Variance Reduction with Efficient Important Sampling

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1 Default Probability Estimation

1.1 Algorithm for estimating $P(X > c)$

Algorithm 1 Efficient Importance Sampling for $P(X > c)$

Require: N: sample size, c: threshold, μ: twisting parameter
Ensure: Estimated $\hat{p}$, standard error SE

// Basic Monte Carlo
$X \sim N(0, 1)$ ▷ Generate standard normal samples
$p_{\text{BMC}} = \frac{1}{N} \sum_{i=1}^{N} I(X_i > c)$
$\sigma^2_{\text{BMC}} = \text{Var}\big(I(X > c)\big)$
$SE_{\text{BMC}} = \sigma_{\text{BMC}}/\sqrt{N}$
// Efficient Importance Sampling ($\mu = c$)
$Z \sim N(c, 1)$ ▷ Generate samples under new measure
$p_{\text{IS}} = \frac{1}{N} \sum_{i=1}^{N} I(Z_i > c) \cdot \exp\big(\frac{c^2}{2} - cZ_i\big)$
$\sigma^2_{\text{IS}} = \text{Var}\big(I(Z > c) \cdot \exp(\frac{c^2}{2} - cZ)\big)$
$SE_{\text{IS}} = \sigma_{\text{IS}}/\sqrt{N}$

1.2 Results

Table 1: Default Probability Estimation Comparison (BMC vs. EIS)

c	True DP	BMC Mean (SE)	EIS Mean (SE)	CPU Time (s)	GPU Time (s)
1	0.1587	0.158855 (0.000365)	0.158891 (0.000192)	0.042	0.006
2	0.0228	0.022784 (0.000149)	0.0227618 (3.48e-05)	0.039	0.006
3	0.0013	0.001324 (3.67e-05)	0.00134977 (2.48e-06)	0.040	0.006
4	3.17e-05	3.1e-05 (5.63e-06)	3.16525e-05 (6.73e-08)	0.042	0.006

Note. GPU times and CPU times are the sum of BMC and EIS methods.

2 CVaR Estimation

2.1 Algorithm for estimating $CVaR$

Algorithm 2 Importance Sampling for $E[X \mid X > c]$

Require: N (sample size), c (threshold), proposal density g (e.g. $N(c,1)$)
Ensure: Estimates of $\hat{m} = E[X \mid X > c]$, $\hat{p} = P(X > c)$, and standard errors

`// Target & likelihood ratio`

$f(x) = \phi(x)$ is the $N(0,1)$ density; $w(x) = \dfrac{f(x)}{g(x)}$ ▷ If $g = N(c,1)$ then $w(x) = \exp\big(\frac{c^2}{2} - cx\big)$

`// 1) Basic Monte Carlo (BMC) under` f

Draw $X_i \sim N(0,1)$ for $i = 1,\dots,N$

$\hat{p}_{\text{BMC}} = \dfrac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{X_i > c\}$

$\hat{m}_{\text{BMC}} = \dfrac{\sum_{i=1}^{N} X_i \,\mathbf{1}\{X_i > c\}}{\sum_{i=1}^{N} \mathbf{1}\{X_i > c\}}$

$r_i^{\text{BMC}} = (X_i - \hat{m}_{\text{BMC}})\,\mathbf{1}\{X_i > c\}$; $s^2_{\text{BMC}} = \dfrac{1}{N-1}\sum_{i=1}^{N} \big(r_i^{\text{BMC}}\big)^2$

$SE_{\text{BMC}} = \sqrt{s^2_{\text{BMC}}/(N\,\hat{p}^2_{\text{BMC}})}$

`// 2) Importance Sampling (IS) under` g

Draw $Z_i \sim g$ for $i = 1,\dots,N$

$b_i = \mathbf{1}\{Z_i > c\}\,w(Z_i)$; $\quad a_i = Z_i\,b_i$

$\hat{p}_{\text{IS}} = \dfrac{1}{N}\sum_{i=1}^{N} b_i$

(Self-normalized IS) $\hat{m}_{\text{IS}} = \dfrac{\sum_{i=1}^{N} a_i}{\sum_{i=1}^{N} b_i} = \dfrac{\sum_{i=1}^{N} Z_i\,\mathbf{1}\{Z_i > c\}\,w(Z_i)}{\sum_{i=1}^{N} \mathbf{1}\{Z_i > c\}\,w(Z_i)}$

SE for ratio (delta): $r_i^{\text{IS}} = a_i - \hat{m}_{\text{IS}}\,b_i$; $s^2_{\text{IS}} = \dfrac{1}{N-1}\sum_{i=1}^{N}\big(r_i^{\text{IS}}\big)^2$

$SE_{\text{IS}} = \sqrt{s^2_{\text{IS}}/(N\,\hat{p}^2_{\text{IS}})}$

`// 3) Comparison`

Output $(\hat{m}_{\text{BMC}}, SE_{\text{BMC}})$ and $(\hat{m}_{\text{IS}}, SE_{\text{IS}})$

2.2 Results

Table 2: $CVaR$ Estimation Comparison (BMC vs. IS)

c	BMC $CVaR$ (SE)	True $CVaR$	IS $CVaR$ (SE)
3	3.2936 (7.303e-03)	3.283	3.2824 (4.145e-04)
4	4.2244 (3.476e-02)	4.226	4.2262 (3.740e-04)
5	— (—)	5.187	5.1864 (3.391e-04)

3 Importance Sampling to $E[XI(X > c)]$ Estimation

3.1 Algorithm for estimating $E[\,X\,\mathbf{1}\{X > c\}]$

Algorithm 3 Efficient Importance Sampling for $E[\,X\,\mathbf{1}\{X > c\}]$

Require: N: sample size, c: threshold, μ: twisting parameter
Ensure: Estimated $\hat{m}$, standard error SE

`// Basic Monte Carlo (BMC) under` $f(x) = \phi(x)$, $X \sim N(0,1)$

$X \sim N(0,1)$ ▷ Generate standard normal samples

$\hat{m}_{\text{BMC}} = \dfrac{1}{N} \sum_{i=1}^{N} X_i\,\mathbf{1}\{X_i > c\}$

$\sigma^2_{\text{BMC}} = \text{Var}\big(X\,\mathbf{1}\{X > c\}\big)$

$SE_{\text{BMC}} = \sigma_{\text{BMC}}/\sqrt{N}$

`// Efficient Importance Sampling (`$\mu = c$`) with` $g(x) = \phi_\mu(x) = \mathcal{N}(\mu, 1)$

$Z \sim N(c, 1)$ ▷ Generate samples under new measure g

$w(Z) = \dfrac{f(Z)}{g(Z)} = \exp\big(\frac{c^2}{2} - cZ\big)$ ▷ Likelihood ratio

$\hat{m}_{\text{IS}} = \dfrac{1}{N} \sum_{i=1}^{N} Z_i\,\mathbf{1}\{Z_i > c\}\,w(Z_i)$

$\sigma^2_{\text{IS}} = \text{Var}\big(Z\,\mathbf{1}\{Z > c\}\,w(Z)\big)$

$SE_{\text{IS}} = \sigma_{\text{IS}}/\sqrt{N}$

3.2 Results

Table 3: $E[X\,\mathbf{1}\{X > c\}]$ Estimation Comparison (BMC vs. EIS)

c	True Value	BMC Mean (SE)	EIS Mean (SE)	CPU Time (s)	GPU Time (s)
1	0.2420	0.241924 (5.845184e-04)	0.242071 (2.552147e-04)	0.049042	0.008566
2	0.0540	0.053713 (3.567679e-04)	0.053971 (7.568887e-05)	0.044671	0.008445
3	0.0044	0.004453 (1.210529e-04)	0.004429 (7.769168e-06)	0.049960	0.009137
4	1.39-04	0.000147 (2.479159e-05)	0.000134 (2.758473e-07)	0.046925	0.008501

Note. For $X \sim N(0,1)$, the theoretical value is $E[X\,\mathbf{1}\{X > c\}] = \int_c^{\infty} x\,\phi(x)\,dx = \phi(c)$, where ϕ denotes the standard normal density.

References

[1] Chuan-Hsiang Han. *Stochastic Computation in Finance.* Taipei, Taiwan: Shin-Lu Publishing, 2012. (in Chinese)