Burning Spiders

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Abstract

Burning a graph is a process that models the spread of information in networks. The process takes place over a discrete number of rounds, and the minimum number of rounds required to burn a graph is the burning number. The burning number conjecture is that a connected graph with n vertices has a burning number at most $\lceil \sqrt{n}\, \rceil$. It has been shown that resolving the burning number conjecture for all graphs is equivalent to resolving it for trees. A tree is called a spider if it has exactly one vertex with degree strictly greater than two. This work summarizes recent progress that resolves the burning number conjecture for spiders.

1 Introduction

The process of burning a graph was introduced in [4] to model the spread of information or epidemics. The burning process take place on graph G and in discrete rounds $t \geq 0$. At $t = 0$, all the vertices of the graph are unburned. In each round $t \geq 1$ there are two actions. First, every burned vertex burns all of its neighbors. Second, an unburned vertex is selected to be burned. The process ends when all vertices are burned.

The *burning number* of a graph, denoted $b(G)$, is defined to be the minimum number of rounds required to burn the graph G. We call the sequence of vetices chosen in the second action a *burning sequence*, and the burning number can be equivalently defined as the length of the shortest burning sequence. Let n be the function counting the number of vertices of a graph, defined as $n(G) = |V(G)|$. The *burning number conjecture* [4] is that for all connected graphs, $b(G) \leq \lceil \sqrt{n(G)} \rceil$. This conjecture is currently open, although there has been progress. We direct the interested reader to the survey [2] for a general discussion. Norin and Turcotte [5] recently showed that the burning number conjecture hold asymptotically, and the best current bound is $\sqrt{4n(G)/3} + 1$, proved in [1]. It was shown in [4] that if the burning number conjecture is satisfied in trees, then it is satisfied for all graphs. Paths acheive the bound $\lceil \sqrt{n} \rceil$, showing that if true, the burning number conjecture would be tight.

This report summarizes the contribution of [3] towards the burning number conjecture. A graph G is a *spider* if it is a connected tree and there is a single vertex with degree strictly greater than two. The main result is Theorem 3.2 (Theorem 7 of [3]), which confirms that spider graphs satisfy the burning number conjecture.

2 Preliminaries

For this report, we consider all graph to be simple, finite, and undirected. A *tree* is a graph that does not contain any cycles, a *forest* is a disjoint union of trees, and a *path-forest* is a disjoint union of paths. Let G be a graph, $v \in V(G)$, and r a positive integer. Let $N_r[v]$ denote the closed neighborhood of v with radius r, defined as the set of vertices u such that there is a path from v to u with length less than or equal to r. We let each vertex have a path of length 0 to itself in order to satisfy the requirements of a metric. We use $G - N_r[v]$ to represent the graph induced by the set of vertices $V(G) \setminus N_r[v]$.

Lemma 2.1. *A graph G satisfies $b(G) \leq M$ for some integer M if and only if there exists an integer k, satisfying $1 \leq k \leq M$, such that*

$$V(G) = \bigcup_{i=1}^{k} N_{M-i}[v_i] \tag{1}$$

for some sequence of vertices $(v_i)_1^k$.

Proof. Let G be a graph and $(v_i)_1^M$ a burning sequence for G. Then the fire started at vertex v_i in round i will spread to $N_{M-i}[v_i]$ by the end of round M. Since $(v_i)_1^M$ burns the graph, let $k = M$ and $V(G) = \cup_{i=1}^{M} N_{M-i}[v_i]$. This concludes the forward implication.

Let $(v_i)_1^k$ be a sequence in $V(G)$ that satisfies equation 1. Let $(u_j)_1^M$ be a sequence in $V(G)$ where $u_j = v_i$ if $j \leq k$ and the other u_j are any vertex that does not appear earlier in the sequence. The fire started at vertex u_j in round j will spread to $N_{M-j}[u_j]$ by round M. Since $(u_j)_1^M$ burns at least $\cup_{j=1}^{M} N_{M-j}[u_j] = V(G)$, it is a burning sequnce. Thus, $b(G) \leq M$. □

Corollary 2.2. *Let G be a graph and $v \in V(G)$. Let α be a positive integer. If $b(G - N_\alpha[v]) \leq \alpha - 1$, then $b(G) \leq \alpha$.*

Proof. By applying the forward direction of Lemma 2.1, there exists some sequence of vertices such that $V(G - N_\alpha[v]) = \cup_{i=1}^{\alpha-1} N_{\alpha-1-i}[v_i]$. Let $v_\alpha = v$ and add $N_\alpha[v]$ to both sides to get $V(G) = \cup_{i=1}^{\alpha} N_{\alpha-i}[v_i]$. Thus, by the reverse direction of Lemma 2.1, $b(G) \leq \alpha$. □

Corollary 2.2 means that we can bound the burning number of a graph by removing a "large" neighborhood with radius α and proving the leftovers can be burned in $\alpha - 1$ rounds. This technique is central to proving an induction step in the following statements. For proofs the upcoming Lemma 2.3 and Lemma 2.4, we refer to [3] (Lemmas 2 and 5 respectively).

Lemma 2.3. *If G is a path-forest with t components then $b(G) \leq \lfloor \frac{n(G)}{2t} \rfloor + t$.*

Lemma 2.4. *Let $\mathcal{G}$ be a set of connected graphs. Let $\hat{\mathcal{G}} \subseteq G$ with the following condition. For every $G \in \hat{\mathcal{G}}$, there exists $v \in V(G)$ and $r \leq \lceil \sqrt{n(G)} \rceil - 1$ such that at least one of the followng conditions are satisfied.*

1. *$N_r[v] = V(G)$.*
2. *$|N_r[v]| \geq 2\lceil \sqrt{n(G)} \rceil - 1$ and the induced subgraph $G[V(G) \setminus N_r[v]]$ is in $\mathcal{G}$.*

If $b(G) \leq \lceil \sqrt{n(G)} \rceil$ for all $G \in \mathcal{G} \backslash \hat{\mathcal{G}}$, then $b(G) \leq \lceil \sqrt{n(G)} \rceil$ for all $G \in \mathcal{G}$.

3 Spiders

Recall that a graph G is a spider if it is a connected tree that has a single vertex with degree strictly greater than two. We call this vertex the *head* of the spider. An *arm* of a spider is a component of the path forest induced by deleting the head.

Lemma 3.1. *If G is a spider and $n(G) \leq 25$, then $b(G) \leq \lceil \sqrt{n(G)} \rceil$.*

Proof. Let G be a spider with head h. Let $\alpha = \lceil \sqrt{n(G)} \rceil$. If $\sqrt{n(G)}$ is not an integer, G is an induced graph of a spider H with $n(H) = \alpha^2$. If $(v_i)_1^\alpha$ is a burning sequnce for H, then $(u_j)_1^\alpha$ is a burning sequence for G, where $v_i = u_j$ if $v_i \in V(G)$, and u_j is the closest vertex to v_i that is in $V(G)$ otherwise. The chosen vertices are shifted up the arms towards the head, so the neighborhoods that they burn may have more overlap but the they will still cover the graph. Thus, we may assume $n(G)$ is a square integer: $4, 9, 16$, or 25.

The unique spider with $n(G) = 4$ is burned in two rounds by choosing the head to burn in round 1.

Suppose $n(G) = 9$. If $G - N_2[h]$ has only one component, the component has length at most 4 which can be burned in two rounds. So, by Corollary 2.2 $b(G) \leq 3$. If $G - N - 2[h]$ has two components, the Lemma 2.3 implies $G - N_2[h]$ can be burned two rounds, and Corollary 2.2 shows $b(G) \leq 3$. Finally, $G - N_2[h]$ cannot have 3 or more components or else $n(G)$ would be at least 10.

Suppose $n(G) = 16$. If the head is burned first, every arm with length less than 4 will be burned by round 4. If an arm is length 7 or greater, since $2\lceil \sqrt{n(G)}\ \rceil - 1 = 7$, we apply Lemma 2.4 on the family of smaller spiders. The path-forest $G - N_3[h]$ cannot have more than 3 components or else it would have at least $4 \cdot 4 + 1 = 17$ vertices. Since each component of the path-forest $G - N_3[h]$ has at most 3 vertices, if $G - N_3[h]$ has 1 or 2 components then Lemma 2.3 and Corollary 2.2 show $b(G) \leq 4$. Thus, we must only consider when $G - N_3[h]$ has 3 components. If $G - N_3[h]$ has less than 6 vertices, apply Lemma 2.3 and Corollary 2.2. There are only two spiders with 3 arms with lengths at least 4 and at most 6 where $n(G - N_3[h]) = 6$. Burning sequences are shown in Figure 1.

Suppose $n(G) = 25$. We use an analogus argument to $n(G) = 16$. If one arm has length greater than 8 then $2\lceil \sqrt{n(G)} \rceil - 1 = 9$ so Lemma 2.4 applies. The path-forest $G - N_4[h]$ cannot have more than 4 components since $5 * 5 + 1 > n(G)$. Since each component of $G - N_4[h]$ has at most 4 vertices, if $G - N_4[h]$ has only 1 or 2 components then Lemma 2.3 and Corollary 2.2 apply. If $G - N_4[h]$ has 3 components and less than 12 vertices, Lemma 2.3 and Corollary 2.2 apply. There is only one spider where $G - N_4[h]$ has 3 components

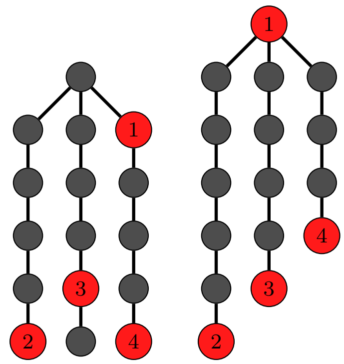

Figure 1: Burning sequences for the two spiders with $n(G) = 16$ and three arms with lengths at least 4 and at most 6. Each labeled vertex is chosen to burn during action two of the round corresponding to its label.

and $n(G - N_4[h]) = 12$. If $G - N_4[h]$ has 4 components and less than 8 vertices, Lemma 2.3 and Corollary 2.2 apply. There are four spiders where $G - N_4[h]$ has 4 components and $n(G - N_4[h]) = 8$. Burning sequences for these spiders are shown in Figure 2.

□

Theorem 3.2. *If G is a spider then $b(G) \leq \lceil \sqrt{n(G)}\, \rceil$.*

Proof. Let G be a spider and let $\alpha = \lceil \sqrt{n(G)}\, \rceil$. If $\alpha \leq 5$, apply Lemma 3.1.

Suppose one arm of G has length ℓ and $\ell > 2\alpha - 2$. Then for $r = \lceil \frac{\ell}{2} \rceil$ and v the vertex on the arm at a distance r from the head, $|N_r[v]| = 2r + 1 = 2(\lceil \frac{\ell}{2} \rceil) + 1 \geq 2\alpha - 1$ and $G - N_r[v]$ is a spider with one fewer arms. We can then inductively apply Lemma 2.4 to the family of spiders with fewer arms. Thus, we must only consider spiders with every arm having length at most $2\alpha - 2$.

Suppose G has $\alpha - 1$ arms of length $\alpha + 1$. Let v be a vertex adjacent to the head. Then $G' = G - N_{\alpha-1}[v]$ is a path forest with $\alpha - 2$ paths of length 2 and one path of length 1. By applying Lemma 2.3 we conclude $b(G') \leq \alpha - 1$ so Corollary 2.2 implies $b(G) \leq \alpha$.

Suppose otherwise; that is, G does not have $\alpha - 1$ arms of length $\alpha + 1$. Let

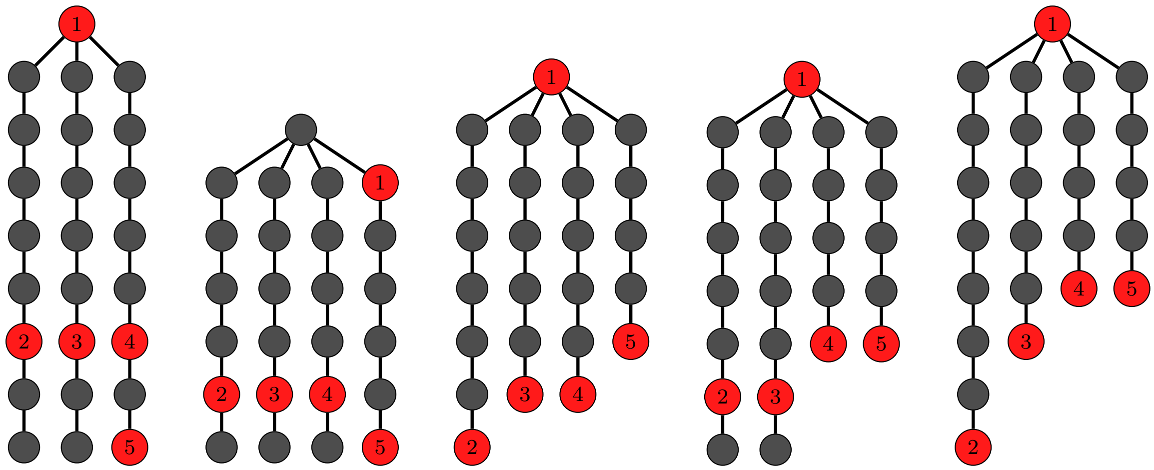

Figure 2: Burning sequnces for the five spiders with $n(G) = 25$ and at least 3 arms with length at least 5 and at most 8. Each labeled vertex is chosen to burn during action two of the round corresponding to its label.

$G' = G - N_{\alpha-1}[h]$ where h is the head of the spider. The graph G' is a path forest, and since the arms of G have length at most $2\alpha - 2$, the components of G' have length at most $\alpha - 1$. Label the components $G'_1, G'_2, \cdots, G'_t$ such that $i < j \implies n(G'_i) \leq n(G'_j)$. For each $1 \leq i \leq t$, let v_i be a centre of G'_i. If $t \leq \frac{\alpha}{2}$, then each G'_i is covered by $N_{\alpha-i}[v_i]$, since every $v \in V(G'_i)$ is distance at most $\lfloor \frac{\alpha-1}{2} \rfloor$ from v_i. Furthermore, if α is odd and $t = \frac{\alpha+1}{2}$, for each $1 \leq i \leq t-1$ G'_i can be covered by $N_{\alpha-i}[v_i]$ and G'_t can be covered by $N_{(\alpha-1)/2}[v_t] \cup N_1[x]$, where x is the end of the path G'_t furtherst from v_t. Since $\alpha \geq 5$, $\frac{\alpha-1}{2} \neq 1$ so these distances are distinct.

Thus, we need only consider $t \geq \lfloor \frac{\alpha+1}{2} \rfloor + 1 = \lfloor \frac{\alpha}{2} + \frac{3}{2} \rfloor$. Since we removed $\alpha - 1$ vertices from each of the t arms longer than $\alpha - 1$ (and possibly others from shorter arms),

$$
\begin{aligned}
n(G') &= n(G) - |N_{\alpha-1}[h]| \\
&\leq n(G) - (t(\alpha-1)+1) \\
&= \left(\sqrt{n(G)}\right)^2 - 1 - t(\alpha-1) \\
&= (\sqrt{n(G)}+1)(\sqrt{n(G)}-1) - t(\alpha-1) \\
&\leq (\alpha+1)(\alpha-1) - t(\alpha-1) \\
&= (\alpha-1)(\alpha+1-t) \qquad (2)
\end{aligned}
$$

Since $n(G') > 0$, then $t < \alpha + 1$. Furthermore, $t \neq \alpha$ since $n(G') \geq t$ and $n(G') \leq \alpha - 1$ (put in $\alpha = t$ into equation 2).

Suppose $t = \alpha - 1$. From equation 2 we see $n(G') \leq 2(\alpha - 1)$. If each of the t components has order 2, then there were $\alpha - 1$ arms of length $\alpha + 1$, and we have already covered this case.

So, suppose $\lfloor \frac{\alpha}{2} + \frac{3}{2} \rfloor \leq t \leq \alpha - 2$. And define an upper bound $f(t)$ on $b(G')$ as follows;

$$
\begin{aligned}
b(G') &\leq \left\lfloor \frac{n(G')}{2t} \right\rfloor + t \\
&\leq \left\lfloor \frac{(\alpha-1)(\alpha+1-t)}{2t} \right\rfloor + t \\
f(t) &= \left\lfloor \frac{\alpha^2-1}{2t} - \frac{\alpha-1}{2} \right\rfloor + t. \qquad (3)
\end{aligned}
$$

Treat $f(t)$ as a real-valued function on the domain $t \in [\lfloor \frac{\alpha}{2} + \frac{3}{2} \rfloor, \alpha - 2]$. Note that a t maximizing $f(t)$ will also maximise $g(t) = \frac{\alpha^2-1}{2t} - \frac{\alpha-1}{2} + t$. However, $g'(t) = -\frac{\alpha^2-1}{2t^2} + 1$ is monotone increasing on $[\lfloor \frac{\alpha}{2} + \frac{3}{2} \rfloor, \alpha - 2]$, so a critical point would be a local minimum. Thus, the maximum must be one of the endpoints.

Let $t = \alpha - 2$ and plug into equation 3,

$$\begin{aligned} b(G') &\leq \left\lfloor \frac{(\alpha-1)(\alpha+1-(\alpha-2))}{2(\alpha-2)} \right\rfloor + (\alpha - 2) \\ &= \left\lfloor \frac{\alpha+1}{2(\alpha-2)} \right\rfloor + \alpha - 2 \\ &\leq \alpha - 1. \end{aligned}$$

Let $t = \lfloor \frac{\alpha}{2} + \frac{3}{2} \rfloor$. If α is even, then $t = \frac{\alpha}{2} + 1$ and

$$\begin{aligned} b(G') &\leq \left\lfloor \frac{(\alpha-1)(\alpha+1-(\frac{\alpha}{2}+1))}{2(\frac{\alpha}{2}+1)} \right\rfloor + (\frac{\alpha}{2} + 1) \\ &= \left\lfloor \frac{(\alpha-1)\frac{\alpha}{2}}{\alpha+2} \right\rfloor + \frac{\alpha}{2} + 1 \\ &= \left\lfloor \frac{(\alpha+2)\frac{\alpha}{2} - \frac{3\alpha}{2}}{\alpha+2} \right\rfloor + \frac{\alpha}{2} + 1 \\ &= \left\lfloor \frac{-3\alpha}{2\alpha+4} \right\rfloor + \alpha + 1 \\ &\leq \alpha - 1. \end{aligned}$$

If α is odd, then $t = \frac{\alpha+1}{2} + 1$ and

$$\begin{aligned} b(G') &\leq \left\lfloor \frac{(\alpha-1)(\alpha+1-(\frac{\alpha+1}{2}+1))}{2(\frac{\alpha+1}{2}+1)} \right\rfloor + (\frac{\alpha+1}{2} + 1) \\ &= \left\lfloor \frac{(\alpha-1)\frac{\alpha+1}{2}}{\alpha+3} \right\rfloor + \frac{\alpha+1}{2} + 1 \\ &= \left\lfloor \frac{(\alpha+3)\frac{\alpha+1}{2} - \frac{4(\alpha+1)}{2}}{\alpha+3} \right\rfloor \frac{\alpha+1}{2} + 1 \\ &= \left\lfloor \frac{-4(\alpha+1)}{2\alpha+6} \right\rfloor + \alpha + 2 \\ &\leq \alpha - 1. \end{aligned}$$

Thus, by Corollary 2.2, $b(G) \leq \alpha$.

$\square$

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