Burning Spiders

Ryan DeWolfe

March 2025

Abstract

Burning a graph is a process that models the spread of information in networks. The process takes place over a discrete number of rounds, and the smallest number of round to burn a graph is the burning number of the graph. The burning number conjecture is that every graph with n vertices has a burning number at most $\lceil \sqrt{n} \rceil$. It has been shown that resolving the burning conjecture for all graphs is equivalent to resolving it for trees. A tree is called a spider if it has exactly one vertex with degree strictly greater than 2. This work summarizes recent progress by proving the burning number conjecture for spiders.

1 Introduction

The process of burning a graph was introduced in [4] to model the spread of information or epidemics on a graph. The burning process take place on graph G and in discrete rounds $t \geq 0$. At $t = 0$, all the vertices of a graph are unburned. In each round $t \geq 1$ there are two actions. First, every burned vertex burns all of its neighbors. Second, an unburned vertex is selected to be burned. The process ends when all vertices are burned.

The *burning number* of a graph, denoted $b(G)$, is defined to be the minimum number of rounds required to burn the graph G. Let $n : G \to \mathbb{N}$ be a function counting the number of vertices of G, defined as $n(G) = |V(G)|$. The *burning number conjecture* [4] is that for all connected graphs, $b(G) \leq \lceil \sqrt{n(G)}\, \rceil$. This conjecture is currently open, although there has been progress. we direct the reader to a the survey [2] for a general discussion. Norin and Turcotte [5] recently showed that the burning number conjecture hold asymptotically, and the best current bound is $\sqrt{4n(g)/3} + 1$ [1]. It was shown in [4] that if the burning number conjecture is satisfied in trees, then it is satisfied for all graphs. Paths acheive the bound $\lceil \sqrt{n}\, \rceil$, showing that if the burning number conjecture is true it is tight.

This report summarizes the contribution of [3] towards the burning number conjecture. A graph G is a *spider* if it is a connected tree and there is a single vertex with degree strictly greater than two. The main result is Theorem 3.2 (Theorem 7 of [3]), which simply confirms that spider graphs satisfy the burning number conjecture.

2 Preliminaries

For this report, we consider all graph to be simple, finite, and undirected. A *tree* is a graph that does not have any cycles, *forest* is a disjoint union of trees, and a *path-forest* is a disjoint union of paths. Since a path is a tree, a path-forest is also a forest, which is also a tree. Let G be a graph, $v \in V(G)$, and r a positive integer. Let $N_r[v]$ denote the closed neighborhood of v with radius r; defined as the set of vertices u such that there is a path from v to u with length less than or equal to r. By an abuse of notation, we use $G - N_r[v]$ to represent the graph induced by the set of vertices $V(G) \setminus N_r[v]$.

Lemma 2.1. *For any graph G, $b(G) \leq M$ for some integer M if and only if there exists an integer k, satisfying $1 \leq k \leq M$, such that*

$$V(G) = \bigcup_{i=1}^{k} N_{M-i}[v_i] \tag{1}$$

for some sequence of vertices $(v_i)_1^k$.

Proof. Let G be a graph with and $(v_i)_1^M$ a burning sequence for G. Then the fire started at vertex v_i in round i will spread to vertices in the neighborhood $N_{M-i}[v_i]$ by the end of round M. And since $(v_i)_1^M$ burns the graph, let $k = M$ and $V(G) = \cup_{i=1}^{M} N_{M-i}[v_i]$. This concludes the forward implication.

Let $(v_i)_1^k$ be a seeequence in $V(G)$ that satisfies equation 1. Let $(v_j)_1^M$ be a sequence in $V(G)$ where $v_j = v_i$ if $j \leq k$ and v_j is any vertex not yet chosen otherwise. The fire started at vertex v_j in round j will spread to $N_{M-j}[v_j]$ by round M. Since $(v_j)_1^M$ burns at least $\cup_{j=1}^{M} N_{M-j}[v_j] = V(G)$, it is a burning sequnce. Thus, $b(G) \leq M$. $\square$

Corollary 2.2. *Let G be a graph and $v \in V(G)$. Let α be a positive integer. If $b(G - N_\alpha[v]) = \alpha - 1$, then $b(G) \leq \alpha$.*

Proof. By applying the forward direction of Lemma 2.1, there exists some sequence of vertices such that $V(G - N_\alpha[v]) = \cup_{i=1}^{\alpha-1} N_{\alpha-1-i}[v_i]$. Let $v_\alpha = v$ and add $N_\alpha[v]$ to both sides to get $V(G) = \cup_{i=1}^{\alpha} N_{\alpha-i}[v_i]$. Thus, by the reverse direction of Lemma 2.1, $b(G) \leq \alpha$. $\square$

Corollary 2.2 means that we can bound the burning number of a graph by removing a "large" neighborhood with radius α and proving the leftovers can be burned in $\alpha - 1$ rounds. This method is central to proving many of the following statements.

Lemma 2.3. *If G is a path-forest with t components then $b(G) \leq \lfloor \frac{n(G)}{2t} \rfloor + t$.*

TODO: double check the $t \geq 1$ condition is just G being non-empty.

Lemma 2.4. *Let $\mathcal{G}$ be a set of connected graphs. Let $\hat{\mathcal{G}} \subseteq G$ with the following condition. For every $G \in \hat{\mathcal{G}}$, there exists $v \in V(G)$ and $r \leq \lceil \sqrt{n(G)}\,\rceil - 1$ such that at least one of the followng conditions are satisfied.*

1. $N_r[v] = V(G)$.
2. $|N_r[v]| \geq 2\lceil \sqrt{n(G)}\,\rceil - 1$ *and the induced subgraph* $G[V(G) \setminus N_r[v]]$ *is in* $\mathcal{G}$.

If $b(G) \leq \lceil \sqrt{n(G)}\,\rceil$ for all $G \in \mathcal{G} \backslash \hat{\mathcal{G}}$, then $b(G) \leq \lceil \sqrt{n(G)}\,\rceil$ for all $G \in \mathcal{G}$.

For proofs of Lemma 2.3 and Lemma 2.4, we refer to [3] (Lemmas 2 and 5 respectively).

3 Spiders

Recall that a graph G is a spider if it is a connected tree that has a single vertex with degree strictly greater than two. We call this vertex the *head* of the spider. An *arm* of a spider is a component of the path forest induced by deleting the head.

Lemma 3.1. *If G is a spider and $n(G) \leq 25$, then $b(G) \leq \lceil \sqrt{n(G)}\,\rceil$.*

Proof. Let G be a spider with head h. Let $\alpha = \lceil \sqrt{n(G)}\,\rceil$. If $\sqrt{n(G)}$ is not an integer, G is an induced graph of a spider H with $n(H) = \alpha^2$. If $(v_i)_1^\alpha$ is a burning sequnce for H, then $(u_j)_1^\alpha$ is a burning sequence for G, where $v_i = u_j$ if $v_i \in V(G)$, and u_j is the closest vertex to v_i that is in $V(G)$ otherwise. The chosen vertices are shifted up the arms towards the head, so the neighborhoods that they burn will overlap more, but will still cover the graph. Thus, we may assume $n(G)$ is a square integer: $4, 9, 16$, or 25.

The unique spider with $n(G) = 4$ is burned in two rounds by choosing the head to burn in round 1.

Suppose $n(G) = 9$. TODO: Why can we assume each arm has length at least 3? Invoke a path-forest lemma? If $G - N_2[h]$ has only one component, the component has length at most 4 which can be burned in two rounds. So, by Corollary 2.2 $b(G) \leq 3$. If $G - N - 2[h]$ has two components, the Lemma 2.3 implies $G - N_2[h]$ can be burned two rounds, and Corollary 2.2 shows $b(G) \leq 3$. Finally, $G - N_2[h]$ cannot have 3 or more components or else $n(G)$ would be at least 10.

Suppose $n(G) = 16$. If the head is burned first, every arm with length less than 4 will be burned by round 4. If an arm is length 7 or greater, since $2\lceil \sqrt{n(G)}\ \rceil - 1 = 7$, we can burn the end of that arm and apply Lemma 2.4. The path-forest $G - N_3[h]$ cannot have more than 3 components or else it would have at least $4 \cdot 4 + 1 = 17$ vertices. Since each component of the path-forest $G - N_3[h]$ has at most 3 vertices, if $G - N_3[h]$ has 1 or 2 components then Lemma 2.3 and Corollary 2.2 show $b(G) \leq 4$. Thus, we must only consider when $G - N_3[h]$ has 3 components. If $G - N_3[h]$ has less than 6 vertices, apply Lemma 2.3 and Corollary 2.2. And there are only two spiders with 3 arms with lengths between 4 and 6 where $n(G - N_3[h]) = 6$. Burning sequences are shown in Figure 1.

Suppose $n(G) = 25$. We use an analogus argument to $n(G) = 16$. If one arm has length greater than 8, $2\lceil \sqrt{n(G)} \rceil - 1 = 9$ so Lemma 2.4 applies with a smaller spider. The path-forest $G - N_4[h]$ cannot have more than 4 components since $5 * 5 + 1 > n(G)$. Since each component of $G - N_4[h]$ has at most 4 vertices, if $G - N_4[h]$ has only 1 or 2 components then Lemma 2.3 and Corollary 2.2 apply. If $G - N_4[h]$ has 3 components and less than 12 vertices, Lemma 2.3 and Corollary 2.2 apply. And there is only one spider where $G - N_4[h]$ has 3

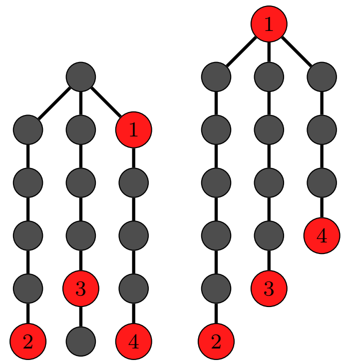

Figure 1: Burning sequences for the two spiders with $n(G) = 16$ and three arms with lengths at least 4 and at most 6.

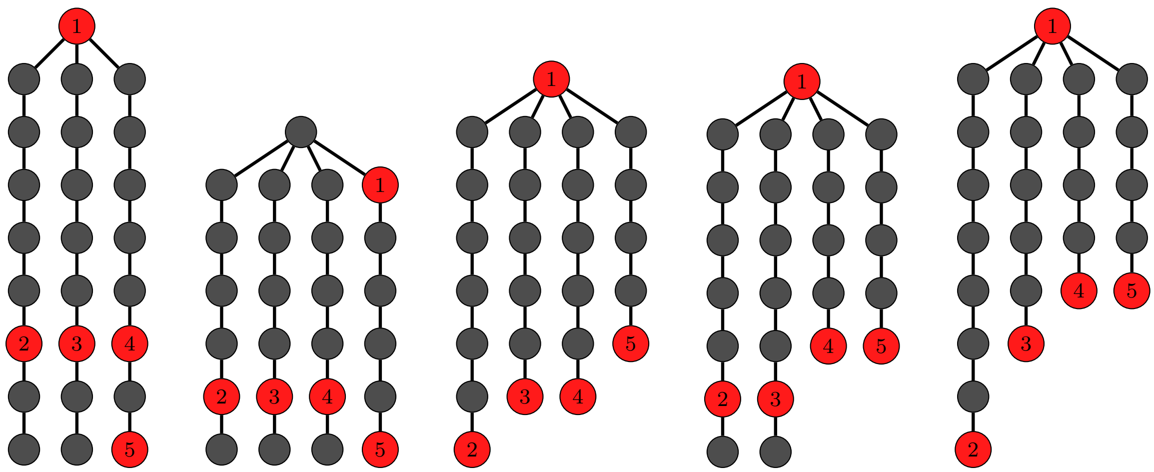

Figure 2: Burning sequnces for the five spiders with $n(G) = 25$ and at least 3 arms with length at least 5 and at most 8.

components and $n(G - N_4[h]) = 12$. If $G - N_4[h]$ has 4 components and less than 8 vertices, Lemma 2.3 and Corollary 2.2 apply. And there are four spiders where $G - N_4[h]$ has 4 components and $n(G - N_4[h]) = 8$. Burning sequences for these spiders are shown in Figure 2.

$\square$

Theorem 3.2. *If G is a spider then $b(G) \leq \lceil \sqrt{n(G)}\, \rceil$.*

Proof. Let G be a spider. If $n(G) \leq 25$, we apply Lemma 3.1 and conclude $b(G) \leq \lceil \sqrt{n(G)}\, \rceil$. So, suppose $n(G) > 25$ and let $\alpha = \lceil \sqrt{n(G)}\, \rceil$.

Suppose one arm of G has length ℓ and $\ell > 2\alpha - 2$. Then for $r = \lceil \frac{\ell}{2} \rceil$ and v the vertex on the arm at a distance r from the head, $|N_r[v]| = 2r + 1 = 2(\lceil \frac{\ell}{2} \rceil) + 1 \geq 2\alpha - 1$ and $G[V(G) \backslash N_r[v]]$ is a spider with one fewer arms. TODO: induct on number of arms until either no arms are long or it's only the head. Apply lemma 2.4 to { spiders with long arms }.

Thus, we must only consider every arm having length at most $2\alpha - 2$.

Suppose G has $\alpha - 1$ arms of length $\alpha + 1$. Let v be a vertex adjacent to the head. Then $G' = G[V(G) - N_{\alpha-1}[v]]$ is a path forest with $\alpha - 2$ paths of length 2 and one path of length 1. By applying Lemma 2.3 we conclude $b(G') \leq \alpha - 1$ which implies $b(G) \leq \alpha$.

Suppose otherwise; that is, G does not have $\alpha - 1$ arms of length $\alpha + 1$. Let $G' = G[V(G) \backslash N_{\alpha-1}(h)]$ where h is the head of the spider. The graph G' is a path forest, and since the arms of G have length at most $2\alpha - 2$, the components of G' have length at most $\alpha - 1$. Label the components $G'_1, G'_2, \cdots, G'_t$ such that $i < j \implies n(G'_i) \leq n(G'_j)$. For each $1 \leq i \leq t$, let v_i be a centre of G'_i. If $t \leq \frac{\alpha}{2}$, then each G'_i is covered by $N_{\alpha-i}[v_i]$, since every $v \in V(G'_i)$ is distance at most $\lfloor \frac{\alpha-1}{2} \rfloor$ from v_i.** Furthermore, if α is odd and $t = \frac{\alpha+1}{2}$, for each $1 \leq i \leq t-1$ G'_i can be covered by $N_{\alpha-i}[v_i]$ and G'_t can be covered by $N_{(\alpha-1)/2}[v_t] \cup N_1[x]$, where x is the end of the path G'_t furtherst from v_t. Since $\alpha \geq 5$, $\frac{\alpha-1}{2} > 1$ so these radii are distinct.

Thus, we need only consider $t \geq \lfloor \frac{\alpha+1}{2} \rfloor + 1 = \lfloor \frac{\alpha}{2} + \frac{3}{2} \rfloor$. We apply Lemma 2.3 to show that $b(G') \leq \alpha - 1$. Since we removed $\alpha - 1$ vertices from each of the t arms longer than $\alpha - 1$,

$$\begin{aligned} n(G') &= n(G) - |N_{\alpha-1}[h]| \\ &\leq n(G) - (t(\alpha-1)+1) \\ &= \left(\sqrt{n(G)}\right)^2 - 1 - t(\alpha-1) \\ &= (\sqrt{n(G)}+1)(\sqrt{n(G)}-1) - t(\alpha-1) \\ &\leq (\alpha+1)(\alpha-1) - t(\alpha-1) \\ &= (\alpha-1)(\alpha+1-t) \end{aligned} \tag{2}$$

Thus, $t < \alpha + 1$. Furthermore, $t \neq \alpha$ since $n(G') \geq t$ and $n(G') \leq \alpha - 1$ (From final line above plugging in $\alpha = t$ into the second bracket).

Suppose $t = \alpha - 1$. Applying equation (2) we see $n(G') \leq 2(\alpha - 1)$. If each of the t components has order 2, then there were $\alpha - 1$ arms of length $\alpha + 1$, and we have already covered this case.

So, suppose $\lfloor \frac{\alpha}{2} + \frac{3}{2} \rfloor \leq t \leq \alpha - 2$. And

$$\begin{aligned} b(G') &\leq \left\lfloor \frac{n(G)}{2t} \right\rfloor + t \\ &\leq \left\lfloor \frac{(\alpha-1)(\alpha+1-t)}{2t} \right\rfloor + t \\ f(t) &= \left\lfloor \frac{\alpha^2-1}{2t} - \frac{\alpha-1}{2} \right\rfloor + t \end{aligned} \tag{3}$$

Treat equation 3 as a real-valued function of t on the domain $t \in [\lfloor \frac{\alpha}{2} + \frac{3}{2} \rfloor, \alpha - 2]$. Note that a t maximizing $f(t)$ will also maximise $g(t) = \frac{\alpha^2-1}{2t} - \frac{\alpha-1}{2} + t$. However, $g'(t) = -\frac{\alpha^2-1}{2t^2} + 1 > 0$ for all $t \in [\lfloor \frac{\alpha}{2} + \frac{3}{2} \rfloor, \alpha - 2]$, so the maximum must be one of the endpoints.

Let $t = \alpha - 2$ and plug into 3, we see

$$\begin{aligned}
b(G') &\leq \left\lfloor \frac{(\alpha-1)(\alpha+1-(\alpha-2))}{2(\alpha-2)} \right\rfloor + (\alpha-2) \\
&= \left\lfloor \frac{\alpha+1}{2(\alpha-2)} \right\rfloor + \alpha - 2 \\
&\leq \alpha - 1
\end{aligned}$$

With the final inequality due to $\alpha > 5$ and $\frac{\alpha+1}{2(\alpha-2)}$ decreasing.

Let $t = \lfloor \frac{\alpha}{2} + \frac{3}{2} \rfloor$. If α is even, then $t = \frac{\alpha}{2} + 1$ and

$$\begin{aligned}
b(G') &\leq \left\lfloor \frac{(\alpha-1)(\alpha+1-(\frac{\alpha}{2}+1))}{2(\frac{\alpha}{2}+1)} \right\rfloor + (\frac{\alpha}{2}+1) \\
&= \left\lfloor \frac{(\alpha-1)\frac{\alpha}{2}}{\alpha+2} \right\rfloor + \frac{\alpha}{2} + 1 \\
&= \left\lfloor \frac{(\alpha+2)\frac{\alpha}{2} - \frac{3\alpha}{2}}{\alpha+2} \right\rfloor + \frac{\alpha}{2} + 1 \\
&= \left\lfloor \frac{-3\alpha}{2\alpha+4} \right\rfloor + \alpha + 1 \\
&\leq \alpha - 1
\end{aligned}$$

If α is odd, then $t = \frac{\alpha+1}{2} + 1$ and

$$\begin{aligned}
b(G') &\leq \left\lfloor \frac{(\alpha-1)(\alpha+1-(\frac{\alpha+1}{2}+1)}{2(\frac{\alpha+1}{2}+1)} \right\rfloor + (\frac{\alpha+1}{2}+1) \\
&= \left\lfloor \frac{(\alpha-1)\frac{\alpha+1}{2}}{\alpha+3} \right\rfloor + \frac{\alpha+1}{2} + 1 \\
&= \left\lfloor \frac{(\alpha+3)\frac{\alpha+1}{2} - \frac{4(\alpha+1)}{2}}{\alpha+3} \right\rfloor \frac{\alpha+1}{2} + 1 \\
&= \left\lfloor \frac{-4(\alpha+1)}{2\alpha+6} \right\rfloor \alpha + 2 \\
&\leq \alpha - 1
\end{aligned}$$

Thus, by Corollary 2.2, $b(G) \leq \alpha$.

□

References

[1] P. Bastide, M. Bonamy, A. Bonato, P. Charbit, S. Kamali, T. Pierron, and M. Rabie. Improved pyrotechnics : Closer to the burning graph conjecture. *arXiv*, 2022.

[2] A. Bonato. A survey of graph burning. *Contributions to Discrete Mathematics*, 16(1):185–197, 2022.

[3] A. Bonato and T. Lidbetter. Bounds on the burning numbers of spiders and path-forests. *Theoretical Computer Science*, 794:12–19, 2019.

[4] A. Bonato, J. Janssen, and E. Roshanbin. How to burn a graph. *Internet Mathematics*, 12(1-2):85–100, 2016.

[5] S. Norin and J. Turcotte. The burning number conjecture holds asymptotically. *Journal of Combinatorial Theory, Series B*, 168:208–235, 2024.