$

□

4. For a set $S \subseteq \mathbb{R}^n$, define the map $\chi_S : \mathbb{R}^n \to \mathbb{R}$ by $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \in \mathbb{R}^n \setminus S$.

(4a) Orisa is trying to prove that if $S \neq \varnothing$ or $S \neq \mathbb{R}^n$, then χ_S is not continuous. They write the following.

> 1. *Let us proceed by contrapositive. Assume that χ_S is continuous.*
> 2. *The sets $\chi_S^{-1}(\{0\})$ and $\chi_S^{-1}(\{1\})$ must therefore both be closed.*
> 3. *This implies that $S = \varnothing$ or $S = \mathbb{R}^n$.*

There is no serious error, but lines 2 and 3 are missing justifications. Justify both lines.

Line 2: Theorem 2.7.25 states 'χ_s is continuous on $\mathbb{R}^n$' is equivalent to 'χ_s preserves closedness for sets under its preimage'. Since $\{0\}$ and $\{1\}$ are both closed, $\chi_s^{-1}(\{0\})$ and $\chi_s^{-1}(\{1\})$ are closed too.

Line 3: $\chi_s^{-1}(\{0\})$ is closed, and since $\chi_s^{-1}(\{0\}) = \{x \in \mathbb{R}^n : \chi_s(x) = 0\} = \mathbb{R}^n \setminus S$, $\mathbb{R}^n \setminus S$ is closed. $\mathbb{R}^n \setminus S$ is the complement of S, so S is open by Lemma 2.4.14 ($A \subseteq \mathbb{R}^n$ is open iff $A^c \subseteq \mathbb{R}^n$ is closed). But we know $\chi_s^{-1}(\{1\}) = \{x \in \mathbb{R}^n : \chi_s(x) = 1\} = S$ is closed too. That means S is clopen, so $S = \phi$ or $S = \mathbb{R}^n$.

(4b) Using the sequential definition of continuity, prove that χ_S is discontinuous at every point of ∂S.

Proof: Suppose that $\exists p \in \partial S$ at which χ_s is continuous. Take this p.

- By definition of boundary point,

  $\exists \{x(k)\}_k \in S$, $\{y(k)\}_k \in S^c$ s.t. $x(k) \to p$ and $y(k) \to p$.

  Take these sequences $\{x(k)\}_k$ and $\{y(k)\}_k$.

- Since $\{x(k)\}_k \in S$, $\chi_s(x(k)) = 1$ for all $k \in \mathbb{N}^+$, so $\chi_s(x(k)) \to 1$.
- Since $\{y(k)\}_k \in S^c$, $\chi_s(y(k)) = 0$ for all $k \in \mathbb{N}^+$, so $\chi_s(y(k)) \to 0$.
- By the definition of continuity of χ_s,
  - $x(k) \to p \Rightarrow \chi_s(x(k)) \to \chi_s(p)$, implying $\chi_s(p) = 1$ since $\chi_s(x(k)) \to 1$.
  - $y(k) \to p \Rightarrow \chi_s(y(k)) \to \chi_s(p)$, implying $\chi_s(p) = 0$ since $\chi_s(y(k)) \to 0$.

- Since $\chi_s(p) = 1$ and $\chi_s(p) = 0$, this creates a contradiction. Thus, χ_s must be discontinuous for p.
- Since p is arbitrary, χ_s is discontinuous for all points in ∂S. $\square$

5. There are many ways to establish that multivariable maps are continuous. One effective strategy is to verify that simple maps are continuous, and then express a more complicated map in terms of those simple maps.

(5a) For $i \in \{1, \ldots, n\}$, define the projection map $\pi_i : \mathbb{R}^n \to \mathbb{R}$ by $\pi_i(x) = x_i$. Prove that π_i is continuous everywhere by the ε-δ definition of continuity.

Proof: We prove continuity assuming x is a limit point of $\mathbb{R}^n$ since the set $\mathbb{R}^n$ doesn't have isolated points, so x cannot be an isolated point.

- Let $a \in \mathbb{R}^n$. Let $\varepsilon > 0$. Take $\delta = \varepsilon$. Let $x \in \mathbb{R}^n$.
- Assume $\|x-a\| < \delta$. Thus, for $i \in \{1, \ldots, n\}$,

$$|x_i - a_i| \leq \left| \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} \right| = \|x-a\| < \delta.$$

- For $i \in \{1, \ldots, n\}$,

$$\|\pi_i(x) - \pi_i(a)\| = |x_i - a_i| < \delta = \varepsilon,$$

as needed.

□

(5b) Prove that the norm function $N : \mathbb{R}^n \to \mathbb{R}$ given by $N(x) = ||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$ is continuous by expressing N in terms of the projection maps in (5a) and continuous maps from $\mathbb{R}$ to $\mathbb{R}$, and applying properties of continuous maps. Do **not** verify continuity directly by one of its definitions.

Proof: First, notice $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \sqrt{x}$, is continuous on $[0, \infty)$.

- From part a, we know that $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is continuous.
- By Theorem 2.7.16, which states that the dot product of two continuous functions is continuous, $\pi_i(x) \cdot \pi_i(x) = x_i x_i = x_i^2$, $i \in \{1, \ldots, n\}$, is continuous on $\mathbb{R}^n$.
- By Theorem 2.7.24, which states that all polynomials in n variables are continuous on $\mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$ where $g(x) = \sum_{i=1}^{n} x_i^2$ is continuous on $\mathbb{R}^n$ since it is a polynomial in n variables.
- By Corollary 2.7.18, which states that the composition of continuous functions is continuous, $N(x) = (f \circ g)(x) = \sqrt{g(x)} = \sqrt{x_1^2 + \ldots + x_n^2}$ is continuous since f and g are continuous on $\mathbb{R}^{\geq 0}$ and $\mathbb{R}^n$ respectively and g is always ≥ 0.

6. Mei is trying to prove the lemma below. She writes a poorly justified argument.

Lemma. *Let $A, B \subseteq \mathbb{R}^n$ be path-connected sets. If $A \cap B$ is non-empty, then $A \cup B$ is path-connected.*

> 1. *Let $a \in A$, $b \in B$. Fix $c \in A \cap B$.*
> 2. *There exists a continuous map $\gamma_A : [0,1] \to \mathbb{R}^n$ such that $\gamma_A([0,1]) \subseteq A$, $\gamma_A(0) = a$ and $\gamma_A(1) = c$.*
> 3. *There exists a continuous map $\gamma_B : [1,2] \to \mathbb{R}^n$ such that $\gamma_B([1,2]) \subseteq B$, $\gamma_B(1) = c$ and $\gamma_B(2) = b$.*
> 4. *Define the map $\gamma : [0,2] \to \mathbb{R}^n$ by $\gamma(t) = \gamma_A(t)$ for $0 \leq t \leq 1$ and $\gamma(t) = \gamma_B(t)$ for $1 < t \leq 2$.*
> 5. *Notice that $\gamma(0) = \gamma_A(0) = a$ and $\gamma(2) = \gamma_B(2) = b$.*
> 6. *This proves that $A \cup B$ is path-connected.*

(6a) Mei does not state when she uses that $A \cap B$ is non-empty. In which line is it used?

● Line 1 ○ Line 2 ○ Line 3 ○ Line 4 ○ Line 5 ○ Line 6

(6b) Mei does not state when she uses that B is path-connected. In which line is it used?

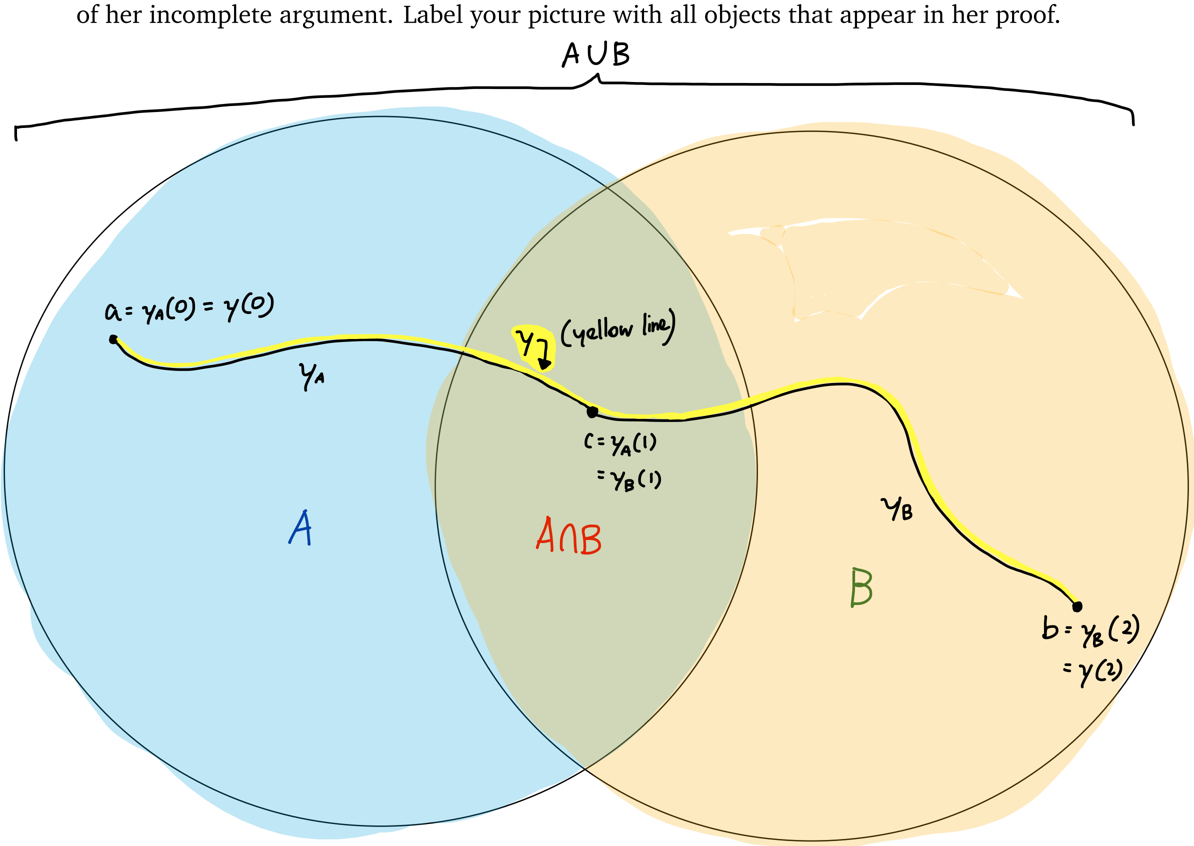
○ Line 1 ○ Line 2 ● Line 3 ○ Line 4 ○ Line 5 ○ Line 6

(6c) Mei also forgets to verify two key facts. Identify them with a brief sentence each. Do not justify them.

She does not verify if γ is continuous on $[0, 2]$.

She does not verify if $img(\gamma) \subseteq A \cup B$.

(6d) Although Mei's argument lacks details, you can still sketch a "picture proof" illustrating the main ideas of her incomplete argument. Label your picture with all objects that appear in her proof.



7. Let $A \subseteq \mathbb{R}^n$ be a closed and unbounded set. Let $f : A \to \mathbb{R}$ be a real-valued function. Assume f is continuous. Prove that if $f(x) \to \infty$ as $\|x\| \to \infty$, then f attains a minimum on A. *Hint:* Use the EVT for compact sets.

Proof: Assume $f(x) \to \infty$ as $\|x\| \to \infty$. By definition, this means

$$\forall N > 0, \exists M > 0 \text{ s.t. } \forall x \in A, \|x\| > M \Rightarrow f(x) > N.$$

Take this M associated with each N.

- Consider $\overline{B_M(0)}$ where 0 is the origin. $\overline{B_M(0)} \subseteq A$ since $x \in A$ and $\|x\| > M$.
- $\overline{B_M(0)}$ is closed and bounded, and by the B-W Theorem it is compact.
- By the EVT for compact sets, since f is continuous, f must have a minimum on $\overline{B_M(0)}$.
- Call this minimum m_0 so that $f(a) = m_0$ for some a. Note $a \in \overline{B_M(0)}$ so $\|a\| \leq M$.
- From the assumption, $f(x) > N$ when $\|x\| > M$ and $f(x) \leq N$ when $\|x\| \leq M$. Since $\|a\| \leq M$, $f(a) = m_0 \leq N$.
- So, $\forall x_1 \in A$ s.t. $\|x_1\| > M$ and $f(x_1) > N$,

$$\|a\| \leq M < \|x_1\| \quad \text{and} \quad f(a) = m_0 \leq N < f(x_1).$$

- Thus, since x_1 was arbitrary, m_0 is also smaller than any value of f outside the ball, so it is the minimum of f in A, as needed.

$\square$