Burning Spiders

AM8204 - Topics in Discrete Mathematics

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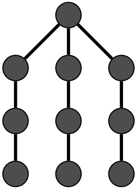
Motivation

- The burning number conjecture is true if it true for trees.
- Spiders are a subset of trees.
- We are following a proof from the paper

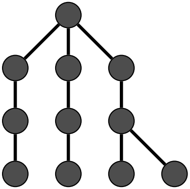
  A. Bonato and T. Lidbetter. Bounds on the burning numbers of spiders and path-forests. *Theoretical Computer Science*, 794:12-19, 2019.

Preliminaries

- A tree is a *spider* if it has exactly one vertex with degree strictly greater than 3.
- The unique vertex with degree strictly greater than 3 is called the *head* and is denoted h.



Yes



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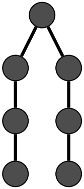

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Preliminaries

Lemma (Leftover Lemma)

Let G be a graph and $v \in V(G)$. Let k a positive integer. If $b(G - N_{k-1}[v]) \leq k - 1$ then $b(G) \leq k$.

This lemma is used in an inductive step in the proofs of the following lemmas and is used often in the main theorem.

Preliminaries

Lemma (Path-Forest Lemma)

If G is a path-forest with t components then $b(G) \leq \lfloor \frac{n(G)}{2t} \rfloor + t$.

Preliminaries

Lemma (Graph Family Lemma)

Let $\mathcal{G}$ be a set of connected graphs. Let $\hat{\mathcal{G}} \subseteq G$ with the following condition. For every $G \in \hat{\mathcal{G}}$, there exists $v \in V(G)$ and $r \leq \lceil \sqrt{n(G)} \, \rceil - 1$ such that at least one of the followng conditions are satisfied.

1. $N_r[v] = V(G)$.
2. $|N_r[v]| \geq 2\lceil \sqrt{n(G)} \, \rceil - 1$ *and the induced subgraph* $G[V(G) \setminus N_r[v]]$ *is in* $\mathcal{G}$.

If $b(G) \leq \lceil \sqrt{n(G)} \, \rceil$ for all $G \in \mathcal{G} \backslash \hat{\mathcal{G}}$, then $b(G) \leq \lceil \sqrt{n(G)} \, \rceil$ for all $G \in \mathcal{G}$.

Main Theorem

Theorem (Theorem 7 in Bonato and Lidbetter (2019))

If G is a spider then $b(G) \leq \lceil \sqrt{n(G)}\, \rceil$.

Proof of Main Theorem

Let $\alpha = \lceil \sqrt{n(G)} \rceil$. The proof procedes as follows:

1. Check all spides with at most 25 vertices.
2. Show the bound holds for spiders with a long arm(s) if it hold for spiders with short arms.
3. Remove $N_{\alpha[h]}$ and argue the remaining path-forest can be burned in $\alpha - 1$ rounds.

Check all spiders with at most 25 vertices.

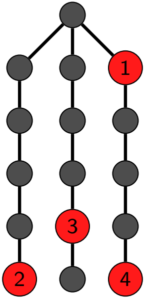
- If $\sqrt{n(G)}$ is not an integer, then G is a subgraph of a spider H with $n(H) = \alpha^2$.
- Given a burning sequence for H we can construct a burning sequence for G by "sliding" the neighborhoods up the arms (if necessary).

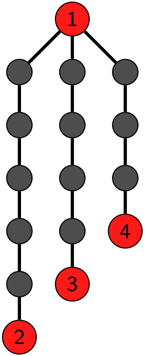
Check all spiders with at most 25 vertices.

- If $n(G) = 4$ there is only one spider and it can be burned in 2 round by choosing the head in round 1.
- Suppose $n(G) = 9$.
- If G has an arm longer than 5, that arm can be used as a neighborhood for the Graph Family lemma, and we take the family to include smaller spiders.
- If $G - N_2[h]$ cannot have more than 2 components or else it would need at least 10 vertices.
- If $G - N_2[h]$ has 1 or 2 compoents Path-Forest Lemma applies.

Check all spiders with at most 25 vertices.

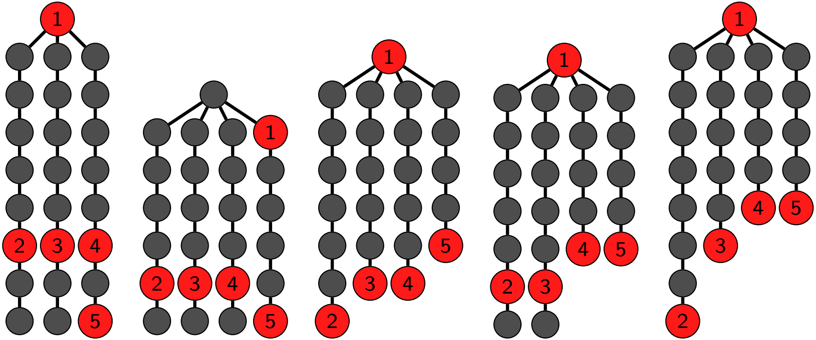
- Applying the same arguments as the $n(G) = 9$ case when $n(G) = 16$ covers all spiders except those with 3 arms with length at least 4 and at most 6.





Check all spiders with at most 25 vertices.

- The same argument when $n(G) = 25$ covers all cases except for at least 3 arms with lengths at least 5 and at most 8.
- If there are 3 arms, $G - N_4[h]$ must have 12 vertices, and if there are 4 arms $G - N_4[h]$ must have 8 vertices.



Long Arms

- If one arm of G has length $\ell > 2\alpha - 2$, then there is a neighborhood on that arm to satisfy the Graph Family Lemma.
- $|N_{\lceil \ell/2 \rceil}[v]| = 2(\lceil \frac{\ell}{2} \rceil) + 1 \geq 2\alpha - 1$ for vertex v that is distance $\lceil \frac{\ell}{2} \rceil$ from the end of the arm.
- We can inductively remove a long arm until we are left with $n(G) \leq 25$ or no long arms.

Remove $N_\alpha[h]$

- Let $G' = G - N_\alpha[h]$.
- G' is a path forest whose components have length at most $\alpha - 1$.
- If there are $\leq \frac{\alpha}{2}$ components, then each component can be covered by a neighborhood of size $\alpha - i$, $1 \leq i \leq t$.
- If α is odd, and $t = \frac{\alpha+1}{2}$, the tth compnent can be covered with neighborhood of size $\frac{\alpha-1}{2}$ and 1.

Remove $N_\alpha[h]$

- So, $t \geq \lfloor \frac{\alpha+1}{2} \rfloor + 1$.
- Since at least $\alpha - 1$ arms got removed for each compnent in G':

$$\begin{aligned} n(G') &= n(G) - |N_{\alpha-1}[h]| \\ &\leq n(G) - (t(\alpha-1)+1) \\ &\leq (\alpha-1)(\alpha+1-t). \end{aligned}$$

Remove $N_\alpha[h]$

$$n(G') \leq (\alpha - 1)(\alpha + 1 - t)$$

- Since $n(G') > 0$, $t < \alpha + 1$.
- Since $n(G') \geq t$, $t \neq \alpha$.
- If $t = \alpha - 1$ we can apply the Path-Forest Lemma to G.
- So $\lfloor \frac{\alpha}{2} \rfloor + 1 \leq t \leq \alpha - 2$.

Remove $N_\alpha[h]$

- From the Path-Forest Lemma,

$$\begin{aligned} b(G') &\leq \left\lfloor \frac{n(G')}{2t} \right\rfloor + t \\ &\leq \lfloor \frac{\alpha^2 - 1}{2t} - \frac{\alpha - 1}{2} \rfloor + t \end{aligned}$$

- We treat this as a real valued function and maximize, the maximum is at an endpoint of the interval.
- Both endpoints are bounded above by $\alpha - 1$.
- So, $b(G') \leq \alpha - 1$ and we apply the Leftover Lemma.

Conclusion

- We showed that spiders satisfy the burning number conjecture.
- The proof was originally published in the following paper:

  A. Bonato and T. Lidbetter. Bounds on the burning numbers of spiders and path-forests. *Theoretical Computer Science*, 794:12-19, 2019.