Portfolio Risk Optimization via VaR, CVaR, and Quantile-Based Large Deviations

Ryan Dai
Quantitative Finance, National Tsing Hua University

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Abstract

This paper studies portfolio risk optimization under Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) criteria. We first revisit the classical one-period model with normally distributed asset returns, showing that for high confidence levels $\alpha \uparrow 1$, minimizing both VaR and CVaR is asymptotically equivalent to minimizing the portfolio variance. To provide a more general theoretical foundation, we incorporate results from large deviation theory and order statistics, which allow us to represent quantiles as random variables and link their tail probabilities to relative entropy rate functions. This large deviation perspective reveals that minimizing VaR can be formulated as a stochastic optimization problem of maximizing a CDF or equivalently maximizing a rate function. For the normal distribution case, this formulation again reduces to variance minimization as $\alpha \uparrow 1$, consistent with the classical analysis. The paper concludes with potential research directions, including extending the framework to CVaR under large deviations, relaxing the normality assumption, and conducting numerical experiments.

1 Previous works on VaR and CVaR optimization

Consider a single period $t \in \{0, 1\}$ with stock prices $\mathbf{S}(t) = [S_1(t)\ S_2(t)\ \dots\ S_n(t)]^\top$ and portfolio value $P_t = \mathbf{N}^\top \mathbf{S}(t)$, where $\mathbf{N} = [N_1, \dots, N_n]^\top$.

Define:

- Portfolio weights $\mathbf{w} = [w_1, \dots, w_n]^\top$ with $w_i = \frac{N_i S_i(0)}{P_0}$.
- Simple returns $\mathbf{r} = [r_1, \dots, r_n]^\top$ with $r_i = \frac{S_i(1) - S_i(0)}{S_i(0)}$.
- Portfolio simple return

$$r_p = \frac{P_1 - P_0}{P_0} = \frac{\mathbf{N}^\top(\mathbf{S}(1) - \mathbf{S}(0))}{P_0} = \sum_{i=1}^n \frac{N_i[S_i(1) - S_i(0)]}{P_0} \frac{S_i(0)}{S_i(0)} = \sum_{i=1}^n w_i r_i = \mathbf{w}^\top \mathbf{r}.$$

Assume $r_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ and $\mathbf{r} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ with $\boldsymbol{\mu} = [\mu_1, \dots, \mu_n]^\top$, $\Sigma = \mathbb{E}[(\mathbf{r} - \boldsymbol{\mu})(\mathbf{r} - \boldsymbol{\mu})^\top]$, and

$$r_p = \mathbf{w}^\top \mathbf{r} \sim \mathcal{N}(\mu_p, \sigma_p^2), \quad \mu_p = \mathbf{w}^\top \boldsymbol{\mu},\ \sigma_p^2 = \mathbf{w}^\top \Sigma \mathbf{w}.$$

1.1 VaR optimization

Define $\mathrm{VaR}_\alpha = \inf\{l : \mathbb{P}(P_0 - P_1 \geq l) \leq 1 - \alpha\}$ with $\mathbb{P}(P_0 - P_1 \geq \mathrm{VaR}_\alpha) = 1 - \alpha$. Then

$$\begin{aligned}
\mathbb{P}\left(\frac{P_1 - P_0}{P_0} \leq \frac{-\mathrm{VaR}_\alpha}{P_0}\right) &= \mathbb{P}\left(r_p \leq \frac{-\mathrm{VaR}_\alpha}{P_0}\right) \\
&= \mathbb{P}\left(\frac{r_p - \mu_p}{\sigma_p} \leq \frac{\frac{-\mathrm{VaR}_\alpha}{P_0} - \mu_p}{\sigma_p}\right) \\
&= \Phi\left(\frac{\frac{-\mathrm{VaR}_\alpha}{P_0} - \mu_p}{\sigma_p}\right) = 1 - \alpha,
\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal CDF. Hence

$$\frac{\frac{-\mathrm{VaR}_\alpha}{P_0} - \mu_p}{\sigma_p} = \Phi^{-1}(1 - \alpha) \quad \Rightarrow \quad \mathrm{VaR}_\alpha = P_0[-\mu_p - \Phi^{-1}(1 - \alpha)\sigma_p].$$

For $\alpha > 0.5$, $\Phi^{-1}(1-\alpha) < 0$, so

$$\begin{aligned}
\arg\min_{\mathbf{w}} \mathrm{VaR}_\alpha &= \arg\min_{\mathbf{w}} [-\mu_p - \Phi^{-1}(1-\alpha)\sigma_p] \\
&= \arg\max_{\mathbf{w}} [\mu_p + \Phi^{-1}(1-\alpha)\sigma_p] \\
&= \arg\min_{\mathbf{w}} \left[\frac{\mu_p}{\Phi^{-1}(1-\alpha)} + \sigma_p\right].
\end{aligned}$$

Since $\lim_{\alpha\uparrow 1} \frac{\mu_p}{\Phi^{-1}(1-\alpha)} = 0$, we get $\arg\min_{\mathbf{w}} \mathrm{VaR}_\alpha = \arg\min_{\mathbf{w}} \sigma_p$ as $\alpha \uparrow 1$.

1.2 CVaR optimization

Define $\mathrm{CVaR}_\alpha = \mathbb{E}[P_0 - P_1 \mid P_0 - P_1 \geq \mathrm{VaR}_\alpha]$. From above, $\mathrm{VaR}_\alpha = P_0[-\mu_p - \Phi^{-1}(1-\alpha)\sigma_p]$. Therefore

$$\begin{aligned}
\mathrm{CVaR}_\alpha &= \mathbb{E}[P_0 - P_1 \mid P_0 - P_1 \geq P_0(-\mu_p - \Phi^{-1}(1-\alpha)\sigma_p)] \\
&= -P_0\, \mathbb{E}\left[\frac{P_1 - P_0}{P_0} \,\middle|\, \frac{P_1 - P_0}{P_0} \leq \mu_p + \Phi^{-1}(1-\alpha)\sigma_p\right] \\
&= -P_0\, \mathbb{E}[r_p \mid r_p \leq \mu_p + \Phi^{-1}(1-\alpha)\sigma_p] \\
&= -P_0\, \mathbb{E}\left[r_p \,\middle|\, \frac{r_p - \mu_p}{\sigma_p} \leq \Phi^{-1}(1-\alpha)\right] \\
&= -P_0\sigma_p\, \mathbb{E}\left[\frac{r_p - \mu_p}{\sigma_p} \,\middle|\, \frac{r_p - \mu_p}{\sigma_p} \leq \Phi^{-1}(1-\alpha)\right] - P_0\mu_p \\
&= -P_0\sigma_p \frac{1}{1-\alpha} \int_{-\infty}^{\Phi^{-1}(1-\alpha)} x\,\phi(x)\,dx - P_0\mu_p,
\end{aligned}$$

where $\phi(\cdot)$ is the standard normal PDF. Noting $\frac{d}{dx}\phi(x) = -x\phi(x)$,

$$\begin{aligned}
\Rightarrow\ \mathrm{CVaR}_\alpha &= -\frac{P_0\sigma_p}{1-\alpha}\,[-\phi(x)]_{-\infty}^{\Phi^{-1}(1-\alpha)} - P_0\mu_p \\
&= P_0\left[\frac{\phi(\Phi^{-1}(1-\alpha))}{1-\alpha}\,\sigma_p - \mu_p\right].
\end{aligned}$$

Thus

$$\begin{aligned}
\arg\min_{\mathbf{w}} \mathrm{CVaR}_\alpha &= \arg\min_{\mathbf{w}} \left[\frac{\phi(\Phi^{-1}(1-\alpha))}{1-\alpha}\,\sigma_p - \mu_p\right] \\
&= \arg\min_{\mathbf{w}} \left[\sigma_p - \frac{1-\alpha}{\phi(\Phi^{-1}(1-\alpha))}\,\mu_p\right].
\end{aligned}$$

By L'Hôpital's rule,

$$\lim_{\alpha\uparrow 1} \frac{1-\alpha}{\phi(\Phi^{-1}(1-\alpha))} = \lim_{\alpha\uparrow 1} \frac{-1}{\Phi^{-1}(1-\alpha)} = 0,$$

hence $\arg\min_{\mathbf{w}} \mathrm{CVaR}_\alpha = \arg\min_{\mathbf{w}} \sigma_p$ as $\alpha \uparrow 1$.

1.3 Conclusion

Under the one-period normal-return assumption,

$$\arg\min_{\mathbf{w}} \mathrm{VaR}_\alpha = \arg\min_{\mathbf{w}} \mathrm{CVaR}_\alpha = \arg\min_{\mathbf{w}} \sigma_p \quad \text{for } \alpha \uparrow 1.$$

2 Order statistics and large deviations

In [1] and [2], we have the following theoretical results.

Condition 2.1. *(i) $\{X_n : n \geq 1\}$ is a sequence of i.i.d. random variables with CDF F continuous and strictly increasing on (a, b), where $-\infty \leq a < b \leq \infty$.*

(ii) $\{k_n : n \geq 1\}$ *is a sequence satisfying* $k_n \in \{1, \dots, n\}$ *and* $\lim_{n\to\infty} \frac{k_n}{n} = \alpha \in (0,1)$. *For example, taking* $k_n = \lfloor 0.99n \rfloor$ *yields* $\alpha = 0.99$. *Here* k_n *is the order and* α *the quantile, and they align as* $n \to \infty$.

Proposition 2.1. *Under Condition 2.1, let* $X_{1:n} \leq \cdots \leq X_{n:n}$ *be order statistics, and consider a sequence* $\{X_{k_n:n} : n \geq 1\}$. *Then* $\{X_{k_n:n} : n \geq 1\}$ *satisfies an LDP with rate function*

$$I_{\alpha,F}(x) = H(\alpha \mid F(x)), \quad x \in (a,b),$$

where $H(p \mid q) = p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q}$ *for* $p, q \in (0,1)$. *That is, for* $x > F^{-1}(\alpha)$, *we have*

$$\mathbb{P}(X_{k_n:n} \geq x) \approx e^{-n \cdot I_{\alpha,F}(x)}.$$

3 Using large deviations for VaR optimization

In section 1, we define $\mathrm{VaR}_\alpha = \inf\{l : \mathbb{P}(P_0 - P_1 \geq l) \leq 1 - \alpha\}$. Let X be a random variable representing the portfolio loss with CDF F and assume Condition 2.1. Then we can define

$$\mathrm{VaR}_\alpha = \inf\{l : \mathbb{P}(X \geq l) \leq 1 - \alpha\} = \inf\{l : F(l) \geq \alpha\} = F^{-1}(\alpha).$$

By Proposition 2.1, for $x > F^{-1}(\alpha) = \mathrm{VaR}_\alpha$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_{k_n:n} \geq x) = -I_{\alpha,F}(x) = -H(\alpha \mid F(x)).$$

Minimizing VaR_α is similar to minimizing $\mathbb{P}(X_{k_n:n} \geq x)$:

$$\begin{aligned}\arg\min_{\mathbf{w}} \mathbb{P}(X_{k_n:n} \geq x) &= \arg\min_{\mathbf{w}} \frac{1}{n} \log \mathbb{P}(X_{k_n:n} \geq x) \\ &= \arg\max_{\mathbf{w}} H(\alpha \mid F(x)),\end{aligned}$$

with $F(x)$ depending on $\mathbf{w}$.

Since $x > F^{-1}(\alpha)$ implies $F(x) > \alpha$ and note that

$$\frac{\partial}{\partial q} H(\alpha \mid q) = \frac{q - \alpha}{q(1-q)},$$

so for $q > \alpha$ the mapping

$$q \longmapsto H(\alpha \mid q)$$

is strictly increasing,

$$\arg\max_{\mathbf{w}} H(\alpha \mid F(x)) = \arg\max_{\mathbf{w}} F(x)$$

Example 3.1. *(Normal distribution) If* $X \sim \mathcal{N}(-\mu_p, \sigma_p^2)$ *with* $\mu_p = \mathbf{w}^\top \boldsymbol{\mu}$ *and* $\sigma_p^2 = \mathbf{w}^\top \Sigma \mathbf{w}$, *then*

$$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}\left(\frac{X + \mu_p}{\sigma_p} \leq \frac{x + \mu_p}{\sigma_p}\right) = \Phi\left(\frac{x + \mu_p}{\sigma_p}\right),$$

and

$$\arg\min_{\mathbf{w}} \mathbb{P}(X_{k_n:n} \geq x) = \arg\max_{\mathbf{w}} F(x) = \arg\max_{\mathbf{w}} \Phi\left(\frac{x + \mu_p}{\sigma_p}\right) = \arg\max_{\mathbf{w}} \frac{x + \mu_p}{\sigma_p}.$$

For α *close to 1, we have* $x > 0$ *and*

$$\arg\min_{\mathbf{w}} \mathbb{P}(X_{k_n:n} \geq x) = \arg\max_{\mathbf{w}} \frac{x + \mu_p}{\sigma_p} = \arg\max_{\mathbf{w}} \frac{1}{\sigma_p} + \frac{\mu_p}{x\sigma_p}.$$

Since $\lim_{\alpha\uparrow 1} VaR_\alpha = \infty$ *and* $\lim_{x\uparrow\infty} \frac{\mu_p}{x\sigma_p} = 0$, *we get*

$$\arg\min_{\mathbf{w}} \mathbb{P}(X_{k_n:n} \geq x) \approx \arg\min_{\mathbf{w}} \sigma_p$$

for α *very close to 1 .*

4 Conclusion

We first show that

$$\arg\min_{\mathbf{w}} \mathrm{VaR}_\alpha = \arg\min_{\mathbf{w}} \mathrm{CVaR}_\alpha = \arg\min_{\mathbf{w}} \sigma_p \quad \text{for } \alpha \uparrow 1.$$

in one-period normal-return assumption. Then we provide a large deviation point of view on this problem. We use order statistics (a type of random variable) to represent VaR, and formulate the problem into stochastic optimization . It is equivalent to maximize a rate function which is a relative entropy function. Finally, the problem becomes maximizing CDF of a random variable of portfolio loss given a large loss level x. In normal case, we have observed that the problem has the same conclusion where the problem is equivalent to minimizing variance for $\alpha \uparrow 1$. The research direction includes following aspects:

(i) Research the large deviation properties for $CVaR$.

(ii) Relax the normal assumption to generalize the optimization setup.

(iii) Conduct numerical experiments to verify the conclusion.

References

[1] Enkelejd Hashorva, Claudio Macci, and Barbara Pacchiarotti (2013). *Large Deviations for Proportions of Observations Which Fall in Random Sets Determined by Order Statistics*. Methodology and Computing in Applied Probability, 15:875–896. doi:10.1007/s11009-012-9290-y.

[2] Valeria Bignozzi, Claudio Macci, and Lea Petrella (2020). *Large deviations for method-of-quantiles estimators of one-dimensional parameters*. Communications in Statistics – Theory and Methods, 49(5):1132–1157. doi:10.1080/03610926.2018.1554134.