STA355H1 - Assignment 4

1. (a) The NP test statistic is

$$T(X_1,\ldots,X_n) = \frac{f(X_1,\ldots,X_n;\mu=0,\sigma^2=3)}{f(X_1,\ldots,X_n;\mu=0,\sigma^2=2)} = \frac{2^n\exp(-\sum_{i=1}^n X_i^2/6)}{3^n\exp(-\sum_{i=1}^n X_i^2/4)} = \frac{2^n}{3^n}\exp(\frac{1}{12}\sum_{i=1}^n X_i^2)$$

and by the NP lemma, the MP test rejects H_0 at level α if $\sum_{i=1}^n T(X_1,\ldots,X_n) \geq \eta$ where $\mathbb{P}_{\sigma^2=2}(\sum_{i=1}^n T(X_1,\ldots,X_n)\geq\eta)=\alpha$ for some η. Notice that since

$$\mathbb{P}_{\sigma^2=2}(\sum_{i=1}^n T(X_1,\ldots,X_n)\geq\eta) = \mathbb{P}_{\sigma^2=2}(\exp(\frac{1}{12}\sum_{i=1}^n X_i^2)\geq\frac{3^n\eta}{2^n}) = \mathbb{P}_{\sigma^2=2}(\sum_{i=1}^n X_i^2 \geq 12\ln(\frac{3^n\eta}{2^n}))$$

this is equivalent to rejecting H_0 at level α if $\sum_{i=1}^n X_i^2 \geq k := 12\ln(3^n\eta/2^n)$.

(b) Define $Z_i\sim\mathcal{N}(0,1)$ and note that $\sum_{i=1}^{20} Z_i^2\sim\chi^2(20)$. Since $Z_i = X_i/\sqrt{2}$ for $X_i\sim\mathcal{N}(0,2)$, we have $\sum_{i=1}^{20} X_i^2 = \sum_{i=1}^{20} 2Z_i^2 \sim 2\chi^2(20)$. Then,

$$\mathbb{P}_{\sigma^2=2}(\sum_{i=1}^n X_i^2\geq k) = \mathbb{P}_{\sigma^2=2}(2\chi^2(20)\geq k) = 1-\mathbb{P}_{\sigma^2=2}(\chi^2(20)<k/2) = 0.01$$
$$\implies \mathbb{P}_{\sigma^2=2}(\chi^2(20)<k/2) = 0.99$$

which gives $k = 2(37.57) = 75.14$ using R; see the attached code and output.

(c) Denote σ_1^2 as the σ^2 under H_1. The test is UMP since $\mathbb{P}_{\sigma^2=2}(\sum_{i=1}^n X_i^2\geq k)=\alpha$ stays the same and $T(X_1,\ldots,X_n)$ increases with σ_1^2 for the same k, thereby increasing the power.

2. (a) The prior distribution function is

$$\int_{-\infty}^x \pi(\theta)d\theta = \beta\lambda^\beta\int_\lambda^x \theta^{-\beta-1}d\theta = -\lambda^\beta\theta^{-\beta}|_\lambda^x = -\lambda^\beta x^{-\beta}+\lambda^\beta\lambda^{-\beta} = 1-\lambda^\beta x^{-\beta} = F(x)$$

for $x>\lambda$ and to find the quantile function,

$$y = 1-\lambda^\beta x^{-\beta} \implies x = (\frac{1-y}{\lambda^\beta})^{-1/\beta} = \lambda(1-y)^{-1/\beta} = F^{-1}(y)$$

for $y\in(0,1)$. For the one-sided credible intervals for θ, note that for $p\in(0,1)$, $\mathbb{P}(\lambda<\theta\leq F^{-1}(p)) = p$ and $\mathbb{P}(\theta>F^{-1}(1-p)) = 1-\mathbb{P}(\theta\leq F^{-1}(1-p)) = 1-(1-p) = p$, so the $100p\%$ intervals are $(\lambda, F^{-1}(p)] = (\lambda,\lambda(1-p)^{-1/\beta}]$ and $[F^{-1}(1-p),\infty) = [\lambda p^{-1/\beta},\infty)$.

For a two-sided credible interval for θ, note that $\mathbb{P}(F^{-1}(\frac{1-p}{2})\leq\theta\leq F^{-1}(\frac{1+p}{2})) = \mathbb{P}(\theta\leq F^{-1}(\frac{1+p}{2})) - \mathbb{P}(\theta<F^{-1}(\frac{1-p}{2})) = \frac{1+p}{2}-\frac{1-p}{2} = p$, so the $100p\%$ interval is $[F^{-1}(\frac{1-p}{2}), F^{-1}(\frac{1+p}{2})] = [\lambda(\frac{1+p}{2})^{-1/\beta}, \lambda(\frac{1-p}{2})^{-1/\beta}]$. In fact, this can be generalized to $[F^{-1}(o), F^{-1}(o+p)] = [\lambda(1-o)^{-1/\beta}, \lambda(1-o-p)^{-1/\beta}]$ for any $o\in(0,1-p)$.

The prior mean is $\mathbb{E}(\theta|\beta>1) = \int_{-\infty}^\infty \theta\pi(\theta)d\theta = \beta\lambda^\beta\int_\lambda^\infty\theta^{-\beta}d\theta = \beta\lambda^\beta(-\frac{\theta^{1-\beta}}{1-\beta})|_\lambda^\infty = 0+\beta\lambda^\beta\frac{\lambda^{1-\beta}}{1-\beta} = \beta\lambda/(1-\beta)$.

3. (a) Assume that $X_1, \dots, X_n$ are independent. To find $\pi(\theta|x_1, \dots, x_n)$,

$$
\begin{aligned}
\pi(\theta)p(x_1, \dots, x_n; \theta) &= \frac{\lambda^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\theta} \prod_{i=1}^n \frac{\theta^{x_i}}{x_i!} e^{-\theta} = \frac{\lambda^\alpha \theta^{\alpha-1+\sum_{i=1}^n x_i}}{\Gamma(\alpha)\prod_{i=1}^n x_i!} e^{-\theta(\lambda+n)} \\
\int_{\mathbb{R}} \pi(s)p(x_1, \dots, x_n; \theta) ds &= \int_0^\infty \frac{\lambda^\alpha s^{\alpha-1+\sum_{i=1}^n x_i}}{\Gamma(\alpha)\prod_{i=1}^n x_i!} e^{-s(\lambda+n)} ds \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)\prod_{i=1}^n x_i!} \int_0^\infty s^{\alpha-1+\sum_{i=1}^n x_i} e^{-s(\lambda+n)} ds \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)\prod_{i=1}^n x_i!} \int_0^\infty \left(\frac{u}{\lambda+n}\right)^{\alpha-1+\sum_{i=1}^n x_i} e^{-u} \frac{du}{\lambda+n} \qquad (1) \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda+n)^{\alpha+\sum_{i=1}^n x_i}\prod_{i=1}^n x_i!} \int_0^\infty u^{\alpha-1+\sum_{i=1}^n x_i} e^{-u} du \\
&= \frac{\lambda^\alpha \Gamma(\alpha+\sum_{i=1}^n x_i)}{\Gamma(\alpha)(\lambda+n)^{\alpha+\sum_{i=1}^n x_i}\prod_{i=1}^n x_i!} \qquad (2) \\
\pi(\theta|x_1, \dots, x_n) &= \frac{\pi(\theta)p(x_1, \dots, x_n; \theta)}{\int_{\mathbb{R}} \pi(s)p(x_1, \dots, x_n; \theta) ds} = \frac{\theta^{\alpha-1+\sum_{i=1}^n x_i}(\lambda+n)^{\alpha+\sum_{i=1}^n x_i}}{\Gamma(\alpha+\sum_{i=1}^n x_i)} e^{-\theta(\lambda+n)} \\
&= g(\theta; \alpha + \sum_{i=1}^n x_i, \lambda+n)
\end{aligned}
$$

where we used $u = s(\lambda+n)$ such that $du = (\lambda+n)ds$ in (1) and the definition of the Gamma function in (2). Note that this is only a valid density when $\sum_{i=1}^n x_i > \alpha$. Furthermore,

$$
\begin{aligned}
\lim_{\alpha\to 0} \pi(\theta|x_1, \dots, x_n) &= e^{-\theta(\lambda+n)} \lim_{\alpha\to 0} \frac{\theta^{\alpha-1+\sum_{i=1}^n x_i}(\lambda+n)^{\alpha+\sum_{i=1}^n x_i}}{\Gamma(\alpha+\sum_{i=1}^n x_i)} \\
&= e^{-\theta(\lambda+n)} \frac{\theta^{-1+\sum_{i=1}^n x_i}(\lambda+n)^{\sum_{i=1}^n x_i}}{\Gamma(\sum_{i=1}^n x_i)} = g(\theta; \sum_{i=1}^n x_i, \lambda+n)
\end{aligned}
$$

which is a proper density when $\sum_{i=1}^n x_i > 0$.

(b) First, noting that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$,

$$
\begin{aligned}
\int_0^\infty \theta g(\theta; \alpha, \lambda) d\theta &= \int_0^\infty \frac{\lambda^\alpha \theta^{\alpha-1+1}}{\Gamma(\alpha)} e^{-\lambda\theta} d\theta = \frac{\alpha}{\lambda} \int_0^\infty \frac{\lambda^{\alpha+1}\theta^{\alpha-1+1}}{\Gamma(\alpha+1)} e^{-\lambda\theta} d\theta \\
&= \frac{\alpha}{\lambda} \int_0^\infty g(\theta; \alpha+1, \lambda) d\theta = \frac{\alpha}{\lambda}
\end{aligned}
$$

so $\mathbb{E}(\theta) = \alpha/\lambda$ since the prior density is $g(\theta; \alpha, \lambda)$ and similarly $\mathbb{E}(\theta|x_1, \dots, x_n) = (\alpha + \sum_{i=1}^n x_i)/(\lambda+n)$ since the posterior density is $g(\theta; \alpha + \sum_{i=1}^n x_i, \lambda+n)$.