Problems

1. No justification is necessary for any part of this question.

   (1a) Find a bounded set $S \subseteq \mathbb{R}^n$ such that f is integrable on S for all bounded functions $f : \mathbb{R}^n \to \mathbb{R}$.

Let $S = \{0\} \subseteq \mathbb{R}^n$.

   (1b) Find a bounded function $f : \mathbb{R}^n \to \mathbb{R}$ such that f is integrable on S for all bounded sets $S \subseteq \mathbb{R}^n$.

Let $f(x) = 0, \forall x \in \mathbb{R}^n$.

   (1c) Find a bounded set $S \subseteq \mathbb{R}^n$ and a bounded function $f : \mathbb{R}^n \to \mathbb{R}$ such that

   - f is not integrable on S.
   - f is integrable on ∂S, is integrable on $\overline{S}$, and is integrable on S^o.

Let $S = (\mathbb{Q} \cap [0,1])^n$. Let $f(x) = 1$, $\forall x \in \mathbb{R}^n$.

   (1d) Find a bounded set $S \subseteq \mathbb{R}^n$ and a bounded function $f : \mathbb{R}^n \to \mathbb{R}$ such that

   - f is integrable on S, and is integrable on S^o.
   - f is not integrable on ∂S, and is not integrable on $\overline{S}$.

Let $S = \mathbb{Q}^n \cap [0,1]^n$. Let $f(x) = 0$ for $x \in \mathbb{Q}^n$, and $f(x) = 1$ for $x \in \mathbb{R}^n \setminus \mathbb{Q}^n$.

2. For each part, decide which statement is true by filling EXACTLY ONE circle. No justification is necessary.
Let $S \subseteq \mathbb{R}^n$ be a Jordan measurable set. Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded. Assume f is integrable on S.

(2a) Is f integrable on the boundary ∂S?

- ○ We cannot determine whether f is integrable on ∂S without more information.
- ○ f is not integrable on ∂S.
- ○ f is integrable on ∂S but we cannot determine the value of $\int_{\partial S} f\,dV$ without more information.
- ● **f is integrable on ∂S and $\int_{\partial S} f\,dV = 0$.**
- ○ f is integrable on ∂S and $\int_{\partial S} f\,dV = \int_S f\,dV$.

(2b) Is f integrable on the interior S^o?

- ○ We cannot determine whether f is integrable on S^o without more information.
- ○ f is not integrable on S^o.
- ○ f is integrable on S^o but we cannot determine the value of $\int_{S^o} f\,dV$ without more information.
- ○ f is integrable on S^o and $\int_{S^o} f\,dV = 0$.
- ● **f is integrable on S^o and $\int_{S^o} f\,dV = \int_S f\,dV$.**

(2c) Is f integrable on the closure $\overline{S}$?

- ○ We cannot determine whether f is integrable on $\overline{S}$ without more information.
- ○ f is not integrable on $\overline{S}$.
- ○ f is integrable on $\overline{S}$ but we cannot determine the value of $\int_{\overline{S}} f\,dV$ without more information.
- ○ f is integrable on $\overline{S}$ and $\int_{\overline{S}} f\,dV = 0$.
- ● **f is integrable on $\overline{S}$ and $\int_{\overline{S}} f\,dV = \int_S f\,dV$.**

(2d) Is f integrable on the complement S^c?

- ● **We cannot determine whether f is integrable on S^c without more information.**
- ○ f is not integrable on S^c.
- ○ f is integrable on S^c but we cannot determine the value of $\int_{S^c} f\,dV$ without more information.
- ○ f is integrable on S^c and $\int_{S^c} f\,dV = 0$.
- ○ f is integrable on S^c and $\int_{S^c} f\,dV = -\int_S f\,dV$.

3. For a system of point masses at positions $x_1, \dots, x_N \in \mathbb{R}^n$ with masses $m_1, \dots, m_N > 0$ respectively, the centre of mass of this system is given by

$$\frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} \in \mathbb{R}^n. \qquad (\star)$$

You may take this physical principle for granted.

Let $\delta : R \to [0, \infty)$ be the density of a rectangular object $R \subseteq \mathbb{R}^n$ with positive mass. Assume δ is continuous on R aside from a set of zero Jordan measure. SpongeBob attempts to derive an integral formula for the centre of mass, but skips some crucial explanation and makes a mathematical error.

> 1. *Let P be a partition of R with subrectangles $\{R_i\}_{i \in I}$.*
> 2. *For each $i \in I$, let $x_i^* \in R_i$ be a sample point.*
> 3. *By $(\star)$, the centre of mass of R is approximately equal to* $\dfrac{\sum_{i \in I} x_i^* \delta(x_i^*) \operatorname{vol}(R_i)}{\sum_{i \in I} \delta(x_i^*) \operatorname{vol}(R_i)} \in \mathbb{R}^n.$
> 4. *Taking $|I| \to \infty$, the centre of mass is therefore equal to* $\dfrac{\int_R x \delta(x) dV}{\int_R \delta dV} \in \mathbb{R}^n.$

(3a) Line 3 applies $(\star)$ without enough detail. What objects are treated as point masses? How are masses approximated? Under what mathematical or physical assumptions can these two heuristics be considered reasonable? Explain in 50 to 100 words. Use full sentences, be concise, and use symbols sparingly.

The subrectangles are treated as point masses. For each subrectangle, its mass is approximated by a sample point lying inside it. These are reasonable with the following assumptions: if we assume the subrectangles are arbitrarily small, then their volumes are roughly uniform and approaching that of a point, which is zero. If we assume the density is continuous, then at arbitrarily small regions, it is roughly constant, so the density at a sample point is roughly equal to the density at any point in a given subrectangle.

(3b) Line 4 takes an incorrect limit. What limit should be taken instead? Also, state which theorem could be used to justify that this corrected limit is equal to the corresponding expression. Do not justify it.

For line 1, construct a sequence of partitions $P_1, P_2, P_3, \ldots$ of R such that $||P_N|| \to 0$ as $N \to \infty$. Thus, in line 4, take the limit $N \to \infty$. The theorem is 7.3.18.

4. Define $\Omega = [-2023, 2023]^4$. Let $(\Omega, \Sigma, \mathbb{P})$ be a continuous probability space in $\mathbb{R}^4$ with probability density function ϕ. Choose $(A, B, C, D) \in \Omega$ randomly according to your continuous probability space. You will prove that the 2×2 matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is invertible with probability one.

(4a) Define the set

$$S = \{(a, b, c, d) \in \Omega : ad - bc = 0\}.$$

Assuming S is an event in Σ and $\mathbb{P}(S) = 0$, explain why M will be invertible with probability one.

Assume $\mathbb{P}(\Omega) = 1$. Ω is Jordan measurable since it is a rectangle. Since $S \in \Sigma$, $\Omega \setminus S \in \Sigma$ by Theorem 8.4.3b. Since S and $\Omega \setminus S$ are disjoint, $\mathbb{P}(\Omega) = \mathbb{P}(S \cup \Omega \setminus S) = \mathbb{P}(S) + \mathbb{P}(\Omega \setminus S)$ by Theorem 8.4.7c. $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(S) = 0$ by assumption, so $\mathbb{P}(\Omega \setminus S) = 1$.

$\Omega \setminus S$ is the event that S does not happen, i.e., that M is invertible. Thus, M is invertible with probability 1.

(4b) Prove that if S has zero Jordan measure, then $S \in \Sigma$ and $\mathbb{P}(S) = 0$.

By Lemma 7.6.8a, since S has zero Jordan measure, it is Jordan measurable. Together with the fact that $S \subseteq \Omega$, by definition of Σ, this means $S \in \Sigma$.

$S \subseteq \Omega$ is bounded, and ϕ is bounded by construction. Since S has zero Jordan measure, by Theorem 7.7.8a and the definition of the probability function $\mathbb{P}$, $\mathbb{P}(S) = \int_S \phi \, dV = 0$.

(4c) Fill the remaining gap in the proof by showing S has zero Jordan measure. *Hint:* Proceed by definition. Cover the origin in $\mathbb{R}^4$ with a small rectangle depending on ε and then use Sard's theorem to cover the other pieces which depend on ε.

Let $\varepsilon > 0$. WLOG, assume $\varepsilon < 1$. Take $r = \sqrt[4]{\varepsilon/32}$. Let $(a, b, c, d) \in S$.

Either $(a, b, c, d) \in (-r, r)^4$ or $(a, b, c, d) \notin (-r, r)^4$. For the former case, notice $(-r, r)^4 \subseteq [-r, r]^4$ and $\text{vol}([-r, r]^4) = (\sqrt[4]{\varepsilon/2})^4 = \varepsilon/2$.

Consider the following functions:
Define $f_1 : \mathbb{R} \setminus \{0\} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^4$ by $f(x, y, z) = (x, y, z, yz/x)$ for $y, z \in \mathbb{R}, x \in \mathbb{R} \setminus \{0\}$.
Define $f_2 : \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \to \mathbb{R}^4$ by $f(x, y, w) = (x, y, xw/y, w)$ for $x, w \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}$.
Define $f_3 : \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \to \mathbb{R}^4$ by $f(x, z, w) = (x, xw/z, z, w)$ for $x, w \in \mathbb{R}, z \in \mathbb{R} \setminus \{0\}$.
Define $f_4 : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \setminus \{0\} \to \mathbb{R}^4$ by $f(y, z, w) = (yz/w, y, z, w)$ for $y, z \in \mathbb{R}, w \in \mathbb{R} \setminus \{0\}$.
Note the above functions are all C^1 on their respective domains.

For $(a, b, c, d) \notin (-r, r)^4$, consider the following subcases:

Subcase $a \leq -r$: $(a, b, c, d) \in f_1([-2023, -r] \times [-2023, 2023]^2)$.
Subcase $a \geq r$: $(a, b, c, d) \in f_1([r, 2023] \times [-2023, 2023]^2)$.
Subcase $b \leq -r$: $(a, b, c, d) \in f_2([-2023, 2023] \times [-2023, -r] \times [-2023, 2023])$.
Subcase $b \geq r$: $(a, b, c, d) \in f_2([-2023, 2023] \times [r, 2023] \times [-2023, 2023])$.

The subcases $c \leq -r$, $c \geq r$, $d \leq -r$, $d \geq r$ are similar, using the images of f_3, f_3, f_4, and f_4 respectively.

By Sard's theorem (Theorem 7.5.12), since f_1, f_2, f_3, and f_4 are C^1 on their open domains, their images as specified in the subcases have zero Jordan measure in R^4.

Notice that S is contained in the union of $(-r, r)^2$ and the eight images of f_1, f_2, f_3, and f_4 as specified in the subcases. Since $\text{vol}((-r, r)^4) \leq \varepsilon/2$ and the images have zero volume, this union has volume $< \varepsilon$. By definition 7.5.1, S has zero Jordan measure.

5. Let $S = \{(x, y) \in \mathbb{R}^2 : -2 \le x \le 2, 0 \le y \le \sqrt{4-x^2}\}$ and define $f : S \to \mathbb{R}$ by $f(x, y) = 3 + \sqrt{4-x^2-y^2}$.

(5a) Prove that f is integrable on S.

$S \in B_3(0)$ so it is bounded. Define $\phi : \mathbb{R} \to \mathbb{R}^2$ by $\phi(x) = (2\cos x, 2\sin x)$ for $x \in \mathbb{R}$. Define $h : \mathbb{R} \to \mathbb{R}^2$ by $h(t) = (t, 0)$ for $t \in R$. Notice $\partial S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 2, y \ge 0\} = \phi([0, \pi]) \cup h([-2, 2])$. Since ϕ and h are C^1, by Sard's theorem, ∂S has zero Jordan measure. Hence, by definition 7.6.3, S is Jordan measurable.

Note f is bounded on S since $\forall x \in S, 3 \le f(x) \le 5$. The set of discontinuities of f in S is empty and hence has zero Jordan measure. By theorem 7.7.4, f is integrable on S.

(5b) Formally verify the assumptions of Fubini's theorem to prove that

$$\iint_S f\,dA = \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 3+\sqrt{4-x^2-y^2}\,dy\,dx.$$

Do not calculate any integrals.

By part a, f is integrable on S and $\iint_S f\,dA$ exists. Define $Q=[-2,2]\times[0,2]\supseteq S$. Define $g:\mathbb{R}^2\to\mathbb{R}$ by $g(x,y)=\chi_S(x,y)f(x,y)$ for $(x,y)\in\mathbb{R}^2$. Since $\iint_S f\,dA=\iint_Q g\,dA$ by definition 7.7.1, g is integrable on Q.

Let $x\in[-2,2]$. Define $g^x:[0,2]\to\mathbb{R}$ by $g^x(y)=g(x,y)$ for $y\in[0,2]$.

Note $g^x(y)=\chi_S(x,y)f(x,y)=\chi_{[0,\sqrt{4-x^2}]}(y)f(x,y)$ by definition of S. Thus, for all $x\in[-2,2]$, the set of discontinuities of g^x on $[0,2]$ is $\{(x,\sqrt{4-x^2})\in Q : x\in[-2,2]\}=\phi([0,\pi])$, for ϕ as defined in part a. By Sard's theorem (Theorem 7.5.12), since ϕ is C^1 on $\mathbb{R}$, this set has zero volume. By Theorem 7.7.17, g^x is integrable on $[0,2]$, $\forall x\in[-2,2]$.

By Fubini's theorem (Theorem 9.7.17), the iterated integral $dydx$ exists and $\iint_S f\,dA=\iint_Q g\,dA=\int_{-2}^{2}\int_0^2 g(x,y)dydx=\int_{-2}^{2}\int_0^2\chi_{[0,\sqrt{4-x^2}]}(y)f(x,y)dydx=\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} f(x,y)dydx$.

6. Consider again the integral

$$I = \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 3+\sqrt{4-x^2-y^2}\,dy\,dx$$

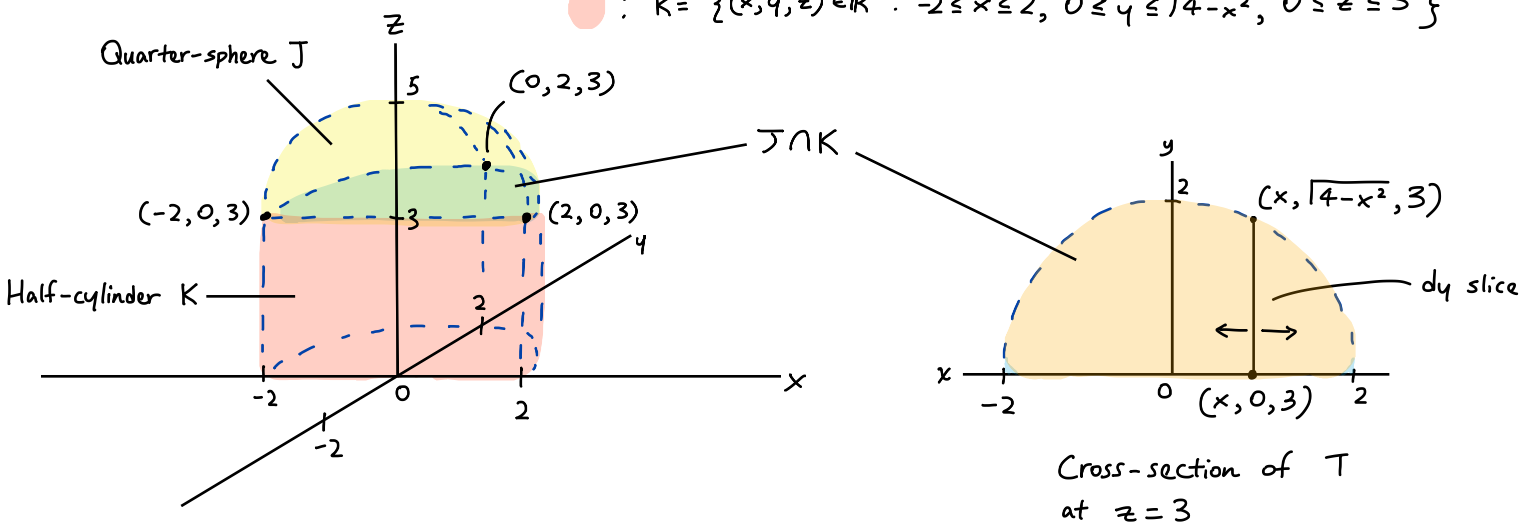
from Question 5. You could evaluate I by direct calculation, but instead you will try two different ways.

(6a) Interpret I as a volume of a solid $T \subseteq \mathbb{R}^3$. Define the solid T using set builder notation, and include sketches in $\mathbb{R}^3$ to support your argument. Compute the volume using geometry of known solids in $\mathbb{R}^3$. Do not evaluate any integrals.

: $J = \{(x,y,z)\in\mathbb{R}^3 : -2\le x\le 2,\ 0\le y\le \sqrt{4-x^2},\ 3\le z\le 3+\sqrt{4-x^2-y^2}\}$

: $J\cap K = \{(x,y,z)\in\mathbb{R}^3 : -2\le x\le 2,\ 0\le y\le \sqrt{4-x^2},\ z=3\}$

: $K = \{(x,y,z)\in\mathbb{R}^3 : -2\le x\le 2,\ 0\le y\le \sqrt{4-x^2},\ 0\le z\le 3\}$

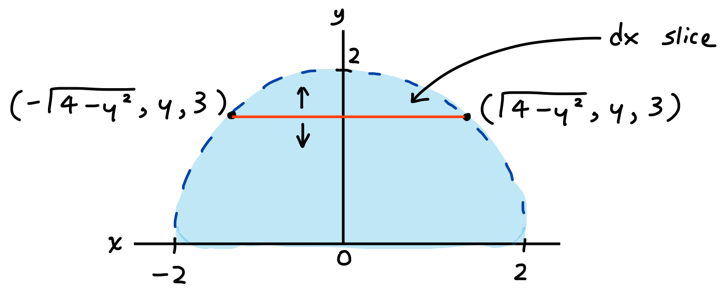


$$T = \{(x,y,z)\in\mathbb{R}^3 : -2\le x\le 2,\ 0\le y\le\sqrt{4-x^2},\ 0\le z\le 3+\sqrt{4-x^2-y^2}\} = J\cup K$$

$I = \mathrm{vol}(T) = \mathrm{vol}(J\cup K) = \mathrm{vol}(J) + \mathrm{vol}(K) - \mathrm{vol}(J\cap K)$ by Lemma 7.6.14 b)

$$= \frac{1}{2}\pi(2)^2(3) + \frac{1}{4}\left(\frac{4}{3}\pi(2)^3\right) - 0 = \frac{26}{3}\pi \approx 27.227$$

(6b) Sketch the region of integration using the $dx\,dy$ order along with a typical slice. Label your figure.



Cross-section of T (from part a) at $z=3$

(6c) Assuming Fubini's theorem applies without justification, evaluate I using the $dx\,dy$ order of integration. Show your steps when calculating single variable integrals. Do not use polar coordinates.

$$I = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left[3 + \sqrt{4-x^2-y^2}\right] dx\, dy$$

Use trig-substitution $x = \sqrt{4-y^2}\sin\theta$, $dx = \sqrt{4-y^2}\cos\theta\, d\theta$,

and bounds $\arcsin\left(\frac{\sqrt{4-y^2}}{\sqrt{4-y^2}}\right) = \frac{\pi}{2}$, $\arcsin\left(-\frac{\sqrt{4-y^2}}{\sqrt{4-y^2}}\right) = \frac{\pi}{2}$:

$$I = \int_0^2 \int_{-\pi/2}^{\pi/2} \left(3 + \sqrt{(4-y^2)-(4-y^2)\sin^2\theta}\right)\sqrt{4-y^2}\cos\theta\, d\theta\, dy$$

$$= \int_0^2 \int_{-\pi/2}^{\pi/2} \left(3 + \sqrt{(4-y^2)\cos^2\theta}\right)\sqrt{4-y^2}\cos\theta\, d\theta\, dy$$

$$= \int_0^2 \int_{-\pi/2}^{\pi/2} \left[3\sqrt{4-y^2}\cos\theta + (4-y^2)\cos^2\theta\right] d\theta\, dy$$

$$= \int_0^2 \int_{-\pi/2}^{\pi/2} \left[3\sqrt{4-y^2}\cos\theta + (4-y^2)\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta\, dy$$

since $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$= \int_0^2 \left[3\sqrt{4-y^2}\sin\theta + (4-y^2)\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right)\right]\Bigg|_{-\pi/2}^{\pi/2} dy$$

$$= \int_0^2 \left[3\sqrt{4-y^2} + (4-y^2)\left(\tfrac{\pi}{4}\right) - 3\sqrt{4-y^2}(-1) - (4-y^2)\left(-\tfrac{\pi}{4}\right)\right] dy$$

$$= \int_0^2 \left[6\sqrt{4-y^2} + \tfrac{\pi}{2}(4-y^2)\right] dy$$

Use trig-substitution $y = 2\sin\theta$, $dy = 2\cos\theta\, d\theta$, and bounds $\arcsin\left(\frac{2}{2}\right)$, $\arcsin\left(\frac{0}{2}\right)$:

$$I = \int_0^{\pi/2} \left(6\sqrt{4-4\sin^2\theta} + \tfrac{\pi}{2}(4-4\sin^2\theta)\right) 2\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \left(12\cos\theta + 2\pi\cos^2\theta\right) 2\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \left[24\cos^2\theta + 4\pi\cos^3\theta\right] d\theta$$

$$= \int_0^{\pi/2} \left[12 + 12\cos 2\theta + 4\pi(1-\sin^2\theta)\cos\theta\right] d\theta$$

since $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$= (12\theta + 6\sin 2\theta)\Big|_0^{\pi/2} + 4\pi\int_0^{\pi/2}(1-\sin^2\theta)\cos\theta\, d\theta$$

$$= 6\pi + 4\pi\int_0^{\pi/2}(1-\sin^2\theta)\cos\theta\, d\theta$$

Use u-substitution $u = \sin\theta$, $du = \cos\theta\, d\theta$:

$$I = 6\pi + 4\pi\int_0^1 \left[1-u^2\right] du$$

$$= 6\pi + 4\pi\left[u - \frac{u^3}{3}\right]\Bigg|_0^1$$

$$= 6\pi + 4\pi\left(1 - \tfrac{1}{3}\right) = 6\pi + \tfrac{8}{3}\pi = \tfrac{26}{3}\pi$$

≈ 27.227