Correlation energy tables for the effective potential

The effective potential is:

$V(r) = \frac{1}{r} + \sum_{i=0}^{imax} \frac{1}{l} c_i \left(\frac{r}{l}\right)^i e^{-\left(\frac{r}{l}\right)^2}$.

Plug this into the Archer-Jain equation:

$$U \simeq \frac{e^2}{2\,\epsilon\,l} \sum_{i \neq j} \frac{\int_0^\infty V^{eff}\, \mathrm{Exp}\left[-\frac{r^2}{4\,l^2}\right]\left[I_0\left(\frac{r\,R_{ij}}{2\,l^2}\right) - J_0\left(\frac{r\,R_{ij}}{2\,l^2}\right)\right] r\,dr}{4\,l\,\sinh\left(\frac{R_{ij}^2}{4\,l^2}\right)}$$

to get the QM two-body energy.

The correlation energy calculation is the sum over all possible R_{ij}.

Reference: "Maki-Zotos Correlation Energy Fig. 2 (7-27-13).nb"

Let a_0 be the distance between electrons, i.e. the lattice constant. The unit cell is a parallelogram with an electron at its center. This is the same parallelogram with an electron at each corner. The area of the unit cell is $a_c = \frac{\sqrt{3}}{2} {a_0}^2$.
Let N be the total number of electrons.

The charge density can be written as: $\rho = \frac{N}{A} = \frac{N}{N \cdot a_c} = \frac{1}{a_c}$. Insert this into the definition of the filling factor, $\nu = 2\,\pi\,l^2\,\rho$.

$\nu = 2\,\pi\,l^2\,\rho = \frac{2\,\pi\,l^2}{a_c}$. Rearranging, $a_c = \frac{2\,\pi\,l^2}{\nu}$.

Then express a_0 in terms of ν.

$\frac{\sqrt{3}}{2} {a_0}^2 = \frac{2\,\pi\,l^2}{\nu}$

$a_0 = \sqrt{\frac{2\,\pi\,l^2}{\nu}\,\frac{2}{\sqrt{3}}} = \sqrt{\frac{4\,\pi\,l^2}{\sqrt{3}\,\nu}}$

Reference: "Bonsall, Maradudin eq. 2.18 derivation (5-17-14).nb"

Using our more accurate calculations, the Bonsall-Maradudin result for the ground-state correlation energy is
$E_G = \frac{1}{2} E_I = \frac{1}{2} (-3.92103157863978\ldots) \frac{e^2}{\sqrt{a_c}} = -0.78213263975429\ldots \frac{e^2}{l} \sqrt{\nu}$. They had -3.921034.
I will use 8 digits, i.e. -0.78213264

The direct lattice vector of a hexagonal lattice is given by eq. 5 in Maki-Zotos:
$\overrightarrow{R_j} = a_0 \left[n + \frac{m}{2}, \frac{\sqrt{3}}{2} m \right]$

Set the origin at <0,0>, i.e. m=0, n=0. The reference electron will be set at the origin. The equation produces all the points of a parallelogram. The total shape will be a parallelogram. The slope of the slant is $\sqrt{3}$. For symmetry, m will run from $-m_{\max} \ldots m_{\max}$ and n will run from $-n_{\max} \ldots n_{\max}$. There are a total of $(2\, m_{\max} + 1)\,(2\, n_{\max} + 1)$ electrons.

$|\overrightarrow{R_i}|$, the distance from the center electron to all other electrons, can be expressed as (where $l \to 1$; the l cancels out with l's in the denominators of the AJ equations):

```
R[m_, n_, ν_] := Sqrt[4 π / (Sqrt[3] ν)] Sqrt[(n + m/2)^2 + (Sqrt[3]/2 m)^2]
```

But in order to have a circular disk shape around the reference electron for maximum symmetry, we will have to limit the points to those within a certain radius. A circle inscribed within a rhombus (where $m_{\max} = n_{\max}$) will have radius $r = a_0 \frac{\sqrt{3}}{2} m_{\max}$.

```
circleradius[msize_, ν_] := Sqrt[4 π / (Sqrt[3] ν)] Sqrt[3]/2 msize
```

Example:

```
circleradius[2, 4 π / Sqrt[3]]
```

$\sqrt{3}$

We count the total number of electrons. We start with count=1 since the reference electron at the origin will not be included in the calculations. Arbitrarily set $\nu = 1$ (ν just scales the distances). Putting the count++ inside the energy summation doesn't work when creating the table (not logical anyway). So we just find the number of electrons inside the circle separately. See "Electron Lattice.nb" in the LateX file (or somewhere else) for a graphical depiction.

```
numberofelectrons[msize_, nsize_] :=
 For[m = -msize, m ≤ msize, m++, For[n = -nsize, n ≤ nsize, n++, If[R[m, n, 1] ≤ cradius, If[m == 0 && n == 0, 0, count++;], 0]]]
```

```
count = 1; cradius = circleradius[2, 1]; numberofelectrons[2, 2]; count
```

```
13
```

```
count = 1; cradius = circleradius[3, 1]; numberofelectrons[3, 3]; count
```

```
19
```

```
count = 1; cradius = circleradius[50, 1]; numberofelectrons[50, 50]; count
```

```
6793
```

```
count = 1; cradius = circleradius[100, 1]; numberofelectrons[100, 100]; count
```

```
27 181
```

```
count = 1; cradius = circleradius[200, 1]; numberofelectrons[200, 200]; count
```

```
108 787
```

```
count = 1; cradius = circleradius[650, 1]; numberofelectrons[650, 650]; count
```

```
1 149 427
```

```
count = 1; cradius = circleradius[1000, 1]; numberofelectrons[1000, 1000]; count
```

```
2 720 557
```

$$\frac{\sqrt{3}}{2} * 650 \text{ // } N$$

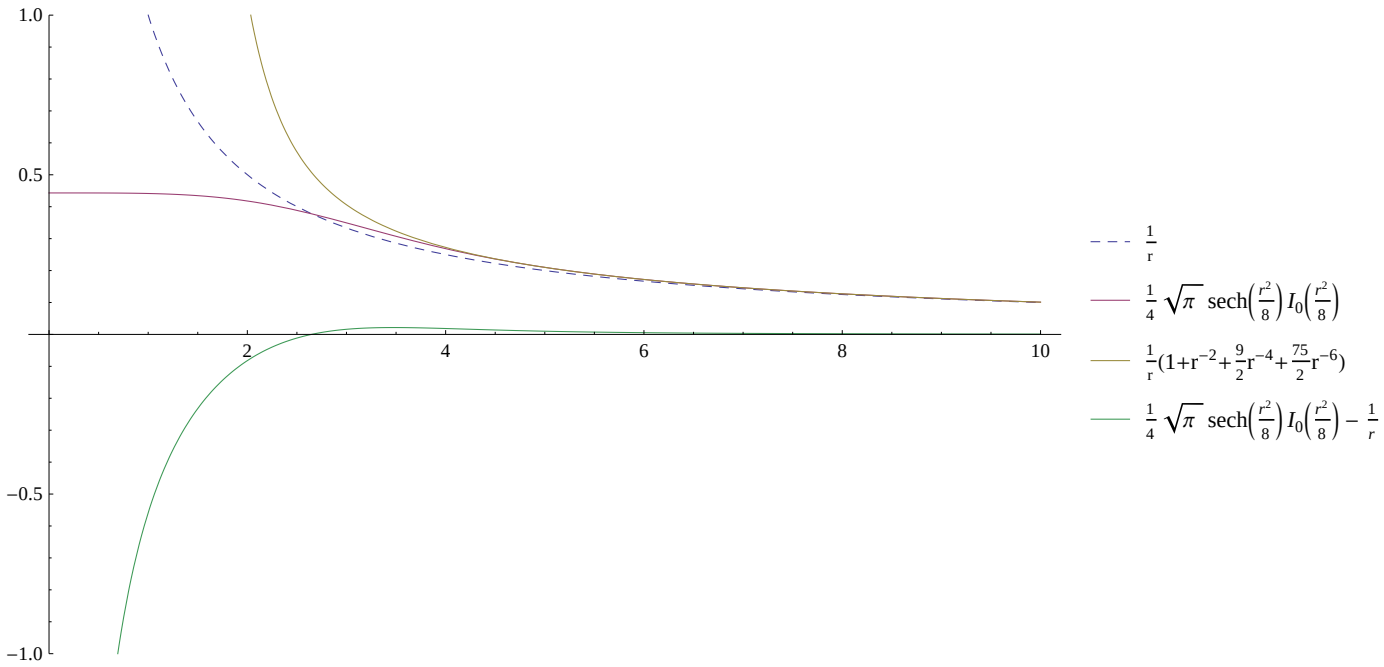
```
562.917
```

```
circleradius[650, ν] // N
```

$$1516.24 \sqrt{\frac{1}{\nu}}$$

Plots

```
Plot[{1/r, Sqrt[π]/4 Sech[r^2/8] BesselI[0, r^2/8], 1/r (1 + r^-2 + 9/2 r^-4 + 75/2 r^-6), Sqrt[π]/4 Sech[r^2/8] BesselI[0, r^2/8] - 1/r},
 {r, 0, 10}, PlotLegends → {1/r, TraditionalForm[Sqrt[π]/4 Sech[r^2/8] BesselI[0, r^2/8]],
   "1/r(1+r^-2+9/2 r^-4+75/2 r^-6)", TraditionalForm[Sqrt[π]/4 Sech[r^2/8] BesselI[0, r^2/8] - 1/r]},
 PlotRange → {-1, 1}, PlotStyle → {Dashed, PointSize[1], PointSize[1], PointSize[1]}]
```

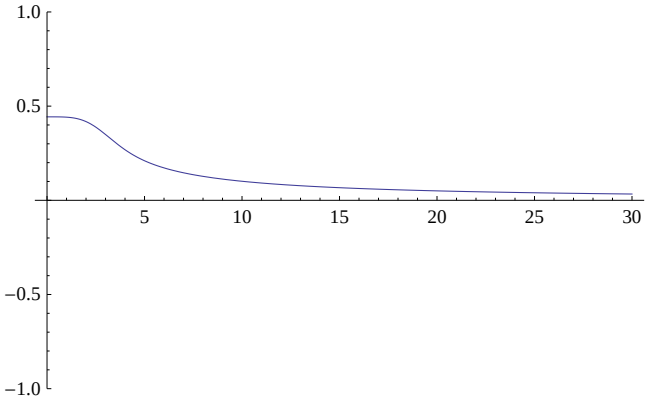


1.) The first term in the effective potential, $\frac{1}{r}$, gives the Maki-Zotos result.

```
Assuming[Rij ∈ Reals && Rij > 0 && l ∈ Reals && l > 0,
  (∫0^∞ (1/r) Exp[-r^2/(4 l^2)] (BesselI[0, r Rij/(2 l^2)] - BesselJ[0, r Rij/(2 l^2)]) r dr) / (4 l Sinh[Rij^2/(4 l^2)])] // FullSimplify
```

$$\frac{1}{4}\sqrt{\pi}\ \text{BesselI}\left[0,\ \frac{R_{ij}^2}{8\,l^2}\right]\ \text{Sech}\left[\frac{R_{ij}^2}{8\,l^2}\right]$$

```
Plot[(1/4) Sqrt[π] BesselI[0, Rij^2/(8 l^2)] Sech[Rij^2/(8 l^2)] /. l → 1, {Rij, 0, 30}, PlotRange → 1]
```



The correlation energy as a function of filling factor is given by Fig. 2 in Maki-Zotos. Their ℝ" is actually R/l. All distances are from the origin. Set l=1.

```
UCorMZ[msize_, nsize_, ν_] := -0.78213264 ν^(1/2) + (1/2) Sum[Sum[If[
    R[m, n, ν] ≤ circleradius[msize, ν], If[m == 0 && n == 0, 0, (R[m, n, ν])^-3 + (9/2) (R[m, n, ν])^-5 + (75/2) (R[m, n, ν])^-7], 0], {n, -nsize, nsize}], {m, -msize, msize}]
```

```
TableMZ200 = Parallelize[Table[{v, UCorMZ[200, 200, v]}, {v, 0.01, 1.0, 0.01}]]
```

```
{{0.01, -0.0779309}, {0.02, -0.109809}, {0.03, -0.133991}, {0.04, -0.154141}, {0.05, -0.171683},
 {0.06, -0.187349}, {0.07, -0.201575}, {0.08, -0.214646}, {0.09, -0.226761}, {0.1, -0.238063}, {0.11, -0.248663},
 {0.12, -0.258645}, {0.13, -0.268076}, {0.14, -0.277012}, {0.15, -0.285497}, {0.16, -0.29357}, {0.17, -0.301261},
 {0.18, -0.308598}, {0.19, -0.315605}, {0.2, -0.322302}, {0.21, -0.328705}, {0.22, -0.334831}, {0.23, -0.340694},
 {0.24, -0.346305}, {0.25, -0.351675}, {0.26, -0.356814}, {0.27, -0.361731}, {0.28, -0.366433},
 {0.29, -0.370926}, {0.3, -0.375219}, {0.31, -0.379315}, {0.32, -0.383221}, {0.33, -0.386941}, {0.34, -0.390479},
 {0.35, -0.39384}, {0.36, -0.397025}, {0.37, -0.40004}, {0.38, -0.402887}, {0.39, -0.405567}, {0.4, -0.408085},
 {0.41, -0.410441}, {0.42, -0.412638}, {0.43, -0.414678}, {0.44, -0.416562}, {0.45, -0.418292}, {0.46, -0.419869},
 {0.47, -0.421293}, {0.48, -0.422567}, {0.49, -0.423691}, {0.5, -0.424666}, {0.51, -0.425492}, {0.52, -0.42617},
 {0.53, -0.426701}, {0.54, -0.427085}, {0.55, -0.427321}, {0.56, -0.427412}, {0.57, -0.427356}, {0.58, -0.427153},
 {0.59, -0.426805}, {0.6, -0.42631}, {0.61, -0.425669}, {0.62, -0.424881}, {0.63, -0.423946}, {0.64, -0.422865},
 {0.65, -0.421636}, {0.66, -0.420259}, {0.67, -0.418734}, {0.68, -0.417061}, {0.69, -0.415239}, {0.7, -0.413267},
 {0.71, -0.411145}, {0.72, -0.408872}, {0.73, -0.406448}, {0.74, -0.403871}, {0.75, -0.401142}, {0.76, -0.39826},
 {0.77, -0.395223}, {0.78, -0.392031}, {0.79, -0.388684}, {0.8, -0.385179}, {0.81, -0.381517}, {0.82, -0.377697},
 {0.83, -0.373717}, {0.84, -0.369576}, {0.85, -0.365275}, {0.86, -0.360811}, {0.87, -0.356183}, {0.88, -0.351391},
 {0.89, -0.346434}, {0.9, -0.34131}, {0.91, -0.336018}, {0.92, -0.330558}, {0.93, -0.324927}, {0.94, -0.319126},
 {0.95, -0.313152}, {0.96, -0.307005}, {0.97, -0.300683}, {0.98, -0.294185}, {0.99, -0.28751}, {1., -0.280657}}
```

```
TableMZ200 = {{0.01`, -0.07793093520809645`}, {0.02`, -0.10980858137349987`}, {0.03`, -0.1339907025500622`},
   {0.04`, -0.15414078889413646`}, {0.05`, -0.17168262396838774`}, {0.060000000000000005`, -0.1873485560780465`},
   {0.07`, -0.2015747635110417`}, {0.08`, -0.21464608283319134`}, {0.09`, -0.22676073986300332`}, {0.1`, -0.23806328340292518`},
   {0.11`, -0.2486629605731652`}, {0.12`, -0.258644706485713`}, {0.13`, -0.2680760837520454`}, {0.14`, -0.27701186047259196`},
   {0.15`, -0.2854971411648267`}, {0.16`, -0.2935695737473291`}, {0.17`, -0.3012609457894361`}, {0.18`, -0.3085983649281623`},
   {0.19`, -0.3156051488111101`}, {0.2`, -0.3223015075409584`}, {0.21000000000000002`, -0.32870507494132806`},
   {0.22000000000000003`, -0.33483132772974455`}, {0.23`, -0.3406939202648525`}, {0.24000000000000002`, -0.346304954802587`},
   {0.25`, -0.351675201856043`}, {0.26`, -0.35681428150017397`}, {0.27`, -0.36173081378116734`},
   {0.28`, -0.36643254444676887`}, {0.29000000000000004`, -0.3709264507859345`}, {0.30000000000000004`, -0.3752188313041423`},
   {0.31000000000000005`, -0.37931538216169597`}, {0.32000000000000006`, -0.38322126269489165`},
   {0.33`, -0.3869411518735364`}, {0.34`, -0.39047929718695984`}, {0.35000000000000003`, -0.3938395571683089`},
   {0.36`, -0.39702543854452854`}, {0.37`, -0.4000401288229616`}, {0.38`, -0.4028865249844855`}, {0.39`, -0.40556725883967826`},
   {0.4`, -0.40808471951269876`}, {0.41000000000000003`, -0.41044107344282077`}, {0.42000000000000004`, -0.412638282232375`},
   {0.43000000000000005`, -0.4146781186194891`}, {0.44000000000000006`, -0.41656218081236795`}, {0.45`, -0.41829190538723876`},
   {0.46`, -0.41986857892319657`}, {0.47000000000000003`, -0.42129334852295136`}, {0.48`, -0.42256723134809404`},
   {0.49`, -0.4236911232802583`}, {0.5`, -0.42466580680494`}, {0.51`, -0.4254919582022811`}, {0.52`, -0.4261701541185108`},
   {0.53`, -0.4267008775826016`}, {0.54`, -0.42708452352488646`}, {0.55`, -0.4273214038476129`},
   {0.56`, -0.42741175209157223`}, {0.5700000000000001`, -0.4273557277378793`}, {0.5800000000000001`, -0.42715342017956587`},
   {0.5900000000000001`, -0.4268048523938097`}, {0.6000000000000001`, -0.42630998434227285`}, {0.61`, -0.4256687161240633`},
   {0.62`, -0.42488089090327286`}, {0.63`, -0.4239462976307508`}, {0.64`, -0.4228646735777781`},
   {0.65`, -0.42163570669753264`}, {0.66`, -0.4202590378286666`}, {0.67`, -0.4187342627539158`},
   {0.68`, -0.41706093412543654`}, {0.6900000000000001`, -0.41523856326744757`}, {0.7000000000000001`, -0.41326662186578206`},
   {0.7100000000000001`, -0.41114454355306335`}, {0.7200000000000001`, -0.4088717253974361`}, {0.73`, -0.40644752930208555`},
   {0.74`, -0.40387128332211863`}, {0.75`, -0.4011422829048445`}, {0.76`, -0.39825979205893225`},
   {0.77`, -0.39522304445751516`}, {0.78`, -0.3920312444798367`}, {0.79`, -0.38868356819568833`},
   {0.8`, -0.38517916429653387`}, {0.81`, -0.3815171549769051`}, {0.8200000000000001`, -0.3776966367693607`},
   {0.8300000000000001`, -0.3737166813360692`}, {0.8400000000000001`, -0.369576336219798`}, {0.85`, -0.3652746255569272`},
   {0.86`, -0.36081055075486346`}, {0.87`, -0.3561830911360987`}, {0.88`, -0.351391204550941`},
   {0.89`, -0.3464338279608501`}, {0.9`, -0.34130987799413287`}, {0.91`, -0.33601825147564635`},
   {0.92`, -0.33055782593203753`}, {0.93`, -0.3249274600739376`}, {0.9400000000000001`, -0.31912599425644717`},
   {0.9500000000000001`, -0.3131522509191286`}, {0.9600000000000001`, -0.30700503500667276`}, {0.97`, -0.30068313437130895`},
   {0.98`, -0.29418532015795423`}, {0.99`, -0.28751034717306134`}, {1.`, -0.28065695423801607`}};
```

```
TableMZ650 = Parallelize[Table[{ν, UCorMZ[650, 650, ν]}, {ν, 0.01, 1.0, 0.01}]]
```

```
{{0.01, -0.0779302}, {0.02, -0.109806}, {0.03, -0.133987}, {0.04, -0.154135}, {0.05, -0.171674},
 {0.06, -0.187338}, {0.07, -0.201561}, {0.08, -0.214629}, {0.09, -0.226741}, {0.1, -0.23804}, {0.11, -0.248636},
 {0.12, -0.258614}, {0.13, -0.268041}, {0.14, -0.276973}, {0.15, -0.285454}, {0.16, -0.293522},
 {0.17, -0.301209}, {0.18, -0.308542}, {0.19, -0.315544}, {0.2, -0.322235}, {0.21, -0.328634}, {0.22, -0.334755},
 {0.23, -0.340612}, {0.24, -0.346218}, {0.25, -0.351582}, {0.26, -0.356716}, {0.27, -0.361627}, {0.28, -0.366323},
 {0.29, -0.370811}, {0.3, -0.375097}, {0.31, -0.379187}, {0.32, -0.383087}, {0.33, -0.3868}, {0.34, -0.390332},
 {0.35, -0.393686}, {0.36, -0.396865}, {0.37, -0.399873}, {0.38, -0.402713}, {0.39, -0.405387}, {0.4, -0.407897},
 {0.41, -0.410246}, {0.42, -0.412436}, {0.43, -0.414469}, {0.44, -0.416346}, {0.45, -0.418068}, {0.46, -0.419637},
 {0.47, -0.421054}, {0.48, -0.42232}, {0.49, -0.423437}, {0.5, -0.424403}, {0.51, -0.425222}, {0.52, -0.425892},
 {0.53, -0.426415}, {0.54, -0.42679}, {0.55, -0.427019}, {0.56, -0.427101}, {0.57, -0.427036}, {0.58, -0.426826},
 {0.59, -0.426469}, {0.6, -0.425965}, {0.61, -0.425315}, {0.62, -0.424519}, {0.63, -0.423575}, {0.64, -0.422485},
 {0.65, -0.421247}, {0.66, -0.419861}, {0.67, -0.418327}, {0.68, -0.416645}, {0.69, -0.414813}, {0.7, -0.412832},
 {0.71, -0.410701}, {0.72, -0.408418}, {0.73, -0.405985}, {0.74, -0.403399}, {0.75, -0.40066}, {0.76, -0.397768},
 {0.77, -0.394722}, {0.78, -0.39152}, {0.79, -0.388162}, {0.8, -0.384648}, {0.81, -0.380976}, {0.82, -0.377146},
 {0.83, -0.373155}, {0.84, -0.369005}, {0.85, -0.364693}, {0.86, -0.360219}, {0.87, -0.355581}, {0.88, -0.350779},
 {0.89, -0.345811}, {0.9, -0.340676}, {0.91, -0.335374}, {0.92, -0.329903}, {0.93, -0.324262}, {0.94, -0.31845},
 {0.95, -0.312465}, {0.96, -0.306307}, {0.97, -0.299974}, {0.98, -0.293465}, {0.99, -0.286779}, {1., -0.279915}}
```

```
TableMZ650 = {{0.01`, -0.07793019304999699`}, {0.02`, -0.1098064822331976`}, {0.03`, -0.13398684618271006`},
   {0.04`, -0.1541348516276189`}, {0.05`, -0.17167432638537689`}, {0.060000000000000005`, -0.18733764862086125`},
   {0.07`, -0.20156101854271632`}, {0.08`, -0.2146292897010321`}, {0.09`, -0.2267407015788203`}, {0.1`, -0.23803981428272286`},
   {0.11`, -0.24863588448752794`}, {0.12`, -0.2586138555200528`}, {0.13`, -0.2680412971536218`}, {0.14`, -0.27697298380504226`},
   {0.15`, -0.28545402561707944`}, {0.16`, -0.2935220755600846`}, {0.17`, -0.30120892577297587`}, {0.18`, -0.3085416880523314`},
   {0.19`, -0.31554368385319265`}, {0.2`, -0.3222351267806091`}, {0.21000000000000002`, -0.3286336538941972`},
   {0.22000000000000003`, -0.3347547449132541`}, {0.23`, -0.34061205699093955`}, {0.24000000000000002`, -0.3462176949932565`},
   {0.25`, -0.35158243187836224`}, {0.26`, -0.3567158900179559`}, {0.27`, -0.36162669162107025`},
   {0.28`, -0.3663225844769226`}, {0.29000000000000004`, -0.37081054780551376`}, {0.30000000000000004`, -0.3750968819425495`},
   {0.31000000000000005`, -0.37918728478622454`}, {0.32000000000000006`, -0.3830869173259023`},
   {0.33`, -0.38680046010631697`}, {0.34`, -0.39033216211955507`}, {0.35000000000000003`, -0.39368588333469856`},
   {0.36`, -0.3968651318526171`}, {0.37`, -0.3998730964969065`}, {0.38`, -0.4027126755109505`},
   {0.39`, -0.40538650191764825`}, {0.4`, -0.4078969660065342`}, {0.41000000000000003`, -0.41024623533826565`},
   {0.42000000000000004`, -0.4124362725952656`}, {0.43000000000000005`, -0.4144688515569357`},
   {0.44000000000000006`, -0.41634557143620776`}, {0.45`, -0.41806786977957844`}, {0.46`, -0.41963703410387976`},
   {0.47000000000000003`, -0.4210542124188056`}, {0.48`, -0.4223204227638208`}, {0.49`, -0.42343656187085105`},
   {0.5`, -0.42440341304951323`}, {0.51`, -0.42522165337921847`}, {0.52`, -0.42589186028183446`},
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Export["/home/ryan/Katsuaki's Files/Physics_MS_Thesis/Calculations/Effective Potentials (new)/TableMZ650.dat", TableMZ650]

/home/ryan/Katsuaki's Files/Physics_MS_Thesis/Calculations/Effective Potentials (new)/TableMZ650.dat
```
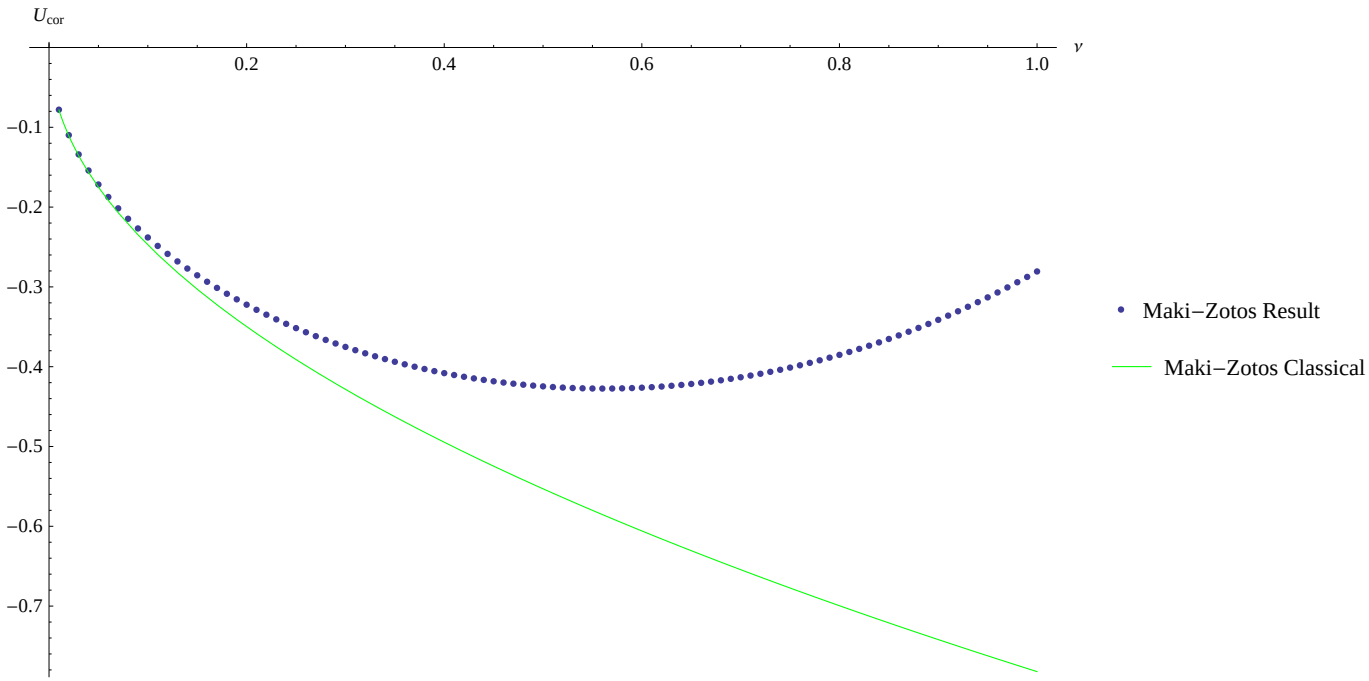
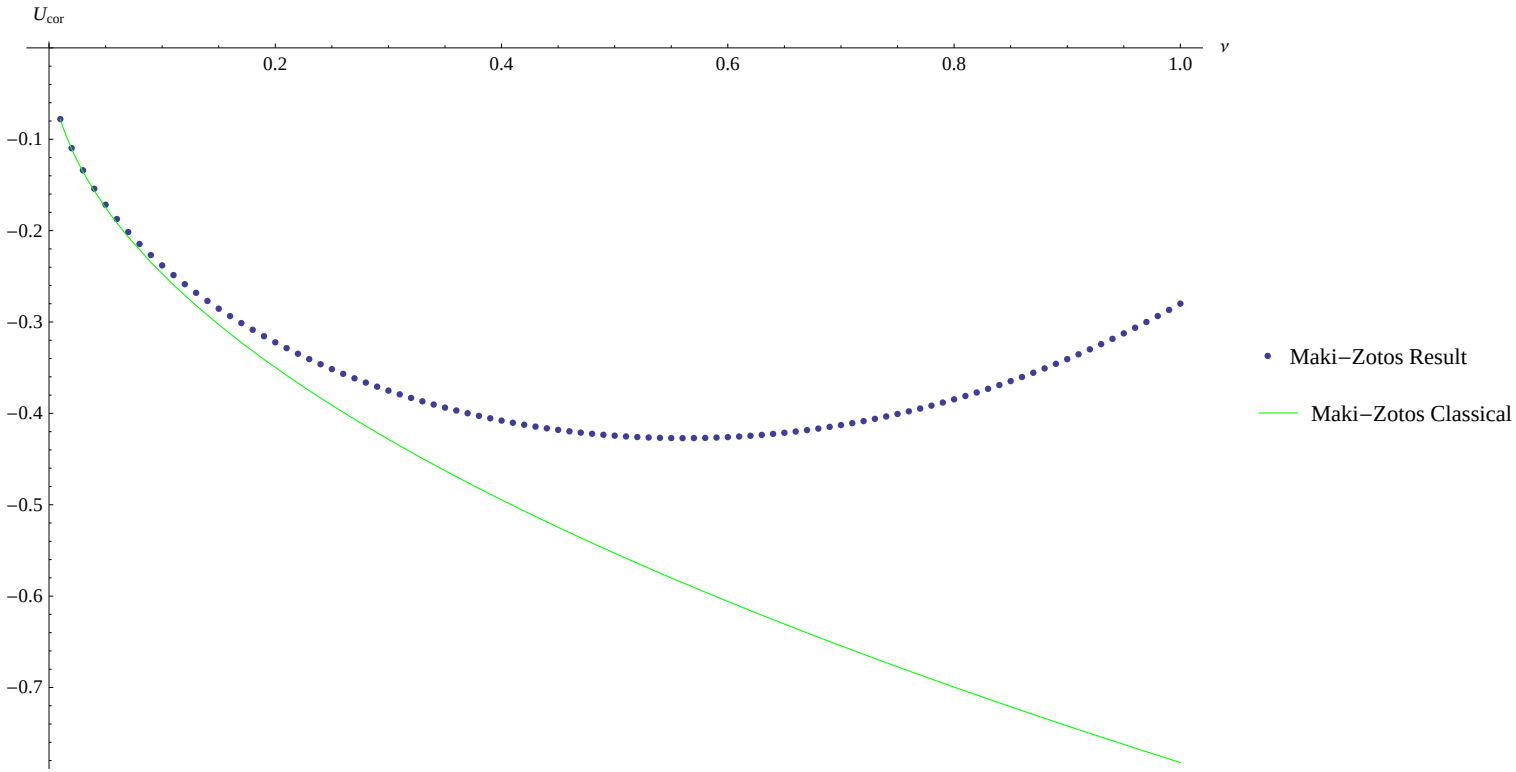

```
Show[ListPlot[TableMZ200, PlotLegends → {"Maki-Zotos Result"}],
 Plot[-0.78213264 √ν , {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```



```
Show[ListPlot[TableMZ650, PlotLegends → {"Maki-Zotos Result"}],
 Plot[-0.78213264 √ν , {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```



(On the plot, the mmax=nmax=200 points are indistinguishable from the mmax=nmax=650 points.)

Take a look at what happens after $\nu = 1.0$:

```
TableMZ200ext = Parallelize[Table[{v, UCorMZ[200, 200, v]}, {v, 0.01, 5.0, 0.02}]]
```

```
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{4.329999999999999`, 24.169341471754375`}, {4.35`, 24.54185255082099`}, {4.369999999999999`, 24.91833424471992`},
{4.39`, 25.298811930745376`}, {4.409999999999999`, 25.683311042456157`}, {4.43`, 26.07185706954055`},
{4.45`, 26.464475557682235`}, {4.47`, 26.86119210842741`}, {4.49`, 27.262032379052464`}, {4.51`, 27.6670220824332`},
{4.53`, 28.076186986914955`}, {4.55`, 28.48955291618326`}, {4.57`, 28.907145749136443`}, {4.59`, 29.328991419758186`},
{4.609999999999999`, 29.75511591699148`}, {4.63`, 30.18554528461385`}, {4.6499999999999995`, 30.620305621113097`},
{4.67`, 31.05942307956363`}, {4.6899999999999995`, 31.502923867504805`}, {4.71`, 31.95083424681901`},
{4.7299999999999995`, 32.4031805336114`}, {4.749999999999999`, 32.85998909809004`}, {4.77`, 33.321286364447126`},
```
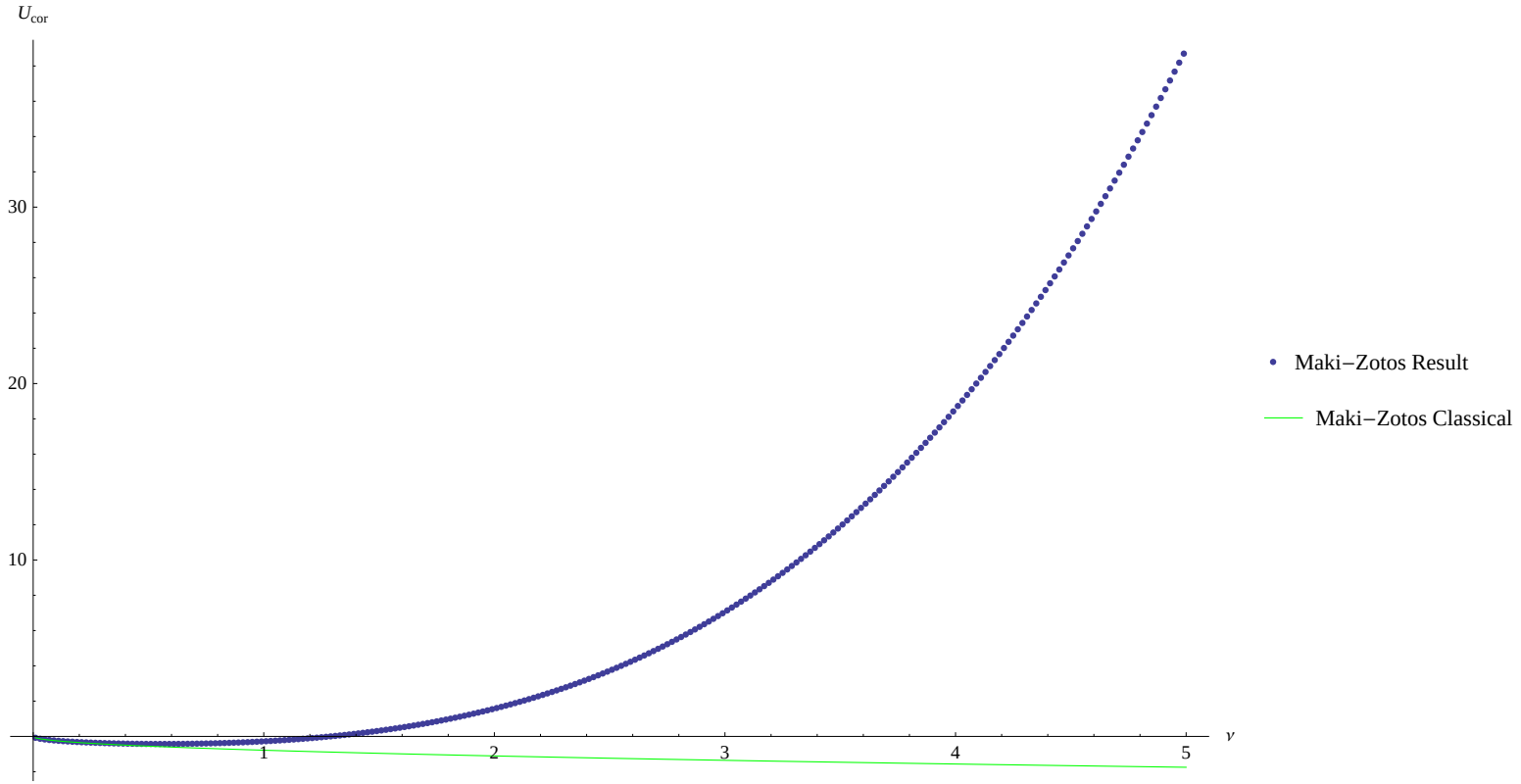

```
{4.789999999999999`, 33.78709881074119`}, {4.81`, 34.25745296877963`}, {4.829999999999999`, 34.732375424002925`},
{4.85`, 35.21189281536891`}, {4.87`, 35.69603183523751`}, {4.89`, 36.184819229258245`}, {4.91`, 36.67828179625591`},
{4.93`, 37.176446388118826`}, {4.95`, 37.679339909687265`}, {4.97`, 38.18698931864252`}, {4.99`, 38.69942162539723`}};

Show[ListPlot[TableMZ200ext, PlotLegends → {"Maki-Zotos Result"}],
 Plot[-0.78213264 √ν , {ν, 0.01, 5}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```



So the energy rapidly becomes positive.

The Maki-Zotos approximation (eq. 19) to the two-body energy (eq. 9) doesn't make sense. Why do it? They created the approximation just because it mostly fits the two-body energy curve and the 1/R term can be separated out. The energy corresponding to this 1/R is replaced by the Bonsall-Maradudin energy. The curve doesn't fit for small R.

We try a better method:
1.) Use the entire two-body expression. This is the full quantum-mechanical two-body energy.
2.) Add the Bonsall-Maradudin classical energy result. This contains the e-e, e-b, b-b energies. Of course, B-M didn't consider the b-b energy.
3.) Subtract out the 1/R correlation energy. But be careful:

$E_{\text{tot}} = \frac{1}{2} N E_{1/R}$, so $E_{\text{cor}} = \frac{E_{\text{tot}}}{N} = \frac{1}{2} E_{1/R}$, where $E_{1/R} = \sum_i \frac{1}{R_i}$ (the Coulomb energies between the reference electron and all surrounding electrons).

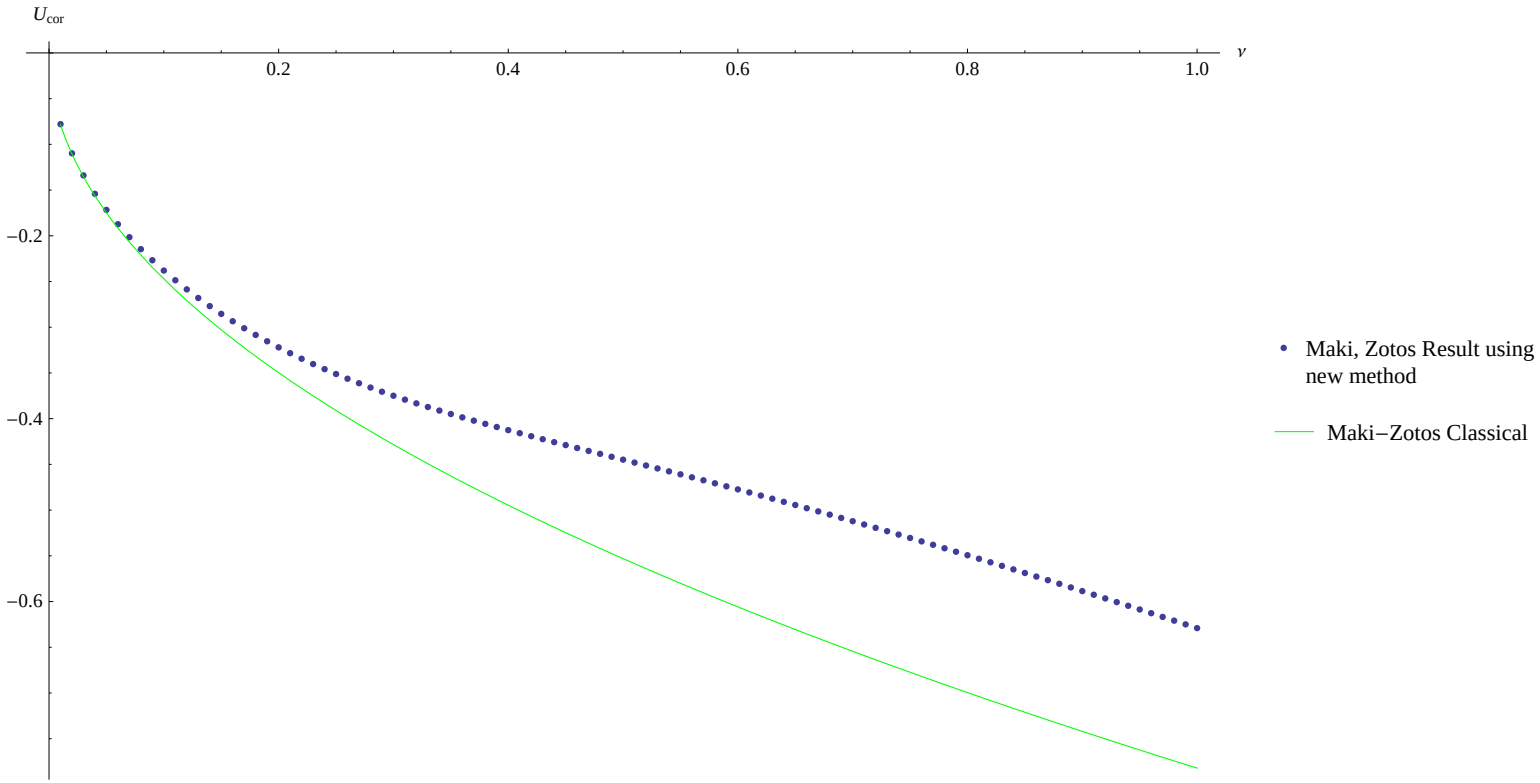
This has the effect of taking the QM e-e energy and adding the classical e-b and b-b energies. The resulting curve is better . . . ?

```
UCorMZnew[msize_, nsize_, ν_] := -0.78213264 ν^(1/2) + 1/2 Sum[Sum[If[
    R[m, n, ν] ≤ circleradius[msize, ν], If[m == 0 && n == 0, 0,
      Sqrt[π] BesselI[0, (R[m, n, ν])^2/8] Sech[(R[m, n, ν])^2/8]/4 - 1/R[m, n, ν]], 0], {n, -nsize, nsize}], {m, -msize, msize}]

TableMZnew650 = Parallelize[Table[{ν, UCorMZnew[650, 650, ν]}, {ν, 0.01, 1, 0.01}]]
```

```
{{0.01, -0.0779302}, {0.02, -0.109806}, {0.03, -0.133987}, {0.04, -0.154135}, {0.05, -0.171674},
 {0.06, -0.187337}, {0.07, -0.20156}, {0.08, -0.214627}, {0.09, -0.226736}, {0.1, -0.238032}, {0.11, -0.248623},
 {0.12, -0.258595}, {0.13, -0.268013}, {0.14, -0.276932}, {0.15, -0.285396}, {0.16, -0.293443},
 {0.17, -0.301103}, {0.18, -0.308403}, {0.19, -0.315369}, {0.2, -0.322019}, {0.21, -0.328375}, {0.22, -0.334453},
 {0.23, -0.34027}, {0.24, -0.345841}, {0.25, -0.351182}, {0.26, -0.356308}, {0.27, -0.361231}, {0.28, -0.365966},
 {0.29, -0.370527}, {0.3, -0.374925}, {0.31, -0.379173}, {0.32, -0.383284}, {0.33, -0.387269}, {0.34, -0.39114},
 {0.35, -0.394906}, {0.36, -0.398579}, {0.37, -0.402168}, {0.38, -0.405683}, {0.39, -0.409133}, {0.4, -0.412526},
 {0.41, -0.41587}, {0.42, -0.419172}, {0.43, -0.42244}, {0.44, -0.425679}, {0.45, -0.428897}, {0.46, -0.432097},
 {0.47, -0.435287}, {0.48, -0.438469}, {0.49, -0.441649}, {0.5, -0.444831}, {0.51, -0.448017}, {0.52, -0.451212},
 {0.53, -0.454418}, {0.54, -0.457637}, {0.55, -0.460872}, {0.56, -0.464124}, {0.57, -0.467396}, {0.58, -0.470689},
 {0.59, -0.474005}, {0.6, -0.477343}, {0.61, -0.480706}, {0.62, -0.484093}, {0.63, -0.487505}, {0.64, -0.490944},
 {0.65, -0.494408}, {0.66, -0.497898}, {0.67, -0.501414}, {0.68, -0.504956}, {0.69, -0.508524}, {0.7, -0.512118},
 {0.71, -0.515736}, {0.72, -0.51938}, {0.73, -0.523048}, {0.74, -0.526739}, {0.75, -0.530454}, {0.76, -0.534192},
 {0.77, -0.537952}, {0.78, -0.541734}, {0.79, -0.545537}, {0.8, -0.54936}, {0.81, -0.553202}, {0.82, -0.557064},
 {0.83, -0.560943}, {0.84, -0.564841}, {0.85, -0.568755}, {0.86, -0.572686}, {0.87, -0.576632}, {0.88, -0.580593},
 {0.89, -0.584569}, {0.9, -0.588558}, {0.91, -0.59256}, {0.92, -0.596575}, {0.93, -0.600602}, {0.94, -0.604639},
 {0.95, -0.608688}, {0.96, -0.612746}, {0.97, -0.616814}, {0.98, -0.620891}, {0.99, -0.624977}, {1., -0.629071}}
```

```
TableMZnew650 = {{0.01`, -0.07793019285933375`}, {0.02`, -0.10980647781694033`}, {0.03`, -0.1339868181283573`},
   {0.04`, -0.15413474662815912`}, {0.05`, -0.17167403213119886`}, {0.060000000000000005`, -0.18733696151956974`},
   {0.07`, -0.20155960310812404`}, {0.08`, -0.2146266277401945`}, {0.09`, -0.22673602835565035`}, {0.1`, -0.23803203837621253`},
   {0.11`, -0.2486234902849187`}, {0.12`, -0.2585947943365395`}, {0.13`, -0.2680128839519396`}, {0.14`, -0.27693182700221974`},
   {0.15`, -0.28539602258430774`}, {0.16`, -0.29344250379903475`}, {0.17`, -0.30110265037353556`},
   {0.18`, -0.30840349295706226`}, {0.19`, -0.31536872105903907`}, {0.2`, -0.32201946644632756`},
   {0.21000000000000002`, -0.32837491071237557`}, {0.22000000000000003`, -0.3344527522884321`}, {0.23`, -0.3402695600681499`},
   {0.24000000000000002`, -0.3458410355915502`}, {0.25`, -0.3511822020235738`}, {0.26`, -0.356307535258009`},
   {0.27`, -0.3612310500353245`}, {0.28`, -0.36596635182997095`}, {0.29000000000000004`, -0.3705266633787011`},
   {0.30000000000000004`, -0.3749248330640831`}, {0.31000000000000005`, -0.3791733309266587`},
   {0.32000000000000006`, -0.38328423684527146`}, {0.33`, -0.3872692243846463`}, {0.34`, -0.3911395429455069`},
   {0.35000000000000003`, -0.3949060001462455`}, {0.36`, -0.3985789457960877`}, {0.37`, -0.4021682583681794`},
   {0.38`, -0.4056833345284637`}, {0.39`, -0.40913308200558013`}, {0.4`, -0.4125259158837048`},
   {0.41000000000000003`, -0.41586975825120404`}, {0.42000000000000004`, -0.41917204103234784`},
   {0.43000000000000005`, -0.42243971175811756`}, {0.44000000000000006`, -0.4256792419875083`}, {0.45`, -0.4288966380669124`},
   {0.46`, -0.4320974539066185`}, {0.47000000000000003`, -0.43528680545668824`}, {0.48`, -0.4384693865758403`},
   {0.49`, -0.44164948600412274`}, {0.5`, -0.44483100517108976`}, {0.51`, -0.44801747659435825`},
   {0.52`, -0.45121208264753987`}, {0.53`, -0.45441767450090065`}, {0.54`, -0.4576367910619461`}, {0.55`, -0.460871677765917`},
   {0.56`, -0.46412430508780667`}, {0.5700000000000001`, -0.467396386667559`}, {0.5800000000000001`, -0.47068939695856443`},
   {0.5900000000000001`, -0.4740045883264198`}, {0.6000000000000001`, -0.477343007539984`}, {0.61`, -0.48070551161042796`},
   {0.62`, -0.4840927829458471`}, {0.63`, -0.48750534379963273`}, {0.64`, -0.4909435699998883`},
   {0.65`, -0.49440770395515304`}, {0.66`, -0.49789786693842586`}, {0.67`, -0.5014140706571291`},
   {0.68`, -0.5049562281215295`}, {0.6900000000000001`, -0.5085241638279598`}, {0.7000000000000001`, -0.5121176232763223`},
   {0.7100000000000001`, -0.5157362818439792`}, {0.7200000000000001`, -0.5193797530399129`}, {0.73`, -0.5230475961644852`},
   {0.74`, -0.5267393234011486`}, {0.75`, -0.5304544063669608`}, {0.76`, -0.5341922821490265`},
   {0.77`, -0.5379523588540768`}, {0.78`, -0.5417340206979867`}, {0.79`, -0.5455366326617573`},
   {0.8`, -0.549359544739823`}, {0.81`, -0.5532020958058159`}, {0.8200000000000001`, -0.5570636171202232`},
   {0.8300000000000001`, -0.5609434355033451`}, {0.8400000000000001`, -0.5648408761961431`}, {0.85`, -0.5687552654305309`},
   {0.86`, -0.572685932729719`}, {0.87`, -0.5766322129581571`}, {0.88`, -0.5805934481396691`},
   {0.89`, -0.5845689890613555`}, {0.9`, -0.588558196679866`}, {0.91`, -0.5925604433456697`},
   {0.92`, -0.596575113860061`}, {0.93`, -0.6006016063786968`}, {0.9400000000000001`, -0.6046393331745707`},
   {0.9500000000000001`, -0.6086877212726389`}, {0.9600000000000001`, -0.612746212967245`}, {0.97`, -0.6168142662330561`},
   {0.98`, -0.6208913550391789`}, {0.99`, -0.6249769695756979`}, {1.`, -0.6290706164010608`}};

Export[
 "/home/ryan/Katsuaki's Files/Physics_MS_Thesis/Calculations/Effective Potentials (new)/TableMZnew650.dat", TableMZnew650]

/home/ryan/Katsuaki's Files/Physics_MS_Thesis/Calculations/Effective Potentials (new)/TableMZnew650.dat
```

```
Show[ListPlot[TableMZnew650, PlotLegends → {"Maki, Zotos Result using
new method"}], Plot[-0.78213264 √ν, {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```



Try just adding the e-b and b-b parts instead:

$$E_{(e-b)} = \int \frac{-e\rho_+}{|\vec{r}|}\, d^2 r = -\frac{e^2}{a_c} \int \frac{1}{|\vec{r}|}\, d^2 r$$

$$E_{(b)} = \frac{1}{2} \int \frac{e\rho_+}{|\vec{r}|}\, d^2 r = \frac{e^2}{2\, a_c} \int \frac{1}{|\vec{r}|}\, d^2 r$$

$$E_{(e-b)} + E_{(b)} = -\frac{e^2\, \nu}{2 * 2\, \pi\, l^2}\, 2\, \pi\, R_{\max} = -\frac{e^2\, \nu}{2\, l^2} \sqrt{\frac{4\, \pi\, l^2}{\sqrt{3}\ \nu}}\ \ \frac{\sqrt{3}}{2}\, m_{\max} = -\frac{e^2 \sqrt{\nu}}{l}\ \frac{1}{2} \sqrt{\frac{4\, \pi}{\sqrt{3}}}\ \ \frac{\sqrt{3}}{2}\, m_{\max}$$

Use $m_{\max} = 650$:

```
-1/2 Sqrt[(4 π)/Sqrt[3]]  Sqrt[3]/2 * 650 // N
```

```
-758.121
```

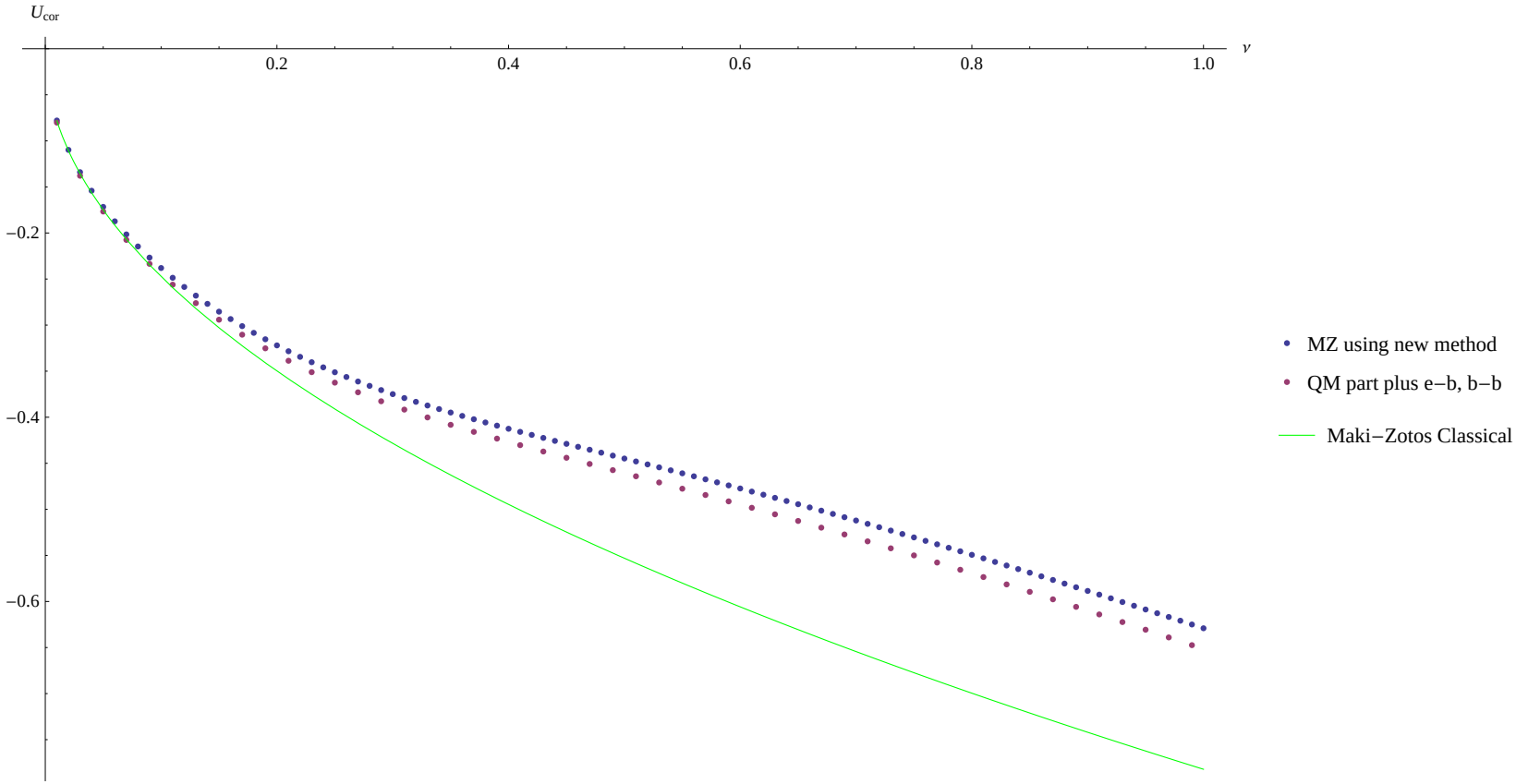
```
UCorMZebb[msize_, nsize_, ν_] := -758.1211470086542` ν^1/2 +
  1/2 Sum[Sum[If[R[m, n, ν] ≤ circleradius[msize, ν], If[m == 0 && n == 0, 0, Sqrt[π] BesselI[0, (R[m, n, ν])^2/8] Sech[(R[m, n, ν])^2/8]/4], 0], {n, -nsize, nsize}], {m, -msize, msize}]
```

```
TableMZnewebb = Parallelize[Table[{ν, UCorMZebb[650, 650, ν]}, {ν, 0.01, 1, 0.02}]]
```

```
{{0.01, -0.0801842}, {0.03, -0.137891}, {0.05, -0.176714}, {0.07, -0.207523}, {0.09, -0.233498},
 {0.11, -0.256099}, {0.13, -0.27614}, {0.15, -0.294126}, {0.17, -0.310396}, {0.19, -0.325194}, {0.21, -0.338704},
 {0.23, -0.35108}, {0.25, -0.362452}, {0.27, -0.372943}, {0.29, -0.382665}, {0.31, -0.391723}, {0.33, -0.400218},
 {0.35, -0.408241}, {0.37, -0.415879}, {0.39, -0.42321}, {0.41, -0.430303}, {0.43, -0.43722}, {0.45, -0.444017},
 {0.47, -0.45074}, {0.49, -0.457428}, {0.51, -0.464115}, {0.53, -0.470827}, {0.55, -0.477588}, {0.57, -0.484414},
 {0.59, -0.491318}, {0.61, -0.49831}, {0.63, -0.505396}, {0.65, -0.51258}, {0.67, -0.519864}, {0.69, -0.527248},
 {0.71, -0.534729}, {0.73, -0.542306}, {0.75, -0.549975}, {0.77, -0.557731}, {0.79, -0.565571},
 {0.81, -0.573488}, {0.83, -0.581479}, {0.85, -0.589536}, {0.87, -0.597657}, {0.89, -0.605834},
 {0.91, -0.614063}, {0.93, -0.622339}, {0.95, -0.630657}, {0.97, -0.639014}, {0.99, -0.647404}}
```

```
TableMZnewebb = {{0.01`, -0.08018423340124059`}, {0.03`, -0.13789093086921866`}, {0.05`, -0.17671422000699977`},
   {0.06999999999999999`, -0.20752323382714621`}, {0.09`, -0.23349814998132956`}, {0.11`, -0.25609929702469003`},
   {0.13`, -0.2761399427022866`}, {0.15`, -0.29412588406438545`}, {0.17`, -0.31039629761193055`},
   {0.19`, -0.32519385599545103`}, {0.21000000000000002`, -0.33870422211481355`}, {0.23`, -0.3510795587543498`},
   {0.25`, -0.3624524047324371`}, {0.27`, -0.3729433882575677`}, {0.29000000000000004`, -0.3826650431785197`},
   {0.31000000000000005`, -0.3917232975287561`}, {0.33`, -0.40021770148547375`}, {0.35000000000000003`, -0.40824108382582835`},
   {0.37000000000000005`, -0.41587905171707007`}, {0.39`, -0.42320956067817406`}, {0.41000000000000003`, -0.4303026598768156`},
   {0.43000000000000005`, -0.4372204440420546`}, {0.45`, -0.44401720169395276`}, {0.47000000000000003`, -0.4507397288668926`},
   {0.49000000000000005`, -0.4574277697954585`}, {0.51`, -0.46411454579924794`}, {0.53`, -0.47082733734032445`},
   {0.55`, -0.4775880898213245`}, {0.5700000000000001`, -0.48441401956915797`}, {0.59`, -0.49131820224920375`},
   {0.61`, -0.4983101310224356`}, {0.63`, -0.5053962359551178`}, {0.65`, -0.5125803597782124`},
   {0.67`, -0.5198641876535248`}, {0.6900000000000001`, -0.5272476307809484`}, {0.7100000000000001`, -0.5347291650442685`},
   {0.7300000000000001`, -0.5423061269963227`}, {0.7500000000000001`, -0.5499749700718439`}, {0.77`, -0.5577314843363865`},
   {0.79`, -0.565570983222301`}, {0.81`, -0.5734884606830519`}, {0.8300000000000001`, -0.581478722145448`},
   {0.8500000000000001`, -0.5895364924152773`}, {0.8700000000000001`, -0.5976565034932264`}, {0.89`, -0.6058335650027402`},
   {0.91`, -0.614062619688525`}, {0.93`, -0.6223387861655283`}, {0.9500000000000001`, -0.6306573908789233`},
   {0.9700000000000001`, -0.6390139910097332`}, {0.9900000000000001`, -0.647404389795156`}};
```

```
Show[ListPlot[{TableMZnew650, TableMZnewebb}, PlotLegends → {"MZ using new method", "QM part plus e-b, b-b"}],
 Plot[-0.78213264 √ν , {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```



Pretty good result.

Try to recreate the classical curve (e-e+e-b+b):

$$E_{(e-b)} = \int \frac{-e\rho_+}{|\vec{r}|}\, d^2 r = -\frac{e^2}{a_c}\int \frac{1}{|\vec{r}|}\, d^2 r$$

$$E_{(b)} = \frac{1}{2}\int \frac{e\rho_+}{|\vec{r}|}\, d^2 r = \frac{e^2}{2\,a_c}\int \frac{1}{|\vec{r}|}\, d^2 r$$

$$E_{(e-b)} + E_{(b)} = -\frac{e^2\,\nu}{2*2\,\pi\,l^2}\,2\,\pi\,R_{\max} = -\frac{e^2\,\nu}{2\,l^2}\sqrt{\frac{4\,\pi\,l^2}{\sqrt{3}\ \nu}}\ \ \frac{\sqrt{3}}{2}\,m_{\max} = -\frac{e^2\,\sqrt{\nu}}{l}\ \frac{1}{2}\sqrt{\frac{4\,\pi}{\sqrt{3}}}\ \ \frac{\sqrt{3}}{2}\,m_{\max}$$

```
-1/2 Sqrt[(4 π)/Sqrt[3]]  Sqrt[3]/2 * 650 // N
```

```
-758.121
```

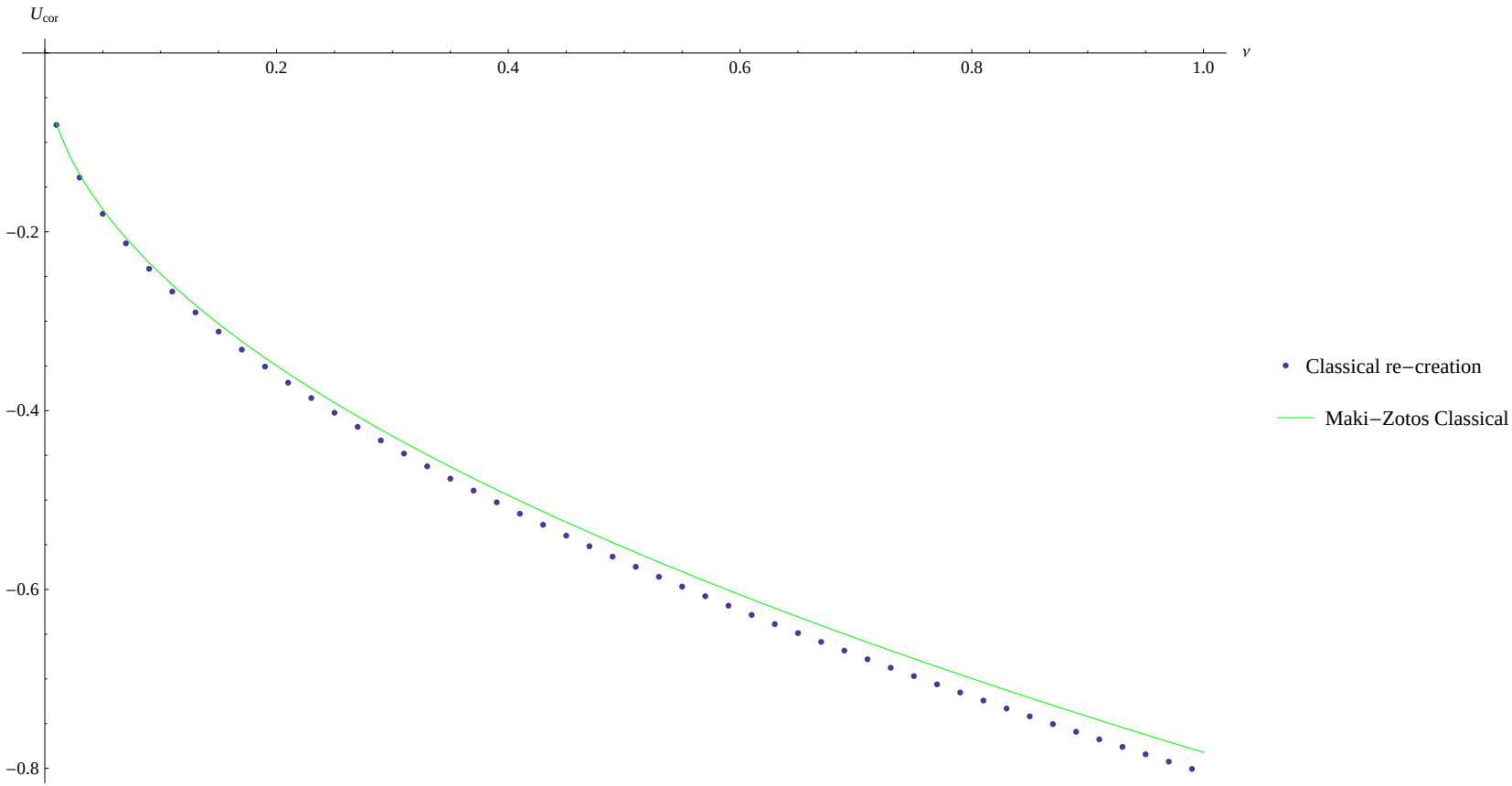
```
UCorClassical[msize_, nsize_, ν_] :=
 -758.1211470086542` ν^1/2 + 1/2 Sum[Sum[If[R[m, n, ν] ≤ circleradius[msize, ν], If[m == 0 && n == 0, 0, 1/R[m, n, ν]], 0], {n, -nsize, nsize}], {m, -msize, msize}]
```

```
TableClassical650 = Parallelize[Table[{ν, UCorClassical[650, 650, ν]}, {ν, 0.01, 1, 0.02}]]
```

```
{{0.01, -0.0804673}, {0.03, -0.139373}, {0.05, -0.17993}, {0.07, -0.212896}, {0.09, -0.241402},
 {0.11, -0.26688}, {0.13, -0.290129}, {0.15, -0.311649}, {0.17, -0.331775}, {0.19, -0.350749}, {0.21, -0.368748},
 {0.23, -0.385908}, {0.25, -0.402337}, {0.27, -0.41812}, {0.29, -0.43333}, {0.31, -0.448023}, {0.33, -0.462249},
 {0.35, -0.476051}, {0.37, -0.489464}, {0.39, -0.502518}, {0.41, -0.515242}, {0.43, -0.527659},
 {0.45, -0.539791}, {0.47, -0.551656}, {0.49, -0.563271}, {0.51, -0.574651}, {0.53, -0.585811},
 {0.55, -0.596762}, {0.57, -0.607515}, {0.59, -0.618081}, {0.61, -0.62847}, {0.63, -0.638689}, {0.65, -0.648748},
 {0.67, -0.658653}, {0.69, -0.668412}, {0.71, -0.67803}, {0.73, -0.687513}, {0.75, -0.696867}, {0.77, -0.706098},
 {0.79, -0.715209}, {0.81, -0.724206}, {0.83, -0.733092}, {0.85, -0.741872}, {0.87, -0.750549},
 {0.89, -0.759127}, {0.91, -0.767609}, {0.93, -0.775999}, {0.95, -0.784298}, {0.97, -0.792511}, {0.99, -0.80064}}
```

```
TableClassical650 = {{0.01`, -0.08046730454186957`}, {0.03`, -0.13937345981437943`}, {0.05`, -0.17993036292179454`},
   {0.06999999999999999`, -0.21289647648876553`}, {0.09`, -0.24140191362536711`}, {0.11`, -0.26687985705652295`},
   {0.13`, -0.2901289925243873`}, {0.15`, -0.3116485304049661`}, {0.17`, -0.33177519603503924`},
   {0.19`, -0.350748848757064`}, {0.21000000000000002`, -0.36874751403126993`}, {0.23`, -0.38590763571755815`},
   {0.25`, -0.4023365227090494`}, {0.27`, -0.4181203794433941`}, {0.29000000000000004`, -0.4333296965442628`},
   {0.31000000000000005`, -0.44802299060029327`}, {0.33`, -0.46224947193877597`}, {0.35000000000000003`, -0.4760509936003814`},
   {0.37000000000000005`, -0.4894635049814724`}, {0.39`, -0.5025181558005443`}, {0.41000000000000003`, -0.5152421480322573`},
   {0.43000000000000005`, -0.5276594027499755`}, {0.45`, -0.5397910887647299`}, {0.47000000000000003`, -0.551656046564517`},
   {0.49000000000000005`, -0.5632711317929306`}, {0.51`, -0.5746514962239644`}, {0.53`, -0.5858108195585601`},
   {0.55`, -0.5967615022040036`}, {0.5700000000000001`, -0.607514826743909`}, {0.59`, -0.6180810941243635`},
   {0.61`, -0.6284697392204635`}, {0.63`, -0.6386894294674903`}, {0.65`, -0.6487481495275915`},
   {0.67`, -0.6586532742770714`}, {0.6900000000000001`, -0.6684116320930116`}, {0.7100000000000001`, -0.6780295599149895`},
   {0.7300000000000001`, -0.6875129513823595`}, {0.7500000000000001`, -0.6968672990727782`}, {0.77`, -0.7060977317033803`},
   {0.79`, -0.7152090470058283`}, {0.81`, -0.7242057408761866`}, {0.8300000000000001`, -0.7330920333215545`},
   {0.8500000000000001`, -0.7418718915827185`}, {0.8700000000000001`, -0.7505490508419825`}, {0.89`, -0.7591270327957318`},
   {0.91`, -0.7676091623476395`}, {0.93`, -0.7759985826802449`}, {0.9500000000000001`, -0.7842982688500797`},
   {0.9700000000000001`, -0.7925110401266693`}, {0.9900000000000001`, -0.8006395711697678`}};
```

```
Show[ListPlot[TableClassical650, PlotLegends → {"Classical re-creation"}],
 Plot[-0.78213264 √ν , {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```



Pretty close

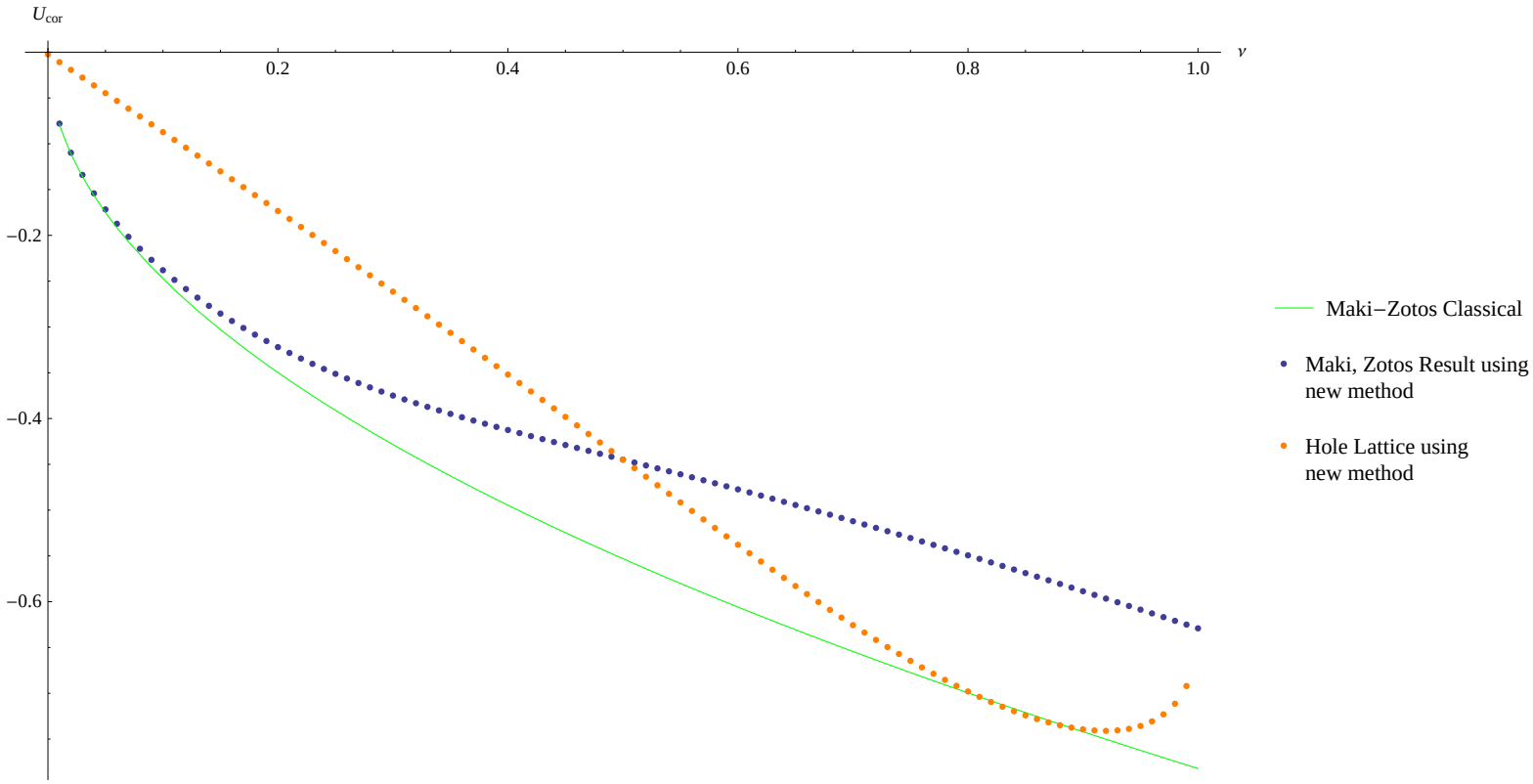
Hole Lattice (MZ eq. 23)

```
Dataholenew650 = AbsoluteTiming[Parallelize[Table[{v, UCorMZnew[650, 650, 1 - v] - 1/2 Sqrt[π/2] (2 v - 1)}, {v, 0, 0.99, 0.01}]]]
```

```
{3009.202097, {{0., -0.00241355}, {0.01, -0.010853}, {0.02, -0.0193006}, {0.03, -0.0277566}, {0.04, -0.0362217},
  {0.05, -0.0446964}, {0.06, -0.0531811}, {0.07, -0.0616765}, {0.08, -0.0701832}, {0.09, -0.0787016},
  {0.1, -0.0872325}, {0.11, -0.0957765}, {0.12, -0.104334}, {0.13, -0.112906}, {0.14, -0.121493}, {0.15, -0.130095},
  {0.16, -0.138714}, {0.17, -0.14735}, {0.18, -0.156003}, {0.19, -0.164675}, {0.2, -0.173365}, {0.21, -0.182076},
  {0.22, -0.190806}, {0.23, -0.199558}, {0.24, -0.208331}, {0.25, -0.217126}, {0.26, -0.225944}, {0.27, -0.234785},
  {0.28, -0.243651}, {0.29, -0.25254}, {0.3, -0.261455}, {0.31, -0.270394}, {0.32, -0.27936}, {0.33, -0.288351},
  {0.34, -0.297368}, {0.35, -0.306411}, {0.36, -0.31548}, {0.37, -0.324575}, {0.38, -0.333695}, {0.39, -0.342841},
  {0.4, -0.352012}, {0.41, -0.361206}, {0.42, -0.370424}, {0.43, -0.379664}, {0.44, -0.388925}, {0.45, -0.398206},
  {0.46, -0.407504}, {0.47, -0.416818}, {0.48, -0.426146}, {0.49, -0.435484}, {0.5, -0.444831}, {0.51, -0.454183},
  {0.52, -0.463536}, {0.53, -0.472886}, {0.54, -0.48223}, {0.55, -0.491562}, {0.56, -0.500878}, {0.57, -0.510172},
  {0.58, -0.519437}, {0.59, -0.528668}, {0.6, -0.537857}, {0.61, -0.546998}, {0.62, -0.556081}, {0.63, -0.565099},
  {0.64, -0.574043}, {0.65, -0.582903}, {0.66, -0.59167}, {0.67, -0.600333}, {0.68, -0.608881}, {0.69, -0.617303},
  {0.7, -0.625588}, {0.71, -0.633723}, {0.72, -0.641695}, {0.73, -0.649493}, {0.74, -0.657103}, {0.75, -0.664511},
  {0.76, -0.671703}, {0.77, -0.678664}, {0.78, -0.685381}, {0.79, -0.691836}, {0.8, -0.698014}, {0.81, -0.703896},
  {0.82, -0.709464}, {0.83, -0.714696}, {0.84, -0.719569}, {0.85, -0.724056}, {0.86, -0.728125}, {0.87, -0.731739},
  {0.88, -0.734854}, {0.89, -0.737416}, {0.9, -0.739358}, {0.91, -0.740595}, {0.92, -0.741019}, {0.93, -0.740485},
  {0.94, -0.738795}, {0.95, -0.735665}, {0.96, -0.730659}, {0.97, -0.723044}, {0.98, -0.711397}, {0.99, -0.692054}}}
```

```
Dataholenew650 = {{0.`, -0.0024135477433107067`}, {0.01`, -0.010853042291102843`}, {0.02`, -0.01930056912773892`},
   {0.03`, -0.02775662169477111`}, {0.04`, -0.0362217098021157`}, {0.05`, -0.04469635948066386`},
   {0.06000000000000005`, -0.05318111275575066`}, {0.07`, -0.061676527333031195`}, {0.08`, -0.07018317618755099`},
   {0.09`, -0.07870164704631466`}, {0.1`, -0.0872325417536659`}, {0.11`, -0.09577647550831042`},
   {0.12000000000000001`, -0.104334075959779`}, {0.13`, -0.11290598215142211`}, {0.14`, -0.12149284329613896`},
   {0.15`, -0.13009531737010588`}, {0.16`, -0.138714069508873`}, {0.16999999999999998`, -0.14734977018923007`},
   {0.18`, -0.1560030931792632`}, {0.19`, -0.16467471323801086`}, {0.2`, -0.17336530354517293`},
   {0.21000000000000002`, -0.18207553284026234`}, {0.22`, -0.19080606224964664`}, {0.23`, -0.19955754177889173`},
   {0.24`, -0.2083306064469964`}, {0.25`, -0.2171258720380858`}, {0.26`, -0.2259439304454286`},
   {0.27`, -0.23478534458192013`}, {0.28`, -0.24365064283050436`}, {0.29000000000000004`, -0.2525403130077256`},
   {0.30000000000000004`, -0.2614547958132212`}, {0.31000000000000005`, -0.2703944777380141`}, {0.32`, -0.27935968340473916`},
   {0.33`, -0.28835066731349274`}, {0.34`, -0.297367604967946`}, {0.35`, -0.306410583357828`}, {0.36`, -0.3154795907757183`},
   {0.37`, -0.3245745059486177`}, {0.38`, -0.3336950864679871`}, {0.39`, -0.342840956505723`}, {0.4`, -0.3520115938084334`},
   {0.41000000000000003`, -0.3612063159680248`}, {0.42000000000000004`, -0.37042426597332434`},
   {0.43000000000000005`, -0.37966439705547317`}, {0.44`, -0.3889254568488767`}, {0.45`, -0.398205970900142`},
   {0.46`, -0.40750422556932614`}, {0.47`, -0.4168182503814356`}, {0.48`, -0.42614579990122986`},
   {0.49`, -0.4354843352212032`}, {0.5`, -0.44483100517108976`}, {0.51`, -0.45418262737727777`},
   {0.52`, -0.4635356693221503`}, {0.53`, -0.47288622957615317`}, {0.54`, -0.48223001939923915`},
   {0.55`, -0.49156234493268813`}, {0.56`, -0.5008780902264378`}, {0.5700000000000001`, -0.5101717013702027`},
   {0.5800000000000001`, -0.5194371720175892`}, {0.5900000000000001`, -0.5286680306095989`}, {0.6`, -0.5378573296152548`},
   {0.61`, -0.5469976371102852`}, {0.62`, -0.5560810310063238`}, {0.63`, -0.5650990962191944`}, {0.64`, -0.5740429250202578`},
   {0.65`, -0.5829031207435703`}, {0.66`, -0.5916698049159865`}, {0.67`, -0.6003326277282813`},
   {0.68`, -0.608880781562062`}, {0.6900000000000001`, -0.6173030170166044`}, {0.7000000000000001`, -0.625587660527184`},
   {0.7100000000000001`, -0.6337226322149558`}, {0.72`, -0.6416954620393809`}, {0.73`, -0.6494933016178895`},
   {0.74`, -0.6571029282137291`}, {0.75`, -0.6645107363524488`}, {0.76`, -0.671702711293581`}, {0.77`, -0.6786643771433357`},
   {0.78`, -0.6853807107367721`}, {0.79`, -0.69183601053387`}, {0.8`, -0.6980137076409775`}, {0.81`, -0.7038961036268445`},
   {0.8200000000000001`, -0.7094640168980224`}, {0.8300000000000001`, -0.7146963156876509`}, {0.84`, -0.7195693104863052`},
   {0.85`, -0.7240559706447325`}, {0.86`, -0.7281249164357998`}, {0.87`, -0.7317391147586747`}, {0.88`, -0.7348541665164295`},
   {0.89`, -0.7374160038379635`}, {0.9`, -0.7393576933024129`}, {0.91`, -0.7405948246550054`}, {0.92`, -0.7410185654127042`},
   {0.93`, -0.740484682153789`}, {0.9400000000000001`, -0.7387951819383901`}, {0.9500000000000001`, -0.7356653939231739`},
   {0.96`, -0.7306592497932892`}, {0.97`, -0.7230444626666424`}, {0.98`, -0.7113972637283804`}, {0.99`, -0.6920541201439289`}};
```

```
Show[ListPlot[TableMZnew650, PlotLegends → {"Maki, Zotos Result using
new method"}], Plot[-0.78213264 √ν, {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 ListPlot[Dataholenew650, PlotStyle → Orange, PlotLegends → {"Hole Lattice using
new method"}], AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```



Plot from $\nu = 0.01 \ldots 5.$

```
TableMZnewext = Parallelize[Table[{ν, UCorMZnew[50, 50, ν]}, {ν, 0.01, 5, 0.02}]]
```

```
{{0.01, -0.0779342}, {0.03, -0.134007}, {0.05, -0.171718}, {0.07, -0.201633}, {0.09, -0.226843}, {0.11, -0.248768},
 {0.13, -0.268198}, {0.15, -0.285626}, {0.17, -0.30138}, {0.19, -0.315697}, {0.21, -0.328756}, {0.23, -0.340706},
 {0.25, -0.351677}, {0.27, -0.361787}, {0.29, -0.371145}, {0.31, -0.379857}, {0.33, -0.38802}, {0.35, -0.395726},
 {0.37, -0.40306}, {0.39, -0.410098}, {0.41, -0.416909}, {0.43, -0.423556}, {0.45, -0.430092}, {0.47, -0.436563},
 {0.49, -0.443008}, {0.51, -0.44946}, {0.53, -0.455946}, {0.55, -0.462487}, {0.57, -0.469101}, {0.59, -0.475799},
 {0.61, -0.482592}, {0.63, -0.489486}, {0.65, -0.496483}, {0.67, -0.503586}, {0.69, -0.510794}, {0.71, -0.518106},
 {0.73, -0.525518}, {0.75, -0.533027}, {0.77, -0.540628}, {0.79, -0.548317}, {0.81, -0.556089}, {0.83, -0.563938},
 {0.85, -0.571859}, {0.87, -0.579846}, {0.89, -0.587894}, {0.91, -0.595998}, {0.93, -0.604154}, {0.95, -0.612355},
 {0.97, -0.620598}, {0.99, -0.628878}, {1.01, -0.637192}, {1.03, -0.645535}, {1.05, -0.653904}, {1.07, -0.662297},
 {1.09, -0.670709}, {1.11, -0.679138}, {1.13, -0.687581}, {1.15, -0.696037}, {1.17, -0.704504}, {1.19, -0.712978},
 {1.21, -0.72146}, {1.23, -0.729946}, {1.25, -0.738436}, {1.27, -0.746929}, {1.29, -0.755424}, {1.31, -0.763919},
 {1.33, -0.772414}, {1.35, -0.780908}, {1.37, -0.7894}, {1.39, -0.797891}, {1.41, -0.806379}, {1.43, -0.814865},
 {1.45, -0.823347}, {1.47, -0.831826}, {1.49, -0.840302}, {1.51, -0.848774}, {1.53, -0.857243}, {1.55, -0.865708},
 {1.57, -0.874169}, {1.59, -0.882626}, {1.61, -0.89108}, {1.63, -0.899531}, {1.65, -0.907977}, {1.67, -0.916421},
 {1.69, -0.924861}, {1.71, -0.933298}, {1.73, -0.941732}, {1.75, -0.950163}, {1.77, -0.958592}, {1.79, -0.967018},
 {1.81, -0.975441}, {1.83, -0.983862}, {1.85, -0.992281}, {1.87, -1.0007}, {1.89, -1.00911}, {1.91, -1.01753},
 {1.93, -1.02594}, {1.95, -1.03435}, {1.97, -1.04276}, {1.99, -1.05117}, {2.01, -1.05957}, {2.03, -1.06798}, {2.05, -1.07639},
 {2.07, -1.08479}, {2.09, -1.09319}, {2.11, -1.1016}, {2.13, -1.11}, {2.15, -1.11841}, {2.17, -1.12681}, {2.19, -1.13521},
 {2.21, -1.14362}, {2.23, -1.15202}, {2.25, -1.16042}, {2.27, -1.16883}, {2.29, -1.17723}, {2.31, -1.18563}, {2.33, -1.19404},
 {2.35, -1.20244}, {2.37, -1.21085}, {2.39, -1.21926}, {2.41, -1.22766}, {2.43, -1.23607}, {2.45, -1.24448}, {2.47, -1.25289},
 {2.49, -1.2613}, {2.51, -1.26971}, {2.53, -1.27812}, {2.55, -1.28653}, {2.57, -1.29495}, {2.59, -1.30336}, {2.61, -1.31178},
 {2.63, -1.32019}, {2.65, -1.32861}, {2.67, -1.33703}, {2.69, -1.34544}, {2.71, -1.35386}, {2.73, -1.36228}, {2.75, -1.3707},
 {2.77, -1.37913}, {2.79, -1.38755}, {2.81, -1.39597}, {2.83, -1.4044}, {2.85, -1.41283}, {2.87, -1.42125}, {2.89, -1.42968},
 {2.91, -1.43811}, {2.93, -1.44654}, {2.95, -1.45497}, {2.97, -1.4634}, {2.99, -1.47183}, {3.01, -1.48027}, {3.03, -1.4887},
 {3.05, -1.49714}, {3.07, -1.50557}, {3.09, -1.51401}, {3.11, -1.52245}, {3.13, -1.53089}, {3.15, -1.53933}, {3.17, -1.54777},
 {3.19, -1.55621}, {3.21, -1.56465}, {3.23, -1.5731}, {3.25, -1.58154}, {3.27, -1.58999}, {3.29, -1.59843}, {3.31, -1.60688},
 {3.33, -1.61533}, {3.35, -1.62377}, {3.37, -1.63222}, {3.39, -1.64067}, {3.41, -1.64912}, {3.43, -1.65758}, {3.45, -1.66603},
 {3.47, -1.67448}, {3.49, -1.68293}, {3.51, -1.69139}, {3.53, -1.69985}, {3.55, -1.7083}, {3.57, -1.71676}, {3.59, -1.72522},
 {3.61, -1.73367}, {3.63, -1.74213}, {3.65, -1.75059}, {3.67, -1.75905}, {3.69, -1.76752}, {3.71, -1.77598}, {3.73, -1.78444},
 {3.75, -1.7929}, {3.77, -1.80137}, {3.79, -1.80983}, {3.81, -1.8183}, {3.83, -1.82676}, {3.85, -1.83523}, {3.87, -1.8437},
 {3.89, -1.85217}, {3.91, -1.86064}, {3.93, -1.86911}, {3.95, -1.87758}, {3.97, -1.88605}, {3.99, -1.89452}, {4.01, -1.90299},
 {4.03, -1.91146}, {4.05, -1.91994}, {4.07, -1.92841}, {4.09, -1.93689}, {4.11, -1.94536}, {4.13, -1.95384}, {4.15, -1.96232},
 {4.17, -1.97079}, {4.19, -1.97927}, {4.21, -1.98775}, {4.23, -1.99623}, {4.25, -2.00471}, {4.27, -2.01319}, {4.29, -2.02167},
 {4.31, -2.03015}, {4.33, -2.03864}, {4.35, -2.04712}, {4.37, -2.0556}, {4.39, -2.06409}, {4.41, -2.07257}, {4.43, -2.08106},
 {4.45, -2.08954}, {4.47, -2.09803}, {4.49, -2.10652}, {4.51, -2.11501}, {4.53, -2.12349}, {4.55, -2.13198}, {4.57, -2.14047},
 {4.59, -2.14896}, {4.61, -2.15745}, {4.63, -2.16595}, {4.65, -2.17444}, {4.67, -2.18293}, {4.69, -2.19142}, {4.71, -2.19992},
 {4.73, -2.20841}, {4.75, -2.21691}, {4.77, -2.2254}, {4.79, -2.2339}, {4.81, -2.2424}, {4.83, -2.25089}, {4.85, -2.25939},
 {4.87, -2.26789}, {4.89, -2.27639}, {4.91, -2.28489}, {4.93, -2.29339}, {4.95, -2.30189}, {4.97, -2.31039}, {4.99, -2.31889}}
```

```
TableMZnewext = {{0.01`, -0.07793415279434164`}, {0.03`, -0.13400739460344915`}, {0.05`, -0.1717183057622552`},
   {0.06999999999999999`, -0.2016329426568981`}, {0.09`, -0.22684294762350565`}, {0.11`, -0.24876796181695476`},
   {0.13`, -0.2681984973482607`}, {0.15000000000000002`, -0.2856260778700636`}, {0.17`, -0.30138021859846703`},
   {0.19`, -0.31569668629096126`}, {0.21000000000000002`, -0.3287560005652666`}, {0.23`, -0.34070636872382387`},
   {0.25`, -0.3516772081034747`}, {0.27`, -0.3617866308048698`}, {0.29000000000000004`, -0.3711451062590337`},
   {0.31000000000000005`, -0.37985684297870814`}, {0.33`, -0.3880199402330218`}, {0.35000000000000003`, -0.3957259886803648`},
   {0.37`, -0.4030595284696013`}, {0.39`, -0.41009758748890346`}, {0.41000000000000003`, -0.41690940216190775`},
   {0.43000000000000005`, -0.42355635013877707`}, {0.45`, -0.4300920832795587`}, {0.47000000000000003`, -0.4365628291312655`},
   {0.49`, -0.44300782166944713`}, {0.51`, -0.4494598220374765`}, {0.53`, -0.45594569389255946`},
   {0.55`, -0.46248700358337136`}, {0.57`, -0.4691006214407869`}, {0.59`, -0.47579930622965805`},
   {0.61`, -0.4825922599175353`}, {0.63`, -0.4894856442190889`}, {0.65`, -0.4964830538581546`},
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```

```
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{4.7299999999999995`, -2.2084118987942167`}, {4.749999999999999`, -2.216906952191932`}, {4.77`, -2.225402590216637`},
{4.789999999999999`, -2.2338988116875207`}, {4.81`, -2.24239561544211`}, {4.829999999999999`, -2.250893000335311`},
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{4.93`, -2.293388602911849`}, {4.95`, -2.3018894514572836`}, {4.97`, -2.3103908735431746`}, {4.99`, -2.3188928681348493`}};

Show[ListPlot[TableMZnewext, PlotStyle → PointSize[0.004], PlotLegends → {"Maki, Zotos Result using
new method"}], Plot[-0.78213264 √ν, {ν, 0.01, 5}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```
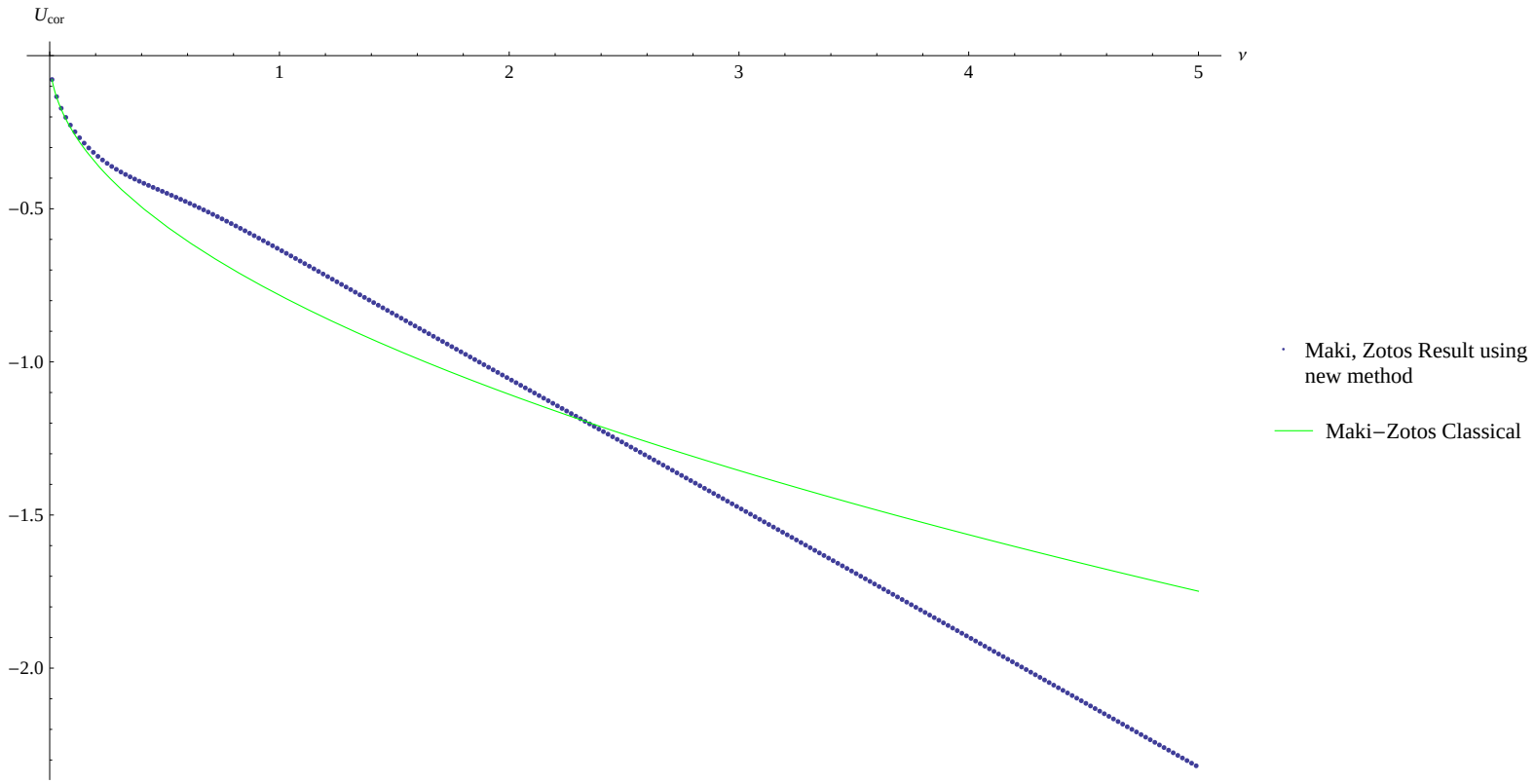



Ok, so Iṁ not sure if this is ok, but at least itṡ negative?

2.) The rest of the effective potential terms.

```
Clear[l]
```

```
Assuming[Rij ∈ Reals && Rij > 0 && l ∈ Reals && l > 0 && i ∈ Integers && i ≥ 0,
  (∫0∞ 1/l (r/l)^i Exp[-(r/l)^2] Exp[-r^2/(4 l^2)] (BesselI[0, r Rij/(2 l^2)] - BesselJ[0, r Rij/(2 l^2)]) r dr) / (4 l^2 Sinh[Rij^2/(4 l^2)])] // FullSimplify
```

$$\frac{2^{-2+i}\,5^{-1-\frac{i}{2}}\,i\,\text{Csch}\left[\frac{R_{ij}^2}{4\,l^2}\right]\,\Gamma\left[\frac{i}{2}\right]\left(-\text{Hypergeometric1F1}\left[1+\frac{i}{2},\,1,\,-\frac{R_{ij}^2}{20\,l^2}\right]+\text{Hypergeometric1F1}\left[1+\frac{i}{2},\,1,\,\frac{R_{ij}^2}{20\,l^2}\right]\right)}{l}$$

Try $\frac{1}{r^i}$

```
TryNegative[i_] := Assuming[Rij ∈ Reals && Rij > 0 && l ∈ Reals && l > 0 && i ∈ Integers && i ≥ 0,
  (∫0∞ 1/l (r/l)^-i Exp[-(r/l)^2] Exp[-r^2/(4 l^2)] (BesselI[0, r Rij/(2 l^2)] - BesselJ[0, r Rij/(2 l^2)]) r dr) / (4 l^2 Sinh[Rij^2/(4 l^2)])] // FullSimplify

l = 1;

TryNegative[4]
```

Integrate::idiv : Integral of $\frac{e^{-\frac{5r^2}{4}}\,\text{BesselI}\left[0,\,\frac{r\,R_{ij}}{2}\right]}{r^3}-\frac{e^{-\frac{5r^2}{4}}\,\text{BesselJ}\left[0,\,\frac{r\,R_{ij}}{2}\right]}{r^3}$ does not converge on $\{0,\infty\}$. >>

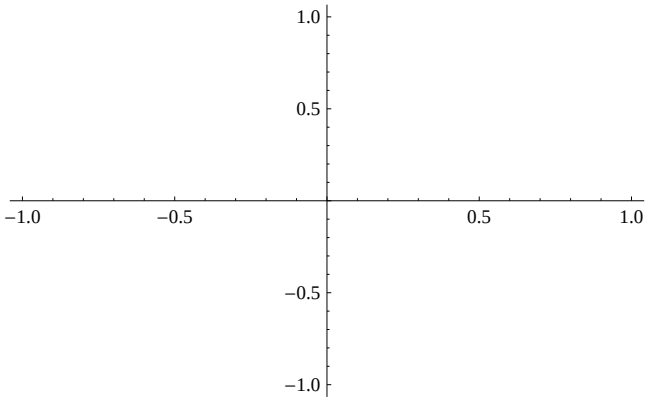
$$\frac{1}{4}\,\text{Csch}\left[\frac{R_{ij}^2}{4}\right]\int_0^\infty \frac{e^{-\frac{5r^2}{4}}\left(\text{BesselI}\left[0,\,\frac{r\,R_{ij}}{2}\right]-\text{BesselJ}\left[0,\,\frac{r\,R_{ij}}{2}\right]\right)}{r^3}\,dr$$

```
NIntegrate[
  2 π (1/l (r/l)^-i Exp[-(r/l)^2] Exp[-r^2/(4 l^2)] (BesselI[0, r Rij/(2 l^2)] - BesselJ[0, r Rij/(2 l^2)])) / (4 l^2 Sinh[Rij^2/(4 l^2)]) /. l → 1 /. Rij → 1 /. i → 4, {r, 0, ∞}]
```

NIntegrate::ncvb : NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in r near $\{r\} = \{1.055041819625699880108516602233643541218559472244530708392720689 90 \times 10^{-29}\}$. NIntegrate obtained 1.3413423956934553`*^31 and 1.233608342398944`*^31 for the integral and error estimates. >>

1.34134×10^{31}

```
Plot[(1/l (r/l)^-i Exp[-(r/l)^2] Exp[-r^2/(4 l^2)] (BesselI[0, r R_ij/(2 l^2)] - BesselJ[0, r R_ij/(2 l^2)]))/(4 l^2 Sinh[R_ij^2/(4 l^2)]) /. l → 1 /. R_ij → 1,
 {r, -5, 5}, PlotRange → All]
```



Doesnt work.

A bit complicated. Input i beforehand.

```
U[i_] := Assuming[R_ij ∈ Reals && R_ij > 0 && l ∈ Reals && l > 0 && i ∈ Integers && i ≥ 0,
  (∫_0^∞ 1/l (r/l)^i Exp[-(r/l)^2] Exp[-r^2/(4 l^2)] (BesselI[0, r R_ij/(2 l^2)] - BesselJ[0, r R_ij/(2 l^2)]) r dr)/(4 l Sinh[R_ij^2/(4 l^2)])] // FullSimplify
```

i=0:

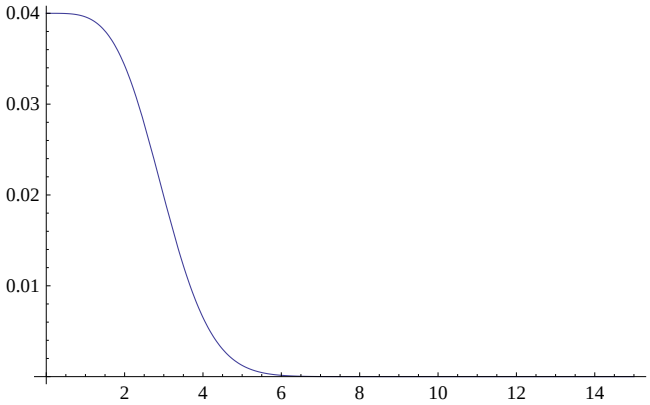
```
U[0]
```

$$\frac{1}{5}\,\text{Csch}\left[\frac{R_{ij}^2}{4\,l^2}\right]\,\text{Sinh}\left[\frac{R_{ij}^2}{20\,l^2}\right]$$

$$\frac{1}{5}\,\text{Csch}\left[\frac{R_{ij}^2}{4\,l^2}\right]\,\text{Sinh}\left[\frac{R_{ij}^2}{20\,l^2}\right]\ /.\ l\to 1$$

$$\frac{1}{5}\,\text{Csch}\left[\frac{R_{ij}^2}{4}\right]\,\text{Sinh}\left[\frac{R_{ij}^2}{20}\right]$$

$$\text{Plot}\left[\frac{1}{5}\,\text{Csch}\left[\frac{R_{ij}^2}{4}\right]\,\text{Sinh}\left[\frac{R_{ij}^2}{20}\right],\ \{R_{ij},\ 0,\ 15\}\right]$$




```
UCori0[msize_, nsize_, ν_] :=
 1/2 Sum[Sum[If[R[m, n, ν] ≤ circleradius[msize, ν], If[m == 0 && n == 0, 0, 1/5 Csch[(R[m, n, ν])^2/4] Sinh[(R[m, n, ν])^2/20]], 0], {n, -nsize, nsize}], {m, -msize, msize}]
```

```
Tablei0 = AbsoluteTiming[Parallelize[Table[{ν, UCori0[650, 650, ν]}, {ν, 0.01, 1, 0.01}]]]
```

```
{2413.931527, {{0.01, 5.75847 × 10^-64}, {0.02, 1.85878 × 10^-32}, {0.03, 5.91838 × 10^-22}, {0.04, 1.05606 × 10^-16}, {0.05, 1.4948 × 10^-13},
  {0.06, 1.88441 × 10^-11}, {0.07, 5.9647 × 10^-10}, {0.08, 7.95922 × 10^-9}, {0.09, 5.97079 × 10^-8}, {0.1, 2.99268 × 10^-7},
  {0.11, 1.11857 × 10^-6}, {0.12, 3.35455 × 10^-6}, {0.13, 8.49151 × 10^-6}, {0.14, 0.0000188118}, {0.15, 0.0000374544},
  {0.16, 0.0000683675}, {0.17, 0.000116169}, {0.18, 0.000185942}, {0.19, 0.000282995}, {0.2, 0.000412629}, {0.21, 0.00057991},
  {0.22, 0.000789491}, {0.23, 0.00104547}, {0.24, 0.00135129}, {0.25, 0.00170968}, {0.26, 0.00212265}, {0.27, 0.00259148},
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  {0.34, 0.0074144}, {0.35, 0.00830583}, {0.36, 0.00924092}, {0.37, 0.0102169}, {0.38, 0.0112308}, {0.39, 0.0122798},
  {0.4, 0.013361}, {0.41, 0.0144714}, {0.42, 0.0156082}, {0.43, 0.0167687}, {0.44, 0.0179502}, {0.45, 0.0191502},
  {0.46, 0.0203662}, {0.47, 0.0215959}, {0.48, 0.0228373}, {0.49, 0.0240882}, {0.5, 0.0253467}, {0.51, 0.0266112},
  {0.52, 0.0278799}, {0.53, 0.0291514}, {0.54, 0.0304243}, {0.55, 0.0316973}, {0.56, 0.0329694}, {0.57, 0.0342396},
  {0.58, 0.0355068}, {0.59, 0.0367704}, {0.6, 0.0380296}, {0.61, 0.0392837}, {0.62, 0.0405323}, {0.63, 0.0417749},
  {0.64, 0.0430111}, {0.65, 0.0442405}, {0.66, 0.0454629}, {0.67, 0.0466781}, {0.68, 0.047886}, {0.69, 0.0490863},
  {0.7, 0.0502792}, {0.71, 0.0514644}, {0.72, 0.0526421}, {0.73, 0.0538123}, {0.74, 0.054975}, {0.75, 0.0561303},
  {0.76, 0.0572784}, {0.77, 0.0584193}, {0.78, 0.0595533}, {0.79, 0.0606804}, {0.8, 0.0618009}, {0.81, 0.0629149},
  {0.82, 0.0640226}, {0.83, 0.0651242}, {0.84, 0.06622}, {0.85, 0.06731}, {0.86, 0.0683946}, {0.87, 0.069474}, {0.88, 0.0705483},
  {0.89, 0.0716177}, {0.9, 0.0726825}, {0.91, 0.0737429}, {0.92, 0.0747991}, {0.93, 0.0758512}, {0.94, 0.0768995},
  {0.95, 0.0779442}, {0.96, 0.0789855}, {0.97, 0.0800234}, {0.98, 0.0810583}, {0.99, 0.0820903}, {1., 0.0831196}}}
```
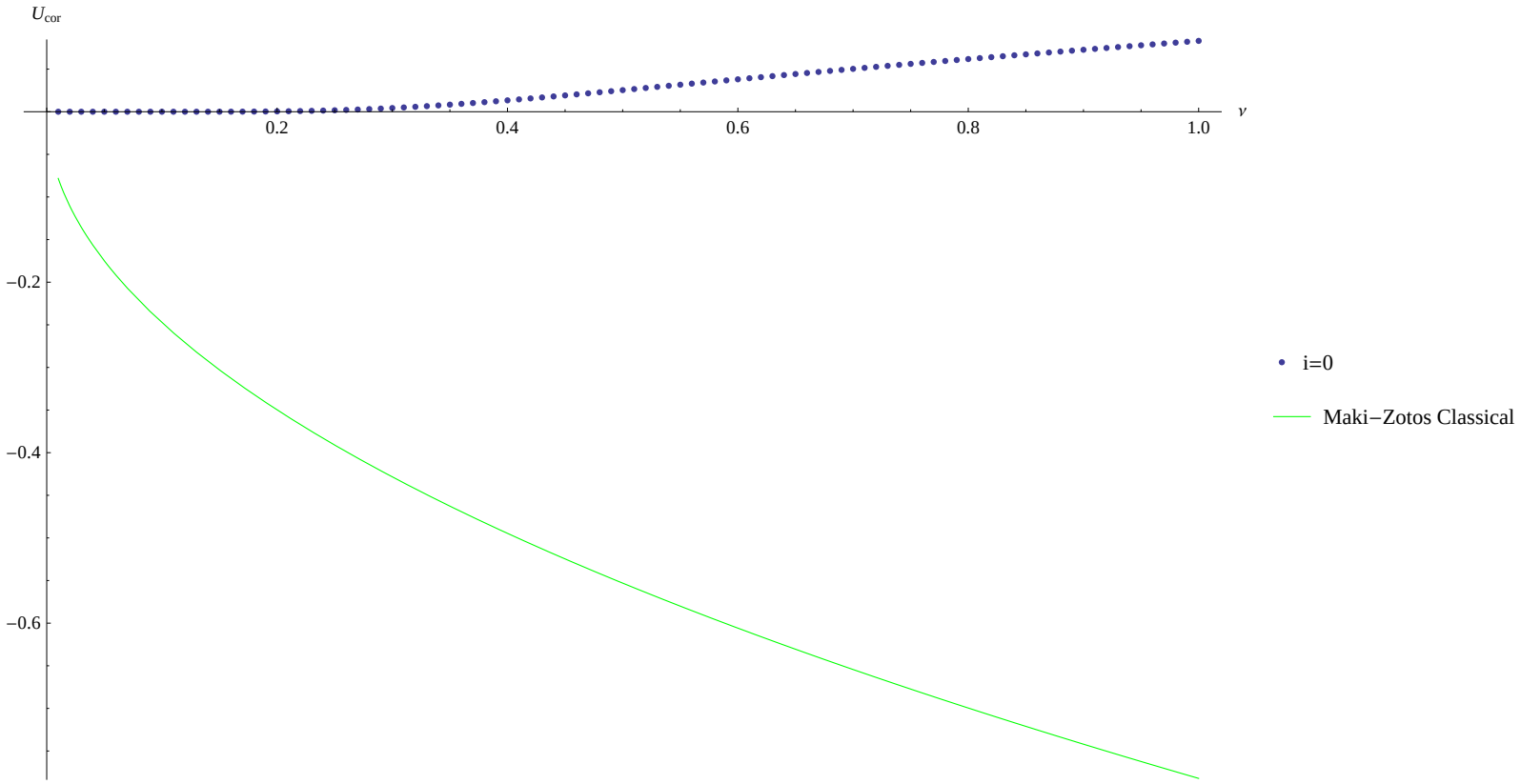

```
Tablei0 = {{0.01`, 5.7584663991852296`*^-64}, {0.02`, 1.8587845059369137`*^-32}, {0.03`, 5.918383628128748`*^-22},
   {0.04`, 1.0560637641590488`*^-16}, {0.05`, 1.4947967707087756`*^-13}, {0.060000000000000005`, 1.8844071682612047`*^-11},
   {0.07`, 5.964696560864723`*^-10}, {0.08`, 7.959223001185128`*^-9}, {0.09`, 5.970785828141957`*^-8},
   {0.1`, 2.9926771981183353`*^-7}, {0.11`, 1.1185683135197586`*^-6}, {0.12`, 3.354554748443657`*^-6},
   {0.13`, 8.49150693535728`*^-6}, {0.14`, 0.00001881184152061707`}, {0.15`, 0.00003745440360987338`},
   {0.16`, 0.00006836751815671624`}, {0.17`, 0.00011616903760157797`}, {0.18`, 0.00018594154934833287`},
   {0.19`, 0.00028299531422356 56`}, {0.2`, 0.00041262881537652314`}, {0.21000000000000002`, 0.0005799097499547003`},
   {0.22000000000000003`, 0.0007894911675527901`}, {0.23`, 0.0010454700224729863`},
   {0.24000000000000002`, 0.0013512895448641847`}, {0.25`, 0.0017096827571765351`}, {0.26`, 0.0021226519988504114`},
   {0.27`, 0.0025914781431855 85`}, {0.28`, 0.0031167529353580297`}, {0.29000000000000004`, 0.0036984282277930 35`},
   {0.30000000000000004`, 0.0043358765825683605`}, {0.31000000000000005`, 0.005027958563128325`},
   {0.32000000000000006`, 0.0057730929227327 34`}, {0.33`, 0.0065693267360624995`}, {0.34`, 0.007414403269861598`},
   {0.35000000000000003`, 0.008305826029396 32`}, {0.36`, 0.009240917946639454`}, {0.37`, 0.01021687509966642`},
   {0.38`, 0.011230814682030554`}, {0.39`, 0.012279817189182819`}, {0.4`, 0.013360962969859268`},
   {0.41000000000000003`, 0.014471363416483097`}, {0.42000000000000004`, 0.015608187151332543`},
   {0.43000000000000005`, 0.016768681614235854`}, {0.44000000000000006`, 0.017950190480996865`}, {0.45`, 0.019150167346272177`},
   {0.46`, 0.020366186095539373`}, {0.47000000000000003`, 0.021595948372313 46`}, {0.48`, 0.022837288522138118`},
   {0.49`, 0.024088176366580172`}, {0.5`, 0.025346718130343195`}, {0.51`, 0.026611155814049534`}, {0.52`, 0.027879865275188525`},
   {0.53`, 0.029151353250860584`}, {0.54`, 0.030424253528698705`}, {0.55`, 0.031697322446987 59`},
   {0.56`, 0.032969433881657 57`}, {0.5700000000000001`, 0.034239573856550 27`}, {0.5800000000000001`, 0.035506834894104 09`},
   {0.5900000000000001`, 0.036770410206328 55`}, {0.6000000000000001`, 0.038029587810510454`},
   {0.61`, 0.039283744640412 95`}, {0.62`, 0.040532340711655 18`}, {0.63`, 0.041774913389358 1`}, {0.64`, 0.043011071796884 3`},
   {0.65`, 0.044240491396451 81`}, {0.66`, 0.045462908765441 55`}, {0.67`, 0.046678116586234 41`},
   {0.68`, 0.047885958862292 19`}, {0.6900000000000001`, 0.049086326368845 26`}, {0.7000000000000001`, 0.050279152342874 45`},
   {0.7100000000000001`, 0.051464408413993 88`}, {0.7200000000000001`, 0.052642100775284 42`}, {0.73`, 0.053812266591021 48`},
   {0.74`, 0.054974970636535746`}, {0.75`, 0.056130302164076 19`}, {0.76`, 0.057278371987477 65`},
   {0.77`, 0.058419309777615 51`}, {0.78`, 0.059553261560031 78`}, {0.79`, 0.060680387405699 1`},
   {0.8`, 0.061800859305631006`}, {0.81`, 0.062914859219914 74`}, {0.8200000000000001`, 0.064022577291724 14`},
   {0.8300000000000001`, 0.065124210216938 72`}, {0.8400000000000001`, 0.066219959760139 34`}, {0.85`, 0.067310031407950 8`},
   {0.86`, 0.068394633150954 87`}, {0.87`, 0.069473974385680 04`}, {0.88`, 0.070548264928490 8`},
   {0.89`, 0.071617714133531 92`}, {0.9`, 0.072682530107230 15`}, {0.91`, 0.073742919012211 97`},
   {0.92`, 0.074799084453854 9`}, {0.93`, 0.075851226943047 02`}, {0.9400000000000001`, 0.076899543429085 09`},
   {0.9500000000000001`, 0.077944226896991 65`}, {0.9600000000000001`, 0.078985466023869 37`}, {0.97`, 0.080023444889244 86`},
   {0.98`, 0.081058342734673 06`}, {0.99`, 0.082090333768180 9`}, {1.`, 0.083119587009426 01`}};
```

```
Show[ListPlot[Tablei0, PlotLegends → {"i=0"}],
 Plot[-0.78213264 √ν , {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```

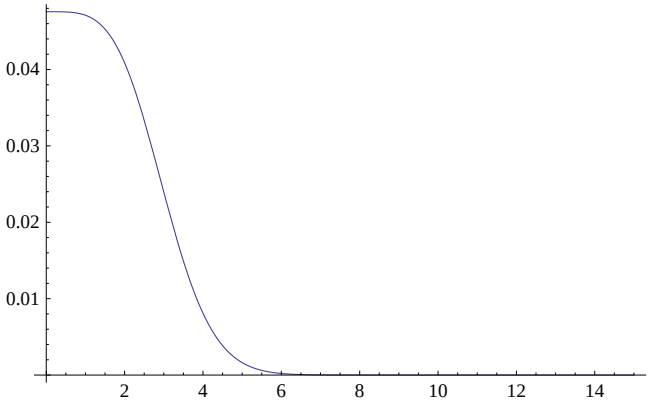


i=1:

```
U[1]
```

$$\frac{1}{100}\sqrt{\frac{\pi}{5}}\ \text{Csch}\left[\frac{R_{ij}^2}{4}\right]\left(\text{BesselI}\left[1,\ \frac{R_{ij}^2}{40}\right]\text{Sinh}\left[\frac{R_{ij}^2}{40}\right]R_{ij}^2+\text{BesselI}\left[0,\ \frac{R_{ij}^2}{40}\right]\left(20\,\text{Sinh}\left[\frac{R_{ij}^2}{40}\right]+\text{Cosh}\left[\frac{R_{ij}^2}{40}\right]R_{ij}^2\right)\right)$$

```
Plot[1/100 Sqrt[π/5] Csch[R_ij^2/4] (BesselI[1, R_ij^2/40] Sinh[R_ij^2/40] R_ij^2 + BesselI[0, R_ij^2/40] (20 Sinh[R_ij^2/40] + Cosh[R_ij^2/40] R_ij^2)), {R_ij, 0, 15}]
```



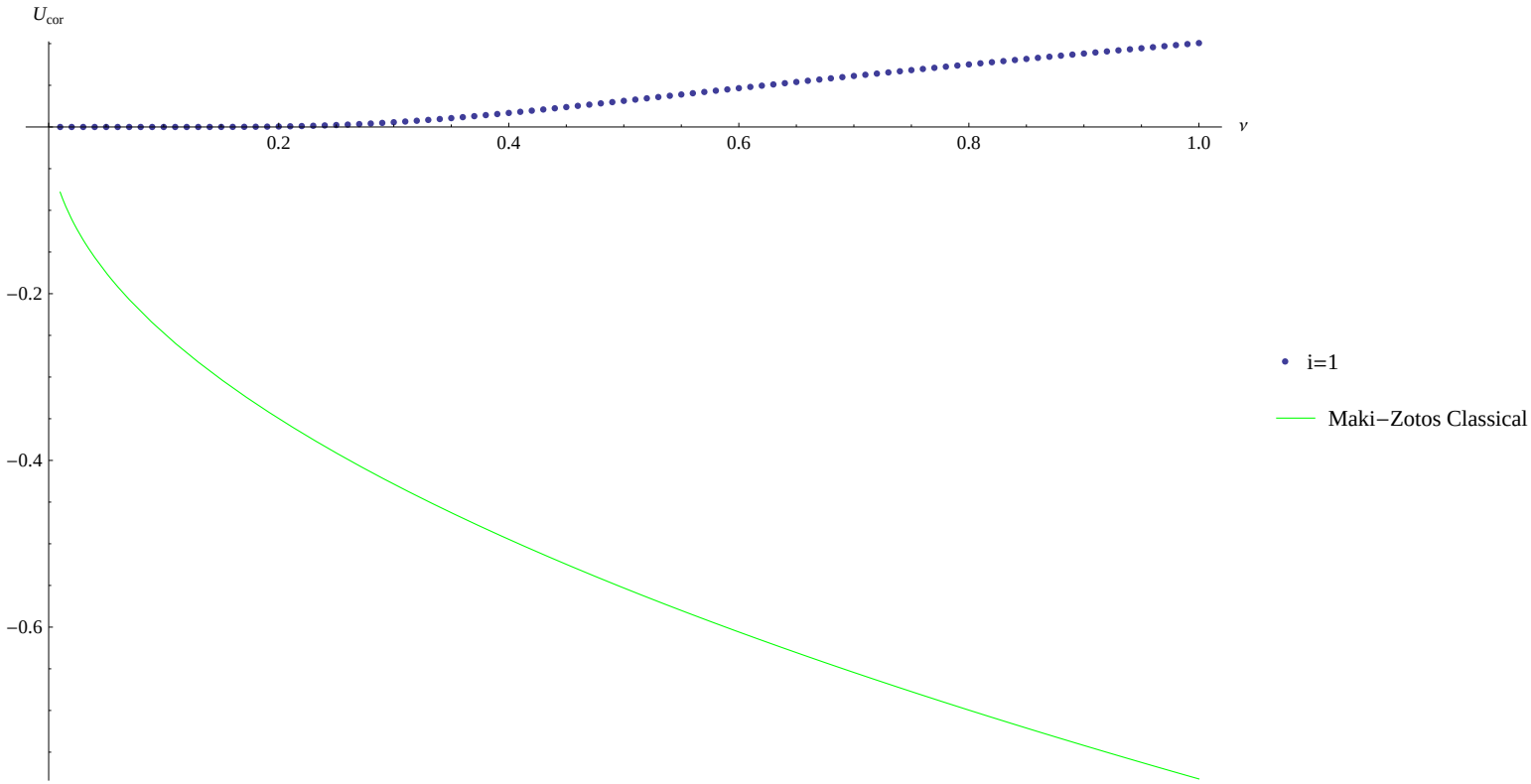

```
UCori1[msize_, nsize_, ν_] :=
 1/2 Sum[Sum[If[R[m, n, ν] ≤ circleradius[msize, ν], If[m == 0 && n == 0, 0,
    1/100 Sqrt[π/5] Csch[(R[m, n, ν])^2/4]
     (BesselI[1, (R[m, n, ν])^2/40] Sinh[(R[m, n, ν])^2/40] (R[m, n, ν])^2 +
      BesselI[0, (R[m, n, ν])^2/40]
       (20 Sinh[(R[m, n, ν])^2/40] + Cosh[(R[m, n, ν])^2/40] (R[m, n, ν])^2))], 0],
  {n, -nsize, nsize}], {m, -msize, msize}]
```

```
Tablei1 = Parallelize[Table[{v, UCori1[650, 650, v]}, {v, 0.01, 1, 0.01}]]
```

$$\{\{0.01, 3.12359\times10^{-63}\}, \{0.02, 7.17887\times10^{-32}\}, \{0.03, 1.87924\times10^{-21}\}, \{0.04, 2.92417\times10^{-16}\}, \{0.05, 3.72779\times10^{-13}\},$$
$$\{0.06, 4.32003\times10^{-11}\}, \{0.07, 1.27497\times10^{-9}\}, \{0.08, 1.603\times10^{-8}\}, \{0.09, 1.14229\times10^{-7}\}, \{0.1, 5.47423\times10^{-7}\},$$
$$\{0.11, 1.96698\times10^{-6}\}, \{0.12, 5.69683\times10^{-6}\}, \{0.13, 0.0000139814\}, \{0.14, 0.0000301328\}, \{0.15, 0.0000585383\},$$
$$\{0.16, 0.000104531\}, \{0.17, 0.000174152\}, \{0.18, 0.000273861\}, \{0.19, 0.000410219\}, \{0.2, 0.0005896\}, \{0.21, 0.000817937\},$$
$$\{0.22, 0.00110053\}, \{0.23, 0.00144189\}, \{0.24, 0.00184567\}, \{0.25, 0.00231463\}, \{0.26, 0.00285063\}, \{0.27, 0.00345466\},$$
$$\{0.28, 0.00412689\}, \{0.29, 0.00486678\}, \{0.3, 0.00567313\}, \{0.31, 0.00654417\}, \{0.32, 0.00747764\}, \{0.33, 0.00847093\},$$
$$\{0.34, 0.00952106\}, \{0.35, 0.0106249\}, \{0.36, 0.011779\}, \{0.37, 0.0129799\}, \{0.38, 0.0142242\}, \{0.39, 0.0155083\},$$
$$\{0.4, 0.0168286\}, \{0.41, 0.0181818\}, \{0.42, 0.0195645\}, \{0.43, 0.0209735\}, \{0.44, 0.0224056\}, \{0.45, 0.023858\},$$
$$\{0.46, 0.0253277\}, \{0.47, 0.0268122\}, \{0.48, 0.028309\}, \{0.49, 0.0298158\}, \{0.5, 0.0313303\}, \{0.51, 0.0328507\},$$
$$\{0.52, 0.0343751\}, \{0.53, 0.0359019\}, \{0.54, 0.0374294\}, \{0.55, 0.0389563\}, \{0.56, 0.0404814\}, \{0.57, 0.0420036\},$$
$$\{0.58, 0.0435218\}, \{0.59, 0.0450352\}, \{0.6, 0.0465431\}, \{0.61, 0.0480446\}, \{0.62, 0.0495394\}, \{0.63, 0.0510268\},$$
$$\{0.64, 0.0525065\}, \{0.65, 0.0539781\}, \{0.66, 0.0554415\}, \{0.67, 0.0568962\}, \{0.68, 0.0583424\}, \{0.69, 0.0597797\},$$
$$\{0.7, 0.0612083\}, \{0.71, 0.062628\}, \{0.72, 0.0640389\}, \{0.73, 0.0654411\}, \{0.74, 0.0668347\}, \{0.75, 0.0682198\},$$
$$\{0.76, 0.0695965\}, \{0.77, 0.070965\}, \{0.78, 0.0723255\}, \{0.79, 0.0736782\}, \{0.8, 0.0750233\}, \{0.81, 0.076361\}, \{0.82, 0.0776915\},$$
$$\{0.83, 0.0790151\}, \{0.84, 0.080332\}, \{0.85, 0.0816424\}, \{0.86, 0.0829467\}, \{0.87, 0.0842449\}, \{0.88, 0.0855375\},$$
$$\{0.89, 0.0868245\}, \{0.9, 0.0881063\}, \{0.91, 0.0893831\}, \{0.92, 0.0906552\}, \{0.93, 0.0919227\}, \{0.94, 0.0931858\},$$
$$\{0.95, 0.0944449\}, \{0.96, 0.0957001\}, \{0.97, 0.0969517\}, \{0.98, 0.0981997\}, \{0.99, 0.0994445\}, \{1., 0.100686\}\}$$

```
Tablei1 = {{0.01`, 3.1235944158886063`*^-63}, {0.02`, 7.178873516155173`*^-32}, {0.03`, 1.879239046985583`*^-21},
   {0.04`, 2.924166692877914`*^-16}, {0.05`, 3.72779027184479`*^-13}, {0.060000000000000005`, 4.320027303149889`*^-11},
   {0.07`, 1.2749736888209213`*^-9}, {0.08`, 1.6030036733194873`*^-8}, {0.09`, 1.1422857760977053`*^-7},
   {0.1`, 5.474234619567345`*^-7}, {0.11`, 1.9669756406572642`*^-6}, {0.12`, 5.696826072938237`*^-6},
   {0.13`, 0.00001398136606799 5643`}, {0.14`, 0.00003013277051481678`}, {0.15`, 0.00005853834942223001`},
   {0.16`, 0.00010453056185385256`}, {0.17`, 0.00017415209334431278`}, {0.18`, 0.00027386092893926 84`},
   {0.19`, 0.0004102191331901728`}, {0.2`, 0.0005896000619966219`}, {0.21000000000000002`, 0.0008179371309223661`},
   {0.22000000000000003`, 0.0011005263235981536`}, {0.23`, 0.0014418859037254855`},
   {0.24000000000000002`, 0.001845670695371994`}, {0.25`, 0.0023146345836592367`}, {0.26`, 0.0028506330638277265`},
   {0.27`, 0.003454657190195986`}, {0.28`, 0.004126890671498896`}, {0.29000000000000004`, 0.004866782752393695`},
   {0.30000000000000004`, 0.005673130643994312`}, {0.31000000000000005`, 0.006544166439967541`},
   {0.32000000000000006`, 0.007477644569137776`}, {0.33`, 0.00847092683089286`}, {0.34`, 0.009521062910152707`},
   {0.35000000000000003`, 0.010624864970107025`}, {0.36`, 0.011778975481971885`}, {0.37`, 0.012979927886917033`},
   {0.38`, 0.014224200013999167`}, {0.39`, 0.015508260417517211`}, {0.4`, 0.01682860796459208`},
   {0.41000000000000003`, 0.018181805113981087`}, {0.42000000000000004`, 0.019564505393020293`},
   {0.43000000000000005`, 0.02097347561186899`}, {0.44000000000000006`, 0.02240561336165109`}, {0.45`, 0.023857960332649555`},
   {0.46`, 0.025327711965917127`}, {0.47000000000000003`, 0.026812223920784365`}, {0.48`, 0.028309015805019194`},
   {0.49`, 0.029815772576246093`}, {0.5`, 0.03133034398444208`}, {0.51`, 0.03285074238715781`},
   {0.52`, 0.03437513923243398`}, {0.53`, 0.03590186046977031`}, {0.54`, 0.03742938111729301`}, {0.55`, 0.0389563191836427`},
   {0.56`, 0.04048142911611163`}, {0.5700000000000001`, 0.042003594922168756`}, {0.5800000000000001`, 0.04352182308963094`},
   {0.5900000000000001`, 0.045035235411231576`}, {0.6000000000000001`, 0.046543061802055224`}, {0.61`, 0.0480446331830683`},
   {0.62`, 0.04953937449062384`}, {0.63`, 0.051026797860163745`}, {0.64`, 0.05250649602223431`},
   {0.65`, 0.053978135940202815`}, {0.66`, 0.0554414527115659`}, {0.67`, 0.05689624374834787`},
   {0.68`, 0.05834236324665626`}, {0.6900000000000001`, 0.05977971695089372`}, {0.7000000000000001`, 0.061208257214302876`},
   {0.7100000000000001`, 0.06262797835435767`}, {0.7200000000000001`, 0.06403891229892336`}, {0.73`, 0.06544112451700984`},
   {0.74`, 0.06683471022627757`}, {0.75`, 0.06821979086815705`}, {0.76`, 0.06959651084046078`},
   {0.77`, 0.07096503447665443`}, {0.78`, 0.07232554326047036`}, {0.79`, 0.07367823326425425`},
   {0.8`, 0.07502331279930734`}, {0.81`, 0.0763610002664871`}, {0.8200000000000001`, 0.07769152219544496`},
   {0.8300000000000001`, 0.07901511146108112`}, {0.8400000000000001`, 0.08033200566607002`}, {0.85`, 0.08164244567864214`},
   {0.86`, 0.08294667431517856`}, {0.87`, 0.08424493515758388`}, {0.88`, 0.08553747149582738`},
   {0.89`, 0.0868245253864882`}, {0.9`, 0.08810633681859008`}, {0.91`, 0.08938314297846281`},
   {0.92`, 0.09065517760582148`}, {0.93`, 0.0919226704336963`}, {0.9400000000000001`, 0.09318584670528356`},
   {0.9500000000000001`, 0.09444492676121279`}, {0.9600000000000001`, 0.09570012569113491`}, {0.97`, 0.09695165304393344`},
   {0.98`, 0.09819971259124279`}, {0.99`, 0.09944450213931853`}, {1.`, 0.10068621338465536`}};
```

```
Show[ListPlot[Tablei1, PlotLegends → {"i=1"}],
 Plot[-0.78213264 √ν , {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```

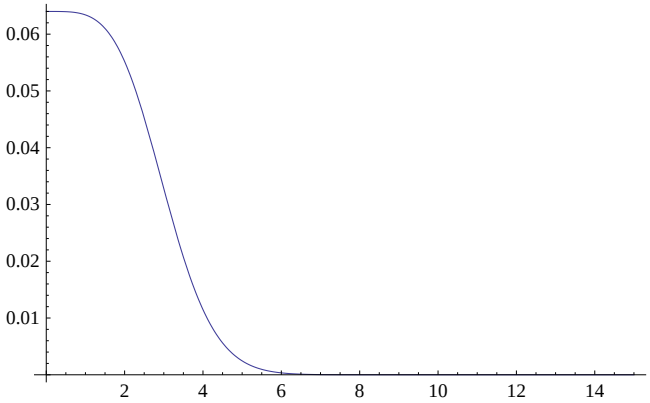


i=2:

```
U[2]
```

$$\frac{1}{125}\,\text{Csch}\left[\frac{R_{ij}^2}{4}\right]\left(20\,\text{Sinh}\left[\frac{R_{ij}^2}{20}\right]+\text{Cosh}\left[\frac{R_{ij}^2}{20}\right]R_{ij}^2\right)$$

```
Plot[1/125 Csch[R_ij^2/4] (20 Sinh[R_ij^2/20] + Cosh[R_ij^2/20] R_ij^2), {R_ij, 0, 15}]
```



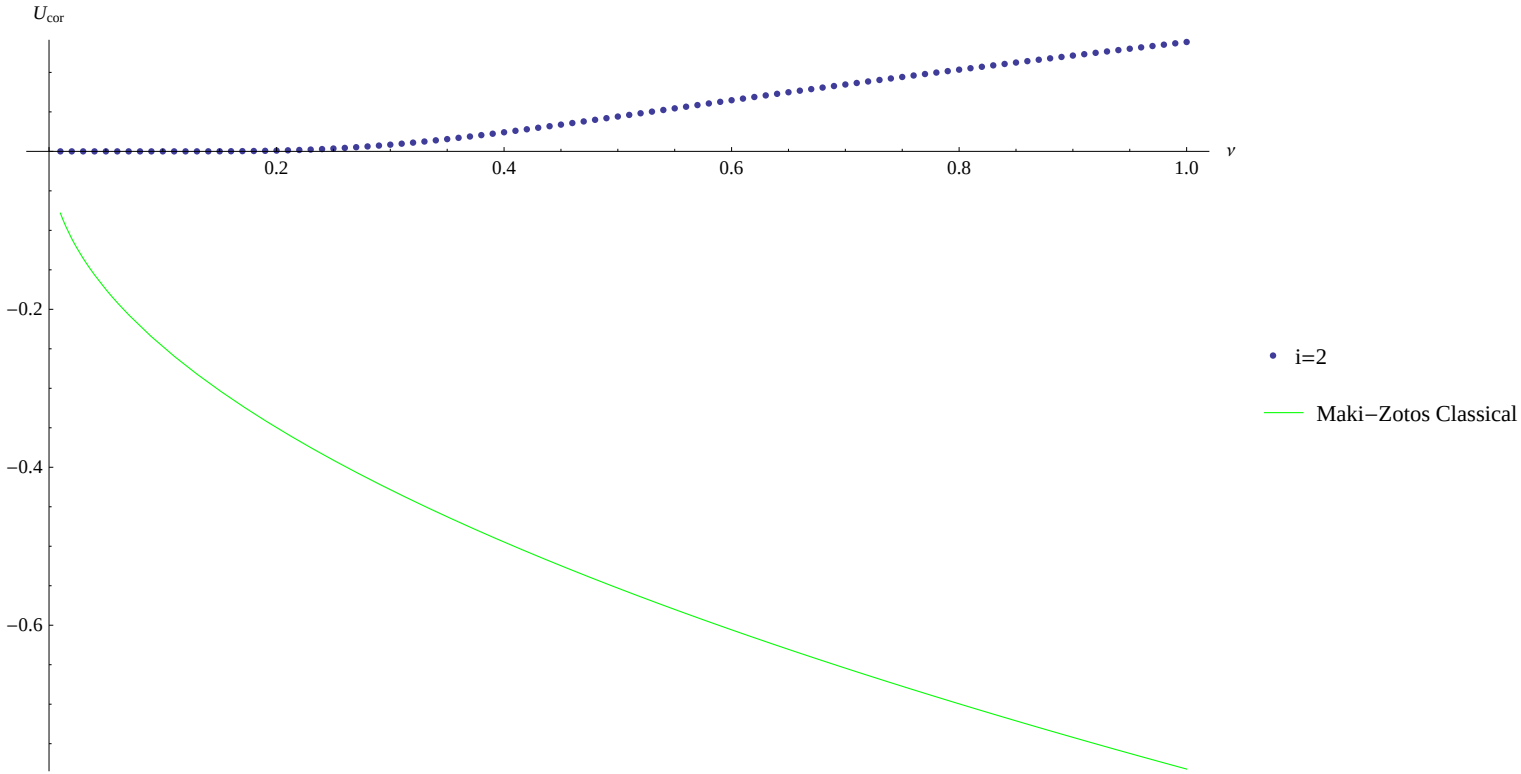

```
UCori2[msize_, nsize_, ν_] := 1/2 Sum_{m=-msize}^{msize} Sum_{n=-nsize}^{nsize} (If[R[m, n, ν] ≤ circleradius[msize, ν],
    If[m == 0 && n == 0, 0, 1/125 Csch[(R[m, n, ν])^2/4] (20 Sinh[(R[m, n, ν])^2/20] + Cosh[(R[m, n, ν])^2/20] (R[m, n, ν])^2)], 0])
```

```
Tablei2 = Parallelize[Table[{ν, UCori2[650, 650, ν]}, {ν, 0.01, 1, 0.01}]]
```

```
{{0.01, 1.71722 × 10^-62}, {0.02, 2.84587 × 10^-31}, {0.03, 6.19868 × 10^-21}, {0.04, 8.5068 × 10^-16}, {0.05, 9.87188 × 10^-13},
 {0.06, 1.06221 × 10^-10}, {0.07, 2.95019 × 10^-9}, {0.08, 3.52469 × 10^-8}, {0.09, 2.40418 × 10^-7}, {0.1, 1.10914 × 10^-6},
 {0.11, 3.854 × 10^-6}, {0.12, 0.0000108348}, {0.13, 0.0000258928}, {0.14, 0.0000544852}, {0.15, 0.000103586},
 {0.16, 0.00018139}, {0.17, 0.000296885}, {0.18, 0.000459383}, {0.19, 0.000678057}, {0.2, 0.000961542}, {0.21, 0.00131762},
 {0.22, 0.00175298}, {0.23, 0.0022731}, {0.24, 0.00288216}, {0.25, 0.00358305}, {0.26, 0.0043774}, {0.27, 0.00526569},
 {0.28, 0.00624734}, {0.29, 0.00732081}, {0.3, 0.00848376}, {0.31, 0.00973318}, {0.32, 0.0110655}, {0.33, 0.0124766},
 {0.34, 0.0139621}, {0.35, 0.0155175}, {0.36, 0.0171379}, {0.37, 0.0188185}, {0.38, 0.0205543}, {0.39, 0.0223407},
 {0.4, 0.0241728}, {0.41, 0.0260459}, {0.42, 0.0279558}, {0.43, 0.0298979}, {0.44, 0.0318684}, {0.45, 0.0338633},
 {0.46, 0.035879}, {0.47, 0.037912}, {0.48, 0.0399593}, {0.49, 0.0420178}, {0.5, 0.0440848}, {0.51, 0.0461577},
 {0.52, 0.0482344}, {0.53, 0.0503127}, {0.54, 0.0523908}, {0.55, 0.0544668}, {0.56, 0.0565393}, {0.57, 0.058607},
 {0.58, 0.0606685}, {0.59, 0.062723}, {0.6, 0.0647694}, {0.61, 0.0668069}, {0.62, 0.068835}, {0.63, 0.070853},
 {0.64, 0.0728604}, {0.65, 0.074857}, {0.66, 0.0768424}, {0.67, 0.0788165}, {0.68, 0.080779}, {0.69, 0.0827299},
 {0.7, 0.0846693}, {0.71, 0.0865971}, {0.72, 0.0885134}, {0.73, 0.0904183}, {0.74, 0.092312}, {0.75, 0.0941947},
 {0.76, 0.0960666}, {0.77, 0.097928}, {0.78, 0.099779}, {0.79, 0.10162}, {0.8, 0.103451}, {0.81, 0.105273},
 {0.82, 0.107085}, {0.83, 0.108889}, {0.84, 0.110684}, {0.85, 0.112471}, {0.86, 0.11425}, {0.87, 0.116022},
 {0.88, 0.117786}, {0.89, 0.119543}, {0.9, 0.121294}, {0.91, 0.123038}, {0.92, 0.124777}, {0.93, 0.126509},
 {0.94, 0.128236}, {0.95, 0.129958}, {0.96, 0.131675}, {0.97, 0.133388}, {0.98, 0.135096}, {0.99, 0.1368}, {1., 0.1385}}
```

```
Tablei2 = {{0.01`, 1.717220162202486`*^-62}, {0.02`, 2.8458724845683855`*^-31}, {0.03`, 6.1986762703401245`*^-21},
   {0.04`, 8.50680235071783`*^-16}, {0.05`, 9.871882661136219`*^-13}, {0.060000000000000005`, 1.0622125290627001`*^-10},
   {0.07`, 2.950191744271645`*^-9}, {0.08`, 3.524689767669546`*^-8}, {0.09`, 2.404177257437141`*^-7},
   {0.1`, 1.1091407794120706`*^-6}, {0.11`, 3.853996792688027`*^-6}, {0.12`, 0.000010834797750296665`},
   {0.13`, 0.00002589281345800404`}, {0.14`, 0.00005448517675744965`}, {0.15`, 0.00010358609859441083`},
   {0.16`, 0.00018138956915317677`}, {0.17`, 0.00029688496487525275`}, {0.18`, 0.00045938309471372054`},
   {0.19`, 0.0006780571480106223`}, {0.2`, 0.0009615424459451618`}, {0.21000000000000002`, 0.0013176186522166065`},
   {0.22000000000000003`, 0.001752981922796069`}, {0.23`, 0.0022731033434406776`},
   {0.24000000000000002`, 0.002882163422572079`}, {0.25`, 0.00358304932651573`}, {0.26`, 0.004377400866166206`},
   {0.27`, 0.005265692045400146`}, {0.28`, 0.006247336573501093`}, {0.29000000000000004`, 0.007320807656608989`},
   {0.30000000000000004`, 0.00848376432127572`}, {0.31000000000000005`, 0.009733178317930176`},
   {0.32000000000000006`, 0.011065457220805038`}, {0.33`, 0.01247656065581433`}, {0.34`, 0.013962107653659604`},
   {0.35000000000000003`, 0.015517473964200508`}, {0.36`, 0.01713787880977922`}, {0.37`, 0.01881846103158428`},
   {0.38`, 0.020554344924730628`}, {0.39`, 0.022340696291880017`}, {0.4`, 0.024172769395520922`},
   {0.41000000000000003`, 0.0260459455752311`}, {0.42000000000000004`, 0.0279557643345166`},
   {0.43000000000000005`, 0.02989794770513576`}, {0.44000000000000006`, 0.031868418675413464`}, {0.45`, 0.0338633144308895`},
   {0.46`, 0.03587899510680947`}, {0.47000000000000003`, 0.03791204869704013`}, {0.48`, 0.039959292706354996`},
   {0.49`, 0.04201777307513361`}, {0.5`, 0.04408476084906471`}, {0.51`, 0.04615774701259525`}, {0.52`, 0.04823443585436015`},
   {0.53`, 0.05031273718608144`}, {0.54`, 0.05239075769362801`}, {0.55`, 0.054466791660106505`},
   {0.56`, 0.05653931126591371`}, {0.5700000000000001`, 0.058606956639450945`}, {0.5800000000000001`, 0.06066852580446767`},
   {0.5900000000000001`, 0.0627229646455067`}, {0.6000000000000001`, 0.06476935699141437`}, {0.61`, 0.06680691489808463`},
   {0.62`, 0.06883496919527428`}, {0.63`, 0.07085296034820347`}, {0.64`, 0.07286042967251198`},
   {0.65`, 0.0748570109307593`}, {0.66`, 0.07684242232982877`}, {0.67`, 0.07881645893114916`},
   {0.68`, 0.08077898547940353`}, {0.6900000000000001`, 0.0827299296502103`}, {0.7000000000000001`, 0.08466927571299918`},
   {0.7100000000000001`, 0.0865970586018379`}, {0.7200000000000001`, 0.08851335838419587`}, {0.73`, 0.09041829511545045`},
   {0.74`, 0.09231202406527593`}, {0.75`, 0.0941947313008214`}, {0.76`, 0.09606662961072145`},
   {0.77`, 0.0979279547534285`}, {0.78`, 0.09977896201306681`}, {0.79`, 0.10161992304592866`},
   {0.8`, 0.1034511230008401`}, {0.81`, 0.10527285789686693`}, {0.8200000000000001`, 0.10708543224219781`},
   {0.8300000000000001`, 0.10888915687849841`}, {0.8400000000000001`, 0.11068434703555462`}, {0.85`, 0.1124713205816047`},
   {0.86`, 0.11425039645537888`}, {0.87`, 0.11602189326650467`}, {0.88`, 0.11778612805159781`},
   {0.89`, 0.1195434151740173`}, {0.9`, 0.12129406535592736`}, {0.91`, 0.12303838483195971`},
   {0.92`, 0.12477667461441`}, {0.93`, 0.12650922986052907`}, {0.9400000000000001`, 0.12823633933306763`},
   {0.9500000000000001`, 0.12995828494582196`}, {0.9600000000000001`, 0.13167534138648262`}, {0.97`, 0.13338777580962288`},
   {0.98`, 0.1350958475931761`}, {0.99`, 0.13679980815222956`}, {1.`, 0.13849990080442348`}};
```

```
Show[ListPlot[Tablei2, PlotLegends → {"i=2"}],
 Plot[-0.78213264 √ν , {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```

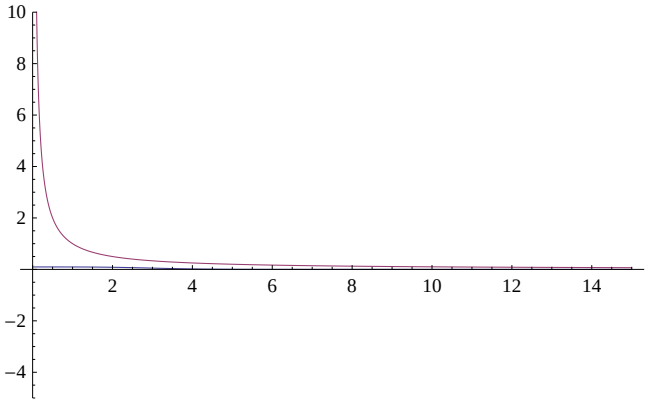


i=3:

```
U[3]
```

$$\frac{1}{2500}\sqrt{\frac{\pi}{5}}\ \text{Csch}\left[\frac{R_{ij}^2}{4}\right]\left(\text{BesselI}\left[1,\frac{R_{ij}^2}{40}\right]R_{ij}^2\left(40\,\text{Sinh}\left[\frac{R_{ij}^2}{40}\right]+\text{Cosh}\left[\frac{R_{ij}^2}{40}\right]R_{ij}^2\right)+\text{BesselI}\left[0,\frac{R_{ij}^2}{40}\right]\left(60\,\text{Cosh}\left[\frac{R_{ij}^2}{40}\right]R_{ij}^2+\text{Sinh}\left[\frac{R_{ij}^2}{40}\right]\left(600+R_{ij}^4\right)\right)\right)$$

```
Plot[
 {1/2500 Sqrt[π/5] Csch[R_ij^2/4] (BesselI[1, R_ij^2/40] R_ij^2 (40 Sinh[R_ij^2/40] + Cosh[R_ij^2/40] R_ij^2) + BesselI[0, R_ij^2/40] (60 Cosh[R_ij^2/40] R_ij^2 + Sinh[R_ij^2/40] (600 + R_ij^4))),
  1/R_ij}, {R_ij, 0, 15}, PlotRange → {-5, 10}]
```



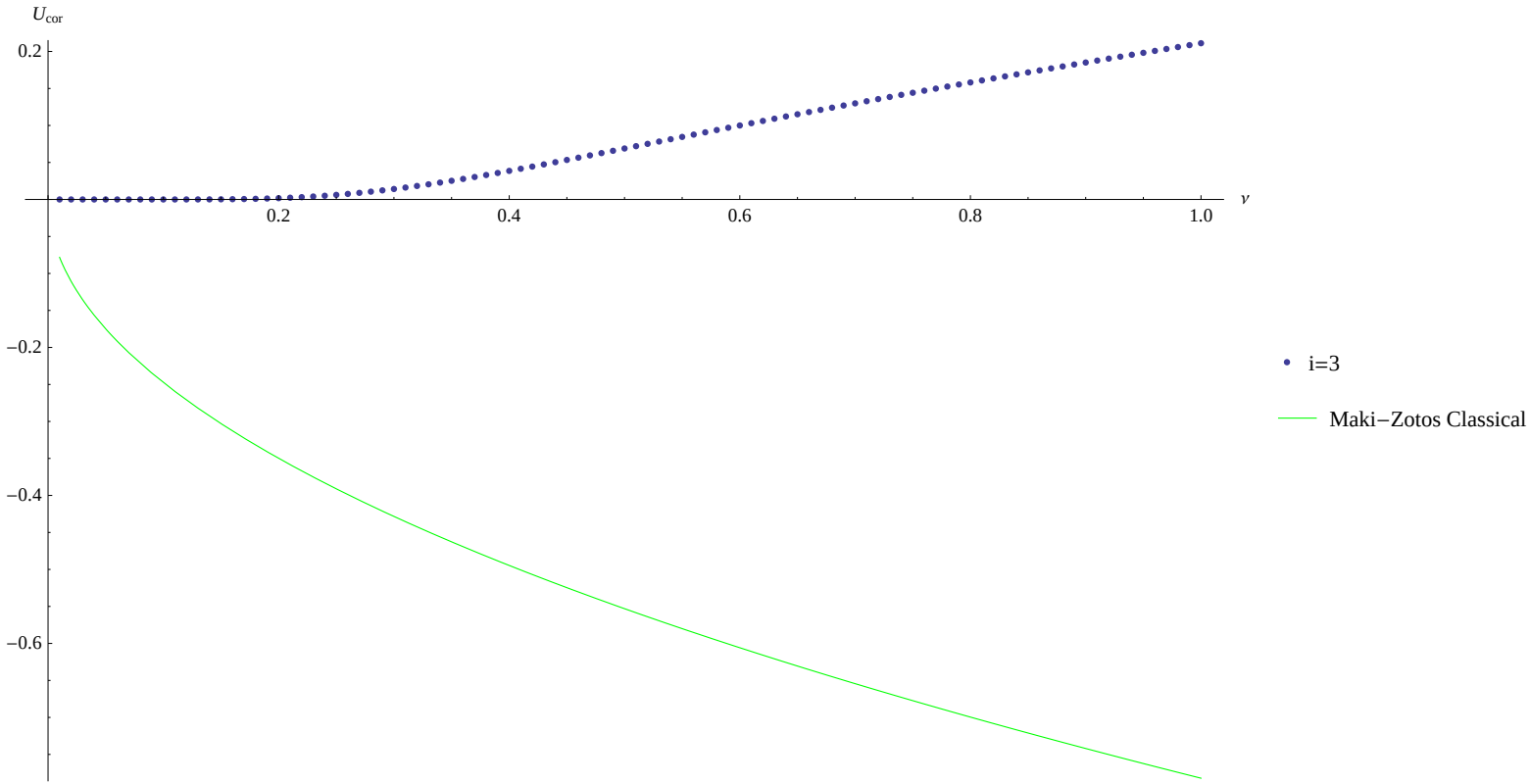

```
UCori3[msize_, nsize_, ν_] :=
 1/2 Sum[Sum[If[R[m, n, ν] ≤ circleradius[msize, ν], If[m == 0 && n == 0, 0, 1/2500 Sqrt[π/5] Csch[(R[m, n, ν])^2/4]
 (BesselI[1, (R[m, n, ν])^2/40] (R[m, n, ν])^2 (40 Sinh[(R[m, n, ν])^2/40] + Cosh[(R[m, n, ν])^2/40] (R[m, n, ν])^2) +
 BesselI[0, (R[m, n, ν])^2/40] (60 Cosh[(R[m, n, ν])^2/40] (R[m, n, ν])^2 + Sinh[(R[m, n, ν])^2/40] (600 + (R[m, n, ν])^4)))], 0], {n, -nsize, nsize}], {m, -msize, msize}]
```

```
Tablei3 = Parallelize[Table[{v, UCori3[650, 650, v]}, {v, 0.01, 1, 0.01}]]
```

$$\{\{0.01, 9.56297\times10^{-62}\}, \{0.02, 1.15575\times10^{-30}\}, \{0.03, 2.11546\times10^{-20}\}, \{0.04, 2.58293\times10^{-15}\}, \{0.05, 2.74975\times10^{-12}\},$$
$$\{0.06, 2.76618\times10^{-10}\}, \{0.07, 7.27507\times10^{-9}\}, \{0.08, 8.30572\times10^{-8}\}, \{0.09, 5.45041\times10^{-7}\}, \{0.1, 2.43169\times10^{-6}\},$$
$$\{0.11, 8.20484\times10^{-6}\}, \{0.12, 0.0000224725\}, \{0.13, 0.0000524643\}, \{0.14, 0.000108097\}, \{0.15, 0.000201621\}, \{0.16, 0.000346963\},$$
$$\{0.17, 0.000558912\}, \{0.18, 0.000852289\}, \{0.19, 0.00124121\}, \{0.2, 0.00173851\}, \{0.21, 0.00235527\}, \{0.22, 0.00310062\},$$
$$\{0.23, 0.00398153\}, \{0.24, 0.0050029\}, \{0.25, 0.00616755\}, \{0.26, 0.0074764\}, \{0.27, 0.00892867\}, \{0.28, 0.010522\},$$
$$\{0.29, 0.0122529\}, \{0.3, 0.0141165\}, \{0.31, 0.0161073\}, \{0.32, 0.018219\}, \{0.33, 0.0204448\}, \{0.34, 0.0227774\},$$
$$\{0.35, 0.0252094\}, \{0.36, 0.0277334\}, \{0.37, 0.0303418\}, \{0.38, 0.0330272\}, \{0.39, 0.0357822\}, \{0.4, 0.0385997\},$$
$$\{0.41, 0.0414729\}, \{0.42, 0.0443954\}, \{0.43, 0.0473608\}, \{0.44, 0.0503633\}, \{0.45, 0.0533974\}, \{0.46, 0.0564579\},$$
$$\{0.47, 0.05954\}, \{0.48, 0.0626393\}, \{0.49, 0.0657516\}, \{0.5, 0.0688731\}, \{0.51, 0.0720005\}, \{0.52, 0.0751306\},$$
$$\{0.53, 0.0782606\}, \{0.54, 0.0813879\}, \{0.55, 0.0845103\}, \{0.56, 0.0876257\}, \{0.57, 0.0907324\}, \{0.58, 0.0938289\},$$
$$\{0.59, 0.0969136\}, \{0.6, 0.0999856\}, \{0.61, 0.103044\}, \{0.62, 0.106087\}, \{0.63, 0.109116\}, \{0.64, 0.112128\}, \{0.65, 0.115125\},$$
$$\{0.66, 0.118105\}, \{0.67, 0.121068\}, \{0.68, 0.124014\}, \{0.69, 0.126944\}, \{0.7, 0.129856\}, \{0.71, 0.132753\}, \{0.72, 0.135632\},$$
$$\{0.73, 0.138496\}, \{0.74, 0.141343\}, \{0.75, 0.144175\}, \{0.76, 0.146992\}, \{0.77, 0.149793\}, \{0.78, 0.152581\}, \{0.79, 0.155354\},$$
$$\{0.8, 0.158113\}, \{0.81, 0.160859\}, \{0.82, 0.163592\}, \{0.83, 0.166313\}, \{0.84, 0.169023\}, \{0.85, 0.17172\}, \{0.86, 0.174407\},$$
$$\{0.87, 0.177083\}, \{0.88, 0.179749\}, \{0.89, 0.182406\}, \{0.9, 0.185053\}, \{0.91, 0.187692\}, \{0.92, 0.190323\}, \{0.93, 0.192945\},$$
$$\{0.94, 0.19556\}, \{0.95, 0.198168\}, \{0.96, 0.200769\}, \{0.97, 0.203364\}, \{0.98, 0.205953\}, \{0.99, 0.208536\}, \{1., 0.211114\}\}$$

```
Tablei3 = {{0.01`, 9.562970336710711`*^-62}, {0.02`, 1.1557526060258763`*^-30}, {0.03`, 2.1154634411186947`*^-20},
   {0.04`, 2.5829301324064986`*^-15}, {0.05`, 2.7497474757367863`*^-12}, {0.060000000000000005`, 2.7661780472996184`*^-10},
   {0.07`, 7.27506680149373`*^-9}, {0.08`, 8.305720814517375`*^-8}, {0.09`, 5.450407788327194`*^-7},
   {0.1`, 2.4316931461902`*^-6}, {0.11`, 8.204835544057543`*^-6}, {0.12`, 0.00002472454920364682`},
   {0.13`, 0.00005246426412300 1984`}, {0.14`, 0.00010809665120058745`}, {0.15`, 0.00020162077624059558`},
   {0.16`, 0.00034696312975826505`}, {0.17`, 0.0005589117799237435`}, {0.18`, 0.000852289438516899`},
   {0.19`, 0.0012412149338493902`}, {0.2`, 0.0017385103811996354`}, {0.21000000000000002`, 0.002355274493873403`},
   {0.22000000000000003`, 0.0031006170243264943`}, {0.23`, 0.003981534576696532`},
   {0.24000000000000002`, 0.005002901474134537`}, {0.25`, 0.0061675482535864`}, {0.26`, 0.007476402421387729`},
   {0.27`, 0.008928669702003521`}, {0.28`, 0.01052203809640705`}, {0.29000000000000004`, 0.01225289102397982`},
   {0.30000000000000004`, 0.014116519342164438`}, {0.31000000000000005`, 0.016107325000971288`},
   {0.32000000000000006`, 0.01821901148427686`}, {0.33`, 0.020444758063274438`}, {0.34`, 0.022777376310701283`},
   {0.35000000000000003`, 0.025209448374690455`}, {0.36`, 0.027733447261373643`}, {0.37`, 0.030341839890175498`},
   {0.38`, 0.03302717401953807`}, {0.39`, 0.03578215033816862`}, {0.4`, 0.03859968111348907`},
   {0.41000000000000003`, 0.04147293681284218`}, {0.42000000000000004`, 0.04439538208589343`},
   {0.43000000000000005`, 0.04736080243515102`}, {0.44000000000000006`, 0.05036332281809196`}, {0.45`, 0.05339741932830526`},
   {0.46`, 0.05645792500109228`}, {0.47000000000000003`, 0.05954003068593349`}, {0.48`, 0.06263928182749325`},
   {0.49`, 0.06575157190063305`}, {0.5`, 0.0688731331546562`}, {0.51`, 0.07200052523851802`}, {0.52`, 0.07513062220236584`},
   {0.53`, 0.07826059830157536`}, {0.54`, 0.08138791296724626`}, {0.55`, 0.08451029525160611`},
   {0.56`, 0.08762572800753382`}, {0.5700000000000001`, 0.09073243201799565`}, {0.5800000000000001`, 0.09382885025311659`},
   {0.5900000000000001`, 0.09691363239939328`}, {0.6000000000000001`, 0.09998561977673295`}, {0.61`, 0.10304383073411387`},
   {0.62`, 0.10608744659329931`}, {0.63`, 0.109115798191785`}, {0.64`, 0.11212835306067985`},
   {0.65`, 0.11512470326016627`}, {0.66`, 0.11810455388425821`}, {0.67`, 0.12106771223752293`},
   {0.68`, 0.12401407767898266`}, {0.6900000000000001`, 0.12694363212238 18`}, {0.7000000000000001`, 0.1298564311771707`},
   {0.7100000000000001`, 0.1327525959107748`}, {0.7200000000000001`, 0.13563230520982025`}, {0.73`, 0.13849578871584203`},
   {0.74`, 0.14134332030950766`}, {0.75`, 0.14417521211642362`}, {0.76`, 0.1469918090070827`},
   {0.77`, 0.1497934835633742`}, {0.78`, 0.15258063148424397`}, {0.79`, 0.15535366740350892`},
   {0.8`, 0.15811302109344672`}, {0.81`, 0.16085913402854946`}, {0.8200000000000001`, 0.16359245628471863`},
   {0.8300000000000001`, 0.16631344375016152`}, {0.8400000000000001`, 0.1690225556252838`}, {0.85`, 0.1717202521899569`},
   {0.86`, 0.1744069928176411`}, {0.87`, 0.1770832342169525`}, {0.88`, 0.17974942888236833`},
   {0.89`, 0.18240602373685097`}, {0.9`, 0.18505345895023778`}, {0.91`, 0.1876921669182728`},
   {0.92`, 0.19032257138816397`}, {0.93`, 0.19294508671750013`}, {0.9400000000000001`, 0.19556011725428873`},
   {0.9500000000000001`, 0.1981680568267496`}, {0.9600000000000001`, 0.20076928833233113`}, {0.97`, 0.20336418341620519`},
   {0.98`, 0.2059531022302445`}, {0.99`, 0.20853639326418155`}, {1.`, 0.21111439324131442`}};
```

```
Show[ListPlot[Tablei3, PlotLegends → {"i=3"}],
 Plot[-0.78213264 √ν , {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```

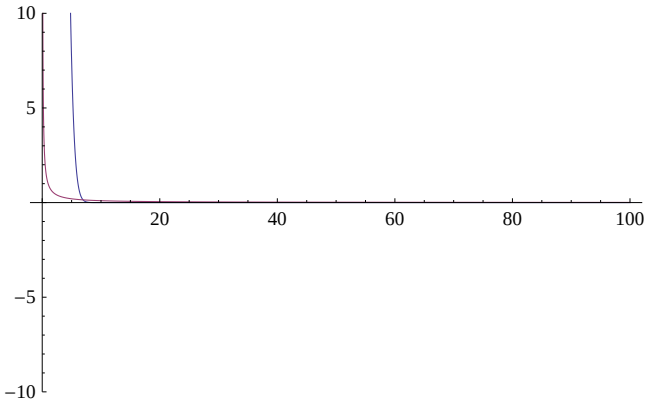


i=4:

```
U[4]
```

$$\frac{\text{Csch}\left[\frac{R_{ij}^2}{4}\right]\left(80\,\text{Cosh}\left[\frac{R_{ij}^2}{20}\right]R_{ij}^2+\text{Sinh}\left[\frac{R_{ij}^2}{20}\right]\left(800+R_{ij}^4\right)\right)}{3125}$$

$$\text{Plot}\left[\left\{1000\,\frac{\text{Csch}\left[\frac{R_{ij}^2}{4}\right]\left(80\,\text{Cosh}\left[\frac{R_{ij}^2}{20}\right]R_{ij}^2+\text{Sinh}\left[\frac{R_{ij}^2}{20}\right]\left(800+R_{ij}^4\right)\right)}{3125},\ \frac{1}{R_{ij}}\right\},\ \{R_{ij},\ 0,\ 100\},\ \text{PlotRange}\to\{-10,\ 10\}\right]$$



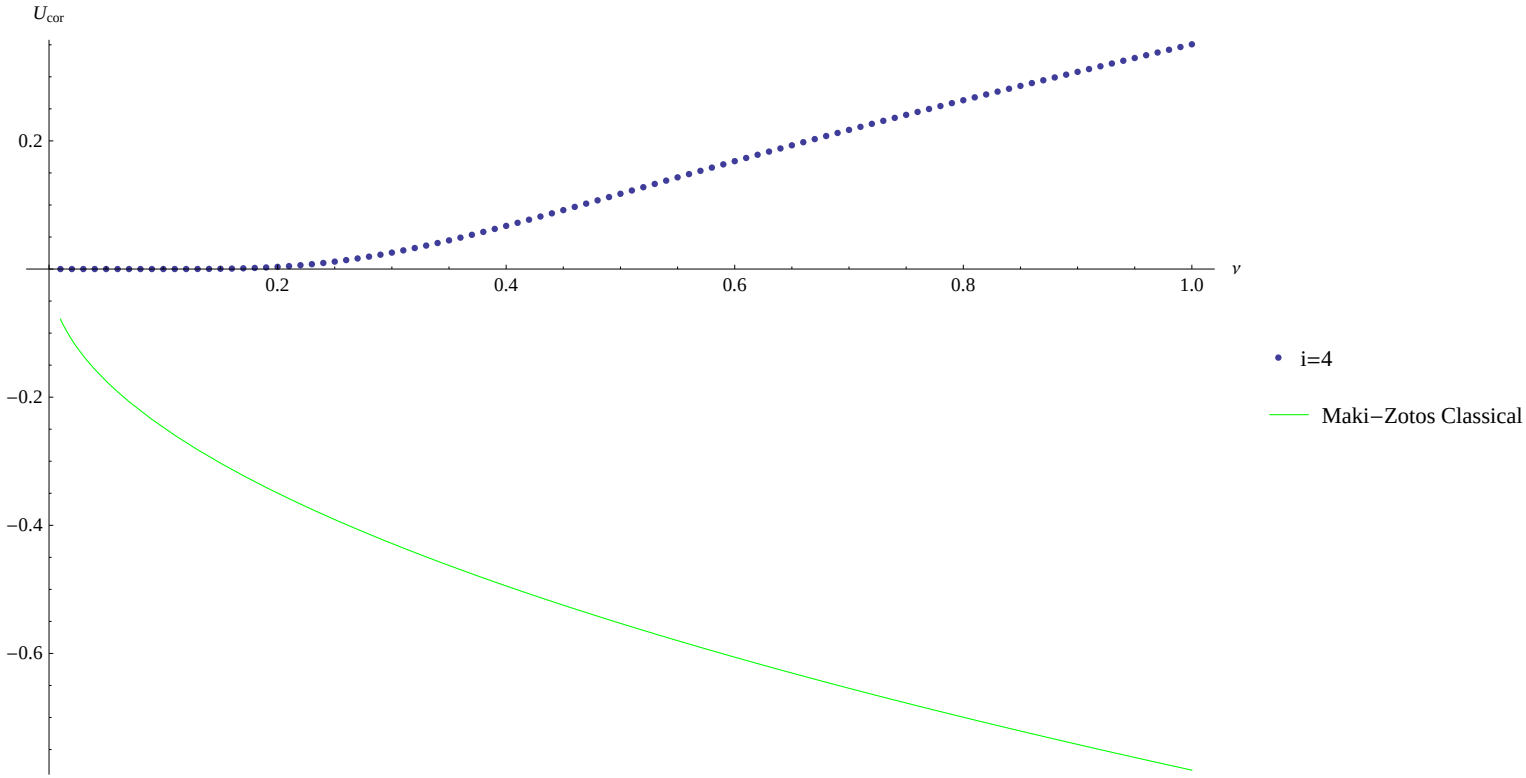
$$\text{UCori4}[\text{msize\_},\ \text{nsize\_},\ \nu\_] := \frac{1}{2}\sum_{m=-\text{msize}}^{\text{msize}}\sum_{n=-\text{nsize}}^{\text{nsize}}\left(\text{If}\left[R[m,\ n,\ \nu] \le \text{circleradius}[\text{msize},\ \nu],\ \text{If}\left[m == 0\ \&\&\ n == 0,\ 0,\right.\right.$$
$$\left.\left.\frac{1}{3125}\,\text{Csch}\left[\frac{(R[m,\ n,\ \nu])^2}{4}\right]\left(80\,\text{Cosh}\left[\frac{(R[m,\ n,\ \nu])^2}{20}\right](R[m,\ n,\ \nu])^2+\text{Sinh}\left[\frac{(R[m,\ n,\ \nu])^2}{20}\right]\left(800+(R[m,\ n,\ \nu])^4\right)\right),\ 0\right]\right)$$

$$\text{Tablei4} = \text{Parallelize}[\text{Table}[\{\nu,\ \text{UCori4}[650,\ 650,\ \nu]\},\ \{\nu,\ 0.01,\ 1,\ 0.01\}]]$$

$$\{\{0.01,\ 5.39196\times10^{-61}\},\ \{0.02,\ 4.80059\times10^{-30}\},\ \{0.03,\ 7.44615\times10^{-20}\},\ \{0.04,\ 8.1459\times10^{-15}\},\ \{0.05,\ 8.00338\times10^{-12}\},$$
$$\{0.06,\ 7.56637\times10^{-10}\},\ \{0.07,\ 1.89292\times10^{-8}\},\ \{0.08,\ 2.07341\times10^{-7}\},\ \{0.09,\ 1.31373\times10^{-6}\},\ \{0.1,\ 5.68664\times10^{-6}\},$$
$$\{0.11,\ 0.0000186867\},\ \{0.12,\ 0.0000499972\},\ \{0.13,\ 0.000114305\},\ \{0.14,\ 0.000231107\},\ \{0.15,\ 0.000423731\},$$
$$\{0.16,\ 0.000717856\},\ \{0.17,\ 0.00113988\},\ \{0.18,\ 0.00171536\},\ \{0.19,\ 0.00246777\},\ \{0.2,\ 0.00341757\},\ \{0.21,\ 0.0045816\},$$
$$\{0.22,\ 0.0059728\},\ \{0.23,\ 0.00760024\},\ \{0.24,\ 0.00946917\},\ \{0.25,\ 0.0115814\},\ \{0.26,\ 0.0139356\},\ \{0.27,\ 0.0165276\},$$
$$\{0.28,\ 0.0193511\},\ \{0.29,\ 0.0223978\},\ \{0.3,\ 0.0256578\},\ \{0.31,\ 0.0291202\},\ \{0.32,\ 0.0327731\},\ \{0.33,\ 0.0366039\},$$
$$\{0.34,\ 0.0405999\},\ \{0.35,\ 0.0447483\},\ \{0.36,\ 0.0490361\},\ \{0.37,\ 0.0534506\},\ \{0.38,\ 0.0579796\},\ \{0.39,\ 0.062611\},$$
$$\{0.4,\ 0.0673334\},\ \{0.41,\ 0.0721359\},\ \{0.42,\ 0.0770081\},\ \{0.43,\ 0.0819403\},\ \{0.44,\ 0.0869235\},\ \{0.45,\ 0.091949\},$$
$$\{0.46,\ 0.0970091\},\ \{0.47,\ 0.102096\},\ \{0.48,\ 0.107204\},\ \{0.49,\ 0.112327\},\ \{0.5,\ 0.117458\},\ \{0.51,\ 0.122593\},\ \{0.52,\ 0.127728\},$$
$$\{0.53,\ 0.132858\},\ \{0.54,\ 0.13798\},\ \{0.55,\ 0.14309\},\ \{0.56,\ 0.148186\},\ \{0.57,\ 0.153266\},\ \{0.58,\ 0.158327\},\ \{0.59,\ 0.163367\},$$
$$\{0.6,\ 0.168385\},\ \{0.61,\ 0.17338\},\ \{0.62,\ 0.178351\},\ \{0.63,\ 0.183296\},\ \{0.64,\ 0.188216\},\ \{0.65,\ 0.19311\},\ \{0.66,\ 0.197977\},$$
$$\{0.67,\ 0.202818\},\ \{0.68,\ 0.207633\},\ \{0.69,\ 0.212421\},\ \{0.7,\ 0.217183\},\ \{0.71,\ 0.221919\},\ \{0.72,\ 0.22663\},\ \{0.73,\ 0.231316\},$$
$$\{0.74,\ 0.235977\},\ \{0.75,\ 0.240615\},\ \{0.76,\ 0.245229\},\ \{0.77,\ 0.24982\},\ \{0.78,\ 0.25439\},\ \{0.79,\ 0.258939\},\ \{0.8,\ 0.263466\},$$
$$\{0.81,\ 0.267974\},\ \{0.82,\ 0.272463\},\ \{0.83,\ 0.276934\},\ \{0.84,\ 0.281386\},\ \{0.85,\ 0.285822\},\ \{0.86,\ 0.290241\},$$
$$\{0.87,\ 0.294645\},\ \{0.88,\ 0.299034\},\ \{0.89,\ 0.303409\},\ \{0.9,\ 0.30777\},\ \{0.91,\ 0.312118\},\ \{0.92,\ 0.316454\},\ \{0.93,\ 0.320779\},$$
$$\{0.94,\ 0.325092\},\ \{0.95,\ 0.329396\},\ \{0.96,\ 0.333689\},\ \{0.97,\ 0.337973\},\ \{0.98,\ 0.342248\},\ \{0.99,\ 0.346515\},\ \{1.,\ 0.350774\}\}$$

```
Tablei4 = {{0.01`, 5.391955961957999`*^-61}, {0.02`, 4.800586538019848`*^-30}, {0.03`, 7.446154024402039`*^-20},
   {0.04`, 8.145897430190758`*^-15}, {0.05`, 8.003377149515402`*^-12}, {0.060000000000000005`, 7.566374531034197`*^-10},
   {0.07`, 1.8929182782176464`*^-8}, {0.08`, 2.0734144360623982`*^-7}, {0.09`, 1.3137295644534753`*^-6},
   {0.1`, 5.686639229988056`*^-6}, {0.11`, 0.000018686685237686292`}, {0.12`, 0.00004997154318362595`},
   {0.13`, 0.00011430503488927854`}, {0.14`, 0.0002311073172832656`}, {0.15`, 0.0004237309167764051`},
   {0.16`, 0.0007178563415493659`}, {0.17`, 0.0011398764165042 71`}, {0.18`, 0.0017153579439620583`},
   {0.19`, 0.0024677728990093136`}, {0.2`, 0.0034175709219577254`}, {0.21000000000000002`, 0.004581595519314924`},
   {0.22000000000000003`, 0.005972804877451621`}, {0.23`, 0.007600239155435728`}, {0.24000000000000002`, 0.009469172255389638`},
   {0.25`, 0.01158139094422151`}, {0.26`, 0.013935553152945332`}, {0.27`, 0.016527587345411855`},
   {0.28`, 0.01935110437410539`}, {0.29000000000000004`, 0.02239780145627514`}, {0.30000000000000004`, 0.02565784457327154`},
   {0.31000000000000005`, 0.029120220773624306`}, {0.32000000000000006`, 0.03277305573416722`},
   {0.33`, 0.03660389473440404`}, {0.34`, 0.04059994715322888`}, {0.35000000000000003`, 0.044748295902911066`},
   {0.36`, 0.04903607403764006`}, {0.37`, 0.05345061124395045`}, {0.38`, 0.0579795531394111`}, {0.39`, 0.06261095635083277`},
   {0.4`, 0.06733336227103315`}, {0.41000000000000003`, 0.07213585224575403`}, {0.42000000000000004`, 0.07700808674985014`},
   {0.43000000000000005`, 0.08194033089582005`}, {0.44000000000000006`, 0.08692346839308238`}, {0.45`, 0.09194900585329664`},
   {0.46`, 0.0970090691221741`}, {0.47000000000000003`, 0.1020963931158113`}, {0.48`, 0.10720430645199046`},
   {0.49`, 0.11232671199526356`}, {0.5`, 0.11745806427922084`}, {0.51`, 0.12259334462982172`}, {0.52`, 0.12772803468932012`},
   {0.53`, 0.13285808893021794`}, {0.54`, 0.13797990665174822`}, {0.55`, 0.14309030386652263`},
   {0.56`, 0.14818648541104054`}, {0.5700000000000001`, 0.15326601754966143`}, {0.5800000000000001`, 0.15832680128636406`},
   {0.5900000000000001`, 0.16336704655116774`}, {0.6000000000000001`, 0.16838524738759297`}, {0.61`, 0.17338015823315464`},
   {0.62`, 0.1783507713558855`}, {0.63`, 0.18329629548558202`}, {0.64`, 0.18821613565826978`},
   {0.65`, 0.1931098742757491`}, {0.66`, 0.19797725336851524`}, {0.67`, 0.20281815803945236`},
   {0.68`, 0.20763260105706466`}, {0.6900000000000001`, 0.21242070856033296`}, {0.7000000000000001`, 0.21718270683225485`},
   {0.7100000000000001`, 0.22191891009550346`}, {0.7200000000000001`, 0.22662970928118956`},
   {0.73`, 0.23131556172024542`}, {0.74`, 0.2359769817063031`}, {0.75`, 0.2406145318789544`},
   {0.76`, 0.2452288153768484`}, {0.77`, 0.249820468711081`}, {0.78`, 0.25439015531067216`}, {0.79`, 0.2589385596935416`},
   {0.8`, 0.2634663822181966`}, {0.81`, 0.267974334373296`}, {0.8200000000000001`, 0.2724631345642947`},
   {0.8300000000000001`, 0.27693350435848074`}, {0.8400000000000001`, 0.28138616515182446`},
   {0.85`, 0.28582183522318333`}, {0.86`, 0.2902412271434975`}, {0.87`, 0.29464504550964576`},
   {0.88`, 0.2990339849746366`}, {0.89`, 0.30340872854771317`}, {0.9`, 0.30776994613981135`}, {0.91`, 0.3121182933315631`},
   {0.92`, 0.3164544103427263`}, {0.93`, 0.32077892118351026`}, {0.9400000000000001`, 0.32509243296977386`},
   {0.9500000000000001`, 0.3293955353854983`}, {0.9600000000000001`, 0.3336888002772586`}, {0.97`, 0.3379727813666816`},
   {0.98`, 0.34224801406803934`}, {0.99`, 0.34651501539921686`}, {1.`, 0.3507742839753128`}};
```

```
Show[ListPlot[Tablei4, PlotLegends → {"i=4"}],
 Plot[-0.78213264 √ν , {ν, 0.01, 1}, PlotStyle → Green, PlotLegends → {"Maki-Zotos Classical"}],
 AxesLabel → {ν, "U_cor"}, PlotRange → All, AxesOrigin → {0, 0}]
```



Notice that as *i* increases, the value of the two-body potential at R=0 increases, and the curve is larger overall. Thus the energy contribution increases as *i* increases. ν and R are inversely related. The value of i_{max} cannot be too high.