Variance Reduction with Efficient Important Sampling

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1 Default Probability Estimation

1.1 Algorithm for estimating $P(X > c)$

Algorithm 1 Efficient Importance Sampling for $P(X > c)$

Require: N: sample size, c: threshold, μ: twisting parameter
Ensure: Estimated $\hat{p}$, standard error SE

`// Basic Monte Carlo`
$X \sim N(0, 1)$ ▷ Generate standard normal samples
$p_{\text{BMC}} = \frac{1}{N} \sum_{i=1}^{N} I(X_i > c)$
$\sigma^2_{\text{BMC}} = \text{Var}\big(I(X > c)\big)$
$SE_{\text{BMC}} = \sigma_{\text{BMC}} / \sqrt{N}$
`// Efficient Importance Sampling` ($\mu = c$)
$Z \sim N(c, 1)$ ▷ Generate samples under new measure
$p_{\text{IS}} = \frac{1}{N} \sum_{i=1}^{N} I(Z_i > c) \cdot \exp\big(\frac{c^2}{2} - cZ_i\big)$
$\sigma^2_{\text{IS}} = \text{Var}\big(I(Z > c) \cdot \exp(\frac{c^2}{2} - cZ)\big)$
$SE_{\text{IS}} = \sigma_{\text{IS}} / \sqrt{N}$

1.2 Results

Table 1: Default Probability Estimation Comparison (BMC vs. EIS)

c	True DP	BMC Mean (SE)	EIS Mean (SE)	CPU Time (s)	GPU Time (s)
1	0.1587	0.158855 (0.000365)	0.158891 (0.000192)	0.042	0.006
2	0.0228	0.022784 (0.000149)	0.0227618 (3.48e-05)	0.039	0.006
3	0.0013	0.001324 (3.67e-05)	0.00134977 (2.48e-06)	0.040	0.006
4	3.17e-05	3.1e-05 (5.63e-06)	3.16525e-05 (6.73e-08)	0.042	0.006

Note. GPU times and CPU times are the sum of BMC and EIS methods.

2 CVaR Estimation

2.1 Algorithm for estimating $CVaR$

Algorithm 2 Importance Sampling for $E[X \mid X > c]$

Require: N (sample size), c (threshold), proposal density g (e.g. $N(c, 1)$)

Ensure: Estimates of $\hat{m} = E[X \mid X > c]$, $\hat{p} = P(X > c)$, and standard errors

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// Target & likelihood ratio
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$f(x) = \phi(x)$ is the $N(0,1)$ density; $w(x) = \dfrac{f(x)}{g(x)}$ ▷ If $g = N(c, 1)$ then $w(x) = \exp\big(\frac{c^2}{2} - cx\big)$

`// 1) Basic Monte Carlo (BMC) under` f

Draw $X_i \sim N(0,1)$ for $i = 1, \ldots, N$

$\hat{p}_{\text{BMC}} = \dfrac{1}{N} \sum_{i=1}^{N} \mathbf{1}\{X_i > c\}$

$\hat{m}_{\text{BMC}} = \dfrac{\sum_{i=1}^{N} X_i \, \mathbf{1}\{X_i > c\}}{\sum_{i=1}^{N} \mathbf{1}\{X_i > c\}}$

$r_i^{\text{BMC}} = (X_i - \hat{m}_{\text{BMC}}) \, \mathbf{1}\{X_i > c\}$; $s^2_{\text{BMC}} = \dfrac{1}{N-1} \sum_{i=1}^{N} \big(r_i^{\text{BMC}}\big)^2$

$SE_{\text{BMC}} = \sqrt{s^2_{\text{BMC}} / (N \, \hat{p}^2_{\text{BMC}})}$

`// 2) Importance Sampling (IS) under` g

Draw $Z_i \sim g$ for $i = 1, \ldots, N$

$b_i = \mathbf{1}\{Z_i > c\} \, w(Z_i)$; $\quad a_i = Z_i \, b_i$

$\hat{p}_{\text{IS}} = \dfrac{1}{N} \sum_{i=1}^{N} b_i$

(Self-normalized IS) $\hat{m}_{\text{IS}} = \dfrac{\sum_{i=1}^{N} a_i}{\sum_{i=1}^{N} b_i} = \dfrac{\sum_{i=1}^{N} Z_i \, \mathbf{1}\{Z_i > c\} \, w(Z_i)}{\sum_{i=1}^{N} \mathbf{1}\{Z_i > c\} \, w(Z_i)}$

SE for ratio (delta): $r_i^{\text{IS}} = a_i - \hat{m}_{\text{IS}} \, b_i$; $s^2_{\text{IS}} = \dfrac{1}{N-1} \sum_{i=1}^{N} \big(r_i^{\text{IS}}\big)^2$

$SE_{\text{IS}} = \sqrt{s^2_{\text{IS}} / (N \, \hat{p}^2_{\text{IS}})}$

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// 3) Comparison
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Output $(\hat{m}_{\text{BMC}}, SE_{\text{BMC}})$ and $(\hat{m}_{\text{IS}}, SE_{\text{IS}})$

Table 2: $CVaR$ Estimation Comparison (BMC vs. IS)

c	BMC $CVaR$ (SE)	True $CVaR$	IS $CVaR$ (SE)
3	3.2936 (7.303e-03)	3.283	3.2824 (4.145e-04)
4	4.2244 (3.476e-02)	4.226	4.2262 (3.740e-04)
5	— (—)	5.187	5.1864 (3.391e-04)

2.2 Results

3 Importance Sampling to $E[XI(X > c)]$ Estimation

3.1 Algorithm for estimating $E[\,X\,\mathbf{1}\{X > c\}]$

Algorithm 3 Efficient Importance Sampling for $E[\,X\,\mathbf{1}\{X > c\}]$

Require: N: sample size, c: threshold, μ: twisting parameter
Ensure: Estimated $\hat{m}$, standard error SE

`// Basic Monte Carlo (BMC) under` $f(x) = \phi(x)$, $X \sim N(0,1)$

1: $X \sim N(0,1)$ ▷ Generate standard normal samples
2: $\hat{m}_{\text{BMC}} = \dfrac{1}{N}\sum_{i=1}^{N} X_i\,\mathbf{1}\{X_i > c\}$
3: $\sigma^2_{\text{BMC}} = \text{Var}\big(X\,\mathbf{1}\{X > c\}\big)$
4: $SE_{\text{BMC}} = \sigma_{\text{BMC}}/\sqrt{N}$

`// Efficient Importance Sampling (`$\mu = c$`) with` $g(x) = \phi_\mu(x) = \mathcal{N}(\mu, 1)$

5: $Z \sim N(c,1)$ ▷ Generate samples under new measure g
6: $w(Z) = \dfrac{f(Z)}{g(Z)} = \exp\big(\frac{c^2}{2} - cZ\big)$ ▷ Likelihood ratio
7: $\hat{m}_{\text{IS}} = \dfrac{1}{N}\sum_{i=1}^{N} Z_i\,\mathbf{1}\{Z_i > c\}\,w(Z_i)$
8: $\sigma^2_{\text{IS}} = \text{Var}\big(Z\,\mathbf{1}\{Z > c\}\,w(Z)\big)$
9: $SE_{\text{IS}} = \sigma_{\text{IS}}/\sqrt{N}$

3.2 Results

Note. For $X \sim N(0,1)$, the theoretical value is $E[X\,\mathbf{1}\{X > c\}] = \int_c^\infty x\,\phi(x)\,dx = \phi(c)$, where ϕ denotes the standard normal density.

Table 3: $E[X\,\mathbf{1}\{X > c\}]$ Estimation Comparison (BMC vs. EIS)

c	True Value	BMC Mean (SE)	EIS Mean (SE)	CPU Time (s)	GPU Time (s)
1	0.2420	0.241924 (5.845184e-04)	0.242071 (2.552147e-04)	0.049042	0.008566
2	0.0540	0.053713 (3.567679e-04)	0.053971 (7.568887e-05)	0.044671	0.008445
3	0.0044	0.004453 (1.210529e-04)	0.004429 (7.769168e-06)	0.049960	0.009137
4	1.39-04	0.000147 (2.479159e-05)	0.000134 (2.758473e-07)	0.046925	0.008501

References

[1] Chuan-Hsiang Han. *Stochastic Computation in Finance*. Taipei, Taiwan: Shin-Lu Publishing, 2012. (in Chinese)