Problems

1. (Revised 2023-01-20) Define the set

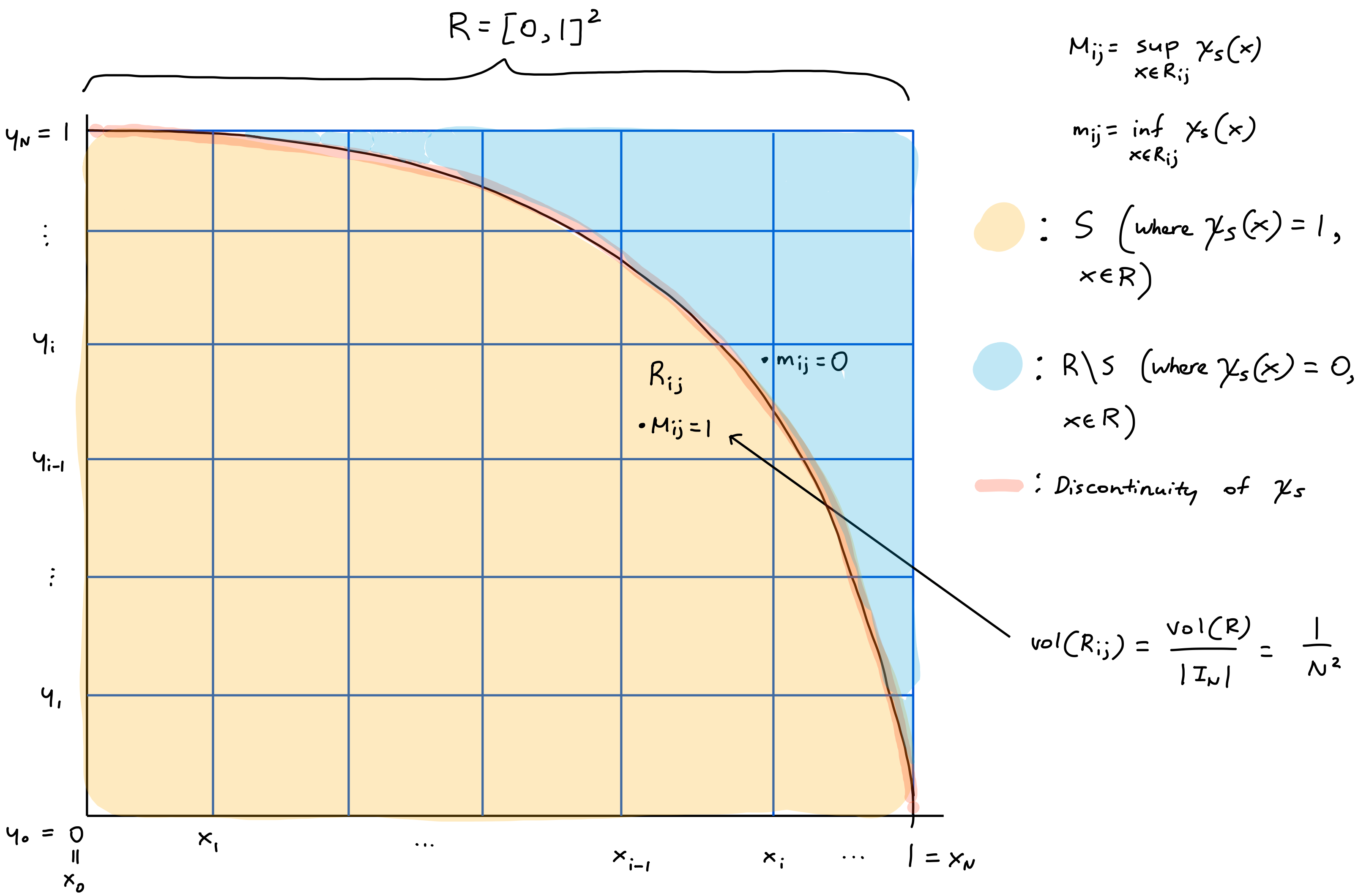
$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}.$$

You will prove that the indicator function χ_S is integrable on the rectangle $R = [0,1]^2$ containing S. Fix $N \in \mathbb{N}^+$. Let $\{x_0, x_1, \ldots, x_N\}$ and $\{y_0, y_1, \ldots, y_N\}$ be regular partitions of $[0,1]$. Let

$$P_N = (\{x_0, x_1, \ldots, x_N\}, \{y_0, y_1, \ldots, y_N\})$$

so P_N is a regular partition of R with index set I_N and subrectangles $\{R_{ij} : (i, j) \in I_N\}$.

(1a) Provide a "picture proof" by drawing a two-dimensional sketch which illustrates the formal argument. Label your diagram with discontinuities of χ_S, the partition P_N of R, and all relevant quantities that you will use in your proof. Add some expressions to explain your picture, but do not write formal arguments, sentences, or calculations. You want a reader (who is familiar with standard notation for integration) to understand the key steps of your formal proof using only your well labelled illustration.



(1b) Define the quarter circle $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x \geq 0, y \geq 0\}$. Show that if the index $(i,j) \in I_N$ satisfies $C \cap R_{ij} \neq \varnothing$, then $(i-1)^2 + (j-1)^2 \leq N^2$ and $i^2 + j^2 \geq N^2$.

Let $(i,j) \in I_N$ and assume $C \cap R_{ij} \neq \phi$.

- The top-right corner of R_{ij} has coordinates $(i/N, j/N)$.

  If this corner touches C, $(i/N)^2 + (j/N)^2 = 1$.

  If it does not, it has to be outside S since the rest of R_{ij} must intersect C, so $(i/N)^2 + (j/N)^2 > 1$.

  Thus, $(i/N)^2 + (j/N)^2 \geq 1$, so $i^2 + j^2 \geq N^2$ in all cases.

- The bottom-left corner of R_{ij} has coordinates $\left(\frac{i-1}{N}, \frac{j-1}{N}\right)$. $i-1$ and $j-1 \geq 0$ by definition of the subrectangle.

  If this corner touches C, $\left(\frac{i-1}{N}\right)^2 + \left(\frac{j-1}{N}\right)^2 = 1$.

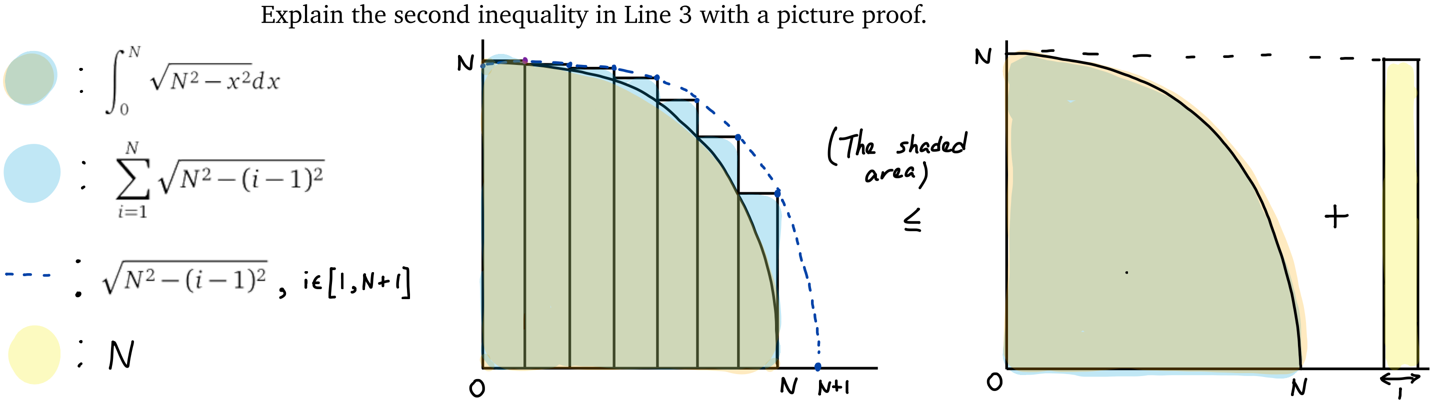
  If it does not, it has to be within S since the rest of R_{ij} must intersect C, so $\left(\frac{i-1}{N}\right)^2 + \left(\frac{j-1}{N}\right)^2 < 1$.

  Thus, $\left(\frac{i-1}{N}\right)^2 + \left(\frac{j-1}{N}\right)^2 \leq 1$, so $(i-1)^2 + (j-1)^2 \leq N^2$ in all cases.

(1c) The subset of indices $J_N = \{(i,j) \in I_N : C \cap R_{ij} \neq \varnothing\}$ has at most $4N$ elements. A concise proof is below.

1. *Notice $(i,j) \in J_N$ implies that $i \in \{1,\dots,N\}$ and, by (1b), $\sqrt{N^2 - i^2} \leq j \leq \sqrt{N^2 - (i-1)^2} + 1$.*
2. *It follows that $|J_N| \leq \sum_{i=1}^{N} \left(\sqrt{N^2 - (i-1)^2} - \sqrt{N^2 - i^2} + 2 \right)$.*
3. *Notice $\sum_{i=1}^{N} \sqrt{N^2 - i^2} \geq -N + \int_0^N \sqrt{N^2 - x^2}\,dx$ and $\sum_{i=1}^{N} \sqrt{N^2 - (i-1)^2} \leq N + \int_0^N \sqrt{N^2 - x^2}\,dx$.*
4. *Hence, $|J_N| \leq \sum_{i=1}^{N} \sqrt{N^2 - (i-1)^2} - \sum_{i=1}^{N} \sqrt{N^2 - i^2} + \sum_{i=1}^{N} 2 \leq 2N + 2N = 4N$, as required.*

Explain the second inequality in Line 3 with a picture proof.



(1d) Let χ_S be the indicator function on S. Prove that $U_{P_N}(\chi_S) - L_{P_N}(\chi_S) \leq \frac{4}{N}$.

- Define $M_i = \sup_{x \in R_i} \chi_S(x)$ and $m_i = \inf_{x \in R_i} \chi_S(x)$.

- For $i \in I_N$, there are 3 cases: $R_i \subseteq S$, $R_i \subseteq R \setminus S$, or R_i is not fully contained in either (i.e., R_i intersects both S and $R \setminus S$ at the same time).

- If $R_i \subseteq S$, $M_i = m_i = 1$. If $R_i \subseteq R \setminus S$, $M_i = m_i = 0$. If R_i is not fully contained in either, $M_i = 1$ and $m_i = 0$.

- Note also that $\text{vol}(R_i) = \dfrac{\text{vol}(R)}{|I_N|} = \dfrac{1}{N^2}$ since P_N is regular, $\text{vol}(R) = 1$, and $|I_N| = N \cdot N = N^2$

- For $i \in J_N$, all three cases are possible since R_i may or may not intersect C at the top-right or bottom-left corners. Thus,

$$\sum_{i \in J_N} (M_i - m_i)\,\text{vol}(R_i) \leq \sum_{i \in J_N} (1-0)\frac{1}{N^2} = \sum_{i \in J_N} \frac{1}{N^2} = \frac{|J_N|}{N^2} \leq \frac{4}{N}$$

- For $i \in I_N \setminus J_N$, R_i must be entirely contained in either S or $R \setminus S$. Thus,

$$\sum_{i \in I_N \setminus J_N} (M_i - m_i)\,\text{vol}(R_i) = \sum_{i \in I_N \setminus J_N} (1-1)\,\text{vol}(R_i) = 0 \quad \text{or} \sum_{i \in I_N \setminus J_N} (0-0)\,\text{vol}(R_i) = 0$$

- Since $I_N = J_N \cup I_N \setminus J_N$,

$$U_P(\chi_S) - L_P(\chi_S) = \sum_{i \in I_N} (M_i - m_i)\,\text{vol}(R_i) = \sum_{i \in J_N} (M_i - m_i)\,\text{vol}(R_i) + \sum_{i \in I_N \setminus J_N} (M_i - m_i)\,\text{vol}(R_i)$$

$$\leq \frac{4}{N} + 0 = 4/N.$$

(1e) Conclude that the indicator function χ_S is integrable on R.

Fix $\varepsilon = 4/N$. Since $N \in \mathbb{N}^+$ was arbitrary, $0 < \varepsilon \leq 4$ is too.

This satisfies the ε-characterization of integrability so the indicator function χ_S is integrable on R.

2. (Revised 2023-01-22) Let R be a rectangle in $\mathbb{R}^n$. Let f and g be integrable functions on R.
Asif attempts to prove $f+g$ is integrable on R. Below is part of his **INCOMPLETE** argument.

> 1. *For any partition P of the rectangle R, notice that*
>
> $$U_P(f+g) \le U_P(f) + U_P(g) \qquad \textit{and} \qquad L_P(f+g) \ge L_P(f) + L_P(g).$$
>
> 2. *Taking suprema and infima over all partitions P gives*
>
> $$\underline{I_R}(f+g) = \sup\{L_P(f+g) : P \textit{ partition of } R\} \ge \sup\{L_P(f)+L_P(g) : P \textit{ partition of } R\} = \underline{I_R}(f)+\underline{I_R}(g)$$
>
> $$\overline{I_R}(f+g) = \inf\{U_P(f+g) : P \textit{ partition of } R\} \le \inf\{U_P(f)+U_P(g) : P \textit{ partition of } R\} = \overline{I_R}(f)+\overline{I_R}(g)$$

(2a) Line 1 applies one TRUE claim about suprema/infima of functions. Formally state that claim.

TRUE claim. *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be bounded functions with domain $D \subseteq \mathbb{R}^n$. Then*

$$\sup_{x\in D}(f+g)(x) \le \sup_{x\in D} f(x) + \sup_{x\in D} g(x) \qquad \text{and}$$

$$\inf_{x\in D}(f+g)(x) \ge \inf_{x\in D} f(x) + \inf_{x\in D} g(x)$$

(2b) Line 2 applies three steps to overall deduce that $\overline{I_R}(f+g) \le \overline{I_R}(f) + \overline{I_R}(g)$ and $\underline{I_R}(f+g) \ge \underline{I_R}(f) + \underline{I_R}(g)$. This pair of inequalities is true, but the argument is not complete. For each sentence below, select the most accurate corresponding statement. Fill in EXACTLY ONE circle per sentence.
Hint: Exactly one of these should be "None of the above".
The claimed equality $\overline{I_R}(f+g) = \inf\{U_P(f+g) : P \text{ partitions } R\}$ is ...

- ● TRUE by definition of the upper integral.
- ◯ TRUE by definition of upper sums.
- ◯ TRUE by a property of infimum only.
- ◯ TRUE by Line 1 only.
- ◯ TRUE by a property of infimum and Line 1.
- ◯ None of the above.

The claimed inequality $\inf\{U_P(f+g) : P \text{ partitions } R\} \le \inf\{U_P(f) + U_P(g) : P \text{ partitions } R\}$ is ...

- ◯ TRUE by the triangle inequality.
- ◯ TRUE by definition of upper sums.
- ◯ TRUE by a property of infimum only.
- ◯ TRUE by Line 1 only.
- ● TRUE by a property of infimum and Line 1.
- ◯ None of the above.

The claimed equality $\inf\{U_P(f) + U_P(g) : P \text{ partitions } R\} = \overline{I_R}(f) + \overline{I_R}(g)$ is ...

- ◯ TRUE by definition of the upper integral.
- ◯ TRUE by definition of upper sums.
- ◯ TRUE by a property of infimum only.
- ◯ TRUE by Line 1 only.
- ◯ TRUE by a property of infimum and Line 1.
- ● None of the above.

(2c) Find a different argument and prove that $\overline{I_R}(f+g) \le \overline{I_R}(f) + \overline{I_R}(g)$.
Hint: Carefully apply the definition of infimum, and introduce two partitions.

- $\overline{I}_R(f) = \inf_P U_P(f)$ and $\overline{I}_R(g) = \inf_P U_P(g)$.
- By definition of infimum, $\forall \varepsilon > 0$, $\exists P'$ that partitions R s.t. $U_{P'}(f) < \overline{I_R}(f) + \varepsilon/2$ and $\forall \varepsilon > 0$, $\exists P''$ that partitions R s.t. $U_{P''}(g) < \overline{I_R}(g) + \varepsilon/2$.
- Let $\varepsilon > 0$ and take such a P' and P''.
- Let P be the common refinement of P' and P''. By Lemma 7.2.7, $U_P(f) \le U_{P'}(f)$ and $U_P(g) \le U_{P''}(g)$ since f and g are bounded (since they are integrable).
- By Lemma 7.2.9, since f and g are bounded,

$$U_P(f+g) \le U_P(f) + U_P(g) \le U_{P'}(f) + U_{P''}(g) < \overline{I_R}(g) + \varepsilon/2 + \overline{I_R}(f) + \varepsilon/2$$
$$= \overline{I_R}(g) + \overline{I_R}(f) + \varepsilon \qquad (1)$$

- Since $\overline{I}_R(f+g) = \inf_P U_P(f+g) \le U_P(f+g)$ for any partition P of R, by (1),

$$\overline{I}_R(f+g) < \overline{I_R}(g) + \overline{I_R}(f) + \varepsilon \qquad (2)$$

- By the below lemma where $c = \overline{I}_R(f+g)$ and $a+b = \overline{I_R}(g) + \overline{I_R}(f)$, (2) $\Rightarrow$ $\overline{I}_R(f+g) \le \overline{I_R}(g) + \overline{I_R}(f)$ since ε was arbitrary.

- Lemma: Let $a, b, c \in \mathbb{R}$. $\forall \varepsilon > 0$, $c < a+b+\varepsilon \Rightarrow c \le a+b$.
  - We prove by contrapositive, which is $c > a+b \Rightarrow \exists \varepsilon > 0$, $c \ge a+b+\varepsilon$.
  - Assume $c > a+b$. Take $\varepsilon = c-a-b > 0$. Then $a+b+\varepsilon = c \le c$ as needed.

(2d) Conclude that $f+g$ is integrable on R. You may assume without proof that $\underline{I_R}(f+g) \ge \underline{I_R}(f) + \underline{I_R}(g)$.

- Since f and g are integrable on R, $\overline{I_R}(f) = \underline{I}_R(f)$ and $\overline{I_R}(g) = \underline{I}_R(g)$. Thus,

$$\overline{I_R}(g) + \overline{I_R}(f) = \underline{I}_R(f) + \underline{I}_R(g) \qquad (1)$$

- Since f and g are bounded, $f+g$ is bounded, so by Lemma 7.3.3, $\underline{I}_R(f+g) \le \overline{I}_R(f+g)$. Thus,

$$\underline{I}_R(g) + \underline{I}_R(f) \le \underline{I}_R(f+g) \le \overline{I}_R(f+g) \le \overline{I_R}(f) + \overline{I}_R(g) \qquad \text{by assumption}$$

but by (1) and the squeeze theorem this means $\underline{I}_R(f+g) = \overline{I}_R(f+g)$.

- Thus, $f+g$ is integrable on R.

3. Let $A \subseteq \mathbb{R}^n$ be closed and unbounded. Prove that if $f : A \to \mathbb{R}$ is continuous and $f(x) \to 237$ as $||x|| \to \infty$, then f is uniformly continuous. *Hint:* Use ideas from PS2 Q7.

- Let $\varepsilon > 0$. By assumption, $\exists R > 0$ s.t. $\forall x \in A,\ \|x\| \geq R \Rightarrow |f(x) - 237| < \varepsilon/3$ (1)
- Define $K = \overline{B_R(0)} \cap A$, as in PS2 Q7. K is compact by PS2 Q7.
- WTS $\exists \delta > 0$ s.t. $\forall x, y \in A,\ \|x-y\| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. There are 3 cases.
- Case $x, y \in K$:
  - By Theorem 7.4.8, since K is compact and $f|_K$ is continuous, $f|_K$ is uniformly continuous. This means $\exists \delta' > 0$ s.t. $\forall x, y \in K,\ \|x-y\| < \delta' \Rightarrow |f(x) - f(y)| < \varepsilon/3 < \varepsilon$ (2)
- Case $x, y \in \overline{B_R(0)}^c \cap A$:
  - Take the δ' in (2). Let $x, y \in \overline{B_R(0)}^c \cap A$ and assume $\|x-y\| < \delta'$.
  - Since $x \in \overline{B_R(0)}^c \cap A \subseteq A$ and $\|x\| \geq R$, $|f(x) - 237| < \varepsilon/3$ by (1), and similarly $|f(y) - 237| < \varepsilon/3$, so

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - 237 + 237 - f(y)| \\ &\leq |f(x) - 237| + |237 - f(y)| && \text{by the triangle inequality} \\ &= \varepsilon/3 + \varepsilon/3 < \varepsilon. && \text{by assumption} \end{aligned}$$

- Case $x \in K$, $y \in \overline{B_R(0)}^c \cap A$ (the case $y \in K$, $x \in \overline{B_R(0)}^c \cap A$ is the same):
  - Let $x \in K$, $y \in \overline{B_R(0)}^c \cap A$ and assume $\|x-y\| < \delta'$ from (2). Pick $z \in \partial B_R(0) \cap A$ s.t. $\|x-z\| < \|x-y\| < \delta'$. Such a z exists since $\|y\| > R$.
  - This implies $|f(x) - f(z)| < \varepsilon/3$ by (2) since $z \in \partial B_R(0) \cap A \subseteq \overline{B_R(0)} \cap A = K$.
  - Notice further that since $\|y\| \geq R$, $|f(y) - 237| < \varepsilon/3$ and since $\|z\| \geq R$, $|f(z) - 237| < \frac{\varepsilon}{3}$.
  - Thus,

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(z) + f(z) - 237 + 237 - f(y)| \\ &\leq |f(x) - f(z)| + |f(z) - 237| + |f(y) - 237| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

- Since $A = K \cup (\overline{B_R(0)}^c \cap A)$, the choice of $\delta = \delta'$ works for all $x, y \in A$, and since ε was arbitrary, this completes the proof.

4. (Revised 2023-01-16) Return to the set

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}.$$

You will give two different proofs that ∂S has zero Jordan measure – one by definition and one with a hammer.

(4a) Show by definition that ∂S has zero Jordan measure using Question 1, especially (1c).

- Let $\varepsilon > 0$. Set $N = \lceil 12/\varepsilon \rceil$.
- Let P_N be a partition of $[0,1]^2$ in the format specified in question 1 with the corresponding index set I_N and subrectangles.
- By 1d, for $i \in I_N$, $\text{vol}(R_i) = \frac{1}{|I_N|} = \frac{1}{N^2}$.
- Take J_N from 1c. Notice $\sum_{i \in J_N} \text{vol}(R_i) = \sum_{i \in J_N} \frac{1}{N^2} \leq \frac{4}{N} = \frac{4}{\lceil 12/\varepsilon \rceil} < \frac{4}{12/\varepsilon} = \varepsilon/3$
- Define the rectangles $R_x = [0, 1] \times [-\varepsilon/6, \varepsilon/6]$ and $R_y = [-\varepsilon/6, \varepsilon/6] \times [0, 1]$
- $\text{vol}(R_x) = \text{vol}(R_y) = (1-0)(\varepsilon/6 + \varepsilon/6) = \varepsilon/3$
- For $i \in J_N$, uniquely assign a number from 1 to $|J_N|$ to R_i, and assign $|J_N|+1$ to R_x and $|J_N|+2$ to R_y.
- Thus, $\partial S \subseteq \bigcup_{i=1}^{|J_N|+2} \text{vol}(R_i)$.
- Moreover, $\sum_{i=1}^{|J_N|+2} \text{vol}(R_i) = \sum_{j \in |J_N|} \text{vol}(R_j) + \text{vol}(R_x) + \text{vol}(R_y)$

  $< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ since $\sum_{j \in |J_N|} \text{vol}(R_j) < \varepsilon/3$

as needed.

(4b) Use Sard's theorem to again prove that ∂S has zero Jordan measure.

$\partial S = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1,\ x \geq 0,\ y \geq 0\}$

Define $g: \mathbb{R} \to \mathbb{R}^2$ by $g(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in \mathbb{R}$. g is C^1 since

$g_1'(\theta) = -\sin\theta$ and $g_2'(\theta) = \cos\theta$ are continuous for $\theta \in \mathbb{R}$.

Define $h: \mathbb{R} \to \mathbb{R}^2$ by $h(t) = (0, t)$ for $t \in \mathbb{R}$. h is C^1 since

$h_1'(t) = 0$ and $h_2'(t) = 1$ are continuous for $t \in \mathbb{R}$.

Define $f: \mathbb{R} \to \mathbb{R}^2$ by $f(t) = (t, 0)$ for $t \in \mathbb{R}$. f is C^1 since

$f_1'(t) = 1$ and $f_2'(t) = 0$ are continuous for $t \in \mathbb{R}$.

Notice $\partial S = g([0, \pi/2]) \cup h([0,1]) \cup f([0,1])$ and $[0, \pi/2] \subseteq \mathbb{R}$, $[0,1] \subseteq \mathbb{R}$.

By Sard's, ∂S has zero Jordan measure in $\mathbb{R}^2$.

(4c) Conclude that S is Jordan measurable and hence χ_S is integrable on any rectangle containing S.

$S \subseteq B_2(0)$ so S is bounded. ∂S has zero Jordan measure in $\mathbb{R}^2$ as shown above. By definition, $S \subseteq \mathbb{R}^2$ is Jordan measurable.

By theorem 7.6.9, this implies χ_S is integrable on any rectangle containing S.

5. Let $S, T \subseteq \mathbb{R}^n$ be Jordan measurable sets. Assume $T \subseteq S$. Use the integral definition of Jordan measurability to prove that the set $S \setminus T$ is Jordan measurable, and $\mathrm{vol}(S \setminus T) = \mathrm{vol}(S) - \mathrm{vol}(T)$.

- $\partial(S\setminus T) = \partial(S \cap T^c) \subseteq \partial S \cup \partial T^c = \partial S \cup \partial T$.
  - Since S, T are Jordan measurable, ∂S and ∂T have zero Jordan measure.
  - $\partial S \cup \partial T$ has zero Jordan measure by Lemma 7.5.7b (a finite union of zero volume sets has zero volume).
  - $\partial(S \setminus T)$ has zero volume since $\partial(S\setminus T) \subseteq \partial S \cup \partial T$ and by Lemma 7.5.7a (any subset of a zero volume set has zero volume).
- $S \setminus T = S \cap T^c \subseteq S$ which is bounded since S is Jordan measurable. Thus, $S\setminus T$ is bounded.
- Since $S\setminus T$ is bounded and $\partial(S\setminus T)$ has zero volume, $S \setminus T$ is Jordan measurable.
- $\chi_{S\setminus T} : \mathbb{R}^n \to \{0,1\}$ is defined as $\chi_{S\setminus T} = \begin{cases} 1 & \text{if } x \in S, x \notin T \\ 0 & \text{otherwise} \end{cases}$
- Notice that $\chi_{S\setminus T} = \chi_S - \chi_T$ everywhere since:
  - Case $x \in S, x \in T$: $\chi_S = 1, \chi_T = 1$, and $\chi_{S\setminus T} = 0 = 1 - 1 = \chi_S - \chi_T$
  - Case $x \in S, x \notin T$: $\chi_S = 1, \chi_T = 0$, and $\chi_{S\setminus T} = 1 = 1 - 0 = \chi_S - \chi_T$
  - Case $x \notin S$: $\chi_S = 0, \chi_T = 0$, and $\chi_{S\setminus T} = 0 = 0 - 0 = \chi_S - \chi_T$
- Note it is impossible for $x \notin S$ and $x \in T$ since $T \subseteq S$
- Since $S\setminus T$, S, and T are Jordan measurable, by theorem 7.6.9, $\chi_{S\setminus T}$, χ_S, and χ_T are integrable on any rectangle containing S.
- Let $R \subseteq \mathbb{R}^n$ be a rectangle containing S (and $S\setminus T$ and T by extension), so

$$\mathrm{vol}(S\setminus T) = \int_R \chi_{S\setminus T}\, dV = \int_R (\chi_S - \chi_T)\, dV \quad \text{since } \chi_{S\setminus T} = \chi_S - \chi_T$$

$$= \int_R \chi_S\, dV - \int_R \chi_T\, dV \quad \text{by linearity (theorem 7.3.13) since } \chi_S \text{ and } \chi_T \text{ are bounded and integrable on } R$$

$$= \mathrm{vol}(S) - \mathrm{vol}(T) \quad \text{by definition of volume of sets.}$$