STA347H1 - Assignment 2

5.1: $\mathbb{E}e^{\lambda S} = e^{\lambda}\mathbb{P}(S=1) + e^{-\lambda}\mathbb{P}(S=-1) = \frac{1}{2}(e^{\lambda}+e^{-\lambda}) = \frac{1}{2}(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\ldots+1-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+\ldots) = \frac{1}{2}(1+1+\frac{\lambda^2}{2!}+\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}+\frac{\lambda^4}{4!}+\ldots) = \Sigma_{n=0}^{\infty}\frac{\lambda^{2n}}{(2n)!} \leq \Sigma_{n=0}^{\infty}\frac{\lambda^{2n}}{2^n n!} = \Sigma_{n=0}^{\infty}\frac{(\lambda^2/2)^n}{n!} = e^{\lambda^2/2}$ since $(2n)! = \Pi_{i=1}^{n}(2i)\Pi_{i=1}^{n}(2i-1) = (2^n n!)\Pi_{i=1}^{n}(2i-1) \geq 2^n n!$ for $n \geq 0$. Notice that $\mathbb{E}(Z_n) = \Sigma_{i=1}^{n}\mathbb{E}(S) = 0$, so by Hoeffding's inequality, $\mathbb{P}(Z \geq t) = \mathbb{P}(Z - \mathbb{E}Z \geq t) \leq \exp(\frac{-2t^2}{\Sigma_{i=1}^{n}(1-(-1))^2}) = \exp(\frac{-t^2}{2n})$ for $t > 0$. Finally, $\mathbb{P}(Z \geq 0) = \frac{1}{2} \leq e^0 = 1$, so the relation also holds for $t = 0$.

Revised: Chernoff's inequality can also be used since $\mathbb{P}(Z \geq t) = \mathbb{P}(Z - \mathbb{E}Z \geq t) \leq \inf_{\lambda>0} M_{Z-\mathbb{E}Z}(\lambda)e^{-\lambda t} = \inf_{\lambda>0} M_Z(\lambda)e^{-\lambda t} = \inf_{\lambda>0}\Pi_{i=1}^{n}M_{S_i}(\lambda)e^{-\lambda t} = \inf_{\lambda>0} e^{\lambda^2 n/2}e^{-\lambda t} = \exp(\frac{t^2 n}{2n^2} - \frac{t^2}{n}) = \exp(\frac{-t^2}{2n})$ for $t \geq 0$, where the minimum of $\exp(\frac{\lambda^2 n}{2} - \lambda t)$ is achieved at $\lambda = \frac{t}{n}$.

5.7:

- $p \geq 0$: $\mathbb{E}|X+Y|^p \leq \mathbb{E}|2\max(X,Y)|^p = 2^p\mathbb{E}\max(|X|^p,|Y|^p) \leq 2^p\mathbb{E}(|X|^p+|Y|^p) = 2^p(\mathbb{E}|X|^p+\mathbb{E}|Y|^p)$.
- $p = 0$: $\mathbb{E}|X+Y|^0 = 1 \leq \mathbb{E}|X|^0 + \mathbb{E}|Y|^0 = 2$.
- $p \in (0,1)$: $|X+Y|^p = |X+Y|^{1-(1-p)} = |X+Y||X+Y|^{-(1-p)} \leq |X||X+Y|^{-(1-p)} + |Y||X+Y|^{-(1-p)}$ by the triangle inequality $\leq |X||X|^{-(1-p)} + |Y||Y|^{-(1-p)} = |X|^p + |Y|^p$ since $|X+Y| \geq X$ and $|X+Y| \geq Y$. Alternatively, since $|\frac{X}{X+Y}|, |\frac{Y}{X+Y}| \in [0,1]$, $|\frac{X}{X+Y}|^p + |\frac{Y}{X+Y}|^p \geq |\frac{X}{X+Y}| + |\frac{Y}{X+Y}| \geq |\frac{X}{X+Y} + \frac{Y}{X+Y}|$ by the triangle inequality $= 1 \implies |X|^p + |Y|^p \geq |X+Y|^p$. In both cases, taking the expectation yields the result.

  Revised: The above is not true if either $X < 0$ or $Y < 0$. Instead, notice that $(\frac{|X|}{|X|+|Y|})^p + (\frac{|Y|}{|X|+|Y|})^p \geq \frac{|X}{|X|+|Y|} + \frac{|Y|}{|X|+|Y|} = 1$.
- $p = 1$: $\mathbb{E}|X+Y| \leq \mathbb{E}(|X|+|Y|)$ by the triangle inequality $= 2^0(\mathbb{E}|X| + \mathbb{E}|Y|)$.
- $p > 1$: Since $|x|^p$ is convex, $|\frac{1}{2}X + \frac{1}{2}Y|^p \leq \frac{1}{2}|X|^p + \frac{1}{2}|Y|^p \implies |X+Y|^p \leq \frac{2^p}{2}(|X|^p + |Y|^p) \implies \mathbb{E}(|X+Y|^p) \leq \mathbb{E}(2^{p-1}(|X|^p + |Y|^p)) = 2^{p-1}(\mathbb{E}|X|^p + \mathbb{E}|Y|^p)$.

5.10: $\forall \varepsilon > 0, \mathbb{P}(|X_n - \mathbb{E}X_n| > \varepsilon) \leq \mathbb{P}(|X_n - \mathbb{E}X_n| \geq \varepsilon) \leq \frac{\mathrm{Var}(X_n)}{\varepsilon^2}$ by Chebyshev's inequality $= \frac{1}{\varepsilon^2\sqrt{n}} \to 0$ as $n \to \infty$. Alternatively, $\mathbb{E}|X_n - \mathbb{E}X_n|^2 = \mathrm{Var}(X_n) = \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, so $X_n \xrightarrow{L^2} \mathbb{E}X_n$. Both cases imply $X_n \xrightarrow{p} m$.

6.1: $\mathbb{E}(\frac{1}{n}S_n - \mathbb{E}(\frac{1}{n}S_n))^2 = \mathrm{Var}(\frac{1}{n}S_n) = \frac{1}{n^2}\mathrm{Var}(S_n) = \frac{1}{n^2}\Sigma_{i=1}^{n}\mathrm{Var}(X_i)$ since the X_i's are uncorrelated $= \frac{1}{n}\Sigma_{i=1}^{n}\frac{\mathrm{Var}(X_i)}{n} \leq \frac{1}{n}\Sigma_{i=1}^{n}\frac{\mathrm{Var}(X_i)}{i} \leq \frac{1}{n}\Sigma_{i=1}^{\infty}\frac{\mathrm{Var}(X_i)}{i} \to 0$ as $n \to \infty$ since $\frac{\mathrm{Var}(X_i)}{i} \to 0$ as $i \to \infty \implies \Sigma_{i=1}^{\infty}\frac{\mathrm{Var}(X_i)}{i} < \infty$.

Revised: For the last step, $\frac{\mathrm{Var}(X_i)}{i} \to 0$ as $i \to \infty \not\Rightarrow \Sigma_{i=1}^{\infty}\frac{\mathrm{Var}(X_i)}{i} < \infty$ since $\frac{1}{n} \to 0$ but $\Sigma_{n=1}^{\infty}\frac{1}{n} = \infty$. Instead, use the Cesàro mean theorem: $a_i \to a \implies \frac{1}{n}\Sigma_{i=1}^{n}a_i \to a$.

6.2: Define $\mathcal{X} = \{(-1)^k k\}_{k=2}^{\infty}$ and notice that $C > 0$. First, $\mathbb{E}|X_1| = \Sigma_{x\in\mathcal{X}}|x|\mathbb{P}(X_1 = x) = \Sigma_{k=2}^{\infty}k\mathbb{P}(X_1 = (-1)^k k) = \Sigma_{k=2}^{\infty}\frac{C}{k\log k} \geq \int_2^{\infty}\frac{1}{y\log y}dy = \log|\log\infty| - \log|\log 2| = \infty$ since $\Sigma_{k=x}^{\infty}f(k) \geq \int_x^{\infty}f(y)dy$ for a positive and decreasing function f. Next, $x\mathbb{P}(|X_1| > x) = x\mathbb{P}(\cup_{k=x+1}^{\infty}\{X_1 = k\}) \leq x\Sigma_{k=x+1}^{\infty}\mathbb{P}(X_1 = k) = x\Sigma_{k=x+1}^{\infty}\frac{C}{k^2\log k} \leq \frac{x}{\log x}\Sigma_{k=x+1}^{\infty}\frac{C}{k^2} \leq \frac{Cx}{\log x}\int_x^{\infty}\frac{1}{y^2}dy = \frac{Cx}{\log x}(0 + \frac{1}{x}) = \frac{C}{\log x} \to 0$ as $x \to \infty$. Thus, $\frac{1}{n}S_n - \mu_n \xrightarrow{p} 0$ by the WLLN $\implies \frac{1}{n}S_n \xrightarrow{p} \mu$ since $\frac{(-1)^k}{k^2\log k}$ is an alternating series converging to 0 and $\mu_n = \mathbb{E}(X_1\mathbb{I}(|X_1| \leq n)) = \Sigma_{k=2}^{n}\frac{(-1)^k C}{k^2\log k} \to$ some μ as $n \to \infty$.

6.7: Since $\Sigma_{n=1}^{\infty}\mathbb{P}(X_n \leq \log n) = \Sigma_{n=1}^{\infty}(1 - \mathbb{P}(X_n > \log n)) = \Sigma_{n=1}^{\infty}(1 - e^{-\log n}) = \Sigma_{n=1}^{\infty}(1 - \frac{1}{n}) = \infty$ and $\{X_n \leq \log n\}_n$ are independent, $\mathbb{P}(X_n \leq \log n \text{ i.o.}) = 1$ by Borel-Cantelli. Similarly, since $\Sigma_{n=1}^{\infty}\mathbb{P}(X_n \geq$

$\log n) \geq \Sigma_{n=1}^{\infty}\mathbb{P}(X_n > \log n) = \Sigma_{n=1}^{\infty}e^{-\log n} = \Sigma_{n=1}^{\infty}\frac{1}{n} = \infty$ and $\{X_n \leq \log n\}_n$ are independent, $\mathbb{P}(X_n \geq \log n \text{ i.o.}) = 1$ by Borel-Cantelli. Thus, $\mathbb{P}(X_n = \log n \text{ i.o.}) = 1 \implies \limsup_{n\to\infty}\frac{X_n}{\log n} = 1$ a.s.

7.2: The characteristic function of X_n is $\varphi_{X_n}(t) = e^{-(\sigma_n t)^2/2}$. By Theorem 7.2.9, $X_n \xrightarrow{D} X \implies \varphi_{X_n}(t) \to \varphi_X(t)$ for all $t \in \mathbb{R}$. Notice that $\varphi_{X_n}(\sqrt{2}) \to \varphi_X(\sqrt{2}) \implies -\log\varphi_{X_n}(\sqrt{2}) = \sigma_n \to \sigma$ for some $\sigma \in [0,\infty]$ since $\varphi_{X_n}(t) \in (0,1] \implies -\log\varphi_{X_n}(t) \in [0,\infty)$. Next, $\sigma = \infty \implies \varphi_{X_n}(t) \to e^{-\infty} = 0$ for all $t \neq 0$, but since $\varphi_{X_n}(0) = 1$, this implies φ_{X_n} is discontinuous at $t = 0$. This yields a contradiction, so $\sigma \neq \infty$.

7.8: Let $X_n \sim \mathcal{N}(0,1)$ and $X \sim \mathcal{N}(0,1)$. $X_n \xrightarrow{D} X$ and $X_n \xrightarrow{D} -X$, but $X_n + X_n \not\xrightarrow{D} X - X = 0$.

7.11: $\varphi_{S_n/n}(t) = \Pi_{i=1}^{n}\varphi_{X_i}(\frac{t}{n})$ by Proposition 7.2.5 $= (\varphi(\frac{t}{n}))^n$ and $\varphi_a(t) = \mathbb{E}(e^{iat}) = e^{iat}$. Notice that $\varphi'(0) = ia \implies \lim_{n\to\infty}\frac{\varphi(t/n)-\varphi(0)}{t/n} = ia \implies \lim_{n\to\infty} n(\varphi(\frac{t}{n}) - 1) = iat \implies (\varphi(\frac{t}{n}))^n = [1 + \frac{n(\varphi(t/n)-1)}{n}]^n \to \exp[n(\varphi(\frac{t}{n}) - 1)] \to e^{iat}$ as $n \to \infty$ since $(1 + \frac{x}{n})^n \to e^x$ as $n \to \infty$. Thus, $\varphi_{S_n/n}(t) \to \varphi_a(t)$ for all $t \in \mathbb{R}$, so $\frac{S_n}{n} \xrightarrow{D} a$ by Theorem 7.2.9 $\implies \frac{S_n}{n} \xrightarrow{p} a$.

10: Notice that $j^2 = \frac{2j(j+1)}{2} - j = 2\Sigma_{k=1}^{j}k - \Sigma_{k=1}^{j}1 = \Sigma_{k=1}^{j}(2k-1)$. Thus, $\mathbb{E}X^2 = \Sigma_{j=0}^{\infty}j^2\mathbb{P}(X = j) = \Sigma_{j=1}^{\infty}j^2\mathbb{P}(X = j) = \Sigma_{j=1}^{\infty}\Sigma_{k=1}^{j}(2k-1)\mathbb{P}(X = j) = \Sigma_{k=1}^{\infty}\Sigma_{j=k}^{\infty}(2k-1)\mathbb{P}(X = j) = \Sigma_{k=1}^{\infty}(2k-1)\Sigma_{j=k}^{\infty}\mathbb{P}(X = j) = \Sigma_{k=1}^{\infty}(2k-1)\mathbb{P}(X \geq k)$.

Revised: Define $i = X(w) \in \mathbb{N}$ and $X_n = \Sigma_{k=1}^{n}(2k-1)\mathbb{I}(X \geq k)$. $X_n \leq X_{n+1}$ and $\forall n \geq i$, $X_n = \Sigma_{k=1}^{i}(2k-1) = i^2 = X^2$. By the DCT, $\mathbb{E}X_n = \Sigma_{k=1}^{n}(2k-1)\mathbb{P}(X \geq k) \uparrow \mathbb{E}X^2$.

11: $F_X(x) = \int_0^x 2y\,dy = x^2$ for $x \in (0,1)$. $F_{Z_n}(z) = 1 - \mathbb{P}(\sqrt{n}\min[X_1,\ldots,X_n] > z) = 1 - \mathbb{P}(\cap_{i=1}^{n}\{X_i > z/\sqrt{n}\}) = 1 - \Pi_{i=1}^{n}\mathbb{P}(X_i > z/\sqrt{n})$ since the X_i's are independent $= 1 - [\mathbb{P}(X_i > z/\sqrt{n})]^n = 1 - (1 - F_X(z/\sqrt{n}))^n = 1 - (1 - \frac{z^2}{n})^n$ for $z \in (0,\sqrt{n})$. As $n \to \infty$, $1 - (1 - \frac{z^2}{n})^n \to 1 - e^{-z^2}$ for $z > 0$, which is the CDF of some $Z^2 \sim$ Exponential(1) or $Z \sim$ Rayleigh($\frac{1}{\sqrt{2}}$). The PDF of Z is $f_Z(z) = \frac{d}{dz}(1 - e^{-z^2}) = 2ze^{-z^2}$ for $z > 0$.