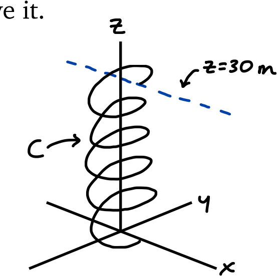
Problems

1. A 70 kilogram person slides down a frictionless slide from a point 30 metres above ground to a point on the ground under the influence of the gravitational force $F(x,y,z) = (0,0,-9.8\cdot 70)$ measured in Newtons. The slide follows a spiral path which circles around the z-axis and, when viewed from above, looks like a circle of radius 2 metres. Starting 30 metres up, the slide circles around five full times. (Revised 2023-03-13)

   (1a) Choose a parametrization of the path the person will follow starting at time $t = 0$ at the top of the slide. It should be a simple regular parametrization, but you do not need to prove it.

Define C as the path of the slide. A picture of C is shown to the right:



Suppose it takes 30 seconds to slide down C, so the height travelled is proportional to the time elapsed. Since the z component should trace the height of the slide, it should be t. The x and y components should trace a circle of radius 2, so they should have a 2cos and 2sin term respectively. Since C circles around the z-axis five times, the variable inside the 2cos and 2sin should be $\frac{\pi t}{3}$ since $30\frac{\pi}{3} = 10\pi$, or five rotations.

Thus, define $\gamma : [0,30] \to \mathbb{R}^3$ by $\gamma(t) = (2\cos(\frac{\pi t}{3}), 2\sin(\frac{\pi t}{3}), t)$ for $t \in [0,30]$. Notice it is a simple and regular parametrizataion of C.

   (1b) By calculating from definition, find the distance travelled (with units) by the person on the slide.

The distance is the arc length. By definition 11.2.1, the arc length of C is $\ell(C) = \int_0^{30} \|\gamma'(t)\| dt = \int_0^{30} \|(-\frac{2\pi}{3}\sin(\frac{\pi t}{3}), \frac{2\pi}{3}\cos(\frac{\pi t}{3}), 1)\| dt = \int_0^{30} \sqrt{(\frac{2\pi}{3})^2 + 1}\, dt = \int_0^{30} \frac{\sqrt{4\pi^2+9}}{3} dt = 10\sqrt{4\pi^2+9} \approx 69.63$. Thus, the person traveled 69.63 meters on the slide.

(1c) By calculating from definition, find the amount of work done (with units) on the person by F.

By definition 11.3.11, the work done by F on the person along C is $\int_C F \cdot T ds = \int_0^{30} F(\gamma(t)) \cdot T(t) \|\gamma'(t)\| dt = \int_0^{30} F(\gamma(t)) \cdot \gamma'(t) dt = \int_0^{30} (0, 0, -9.8 \cdot 70) \cdot (-\frac{2\pi}{3} \sin(\frac{\pi t}{3}), \frac{2\pi}{3} \cos(\frac{\pi t}{3}), 1) dt = \int_0^{30} -9.8 \cdot 70 dt = 30 \cdot -9.8 \cdot 70 = -20580$. Thus, the work done is -20,580 Joules.

(1d) If possible, use the fundamental theorem of line integrals to calculate the work done (with units) by F. If not, explain why not.

Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = (-9.8 \cdot 70 \cdot z)$. Notice f is C^1. Since $F = \nabla f$, by the fundamental theorem of line integrals, $\int_C F \cdot d\gamma = \int_C \nabla f \cdot d\gamma = f(\gamma(30)) - f(\gamma(0)) = 30 \cdot -9.8 \cdot 70 = -20580$ Joules, the same answer as in c).

2. Let C be an oriented curve that is parametrized by $\gamma_1 : [a,b] \to \mathbb{R}^n$ and also by $\gamma_2 : [c,d] \to \mathbb{R}^n$.
   Let F be a vector field in $\mathbb{R}^n$ that is continuous on C. Prove that

$$\int_a^b F(\gamma_1(t)) \cdot \gamma_1'(t) dt = \int_c^d F(\gamma_2(t)) \cdot \gamma_2'(t) dt.$$

   (Hence, the line integral of F along C is well-defined.)

Since γ_1 and γ_2 parameterize the same oriented curve, they are reparametrizations of each other with the same orientation. By definition 11.1.24, this means there exists a continuous invertible $\phi : [a,b] \to [c,d]$ such that ϕ is C^1 on (a,b) with $\phi' > 0$ and $\gamma_1 = \gamma_2 \circ \phi$.

Then, $F(\gamma_1(t)) \cdot \gamma_1'(t) = F(\gamma_2(\phi(t))) \cdot (\gamma_2'(\phi(t))\phi'(t))$ since $\gamma_1'(t) = \gamma_2'(\phi(t))\phi'(t)$ by the chain rule.

Since $\phi' > 0$ on (a,b) and ϕ is continuous and invertible on [a, b], $\phi(a) = c$ and $\phi(b) = d$. Since F, γ_2, ϕ, and ϕ' are continuous on their respective domains, $(F(\gamma_2(\phi(t))) \cdot \gamma_2'(\phi(t)))\phi'(t)$ is integrable on $t \in [a,b]$. By 1D substitution of integrals, this means $(F \circ \gamma_2) \cdot \gamma_2'$ is integrable on $[c,d]$.

Thus, using the substitution $u = \phi(t)$ and $du = \phi'(t)dt$, we get $\int_a^b F(\gamma_1(t)) \cdot \gamma_1'(t)dt = \int_a^b F(\gamma_2(\phi(t))) \cdot (\gamma_2'(\phi(t))\phi'(t))dt = \int_a^b (F(\gamma_2(\phi(t))) \cdot \gamma_2'(\phi(t)))\phi'(t)dt = \int_{\phi(a)}^{\phi(b)} F(\gamma_2(u)) \cdot \gamma_2'(u)du = \int_c^d F(\gamma_2(u)) \cdot \gamma_2'(u)du = \int_c^d F(\gamma_2(t)) \cdot \gamma_2'(t)dt$ as needed.

3. (Revised 2023-03-13) Consider the following true theorem.

Theorem A. *Let $U \subseteq \mathbb{R}^n$ be a non-empty C^1 path-connected open set. Let F be a vector field in $\mathbb{R}^n$ that is continuous on U. If $\int_C F \cdot d\gamma = 0$ for any closed piecewise curve C lying in U, then there exists a C^1 function $f : U \to \mathbb{R}$ such that $F = \nabla f$.*

Here is a WRONG proof of Theorem A.

1. *Fix $a \in U$. For each $x \in U$, choose a curve C_x from a to x lying inside U.*
2. *Define $f : U \to \mathbb{R}$ by $f(x) = \int_{C_x} F \cdot d\gamma$ for $x \in U$.*
3. *Fix $j \in \{1, \ldots, n\}$ and let $\{e_1, \ldots, e_n\}$ be the standard basis in $\mathbb{R}^n$.*
4. *Let $\varepsilon > 0$ be such that $B_\varepsilon(a) \subseteq U$.*
5. *For $h \in (-\varepsilon, \varepsilon)$ with $h \neq 0$, define L_{a+he_j} to be the straight line segment from a to $a + he_j$, so that*
$$\frac{f(a + he_j) - f(a)}{h} = \frac{1}{h}\int_{C_{a+he_j}} F \cdot d\gamma = \frac{1}{h}\int_{L_{a+he_j}} F \cdot d\gamma.$$
6. *Defining $\gamma : [0, 1] \to \mathbb{R}^n$ by $\gamma(t) = a + the_j$, it follows that*
$$\frac{1}{h}\int_{L_{a+he_j}} F \cdot d\gamma = \frac{1}{h}\int_0^1 F(a + the_j) \cdot (he_j)dt = \int_0^1 F_j(a + the_j)dt = \frac{1}{h}\int_0^h F_j(a + te_j)dt$$
7. *By the fundamental theorem of calculus, $\partial_j f(a) = \lim_{h\to 0}\left[\frac{1}{h}\int_0^h F_j(a + te_j)dt\right] = F_j(a)$.*
8. *Since a and j were arbitrary, this proves that $F = \nabla f$.*

You will identify when specific assumptions are required, and you will also identify the fatal error.

(3a) Which line(s) require that U is open?

☐ Line 1 ☐ Line 2 ☐ Line 3 ■ **Line 4** ☐ Line 5 ☐ Line 6 ☐ Line 7 ☐ Line 8

(3b) Which line(s) require that U is path-connected?

■ **Line 1** ☐ Line 2 ☐ Line 3 ☐ Line 4 ☐ Line 5 ☐ Line 6 ☐ Line 7 ☐ Line 8

(3c) Which line(s) require that the integral of F along any piecewise closed curve is zero?

☐ Line 1 ☐ Line 2 ☐ Line 3 ☐ Line 4 ■ **Line 5** ☐ Line 6 ☐ Line 7 ☐ Line 8

(3d) Which line has the false claim in this argument? Identify the line and describe the flaw in ≤ 100 words.

The false line is line 8. Instead of showing $\exists f$ such that $F(a) = \nabla f(a)$, $\forall a \in U$, the proof shows $\forall a \in U$, $\exists f$ such that $F(a) = \nabla f(a)$. This is because in the proof, the line integral of f only depends on a single curve C_{a+he_j}, which in turn depends on the point a. This potentially allows for multiple functions f in U for different curves and different a. Thus, the proof does not conclude that there exists a single f satisfying $F(a) = \nabla f(a)$ everywhere in U.

4. Irrotational vector fields are gradient vector fields in some cases.

> **Theorem B.** *If $U \subseteq \mathbb{R}^2$ is an open simply-connected set and F is a C^1 irrotational vector field on U, then F is a gradient vector field on U. That is, $F = \nabla f$ on U for some C^2 scalar function f on U.*

On the other hand, you can verify that the vector field $F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ is irrotational and yet F is not a gradient vector field. You may assume these facts without proof.

(4a) Explain why F does not contradict Theorem B in at most 2 full sentences.

F is not defined at the point $(0,0)$, so its domain is $\mathbb{R} \setminus \{(0,0)\}$, which is not a simply-connected set. Thus, Theorem B does not apply to F.

(4b) Let $V = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. Find all potential functions of the vector field $F|_V$.

Suppose $F|_V$ has a potential $f : V \to \mathbb{R}^2$, so $F|_V(x,y) = \nabla f(x,y)$ for $(x,y) \in V$. This means f must satisfy $\frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}$ and $\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$ for $(x,y) \in V$. (*)

Integrate the first equation with respect to x, holding y fixed: $f(x,y) = \int \frac{-y}{x^2+y^2} dx$. Using the u-substitution $u = \frac{x}{y}$ and $du = \frac{dx}{y}$, this equals: $-\int \frac{y^2}{y^2u^2+y^2} du = -\int \frac{1}{u^2+1} du = -\arctan(u) + \phi(y) = -\arctan(\frac{x}{y}) + \phi(y)$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^2$ is an arbitrary function of y.

Taking $f(x,y) = -\arctan(\frac{x}{y}) + \phi(y)$ as solved above, $\frac{\partial f}{\partial y} = (-\frac{y^2}{y^2+x^2})(-\frac{x}{y^2}) + \phi'(y) = \frac{x}{y^2+x^2} + \phi'(y)$. By (*), this means $\phi'(y) = 0$, so $\phi(y) = C$ for some $C \in \mathbb{R}$ by the MVT.

Since $\frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}$ and $\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$ by direct calculation, all potential functions of $F|_V$ are represented by $f(x,y) = -\arctan(\frac{x}{y}) + C$ for $(x,y) \in V$, $C \in \mathbb{R}$.

(4c) Let $W = \{(x, y) \in \mathbb{R}^2 : y < 0\}$. State (without proof) all potential functions of the vector field $F|_W$.

$f(x, y) = -\arctan(\frac{x}{y}) + C$ for $(x, y) \in W$, $C \in \mathbb{R}$.

(4d) Let $U = V \cup W \cup \{(x, y) \in \mathbb{R}^2 : x > 0, y = 0\}$. Find a potential function $\phi : U \to \mathbb{R}$ of the vector field $F|_U$. Use (4b) and (4c) and additional arguments to justify that $F|_U = \nabla\phi$. Can you extend your function ϕ to be continuous on a larger set containing U? Briefly explain why or why not. (Revised 2023-03-13)

Define $\phi : U \to \mathbb{R}$ by

$$\phi(x, y) = \begin{cases} f(x, y) + \frac{\pi}{2} & x \in \mathbb{R}, y > 0 \\ f(x, y) - \frac{\pi}{2} & x \in \mathbb{R}, y < 0 \\ \arctan(\frac{y}{x}) & x > 0, y = 0 \end{cases} \tag{1}$$

for f as defined in 4b and 4c. Notice ϕ is continuous on U since f is continuous on V and W, and for $x > 0$, $\lim_{y \to 0^-} f(x, 0) - \frac{\pi}{2} = 0 = \lim_{y \to 0^+} f(x, 0) + \frac{\pi}{2} = \phi(x, 0)$.

Since

- $\nabla\phi|_V = \nabla(f(x, y) + \frac{\pi}{2}) = \nabla f(x, y) = F|_V$ by 4b,
- $\nabla\phi|_W = \nabla(f(x, y) - \frac{\pi}{2}) = \nabla f(x, y) = F|_W$ by 4c, and
- $\nabla(\arctan(\frac{y}{x})) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}) = F(x, y)$ for $x > 0, y = 0$,

$\nabla\phi = F|_U$.

ϕ cannot be extended to be continuous on a larger set containing U since for $x < 0, y = 0$, $\lim_{y \to 0^-} f(x, 0) - \frac{\pi}{2} = -\pi \neq \pi = \lim_{y \to 0^+} f(x, 0) + \frac{\pi}{2}$. Hence, it is impossible to join the pieces of ϕ with a function at the negative x-axis.

5. Let $F = (f, g)$ be a vector field in $\mathbb{R}^2$ with C^1 components f and g. Fix a point $p = (x, y) \in \mathbb{R}^2$ in the domain of F. For $\varepsilon > 0$, let $B_\varepsilon(p) \subseteq \mathbb{R}^2$ be the disk of radius ε centred at p. Orient its boundary $\partial B_\varepsilon(p)$ counterclockwise. **Do not use Green's theorem for any part of this question.**

   (5a) For $\varepsilon > 0$, show that the flux of F across $\partial B_\varepsilon(p)$ may be expressed as

$$\oint_{\partial B_\varepsilon(p)} (F \cdot n)\, ds = \int_0^{2\pi} f(x + \varepsilon \cos t, y + \varepsilon \sin t) \cdot \varepsilon \cos t + g(x + \varepsilon \cos t, y + \varepsilon \sin t) \cdot \varepsilon \sin t\, dt.$$

Let $\varepsilon > 0$. Parametrize $\partial B_\varepsilon(p)$ by $\gamma : [0, 2\pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (x + \varepsilon \cos t, y + \varepsilon \sin t)$ for $t \in [0, 2\pi]$. Notice $||\gamma'(t)|| = ||(-\varepsilon \sin t, \varepsilon \cos t)|| = \sqrt{(-\varepsilon \sin t)^2 + (\varepsilon \cos t)^2} = \sqrt{\varepsilon^2} = \varepsilon$.

The unit tangent of $\partial B_\varepsilon(p)$ is $T(t) = \frac{\gamma'(t)}{||\gamma'(t)||} = \frac{(-\varepsilon \sin t, \varepsilon \cos t)}{\varepsilon} = (-\sin t, \cos t)$.

Define $n : [0, 2\pi] \to \mathbb{R}^2$ by $n(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Notice that for all $t \in [0, 2\pi]$:

- $T(t) \cdot n(t) = (-\sin t)(\cos t) + (\cos t)(\sin t) = 0$.
- $||n(t)|| = \sqrt{\cos^2 t + \sin^2 t} = 1$.
- The matrix $(n(t), T(t)) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ has determinant $1 > 0$, so $\{n(t), T(t)\}$ is a positively oriented basis in $\mathbb{R}^2$.

Thus, n is the unit normal of $\partial B_\varepsilon(p)$.

By definition, the flux of F across $\partial B_\varepsilon(p)$ is $\oint_{\partial B_\varepsilon(p)} (F \cdot n)\, ds = \int_0^{2\pi} F(\gamma(t)) \cdot n(t) ||\gamma'(t)|| dt = \int_0^{2\pi} F(x + \varepsilon \cos t, y + \varepsilon \sin t) \cdot (\cos t, \sin t)(\varepsilon) dt = \int_0^{2\pi} f(x + \varepsilon \cos t, y + \varepsilon \sin t) \cdot \varepsilon \cos t + g(x + \varepsilon \cos t, y + \varepsilon \sin t) \cdot \varepsilon \sin t\, dt$ as needed.

(5b) Since f is C^1 on U, differentiability implies that there exists $\delta_f > 0$ and $E_f : B_{\delta_f}((0,0)) \to \mathbb{R}$ such that

$$\forall(\Delta x, \Delta y) \in B_{\delta_f}(0,0), \quad f(x+\Delta x, y+\Delta y) = f(x,y) + \partial_1 f(x,y)\Delta x + \partial_2 f(x,y)\Delta y + E_f(\Delta x, \Delta y),$$

where $\lim\limits_{(a,b)\to(0,0)} \frac{|E_f(a,b)|}{||(a,b)||} = 0$. The analogous statement holds for g with $\delta_g > 0$ and $E_g : B_{\delta_g}((0,0)) \to \mathbb{R}$.

Prove that for $0 < \varepsilon < \frac{\min\{\delta_f, \delta_g\}}{2}$,

$$\frac{1}{\text{area}(B_\varepsilon(p))} \oint_{\partial B_\varepsilon(p)} (F \cdot n)\, ds = (\text{div}\, F)(p) + \frac{1}{\pi\varepsilon} \int_0^{2\pi} E_f(\varepsilon\cos t, \varepsilon \sin t)\cdot \cos t + E_g(\varepsilon \cos t, \varepsilon \sin t) \cdot \sin t \, dt.$$

Let $0 < \varepsilon < \frac{\min\{\delta_f, \delta_g\}}{2}$ for δ_f, δ_g as defined in the question. Notice $\text{area}(B_\varepsilon(p)) = \pi\varepsilon^2$. Thus,

$$\frac{1}{\text{area}(B_\varepsilon(p))} \oint_{\partial B_\varepsilon(p)} (F \cdot n)\, ds = \frac{1}{\pi\varepsilon^2} \int_0^{2\pi} f(x+\varepsilon\cos t, y+\varepsilon\sin t)\cdot\varepsilon\cos t + g(x+\varepsilon\cos t, y+\varepsilon \sin t)\cdot \varepsilon\sin t\, dt$$ by part a.

Since $\varepsilon\cos t \le \varepsilon$ and $\varepsilon \sin t \le \varepsilon$ for all $t \in \mathbb{R}$, and since $\varepsilon < \delta_f$ and $\varepsilon < \delta_g$, $(\varepsilon\cos t, \varepsilon \sin t) \in B_{\delta_f}(0,0)$ and $(\varepsilon \cos t, \varepsilon\sin t) \in B_{\delta_g}(0,0)$. Thus, the above is equal to

$$\frac{1}{\pi\varepsilon}\int_0^{2\pi} \left(f(x,y) + \partial_1 f(x,y)\varepsilon\cos t + \partial_2 f(x,y)\varepsilon \sin t + E_f(\varepsilon\cos t, \varepsilon \sin t)\right)\cdot \cos t$$
$$+ \left(g(x,y) + \partial_1 g(x,y)\varepsilon\cos t + \partial_2 g(x,y)\varepsilon\sin t + E_f(\varepsilon\cos t, \varepsilon\sin t)\right)\cdot \sin t\, dt$$

$$= \frac{1}{\pi\varepsilon}\left[\int_0^{2\pi} f(x,y)\cos t\, dt + \int_0^{2\pi} g(x,y)\sin t\, dt + \int_0^{2\pi}\partial_1 f(x,y)\varepsilon\cos^2 t + \int_0^{2\pi}\partial_2 f(x,y)\varepsilon\sin t\cos t\, dt + \int_0^{2\pi}\partial_1 g(x,y)\varepsilon\cos t \sin t\, dt\right.$$
$$\left. + \int_0^{2\pi}\partial_2 g(x,y)\varepsilon\sin^2 t\, dt + \int_0^{2\pi} E_f(\varepsilon\cos t, \varepsilon\sin t)\cdot\cos t + E_g(\varepsilon\cos t, \varepsilon\sin t)\cdot \sin t\, dt\right]$$

$$= \frac{1}{\pi\varepsilon}\left[f(x,y)\sin t\Big|_{t=0}^{t=2\pi} - g(x,y)\cos t\Big|_{t=0}^{t=2\pi} + \partial_1 f(x,y)\varepsilon\int_0^{2\pi}\left(\frac{1}{2} + \frac{\cos 2t}{2}\right)dt + \varepsilon\left(\partial_2 f(x,y) + \partial_1 g(x,y)\right)\int_0^0 u\,du\right.$$
$$\left. + \partial_2 g(x,y)\varepsilon\int_0^{2\pi}\left(\frac{1}{2} - \frac{\cos 2t}{2}\right)dt + \int_0^{2\pi} E_f(\varepsilon\cos t, \varepsilon\sin t)\cdot\cos t + E_g(\varepsilon\cos t, \varepsilon\sin t)\cdot\sin t\, dt\right]$$

using u-sub $u = \sin t$, $du = \cos t\, dt$, and trig identities for $\sin^2 t$ and $\cos^2 t$

$$= \frac{1}{\pi\varepsilon}\left[\partial_1 f(x,y)\varepsilon\left(\pi + \frac{1}{4}\sin(2t)\Big|_{t=0}^{t=2\pi}\right) + \partial_2 g(x,y)\varepsilon\left(\pi - \frac{1}{4}\sin(2t)\Big|_{t=0}^{t=2\pi}\right) + \int_0^{2\pi} E_f(\varepsilon\cos t, \varepsilon\sin t)\cdot\cos t + E_g(\varepsilon\cos t, \varepsilon\sin t)\cdot\sin t\, dt\right]$$

$$= \frac{1}{\pi\varepsilon}\left[\partial_1 f(x,y)\varepsilon\pi + \partial_2 g(x,y)\varepsilon\pi + \int_0^{2\pi} E_f(\varepsilon\cos t, \varepsilon\sin t)\cdot\cos t + E_g(\varepsilon\cos t, \varepsilon\sin t)\cdot\sin t\, dt\right]$$

$$= \partial_1 F_1(x,y) + \partial_2 F_2(x,y) + \frac{1}{\pi\varepsilon}\int_0^{2\pi} E_f(\varepsilon\cos t, \varepsilon\sin t)\cdot\cos t + E_g(\varepsilon\cos t, \varepsilon\sin t)\cdot\sin t\, dt$$

$$= (\text{div}\, F)(p) + \frac{1}{\pi\varepsilon}\int_0^{2\pi} E_f(\varepsilon\cos t, \varepsilon\sin t)\cdot\cos t + E_g(\varepsilon\cos t, \varepsilon\sin t)\cdot\sin t\, dt$$ by definition of divergence.

(5c) Use the definition of a limit to prove that $\lim_{\varepsilon\to0^+}\frac{1}{\pi\varepsilon}\int_0^{2\pi}E_f(\varepsilon\cos t,\varepsilon\sin t)\cos t\,dt=0$. Assuming without proof that a similar identity holds for E_g, conclude that

$$(\operatorname{div}F)(p)=\lim_{\varepsilon\to0^+}\frac{1}{\operatorname{area}(B_\varepsilon(p))}\oint_{\partial B_\varepsilon(p)}(F\cdot n)\,ds.$$

WTS: $\forall\mu>0,\ \exists\delta>0$, s.t. $\forall\varepsilon\in(0,\frac{\min\{\delta_f,\delta_g\}}{2})$, $0<\varepsilon<\delta\Rightarrow\left|\frac{1}{\pi\varepsilon}\int_0^{2\pi}E_f(\varepsilon\cos t,\varepsilon\sin t)\cos t\,dt\right|<\mu$

Pf: Let $\mu>0$. Assume $\lim_{(a,b)\to(0,0)}\frac{|E_f(a,b)|}{\|(a,b)\|}=0$; that is,

$$\forall\varepsilon_1>0,\ \exists\delta_1>0 \text{ s.t. } \forall(a,b)\in\mathbb{R}^2,\ 0<\|(a,b)\|<\delta_1\Rightarrow\frac{|E_f(a,b)|}{\|(a,b)\|}<\varepsilon_1 \qquad (1)$$

Take the δ_1 above corresponding to $\varepsilon_1=\frac{\mu}{4}$. Take $\delta=\min\{\frac{\delta_1}{2},\frac{\min\{\delta_f,\delta_g\}}{2}\}$. Let $0<\varepsilon<\delta$ so $\varepsilon<\delta_1$.

Notice $\int_0^{2\pi}E_f(\varepsilon\cos t,\varepsilon\sin t)\cos t\,dt$ is integrable. Then,

$$\left|\frac{1}{\pi\varepsilon}\int_0^{2\pi}E_f(\varepsilon\cos t,\varepsilon\sin t)\cos t\,dt\right|=\frac{1}{\pi\varepsilon}\left|\int_0^{2\pi}E_f(\varepsilon\cos t,\varepsilon\sin t)\cos t\,dt\right|$$

$$\le\frac{1}{\pi\varepsilon}\int_0^{2\pi}|E_f(\varepsilon\cos t,\varepsilon\sin t)\cos t|\,dt$$ by the triangle inequality for integrals

$$\le\frac{1}{\pi\varepsilon}\int_0^{2\pi}|E_f(\varepsilon\cos t,\varepsilon\sin t)|\,dt$$ by monotonicity and $\cos t\le1$

$$<\frac{1}{\pi\varepsilon}\int_0^{2\pi}\varepsilon_1\varepsilon\,dt$$ by (1) and monotonicity since $\|(\varepsilon\cos t,\varepsilon\sin t)\|=\varepsilon<\delta_1$

$$=\frac{1}{\pi\varepsilon}\varepsilon_1\varepsilon(2\pi)=2\varepsilon_1=\frac{\mu}{2}$$

$$<\mu.$$

Thus, the limit holds. Taking limits in the equation for $(\operatorname{div}F)(p)$ from part b,

$$(\operatorname{div}F)(p)=\lim_{\varepsilon\to0^+}(\operatorname{div}F)(p)$$

$$=\lim_{\varepsilon\to0^+}\frac{1}{\operatorname{area}(B_\varepsilon(p))}\oint_{\partial B_\varepsilon(p)}(F\cdot n)\,ds+\lim_{\varepsilon\to0^+}\left[\frac{1}{\pi\varepsilon}\int_0^{2\pi}E_f(\varepsilon\cos t,\varepsilon\sin t)\cdot\cos t+E_g(\varepsilon\cos t,\varepsilon\sin t)\cdot\sin t\,dt\right]$$

$$=\lim_{\varepsilon\to0^+}\frac{1}{\operatorname{area}(B_\varepsilon(p))}\oint_{\partial B_\varepsilon(p)}(F\cdot n)\,ds+\lim_{\varepsilon\to0^+}\frac{1}{\pi\varepsilon}\int_0^{2\pi}E_f(\varepsilon\cos t,\varepsilon\sin t)\cos t\,dt+\lim_{\varepsilon\to0^+}\frac{1}{\pi\varepsilon}\int_0^{2\pi}E_g(\varepsilon\cos t,\varepsilon\sin t)\cdot\sin t\,dt$$

by the limit sum law, and this equals $\lim_{\varepsilon\to0^+}\frac{1}{\operatorname{area}(B_\varepsilon(p))}\oint_{\partial B_\varepsilon(p)}(F\cdot n)\,ds$.