Coefficients for higher Landau levels

Fourier Transform of $\frac{1}{r}$:

```
Assuming[r ∈ Reals && r > 0 && k ∈ Reals && k > 0 && d ∈ Reals && d > 0, 1/(2 π) ∫_0^(2 π) 1/r ⅇ^(-ⅈ (k r Cos[θ])) r ⅆθ]
```

```
BesselJ[0, k r]
```

```
Assuming[r ∈ Reals && r > 0 && k ∈ Reals && k > 0 && d ∈ Reals && d > 0, ∫_0^∞ BesselJ[0, k r] ⅆr]
```

$$\frac{1}{k}$$

r has units of length, while k has units of 1/length.

Haldane Pseudopotential

Use Haldane pseudopotential. Jain's E3.28 does not have l's and omits e, the electron charge. We can view q as the unitless $q = k\, l$. We expect the energy to have units of $\frac{e^2}{l}$.

See "Jain problem 3.23--Vm_Coulomb.nb" for an example that shows our k, l placements are correct.

$$V_m{}^{(n)} = \frac{e^2}{(2\pi)} \int V(q)\, e^{-(q)^2} L_m\big((q)^2\big) \left(L_n\!\left(\frac{(q)^2}{2}\right)\right)^2 d^2 \vec{q}$$

$$= \frac{e^2}{(2\pi)} \int V(k)\, e^{-(k\, l)^2} L_m\big((k\, l)^2\big) \left(L_n\!\left(\frac{(k\, l)^2}{2}\right)\right)^2 d^2 \vec{k}$$

Try m=3, n=0:

```
Assuming[k ∈ Reals && k > 0 && d ∈ Reals && d ≥ 0 && l ∈ Reals && l > 0,
  1/(2 π) ∫_0^(2 π) ∫_0^∞ 1/k Exp[-(k l)^2] LaguerreL[3, (k l)^2] (LaguerreL[0, (k l)^2/2])^2 k ⅆk ⅆθ] // Simplify
```

$$\frac{5\sqrt{\pi}}{32\,l}$$

Here it is apparent that we have the correct units of 1/length--specifically, 1/l.

General equation:

```
Vmn[m_, n_] := Assuming[k ∈ Reals && k > 0 && d ∈ Reals && d ≥ 0 && l ∈ Reals && l > 0,
  1/(2 π) ∫_0^(2 π) ∫_0^∞ 1/k Exp[-(k l)^2] LaguerreL[m, (k l)^2] (LaguerreL[n, (k l)^2/2])^2 k ⅆk ⅆθ] // Simplify
```

Find c_i for n.

We find the constants c_i of the effective potential by equating the Haldane pseudopotential of $V_{\text{eff}}(k) = \frac{1}{k} + \frac{1}{l}\sum_{i=0}^{\text{imax}} c_i\, f_i(k)$ to the Haldane pseudopotential of $V_{\text{ZDS}}(k)$.

Reference: "Frame potential--Haldane pseudopotential (9-19-13).nb"

The effective potential is:

$V(r)^{\text{eff}} = \frac{1}{r} + \frac{1}{l}\sum_{i=0}^{\text{imax}} c_i \left(\frac{r}{l}\right)^i e^{-\left(\frac{r}{l}\right)^2}$. The placement of the l's ensures that the second part has units of 1/length.

Take the 2-D Fourier transform using the formula (see "2-D Fourier transform integral form.nb," "Higher Mathematics for Physics and Engineering" by Shima, Nakayama and p. 690, 680; Arfken and Weber)

$F(\vec{k}) = \int_0^\infty r\, f(r)\, J_0(k\, r)\, dr$

so that:

$$
\begin{aligned}
V(k)\\
&= \int_0^\infty r\, V(r)\, J_0(k\, r)\, dr\\
&= \int_0^\infty r \left(\frac{1}{r} + \sum_{i=0}^{\text{imax}} c_i\, r^i\, e^{-r^2}\right) J_0(k\, r)\, dr\\
&= \int_0^\infty J_0(k\, r)\, dr + \sum_{i=0}^{\text{imax}} c_i \int_0^\infty r^{i+1}\, e^{-r^2}\, J_0(k\, r)\, dr\,.
\end{aligned}
$$

The Fourier transform of $\frac{1}{r}$ is $\frac{1}{k}$:

```
Assuming[k ∈ Reals && k > 0, ∫_0^∞ BesselJ[0, k r] ⅆr]
```

$\frac{1}{k}$

The Fourier transform of the other terms:

Method 1:

```
Assuming[r ∈ Reals && r > 0 && k ∈ Reals && k > 0 && l ∈ Reals && l > 0, 1/(2 π) ∫_0^(2 π) 1/l (r/l)^i Exp[-(r/l)^2] ⅇ^(-ⅈ (k r Cos[θ])) r ⅆθ]
```

$e^{-\frac{r^2}{l^2}} \left(\frac{r}{l}\right)^{1+i}$ BesselJ[0, k r]

```
Assuming[r ∈ Reals && r > 0 && k ∈ Reals && k > 0 && l ∈ Reals && l > 0, ∫_0^∞ ⅇ^(-r^2/l^2) (r/l)^(1+i) BesselJ[0, k r] ⅆr]
```

ConditionalExpression$\left[\frac{1}{2}\, l\, \text{Gamma}\left[1+\frac{i}{2}\right] \text{LaguerreL}\left[-1-\frac{i}{2}, -\frac{1}{4} k^2 l^2\right], \text{Re}[i] > -2\right]$

Method 2:

```
Assuming[k ∈ Reals && k > 0 && i ∈ Integers && i ≥ 0 && l ∈ Reals && l > 0, ∫_0^∞ (1/l) (r/l)^i Exp[-(r/l)^2] BesselJ[0, k r] r ⅆr]
```

$$\frac{1}{2}\,l\,\mathrm{Gamma}\left[1+\frac{i}{2}\right]\,\mathrm{LaguerreL}\left[-1-\frac{i}{2},\,-\frac{1}{4}k^2\,l^2\right]$$

```
Vkiterm[i_] := (1/2) l Gamma[1 + i/2] LaguerreL[-1 - i/2, -(1/4) k^2 l^2];
```

Example:

```
Vkiterm[1] // FullSimplify
```

$$\frac{1}{4}\,l\,\sqrt{\pi}\,\mathrm{LaguerreL}\left[-\frac{3}{2},\,-\frac{1}{4}k^2\,l^2\right]$$

The Psuedopotential energies

$\frac{1}{k}$ term:

```
Vmnterm1[m_, n_] := Assuming[k ∈ Reals && k > 0 && m ∈ Integers && m ≥ 0 && n ∈ Integers && n ≥ 0 && l ∈ Reals && l > 0,
   1/(2 π) ∫_0^(2 π) ∫_0^∞ (1/k) Exp[-(k l)^2] LaguerreL[m, (k l)^2] (LaguerreL[n, (k l)^2/2])^2 k ⅆk ⅆθ] // Simplify
Vmnterm1[2, 2]
```

$$\frac{3451\sqrt{\pi}}{16\,384\,l}$$

i=0 to imax terms:

$$\text{Vmniterm}[m\_, n\_, i\_] := \text{Assuming}\Big[k \in \text{Reals} \,\&\&\, k > 0 \,\&\&\, m \in \text{Integers} \,\&\&\, m \ge 0 \,\&\&\, n \in \text{Integers} \,\&\&\, n \ge 0 \,\&\&\, i \in \text{Integers} \,\&\&\, i \ge 0, \frac{1}{(2\,\pi)} \int_0^{2\pi}\int_0^{\infty} \frac{1}{2}\, l\, \text{Gamma}\Big[1+\frac{i}{2}\Big]\, \text{LaguerreL}\Big[-1-\frac{i}{2}, -\frac{1}{4} k^2 l^2\Big]\, \text{Exp}\big[-(k\, l)^2\big]\, \text{LaguerreL}\big[m, (k\, l)^2\big] \left(\text{LaguerreL}\Big[n, \frac{(k\, l)^2}{2}\Big]\right)^2 k\, dk\, d\theta\Big] \;//\; \text{Simplify}$$

<u>Example</u>:

```
Vmniterm[2, 1, 3]
```

$$\frac{9159\sqrt{\frac{\pi}{5}}}{200\,000\; l}$$

$$(*\ \text{Numerical integation:}\ \ \text{Vmniterm}[m\_,n\_,i\_]{:=}\text{Assuming}\Big[k{\in}\text{Reals}\&\&k{>}0\&\&m{\in}\text{Integers}\&\&m{\ge}0\&\&n{\in}\text{Integers}\&\&n{\ge}0\&\&i{\in}\text{Integers}\&\&i{\ge}0, \frac{1}{(2\pi)} \text{NIntegrate}\Big[\frac{1}{2}\ \text{Gamma}\Big[1+\frac{i}{2}\Big]\ \text{LaguerreL}\Big[-1-\frac{i}{2},-\frac{k^2}{4}\Big]\text{Exp}\big[-(k)^2\big]\text{LaguerreL}\big[m,(k)^2\big]\left(\text{LaguerreL}\Big[n,\frac{(k)^2}{2}\Big]\right)^2 k,\{k,0,\infty\},\{\theta,0,2\pi\}\Big]\Big]//\text{Simplify}*)$$

Find the effective potential for the n=0 level that corresponds to a V(k) at a higher n:

$V_m{}^0\left(\frac{1}{k} + \sum_{i=0}^{4} c_i\ f_i(k)\right) = V_m{}^n(V(k)).$

We let m run from 0 to 4 so that there will be 5 equations with 5 unknowns (c_0 to c_4):

$$V_0{}^0\left(\frac{1}{k} + \sum_{i=0}^{4} c_i\ f_i(k)\right) = V_0{}^n(V(k))$$
$$\vdots$$
$$V_4{}^0\left(\frac{1}{k} + \sum_{i=0}^{4} c_i\ f_i(k)\right) = V_4{}^n(V(k))\,.$$

We can thus set up a matrix to solve for the c_i. The classical term, i.e. the first term in the effective potential, is moved to the right side so that the left side is composed entirely of the c_i terms.

$$c_0\ V_0{}^0(f_0(k)) + c_1\ V_0{}^0(f_1(k)) \ldots c_4\ V_0{}^0(f_4(k)) = V_0{}^n(V(k)) - V_0{}^0\left(\frac{1}{k}\right)$$
$$\vdots$$
$$c_0\ V_4{}^0(f_0(k)) + c_1\ V_4{}^0(f_1(k)) \ldots c_4\ V_4{}^0(f_4(k)) = V_4{}^n(V(k)) - V_4{}^0\left(\frac{1}{k}\right).$$

Now this can be written in the matrix form:

$\boldsymbol{A}\,\vec{c} = \vec{v}$

where:

$$\boldsymbol{A} = \begin{pmatrix} V_0{}^0\,(f_0\,(k)) & \cdots & \cdots & V_0{}^0\,(f_4\,(k)) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ V_4{}^0\,(f_0\,(k)) & \cdots & \cdots & V_4{}^0\,(f_4\,(k)) \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} c_0 \\ \vdots \\ c_4 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} V_0{}^n(V(k)) - V_0{}^0\left(\frac{1}{k}\right) \\ \vdots \\ V_4{}^n(V(k)) - V_4{}^0\left(\frac{1}{k}\right) \end{pmatrix}.$$

Solve for $\vec{c}$:

See "Solving Linear Systems" in the *Mathematica* tutorial.

```
A[imax_] := Parallelize[Table[Vmniterm[m, 0, i], {m, 0, imax}, {i, 0, imax}]]
```

```
A[4]
```

$$\left\{\left\{\frac{1}{5\,l}, \frac{\sqrt{\frac{\pi}{5}}}{5\,l}, \frac{4}{25\,l}, \frac{6\sqrt{\frac{\pi}{5}}}{25\,l}, \frac{32}{125\,l}\right\}, \left\{\frac{1}{25\,l}, \frac{3\sqrt{\frac{\pi}{5}}}{50\,l}, \frac{8}{125\,l}, \frac{3\sqrt{\frac{\pi}{5}}}{25\,l}, \frac{96}{625\,l}\right\}, \left\{\frac{1}{125\,l}, \frac{3\sqrt{\frac{\pi}{5}}}{200\,l}, \frac{12}{625\,l}, \frac{21\sqrt{\frac{\pi}{5}}}{500\,l}, \frac{192}{3125\,l}\right\}, \left\{\frac{1}{625\,l}, \frac{7\sqrt{\frac{\pi}{5}}}{2000\,l}, \frac{16}{3125\,l}, \frac{63\sqrt{\frac{\pi}{5}}}{5000\,l}, \frac{64}{3125\,l}\right\}, \left\{\frac{1}{3125\,l}, \frac{63\sqrt{\frac{\pi}{5}}}{80\,000\,l}, \frac{4}{3125\,l}, \frac{693\sqrt{\frac{\pi}{5}}}{200\,000\,l}, \frac{96}{15\,625\,l}\right\}\right\}$$

Calculation of v for various d

```
v[imax_, n_] := Parallelize[Table[Vmn[m, n] - Vmnterm1[m, 0], {m, 0, imax}]]
```

```
v[4, 1]
```

$$\left\{-\frac{5\sqrt{\pi}}{32\,l}, -\frac{\sqrt{\pi}}{64\,l}, \frac{17\sqrt{\pi}}{256\,l}, \frac{11\sqrt{\pi}}{512\,l}, \frac{49\sqrt{\pi}}{4096\,l}\right\}$$

```
v[4, 2]
```

$$\left\{-\frac{439\sqrt{\pi}}{2048\,l}, -\frac{191\sqrt{\pi}}{4096\,l}, \frac{379\sqrt{\pi}}{16\,384\,l}, \frac{709\sqrt{\pi}}{32\,768\,l}, \frac{14\,075\sqrt{\pi}}{262\,144\,l}\right\}$$

Calculation of coefficients

```
c[imax_, n_] := LinearSolve[A[imax] /. l → 1, v[imax, n] /. l → 1]
```

```
d[imax_, n_] := LinearSolve[A[imax], v[imax, n]]
```

the d function is the same because the lś cancel out in the calculations

```
d[4, 0]
```

{0, 0, 0, 0, 0}

```
c[4, 0]
```

{0, 0, 0, 0, 0}

As expected.

imax=4, n=1:

```
d[4, 1]
```

$$\left\{\frac{8125\sqrt{\pi}}{16}, -\frac{30\,525\sqrt{5}}{16}, \frac{108\,125\sqrt{\pi}}{32}, -\frac{91\,625\sqrt{5}}{64}, \frac{165\,625\sqrt{\pi}}{512}\right\}$$

```
c[4, 1]
```

$$\left\{\frac{8125\sqrt{\pi}}{16}, -\frac{30\,525\sqrt{5}}{16}, \frac{108\,125\sqrt{\pi}}{32}, -\frac{91\,625\sqrt{5}}{64}, \frac{165\,625\sqrt{\pi}}{512}\right\}$$

$$\left\{\frac{8125\sqrt{\pi}}{16}, -\frac{30525\sqrt{5}}{16}, \frac{108125\sqrt{\pi}}{32}, -\frac{91625\sqrt{5}}{64}, \frac{165625\sqrt{\pi}}{512}\right\} \text{ // N}$$

{900.074, -4266., 5988.96, -3201.25, 573.365}

```
cn1 = {900.0742211629573`, -4265.998438323818`, 5988.955394661216`, -3201.245756850285`, 573.3645880004416`};
```

```
c[8, 1]
```

Parallelize::nopar1 :
LinearSolve[A[8] /. l → 1, v[8, 1] /. l → 1] cannot be parallelized; proceeding with sequential evaluation. >>

$$\left\{\frac{9090375\sqrt{\pi}}{32}, -\frac{18412775\sqrt{5}}{8}, \frac{326664375\sqrt{\pi}}{32}, -\frac{2651522875\sqrt{5}}{192}, \frac{8413115625\sqrt{\pi}}{512}, -\frac{1367618125\sqrt{5}}{192}, \frac{8689446875\sqrt{\pi}}{3072}, -\frac{72153125\sqrt{5}}{192}, \frac{6425609375\sqrt{\pi}}{196608}\right\}$$

$$\left\{\frac{9090375\sqrt{\pi}}{32}, -\frac{18412775\sqrt{5}}{8}, \frac{326664375\sqrt{\pi}}{32}, -\frac{2651522875\sqrt{5}}{192}, \frac{8413115625\sqrt{\pi}}{512}, -\frac{1367618125\sqrt{5}}{192}, \frac{8689446875\sqrt{\pi}}{3072}, -\frac{72153125\sqrt{5}}{192}, \frac{6425609375\sqrt{\pi}}{196608}\right\} \text{ // N}$$

$\{503508., -5.14653\times10^{6}, 1.80937\times10^{7}, -3.08801\times10^{7}, 2.91247\times10^{7}, -1.59275\times10^{7}, 5.01356\times10^{6}, -840309., 57927.9\}$

imax=4, n=2:

```
c[4, 2]
```

$$\left\{\frac{4438075\sqrt{\pi}}{2048}, -\frac{1110375\sqrt{5}}{128}, \frac{66728375\sqrt{\pi}}{4096}, -\frac{29742325\sqrt{5}}{4096}, \frac{111994375\sqrt{\pi}}{65536}\right\}$$

$$\left\{\frac{4438075\sqrt{\pi}}{2048}, -\frac{1110375\sqrt{5}}{128}, \frac{66728375\sqrt{\pi}}{4096}, -\frac{29742325\sqrt{5}}{4096}, \frac{111994375\sqrt{\pi}}{65536}\right\} \text{ // N}$$

{3840.96, -19397.5, 28875.2, -16236.8, 3028.94}

```
cn2 = {3840.9585568151842`, -19397.45297278382`, 28875.235652689782`, -16236.782350803573`, 3028.94380567179`};
```

```
c[4, 2.5] // N
```

```
$Aborted
```

After n=2, the correlation energies stop becoming more positive and jump down below the n=0 level. The curves that are greater than n=3 will be below n=3.

```
c[4, 3] // N
```

```
{2461.9, -12 412.9, 18 439.9, -10 345.8, 1925.12}
```

```
c[4, 4] // N
```

```
{1760.71, -8841.47, 13 072.1, -7296.46, 1349.96}
```

```
c[4, 5] // N
```

```
{1282.66, -6399.65, 9391.39, -5199.41, 953.232}
```

```
cn5 = {1282.658703949136`, -6399.65137024662`, 9391.394237810113`, -5199.410444225987`, 953.2319876007417`};
```

```
c[4, 6] // N
```

```
{921.69, -4552.76, 6602.8, -3607.94, 651.638}
```

```
cn6 = {921.6895110934634`, -4552.763367933509`, 6602.796406745576`, -3607.9364288665274`, 651.6380762878204`};
```

```
c[4, 7]
```

$$\left\{\frac{6\,141\,729\,203\,475\,\sqrt{\pi}}{17\,179\,869\,184},\ -\frac{94\,573\,751\,771\,375\,\sqrt{5}}{68\,719\,476\,736},\ \frac{42\,382\,398\,924\,125\,\sqrt{\pi}}{17\,179\,869\,184},\ -\frac{286\,886\,752\,192\,925\,\sqrt{5}}{274\,877\,906\,944},\ \frac{127\,140\,398\,570\,625\,\sqrt{\pi}}{549\,755\,813\,888}\right\}$$

$$\left\{\frac{6\,141\,729\,203\,475\,\sqrt{\pi}}{17\,179\,869\,184},\ -\frac{94\,573\,751\,771\,375\,\sqrt{5}}{68\,719\,476\,736},\ \frac{42\,382\,398\,924\,125\,\sqrt{\pi}}{17\,179\,869\,184},\ -\frac{286\,886\,752\,192\,925\,\sqrt{5}}{274\,877\,906\,944},\ \frac{127\,140\,398\,570\,625\,\sqrt{\pi}}{549\,755\,813\,888}\right\} \mathbin{//} N$$

```
{633.645, -3077.34, 4372.61, -2333.76, 409.91}
```

```
cn7 = {633.6446140146658`, -3077.3420854236706`, 4372.6087421923385`, -2333.7571465072588`, 409.9101516697673`};
```

c[4, 10]

$$\left\{\frac{1523661083467558675\sqrt{\pi}}{144115188075855872}, \frac{38489450927660925\sqrt{5}}{1125899906842624}, -\frac{65148254720069204625\sqrt{\pi}}{288230376151711744}, \frac{51144259670548626075\sqrt{5}}{288230376151711744}, -\frac{282946662333618450625\sqrt{\pi}}{4611686018427387904}\right\}$$

$$\left\{\frac{1523661083467558675\sqrt{\pi}}{144115188075855872}, \frac{38489450927660925\sqrt{5}}{1125899906842624}, -\frac{65148254720069204625\sqrt{\pi}}{288230376151711744}, \frac{51144259670548626075\sqrt{5}}{288230376151711744}, -\frac{282946662333618450625\sqrt{\pi}}{4611686018427387904}\right\} \text{ // N}$$

{18.7393, 76.4411, -400.625, 396.773, -108.748}

```
cn10 = {18.739308402702555`, 76.44110117412255`, -400.62493238943216`, 396.7730355458878`, -108.74762489253389`};
```

c[4, 25]

$$\left\{-\frac{16189982079673175202579413213454698429626515375\sqrt{\pi}}{22300745198530623141535718272648361505980416}, \frac{67681223489106848042628576973526535486264598875\sqrt{5}}{22300745198530623141535718272648361505980416}, -\frac{266143232952525608835049441216548388920580229375\sqrt{\pi}}{44601490397061246283071436545296723011960832}, \frac{248501861250419059726364692228031259753990056375\sqrt{5}}{89202980794122492566142873090593446023921664}, -\frac{490677959795921413375927980179111208770785971875\sqrt{\pi}}{713623846352979940529142984724747568191373312}\right\}$$

$$\left\{-\frac{16189982079673175202579413213454698429626515375\sqrt{\pi}}{22300745198530623141535718272648361505980416}, \frac{67681223489106848042628576973526535486264598875\sqrt{5}}{22300745198530623141535718272648361505980416}, -\frac{266143232952525608835049441216548388920580229375\sqrt{\pi}}{44601490397061246283071436545296723011960832}, \frac{248501861250419059726364692228031259753990056375\sqrt{5}}{89202980794122492566142873090593446023921664}, -\frac{490677959795921413375927980179111208770785971875\sqrt{\pi}}{713623846352979940529142984724747568191373312}\right\} \text{ // N}$$

{-1286.77, 6786.31, -10576.5, 6229.24, -1218.71}

```
cn25 = {-1286.7729677974512`, 6786.31207947123`, -10576.476120856927`, 6229.2431188327555`, -1218.7149348209432`};
```

```
c[4, 35]
```

```
$Aborted
```

```
c[4, 50]
```

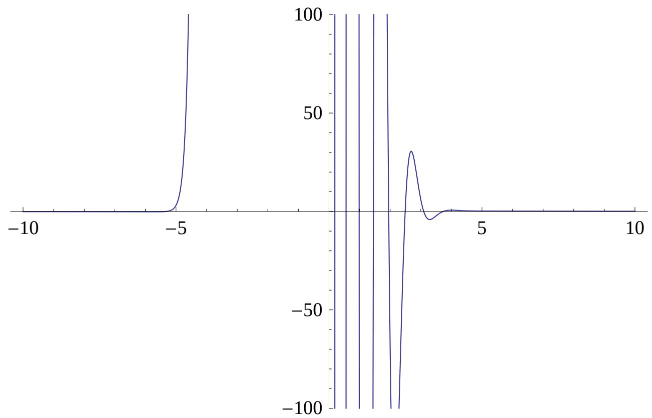
```
$Aborted
```

50 is too high! Didn't calculate even after ~2hr 20min.

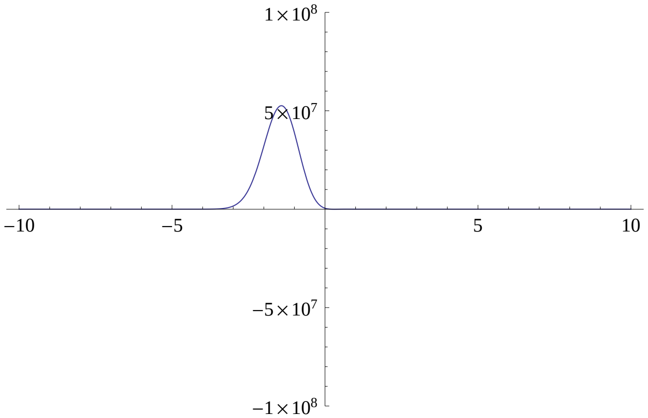
We see that as imax increases, so do the coefficients.

Plot of imax=8, n=1:

```
Plot[1/r + e^(-r^2) (503508.4429664134` - 5.1465270693010865`*^6 r +
    1.80936727944498`*^7 r^2 - 3.0880132252060823`*^7 r^3 + 2.912472497586839`*^7 r^4 - 1.592753695187919`*^7 r^5 +
    5.013555851508024`*^6 r^6 - 840308.8140054143` r^7 + 57927.93823818632` r^8), {r, -10, 10}, PlotRange → 100]
```



```
Plot[1/r + ⅇ^(-r^2) (503508.4429664134` - 5.1465270693010865`*^6 r +
    1.80936727944498`*^7 r^2 - 3.0880132252060823`*^7 r^3 + 2.912472497586839`*^7 r^4 - 1.592753695187919`*^7 r^5 +
    5.013555851508024`*^6 r^6 - 840308.8140054143` r^7 + 57927.93823818632` r^8), {r, -10, 10}, PlotRange → 100 000 000]
```



Plot of imax=4, n=1:

```
Plot[1/r + ⅇ^(-r^2) (900.0742211629573` - 4265.998438323818` r + 5988.955394661216` r^2 - 3201.245756850285` r^3 + 573.3645880004416` r^4),
  {r, -10, 10}, PlotRange → 100]
```