1 Interest and Present Value

1.1 Interest Accumulation

One of the most important financial terms is interest due to its financial power and how it serves as the basis for making money. What exactly is interest?

Suppose you open a savings account at your local bank with an initial deposit of \$100. When you check one year later, the account balance is \$105. This is interest hard at work!

- The initial deposit of \$100 is the **principal**.
- The account balance of \$105 is the **accumulated value (AV)** at time 1, where time is measured in years.
- **Interest** is the difference between these two quantities (in this case, \$5!)

In summary, interest is the compensation a borrower of capital pays to the lender for its use, or in other words, the rent for borrowing the principal. Therefore, every financial transaction can be viewed from two different perspectives: lender of capital and borrower of capital.

For instance, in a savings account, the depositor is the lender and the bank is the borrower. For a car loan, the bank is the lender and the car buyer is the borrower.

Definition 1.1 (Amount Function). Let $A(t)$ represent the accumulated value at time t for an original investment of k. Then

1. $A(0) = k$
2. $A(t)$ is generally increasing
3. If interest accrues continuously the function will be continuous

The amount of interest earned for period n is

$$I_n = A(n) - A(n-1)$$

Example 1.2. An investment of 100 is made into a fund at time $t = 0$. The fund develops the following balances over the next 4 years.

t	0	1	2	3	4
$A(t)$	\$100.00	\$105.00	\$110.25	\$115.76	\$121.55

Then the amount of interest earned in the third year is

$$I_3 = A(3) - A(2) = 115.76 - 110.25 = \boxed{\$5.51}$$

where $A(3)$ is the accumulated value at time 3.

Definition 1.3 (Accumulation Function). The **accumulation function** is a special case of the amount function where the original investment is one unit. $a(t)$ is the AV at time t of an original investment of k, thus

$$A(t) = k \cdot a(t)$$

The second and third properties from Definition 1.1 also apply to $a(t)$. Generally, $a(t)$ and $A(t)$ are interchangeable.

Example 1.4. Given $A(t) = t^2 + 3t + 5$, find the corresponding $a(t)$. First find k

$$A(0) = k \iff 0^2 + 3(0) + 5 = k \iff k = 5$$

and by using $A(t) = k \cdot a(t)$

$$A(t) = 5a(t), \quad a(t) = \frac{1}{5}A(t) = \frac{1}{5}\left(t^2 + 3t + 5\right)$$

$$a(t) = \frac{1}{5}t^2 + \frac{3}{5}t + 1 \implies \boxed{a(0) = 1}$$

Example 1.5. You are given

1. $a(t) = 1 + 0.1t$
2. \$1000 is invested at time 0.

Determine the amount of interest earned during year 8.

$$A(t) = 1000(1 + 0.1t) = I_8 = A(8) - A(7)$$

$$= 1000(1 + 0.1(8)) - 1000(1 + 0.1(7)) = 1000(0.1) = \boxed{100}$$

Find the amount of interest during year n.

$$I_n = A(n) - A(n-1) = 1000(1 + 0.1(n)) - 1000(1 + 0.1(n-1))$$

$$= 1000(0.1)(n - (n-1)) = \boxed{100}$$

Example 1.6. You are given:

1. $a(t) = bt^2 + c$
2. \$100 invested at time 0 will be worth \$200 at time 10.

Find the accumulated value at time 15 of \$500 invested at time 0. From bullet (ii):

$$A(0) = 100 \implies A(0) = 100a(0) \implies A(t) = 100a(t) \text{ and}$$

$$A(10) = 200 \implies 100a(10) = 200 \implies a(10) = 2$$

We know that $a(0) = 1$ and $a(10) = 2$:

$$a(0) = 1 \qquad a(10) = 2$$

$$b(0)^2 + c = 1 \qquad b(10)^2 + 1 = 2$$

$$c = 1, b = 0.01$$

The accumulated value of \$500 in 15 years is

$$500a(15) = 500\left[0.01(15)^2 + 1\right] = \boxed{1625}$$

Definition 1.7 (Effective Rate of Interest). The **effective rate of interest**, i, is the amount of money that one unit invested at the beginning of a period will earn during the period, where interest is paid at the end of the period.

If I_1 is the amount of interest and $A(0)$ is the principal investment, then

$$i = \frac{I_1}{A(0)}$$

Remarks:

1. The term "effective" is used for rates of interest in which interest is paid once per measurement period.
2. The effective rate of interest is usually expressed as a percentage, e.g. 5%. 5% is equivalent to 0.05 per unit of principal.
3. Amount of principal remains constant throughout the period.

We can also write the effective rate of interest in terms of the accumulation function

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)}$$

as a recursive function.

Example 1.8. An investment of 1000 is made into a fund at time $t = 0$. The fund develops the following balances over the next 4 years:

t	0	1	2	3	4
$A(t)$	\$1000.00	\$1060.00	\$1113.00	\$1146.00	\$1181.00

If \$5000.00 is invested in the same fund at time 1, find the accumulated value of this deposit at time 4.

$$X = 5000\left(\frac{a(4)}{a(1)}\right) = 5000\left(\frac{A(4)}{A(1)}\right) = 5000\left(\frac{1181}{1060}\right) = \boxed{\$5570.75}$$

If \$5000.00 is invested in the same fund at time 1, find the amount of interest earned on this deposit between time 2 and 3.

$$A^*(2) = 5000\left(\frac{1113}{1060}\right) = 5250, \quad A^*(3) = 5000\left(\frac{1146}{1060}\right) = 5405.66$$

$$I_3^* = 5405.66 - 5250 = \boxed{\$155.66}$$

Example 1.9. Given $A(t) = 10(1.08)^t$, find i_3 and i_7.

i_3 is the effective rate of interest for year 3

$$i_3 = \frac{A(3) - A(2)}{A(2)} = \frac{10(1.08)^3 - 10(1.08)^2}{10(1.08)^2} = 1.08 - 1 = 0.08$$

Similarly,

$$i_7 = \frac{A(7) - A(6)}{A(6)} = \frac{(1.08)^7 - (1.08)^6}{(1.08)^6} = 1.08 - 1 = 0.08$$

As a general observation

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{10(1.08)^n - 10(1.08)^{n-1}}{10(1.08)^{n-1}} = 1.08 - 1 = 0.08$$

This is an assumption for any compound interest problem.

Example 1.10. Given $A(2) = 100$ and $i_n = 0.03n$, determine $A(5)$.

The interest is not uniform over each time period, so we must multiply by factors of $1 + i_3$, $1 + i_4$, and $1 + i_5$.

$$A(5) = 100(1+i_3)(1+i_4)(1+i_5) = 100(1+0.03(3))(1+0.03(4))(1+0.03(5)) = \boxed{140.39}$$

1.2 Present Value

Suppose you need \$110 one year from today. If you can invest money at an effective interest rate of \$10, then how much do you need to invest today to have exactly 110 one year from today?

The amount required is called the **present value (PV)**. Earlier we used this quantity to solve for the AV. Now, we will be given the AV and use that to calculate the PV.

Definition 1.11 (Discount Functions and Factors). Given an accumulation function $a(t)$, the **discount function**

$$a^{-1}(t) = \frac{1}{a(t)}$$

reveals the discounted cash flows from time t to time 0.

The **discount factor** d discounts the value of a cash flow from time t to time $t-1$:

$$v_t = \frac{a(t-1)}{a(t)}$$

Additionally, if $i_t = \frac{a(t)-a(t-1)}{a(t-1)}$ is the effective rate of interest, then

$$d_t = \frac{a(t) - a(t-1)}{a(t)}$$

is the **effective rate of discount**. Consequentially, $v_t = 1 - d_t$.

Example 1.12. Given $a(t) = 1 + 0.08t$, find i_4 and d_4:

$$a(3) = 1 + 0.08(3) = 1.24, \quad a(4) = 1 + 0.08(4) = 1.32$$

$$i_4 = \frac{1.32 - 1.24}{1.24} = \boxed{0.0645}, \quad d_4 = \frac{1.32 - 1.24}{1.32} = \boxed{0.0606}$$

Example 1.13. John can earn the following rates for the next five years:

1. Year 1: effective rate of interest = 4%
2. Year 2: effective rate of discount = 5%
3. Year 3: effective rate of interest = 6%
4. Years 4-5: effective rate of discount = 10%

John needs to have \$1000 in 5 years. How much should he invest now?

$$\text{PV} = 1000(0.9)(0.9)(1.06)^{-1}(.95)(1.04)^{-1} = \boxed{\$698.02}$$

Now, suppose John invests \$1000 today with the same rates. How much will John have in 5 years?

$$\text{AV} = 1000(1.04)(0.95)^{-1}(1.06)[(0.9)^{-1}]^2 = \boxed{1432.62}$$

Example 1.14. Given $a(t) = 1.08^t$, find the discount factor for year 5.

$$d_5 = \frac{a(4)}{a(5)} = (1.08)^{-1} = \boxed{0.9259}$$

Corollary 1.15 (Relationship between i_t and d_t). *Let i_t and d_t be the effective rates of interest and discount, respectively. Then*

$$i_t = \frac{d_t}{1-d_t}, \quad d_t = \frac{i_t}{1+i_t}$$

WARNING: Each term is interpreted differently:

1. The statement "cash flows are discounted at an effective rate of 10%" implies that $i = 10\%$ (described as a verb)
2. The statement "the annual effective discount rate is 10%" implies that $d = 10\%$ (described as an adjective)

Example 1.16. You are told:

1. For years one through three, cash flows are discounted at an annual effective rate of 5%.
2. For years four and five the annual effective discount rate is 6%. Find the present value of \$1000 paid five years from now.

The solution is obtained by multiplying \$1000 by $(1 - 0.06)$ twice and then dividing by $(1 + 0.05)$ three times:

$$\text{PV} = \frac{1000(0.94)^2}{(1.05)^3} \approx \boxed{\$763.29}$$

1.3 Compound and Simple Interest

One of the key principles of growing money is through compound interest, where deposited money grows exponentially. Not only do you earn money through interest, but also through accumulated interest from earlier periods!

Compound interest is defined using the effective rate of interest, and it usually follows that this rate of interest is constant through all periods.

For instance, suppose we deposit \$100 into the bank when the annual interest rate is 10%.

1. In year 1, the money acquired through interest is \$10, resulting in a total balance of \$110.
2. In year 2, the money acquired through interest is $(110)(0.1) = 11$, resulting in a total balance of \$121.
3. In year 3, the money acquired through interest is $(121)(0.1) = 12.1$, resulting in a total balance of \$133.10.

As a result, the accumulation function $a(t)$ becomes

$$\boxed{a(t) = (1+i)^t.}$$

We assume i is the same for all t, so we drop the subscript. Similarly, we drop the subscript for the discount factor d.

Example 1.17. Given $d = \frac{1}{9}$, find $a(t)$.

$$i = \frac{d}{1-d} = \frac{\frac{1}{9}}{1-\frac{1}{9}} = 0.125 \qquad \Longrightarrow \qquad \boxed{a(t) = (1.125)^t}$$

Example 1.18. \$100 invested for 3 years, at an effective rate of interest i, will earn \$36 of interest. Find the accumulated value of \$50 invested at the same rate of compound interest i for 5 years.

$$136 = 100(1+i)^3 \qquad \Longleftrightarrow \qquad i = \left(\frac{136}{100}\right)^{\frac{1}{3}} - 1 \qquad \Longleftrightarrow \qquad i = 0.10793$$

The accumulated value of \$50 for 5 years at i is

$$50(1.10793)^5 \approx \boxed{\$83.47}$$

Example 1.19. At a certain rate of compound interest:

- 1 grows to 3 in x years.
- 3 grows to 14 in y years.
- 1 grows to 21 in z years.

Determine what 5 grows to in $z - x - y$ years.
The idea is to write an expression for what 5 grows to in $z - x - y$ years and simplify. We

know that

$$(1+i)^x = 3, \quad (1+i)^y = \frac{14}{3}, \quad (1+i)^z = 21$$

And so

$$5(1+i)^{z-x-y} = 5 \cdot \frac{(1+i)^z}{(1+i)^x(1+i)^y} = \frac{5(21)}{3 \cdot \frac{14}{3}} = \boxed{\$7.50}$$

Example 1.20. You invested \$100 on January 1, 1997. The investment was worth \$190 on July 1, 2002. The effective rate of return for the first year was 12%.

Determine the annualized effective rate of return from January 1, 1998 to July 1, 2002.

Let i denote the annual rate of return from 1/1/98 to 7/1/02. We will assume that Jan. 1 to Jul. 1 is half a year. From 1/1/97 to 1/1/98, the new amount is \$112. We set up the following compound interest expression to solve for i:

$$190 = 112(1+i)^{4.5} \quad \Longleftrightarrow \quad \boxed{i \approx 0.1246}$$

Definition 1.21 (Simple Interest). As opposed to applying interest from the previous period, **simple interest** adds interest based on the interest earned in the first period. The accumulated value at time t, given interest rate i, is

$$a(t) = 1 + it$$

Compared to the exponential growth from compound interest, simple interest assumes a linear growth because the earned interest is fixed.

Therefore, simple interest assumes the **amount of interest** in each period is constant, whereas compound interest assumes the **interest rate** in each period is constant.

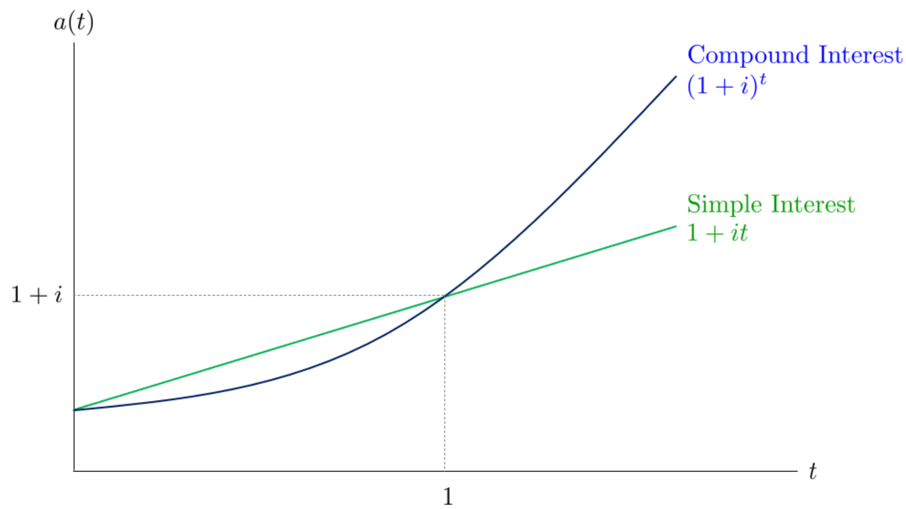
Using the example from earlier, where \$100 into the bank is deposited when the annual interest rate is 10%:

1. In year 1, the money acquired through interest is \$10, resulting in a total balance of \$110.

2. In year 2, the money acquired through interest is \$10 resulting in a total balance of \$120.

3. In year 3, the money acquired through interest is \$10, resulting in a total balance of \$130.

By compounding annually, we will have earned more in our account compared to simple interest. Most banks and insurances will choose to compound money/returns, but sometimes simple interest might be more convenient, especially if the period is monthly.



For less than one year, a borrower would prefer compound interest of i vs. a simple interest of i. For more than one year, a lender would prefer compound interest of i vs. a simple interest of i.

Using the simple interest computations from the previous page, we can compute the annual effective rates of interest:

$$\text{AV}(1) = 110, \quad \text{AV}(2) = 120, \quad \text{AV}(3) = 130$$

$$i_1 = \frac{110 - 100}{100} = 10\%, \quad i_2 = \frac{120 - 110}{110} \approx 9.09\%, \quad i_3 = \frac{130 - 120}{120} \approx 8.33\%$$

More generally,

$$\boxed{i_n = \frac{i}{1 + i_{n-1}}} \qquad i = \text{simple interest}$$

We can also impose a mixture of compound and simple interest. The most common case is if we are dealing with partial years (i.e. 1.5 years or 2.25 years).

Example 1.22. Given $i = 0.10$ and simple interest is only used for partial years, find the accumulated value of \$100 five and a half years from now.

The balance 5 years from now is $100(1.1)^5 \approx \$161.05$. For the remaining half-year, we apply simple interest onto the existing balance:

$$161.05(1 + 0.5(0.1)) \approx \boxed{\$169.10}$$

Compounding for the entire 5.5 year duration would give us \$168.91, a slightly lower amount because the half-year was also compounded.

An important distinction to be made is that, under simple interest, every cash flow is treated separately with its own "time 0."

Example 1.23. \$100 is deposited at the beginning of each year for 3 years. If the deposits earn simple interest of 10%, what is the accumulated value at the end of 3 years?

Solution: The first, second, and third deposits would earn $\$100 + 3(10) = \$130$, $\$100 + 2(10) = \$120$, and $\$100 + 10 = \$110$, respectively. The accumulated value is the sum of the three deposits, \$360.

Instead, suppose \$100 is deposited at the beginning of each year for 3 years. Given $a(t) = 1 + 0.1t$, what is the accumulated value at the end of 3 years?

Solution: This time, we need to compute the effective rates of interest for each year:

$$\text{AV} = 100a(3) + 100\left(\frac{a(3)}{a(1)}\right) + 100\left(\frac{a(3)}{a(2)}\right) \approx \boxed{\$365.52}$$

It is important to understand what you are asked to solve by understanding the nuances between both problems.

Example 1.24. A loan is made for five years at a simple interest rate of 12% per annum. What is the equivalent annual effective rate of discount during the fourth year of the loan?

$$d_4 = \frac{a(4) - a(3)}{a(4)} = \frac{1.48 - 1.36}{1.48} \approx \boxed{8.108\%}$$

Example 1.25. At a rate of simple interest i, \$10 will accumulate to \$15 after x years. What will \$20 accumulate, at a simple rate of $2i$, to after $5x$ years?

Use the first bit of information to express ix as a number:

$$10(1 + ix) = 15 \quad \iff \quad ix = 0.5$$

The second bit asks us to calculate $20(1 + 2i(5x))$:

$$20(1 + 2i(5x)) = 20(1 + 10ix)) = 20(1 + 5) = \boxed{\$120}$$

Example 1.26. An investment earns 10% compound interest for each complete year and 8% simple interest for each partial year. A \$100 investment is made on January 1, 2008. What is the accumulated value of the investment on July 1, 2012?

The accumulated value after 4 years is $\text{AV} = 100(1.1)^4$. The accumulated value after 4.5 years is the accumulated value after 4 years multiplied by the simple interest of 8% over

0.5 years:

$$\text{AV} = 100(1.1)^4(1 + 0.5(0.08)) \approx \boxed{\$152.27}$$

1.4 Nominal Annual Rates

Definition 1.27 (Nominal Annual Rate of Interest). Nominal rates are interest rates in name only. The **nominal annual rate (NAR)** $i^{(m)}$ is the basic interest rate before compounding or inflating. The effective rate of interest per m-th of a year is

$$\frac{i^{(m)}}{m}$$

For instance, if $i^{(12)} = 6\%$, then the effective rate per month is $\frac{0.06}{12} = 0.005$. The equivalent annual effective rate i satisfies

$$(1.005)^{12} = 1 + i \quad \Longrightarrow \quad \boxed{i \approx 0.0617}$$

The NAR is generally called the APR (average percentage rate). In context of the example, a loan at an APR of 6% sounds like a better deal than a loan at an effective rate of 6.17%, but these quantities are equivalent. Additionally, this example assumes that the interest is applied monthly because $m = 12$.

More generally, if i is the equivalent annual effective rate of interest, and $i^{(m)}$ is the nominal annual rate, then

$$\boxed{i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]}$$

By taking the limit $\lim_{m\to\infty} i^{(m)}$, we have that $\frac{1}{m} \to 0$ and therefore $(1+i)^{\frac{1}{m}} \to \frac{1}{m}\ln(1+i)$, therefore making $i^{(m)}$ converge to $\ln(1+i)$. This occurs when we are compounding infinitely many times, or *continuously* and represents the limiting nominal rate as the compounding frequency without bound.

Suppose we had to compute $i^{(2)}, i^{(4)}$, and $i^{(12)}$ given $i = 12\%$. We could apply the formula above:

- $i^{(2)} = 2\left[(1.12)^{\frac{1}{2}} - 1\right] \approx 0.1166$. This is compounding semiannually, or twice per year.
- $i^{(4)} = 4\left[(1.12)^{\frac{1}{4}} - 1\right] \approx 0.1149$. This is compounding quarterly, or four times per year.
- $i^{(12)} = 12\left[(1.12)^{\frac{1}{12}} - 1\right] \approx 0.1139$. This is compounding monthly, or twelve times per year.

Therefore,

$$i^{(\infty)} < \cdots < i^{(12)} < i^{(4)} < i^{(2)} < i$$

We can interpret these calculations using financial terms we already know. Let's use the first one as an example:

1. 0.12 is the annual effective rate of interest
2. Add 1 to get the accumulation factor 1.12 for one year
3. Raise to the $\frac{1}{2}$ power = 1.0583, the accumulation factor for 6 months
4. Subtract 1 to get 0.0583 as the effective rate per 6 months
5. Multiply by 2 to get 0.1166 as the NAR convertible twice per year

We can also convert between two different nominal annual rates. Say $i^{(m)}$ and $i^{(n)}$ are two nominal annual rates. Then

$$\boxed{i^{(n)} = n\left[\left(1+\frac{i^{(m)}}{m}\right)^{\frac{m}{n}} - 1\right]}$$

For example, if $i^{(12)} = 0.12$, then the equivalent rate $i^{(4)}$ is

$$i^{(4)} = 4\left[\left(1+\frac{0.12}{12}\right)^{\frac{12}{4}} - 1\right] \approx \boxed{0.1212}$$

Now, suppose we have rates convertible less frequently than annually. For instance, if we wanted to find the accumulated value of \$1000 in 5 years if the nominal annual rate convertible once every two years is 15%, then we have

$$i^{\left(\frac{1}{2}\right)} = 0.15, \qquad \text{effective rate per 2 years} = \frac{0.15}{0.5} = 0.30$$

The accumulated value follows:

$$\text{AV} = 1000(1+0.30)^{2.5} \approx \boxed{\$1926.90}$$

Example 1.28. Given $i^{(2)} = 0.08$, find the accumulated value of \$500 in 4.5 years.

The effective rate per six months = $\frac{0.08}{2} = 0.04$. So, \$500 accumulated for 4.5 years is

$$500(1+0.04)^{4.5(2)} \approx \boxed{\$711.66}$$

Example 1.29. The nominal annual interest rate convertible once every 4 years is 6%. Find the present value of $400 to be paid in 12 years.

The effective rate per four years is $\frac{0.06}{0.25} = 0.24$. The present value of $400 paid in 12 years is

$$\frac{400}{(1.24)^{12(0.25)}} \approx \boxed{\$209.79}$$

The concept of nominal rates of interest applies one-to-one with nominal rates of discount. We say $d^{(m)}$ is the *nominal rate of discount* convertible m times per year, and $\dfrac{d^{(m)}}{m}$ is the *effective rate of discount* per m-th of a year.

For example, if given $d^{(4)}$, then the effective discount rate per quarter is $\frac{d^{(4)}}{4} = \frac{0.08}{4} = 0.02$. Thus, the effective rate of discount satisfies

$$(1 - 0.02)^4 = (1 - d) \quad \Longrightarrow \quad \boxed{d \approx 0.0776}$$

More generally,

$$d^{(m)} = m\left[1 - (1 - d)^{\frac{1}{m}}\right]$$

As an example, if we know $i = 0.12$, then $d = \frac{0.12}{1.12} \approx 0.10714$ and

$$d^{(2)} = 2\left[1 - (1 - 0.10714)^{\frac{1}{2}}\right] \approx 0.1102$$

Similarly, we can summarize this series of steps into financial terms:

1. 0.12 is the annual effective rate of interest
2. Add 1 to get the accumulation factor 1.12 for one year
3. Raise to the -1 power to get 0.892857 as the discount factor for one year
4. Raise to the $\frac{1}{2}$ power to get 0.9449 as the discount factor for 6 months
5. Subtract from 1 to get 0.055 as the effective rate per 6 months
6. Multiply by 2 to get 0.1102 as the nominal annual discount rate convertible semiannually

We can also convert between $d^{(m)}$ and $d^{(n)}$ using the formula

$$\boxed{d^{(n)} = n\left[1 - \left(1 - \frac{d^{(m)}}{m}\right)^{\frac{m}{n}}\right]}$$

Suppose we are given $d^{(2)} = 0.12$ and we want to find the equivalent rate $d^{(4)}$:

1. 0.12 is the nominal discount rate convertible semiannually
2. Divide by 2 to get 0.06 as the effective discount rate per 6 months
3. Subtract from 1 to get 0.94 as the discount factor for 6 months
4. Raise to the $\frac{1}{2}$ power to get 0.9695 as the discount factor for 3 months (quarterly)
5. Subtract from 1 to get 0.0305 as the effective discount rate per quarter
6. Multiply by 4 to get $\boxed{0.1219}$ as the nominal discount rate convertible quarterly

Lastly, if we are given a nominal annual rate of interest $i^{(n)}$ and want to convert to a nominal annual rate of discount $d^{(m)}$, we can equate the two such that they both accumulate to 1:

$$\boxed{\left(1 + \frac{i^{(n)}}{n}\right)^{n} = \left(1 - \frac{d^{(m)}}{m}\right)^{-m}}$$

Example 1.30. Given $d^{(2)} = 0.08$, find the accumulated value of \$500 in 4.5 years.

$$\text{AV} = 500\left(1 - \frac{0.08}{2}\right)^{-4.5(2)} \approx \boxed{\$721.99}$$

Example 1.31. The nominal annual discount rate convertible once every 4 years is 6%. Find the present value of \$400 to be paid in 10 years.

$$\text{PV} = 400\left(1 - \frac{0.06}{0.25}\right)^{2.5} \approx \boxed{\$201.42}$$

Example 1.32. Fix $i^{(4)} = 0.10$. What is $d^{(2)}$?

Use the expression for equating $i^{(n)}$ and $d^{(m)}$:

$$\left(1 + \frac{0.1}{4}\right)^{4} = \left(1 - \frac{d^{(2)}}{2}\right)^{-2}$$

$$(1.025)^4 = \left(1 - \frac{d^{(2)}}{2}\right)^{-2}$$

$$d^{(2)} = 2(1 - (1.025)^{-2}) \approx \boxed{0.09637}$$

1.5 Force of Interest

Previously, we have measured interest over specified intervals of time (yearly, semiannually, quarterly, monthly). What if we are concerned about the *intensity* of interest for each moment in time, or in other words, over infinitesimally small intervals of time? This quantity is called the force of interest.

Definition 1.33 (Force of Interest). The **force of interest** at time t, denoted as δ_t, is a measure of the intensity of interest at time t. The measurement is expressed as a rate per measure period (typically one year). If $a(t)$ is the accumulation function, then

$$\delta_t = \frac{a'(t)}{a(t)}$$

To conceptually derive this, consider a fund with a balance $A(t)$ at time t.

1. The only force acting on the fund is interest
2. The rate of change on $A(t)$ is measured by the slope
3. The slope of a curve is found through the first derivative. However, $A'(t)$ is not sufficient for the measure of intensity because it cannot depend on the amount invested
4. Therefore, we divide by $A(t)$ to "normalize," or adjust for the amount invested at time t for the effective rate of interest

What if we wanted to find $a(n)$ in terms of δ_t? This requires some simple calculus:

$$\delta_t = \frac{a'(t)}{a(t)} \qquad \Longleftrightarrow \qquad \delta_t = \frac{d}{dt}\ln(a(t))$$

$$\int_0^n \delta_t dt = \int_0^n \frac{d}{dt}\ln(a(t))dt = \ln(a(n)) - \ln(a(0)) = \ln(a(n))$$

By exponentiating, we have

$$\boxed{a(n) = \exp\left[\int_0^n \delta_t dt\right]}$$

This is to say: to accumulate from time 0 to n, add up the force of interest from time 0 to n and take the exponential. n can be any nonnegative real number, not just an integer.

If we wanted to measure the accumulation from time t_1 to t_2, set those as the bounds of the integral:

$$a(t_1, t_2) = \exp\left[\int_{t_1}^{t_2} \delta_t dt\right]$$

Example 1.34. If $\delta_t = 0.15\sqrt{t}$ and an amount of \$5000 is invested at time $t = 1$, what is the accumulated value at time $t = 4$?

$$5000a(1,4) = 5000 \exp\left[\int_1^4 \delta_t dt\right] = 5000 \exp\left[\int_1^4 0.15t^{0.5} dt\right]$$
$$= 5000 \exp\left(\left[\frac{0.15}{1.5}t^{1.5}\right]_1^4\right) \approx \boxed{\$10068.76}$$

The **discount function** using force of interest is $a^{-1}(t_1, t_2)$, or

$$a^{-1}(t_1, t_2) = \exp\left[-\int_{t_1}^{t_2} \delta_t dt\right]$$

Example 1.35. On July 1, 1984, a person invested \$1000 in a fund for which the force of interest at time t is given by $\delta_t = \frac{3+2t}{50}$, where t is the number of years since January 1, 1984. Determine the accumulated value of the investment on January 1, 1985.

Since the investment was made halfway into the year, we have $t_1 = 0.5$ and $t_2 = 1$. Therefore, we simply set up the expression

$$a(0.5, 1) = 1000 \exp\left[\int_{0.5}^1 \frac{3+2t}{50} dt\right] = 1000 \exp\left(\left[\frac{1}{50}(3t + t^2)\right]_{0.5}^1\right) \approx \boxed{\$1046.03}$$

Example 1.36. X is deposited into a savings account at time $t = 0$. No other amounts are deposited. The force of interest for the fund is $\delta_t = \frac{t}{30}$. The balance of the fund after 10 years is \$12,500. Determine X.

We are given the accumulated value, so we need to apply a discount factor to calculate the deposit X:

$$X = 12500a^{-1}(10) = 12500 \exp\left[-\int_0^{10} \frac{t}{30} dt\right] = 12500 \exp\left(\left[-\frac{t^2}{60}\right]_0^{10}\right) \approx \boxed{\$2360.95}$$

Example 1.37 (\$). 000 is deposited into a savings account at time $t = 0$. No other amounts are deposited. The accumulated amount in the account at time t is given by

$$A(t) = 1000\left(1 + \frac{2t}{35}\right)^2$$

Determine the force of interest at time $t = 32.5$

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{1000(2)\left(1 + \frac{2t}{35}\right)\left(\frac{2}{35}\right)}{1000\left(1 + \frac{2t}{35}\right)^2}$$

$$\delta_{32.5} = \frac{\frac{4}{35}}{(1+\frac{65}{35})} = \boxed{0.04}$$

Example 1.38. A deposit of \$1 will accumulate to e in 10 years with a force of interest

$$\delta_t = \begin{cases} kt & 0 < t \leq 5 \\ 0.04kt^2 & 5 < t \leq 10 \end{cases}$$

Determine k.

$$a(10) = a(0,5) \cdot a(5,10)$$
$$e = \exp\left[\int_0^5 ktdt\right] \cdot \exp\left[\int_5^{10} 0.04kt^2dt\right]$$
$$e = \exp\left([0.5kt^2]_0^5\right) \cdot \exp\left(\left[\frac{0.04}{3}kt^3\right]_5^{10}\right) = e^{12.5k}e^{11.667k} = e^{24.167k}$$
$$\boxed{k = 0.0414}$$

This process of finding $a(t)$ can be expedited if the force of interest function can easily help us see $a'(t)$ and $a(t)$. For example, given $\delta_t = \dfrac{2t}{t^2+1}$, $a(t) = t^2+1$ because $a'(t) = 2t$ and therefore

$$\delta_t = \frac{2t}{t^2+1} = \frac{a'(t)}{a(t)}.$$

While convenient, it is important to remember that $a(0) = 1$ always. Suppose we want to find $a(t)$ given $\delta_t = \dfrac{2t}{t^2+8}$. $a(t) \neq t^2+8$ because $a(0) \neq 1$. To fix this, we must normalize $a(t)$ by dividing it by 8. Therefore, we must also divide $a'(t)$ by 8.

$$\delta_t = \frac{\frac{2t}{8}}{\frac{t^2+8}{8}} = \frac{\frac{1}{4}t}{\frac{1}{8}t^2+1}$$

Thus, $a(t) = \dfrac{1}{8}t^2+1$.

What if we wanted to multiply the force of interest by a factor of k? Or, given δ_t implies a certain $a(t)$, what is $a^*(t)$ if $\delta_t^* = k\delta_t$? By extracting the constant out of the integral, the exponential term is raised to the k-th power, so

$$a^*(t) = [a(t)]^k$$

If we have $\delta_t = \frac{4}{1+t}$, then we can extract the constant to get $4\left(\dfrac{1}{1+t}\right)$. We can easily see

$a(t) = 1 + t$ and $A'(t) = 1$, so

$$a^*(t) = (1+t)^4$$

Suppose $a(t) = (1+i)^t$ and we wanted to find δ_t

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{(1+i)^t \ln(1+i)}{(1+i)^t} = \ln(1+i)$$

Because δ_t does not vary with t:

$$\delta = \ln(1+i), \quad e^\delta = 1+i, \quad i = e^\delta - 1$$

So, for a constant force of interest δ,

$$\boxed{a(t) = e^{\delta t}, \quad a(t_1, t_2) = e^{\delta(t_2 - t_1)}, \quad a^{-1}(t_1, t_2) = e^{-\delta(t_2 - t_1)}}$$

which are the formulas for continuous compound interest with a continuous interest rate δ.

Example 1.39. If $\delta = 0.05$ and an amount of \$5000 is invested at time $t = 1$, what is the accumulated value at time $t = 4$?

$$5000e^{0.05(3)} \approx \boxed{\$5809.17}$$

Example 1.40. If $\delta = 0.08$, find the present value of \$1000 to be paid in 4.25 years.

$$1000e^{-0.08(4.25)} \approx \boxed{\$711.77}$$

How can we apply force of interest to simple interest? Given $a(t) = 1 + it$, find δ_t.

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{i}{1+it}$$

In this case, simple interest is not linear and it decreases over time.

For example, if \$5000 is invested at time $t = 1$, find the accumulated value at time $t = 4$, given (1) $\delta_t = \dfrac{0.08}{1+0.08t}$, (2) Simple interest at a rate of 8%.

1. $a(t) = 1 + 0.08t$. So,

$$\text{AV} = 5000a(1,4) = 5000\left(\frac{a(4)}{a(1)}\right) \approx \boxed{\$6111.11}$$

2.

$$5000(1 + 0.08(3)) = \boxed{\$6200}$$

Example 1.41. Given $\delta = 0.06$, find $i^{(12)}$.

1. 0.06 is the force of interest
2. Exponentiate 0.06 to get 1.0618 as the accumulation factor for one year
3. Raise to the $\frac{1}{2}$ power to get 1.005 as the accumulation factor for one month
4. Subtract 1 to get 0.005 as the effective rate of interest per month
5. Multiply by 12 to get 0.06015 as the NAR convertible monthly

Example 1.42. At a force of interest $\delta = 0.05$, the following payments have the same present value:

- X at the end of the year plus $2X$ at the end of year 10
- Y at the end of year 14

Calculate $\frac{Y}{X}$.

Set up an expression that relates X and Y.

$$Xe^{-5(0.05)} + 2Xe^{-10(0.05)} = Ye^{-14(0.05)}$$

$$1.991862X = 0.4965853Y \quad \Longrightarrow \quad \frac{Y}{X} \approx \boxed{4.0111}$$

Example 1.43. On 1/1/97, Kelly deposits X into a bank account. The account is credited with simple interest at a rate of 10% per year. On the same date, Tara deposits X into a different bank account. The account is credited interest using a force of interest:

$$\delta_t = \frac{2t}{t^2 + k}$$

From the end of the 4th year until the end of the 8th year, both accounts earn the same dollar amount of interest. Calculate k.

For Kelly, $a_K(t) = 1 + 0.1t$. For Tara, we need to normalize her accumulation function such that $a_T(0) = 1$. This is done by dividing both sides of the fraction by k:

$$\delta_t = \frac{2t}{t^2 + k} = \frac{\frac{2}{k}t}{\frac{1}{k}t^2 + 1} \quad \Longrightarrow \quad a_T(t) = \frac{1}{k}t^2 + 1$$

We are given the dollar amount of interest earned from the end of the 4th year to the end of the 8th year is equal for Kelly and Tara:

$$X_{a_K}(8) - X_{a_K}(4) = X_{a_T}(8) - X_{a_T}(4)$$

$$1 + 0.1(8) - (1 + 0.1(4)) = \frac{1}{k}(8)^2 + 1 - \left(\frac{1}{k}(4)^2 + 1\right)$$

$$0.4 = \frac{48}{k} \iff k = \boxed{120}$$

1.6 Cash Flows, Present Value, Rate of Return

In the previous sections, we have already witnessed examples of situations that describe a cash flow stream. Here, we will tie in what we have learned with mathematical notation.

Definition 1.44 (Cash Flow Stream). A cash flow stream describes a sequence of cash inflows (money going in, positive) and outflows (money going out, negative) at periodic or irregular intervals. We can express this sequence as

$$(x_0, x_1, x_2, \ldots, x_n)$$

where x_0 is current amount and x_n is the last in/out-flow.

Typically, we will assume that the time interval between each x_i is equivalent.

For example, the cash flow stream $(2, -1, 3, 4)$ (in thousands) could describe a savings account with an initial deposit of \$2000, withdrawal of \$1000 at the first period, followed by deposits of \$3000 and \$4000 in the second and third period. This does not imply, however, that the balance after year 3 is \$8000, since the accumulated value depends on interest and compounding.

Definition 1.45 (Present Value for Frequent and Continuous Compounding). Let $(x_0, x_1, \ldots, x_n)$ be a cash flow stream with interest rate r and compounding frequency m. The present value after n periods is

$$\text{PV} = \sum_{k=0}^{n} \frac{x_k}{\left(1 + \frac{r}{m}\right)^k}$$

If compounding is continuous, then the present value is

$$\text{PV} = \sum_{k=0}^{n} x(t_k)e^{-rt_k}$$

where $x(t_0), \ldots, x(t_n)$ are the cash flows at the given time periods.

For instance, if we have the set of monthly cash flows $(10, -2, 4, -1)$ and the interest rate is 9%,

1. Compounding monthly would yield in a present value of

$$\text{PV} = 10 - \frac{2}{\left(1+\frac{0.09}{12}\right)} + \frac{4}{\left(1+\frac{0.09}{12}\right)^2} - \frac{1}{\left(1+\frac{0.09}{12}\right)^3} \approx \boxed{10.9777}$$

2. Compounding continuously would yield in a present value of

$$\text{PV} = 10 - 2e^{-\frac{0.09}{12}} + 4e^{-\frac{2(0.09)}{12}} - e^{-\frac{3(0.09)}{12}} \approx \boxed{10.9776}$$

Example 1.46. A major lottery advertises that it pays the winner \$10 million. However, this prize money is paid at the rate of \$400,000 each year (with the first payment being immediate) for a total of 25 payments. What is the present value of this prize at 5% interest?

$$\text{PV} = 400000 \sum_{n=0}^{24} \frac{1}{(1.05)^n} \approx \boxed{\$5{,}919{,}456.72}$$

Example 1.47. A cutting-edge pharmaceutical company has developed a new vaccine for the Coronavirus disease 2019 (COVID-19). Production of of the vaccine would require \$10 million in initial capital expenditure. It is anticipated that 1 million units would be sold each year for 5 years, and then herd immunity would be achieved and the mass production of the vaccine would cease. Each year's production would require 10,000 hours of labor and 100 tons of raw material.

In the first year, the average wage rate is \$30 per hour, the cost of the raw material is \$100 per ton and the vaccine will sell for \$3.30 per unit. All three of these unit prices (wage rate per hour, cost of raw material per ton, and vaccine revenue per unit) will increase each year after the first year by the inflation rate which is assumed to be 10% per year. All cash flows (revenues and costs) are assumed to come at the end of each year.

The interest rate is 5% and corporate tax rate is 34% on profit. The initial capital expenditure can be depreciated in a straight line fashion over 5 years (\$2 million per year).

What is the (after-tax) present value of the new vaccine?

Solution: We will build a single expression for the present value step-by-step:

1. Initial deposit: This is a negative cash outflow of -\$10,000,000. This money does not get compounded or discounted.
2. Profit: The net profit per year is the money earned from the vaccines minus the total cost of labor and materials. Each year, raw revenue is $3.30 \times 1000000 = \$3300000$.

The total money spent on wages is $10000 \times 30 = \$300000$. The total money spent on materials is $100 \times 100 = \$10000$. The "raw" net profit per year is $\$2,990,000$. However, we have to adjust for a 10% inflation of each year, excluding the first year, as well as a tax of 34%. The expression for the "true" profit over 5 years is

$$\sum_{i=1}^{5}(1-0.34)(2990000)(1.1)^{i-1}$$

3. Depreciation: The problem assumes the expenditure is depreciated at $2,000,000 each year. *Depreciation* is money *added back* as a positive adjustment on a cash flow statement despite it lowering the net income. The depreciated value is the cash flow (assumed to be $2mil each year) multiplied by the tax rate. The expression is therefore

$$\sum_{i=1}^{5} 200000(0.34)$$

This is added onto the profit.

4. Interest: Now that the "true" income is obtained by combining profit and depreciation, we apply the interest of 5% onto it. Therefore, the change in profit over 5 years is

$$\sum_{i=1}^{5} \frac{0.66(2990000)(1.1)^{i-1} + 2000000(0.34)}{(1.05)^i}$$

The final expression is

$$-10000000 + \sum_{i=1}^{5} \frac{0.66(2990000)(1.1)^{i-1} + 2000000(0.34)}{(1.05)^i}$$

which results in a present value of $\boxed{\$3,279,974.77}$.

Definition 1.48 (Internal Rate of Return). Given a cash flow stream $(x_0, \ldots, x_n)$, the *internal rate of return* is the interest rate r that makes the present value equal to zero.

Mathematically, we have

$$0 = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n}$$

Suppose we have the cash flow $(-2, 2, 3, -1)$. What is the internal rate of return?

The easiest solution is by substituting $u = \frac{1}{(1+r)}$. Then, we are left with the polynomial

$$0 = x_0 + x_1 u + x_2 u^2 + x_3 u^3 = -2 + 2u + 3u^2 - u^3$$

A graphing calculator shows that $u = 0.58579$. Then, we find that

$$0.58579 = \frac{1}{1+r} \qquad \Longrightarrow \qquad \boxed{r \approx 0.707}$$

A solution exists only if $u \in (0.5, 1)$. Otherwise, there is no interest rate that would yield a present value of 0.

Example 1.49. You purchase a commercial office building for \$800,000 and sell it 10 years later for \$1,000,000.

At the end of each year that you own the building, you receive \$24,000 in rental income: this is a total of 10 cash flows of \$24,000 each.

What is the internal rate of return?

Solution: In ten-thousands, the cash flow is

$$(-80, 2.4, 2.4, 2.4, 2.4, 2.4, 2.4, 2.4, 2.4, 2.4, 102.4)$$

The corresponding present value expression, written as a linear combination of polynomials, is

$$0 = \text{PV} = -80 + \left(\sum_{i=1}^{9} 2.4u^i\right) + 102.4u^{10}$$

We get $u \approx 0.95248$, and therefore $\boxed{r \approx 0.0499}$.

2 Annuities Part I

In the world of financial modeling, terms such as interest, mortgage, loan, and annuities are used frequently. In this section, we plan to go over these concepts and model them in an actuarial setting.

2.1 Annuity-Immediates

Before proceeding, we need to review the general geometric series formula.

Suppose we are given the series $S = a + ar + ar^2 + \cdots + ar^{n-1}$. We can exact a value for this series by subtracting rS from S:

$$S - rS = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^{n-1} + ar^n)$$

$$S(1-r) = a - ar^n \quad \Longleftrightarrow \quad \boxed{S = \frac{a(1-r^n)}{1-r}}$$

Example 2.1. Given $i = 0.05$, find $d^2 + d^4 + \cdots + d^{12}$.

$$\frac{d^2 - d^{14}}{1-d^2} = \frac{(1.05)^{-2} - (1.05)^{-14}}{1-(1.05)^{-2}} \approx \boxed{4.32}$$

Example 2.2. Given $i = 0.08$ and $r = 0.05$, find

$$S = (1+i)^{20} + (1+i)^{25}(1+r)^3 + (1+i)^{30}(1+r)^6 + \cdots + (1+i)^{55}(1+r)^{21}$$

The first term is $(1+i)^{20}$ and the final term is $(1+i)^{55}(1+r)^{21}$. The exponent for $(1+i)$ is going up by 5 and the exponent for $(1+r)$ is going up by 3. This makes the ratio $(1+i)^5(1+r)^3$.

$$S = \frac{(1+i)^{20}(1-(1+i)^{40}(1+r)^{24})}{1-(1+i)^5(1+r)^3} \approx \boxed{459.25}$$

Definition 2.3 (Annuities). **Annuities** are series of payments made at equal time intervals (e.g. rent payment, mortgage payment, car payment, paycheck, pensions). **Annuity-Immediates** are annuities with payments at the *end* of each time interval.

Example 2.4. Given $i = 0.05$, find the present value of a 10-year annuity-immediate with annual payments of 100.

This is equivalent to finding the value of the geometric series

$$100v + 100v^2 + \cdots + 100v^{10} = \frac{100d(1-v^{10})}{1-v} \approx \boxed{\$772.17}$$

An alternative expression for the present value of an annuity is

$$\boxed{P = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right]}$$

where A is the annuity payment, r is the interest rate, and n is the time-period. We can also rearrange the terms to solve for the annuity payment:

$$\boxed{A = \frac{r(1+r)^n P}{(1+r)^n - 1}}$$

if we plugged in $r = 0.05$ and $A = 100$ into the previous example, we would end up with the same solution.

We can rewrite A equivalently using important actuarial notation: $a_{\overline{n}|}$. We would use the notation $a_{\overline{n}|i}$ to denote the present value of annuity-immediate payment. At an effective rate of i, we can show that

$$\boxed{\text{PV} = \frac{1 - v^n}{i} = a_{\overline{n}|i}}$$

Example 2.5. Refer to Example 2.4. Compute $100a_{\overline{10}|0.05}$.

$$100a_{\overline{10}|0.05} = 100\left(\frac{1 - v^{10}}{0.05}\right) \approx \boxed{772.17}$$

Example 2.6. Given $i = 0.1$, find the present value of payments at 200 at times 4, 5, 6, and 7.

We will need to multiply $200a_{\overline{4}|}$ by the discount factor v^3 to account for the 3 years without payments.

$$\text{PV} = 200a_{\overline{4}|}v^3 = 200\left(\frac{1 - v^4}{0.1}\right)(1.1)^{-3} = \boxed{476.31}$$

Alternatively, one could compute $200a_{\overline{7}|} - 200a_{\overline{3}|}$ or use the annuity payment formula with $A = 200, r = 0.1, n = 4$:

$$P = \frac{200}{0.1}\left(1 - \frac{1}{(1.1)^4}\right)(1.1)^{-3} \approx \boxed{476.31}$$

WARNING: The i in the $a_{\overline{n}|}$ formula should be interpreted as the effective rate of interest per payment period. And d is the discount factor per payment period.

Example 2.7. Given $i = 0.061677812$, find the present value of a 5-year annuity-immediate with monthly payments of \$100.

Let $j = (1.061677812)^{\frac{1}{12}} - 1 = 0.005$ be the effective rate per month. Then the present value is

$$\text{PV} = 100a_{\overline{60}|j} = 100\left(\frac{1-(1.005)^{-60}}{0.005}\right) \approx \boxed{\$5172.56}$$

Example 2.8. A \$1 payment is to be made each year for 22 years. The effective annual rate of interest is 8.25%. If the first payment is one year from now, the present value of these payments is \$10. Calculate the present value if the first payment is 22 years from now.

We are given $a_{\overline{22}|} = \$10$. We want to find the present value by multiplying the discount value v^{21}:

$$\text{PV} = a_{\overline{22}|}v^{21} = 10(1.0825)^{-21} \approx \boxed{\$1.89}$$

Example 2.9. John invests \$100,000 into a fund earning an annual effective rate of 4%. John will make withdrawals of X at the end of every 5 years for a total of 10 withdrawals. After the last withdrawal the fund's balance will be 0. Calculate X.

Fix $j = (1.04)^5 - 1 = 0.2167$ as the effective rate per 5 years. We can use the formula for computing the annuity payments, using $P = 100000, r = j$, and $n = 10$:

$$X = \frac{100000(0.2167)(1.2167)^{10}}{(1.2167)^{10} - 1} \approx \boxed{\$25{,}216.97}$$

It is important to count the number of payments for annuity problems:

Example 2.10. Given $\delta = 0.05$, find the present value of payments of 200 made at times $6, 9, \ldots, 21$.

While we can use a geometric series from earlier, we have that 6 payments are made, at every 3 time periods. So, the effective rate per 3 years j is $e^{3(0.05)} - 1 \approx 0.161834243$, therefore making the present value

$$\text{PV} = 200a_{\overline{6}|j}v_j = \frac{200}{j}\left(1 - \frac{1}{(1+j)^6}\right)(1+j)^{-1} \approx \boxed{631.27}$$

Sometimes it is helpful to create groups of payments

Example 2.11. Given $a_{\overline{10}|} = 7.722$ and $d^{10} = 0.614$, find $a_{\overline{20}|}$.

We are grouping, or stacking, chunks of annuities. We already have the present value after 10 time periods, as well as the discount factor. We take the present value at 10 years and then add the present value at 10 years multiplied by the discount factor raised to the tenth power:

$$a_{\overline{20}|} = a_{\overline{10}|} + v^{10} a_{\overline{10}|} = 7.722(1 + 0.614) \approx \boxed{12.46}$$

More generally,

$$\boxed{a_{\overline{2n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|}, \qquad a_{\overline{3n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|}}$$

... and so on. This is just a geometric series in powers of v^n.

Example 2.12. A fifteen-year annuity-immediate has payments of \$500 per year. The rate of interest is 3.5% for the first nine years and 5% thereafter. What is the present value of this annuity?

$$\text{PV} = 500 a_{\overline{9}|0.035} + 500 a_{\overline{6}|0.05} v^9_{0.035} \approx \boxed{\$5665.94}$$

2.2 Accumulated Value with Annuities

Accumulated values operate no differently with annuities. We can find the accumulated value of an annuity-immediate using first principles.

Example 2.13. Given $i = 0.05$, find the accumulated value at time 10 of a 10-year annuity immediate with annual payments of \$100.

$$a(10) = 100(1.05)^9 + 100(1.05)^8 + \cdots + 100(1.05) + 100 = 100 \sum_{k=0}^{9} (1.05)^k$$

$$= 100 \left(\frac{1 - (1.05)^{10}}{1 - 1.05} \right) \approx \boxed{\$1257.79}$$

We established notation for the present value of annuity payments over n periods. Similarly, there is also one for the accumulated value of annuity payments, $s_{\overline{n}|}$. This quantity can be expressed as

$$\boxed{s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}}$$

Example 2.14. Given $i = 0.1$, find the accumulated value at time 7 of payments at \$200 at times 4, 5, 6, 7.

The accumulated value measures the annuity-immediates over 4 periods.

$$\text{AV} = 200s_{\overline{4}|} = 200\left(\frac{(1.1)^4 - 1}{.1}\right) = \boxed{\$928.20}$$

Corollary 2.15 (Relationship between $a_{\overline{n}|}$ and $s_{\overline{n}|}$). *For an arbitrary effective interest rate i,*

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$$

As with $a_{\overline{n}|}$, the i in the $s_{\overline{n}|}$ formula should be interpreted as the effective rate of interest per payment period.

Example 2.16. Given $i^{(4)} = 0.12$, find the accumulated value of a 5-year annuity-immediate with monthly payments of \$100.

Let j be the effective rate per month. Then

$$j = \left(1 + \frac{0.12}{4}\right)^{4/12} - 1 \approx 0.009901634$$

The total number of payments is 60 because they are done monthly.

$$\text{AV} = 100s_{\overline{60}|j} = 100\left(\frac{(1+j)^{60} - 1}{j}\right) = \boxed{\$8141.19}$$

Lastly, the value of the payments at time $2n$ is

$$s_{\overline{2n}|} = s_{\overline{n}|}(1+i)^n + s_{\overline{n}|}$$

...incorporating the same idea for annuity-immediates.

Example 2.17. Grandma decides to set up a college fund for her newborn grandson. She will deposit \$1,335.10 into a savings account every year on his birthday, starting on his first birthday and ending on his 18th birthday. She hopes to accumulate \$50,000 to give him on this 18th birthday. What is the minimum annual compound interest rate the savings account must earn in order to reach the goal?

In actuarial notation, this is

$$1335.1s_{\overline{18}|i} = 50000 \quad \Longleftrightarrow \quad 1335.1\left(\frac{(1+i)^{18} - 1}{i}\right) = 50000$$

This cannot be solved analytically, so a calculator must be used. The satisfying interest rate is $\boxed{i = 0.08}$.

Example 2.18. You are given $\delta_t = \dfrac{1}{1+t}$ for $0 \le t \le 5$. Calculate $s_{\overline{5}|}$.

This is a force of interest + annuity problem. Recall that $\delta_t = \frac{a'(t)}{a(t)}$, so $a(t) = 1 + t$. $s_{\overline{5}|}$ is the accumulated value of 5 payments of 1 at the time of the last payment.

$$s_{\overline{5}|} = \frac{a(5)}{a(1)} + \frac{a(5)}{a(2)} + \frac{a(5)}{a(3)} + \frac{a(5)}{a(4)} + 1 \approx 8.7$$

Example 2.19. A person age 40 wishes to accumulate a fund for retirement by depositing an amount X at the end of each year into account paying 4% interest. At age 65, the person will use the entire account balance to purchase a 15-year 5% annuity-immediate with annual payments of \$10,000. Find X.

By equating $Xs_{\overline{25}|0.04} = 10000a_{\overline{15}|0.05}$, we can solve for X:

$$X\left(\frac{(1.04)^{25} - 1}{0.04}\right) = \frac{10000}{0.05}\left(1 - \frac{1}{(1.05)^{15}}\right)$$

$$41.6459X = 103796.58 \quad \Longleftrightarrow \quad X = \boxed{\$2492.36}$$

Example 2.20. At an effective annual interest rate i, you are given

1. The present value of an annuity-immediate with annual payment of 1 for n years is 40.
2. The present value of an annuity-immediate with annual payments of 1 for $3n$ years is 70.

Calculate the accumulated value of annuity-immediate with annual payments of 1 for $2n$ years.

We need to use the equation for $a_{\overline{3n}|}$ to solve for d^n:

$$a_{\overline{3n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|}$$

$$70 = 40 + 40v^n + 40v^{2n} \quad \Longleftrightarrow \quad 40v^{2n} + 40v^n - 30 = 0$$

This is a quadratic equation in v^n:

$$v^n = \frac{-40 + \sqrt{1600 + 4(40)(30)}}{60} = 0.5$$

We can write an expression to find the accumulated value from n to $3n$, or for a total of $2n$ periods. We are adding the present value after n periods to Xv^{3n}, the accumulated value over 2 years times the discount factor of 3 years.

$$a_{\overline{3n}|} = a_{\overline{n}|} + Xv^{3n} \qquad \Longleftrightarrow \qquad 70 = 40 + X(0.5)^3$$

$$\boxed{X = 240}$$

2.3 Annuity-Due

While an annuity-immediate has payments at the end of each time interval, annuity dues are annuities with payments at the *beginning* of each time interval. Mathematically speaking, we are shifting the geometric series by one term.

Example 2.21. Given $i = 0.05$, find the present value of a 10-year annuity-due with annual payments of \$100.

If this was an annuity-immediate, the first payment is made at year 1. For an annuity-due, we start at year 0. Therefore, the present value is

$$\text{PV} = 100 + 100v + 100v^2 + \cdots + 100v^9 = 100\sum_{k=0}^{9} v^k = \frac{100 - 100v^{10}}{1 - v} \approx \boxed{\$810.78}$$

Continuing on with notation, we use $\ddot{a}_{\overline{n}|}$ to represent the present value of an n-year annuity-due with annual payments of 1. Mathematically,

$$\ddot{a}_{\overline{n}|i} = \frac{1 - v^n}{d}$$

Recall the present value of an annuity-immediate

$$a_{\overline{n}|i} = \frac{1 - v^n}{i}$$

IMPORTANT: For the annuity-immediate we divide by i and for the annuity-due we divide by d.

Example 2.22. Given $i = 0.05$, find the present value of a 10-year annuity-due with annual payments of \$100.

$$100\ddot{a}_{\overline{100}|0.05} = 100\left(\frac{1 - v^{10}}{d}\right) = 100\left(\frac{1 - (1.05)^{-10}}{\frac{0.05}{1.05}}\right) = \boxed{\$810.78}$$

Example 2.23. Given $i = 0.1$, find the present value of payments of \$200 at times 4, 5, 6, and 7 using $\ddot{a}_{\overline{n}|}$.

$$\text{PV} = 200\ddot{a}_{\overline{4}|}v^4 = 200\left(\frac{1-v^4}{\frac{0.1}{1.1}}\right)(1.1)^{-4} \approx \boxed{\$476.31}$$

As always, the d in the $\ddot{a}_{\overline{n}|}$ formula should be interpreted as the effective rate of discount per payment period, and v is the discount factor per payment period.

Example 2.24. Given $i = 0.061677812$, find the present value of a 5-year annuity-due with monthly payments of 100.

From earlier, we have already shown the effective rate per month $j = 0.005$, and the total number of payments is 60.

$$\text{PV} = 100\ddot{a}_{\overline{60}|j} = 100\left(\frac{1-v_j^{60}}{d_j}\right) = 100\left(\frac{1-(1.005)^{-60}}{\frac{0.005}{1.005}}\right) \approx \boxed{\$5198.42}$$

Example 2.25. At an annual effective interest rate of 8% the present value of a 30-year annuity-due with annual payment X is 1500. Determine X using first principles and the $\ddot{a}_{\overline{n}|}$ formula.

Using first principles,

$$1500 = X + Xv + Xv^2 + \cdots + Xv^{29} = X\sum_{k=0}^{29} v^k = X\left(\frac{1-v^{30}}{1-v}\right), \qquad v = \frac{1}{1.08} = 0.9259$$

$$X = \frac{1500 \times (1-0.9259)}{1-(0.9259)^{30}} = \boxed{\$123.37}$$

Using actuarial notation,

$$1500 = X\ddot{a}_{\overline{30}|0.08} = X\left(\frac{1-v^{30}}{d}\right), \qquad d = \frac{0.08}{1.08}$$

$$X = \frac{1500(0.08)}{1.08(1-(1.08)^{-30})} \approx \boxed{\$123.37}$$

Example 2.26. Given $d = 0.1$, calculate the present value of 20 payments of \$1000 paid every two years starting immediately.

Use d to find i:

$$v = \frac{i}{1+i} \qquad \Longrightarrow \qquad i = \frac{d}{1-d} = \frac{1}{9}$$

Using first principles,

$$\text{PV} = 1000\sum_{k=0}^{19}\left(1+v^2+v^4+\cdots+v^{38}\right) = 1000\sum_{k=0}^{19} v^{2k}$$

$$= 1000\left(\frac{1-v^{40}}{1-v^2}\right) = 1000\left(\frac{1-(0.9)^{40}}{1-(0.9)^2}\right) \approx \boxed{\$5185.36}$$

Similarly, $\ddot{s}_{\overline{n}|}$ denotes the accumulated value of an annuity-due, and we can express this using first principles.

Example 2.27. Given $i = 0.08$, find the accumulated value at time 15 of a 15-year annuity-due with annual payments of \$100.

We can write the accumulated value using a summation

$$100(1.08) + 100(1.08)^2 + \cdots + 100(1.08)^{15} = 100\sum_{k=1}^{15}(1.08)^k$$

$$= 100\left(\frac{1.08-(1.08)^{16}}{1-1.08}\right) \approx \boxed{\$2932.43}$$

Just like with annuity-immediate and annuity-dues, replace the d in

$$s_{\overline{n}|i} = \frac{(1+i)^n-1}{d}$$

with i:

$$\ddot{s}_{\overline{n}|i} = \frac{(1+i)^n-1}{i}$$

Using the previous example, for instance,

$$100\ddot{s}_{\overline{15}|0.08} = 100\left(\frac{(1.08)^{15}-1}{\frac{0.08}{1.08}}\right) \approx \boxed{\$2932.43}$$

Example 2.28. Given $i = 0.1$, find the accumulated value at time 8 of payments at \$200 at times 4, 5, 6, and 7.

$$\text{AV} = 200\ddot{s}_{\overline{4}|} = 200\left(\frac{(1.1)^4-1}{\frac{0.1}{1.1}}\right) \approx \boxed{\$1021.02}$$

The i in the $\ddot{s}_{\overline{n}|}$ formula should be interpreted as the effective rate of interest per payment period. And d is the effective rate of discount per payment period.

Example 2.29. Given $d = 0.04$, find the accumulated value at time 20 of an annuity with payments of \$250 every two years. The first payment is made immediately and the last payment is made 18 years from now.

Let $j = (1 - 0.04)^{-2} - 1 = 0.08507$ be the effective rate per 2 years. Since there are 10 payments, the accumulated value becomes

$$\text{AV} = 250\ddot{s}_{\overline{10|}} = 250\left(\frac{(1.08507)^{10} - 1}{\frac{0.08507}{1.08507}}\right) \approx \boxed{\$4025.61}$$

Example 2.30. Bill deposits X into a savings account every six months. His first deposit is made today. The savings account earns a rate of $i^{(2)} = 0.06$. Bill's account balance immediately before his 25th deposit is \$1772.96. Determine X.

We are told that the accumulated value over 25 deposits is \$1772.96. If the first payment is made today, then the 25th payment is made 12 years from now. First, we need the nominal rate $\frac{i^{(2)}}{2} = 0.03$.

$$1772.96 = X\ddot{s}_{\overline{24|}0.03} \iff 1772.96 = X\left(\frac{(1.03)^{24} - 1}{\frac{0.03}{1.03}}\right)$$

$$1772.96 = 35.46X \iff \boxed{X = \$50}$$

2.4 Annuity-Immediate vs. Annuity-Due

Just because an annuity is an annuity-immediate doesn't mean you have to value the payment or accumulated value using $a_{\overline{n|}}$ or $s_{\overline{n|}}$, and likewise for $\ddot{a}_{\overline{n|}}$ or $\ddot{s}_{\overline{n|}}$ for annuity-due.

To summarize:

- $a_{\overline{n|}}$: value of n payments of 1 one period before the first payment (time 0).
- $\ddot{a}_{\overline{n|}}$: value of n payments of 1 at time of first payment (time 1).
- $s_{\overline{n|}}$: value of n payments at time of last payment (time n).
- $\ddot{s}_{\overline{n|}}$: value of n payments of 1 one period after last payment (time $n + 1$).

Which quantity is larger, $a_{\overline{n|}}$ or $\ddot{a}_{\overline{n|}}$?.

- With respect to time 0, $a_{\overline{n|}}$ payments are at times 1, 2, . . ., n and $\ddot{a}_{\overline{n|}}$ payments are at times $0, 1, \ldots, n - 1$.

- $\ddot{a}_{\overline{n}|}$ is larger because the discount factor is not applied on first payment, whereas it would be for $a_{\overline{n}|}$.
- More precisely, it is larger by a factor of $1+i$ because each payment comes one year earlier.

These results follow immediately:

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|} \qquad \ddot{s}_n = s_{\overline{n+1}|} - 1$$

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|} \qquad \ddot{s}_{\overline{n}|} = s_{\overline{n}|}(1+i)$$

For instance, if $a_{\overline{8}|} = 10$, then

$$\ddot{a}_{\overline{9}|} = 1 + a_{\overline{8}|} = 1 + 10 = \boxed{11}$$

Example 2.31. On June 1, 1970, a company begins making equal annual deposits into a fund to provide \$15,000 a year for five successive years with which to retire a bond issue. The fund earns interest at an effective rate of 3%. The last deposit into the fund is made on June 1, 1980. The first bonds are redeemed June 1, 1980. Calculate the amount of the initial deposit.

Let X represent the annual deposit. There are 11 total deposits and we want to find X such that the account is funded \$15,000 a year over 5 consecutive years. This is equivalent to solving the equation:

$$Xs_{\overline{11}|} = 15000\ddot{a}_{\overline{5}|}$$

$$X\left(\frac{(1.03)^{11}-1}{0.03}\right) = 15000\left(\frac{1-(1.03)^{-5}}{\frac{0.03}{1.03}}\right)$$

$$\boxed{X = \$5524.49}$$

2.5 Annuities - Miscellaneous

Here, we will learn of different situations regarding annuities.

Given $a_{\overline{20}|i} = 12.46221$ and $s_{\overline{5}|i} = 5.52563$, where i is the effective rate per year, find the present value of 4 payments of 1 paid every 5 years with the first payment 5 years from now.

The first step would typically ask us to exact i, but the scenario is laid out nicely for us to try something else.

If we could split those 4 payments of 1 into smaller payments made annually, then we could find the present value using $a_{\overline{20}|}$. More precisely, we want to form 4 groups of 5 annual payments of $\frac{1}{s_{\overline{5}|}}$ because the value of the first 5 payments at time 5 is $\frac{1}{s_{\overline{5}|}} \cdot s_{\overline{5}|} = 1$.

$$\text{PV} = \left(\frac{1}{s_{\overline{5}|}}\right) a_{\overline{20}|} = \frac{12.46221}{5.52563} = \boxed{2.255}$$

This is commonly referenced as the **fission method**, with the objective of creating a new annuity with more frequent payments to compute its present value.

Definition 2.32 (Fission Method). If annuities are paid between time periods of equal length t, $\sum_{k=1}^{n} d^{kt}$ represents the present value of annuity payments. This geometric series is equal to

$$\frac{a_{\overline{nt}|}}{s_{\overline{t}|}}$$

For instance, the series

$$v^5 + v^{10} + \cdots + v^{100}$$

represents an annuity-immediate with payments every 5 time periods then

$$v^5 + v^{10} + \cdots + v^{100} = \sum_{k=1}^{20} v^{5k} = \frac{a_{\overline{100}|}}{s_{\overline{5}|}}$$

Suppose we had the series

$$v^5 + v^{12} + v^{19} + \cdots + v^{54}.$$

The fission method can still be applied, but requires a translation of $\frac{2}{7}$th of a period by factoring out v^{-2}:

$$\frac{1}{v^2}\left(v^7 + v^{14} + v^{21} + \cdots + v^{56}\right) = \frac{1}{v^2}\sum_{k=1}^{8} v^{7k} = \frac{a_{\overline{56}|}}{v^2 s_{\overline{7}|}}$$

Definition 2.33 (Deferred Annuities). A **deferred annuity** has payments that do not start in the first period. We use

$$\text{“}k|\text{”}$$

to defer the start of the annuity for k periods.

Example 2.34. Refer to Example 2.28. We could describe the annuity in two ways:

1. 3-year deferred 4-year annuity-immediate with payments of \$200.
2. 4-year deferred 4-year annuity-due with payments of \$200.

Their respective notations would be $200_{3|}a_{\overline{4}|}$ and $200_{4|}\ddot{a}_{\overline{4}|}$.

We also showed in this example that we can write the present value of an m-year deferred annuity-immediate with n payments of 1 in terms of non-deferred annuity-immediates:

$$_{m|}a_{\overline{n}|} = v^m a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$$

Similarly,

$$_{m|}\ddot{a}_{\overline{n}|} = v^m \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|}$$

There are multiple ways we can express accumulated value with more than one period after the last payment. For example, what is the accumulated value of a 4-year annuity-immediate 3 years after the last payment?

The following responses are all valid answers:

$$s_{\overline{4}|}(1+i)^3 \qquad \ddot{s}_{\overline{4}|}(1+i)^2 \qquad a_{\overline{4}|}(1+i)^7$$

$$s_{\overline{7}|} - s_{\overline{3}|} \qquad \ddot{s}_{\overline{6}|} - \ddot{s}_{\overline{2}|} \qquad \ddot{a}_{\overline{4}|}(1+i)^6$$

Definition 2.35 (Current Payment). The **current payment** is the value of an annuity at a time during the payments.

For example, what is the current value at time 3 of a 5-year annuity-immediate?

Once again, there are multiple ways to express this quantity:

$$s_{\overline{3}|} + a_{\overline{2}|} \qquad a_{\overline{5}|}(1+i)^3 \qquad s_{\overline{5}|}v^2$$

$$\ddot{s}_{\overline{2}|} + \ddot{a}_{\overline{3}|} \qquad \ddot{a}_{\overline{5}|}(1+i)^2 \qquad \ddot{s}_{\overline{5}|}v^3$$

Example 2.36. An annuity makes 5 payments of \$200 every two years starting today. You are given that $\ddot{a}_{\overline{2}|} = 1.943$ and $\ddot{s}_{\overline{10}|} = 13.972$. Find the accumulated value of the annuity two years after the last payment.

Using the fission method,

$$\text{AV} = 200\left(\frac{\ddot{s}_{\overline{10}|}}{\ddot{a}_{\overline{2}|}}\right) = 200\left(\frac{13.972}{1.943}\right) \approx \boxed{\$1438.19}$$

Example 2.37. Annuities A and B have the following payments:

End of Year	Annuity A	Annuity B
1-5	0	K
6-10	2	0
11-15	1	K

Annuities A and B have equal present values at annual effective interest rate i such that $d^5 = \frac{1}{3}$. Calculate K.

- The present value of Annuity A is the present value after 5 payments times the discount factor d^5 plus the present value after 5 more payments times the discount factor v^{10}.
- The present value of Annuity B is the present value after 5 payments plus the present value of after 5 payments times the discount factor v^{10}.

In actuarial notation, this equates to

$$2a_{\overline{5}|}v^5 + a_{\overline{5}|}v^{10} = Ka_{\overline{5}|} + Ka_{\overline{5}|}v^{10}$$

$$2\left(\frac{1-v^5}{i}\right)v^5 + \left(\frac{1-v^5}{i}\right)v^{10} = K\left(\frac{1-v^5}{i}\right) + K\left(\frac{1-v^5}{i}\right)v^{10}$$

We are told $v^5 = \frac{1}{3}$, so $v^{10} = \left(v^5\right)^2 = \frac{1}{9}$. Rearranging and solving for K gives $\boxed{K = 0.7}$.

Example 2.38. You are given that X is the current value at the end of year two of a twenty-year annuity-due of one per annum and the effective annual interest rate for year t is $\frac{1}{8+t}$. Calculate X.

The idea is to break the payments into 4 parts.

1. To accumulate from time 0 to 2, we must multiply by $\left(1+\frac{1}{9}\right)\left(1+\frac{1}{10}\right) = \frac{11}{9}$.
2. To accumulate from time 1 to 2, add 1 to the interest rate, so $1 + \frac{1}{10} = \frac{11}{10}$.
3. At time $t = 2$, the current value is 1.
4. For time $t \geq 3$, we have to discount by 1 time at $t = 3$, twice for $t = 4$, and up to 17

times for $t = 19$. For instance, at time $t = 3$, we have

$$\frac{1}{1+\frac{1}{11}}$$

and at time 4,

$$\frac{1}{1+\frac{1}{11}} \cdot \frac{1}{1+\frac{1}{12}}$$

yielding $\frac{11}{12}$ and $\frac{11}{13}$, respectively. The last term would yield $\frac{11}{28}$.

This is a telescoping pattern, with the sum

$$\frac{11}{9} + \frac{11}{10} + \cdots + \frac{11}{27} + \frac{11}{28} = \sum_{t=9}^{28} \frac{11}{t}$$

2.6 Perpetuities

Definition 2.39 (Perpetuities). A **perpetuity** is an annuity whose payments continue forever, or infinitely. A **perpetuity-immediate** has payments at the end of each period forever, whereas a **perpetuity-due** has payments at the beginning of each period forever.

The actuarial notation for perpetuities can be expressed algebraically

$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i}$$

where i should be interpreted as the effective rate of interest per payment period.

The actuarial notation for perpetuity-dues can be derived mathematically:

$$\ddot{a}_{\overline{\infty}|} = 1 + v + v^2 + \cdots = 1 + a_{\overline{\infty}|} = 1 + \frac{1}{i}$$
$$= \frac{1+i}{i} = \boxed{\frac{1}{d}}$$

Example 2.40. Given $i = 0.1$, find the present value of a perpetuity with payments of \$300 at times 2, 4, 6, . . .

Let j = effective rate per 2 years = $(1.1)^2 - 1 = 0.21$.

$$300a_{\overline{\infty}|0.21} = \frac{300}{0.21} \approx \boxed{\$1428.57}$$

Example 2.41. Deposits of \$1000 are placed into a fund at the beginning of each year for 30 years. At the end of the 40th year, annual withdrawals commence and continue forever. Interest is at an effective annual rate of 5%. Calculate the annual withdrawal.

Let X represent the annual withdrawal. We want to set the total money deposited equal to the money withdrawn forever. We can compute the accumulated value over 30 years, and multiply by the compounded interest rate (over 10 years) to obtain the real accumulated value. This quantity is set equal to X times the perpetuity:

$$1000 s_{\overline{30}|}(1.05)^{10} = X a_{\overline{\infty}|}$$

$$1000\left(\frac{(1.05)^{30}-1}{\frac{0.05}{1.05}}\right) = 20X \quad \Longleftrightarrow \quad \boxed{X \approx \$5681.65}$$

Example 2.42. A perpetuity of \$1 each year, with the first payment due immediately, has a present value of \$25 at an effective rate of i. The owner is considering exchanging it now for another perpetuity with the first payment due immediately and subsequent payments due at two-year intervals. What should the payment of the second perpetuity be, in order to keep the same interest rate (i) and the same present value?

The first sentence describes a perpetuity-due, where

$$\ddot{a}_{\overline{\infty}|i} = 25 \quad \Longrightarrow \quad \frac{1}{d} = 25 \quad \Longleftrightarrow \quad d = 0.04$$

Fix $j =$ effective rate of discount per 2 years $= 1-(1-0.04)^2 = 0.0784$. Then, if X is the payment value,

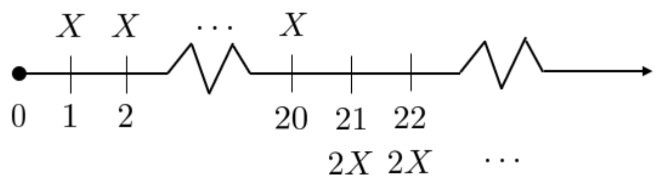
$$\frac{X}{j} = 25 \quad \Longleftrightarrow \quad \boxed{X = 1.96}$$

Example 2.43. John's estate is to be divided into three equal parts and invested, to be paid out as follows:

- John's two children will each receive their share in 20 level annual payments beginning one year after John's death.
- Charity Q will receive its share as equal annual payments in perpetuity beginning 21 years after John's death.

Q's annual payment is twice the annual payment for one child. Determine the effective interest rate at which the estate is invested.

"Divided into three equal parts" implies the present value of each party's payments is equal. Let $X =$payment per child, then $2X$ is Charity Q's payment.



$$X_{\overline{20}|} = 2Xa_{\overline{\infty}|}v^{20}$$

$$\frac{1-v^{20}}{i} = 2\left(\frac{1}{i}\right)v^{20} \qquad \Longleftrightarrow \qquad 1-v^{20} = 2v^{20}$$

$$1 = 3v^{20} \qquad \Longleftrightarrow \qquad i = \left(\left(\frac{1}{3}\right)^{0.05}\right)^{-1}$$

$$\boxed{i \approx 0.0565}$$

3 Annuities Part II

In this section, we will be expanding upon the annuities mentioned previously. More precisely, we will discuss m-thly payments, arithmetic vs. geometric annuities, and continuous annuities.

3.1 Annuities Payable m-thly

Just as money can be compounded m times per year, we can apply the same to payments. A **m-thly annuity** is an annuity with m payments per year.

- $m = 2$: semi-annual payments
- $m = 4$: quarterly payments
- $m = 12$: monthly payments

We have already used the effective rate per payment period method to solve such problems, but we can streamline the process through *fusion*.

As opposed to splitting an annuity into multiple "annual" payments through the fission method, the **fusion method** offers to group m-thly payments into an annual payment.

Example 3.1. Given $i = 0.061677812$, find the present value of a 5-year annuity-immediate with monthly payments of \$100.

Previously, we calculated the effective rate per month as $j = (1.061677812)^{1/12} - 1$ and calculated the total number of payments to find the present value.

Alternatively, we can consider the accumulated value over each year, leaving us with 5 "annual" payments. Fix $j = \frac{i^{(12)}}{12}$ as the effective rate per month, then the accumulated value per year, or the payment per year, is $100s_{\overline{12}|j}$. Now, we multiply by the present value of the annuities over 5 years with rate i.

$$\text{PV} = 100s_{\overline{12}|j}a_{\overline{5}|i} = 100\left(\frac{(1+j)^{12}-1}{j}\right)a_{\overline{5}|i}$$

$$= \frac{1200}{12}\left(\frac{(1+i)-1}{j}\right)a_{\overline{5}|i} = 1200\left(\frac{i}{i^{(12)}}\right)a_{\overline{5}|i} = \$5172.56$$

$$= 1200a^{(12)}_{\overline{5}|i}$$

The coefficient of the m-thly annuity factor is the total annual payment!

Definition 3.2 (Fusion Method). Consider n years of m-thly payments of $\frac{1}{m}$. Then the present value of the annuity is

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}}$$

There are multiple equivalent expressions using annual/monthly annuity-immediates or annuity-dues.

$$a_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} a_{\overline{n}|} \qquad a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}} \qquad a_{\overline{n}|} = \frac{1-v^n}{i}$$

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{d}{d^{(m)}} \ddot{a}_{\overline{n}|} = \frac{i}{d^{(m)}} a_{\overline{n}|} \qquad \ddot{a}_{\overline{n}|}^{(m)} = \frac{1-v^n}{d^{(m)}} \qquad \ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$$

$$s_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} s_{\overline{n}|} \qquad s_{\overline{n}|}^{(m)} = \frac{(1+i)^n-1}{i^{(m)}} \qquad s_{\overline{n}|} = \frac{(1+i)^n-1}{i}$$

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{d}{d^{(m)}} \ddot{s}_{\overline{n}|} = \frac{i}{d^{(m)}} s_{\overline{n}|} \qquad \ddot{s}_{\overline{n}|}^{(m)} = \frac{(1+i)^n-1}{d^{(m)}} \qquad \ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d}$$

Example 3.3. Dave receives twenty payments of \$500 every 6 months with the first payment immediately. He immediately deposits each payment into an account earning an annual effective rate of i. What is an expression that could represent the value of the account at time 10?

The total annual payment is \$1,000 and there are 10 years worth of payments. We can express the accumulated value as

$$1000\ddot{s}_{\overline{10}|i}^{(2)} \quad \text{or} \quad 500\ddot{s}_{\overline{20}|j}$$

The latter expression is equivalent to the former if j is the effective rate per 6 months.

Example 3.4. Given $\ddot{a}_{\overline{3}|}^{(4)} = 2.7732$ and $v^3 = 0.8396$, what is the present value of 12 quarterly payments of \$100 with the first payment one quarter from now?

Using the fusion method, the total annual payment is \$400. The present value of the annuity-immediate is the present value of the annuity-due multiplied by the discount over one quarter:

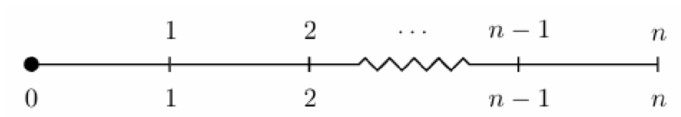
$$400a_{\overline{3}|}^{(4)} = 400\ddot{a}_{\overline{3}|}^{(4)} v^{0.25} = 400(2.7732)(0.8396)^{0.25/3} \approx \boxed{\$1093.24}$$

Alternatively, one could reach the same value through the following expression:

$$400a_{\overline{3}|}^{(4)} = 400\ddot{a}_{\overline{3}|}^{(4)} - 100 + 100v^3$$

3.2 Arithmetic Annuities

So far, we've only looked at annuities whose payments are constant over each time period. Suppose the payments increased or decreased linearly over each time period. For example, instance the following increasing annuity-immediate, where the payment is equal to its corresponding time period.



The present value of this annuity is denoted $(Ia)_{\overline{n}|}$, where I is used to distinguish itself from geometric annuities, which we will discuss later.

Corollary 3.5 (Present Value of Increasing Arithmetic Annuity-Immediates). *Suppose an annuity-immediate has payments equal to its corresponding time period, then*

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

Proof. We will use the same approach as when we proved the geometric series convergence. We have that

$$(Ia)_{\overline{n}|} = v + 2v^2 + \cdots + (n-1)v^{n-1} + nv^n$$

$$(1+i)(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \cdots + nv^{n-1}$$

$$(1+i)(Ia)_{\overline{n}|} - (Ia)_{\overline{n}|} = i(Ia)_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-1} - nv^n$$

$$i(Ia)_{\overline{n}|} = \ddot{a}_{\overline{n}|} - nv^n \iff \boxed{(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}}$$

□

Example 3.6. Given $i = 0.05$, find the present value of payments of \$10, \$20, ..., \$90 at times 1, 2, ..., 9 respectively.

The notation to describe this quantity is $10(Ia)_{\overline{9}|}$, and it is equal to

$$10(Ia)_{\overline{9}|} = 10\left(\frac{\ddot{a}_{\overline{9}|} - 9v^9}{0.05}\right) \approx \boxed{\$332.35}$$

where $\ddot{a}_{\overline{9}|} = \dfrac{1 - \left(\frac{1}{1.05}\right)^9}{\frac{0.05}{1.05}} \approx 7.463$.

Here lies the reoccurring calculation

Corollary 3.7 (Present Value of Increasing Arithmetic Annuity-Dues). *Suppose an annuity-due has payments equal to its corresponding time period plus one, then*

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

Proof.

$$(I\ddot{a})_{\overline{n}|} = (1+i)(Ia)_{\overline{n}|} = (1+i)\left(\frac{\ddot{a}_{\overline{n}|} - nv^n}{i}\right) = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\frac{1}{1+i}} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

□

We divide by d instead of i just like we did for the level annuity-due vs. the level annuity-immediate.

As such, we can also describe the accumulated value of arithmetic annuities.

Corollary 3.8 (Accumulated Value of Arithmetic Annuity-Immediates and Dues). *For an annuity-immediate, we have*

$$(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

For an annuity-due, we have

$$(I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_n - n}{d}$$

Proof. For an arithmetic annuity-immediate,

$$(Is)_{\overline{n}|} = (Is)_{\overline{n}|} = (Ia)_{\overline{n}|}(1+i)^n = \left(\frac{\ddot{a}_{\overline{n}|} - nv^n}{i}\right)(1+i)^n = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

The same procedure is followed for arithmetic annuity-dues. □

Corollary 3.9 (Present Value of Increasing Arithmetic Perpetuities). *he present value of an increasing arithmetic perpetuity-immediate is expressed as*

$$(Ia)_{\overline{\infty}|} = \frac{1}{id}$$

The present value of an increasing arithmetic perpetuity-due is expressed as

$$(I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$$

Proof. This is an increasing annuity-immediate where $n \to \infty$:

$$(Ia)_{\overline{\infty}|} = v + 2v^2 + 3v^3 + 4v^4 + \cdots$$

$$v(Ia)_{\overline{\infty}|} = v^2 + 2v^3 + 3v^4 + \cdots$$

$$(1-v)(Ia)_{\overline{\infty}|} = v + v^2 + v^3 + v^4 + \cdots = a_{\overline{\infty}|}$$

For an increasing arithmetic perpetuity-immediate,

$$(Ia)_{\overline{\infty}|} = \frac{a_{\overline{\infty}|}}{1-v} = \frac{\frac{1}{i}}{1-\frac{1}{1+i}} = \frac{1+i}{i^2} = \frac{1}{id}$$

For an increasing arithmetic perpetuity-due,

$$(I\ddot{a})_{\overline{\infty}|} = (1+i)(Ia)_{\overline{\infty}|} = \frac{(1+i)^2}{i^2} = \frac{1}{d^2}$$

□

Example 3.10. Olga buys a five-year increasing annuity for X. Olga will receive \$2 at the end of the first month, \$4 at the end of the second month, and for each month thereafter the payment increases by \$2. The nominal interest rate is 9% convertible quarterly. Calculate X.

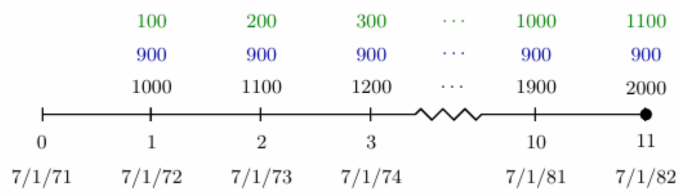
Let $j = \left(1+\frac{0.09}{4}\right)^{1/3} - 1 \approx 0.0074$ be the effective rate per month. We want to compute $100(2Ia)_{\overline{5\times 12}|j}$.

$$(2Ia)_{\overline{5\times 12}|j} = 200(Ia)_{\overline{60}|j} = 2\left(\frac{\ddot{a}_{\overline{60}|j} - 60(1.0074)^{-1}}{\frac{0.0074}{1.0074}}\right) \approx \boxed{\$2733.89}$$

where

$$\ddot{a}_{\overline{60}|j} = \frac{1-\left(\frac{1}{1.0074}\right)^{60}}{\frac{0.0074}{1.0074}} \approx 48.67$$

Example 3.11. Smith makes deposits to a savings account on July 1 of each year. The first deposit was \$1,000 and each subsequent deposit increases by \$100. The account is credited with an effective annual rate of interest of 4%. Determine the value of Smith's account as of July 2, 1982.



The first option is to compute $1000\ddot{s}_{\overline{11}|0.04} + 100(Is)_{\overline{11}|0.04}$:

$$1000\ddot{s}_{\overline{11}|0.04} + 100(Is)_{\overline{11}|0.04} = 1000\ddot{s}_{\overline{11}|} + \frac{\ddot{s}_{\overline{11}|0.04} - 11}{0.04} \approx \boxed{\$19705.88}$$

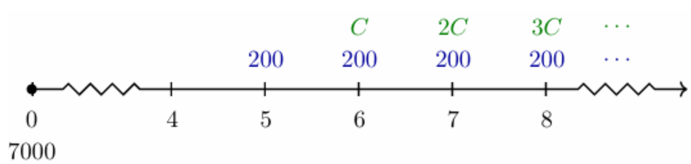
where

$$\ddot{s}_{\overline{11}|0.04} = \frac{(1.04)^{11} - 1}{0.04} \approx 13.49$$

Also, by the timeline, we can divide the payments as a fixed deposit of \$900 each year plus an increasing deposit of \$100 per year, starting with the first payment. So, we can express the current value as the sum of the accumulated value of fixed \$900 over 11 years plus the accumulated value of the increasing \$100 over 11 years. In actuarial notation, this reduces to

$$900s_{\overline{11}|} + 100(Ia)_{\overline{11}|}(1.04)^{11} \approx \boxed{\$19705.88}$$

Example 3.12. Jones purchased a perpetuity today for \$7,000. He will receive the first annual payment of \$200 five years from now. The second annual payment will be \$200 plus an amount C. Each subsequent payment will be the prior amount plus an additional constant amount C. If the effective interest rate is 4%, find C.



There are two parts that will be summed:

1. The present value of the deferred 4-year payment of \$200.
2. The present value of the increasing arithmetic annuity of C dollars, deferred by 5 years.

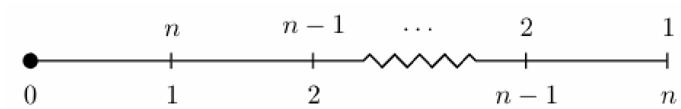
Thus we set this sum equal to the cost of the perpetuity:

$$7000 = 200a_{\overline{\infty}|}v^4 + C(Ia)_{\overline{\infty}|}v^5$$

$$7000 = 200\left(\frac{1}{0.04}\right)(1.04)^{-4} + C\left(\frac{1}{0.04\left(\frac{0.04}{1.04}\right)}\right)(1.04)^{-5}$$

Solving for C gives $\boxed{C \approx \$5.10}$.

Similarly, annuity payments can decrease arithmetically per time period, using $(Da)_{\overline{n}|}$ to describe said quantity.



Corollary 3.13 (Present Value of Arithmetic Decreasing Annuities). *For a decreasing arithmetic annuity-immediate,*

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}.$$

For a decreasing arithmetic annuity-due,

$$(D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d}.$$

These are derived in the same fashion as the proofs for Corollaries 3.7 and 3.8.

To summarize, we have the formulas

$$a_{\overline{n}|} = \frac{1 - v^n}{i} \qquad (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \qquad (Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d} \qquad (I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} \qquad (D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d}$$

Corollary 3.14 (Accumulated Value of Arithmetic Decreasing Annuities). *For a decreasing arithmetic annuity-immediate,*

$$(Ds)_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{i}$$

For a decreasing arithmetic annuity-due,

$$(D\ddot{s})_{\overline{n|}} = \frac{n(1+i)^n - s_{\overline{n|}}}{d}$$

It is typically easier, however, to find the present value (with $(Da)_{\overline{n|}}$) and then accumulate.

Example 3.15. Jane receives a ten-year increasing annuity-immediate paying \$100 the first year and increasing by \$100 each year thereafter. Mary receives a ten-year decreasing annuity-immediate paying X the first year and decreasing by $\frac{X}{10}$ each year thereafter. At an effective interest rate of 5%, both annuities have the same present value. Calculate X.

Set Jane and Mary's present values equal to each other:

$$100(Ia)_{\overline{10|}} = \left(\frac{X}{10}\right)(Da)_{\overline{10|}}$$

Using a TVM calculator, this gives

$$100\left(\frac{8.108 - 10\left(\frac{1}{1.05}\right)^{10}}{0.05}\right) = \frac{X}{10}\left(\frac{10 - 7.722}{0.05}\right)$$

Solving for X gives $\boxed{X \approx \$864.30}$.

Example 3.16. For a decreasing annuity-due with annual payments:

1. The first payment is \$120.
2. Each subsequent payment decreases by \$10.
3. The last payment is \$40.
4. The annual effective rate of interest if 5%.

Find the accumulated value of the annuity payments one year after the last payment.

There are 9 total payments. We can express the accumulated value in two ways:

1. The accumulated value of the decreasing annuity-due payments of \$10 over 9 years + the accumulated value of the fixed \$30 over 9 years.

$$10(D\ddot{s})_{\overline{9}|} + 30\ddot{s}_{\overline{9}|} \approx \boxed{\$963.77}$$

   using a TVM calculator.

2. The present value of the the decreasing annuity-immediate payments of \$10 over 9 years + the present value of the fixed \$30 over 9 years. Since we are starting at time -1 (using annuity-immediate instead of annuity-due), the sum is then accumulated over the interest factor for 10 years instead of 9.

$$(10(Da)_{\overline{9}|} + 30a_{\overline{9}|})(1.05)^{10} \approx \boxed{\$963.77}$$

Example 3.17. For an annuity-immediate with monthly payments:

1. The monthly payments for each year are level.
2. The monthly payments for the first year are \$100.
3. Each subsequent year the monthly payments decrease by \$10. For example, in year two the monthly payments are \$90, in year three the monthly payments are \$80, and so on.
4. There are ten years worth of payments.
5. The annual effective interest rate is 8%.

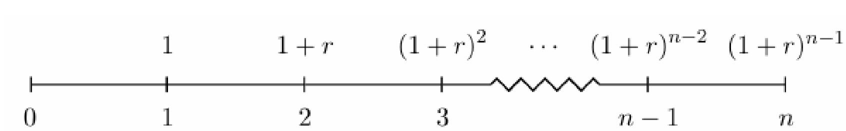
Find the present value of this annuity.

Let $j = (1.08)^{1/12} - 1 \approx 0.00643$ be the effective rate per month. We want to multiply the accumulated value of the monthly payments (with rate j) by the decreasing annuity-immediate payments of \$10 per year (with rate 0.08). In actuarial notation, this becomes

$$10s_{\overline{12}|j}(Da)_{\overline{10}|0.08} \approx \boxed{\$5113.31}$$

3.3 Geometric Annuities

Moving on, we will discuss annuities whose payments compound by some rate $1 + r$ per time period.



While it is not standard actuarial notation, we will use $(Ga)_{\overline{n}|i,r}$ to denote the present value.

Corollary 3.18 (Present Value of Geometric Annuities). *For a geometric annuity-immediate, the present value is*

$$(Ga)_{\overline{n}|i,r} = \frac{1-\left(\frac{1+r}{1+i}\right)^n}{i-r}$$

For a geometric annuity-due, the present value is

$$(G\ddot{a})_{\overline{n}|i,r} = \ddot{a}_{\overline{n}|j}$$

where $j = \frac{i-r}{1+r}$.

Proof. For a geometric annuity-immediate,

$$(Ga)_{\overline{n}|i,r} = v_i + (1+r)v_i^2 + (1+r)^2 v_i^3 + \cdots + (1+r)^{n-1}v_i^n$$

$$= \frac{v_i - (1+r)^n v_i^{n+1}}{1-(1+r)v_i} = \frac{v_i(1-(1+r)^n v_i^n)}{1-(1+r)v_i} = \frac{1-\left(\frac{1+r}{1+i}\right)^n}{1+i-(1+r)} = \boxed{\frac{1-\left(\frac{1+r}{1+i}\right)^n}{i-r}}$$

For a geometric annuity-due,

$$(G\ddot{a})_{\overline{n}|i,r} = (1+i)(Ga)_{\overline{n}|i,r} = (1+i)\left(\frac{1-\left(\frac{1+r}{1+i}\right)^n}{i-r}\right) = \frac{1-\left(\frac{1+r}{1+i}\right)^n}{\frac{i-r}{1+i}}$$

By fixing $v_j = \frac{1+r}{1+i}$, then $d_j = 1 - \frac{1+r}{1+i} = \frac{i-r}{1+i}$ and

$$(G\ddot{a})_{\overline{n}|i,r} = \frac{1-(v_j)^n}{d_j} = \ddot{a}_{\overline{n}|j}$$

where $(1+j)^{-1} = \frac{1+r}{1+i} \implies j = \frac{i-r}{1+r}$. □

Example 3.19. For an annuity-immediate with annual payments:

1. The first payment is \$1000.
2. Each subsequent payment increases by 8%.
3. The last payment is $1000(1.08)^{15}$.
4. The annual effective rate of interest is 6%.

Find the present value of this annuity.

There are 16 total payments. In actuarial notation, we are calculating

$$1000(Ga)_{\overline{16}|0.06,0.08} = 1000\left(\frac{1-\left(\frac{1.08}{1.06}\right)}{0.06-0.08}\right) \approx \boxed{\$17{,}430.48}$$

Alternatively, we can use what we know about geometric series:

$$\text{PV} = \frac{1000(1.06)^{-1} - 1000(1.08)^{16}(1.06)^{-17}}{1-(1.08)(1.06)^{-1}} \approx \boxed{\$17{,}430.48}$$

Example 3.20. For an annuity-due with annual payments:

1. The first payment is \$1000.
2. Each subsequent payment increases by 6%.
3. The last payment is $1000(1.06)^{15}$.
4. The annual effective rate of interest is 8%.

Find the present value of this annuity.

$$1000(G\ddot{a})_{\overline{16}|0.08,0.06} = 1000(1.08)(Ga)_{\overline{16}|0.08,0.06} = 1000(1.08)\left(\frac{1-\left(\frac{1.06}{1.08}\right)}{0.08-0.06}\right) \approx \boxed{\$13{,}958.76}$$

Alternatively, let $j = \frac{i-r}{1+r} = \frac{0.08-0.06}{1.06} \approx 0.01887$. Then

$$1000(G\ddot{a})_{\overline{16}|0.08,0.06} = 1000\ddot{a}_{\overline{16}|0.01887} \approx \boxed{\$13{,}958.76}$$

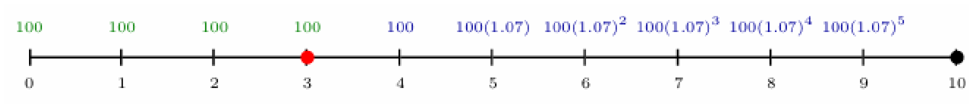
Corollary 3.21 (Accumulated Value of Geometric Annuities). *For a geometric annuity-immediate, the accumulated value over n time periods is*

$$(Gs)_{\overline{n}|i,r} = \frac{(1+i)^n - (1+r)^n}{i-r}$$

For a geometric annuity-due,

$$(G\ddot{s})_{\overline{n}|i,r} = (Ga)_{\overline{n}|i,r}(1+i)^{n+1}$$

Example 3.22. Annual deposits are made into a fund at the beginning of each year for 10 years. The first 5 deposits are \$100 each and deposits increase by 7% per year thereafter. If the fund earns an annual effective rate of 5%, find the accumulated value at the end of 10 years.



The present value at time 3 is

$$100s_{\overline{4}|0.05} + 100(Ga)_{\overline{6}|0.05,0.07} = 431.01 + 100\left(\frac{1-\left(\frac{1.07}{1.05}\right)^6}{0.05-0.07}\right) \approx 1030.35$$

Accumulate to time 10,

$$1030.35(1.05)^7 \approx \boxed{\$1449.81}$$

Example 3.23. Stan elects to receive his retirement benefit over 20 years at the rate of \$2,000 per month beginning one month from now. The monthly benefit increases by 5% each year. At a nominal rate of 6% convertible monthly, calculate the present value of the retirement benefit.

Let $j = \frac{0.06}{12} = 0.005$ be the effective rate per month and $i = (1+j)^{12} - 1 \approx 0.0616778$ be the effective rate per year. Then,

$$\text{PV} = 2000s_{\overline{12}|j}(Ga)_{\overline{20}|i,0.05} = 2000\left(\frac{(1.005)^{12}-1}{0.005}\right)\left(\frac{1-\left(\frac{1.05}{1+i}\right)^{20}}{i-0.05}\right)$$

$$\approx \boxed{\$419{,}243.25}$$

In the special case where $r = i$, then

1. $(Ga)_{\overline{n}|r,r} = nv_r$.
2. $(Gs)_{\overline{n}|r,r} = n(1+r)^{n-1}$.
3. $(G\ddot{a})_{\overline{n}|r,r} = n$.
4. $(G\ddot{s})_{\overline{n}|r,r} = n(1+r)^n$.

Example 3.24. An annuity provides for 12 annual payments. The first payment is \$100, paid at the end of the first year, and each subsequent payment is 5% more than the one preceding it. Calculate the present value of this annuity if $i = 0.05$.

Using the formula $(Ga)_{\overline{n}|r,r} = nv_r$,

$$100(Ga)_{\overline{12}|0.05,0.05} = 100(12)(1.05)^{-1} \approx \boxed{\$1142.86}$$

Corollary 3.25 (Present Value of Geometric Perpetuities). *The limits for annuity-immediates and annuity-dues depend on the values of r and i.*

$$(Ga)_{\overline{\infty}|i,r} = \begin{cases} \frac{1}{i-r} & r < i \\ \text{undefined} & r \geq i \end{cases}$$

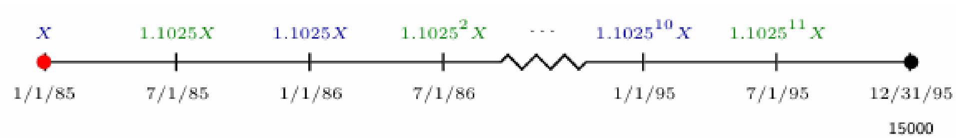
$$(G\ddot{a})_{\overline{\infty}|i,r} = \begin{cases} \frac{1+i}{i-r} & r < i \\ \text{undefined} & r \geq i \end{cases}$$

Example 3.26. An annuity provides payments forever. The first payment is \$100, paid at the beginning of the first year, and each subsequent is 5% more than the one preceding it. Calculate the present value of this annuity if $i = 0.08$.

Using the formula $(G\ddot{a})_{\overline{\infty}|i,r} = \begin{cases} \frac{1+i}{i-r} & r < i \\ \text{undefined} & r > i \end{cases}$

$$100(G\ddot{a})_{\overline{\infty}|0.08,0.05} = 100\left(\frac{1.08}{0.08 - 0.05}\right) = \boxed{\$3600}$$

Example 3.27. Smith will make deposits to a fund each Janury 1 and July 1 from 1985 through 1995. Each July 1, the level deposit is increased 10.25% over the previous level. The effective annual interest rate is 10.25%. The fund balance on December 31, 1995 will be \$15,000. Calculate the amount of Smith's initial deposit on January 1, 1985.



The present value at 1/1/85 is the sum of the **present values** of the January and July payments. We split these separately:

$$X(G\ddot{a})_{\overline{11}|0.1025,0.1025} + 1.1025X(G\ddot{a})_{\overline{11}|0.1025,0.1025}v_{0.1025}^{0.5}$$

$$X(11) + 1.1025(X)(11)(1.1025)^{-0.5} = 22.55X = 15000$$

$$\boxed{X \approx \$227.39}$$

Example 3.28. A perpetuity pays $2X$ one year from today. The annual payments increase by 5% per year thereafter. The effective annual interest rate on this perpetuity is 6%. The present value is \$32,400.

A second perpetuity pays Y one year from now, and the annual payments increase by an amount of X per year thereafter. The effective annual interest rate on this perpetuity is i and the present value is \$24,000.

A third perpetuity pays Y per year, with the first payment one year from now. The effective annual interest rate on this perpetuity is i and the present value is \$4,000.

What is Y?

The respective values for each perpetuity are:

1. $$2X(Ga)_{\overline{\infty}|0.06,0.05} = 32400$$

2. $$Ya_{\overline{\infty}|i} + X(Ia)_{\infty i}v_i = 24000$$

3. $$Ya_{\overline{\infty}|i} = 4000$$

Use (1) to solve for X:

$$X\left(\frac{1}{0.06 - 0.05}\right) = 16200 \quad \Longleftrightarrow \quad \boxed{X = 162}$$

Use (2) and (3) to solve for i:

$$Ya_{\overline{\infty}|i} + X(Ia)_{\overline{\infty}|}v_i = 24000$$

$$4000 + X\left(\frac{1}{id}\right)v_i = 24000$$

$$4000 + X\left(\frac{1+i}{i^2}\right)v_i = 24000$$

$$X\left(\frac{1}{i^2}\right) = 20000 \quad \Longleftrightarrow \quad \boxed{i = 0.09}$$

Plug i into (3) to solve for Y:

$$Y\left(\frac{1}{0.09}\right) = 4000 \quad \Longleftrightarrow \quad \boxed{Y = 360}$$

3.4 Continuous Annuities

A **continuous annuity** is an annuity with payments made continuous, or infinitely many times. It is often used as a theoretical tool and as an approximation for annuities payable with great frequency (e.g. daily). Its corresponding notation is $\overline{a}_{\overline{n}|}$. We will build upon the concepts outlined in forces of interest.

Corollary 3.29 (Present Value of Continuous Annuities). *The present value of a continuous annuity over a time length n is*

$$\overline{a}_{\overline{n}|} = \frac{1 - v^n}{\delta}$$

where $\delta = \ln(1+i)$.

Proof. The amount of the payment between time t and $t + dt$ is $1dt$. The present value of that payment is $v^t dt$.

$$\overline{a}_{\overline{n}|} = \int_0^n v^t dt = \left[\frac{v^t}{\ln(v)}\right]_0^n = \frac{v^n - 1}{\ln(v)} = \frac{v^n - 1}{\ln(1+i)^{-1}} = \boxed{\frac{1 - v^n}{\delta}}$$

□

Example 3.30. The annual effective rate of interest is 5%. Find the present value of an annuity of \$25 per year payable continuously for 15 years.

$$25\overline{a}_{\overline{15}|0.05} = 25\left(\frac{1 - (1.05)^{-15}}{\ln(1.05)}\right) \approx \boxed{\$265.93}$$

Corollary 3.31 (Accumulated Value of Continuous Annuities). *The accumulated value of a continuous annuity over a time length n is*

$$\overline{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

where $\delta = \ln(1+i)$ and $(1+i)^n = e^{-\delta n}$.

Proof.

$$\overline{s}_{\overline{n}|} = (1+i)^n\left(\frac{1 - v^n}{\delta}\right) = \frac{(1+i)^n - 1}{\delta}$$

□

Example 3.32. A ten year annuity pays \$50 per year payable continuously. Given $\delta = 0.04$, find the accumulated value at the end of the ten years.

$$50\bar{s}_{\overline{10}|} = 50\left(\frac{e^{10(0.04)} - 1}{0.04}\right) \approx \boxed{\$614.78}$$

We can establish a relationship between continuous and non-continuous annuities:

$$\boxed{\bar{a}_{\overline{n}|} = \frac{i}{\delta}a_{\overline{n}|} \qquad \bar{s}_{\overline{n}|} = \frac{i}{\delta}s_{\overline{n}|}}$$

If given i, using $\bar{a}_{\overline{n}|} = \frac{i}{\delta}a_{\overline{n}|}$ is easier. If given δ, using $\bar{a} = \frac{1-v^n}{\delta}$ is easier.

Corollary 3.33 (Present Value of Continuous Perpetuities). *The present value of a continuous perpetuity with a rate of payment of 1 per year is*

$$\bar{a}_{\overline{\infty}|} = \frac{1}{\delta}$$

This can be easily shown by taking the limit of $\bar{a}_{\overline{n}|}$ at infinity.

In summary,

$$\boxed{a_{\overline{\infty}|} = \frac{1}{i} \qquad \ddot{a}_{\overline{\infty}|} = \frac{1}{d} \qquad \bar{a}_{\overline{\infty}|} = \frac{1}{\delta}}$$

Example 3.34. A continuous perpetuity has a rate of payment of \$10 per year for the first 5 years and \$20 per year thereafter. If the annual effective rate is 8%, find the present value of this perpetuity.

The present value is

$$10\bar{a}_{\overline{5}|} + 20\bar{a}_{\overline{\infty}|}v^5 = 10\left(\frac{1-(!.08)^{-5}}{\ln(1.08)}\right) + 20\left(\frac{1}{\ln(1.08)}\right)(1.08)^{-5} \approx \boxed{\$218.37}$$

Example 3.35. The present value of a continuous annuity of \$1 per year for a certain number of years is \$4. The continuously compounded rate of interest is 12.5%. What is the accumulated value of a continuous annuity of \$1 per year for twice that number of years?

$\bar{a}_{\overline{n}|} = 4$ implies that

$$\frac{1-v^n}{0.125} = 4 \implies 1 - v^n = 0.5 \implies v^n = (1+i)^{-n} = 0.5$$
$$\implies (1+i)^n = 2$$

Now, we want to compute $\bar{s}_{\overline{2n}|}$:

$$\bar{s}_{\overline{2n}|} = \frac{(1+i)^{2n}-1}{\delta} = \frac{2^2-1}{0.125} = \boxed{\$24}$$

So far we've only made observations for continuous annuities with level payments and constant forces of interest.

Consider an n-year continuous annuity with, at time t, a rate of payment $f(t)$ and force of interest δ_t. Then the amount of payment between time t and $t+dt = f(t)dt$. The present value and accumulated value become:

$$\boxed{\text{PV} = \int_0^n \frac{f(t)}{a(t)}dt = \int_0^n f(t)\exp\left(-\int_0^t \delta_s ds\right)dt}$$

$$\boxed{\text{AV} = \int_0^n f(t)a(t,n)dt = \int_0^n f(t)\exp\left(\int_t^n \delta_s ds\right)dt}$$

With these in mind, now consider continuously increasing/decreasing annuities for n years with continuous payments.

Corollary 3.36 (Continuously Increasing Annuities). *Consider a continuously increasing annuity of n years with continuous payments. The rate of payment at time t is t. If the force of interest is δ, then the present value and accumulated value are given by*

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} \qquad (\bar{I}\bar{s})_{\overline{n}|} = \frac{\bar{s}_{\overline{n}|} - n}{\delta}$$

Proof. We will prove the present value statement:

$$\text{PV} = \int_0^n \frac{f(t)}{a(t)}dt = \int_0^n te^{-\delta t}dt$$

This can be done with integration by parts, setting $u = t$ and $dv = e^{-\delta t}dt$, making $du = dt$ and $v = -\frac{e^{-\delta t}}{\delta}$.

$$= \left[t \cdot \frac{-e^{-\delta t}}{\delta}\right]_0^n - \int_0^n \frac{-e^{-\delta t}}{\delta}dt$$

$$= \frac{-ne^{-\delta n}}{\delta} + \frac{1}{\delta}\int_0^n e^{-\delta t}dt$$

$$= \frac{-ne^{-\delta n}}{\delta} + \frac{\bar{a}_{\overline{n}|}}{\delta} = \boxed{\frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}}$$

□

Corollary 3.37 (Continuously Decreasing Annuities). *Consider a continuous annuity with a rate of payment that starts at n and linearly decreases to 0 by time n. Then the rate of payment at time t is $n - t$. If the force of interest is δ, the present value and accumulated value are given by*

$$(\overline{D}\overline{a})_{\overline{n}|} = \frac{n - \overline{a}_n}{\delta} \qquad (\overline{D}\overline{s})_{\overline{n}|} = (\overline{D}\overline{a})_{\overline{n}|}(1+i)^n$$

The proof adapts a similar methodology to the previous corollary.

Corollary 3.38 (Continuously Increasing Perpetuities). *Consider a continuously increasing perpetuity with continuous payments. The rate of payment at time t is t. If the force of interest is δ, then the present value is given by*

$$(\overline{I}\overline{a})_{\overline{\infty}|} = \frac{1}{\delta^2}$$

This results from taking the limit of $(\overline{I}\overline{a})_{\overline{n}|}$ at infinity.

Example 3.39. Given $\delta = 0.05$, calculate

1. $(\overline{I}\overline{a})_{\overline{10}|}$

$$= \frac{\overline{a}_{\overline{n}|} - 10e^{-10(0.05)}}{0.05} = \frac{\frac{1-e^{-0.5}}{0.05} - 10e^{-0.5}}{0.05} \approx \boxed{36.08}$$

2. $(\overline{D}\overline{a})_{\overline{10}|}$

$$= \frac{10 - \overline{a}_{\overline{n}|}}{0.05} = \frac{10 - \frac{1-e^{-0.5}}{0.05}}{0.05} \approx \boxed{42.61}$$

3. $(\overline{I}\overline{a})_{\overline{\infty}|}$

$$= \frac{1}{(0.05)^2} = \boxed{400}$$

Example 3.40. Payments are made to an account at a continuous rate of $(8k + tk)$, where $0 \leq t \leq 10$. Interest is credited at a force of interest of $\delta_t = \frac{1}{8+t}$. After 10 years, the account is worth \$20,000. Calculate k.

We know that $\delta_t = \frac{a'(t)}{a(t)}$, so $a(t) = 1 + \frac{1}{8}t$. We are tasked to find k such that the accumulated value after 10 years is \$20,000:

$$20000 = \int_0^{10} (8k + tk)a(t, 10)dt = \int_0^{10} (8k + tk)\frac{a(10)}{a(t)}$$

$$20000 = \int_0^{10} k(8+t)\left(\frac{\frac{18}{8}}{1+\frac{1}{8}t}\right) dt = \int_0^{10} k(8+t)\left(\frac{18}{8+t}\right)$$

$$20000 = \int_0^{10} 18k dt = [18kt]_0^{10} = 180k$$

$$\boxed{k \approx 111.11}$$

3.5 Annuity Tricks

Below are some tricks that may simplify problems with annuities:

Corollary 3.41 (Double Dots Cancel). *If you have an annuity-due on both sides on both sides, then you can replace those annuity-dues with annuity-immediates*

$$X\ddot{a}_{\overline{n}|} = Y\ddot{a}_{\overline{m}|} \implies Xa_{\overline{n}|} = Ya_{\overline{m}|}$$

This also applies for the ratio of two annuities. For example,

$$\frac{\ddot{a}_{\overline{20}|}}{\ddot{s}_{\overline{5}|}} = \frac{a_{\overline{20}|}}{s_{\overline{5}|}}$$

Corollary 3.42 (Upper (m)'s Cancel). *If you have m-thly annuities on both sides of an equation, then you can replace those annuities with annual annuities:*

$$Xa^{(m)}_{\overline{p}|} = Ys^{(m)}_{\overline{q}|}$$

This also applies for the ratio of two annuities. For example,

$$\frac{\ddot{a}^{(12)}_{\overline{20}|}}{\ddot{s}^{(12)}_{\overline{5}|}} = \frac{a_{\overline{20}|}}{s_{\overline{5}|}}$$

Corollary 3.43 ($a_{\overline{2n}|}/a_{\overline{n}|}$ Trick).

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$$

Proof.

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = \frac{\frac{1-v^{2n}}{i}}{\frac{1-v^n}{i}} = \frac{1-v^{2n}}{1-v^n} \cdot \frac{1+v^n}{1+v^n} = \frac{(1-v^{2n})(1+v^n)}{1-v^{2n}} = \boxed{1+v^n}$$

□

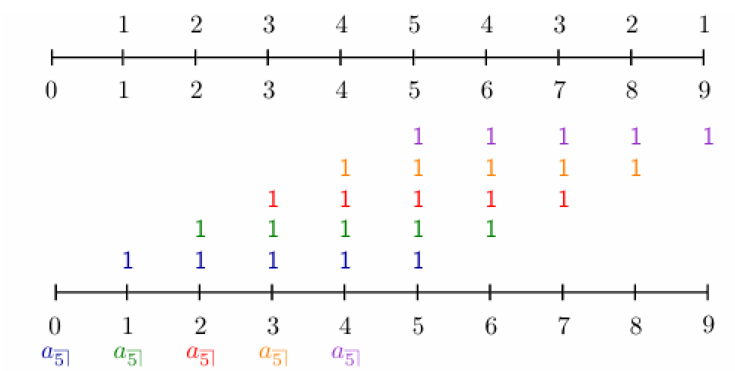
Example 3.44. Given $\ddot{a}_{\overline{20}|} = 13.0853$ and $\ddot{a}_{\overline{10}|} = 8.1078$, find i.

Apply the tricks from Cor 3.41 and 3.43:

$$\frac{\ddot{a}_{\overline{20}|}}{\ddot{a}_{\overline{10}|}} = \frac{a_{\overline{20}|}}{a_{\overline{10}|}} = 1 + v^{10}$$

$$\frac{13.0853}{8.1078} - 1 = (1+i)^{-10} \quad \Longrightarrow \quad \boxed{i = 0.05}$$

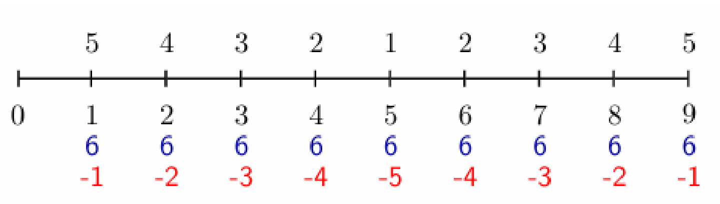
Pyramid Annuities: Consider the following annuity:



Then, the present value is PV $= a_{\overline{5}|}\ddot{a}_{\overline{5}|}$. More generally,

$$\boxed{\text{PV}_{\text{pyramid-imm}} = a_{\overline{n}|}\ddot{a}_{\overline{n}|} \qquad \text{PV}_{\text{pyramid-due}} = \ddot{a}_{\overline{n}|}\ddot{a}_{\overline{n}|}}$$

For an "upside pyramid,"



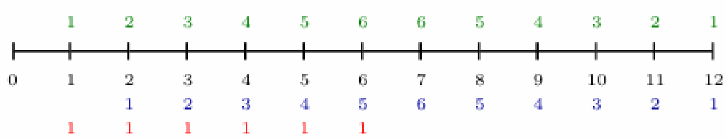
$$\text{PV} = 6a_{\overline{9}|} - a_{\overline{5}|}\ddot{a}_{\overline{5}|}$$

Example 3.45. An 11-year annuity has a series of payments $\{1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1\}$ with the first payment at the end of the second year. The present value of this annuity is \$25 at an interest rate i.

A 12-year annuity has a series of payments $\{1, 2, 3, 4, 5, 6, 6, 5, 4, 3, 2, 1\}$ with the first payment made at the end of the first year.

Calculate the present value of the 12-year annuity at interest rate i.

The series of payments should look like this:



We are given the present value of the 11-year annuity is \$25. Using the formula for annuity-immediate pyramids,

$$a_{\overline{6}|}\ddot{a}_{\overline{6}|}v = a_{\overline{6}|}a_{\overline{6}|} = 25 \quad \Longrightarrow \quad a_{\overline{6}|} = 5$$

The present value of the 12-year annuity is the present value of the 11-year annuity plus the present value of the missing cash flows in red ($a_{\overline{6}|}$). There is no discount for the missing cash flows because they occur at the start.

$$\text{PV}(\text{11-year annuity}) + a_{\overline{6}|} = 25 + 5 = \boxed{\$30}$$

4 Yield Rates and Amortization

This section will serve as a basis for understanding securities and term structures; it is first important to understand how returns and payment schedules operate.

We previously discussed the **net present value** and **internal rate of return** in Section 1.6. To align the definitions for future sections, we will redefine each of the relevant terms:

Net Present Value: For an n-period project and a series of cash flows (CFs) at times $0 \leq t \leq n$, the net present value is

$$\text{NPV} = \sum_{i=0}^{n} \frac{(\text{net CF})_t}{(1+r)^t}$$

where r is the required return, cost of capital, opportunity cost of capital or benchmark interest rate.

If NPV > 0, then the company should do the project. If choosing between projects, do the project with the greatest NPV.

Internal Rate of Return (IRR): The rate such that the present value of the cash inflows is equal to the present value of the cash outflows. In other words, the rate such that the NPV is 0.

If IRR $>$ r, then do the project. If choosing between projects, do the project with the greatest IRR.

Below is a quick example to familiarize ourselves:

Example 4.1. A five-year investment project requires an initial investment of $10,000 at inception and maintenance expenses at the beginning of each year. The maintenance expense is $300 for the first year and is anticipated to increase 6% each year thereafter. Projected annual returns from the project are $3,000 at the end of the first year and decrease by $100 per year thereafter.

1. If the cost of capital is 7%, calculate NPV of the project.

2. Compute the internal rate of return for the project.

The set of inflows and outflows are $(0, 3000, 2900, 2800, 2700, 2600)$ and $(10300, 300(1.06), 300(1.06)^2, 300(1.06)^3, 300(1.06)^4, 0)$, respectively. This makes the net cash flow

$$x = (-10300, 2682, 2562.92, 2442.70, 2321.26, 2600)$$

We can compute the NPV and IRR using a graphing calculator:

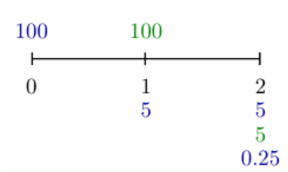
$$\text{NPV} = \sum_{i=0}^{5} \frac{x_i}{(1.07)^i} \approx \boxed{\$63.71} \qquad \boxed{\text{IRR} \approx 0.07234 \text{ or } 7.234\%}$$

4.1 Reinvestment Rates

Unless stated otherwise, we assume interest earned is automatically reinvested at the same rate. This is what we have done so far

Example 4.2. \$100 is invested in a fund at time 0 and time 1. If the fund earns an annual effective interest rate of 5% what is the balance at time 2?

The simple solution is $100\ddot{s}_{\overline{2}|0.05} = \$215.25$. If we separated the deposits from the interest earned, then



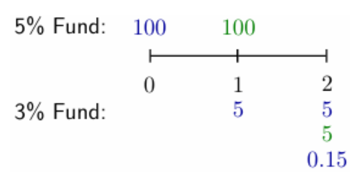
The first deposit at time 0 earns interest of \$5 from time 0 to 1 and from time 1 to 2. Additionally, the interest earned at time 1 gets applied again (interest on interest), resulting in a new interest of 0.25. The second deposit at time 1 earns interest of \$5 from time 1 to 2.

Thus, we can think of this as the sum of deposits and interests separately:

$$100 + 100 + 5 + 5 + 5 + 0.25 = 100(2) + 5(Is)_{\overline{2}|0.05} = 215.25$$

Sometimes an investment does not allow automatic investment.

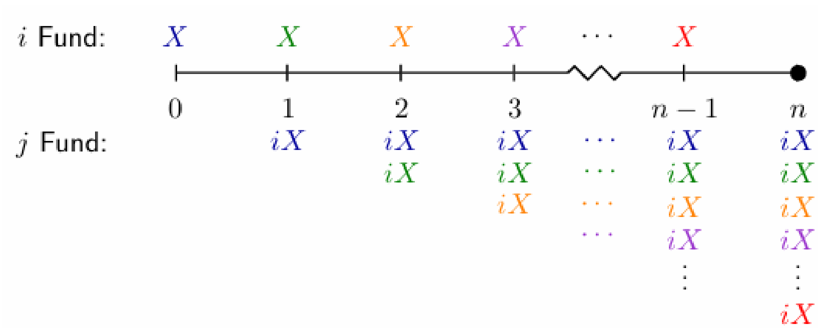
Example 4.3. \$100 is invested in a fund at time 0 and time 1. The fund earns interest at an annual effective interest rate of 5% payable annually. Interest payments are reinvested in a separate fund that earns an annual effective rate of 3%. Calculate the total balance of the funds at time 2.



The interest earned from the deposits is still 5%. However, we now apply a 3% interest to the acquired interest. So, instead of the interest on interest being 0.25, it is now $0.03(5) = 0.15$.

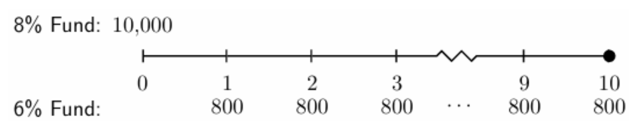
$$100(2) + 5(Is)_{\overline{2}|0.03} = 215.15$$

More generally, suppose X is invested at the beginning of each year for n years in a fund earning an annual effective rate of i payable annually. Interest payments are reinvested in a fund an annual effective rate of j. What is the total balance of the two funds at time n?



$$\boxed{\text{total balance} = \underbrace{nX}_{i\text{ fund}} + \underbrace{(iX)(Is)_{\overline{n}|j}}_{j\text{ fund}}}$$

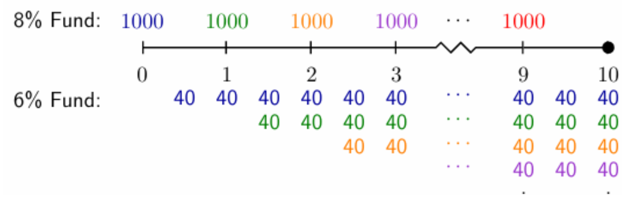
Example 4.4. Phil invests \$10,000 now into a fund earning an annual effective rate of 8% payable annually. The interest payments are reinvested in a separate fund earning an annual effective rate of 6%. Determine the total value of Phil's investment ten years from now.



The value of the 8% fund is just \$10,000. The interest earned is therefore \$800 each year. Now, a 6% interest is applied to this new fund. The accumulated value is

$$\text{AV} = 10000 + 800s_{\overline{10}|0.06} = 10000 + 800\left(\frac{(1.06)^{10} - 1}{0.06}\right) \approx \boxed{\$20{,}544.64}$$

Example 4.5. Tiger invests \$1000 at the beginning of each year into a fund earning a nominal annual rate of 8% convertible semiannually payable semiannually. The interest payments are reinvested in a separated fund earning an annual effective rate of 6%. Determine the total value of Tiger's investment ten years from now.



Let $j = (1.06)^{0.5} - 1$ be the effective rate of interest semiannually for the 6% fund. The total value of the 8% fund is $10(1000) = 10000$. The total value of the 6% reinvestment is the accumulated value over the semiannual payments of \$40 times the increasing accumulated value per year:

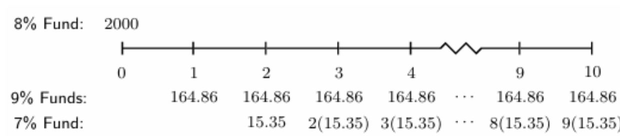
$$\text{AV} = 10000 + 40 s_{\overline{2}|j} (Is)_{\overline{10}|0.06} \approx \boxed{\$15{,}373.80}$$

Example 4.6. At time $t = 0$, Rory invests \$2000 in a fund earning 8% convertible quarterly, but payable annually. He reinvests each interest payment in individual separate funds each earning 9% convertible quarterly, but payable annually.

The interest payments from the separate funds are accumulated in a side fund that earns an annual effective rate of 7%. Determine the total value of all funds at $t = 10$.

First, we need to convert the convertible rates to annual ones:

- 8% convertible quarterly: $\left(1 + \dfrac{0.08}{4}\right)^4 - 1 = 0.08243$.
- 9% convertible quarterly: $\left(1 + \dfrac{0.09}{4}\right)^4 - 1 = 0.09308$.



The total value of the 8% fund is \$2000. The total value of the 9% fund is $2000(0.08243)(10) = 1648.60$ per year. The side fund receives deposits of $2000(0.08243)(0.09308) = 15.35$ and increases by this value each year (starting year 2). Therefore, the side fund has 9 annual payments and not 10.

$$\text{AV} = 2000 + 1648.60 + 15.35 (Is)_{\overline{9}|0.07} \approx \boxed{\$4485.49}$$

4.2 Yield Rates

The **yield rate** y is another term for internal rate of return. These terms can be used interchangeably.

Unless otherwise stated we assume cash flows are reinvested at the yield rate. Thus, the same rules apply:

$$\sum_{i=0}^{n} \frac{(\text{cash inflow})_t}{(1+y)^t} = \sum_{i=0}^{n} \frac{(\text{cash outflow})_t}{(1+y)^t}$$

If we are told cash inflows are reinvested at a rate other than the yield rate (almost always the case), then we use the formula

what I have = what I demand

where "what I have" are the cash inflows accumulated at the reinvestment rate and "what I demand" is the accumulated outflows at the desired yield rate.

WARNING: The reinvestment rate is not always equal to the yield rate!

Example 4.7. Bill agrees to loan Sally \$10,000 in exchange for payments of \$1,295 at the end of each year for 10 years. What is Bill's yield rate?

$$10000 = 1295a_{\overline{10}|y} \quad \Longrightarrow \quad y = 5\%$$

This is an example in which the reinvestment rate matches the yield rate.

Now, suppose Bill agrees to loan Sally \$10,000 in exchange for payments of \$1,295 at the end of each year for 10 years. Bill can invest Sally's payments at an annual effective rate of 3%. Determine Bill's annual effective yield rate over the 10 years.

In this case, what I have is annual payments of \$1295 accumulating over 10 years. What I demand is \$10,000 compounded at a rate (annually for 10 years) such that it matches the accumulated payments. In actuarial notation,

$$1295s_{\overline{10}|0.03} = 10000(1+y)^{10} \quad \Longrightarrow \quad y \approx 4.03\%$$

This is an example in which the reinvestment rate does not match the yield rate.

Example 4.8. Sally lends \$10,000 to Tim. Tim agrees to pay back the loan over 5 years with monthly payment payable at the end of each month.

Sally can reinvest the monthly payments from Tim in a savings account paying interest at 6%, compounded monthly. The yield rate earned on Sally's investment over the five-year period turned out to be 7.45% compounded semiannually.

What nominal rate of interest, compounded monthly, did Sally charge Tim on the loan?

The interest that Sally reinvests in is $\frac{0.06}{12} = 0.005$ per month. What I have is the accumulated value of 60 monthly payments of X at the rate of 0.5%. What I demand is the present value is the accumulated value of \$10,000 at the yield rate of $\frac{0.0745}{2}$ semiannually:

$$Xs_{\overline{60}|0.005} = 10000\left(1 + \frac{0.0745}{2}\right)^{10}$$

$$X\left(\frac{(1.005)^{60} - 1}{0.005}\right) = 14415.66 \qquad \Longrightarrow \qquad X \approx \$206.62$$

Lastly, we want to find the loan interest rate.

$$10000 = 206.62a_{\overline{60}|j} \qquad \Longrightarrow \qquad j \approx 0.007333766$$

$$12j \approx \boxed{0.088}$$

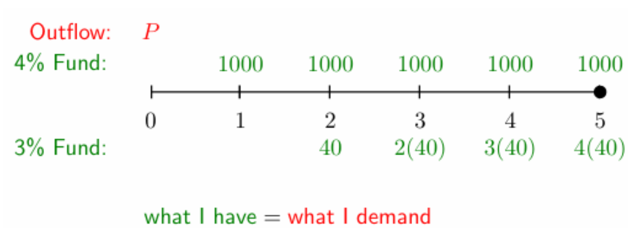
Example 4.9. An investor purchased a 5-year financial investment:

1. The investor receives payments of \$1000 at the end of each year for 5 years.
2. These payments earn interest at an effective rate of 4% per year. At the end of the year, this interest is reinvested at the rate of 3% per year.

Calculate the purchased price to the investor to produce a yield rate of 4%.

This time, we are tasked to compute the initial purchasing price.

What I have is 5 payments of \$1000 plus the annual interest earned (which becomes an increasing annuity of \$40 starting year 2). What I demand is the price of the instrument.



$$5(1000) + 40(Is)_{\overline{4}|0.03} = P(1.04)^5$$

$$5000 + 40\left(\frac{\frac{(1.03)^4 - 1}{\frac{0.03}{1.03}} - 4}{0.03}\right) = P(1.04)^5 \qquad \Longrightarrow \qquad \boxed{P \approx \$4448.42}$$

4.3 Amortization Schedules

Amortization Method: Each loan payment pays interest first; anything left reduces the loan balance.
Suppose a loan at an annual effective interest rate of 10% is repaid with three annual payments of \$500. Calculate the loan amount L.

The loan payment is simply the present value of the annuity after 3 years:

$$L = 500a_{\overline{3}|0.1} \approx 1243.43$$

The interest due after one year is $1243.43(0.1) = 124.34$. If the borrower paid \$124.34 at time 1, then the loan balance would still be \$1243.34. However, he pays \$500 so the extra payment reduces the loan balance. The new loan balance after the first payment is

$$1243.43 - (500 - 124.34) = 1243.43 - 375.66 = 867.77$$

The extra paid, \$375.66, is called the **principal payment**.

Let's organize this information into a table:

Time	Loan Payment	Interest Paid	Principal Paid	Loan Balance
0				1243.43
1	500	124.34	375.66	867.77
2	500			
3	500			

At time 2, the interest paid is $0.1(867.77) = 86.78$, making the principal paid $500 - 86.78 = 413.22$, and the loan balance $867.77 - 413.22 = 454.55$.

At time 3, the interest paid is $0.1(454.55) = 45.46$, making the principal paid $500 - 45.46 = 454.54$, and the loan balance $= 0.01$.

Time	Loan Payment	Interest Paid	Principal Paid	Loan Balance
0				1243.43
1	500	124.34	375.66	867.77
2	500	86.78	413.22	454.55
3	500	45.46	454.54	0.01
Total	1500	256.58	1243.42	

This table is called the **amortization schedule**.

- The total principal equals the original loan balance.
- The total interest paid equals total loan payments minus loan amount.

We built this table using the **retrospective method**. In other words, we looked backwards to find the loan balances.

We can also look forward to find the loan balances, which is called the **prospective method**.

$$500a_{\overline{2}|} = 867.77 \qquad 500v = 454.44$$

It will be easier if we have notation for these items:

- t: time (in years unless otherwise stated)
- R_t: payment amount at time t
- I_t: amount of interest paid in R_t
- P_t: amount of principal paid in R_t
- OB_t: outstanding loan balance **after** payment at time t
- L: original loan amount

The relevant formulas we have already used are

1. $L = \sum_{\text{all } t} R_t v^t$
2. $I_t = OB_{t-1} \cdot i$
3. $P_t = R_t - I_t$
4. $OB_t = OB_{t-1} - P_t$

...in that order.

Example 4.10. A loan is repaid with 20 annual payments of $1000 with the first payment one year from now. The interest rate is 6%. Find the loan balance after the 15th year using both the prospective and retrospective method.

Using the prospective method,

$$OB_{15} = 1000a_{\overline{5}|} = \boxed{\$4212.36}$$

Using the retrospective method,

$$L = 1000a_{20} = 11469.22$$

$$OB_{15} = 11469.22(1.06)^{15} - 1000s_{\overline{15}|} = \boxed{\$4212.36}$$

Example 4.11. A loan of $4,724.80 is repaid in annual payments of $1000, $1500, and $3000 with the first payment 1 year from now. The interest rate is 5% for year 1, 10% for year 2, and 5% for year 3. Build the amortization schedule for this loan.

We have $I_1 = 4724.8(0.05) = 236.24$, $P_1 = 1000 - 236.24 = 763.76$, and $OB_1 = 4724.8 - 736.76 = 3961.04$.

For year 2, $I_2 = 3961.04(0.1) = 396.10$, $P_2 = 1500 - 396.1 = 1103.9$, and $OB_2 = 3961.04 - 1103.9 = 2857.14$.

For year 3, $I_3 = 2857.14(0.05) = 142.86$, $P_3 = 3000 - 142.86 = 2857.14$, and $OB_3 = 0$.

This makes the following amortization schedule:

Time	Loan Payment	Interest Paid	Principal Paid	Loan Balance
0				4,724.80
1	1000	236.24	736.76	3961.04
2	1500	396.10	1103.90	2857.14
3	3000	142.86	2857.14	0.00

Example 4.12. On December 31, 1984, Smith borrowed $5,000 to be repaid in four years with level payments at the end of every quarter. The first payment was made on March 31, 1985. The effective annual interest rate was 4%. What was the amount of interest paid in 1986?

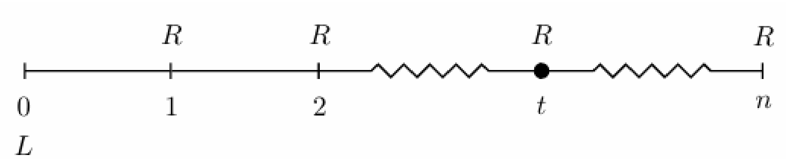
The first four payments (1-4) are in 1985. The next four payments (5-8) are in 1986. The effective quarterly rate of interest is $j = (1.04)^{1/4} - 1 \approx 0.0098534$. Let X be the loan payments. Then

$$Xa_{\overline{16}|j} = 5000$$

$$X\left(\frac{1-(1.0098534)^{-16}}{0.0098534}\right) = 5000 \qquad \Longrightarrow \qquad X \approx \$339.31$$

Using the AMORT worksheet, setting $P_1 = 5$ and $P_2 = 8$ gives us the interest of $\boxed{\$132.71}$ paid in 1986 (months 5 - 8).

The first example in this section described a **level payment**, where loan payments are constant throughout the amortization period.



For level payments, the following hold:

- $L = Ra_{\overline{n}|}$
- $OB_t = Ra_{\overline{n-t}|}$ (prospective) or $OB_t = L(1+i)^t - Rs_{\overline{t}|}$ (retrospective)
- $I_t = R\left(1 - v^{n-t+1}\right)$
- $P_t = Rv^{n-t+1}$

Example 4.13. A loan of \$100,000 is to be repaid in 30 equal payments starting one year from now. If the annual effective rate of interest is 5%, calculate the following:

$$100000 = Ra_{\overline{30}|0.05} \qquad \Longrightarrow \qquad R = 6505.14$$

1. OB_9

$$OB_9 = 6505.14a_{\overline{21}|} \approx \$83{,}403.39$$

2. P_9

$$P_9 = 6505.14v^{22} = 6505.14(1.05)^{-22} \approx \$2223.78$$

3. I_9

$$I_9 = R - P_9 = 6505.14 - 2223.78 \approx \$4281.36$$

4. The first payment in which the principal payment exceeds the interest payment

$$P_t > I_t \qquad \Longrightarrow \qquad Rv^{n-t+1} > R - Rv^{n-t+1}$$

$$2v^{n-t+1} > 1 \qquad \Longrightarrow \qquad v^{31-t} > \frac{1}{2}$$

$$31 - t < \frac{\ln(1/2)}{\ln(v)} \qquad \Longrightarrow \qquad t > 16.793$$

or the 17th payment.

Recall that principal payments are compounded. It follows that for some $0 \leq k \leq n$,

$$\boxed{P_{t+k} = P_t(1+i)^k}$$

We can write the original loan amount in terms of the first principal payment

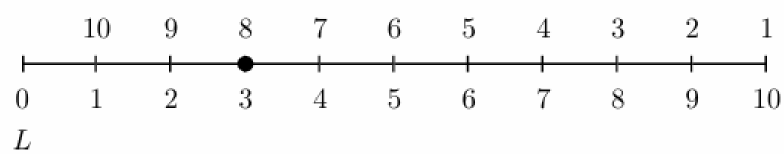
$$\boxed{L = P_1 s_{\overline{n}|}}$$

Example 4.14. For a loan with level payments of \$1018.52, the principal repaid in the 5th payment is \$297.30 and the interest paid in the 16th payment is \$325.33. Find the loan interest rate.

$$P_{16} = P_{16}$$
$$297.3(1+i)^{11} = R - I_{16}$$
$$297.3(1+i)^{11} = 1018.52 - 325.33$$
$$\boxed{i = 0.08}$$

For non-level payments we must rely on first principles.

Example 4.15. A loan has payments of \$10, \$9, . . ., \$1 starting one year from now. The loan interest rate is 5%. Determine the following:



1. OB_3
$$OB_3 = (Da)_{\overline{7}|} = \frac{7 - a_{\overline{7}|}}{0.05} \approx \boxed{24.27}$$

2. I_4
$$I_4 = 0.05(OB_3) = \boxed{1.21}$$

3. P_4
$$P_4 = R_4 - I_4 = 7 - 1.21 = \boxed{5.79}$$

Example 4.16. A loan is repaid with 10 annual payments of \$1295.05 with the first payment one year after the loan is issued. The loan rate is an annual effective rate of 5%. The interest portion of the k-th payment is \$328.67. Determine k.

$$I_k = 328.67 \quad\Longrightarrow\quad P_k = 1295.05 - 328.67 = 966.38$$
$$P_k = Rv^{n-k+1} \quad\Longrightarrow\quad 966.38 = 1295.05v^{11-k}$$
$$\boxed{k = 5}$$

Example 4.17. Joe negotiates an 8-year loan which requires him to pay \$1200 per month for the first 4 years and \$1500 per month for the remaining years. The interest rate is 13%, convertible monthly, and the first payment is due in one month. Determine the amount of principal in the 17th payment.

The effective monthly rate is $\frac{0.13}{12} \approx 0.01083$. We first want to compute OB_{16}. We can compute the present value over 80 months, but then we also must account for the difference of \$300 payments between months 17 and 48.

$$OB_{16} = 1500a_{\overline{96-16}|0.01083} - 300a_{\overline{48-16}|0.01083} = 71{,}911.70$$

$$I_{17} = (0.01083)(71911.7) = 779.04$$

$$P_{17} = 1200 - 779.04 = \boxed{\$420.96}$$

Example 4.18. Larry is repaying a loan with payments of \$2500 at the end of every two years. If the amount of interest in the fourth installment is \$2458, find the amount of principal in the seventh installment. Assume an annual effective interest rate of 13%.

The effective rate of interest per 2 years is $(1.13)^2 - 1 \approx 0.2769$

$$P_4 = 2500 - 2458 = 42$$

$$P_7 = P_4(1.2769)^3 \approx \boxed{\$87.44}$$

4.4 Drop and Balloon Payments

Sometimes a loan with level payments has a smaller or larger final payment.

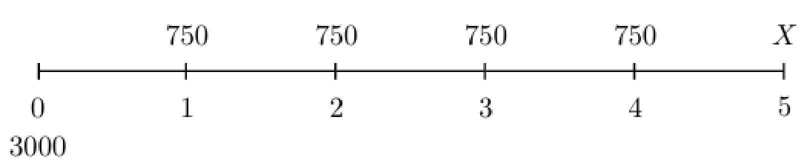
- If the last payment is smaller than the level payment it is called a **drop payment**.
- If the last payment is larger than the level payment it is called a **balloon payment**.

Example 4.19. A \$3000 loan at an annual effective rate of 6% is repaid with annual payments of \$750, plus a smaller final drop payment. Determine the amount of the drop payment.

We proceed by verifying the number of \$750 payments:

$$750a_{\overline{n}|0.06} = 3000 \implies n \approx 4.71.$$

There will be 4 payments of \$750 and a drop payment (say X)



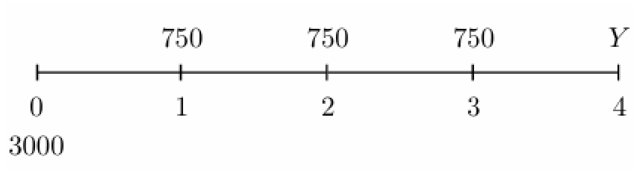
We can apply general properties of annuities to reach the solution

$$3000 = 750a_{\overline{4}|0.06} + Xv_{0.06}^5 \quad \Longrightarrow \quad \boxed{X \approx \$536.86}$$

...or we can use what we know about loans. We take the outstanding balance after 4 years, then multiply by the growth factor to get X:

$$OB_4 = 3000(1.06)^4 - 750s_{\overline{4}|} = 506.47 \quad \Longrightarrow \quad X = 1.06(506.47) \approx \boxed{\$536.86}$$

Example 4.20. A \$3000 loan at an annual effective rate of interest is 6% is repaid with 3 annual payments of \$750, plus a balloon payment at the end of year 4. Determine the amount of the balloon payment.



We can apply general properties of annuities to reach the solution

$$3000 = 750a_{\overline{3}|0.06} + Yv_{0.06}^4 \quad \Longrightarrow \quad \boxed{Y \approx \$1256.47}$$

...or we can use what we know about loans. We take the outstanding balance after 4 years, then add a fourth payment of \$750.

$$OB_4 = 506.47 \quad \Longrightarrow \quad Y = 506.47 + 750 = \boxed{\$1256.47}$$

Example 4.21. A \$1000 loan at 3.5% annual effective rate of interest can be repaid by either:

1. 11 annual payments of \$100 plus a final balloon payment, or
2. 12 annual payments of \$100 plus a final drop payment

Determine the difference between the balloon and drop payment.

If \$100 is paid for 12 years, the outstanding balance is

$$1000(1.035)^{12} - 100s_{\overline{12}|} = 50.87$$

$$\begin{aligned} \text{balloon payment} &= 100 + 50.87 = 150.87 \\ \text{drop payment} &= 50.87(1.035) = 52.65 \\ \text{difference} &= 150.87 - 52.65 = \boxed{98.22} \end{aligned}$$

Now, compute the absolute difference between the total interest paid in the two repayment options.

$$\begin{aligned}\text{total interest} = \text{total loan payments } &- \text{ loan amount}\\ \text{for balloon payment } &: 1250.87 - 1000 = 250.87\\ \text{for drop payment } &: 1252.65 - 1000 = 252.65\\ \text{absolute difference } &= |250.87 - 252.65| = \boxed{1.78}\end{aligned}$$

5 Bonds

In this section, we will be discussing a primary security type: bonds. Securities are tradable financial assets used for investors and/or corporations to generate capital. There are two ways for corporations to raise capital:

1. Issuing debt. Examples include loans from investors and bonds.
2. Issuing equity. Typically involves investment in a company or buying stock.

Bonds are issued by corporations and governments.

In summary, the bond issuer borrows from the bond-holder (investor) and makes regular interest payments, called **coupons**, and a redemption amount upon maturity.

Maturities are typically within 1 to 30 years, given different names based on their length:

- Maturity of one year or less is called a **bill**
- Maturity of 1-10 years is called a **note**
- Maturity of greater than 10 years is called a **bond**

Despite these technicalities, all problems will call them a bond regardless of its time to maturity.

Interest payments are tax deductibles for the corporation. In bankruptcy, bondholders are paid before stockholders.

There are multiple types of bonds:

- Fixed-rate: interest rate is fixed over the life of the bond
- Floating-rate: interest rate is allowed to fluctuate
- Zero-coupon: all interest is paid at maturity (no coupon payments)
- Callable: gives bond-issuer right to redeem bond before maturity
- Convertible: gives bondholder option to convert to fixed number of shares of common stock

We will find that bonds are facilitated very similarly to loans.

Before proceeding to bonds, we will look at one type of security found in the U.S. government: **Treasury bills** (or T-bills). These are short-term debts (less than one year) issued by the government. Face amounts are in round values (100, 1000, 10000, etc...).

There are no coupon payments, and T-bills are quoted on a simple discount yield basis (360 days per year)

$$\boxed{\text{Price} = \text{Face Amount}\left[1 - \text{ discount yield } \cdot \frac{n}{360}\right]}$$

Example 5.1. The quoted yield rate on a \$10,000 three-month T-bill is 2.05%. Determine the price.

$$10000\left[1 - 0.0205\left(\frac{90}{360}\right)\right] \approx \boxed{\$9948.75}$$

5.1 Price of a Bond

As we have discussed before, a bond is a type of loan. They are issued by governments and corporations to borrow money. The bond issuer is borrowing money from the bond investor.

Most bonds have systematic interest payments (called coupons) and a balloon payment (called redemption amount) on redemption of the bond.

Example 5.2. To fund a new road project, the City of New York issues a block of bonds. Each bond pays coupons of \$25 every six months for 10 years and 1000 at the end of ten years.

If the yield rate of the bond is 6% convertible semiannually, calculate the price of the bond.

$$25a_{\overline{20}|0.03} + 1000v_{0.03}^{20} \approx \boxed{\$925.61}$$

Notation for Bonds

- P: price of the bond
- F: face amount or par value. It is the unit in which the bond is issue (e.g. 100, 1000, 10000) and is used in determining the coupon payment
- r: coupon rates per payment period. They are typically payable semiannually. If given the annual nominal rate, divide by 2 for semiannual, 4 for quarterly, or 12 for monthly
- Fr: amount of each coupon payment. For instance, a 1000 par value bond with 6% semiannual coupons, then

$$Fr = 1000(0.03) = 30 \text{ every six months}$$

- C: redemption value of the bond. Unless otherwise stated, assume $C = F$

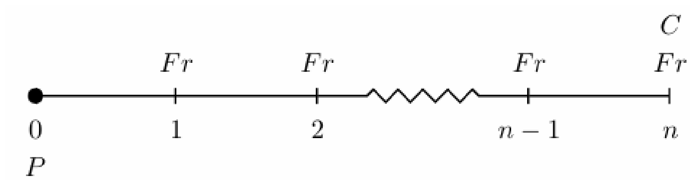
- g: modified coupon rate. It follows that

$$Fr = Cg \quad \Rightarrow \quad g = \frac{Fr}{C}$$

- i: yield rate of the bond. Also called yield-to-maturity. **Always expressed as effective rate per coupon payment period**
- n: number of coupon payments. For example, a 30-year bond with semiannual coupons implies 60 coupon payments
- K: present value of the redemption value

$$K = Cv_i^n$$

The time diagram for a bond is



The basic formula is

$$P = Fra_{\overline{n}|i} + Cv_i^n$$

Recall that $K = Cv_i^n$:

$$\boxed{P = Fra_{\overline{n}|} + K}$$

...which we will refer to as the **Frank formula**.

In the next section, we will discuss premiums and discounts. We will rearrange the basic formula to adapt ourselves for them.

Start with the basic formula:

$$P = Fra_{\overline{n}|i} + Cv_i^n$$

Recall that $a_{\overline{n}|} = \frac{1-v^n}{i} \quad \Rightarrow \quad v^n = 1 - ia_{\overline{n}|}$.

$$P = Fra_{\overline{n}|i} + C(1 - ia_{\overline{n}|i}) \quad \Longrightarrow \quad \boxed{P = C + (Fr - Ci)a_{\overline{n}|i}}$$

It can also be written as

$$P = C + (Cg - Ci)_{\overline{n}|i}$$

since $Fr = Cg$ by construction.

Makeham's Formula: Useful for "serial bonds," where the principal is repaid in installments over a series of schedules dates.

Start with the basic formula:

$$P = Fra_{\overline{n}|i} + Cv_i^n = Cg\left(\frac{1-v^n}{i}\right) + K$$
$$= g\left(\frac{C - Cv^n}{i}\right) + K = \boxed{K + \frac{g}{i}(C-K)}$$

Notice how there are no annuity factors in this formula.

Example 5.3. A \$1000 par value ten-year bond has 8.4% semiannual coupons. The bond is redeemable at \$1050. If the bond is bought to yield 10% convertible semiannually, calculate the price using all three bond formulas:

$$F = 1000 \quad r = \frac{0.084}{2} = 0.042 \quad n = 20 \quad Fr = 42$$
$$C = 1050 \quad i = 0.05 \quad Ci = 52.5 \quad g = \frac{42}{1050} = 0.04 \quad K = 1050(1.05)^{-20} = 395.73$$

1. Basic/Frank's formula:
$$P = 42a_{\overline{20}|0.05} + 395.73 = 919.15$$

2. Premium/Discount formula:
$$P = 1050 + (42 - 52.5)a_{\overline{20}|0.05} = 1050 - 10.5a_{\overline{20}|0.05} = 919.15$$

3. Makeham's formula:
$$P = 395.73 + \frac{0.04}{0.05}(1050 - 395.73) = 919.15$$

Example 5.4. A \$1000 par value twenty-year bond has coupons at 5% convertible semiannually. If the yield rate is 6% convertible semiannually, what is the bond's price?

$$P = 1000(0.025)a_{\overline{40}|0.03} + 1000(1.03)^{-40} \approx \boxed{\$884.43}$$

Example 5.5. A 9% bond with a \$1000 par value and coupons payable semiannually is redeemable at maturity for \$1100. At a purchase price P, the bond yields a nominal annual interest rate of 8%, compounded semiannually, and the present value of the redemption amount is \$190. Determine P.

We are given $r = 0.045$, $i = 0.04$, $F = 1000$, $C = 1100$, and $K = 190$. n is unknown, but we can use K to calculate n:

$$190 = 1100(1.04)^{-n} \qquad \Longrightarrow \qquad n \approx 44.77$$

Assuming \$190 was rounded, let $n = 45$, then

$$P = 45a_{\overline{45}|0.04} + 1100v_{0.04}^{45} \approx \boxed{\$1120.72}$$

Alternatively we could have used Makeham's formula:

$$P = 190 + \frac{\frac{45}{1000}}{0.04}(1100 - 190) \approx \boxed{\$1120.68}$$

5.2 Premium and Discount

If $P > C$, then a **premium** is paid for the bond. If $P < C$, then the bond is bought at a **discount**. Bonds bought at a premium aren't necessarily at a "bad deal" and bonds bought at a discount aren't necessarily a "good deal."

To motivate this, let's compare three bonds (A, B, C) all issued by the same corporation.

Bond A is a two-year \$1000 par value bond with 8% semiannual coupons. The bond is bought to yield 8% convertible semiannually.

Recall the premium/discount formula

$$P = C + (Fr - Ci)_{\overline{n}|i}$$

Think of $Ci = 1000(0.04)$ as the "required payment" for the bond price to equal the redemption amount. For this bond the coupon payment $= Fr = 1000(0.04) = 40$ is equal to the "required payment." Thus

$$P = 1000 + (40 - 40)a_{\overline{20}|0.04} = 1000$$

We say this bond is priced at par.

Bond B is a two-year \$1000 par value bond with 10% semiannual coupons. The bond is bought to yield 8% convertible semiannually.

The "required payment" is still $Ci = 1000(0.4)$, but the coupon payment $Fr = 1000(0.05)$ is larger than required, so the bond issuer should receive more money at issue.

Clearly this bond should be more expensive than Bond A because the bond issuer is paying a larger coupon but the yield rate is the same.

$$P = C + (Fr - Ci)_{\overline{n}|i} = 1000 + (50 - 40)a_{\overline{4}|0.04} = 1036.30$$

$P - C$ is called the premium. So for this bond the premium is \$36.30.

Bond C is a two-year \$1000 par value bond with 5% semiannual coupons. The bond is brought to yield 8% convertible semiannually.

The "required payment" is still $Ci = 1000(0.04) = 40$, but the coupon payment $Fr = 1000(0.025) = 25$ is smaller than required, so the bond issuer receives less money at issue.

Clearly this bond should be cheaper than Bond A because the bond issuer is paying a smaller coupon but the yield rate is the same.

$$P = 1000 + (25 - 40)a_{\overline{4}|0.04} = 1000 - 54.45 = 945.55$$

While we could say this bond has a premium of -\$54.45, we can instead say that the bond is bought at a discount of $C - P = 54.45$.

In short

Bond	Ci	Fr	Price	Premium / Discount
A	40	40	1000.00	priced at par
B	40	50	1036.30	premium = 36.30
C	40	25	945.55	discount = 54.45

Which is the "best buy?" There isn't one. All three bonds have the same yield-to-maturity. The investor would instead buy the bond with the coupon stream that matches their needs the best.

Formulas

If $Fr > Ci$, then

$$\boxed{P - C = (Fr - Ci)a_{\overline{n}|i} = \text{ premium}}$$

If $Ci > Fr$, then

$$\boxed{C - P = (Ci - Fr)a_{\overline{n}|i} = \text{ discount}}$$

We can also write the formula in terms of g

$$\boxed{P = C + (Cg - Ci)a_{\overline{n}|i}}$$

Thus if $g > i$, then bought at premium. If $g < i$, then bought at a discount.

Example 5.6. Determine if each statement is true or false.

1. If $r > i$, then the bond must be a premium bond.
2. If $g < i$, then the bond must be a discount bond.
3. If $Ki > Fr$, then the bond must be a discount bond.
4. If $i = r$ and bond is redeemable at par, then the bond's price is equal to the par amount.

Answers:

1. False. One must look at Fr vs Ci or g vs i, not just r vs i.
2. True.
3. True. $K = \text{PV}(c)$, thus $K < C$ and $Ki < Ci$. So if $Ki > Fr$, then $Ci > Fr$.
4. True.

Example 5.7. A fifteen-year \$1000 bond with quarterly coupon of \$25 is redeemable for \$1200. The bond is priced to yield 8% convertible quarterly. Determine the amount of premium or discount.

We have that coupon payments are $Fr = 25$ and the required payments are $Ci = 1200(0.02) = 24$. Since $Fr > Ci$, the bond is bought at a premium.

$$P - C = a_{\overline{60}|0.02} \approx \boxed{\$34.76}$$

Example 5.8. For a ten-year bond with semiannual coupons you are given: $C = 1200, r = 0.04, g = 0.03, i = 0.04$. Determine the bond premium or discount.

Since $g < i$, this is a discount bond.

$$\text{discount} = (Ci - Fr)_{\overline{n}|i} = (Ci - Cg)_{\overline{n}|i} = 1200(0.04 - 0.03)a_{\overline{20}|0.04} \approx \boxed{\$163.08}$$

5.3 Bond Amortization

Remember a bond is a type of loan. We amortize bonds just like we amortized loans. The only difference is the terminology.

- Instead of *outstanding balance* for bonds we say *book value*.
- Instead of *loan balance* for bonds we say *coupon payment*.

- Instead of *principal payment* for bonds we say *write-down* or *write-up* depending on if it is a premium or discount bond.

For a bond with a premium we amortize $P - C$.

The book value of the bond will start at P and we will "write down" the book value with each coupon payment. After the final coupon payment the book value will equal the redemption amount C.

For example, what if we wanted to create an amortization schedule for a two-year $1000 par value bond with 10% semiannual coupons. The bond is bought to yield 8% convertible semiannually.

$$P = 50a_{\overline{4}|0.04} + 1000v_{0.04}^4 = 1036.60$$

Half Year	Coupon	Interest	Write-Down	Book Value
0				1036.30
1	50			
2	50			
3	50			
4	50			

Half-Year 1

$$I_1 = 1036.30(0.04) = 41.45, P_1 = 50 - 41.45 = 8.55, BV_1 = 1036.30 - 8.55 = 1027.75$$

Half-Year 2

$$I_2 = 1027.75(0.04) = 41.11, P_2 = 50 - 41.11 = 8.89, BV_2 = 1027.75 - 8.89 = 1018.86$$

Half-Year 3

$$I_3 = 1018.86(0.04) = 40.75, P_3 = 50 - 40.75 = 9.25, BV_3 = 1018.86 - 9.25 - 1009.61$$

Half-Year 4

$$I_4 = 1009.61(0.04) = 40.39, P_4 = 50 - 40.39 = 9.61, BV_4 = 1009.61 - 9.61 = 1000$$

We can now complete the table:

Half Year	Coupon	Interest	Write-Down	Book Value
0				1036.30
1	50	41.45	8.55	1027.75
2	50	41.11	8.89	1018.86
3	50	40.75	9.25	1009.61
4	50	40.39	9.61	1000

The total write-downs equal the premium.

Just like the OB on a loan is the present value of future loan payments, book value is the present value of the future bond payments.

Bond Amortization Formulas

The bond amortization formulas are synonymous to those of loans; we just need to translate them in the language of bonds.

- Book Value:
$$\begin{aligned} BV_t &= Fra_{\overline{n-t}|i} + Cv_i^{n-t} \text{ (prospective)} \\ &= P(1+i)^t - Frs_{\overline{t}|i} \text{ (retrospective)} \\ &= BV_{t-1} - P_t \end{aligned}$$

- Interest:
$$I_t = BV_{t-1} \cdot i$$

- Write-up/Write-down:
$$\begin{aligned} P_t &= Fr - I_t \\ &= (Fr - Ci)v^{n-t+1} \end{aligned}$$

$P_t = (Fr - Ci)v^{n-t+1}$ should look familiar. Recall that for a loan with level payments of R, $P_t = Rv^{n-t+1}$.

Just like for loans, each principal payment grows by a factor of $1 + i$:

$$\boxed{P_{t+k} = P_t(1+i)^k}$$

Now, let's do a write-up amortization schedule. Suppose we want to create an amortization schedule for a two-year \$1000 par value bond with 8% semiannual coupons, redeemable at \$1,050. The bond is bought to yield 8% convertible semiannually.

$$P = 40a_{\overline{4}|0.04} + 1050(1.04)^{-4} \approx 1042.74$$

Half Year	Coupon	Interest	Write-Down	Book Value
0				1042.74
1	40			
2	40			
3	40			
4	40			

Half-Year 1

$$I_1 = 1042.74(0.04) = 41.71, P_1 = 40 - 41.71 = -1.71, BV_1 = 1042.74 + 1.71 = 1044.45$$

Half-Year 2

$$I_2 = 1044.45(0.04) = 41.78, P_2 = 40 - 41.78 = -1.78, BV_2 = 1044.45 + 1.78 = 1046.23$$

Half-Year 3

$$I_3 = 1046.23(0.04) = 41.85, P_3 = 40 - 41.85 = -1.85, BV_3 = 1046.23 + 1.85 = 1048.08$$

Half-Year 4

$$I_4 = 1048.08(0.04) = 41.92, P_4 = 40 - 41.92 = -1.92, BV_4 = 1048.08 + 1.92 = 1050$$

We can now complete the table:

Half Year	Coupon	Interest	Write-Down	Book Value
0				1042.74
1	40	41.71	1.71	1044.45
2	40	41.78	1.78	1046.23
3	40	41.85	1.85	1048.08
4	40	41.92	1.92	1050

The total write-ups equals the discount $(1050 - 1042.74 = 7.26)$.
The write-ups are in geometric progression (e.g. $1.71(1.04^2) = 1.85$)

Example 5.9. A \$1000 par value bond redeemable in 20 years at par with 6% annual coupons is purchased to produce an annual yield of 5%. The purchaser values the bond for accounting purposes at its amortized value. Calculate the amortized value of the bond at the end of the 5th year.

The amortized value is the present value of future payments:

$$BV_5 = 60a_{\overline{15}|0.05} + 1000(1.05)^{-15} \approx \boxed{\$1103.80}$$

Alternatively we can use the AMORT worksheet

$$P = 60a_{\overline{20}|0.05} + 1000(1.05)^{-20} = 1124.62$$

and set $P1 = P2 = 5$ to get the same answer.

Example 5.10. A ten-year bond bears semiannual coupons of $4 each and has a redemption value of $100. The bond is purchased to yield 10% compounded semiannually. Find the amount of increase in the book value at the time of the tenth coupon payment.

$$BV_9 = 4a_{\overline{11}|0.05} + 100v_{0.05}^{11} \approx 91.69$$

$$BV_10 = 4a_{\overline{10}|0.05} + 100v_{0.05}^{10} \approx 92.28$$

The increase is therefore $92.28 - 91.69 = 0.584$.

Two other ways to find the increase in book value is

$$P_{10} = \text{coupon} \ - \ \text{interest due} = 4 - 0.05(91.69) \approx -0.58$$

$$P_{10} = (Fr - Ci)v^{n-t+1} = -v_{0.05}^{11} \approx -0.58$$

Example 5.11. A bond with a par value of $1000 and 6% semiannual coupons is redeemable for $1100. You are given:

1. The bond is purchased at P to yield 8% convertible semiannually.
2. The amount of principal adjustment for the 16th semiannual period is 5.

Calculate P.

The required payment is $Ci = 1100(0.04) = 44$ and the coupon payment is $Fr = 1000(0.3) = 30$, thus making it a discount bond.

Next, we want to find the first principal adjustment:

$$P_{16} = -5 = P_1(1.04)^{15} \qquad \Longrightarrow \qquad P_1 \approx -2.7763$$

Use P_1 to find the price:

$$I_1 = P(0.04)$$

$$P_1 = 30 - P(0.04) \qquad \Longrightarrow \qquad \boxed{P = \$819.41}$$

5.4 Callable Bonds

The other type of bonds we'll discuss in this section is a callable bond.

A **callable bond** can be redeemed at the bond issuer's option before maturity.

- Call price: the price paid by bond issuer to redeem before maturity
- Call premium: Call price $-$ Redemption value
- Call protection period: period of time where the bond cannot be called

Call bonds are typically issued when interest rates fall. Then the issuer can call bonds and issue new ones at a lower yield rate.

Callable bonds offer higher yields to compensate for possible reinvestment risk.

Realized yield cannot be predicted because the yield will depend on when the bond is redeemed.

Below is an example of a premium bond with an unknown yield rate:

Example 5.12. Consider a ten-year \$100 par value 4% bond with annual coupons callable at par starting 5 years from now. If the price of the bond is \$104.58, calculate the yield if the bond is redeemed in 5 years, 6 years, and so on.

$$104.58 = 4a_{\overline{5}|i} + 100v_i^5 \quad \Longrightarrow \quad i = 0.03$$

$$104.58 = 4a_{\overline{6}|i} + 100v_i^6 \quad \Longrightarrow \quad i = 0.0315$$

Redemption	Yield
5	0.0300
6	0.0315
7	0.0326
8	0.0334
9	0.0340
10	0.0345

For a premium bond the worst case scenario for the bond investor is the earliest call date.

Below is an example of a discount bond with an unknown yield rate:

Example 5.13. Consider a ten-year \$100 par value 4% bond annual coupons callable at par starting 5 years from now. If the price of the bond is \$95.67, calculate the yield if the bond is redeemed in 5 years, 6 years, and so on.

$$95.67 = 4a_{\overline{5}|i} + 100v_i^5 \quad \Longrightarrow \quad i = 0.05$$

$$95.67 = 4a_{\overline{6}|i} + 100v_i^6 \quad \Longrightarrow \quad i = 0.0485$$

Redemption	Yield
5	0.0500
6	0.0485
7	0.0474
8	0.0466
9	0.0460
10	0.0455

For a discount bond the worst case scenario for the bond investor is the redemption date.

Calculating the Price

Because the yield rate depends on when the bond is redeemed we cannot calculate the price for an exact yield. Instead, we calculate the maximum price the investor can pay to earn a minimum yield. The algorithm is as follows:

1. Calculate the price of a bond for each possible redemption date at the desired yield rate.
2. Return the minimum of the prices found in step 1.

If the call premium is zero, then the Premium / Discount formula is very useful for calculating the price:

$$P = C + (Fr - Ci)a_{\overline{n}|i}$$

- If $Fr - Ci$ is negative, then the minimum price occurs at the latest redemption time.
- If $Fr - Ci$ is positive, then the minimum price occurs at the earliest redemption time.

Example 5.14. An investor bought a 15-year bond with par value of \$100,000 and 8% semiannual coupons. The bond is callable at par on any coupon date beginning with the 24th coupon. Find the highest price paid that will yield a rate not less than $i^{(2)} = 10\%$.

All the possible prices are

$$P = 100000 + (4000 - 5000)a_{\overline{n}|0.05} \quad n = 24, 25, \ldots, 30$$

The minimum occurs at $n = 30$ because $Fr - Ci < 0$.

$$P = 100000 + 1000a_{\overline{30}|0.05} = \boxed{\$84{,}627.55}$$

Why do we find the "minimum price" if we are looking for the "highest price?" We are looking for the "highest price paid that will **yield a rate not less than** $i^{(2)} = 10\%$.

Example 5.15. Refer to Example 5.14. Find the price assuming the bond is redeemed at the time of the 24th coupon.

$$100000 + (4000 - 5000)a_{\overline{24}|0.05} = \$86.201.36$$

This price is higher than \$84,627.55, but if the investor pays \$86,201.36 and the bond is redeemed at the time of the 30th coupon then the investors yield will be

$$86201.36 = 4000a_{\overline{30}|y} + 100000v_y^{30} \quad \Rightarrow \quad y = 4.886\%$$

And this yield rate is less than $i^{(2)} = 10\%$.

Example 5.16. An investor bought a 15-year bond with par value of \$100,000 and 8% semiannual coupons. The bond is callable at par on any coupon date beginning with the 24th coupon. Find the highest price paid that will yield a rate not less than $i^{(2)} = 6\%$.

All the possible prices are

$$P = 100000 + (4000 - 3000)a_{\overline{n}|0.03} \quad n = 24, 25, \ldots, 30$$

The minimum occurs at $n = 24$ because $Fr - Ci > 0$:

$$P = 100000 + 1000a_{\overline{24}|0.03} \approx \boxed{\$116,935.54}$$

Example 5.17. A \$1000 par value bond with coupons of 9% payable semiannually was called for \$1100 prior to maturity. The bond was bought for \$917.44 immediately after a coupon and was held to call. The nominal yield rate convertible semiannually was 10%. Calculate the number of years the bond was held.

Let n represent the number of half-years:

$$917.44 = 45a_{\overline{n}|0.05} + 1100v^n$$

$$917.44 = 900 - 900v^n + 1100v^n \quad \Longrightarrow \quad 17.44 = 200v^n$$
$$v^n = 0.0872 \quad \Longrightarrow \quad (1.05)^n = 114.679$$
$$n = 50 \quad \Longrightarrow \quad \boxed{25 \text{ years}}$$

Example 5.18. A \$1000 par value bond pays annual coupons of \$80. The bond is redeemable at par in 30 years, but is callable any time from the end of the 10th year at \$1,050. Based on her desired yield rate, an investor calculates the following potential purchase prices (P):

1. Assuming the bond is called at the end of the 10th year, $P = 957$.

2. Assuming the bond is held until maturity, $P = 897$.

The investor buys the bond at the highest price that guarantees she will receive at least her desired yield rate regardless of when the bond is called. The investor holds the bond for twenty years, after which time the bond is called. Calculate the annual yield rate the investor earns.

Since the investor held for more than ten years, $P = 897$. Step 2 of the algorithms asks us to find the minimum return price:

$$897 = 80a_{\overline{20}|i} + 1050v_i^{20} \quad \implies \quad \boxed{i \approx 0.0924}$$

6 Interest Rate Behavior

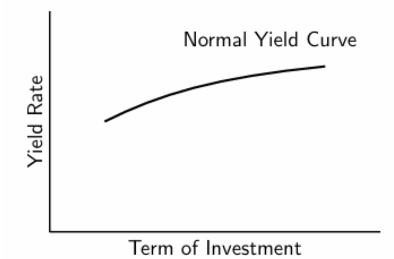
Yield rates typically vary by the term of the investment. For example, if you look at the yield rates for CDs at your bank you will likely see something like

Term	Yield Rate
12 months	0.45%
18 months	1.15%
24 months	2.00%

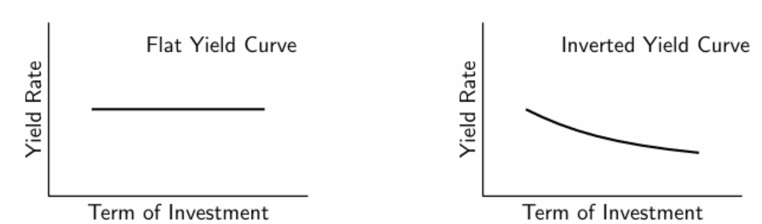
Term structure of interest rates: the phenomenon in which interest rates differ depending on the term of otherwise identical financial instruments.

Yield curves: Graphical representation of the term structure of interest rates.

Most of the time, the yield rates are larger for longer investment terms.

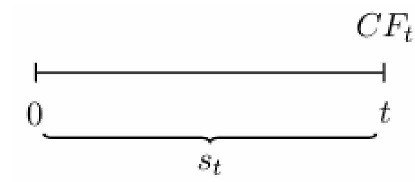


There are two other types of yield curves − flat and inverted. While these are not common, they have occurred before. Most problems will assume a flat yield curve.



6.1 Spot Rates

Definition 6.1 (Spot Rates). The t-year **spot rate**, s_t, is the yield rate for a zero-coupon bond with maturity in t years. s_t is the rate of interest between time 0 and t.

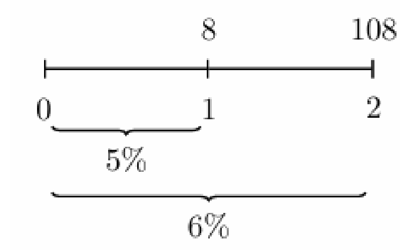


The present value of CF_t is

$$\boxed{\frac{CF_t}{(1+s_t)^t}}$$

Unless otherwise stated, assume that s_t is an annual effective rate of interest.

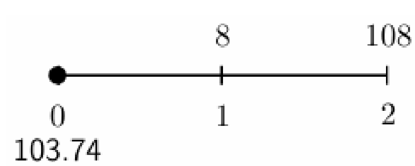
Example 6.2. Given $s_1 = 5\%$ and $s_2 = 6\%$, calculate the price of a two-year bond with \$100 par value and 8% annual coupons.



The price is the present value of the cash flows:

$$P = \frac{8}{1.05} + \frac{108}{(1.06)^2} \approx \boxed{\$103.74}$$

If the bond investor holds the bond to maturity, calculate his yield rate.



$$103.74 = 8a_{\overline{2}|i} + 100v_i^2 \qquad \Longrightarrow \qquad \boxed{i = 0.0596}$$

Example 6.3. Given the following \$100 face amount bonds with annual coupons:

Bond	Maturity	Coupon Rate	Price
A	1	10%	101.85
B	2	10%	102.66

Determine s_2.

Find s_1 using bond A

$$101.85 = \frac{110}{(1+s_1)} \quad \Longrightarrow \quad s_1 = 0.08$$

Find s_2 using bond B

$$102.66 = \frac{10}{1.08} + \frac{110}{(1+s_2)^2} \quad \Longrightarrow \quad \boxed{s_2 = 0.08523}$$

This iterative process is called **bootstrapping**.

Percentage of Price Method

If you know the present value of cash flow A at time t, then you can quickly find the present value of a cash flow B at time t:

$$\boxed{\mathrm{PV}(B) = \mathrm{PV}(A)\left(\frac{B}{A}\right)}$$

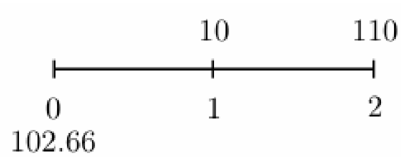
Suppose you are given the price of a 5-year zero coupon bond redeemable at \$100 is \$95.25. What is the present value of \$10 in five years?

The cash flow we want the present value of is $\frac{10}{100} = 10\%$ of the zero coupon bond cash flow, thus its present value is 10% of the bond's price.

$$95.25\left(\frac{10}{100}\right) = 9.525$$

Example 6.4. Rework Example 6.3 using the percentage of price method.

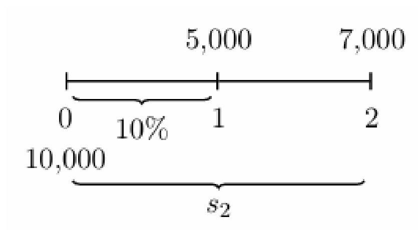
Bond A is the same as a zero-coupon bond with a single cash flow of \$110 at time 1.



$$102.66 = 101.85\left(\frac{10}{110}\right) + \frac{110}{(1+s_2)^2} \quad\Longrightarrow\quad \boxed{s_2 = 0.08523}$$

Example 6.5. Company XYZ invests \$10,000 into a project that expects to return \$5,000 one year after the investment and \$7,000 two years after the investment.

For XYZ's investment the one-year spot rate is 10%, calculate the two-year spot rate.

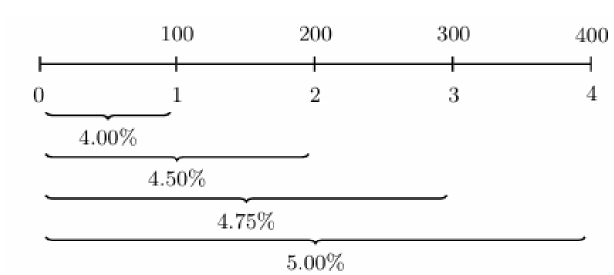


$$10000 = \frac{5000}{1.1} + \frac{7000}{(1+s_2)^2} \quad\Longrightarrow\quad \boxed{s_2 = 0.133}$$

Example 6.6. Given the following annual effective spot rates:

$$s_1 = 4.00\% \qquad s_2 = 4.50\% \qquad s_3 = 4.75\% \qquad s_4 = 5.00\%$$

Calculate $100(Ia)_{\overline{4|}}$.



$$100(Ia)_{\overline{4|}} = \frac{100}{1.04} + \frac{200}{(1.045)^2} + \frac{300}{(1.0475)^2} + \frac{400}{(1.05)^2} \approx \boxed{\$869.39}$$

Example 6.7. Given the prices of zero-coupon bonds with a redemption amount of \$105:

Maturity	Price
1	100.96
2	98.02
3	95.39
4	93.29

Calculate the price of a 4-year \$100 par value with 5% annual coupons.

Use the percentage of price method:

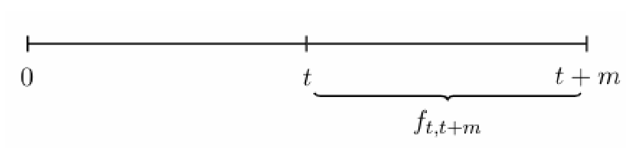
$$P = 100.96\left(\frac{5}{105}\right) + 98.02\left(\frac{5}{105}\right) + 95.39\left(\frac{5}{105}\right) + 93.29\left(\frac{105}{105}\right) \approx \boxed{\$107.31}$$

6.2 Forward Rates

Definition 6.8 (Forward Rates). Forward rates are rates of interest that can be earned on an investment at a future point in time.

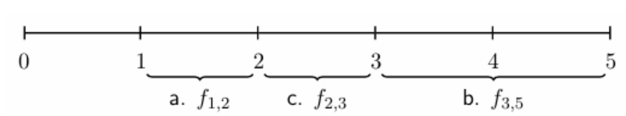
Since forward rates are dependent on spot rates, we can also adapt this definition:
An m-year spot rate that comes into effect t-years in the future will be referred to as the "m-year forward rate, deferred t-years" or as the "m-year forward rate, starting in t years."

We will use $f_{t,t+m}$ to denote this forward rate.



Unless otherwise stated, assume forward rates are annual effective rates.

If we were label forward rates on a time diagram:



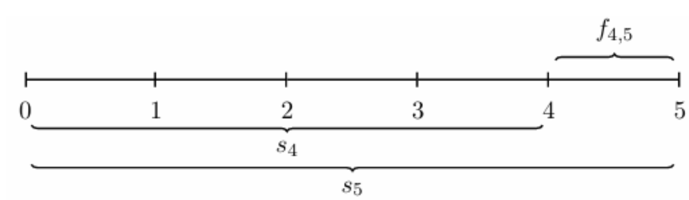
(a) 1-year forward rate, deferred 1 year

(b) 2-year forward rate, starting 3 years

(c) 1-year forward rate, deferred 2 years

Forward Rates are implied by Spot Rates

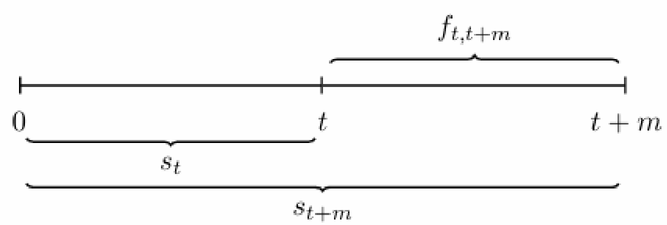
How would we write the 1-year forward rate, starting in 4 years in terms of the 4-year and 5-year spot rates?



We want to accumulate \$1 from time 0 to time 5. The idea is to multiply the accumulated value of the spot rate over 4 years by the forward rate to obtain the accumulated value of the spot rate over 5 years:

$$(1+s_4)^4(1+f_{4,5}) = (1+s_5)^5$$
$$1+f_{4,5} = \frac{(1+s_5)^5}{(1+s_4)^4} \quad\Longrightarrow\quad f_{4,5} = \frac{(1+s_5)^5}{(1+s_4)^4} - 1$$

More generally, how would we write the m-year forward rate, starting in t years in terms of spot rates?



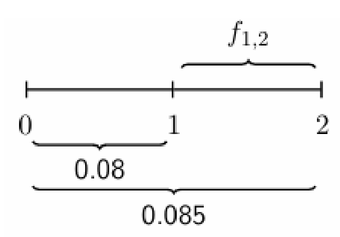
We want to accumulate \$1 from time 0 to time $t+m$. The idea is to multiply the accumulated value of the spot rate over t periods by the geometric forward rate to obtain the accumulated value of the spot rate over $t+m$ periods:

$$(1+s_t)^t(1+f_{t,t+m})^m = (1+s_{t+m})^{t+m}$$
$$(1+f_{t,t+m})^m = \frac{(1+s_{t+m})^{t+m}}{(1+s_t)^t} \quad\Longrightarrow\quad \boxed{f_{t,t+m} = \left[\frac{(1+s_{t+m})^{t+m}}{(1+s_t)^t}\right]^{\frac{1}{m}} - 1}$$

Lock-in Forward Rate

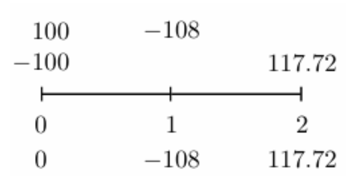
We showed that forward rates are implied by spot rates, but how can we lock in a forward rate as our rate of return in the future?

For example, suppose we are told $s_1 = 0.08$ and $s_2 = 0.085$, and we want to calculate the 1-year forward rate starting in 1 year.



$$f_{1,2} = \frac{(1.085)^2}{1.08} - 1 = 0.09$$

Moreover, if we borrowed \$100 for one year and invested \$100 for two years, what is your rate of return for year 2?



If we borrowed \$100 for one year, then we are repaying $100(1.08) = \$108$. By investing \$100 for two years, we are receiving a payoff of $100(1.085)^2 = \$117.12$. The rate of return for year 2 is therefore

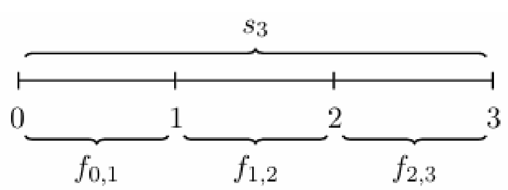
$$\frac{117.72}{108} - 1 = 0.09$$

which is the forward rate $f_{1,2}$ from earlier.

Spot Rates in Terms of Forward Rates

Conversely, we can write spot rates in terms of forward rates.

How would we write s_3 in terms of $f_{0,1}$, $f_{1,2}$, and $f_{2,3}$?



Accumulate \$1 from time 0 to 3 using spot and forward rates:

$$\$1(1+s_3)^3 = \$1(1+f_{0,1})(1+f_{1,2})(1+f_{2,3})$$

$$\Rightarrow \quad s_3 = [(1+f_{0,1})(1+f_{1,2})(1+f_{2,3})]^{\frac{1}{3}} - 1$$

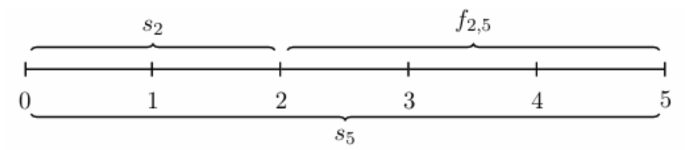
More generally,

$$\boxed{s_t = \left[\prod_{t=0}^{t-1}(1+f_{t,t+1})\right]^{\frac{1}{t}} - 1}$$

Example 6.9. Given the following spot rates:

$$s_1 = 4.00\% \qquad s_2 = 5.00\% \qquad s_3 = 5.50\% \qquad s_4 = 5.75\% \qquad s_5 = 6.00\%$$

Calculate the 3-year forward rate, deferred for two years.



Using forward rate notation, we want to compute $f_{2,5}$:

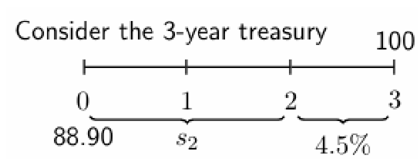
$$(1+s_2)^2(1+f_{2,5})^3 = (1+s_5)^5$$

$$\Rightarrow \quad f_{2,5} = \left[\frac{(1.06)^5}{(1.05)^2}\right]^{1/3} - 1 \quad \Rightarrow \quad \boxed{f_{2,5} \approx 0.06672}$$

Example 6.10. You are given the following information about the Treasury market. Note that Treasuries always have a par value of \$100.

Term	Coupon	Price
1 year	0%	96.62
2 years	0%	x
3 years	0%	88.90

It is known that the one-year forward rate starting in two-years is 4.5%. Calculate x.



Use the par value, price after 3 years, and the forward rate to find s_2:

$$88.9(1.045)(1+s_2)^2 = 100 \quad \Longrightarrow \quad s_2 = \left[\frac{100}{88.9(1.045)}\right]^{0.5} - 1 = 0.0375$$

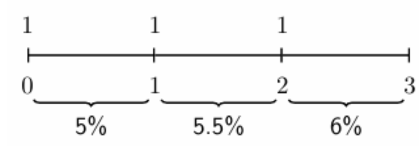
Use the spot rate to find x:

$$x = \frac{100}{(0.0375)^2} = \boxed{\$92.90}$$

Example 6.11. You are given:

1. The one-year spot rate is 5%
2. The one-year forward rate starting in one year is 5.5%
3. The one-year forward rate starting in two years is 6.0%

Calculate $\ddot{a}_{\overline{3}|}$.



We have that

$$\ddot{a}_{\overline{3}|} = 1 + \frac{1}{1+s_1} + \frac{1}{(1+s_2)^2}$$

where $s_1 = 0.05$. We use forward rates to compute $(1+s_2)^2$:

$$(1+s_2)^2 = (1+s_1)(1+f_{1,2}) = (1.05)(1.055)$$

Therefore,

$$\ddot{a}_{\overline{3}|} = 1 + \frac{1}{1.05} + \frac{1}{(1.05)(1.055)} \approx \boxed{2.855}$$

What if we wanted to calculate $a_{\overline{3}|}$?

It follows that $(1+s_3)^3 = (1+s_1)(1+f_{1,2})(1+f_{2,3})$, so

$$a_{\overline{3}|} = \frac{1}{1.05} + \frac{1}{(1.05)(1.055)} + \frac{1}{(1.05)(1.055)(1.06)} \approx \boxed{2.7067}$$

6.3 Real Rates of Interest

Let's say you have \$10,000 in your savings account today. Your savings account earns 1% annual effective rate of interest. That means one year late you have

$$10000(1.01) = 10100$$

But did you really earn 1%? One year later you have 1% more dollars, but can you buy 1% more than you could a year ago?

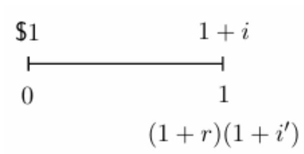
Typically, the answer is no because things today cost more than they did a year ago. This is called **inflation**.

If the inflation rate is 2%, then you need to have $10000(1.02) = 10200$ to have the same purchasing power you had one year ago.

You have \$100 less. Your purchasing power in one year on the same basis as today is $\frac{10000}{1.02} = 9901.96$. Thus your **real rate of return** is

$$\frac{9901.96}{10000} - 1 = -0.0098$$

Let i be the annual rate of interest, r be the rate of inflation, and i' be the inflation-adjusted return or real rate of return,



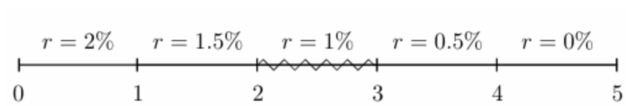
$$(1+i) = (1+r)(1+i') \quad \Rightarrow \quad \frac{1+i}{1+r} = 1+i' \quad \Rightarrow \quad \boxed{i' = \frac{i-r}{1+r}}$$

Geometric annuities can be viewed as real rates of interest problems.

Some people refer to i as the *nominal* rate of interest. They use *nominal* to infer that the rate is a rate in name only. The true rate earned is i'.

We already used the term nominal rate of interest in a different sense. For example, the nominal annual rate of interest is 12% convertible monthly. These two are different concepts all together.

Example 6.12. You expect inflation to be 2% next year and linearly grade to 0% by the fifth year. If your investment returns 8% per year, what is the real rate of interest earned during the third year?



$$i' = \frac{i-r}{1+r} = \frac{0.08 - 0.01}{1.01} = \boxed{0.0693}$$

Example 6.13. The current real rate of interest is 4%. The expected annual inflation rate over the next several years is 5%. Given the following expected nominal cash flows, what is the net present value of the investment using the nominal rate of interest?

Year	0	1	2
Cash Flow	-300	160	160

Find the nominal rate of interest i.

$$1 + i = (1 + r)(1 + i') \quad \Rightarrow \quad 1 + i = (1.05)(1.04)$$

$$i = 0.092$$

The net present value is

$$-300 + \frac{160}{1.092} + \frac{160}{(1.092)^2} \approx \boxed{-\$19.30}$$

7 Duration

In this section, we will elaborate on ways to manage interest-rate risk. The key components are duration, convexity, and immunization (convexity and immunization to be covered next section).

7.1 Price Sensitivity

Recall that the price is the present value of all cash flows.

$$\boxed{P = \sum_{\text{all } t} CF_t (1+i)^{-t}}$$

Suppose we want to calculate the price of a \$100 ten-year zero-coupon bond with an annual effective yield of 8%.

$$P_{0.08} = 100(1+0.08)^{-10} = 46.32$$

Now if the yield is 9% instead of 8%, will the price be higher or lower? Price and yield are inversely related, thus lower.

$$P_{0.09} = 100(1+0.09)^{-10} = 42.24$$

Definition 7.1 (Price Sensitivity). **Price sensitivity** is the percentage change in the price of an asset from a shift in the yield curve.

$$\text{price sensitivity } = \frac{P_{i_1} - P_{i_0}}{P_{i_0}} = \%\Delta P$$

We can apply this to our previous example. Let's calculate the price sensitivity of our ten-year bond from a change in yield from 8% to 9%.

$$\%\Delta P = \frac{42.24 - 46.32}{46.32} \approx \boxed{-0.088}$$

Would a \$100 twenty-year zero-coupon bond be more or less sensitive to yield changes?

It would be more sensitive. If we applied the same Δi (from 8% to 9%) to calculate the price sensitivity of a \$100 20-year bond:

$$\frac{100(1.09)^{-20} - 100(1.08)^{-20}}{100(1.08)^{-20}} \approx -0.168$$

Cash flows further into the future are more sensitive to changes in the yield curve.

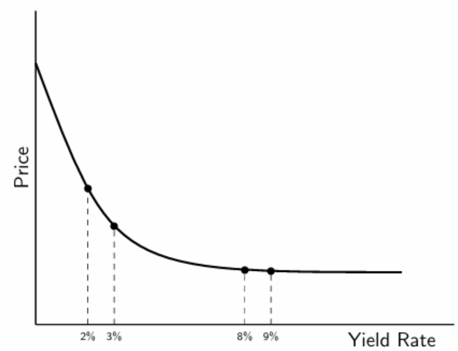
Price sensitivity depends on Δi, but it also depends on the current level of the yield curve. In other words, the price sensitivity for a change from 8% to 9% would be different than the price sensitivity for a change in the yield rate from 2% to 3%.

For our \$100 ten-year bond, calculate the price sensitivity for a change from 2% to 3\$.

$$\frac{100(1.03)^{-10} - 100(1.02)^{-10}}{100(1.02)^{-10}} \approx -0.093$$

Recall we had a price sensitivity of -0.088 for a change in the yield rate from 8% to 9\$.

The lower the yield rate the more sensitive cash flows will be to changes in the yield rate.



Example 7.2. True or false: When interest rates decrease, bond prices go down.

False. Price and interest rates are inversely related. Thus, a decrease in interest rate causes bond prices to go up.

Example 7.3. Which of the following bonds is the most price sensitive to changes in interest rate? A 5-year, 10-year, 20-year zero coupon bond, or are they equally sensitive?

The 20-year coupon bond. Cash flows further into the future are more sensitive to changes in the yield curve.

Example 7.4. Let x be the price change for a 20-year zero coupon bond when interest rates change from 5% to 4%. Let y be the price change for the same bond when interest rates change from 15% to 14%.

True or false: The absolute value of $x >$ absolute value of y.

True. Prices are more sensitive for to changes in yield rate for lower yield rates.

7.2 Macaulay Duration

Recall that price sensitivity depends on when the cash flow occurs.

However, how do we judge the price sensitivity of a *collection* of cash flows occurring at different times, as with a coupon bond, an annuity, or a whole portfolio of investments?

The solution lies in *duration*, a concept we will discuss later.

Definition 7.5 (Macaulay Duration). The **(Macaulay) duration** of a single cash flow is the length of time, in years, until the cash flow occurs.

If we have multiple cash flows, we extend the definition:

Duration is the average time (in years), weighted by the present value of the cash flows.

So, even though there are multiple cash flows, we still have a single value − a particular choice of average − that can function as a "length of time" for the entire group of payments.

For example, an investment with payments of $100 at time 5 and time 10 has an annual effective yield rate of 8%. What is its duration?

Let A_t be the present value of the cash flow occurring at time t.

$$A_5 = 100(1.08)^{-5} = 68.06 \qquad A_{10} = 100(1.08)^{-10} = 46.32$$

Here, 5 gets more weight than 10 in the average.

$$\text{Duration} = 5\left(\frac{68.06}{68.06 + 46.32}\right) + 10\left(\frac{46.32}{68.06 + 46.32}\right) = 7.025$$

which is closer to 5, as expected.

Measuring Duration

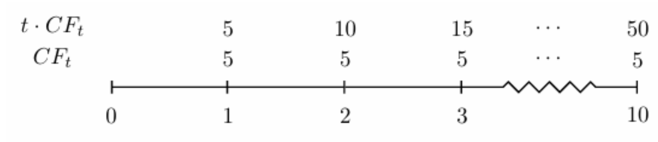
$$\boxed{\text{Macaulay Duration (duration)} = \text{MacD} = \frac{\sum_{\text{all } t} t \cdot A_t}{\sum_{\text{all } t} A_t} = \frac{\sum_{\text{all } t} t \cdot A_t}{\text{Price}}}$$

For a single cash flow at time t, $\text{MacD} = t \cdot \frac{A_t}{A_t} = t$.

Because duration is a weighted average, all else being equal, we get shorter duration with:

- larger coupons, as PVs of earlier cash flows increase
- higher yield, as PVs of later cash flows are more discounted

Example 7.6 (Duration of a Level Annuity-Immediate). Find the duration of a ten-year annuity-immediate with annual payments of \$5. The first payment occurs one year from now and the annual effective rate is 3%.

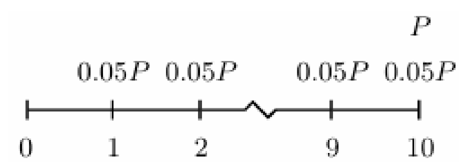


$$\text{MacD} = \frac{\sum_{\text{all } t} t \cdot A_t}{\sum_{\text{all } t} A_t} = \frac{5(Ia)_{\overline{10}|0.03}}{5a_{\overline{10}|0.03}} \approx \boxed{5.256}$$

The duration of a n-year level annuity with m payments per year is

$$\boxed{\frac{(Ia)_{\overline{m\times n}|}}{ma_{\overline{m\times n}|}}}$$

Example 7.7 (Duration of a Par Bond). Calculate the duration of a 10-year bond with annual coupons of 5% priced at par.



$$\begin{aligned}\text{MacD} = \frac{\sum_{\text{all } t} t \cdot A_t}{\sum_{\text{all } t} A_t} &= \frac{\sum_{k=1}^{10} k(0.05P)v^k + 10Pv^{10}}{P} = 0.05(Ia)_{\overline{10}|0.05} + 10v^{10} \\ &= 0.05\left(\frac{\ddot{a}_{\overline{10}|0.05} - 10v^{10}}{0.05}\right) + 10v^{10} \\ &= \ddot{a}_{\overline{10}|0.05} - 10v^{10} + 10v^{10} = \ddot{a}_{\overline{10}|0.05} \approx \boxed{8.107}\end{aligned}$$

The duration of an n-year par bond with m-thly coupons is

$$\boxed{\text{MacD of Par Bond} = \ddot{\overline{n}|}i^{(m)} = \frac{1}{m}\ddot{a}_{\overline{n\times m}|j} \quad \text{where } j = \frac{i^{(m)}}{m}}$$

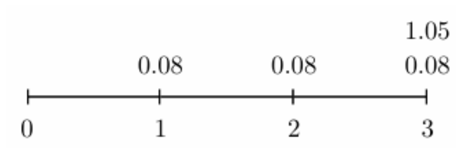
Example 7.8. A 20-year bond pays semiannual coupons of 7.4% and is priced at par. Calculate the duration.

The bond is priced at par, so we can use the duration of a par bond formula:

$$\text{MacD} = \frac{1}{2}\ddot{a}_{\overline{40}|0.037} \approx \boxed{10.72}$$

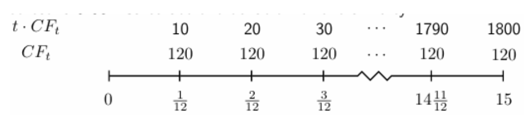
Example 7.9. A three-year bond with 8% annual coupons has a redemption value of 105% of the par value and is priced to yield at an annual effective rate of 6%. Calculate the duration.

Not priced at par $\Rightarrow$ No shortcut. The par value doesn't matter, so let's set it at 1.



$$\text{MacD} = \frac{1 \cdot 0.08v_{0.06} + 2 \cdot 0.08v_{0.06}^2 + 3 \cdot (0.08 + 1.05)v_{0.06}^3}{0.08v_{0.06} + 0.08v_{0.06}^2 + (0.08 + 1.05)v_{0.06}^3} \approx \boxed{2.7972}$$

Example 7.10. An annuity-immediate makes monthly payments of \$120 for 15 years. The constant force of interest is 0.05. Calculate the duration of the annuity.



Let $j = \left(e^{0.05}\right)^{1/12} - 1$ be the effective rate per month. We can apply the shortcut:

$$\text{MacD} = \frac{(Ia)_{\overline{180}|j}}{12a_{\overline{180}|j}} \approx \boxed{6.613}$$

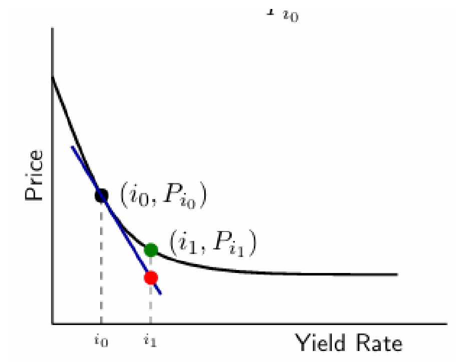
7.3 Modified Duration

Recall our motivation: How do we judge the price sensitivity of a **collection** of cash flows occurring at different times?

Previously, we developed the concept of duration. It is useful, but it isn't exactly what we need in order to estimate price sensitivity. For this, we use modified duration.

We defined price sensitivity as

$$\%\Delta P = \frac{P_{i_1} - P_{i_0}}{P_{i_0}}$$



Our aim is to approximate the new price using the tangent line, and use the approximated new price to estimate $\%\Delta P$.

For the blue line: (i_0, P_{i_0}) and $m = \frac{d}{di}P_{i_0}$

$$P_i - P_{i_0} = \left(\frac{d}{dt}P_{i_0}\right)(i - i_0)$$

$$P_{i_1} \approx \frac{d}{di}P_{i_0}\Delta i + P_0$$

The % change in price is

$$\frac{P_{i_1} - P_{i_0}}{P_{i_0}} \approx \frac{\frac{d}{di}P_{i_0} \cdot \Delta i}{P_{i_0}}$$

$$\%\Delta \approx -\Delta i\left(-\frac{\frac{d}{di}P_{i_0}}{P_{i_0}}\right)$$

Why the extra negative signs? We expect price and interest rates to be inversely related, so a positive change in i should cause a negative (percent) change in P. Mathematically, this would be $\Delta P = -\Delta i \cdot k$, for some constant $k > 0$.

The constant in the parentheses is the **modified duration**.

$$\text{ModD} = -\frac{\frac{d}{di}P_{i_0}}{P_{i_0}} \implies \%\Delta P \approx -\Delta i \cdot \text{ModD}$$

$$\boxed{\Delta P \approx -\Delta i \cdot \text{ ModD } \cdot P_{i_0}}$$

For all of these formulas so far, i could be expressed in any form (e.g. as a nominal annual yield or an effective rate over some period). As long as the form is consistent, everything we've seen so far will work.

Example 7.11 ($\%\Delta P$ Example). A two-year \$1000 par value bond with semiannual coupons of 8% is to be priced to yield 8% convertible semiannually. The modified duration of the bond using the nominal annual yield is 1.815. Estimate the price using the modified duration if the yield changes to 7.8%.

$$\%\Delta P \approx -\Delta i \cdot \text{ModD} = -(-0.002)(1.815) \approx 0.00363$$

The price at 8% is \$1000 because coupon rate equals yield rate and it redeems at par. The estimated new price is

$$1000(1.00363) = \boxed{1003.63}$$

The actual new price is

$$40a_{\overline{4}|0.039} + 1000v_{0.039}^4 = 1003.64$$

Modified duration always underestimates the price.

Example 7.12. A five-year bond with annual coupons of \$100 and a redemption of \$1500 is priced at \$1608.24 to yield 5%.

Your coworker uses the bond's modified duration to estimate the change in price if the yield rises to 6%. The coworker's estimated change is 76.58.

Determine the estimated price of the bond at a 6% yield. Is this estimated price larger or smaller than the actual price at 6%?

An increase in yields means a decrease in the price

$$\text{estimated price } = 1608.24 - 76.58 = \boxed{1531.66}$$

Modified duration always underestimates the price. So the estimated price is $\boxed{\textit{smaller}}$ than the actual price.

$$\text{actual price } = 100a_{\overline{5}|0.06} + 1500v_{0.06}^5 = 1542.12$$

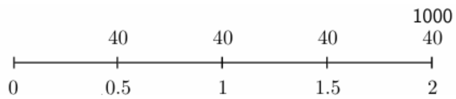
(This estimate was not as close as the previous example because Δi was much larger).

Determine the modified duration of the bond using the effective annual yield.

$$\frac{P_{0.06} - P_{0.05}}{P_{0.05}} = -\Delta i\text{ModD}$$

$$\frac{-76.58}{1608.24} = -0.01\text{ModD} \qquad \Longrightarrow \qquad \text{ModD} \approx \boxed{4.762}$$

Recall Example 7.11. How would we verify that the modified duration is indeed 1.815?



For convenience, let $j = i^{(2)}$.

$$P_j = 40\left(1+\frac{j}{2}\right)^{-1} + 40\left(1+\frac{j}{2}\right)^{-2} + 40\left(1+\frac{j}{2}\right)^{-3} + 1040\left(1+\frac{j}{2}\right)^{-4}$$

$$\frac{d}{dj}P_j = -20\left(1+\frac{j}{2}\right)^{-2} - 40\left(1+\frac{j}{2}\right)^{-3} - 60\left(1+\frac{j}{2}\right)^{-4} - 2080\left(1+\frac{j}{2}\right)^{-5}$$

$$-\frac{d}{dj}P_{0.08} = 20(1.04)^{-2} + 40(1.04)^{-3} + 60(1.04)^{-4} + 2080(1.04)^{-5} \approx 1814.95$$

$$\text{ModD} = -\frac{\frac{d}{dj}P_j}{P_j} = \frac{1814.95}{1000} = \boxed{1.815}$$

Calculating Modified Duration

Let CF_t be the cash flow at time t, where t is measured in years. Let i be the nominal annual rate convertible m-thly.

$$P_i = \sum_{\text{all } t} = \sum_{\text{all } t} CF_t\left(1+\frac{i}{m}\right)^{-mt}$$

$$\frac{d}{di}P_i = \sum_{\text{all } t} -mtCF_t\left(1+\frac{i}{m}\right)^{-mt-1}\cdot\frac{1}{m}$$

$$-\frac{d}{di}P_i = \sum_{\text{all } t} tCF_t\left(1+\frac{i}{m}\right)^{-mt}\left(1+\frac{i}{m}\right)^{-1} = \sum_{\text{all } t} t\cdot A_t\cdot v$$

$$\boxed{\text{ModD} \;=\; \frac{v\sum_{\text{all } t} t\cdot A_t}{\sum_{\text{all } t} A_t} \qquad \text{where } t \text{ is in years}}$$

Now we can see why this volatility measure is called **modified** duration.

$$\boxed{\text{ModD} = v\cdot\text{MacD}}$$

It's the duration, only modified by the discount factor for one period.

Recall Example 7.11 one last time:

$$\text{ModD} = v_{0.04}\text{MacD} = (1.04)^{-1}\ddot{a}^{(2)}_{\overline{2}|0.08} = \frac{1.8875}{1.04} = 1.815$$

When interest is compounded continuously, the Macaulay and Modified duration are equivalent.

Example 7.13. A three-year bond with 8% annual coupons has a redemption value of 105% of the par value and is priced to yield an annual effective rate of 6%. Calculate the modified duration.

The par value is actually not required here—this term will be canceled out in the ModD calculation. Let's fix a par value of \$100.

$$\sum_{\text{all } t} t \cdot A_t \cdot v = 8(1.06)^{-2} + 16(1.06)^{-3} + 3(113)(1.06)^{-4} \approx 289.07$$

$$\sum_{\text{all } t} A_t = 8a_{\overline{3}|0.06} + 105v^3_{0.06} \approx 109.54$$

$$\text{ModD} = \frac{v\sum_{\text{all } t} tA_t}{\sum_{\text{all } t} A_t} = \frac{289.07}{109.54} \approx \boxed{2.639}$$

Example 7.14. A thirty-year bond with 8% coupons payable quarterly is priced to yield x% convertible quarterly. The modified duration is 8.392 and duration is 8.644. Determine x%.

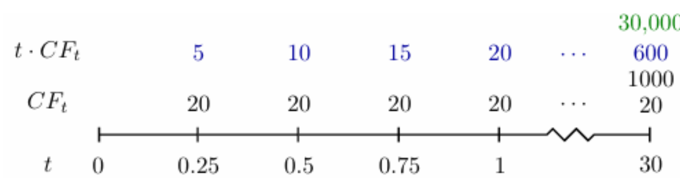
$$\text{ModD} = v\text{MacD} \quad \Longrightarrow \quad v = \frac{8.392}{8.644}$$

$$\left(1 + \frac{x}{4}\right)^{-1} = \frac{8.392}{8.644}$$

$$x = 4\left(\frac{8.644}{8.392} - 1\right) \approx \boxed{12\%}$$

Example 7.15. A thirty-year bond with 8% coupons payable quarterly is priced to yield 12% convertible quarterly. Calculate the modified duration.

Let's use a par value of \$1000. The quarterly coupons are then \$20 and the effective rate per quarter is 3%.



The \$30,000 comes from $t \cdot CF_t$, multiplying the redemption value of \$1000 by 30.

$$v \sum_{\text{all } t} tA_t = v\left(5v_{0.03} + 10v_{0.03}^2 + 15v_{0.03}^3 + \cdots + 600v_{0.03}^{120}\right) + 30000v_{0.03}^{120}$$

$$= v\left(5(Ia)_{\overline{120}|0.03} + 30000v_{0.03}^{120}\right) \approx 5675.21$$

$$\text{ModD} = \frac{v\sum_{\text{all } t} tA_t}{\sum_{\text{all } t} A_t} = \frac{5675.21}{20a_{\overline{120}|0.03} + 1000v_{0.03}^{120}} = \frac{5675.21}{676.27} \approx \boxed{8.392}$$

7.4 Portfolios and Passage of Time

There are two methods in measuring the duration of a portfolio:

1. Determine the total cash flow of the entire portfolio at each time t, and then use MacD or ModD summation formulas.
2. Compute the weighted (by price) average of the MacD or ModD values of the portfolio components

Example 7.16 (Weighted Average). You are given the following information about a portfolio of 3 bonds

Bond	Price	Duration
A	X	4 years
B	$2X$	6 years
C	$\frac{1}{2}X$	Y years

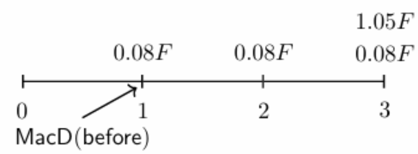
The duration of the entire portfolio is 8 years. Find Y.

$$\text{MacD}_p = \frac{P_A \text{MacD}_A + P_B \text{MacD}_B + P_C \text{MacD}_C}{P_A + P_B + P_C}$$

$$8 = \frac{4X + 12X + 0.5XY}{3.5X} = \frac{X(16.5 + 0.5Y)}{3.5X} = \frac{16.5 + 0.5Y}{3.5}$$

$$\boxed{Y = 24}$$

Example 7.17 (Passage of Time). A three-year bond with 8% annual coupons has a redemption value of 105% of the par value and is priced to yield at an annual effective rate of 6%. The duration at time 0 is 2.7972. Calculate the duration at time 1, both just before and just after the coupon payment.



$$\begin{aligned}\text{MacD(before)} &= \frac{0 \cdot 0.08F + 1 \cdot 0.08F v_{0.06} + 2(0.08F + 1.05F) v_{0.06}^2}{0.08F + 0.08F v_{0.06} + (0.08F + 1.05F) v_{0.06}^2} \\ &= \frac{2.0869}{1.1612} = 1.7972 = \text{MacD}(0) - 1\end{aligned}$$

$$\begin{aligned}\text{MacD(after)} &= \frac{1 \cdot 0.08F v_{0.06} + 2(0.08F + 1.05F) v_{0.06}^2}{0.08F v_{0.06} + (0.08F + 1.05F) v_{0.06}^2} \\ &= \frac{2.0869}{1.0812} = 1.9302\end{aligned}$$

Immediately after a cash flow, the duration increases!

Example 7.18. A portfolio contains three bonds, all of which have par value \$1000. The first has a two-year term and no coupons, the second is a three-year bond with no coupons, and the third is a two-year 6% bond with semiannual coupons. If each bond is purchased to yield 5% convertible semiannually, find the modified duration of the portfolio.

$$P_1 = 1000v^4 = 905.05 \qquad P_2 = 1000v^6 = 862.30 \qquad P_3 = 30a_{\overline{4}|0.025} + 1000v^4 = 1018.81$$

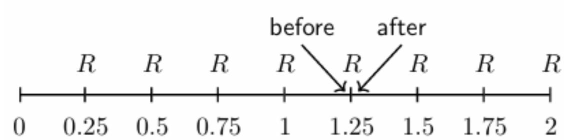
$$\text{MacD}_3 = \frac{0.5 \cdot 30v + 1 \cdot 30v^2 + 1.5 \cdot 30v^3 + 2 \cdot 1030v^4}{1018.81} = 1.9152$$

For zero-coupon bonds, the duration and bond length are equivalent.

$$\text{MacD}_P = \frac{2P_1 + 3P_2 + 1.9152P_3}{P_1 + P_2 + P_3} \approx 2.2784$$

$$\text{ModD}_P = (1.025)^{-1}\text{MacD}_P \approx \boxed{2.2228}$$

Example 7.19. A \$5000 loan is being repaid over a two-year period with level payments at the end of each quarter. The interest rate is 8% convertible quarterly. Find the jump in the duration of the loan following the 5th payment.



$$\text{MacD(before)} = \frac{0R + 0.25Rv + 0.5Rv^2 + 0.75Rv^3}{R(1 + v + v^2 + v^3)} \approx 0.369$$

$$\text{MacD(after)} = \frac{0.25Rv + 0.5Rv^2 + 0.75Rv^3}{R(v + v^2 + v^3)} \approx 0.497$$

$$\text{Jump} = 0.497 - 0.369 = \boxed{0.128}$$

7.5 Approximation Using Duration

Recall:

- An important task is to model how prices change in response to changes in the yield rate
- For a set of cashflows, we graphed price vs. yield and used a tangent line to that curve to produce an approximation

- That approximation lets us easily estimate the price sensitivity ("percent change in price") of cashflows due to a change in yield. The resulting relationship was a negative proportion: $\%\Delta P = -\delta i \cdot k$
- Modified duration was *defined* to be a constant of proportionality k

In other words, ModD was custom-built to play a role in linear approximation of the price-yield curve.

A MacD Approximation?

Could MacD be used to build an approximation too?

- MacD is pretty similar to ModD:

$$\text{ModD} = v\text{MacD}$$

  so it's *conceivable* that it might be similarly useful for approximating the price sensitivity of a set of cashflows
- A MacD approximation wouldn't be a simple linear approximation, however
- MacD is typically easier to calculate, so that is already one advantage for us if such an approximation exists

As is turns out, there **is** an approximation for the price using MacD!

Suppose we start with an **effective annual** yield rate of i_0, and the yield rate changes to i_1. The new price is approximated by

$$\boxed{P_{i_1} \approx P_{i_0}\left(\frac{1+i_0}{1+i_1}\right)^{\text{MacD}}}$$

where MacD is evaluated at the original yield rate i_0.

This, in return, leads to an approximation for the price sensitivity:

$$\boxed{\%\Delta P \approx \left(\frac{1+i_0}{1+i_1}\right)^{\text{MacD}} - 1}$$

Example 7.20. A two-year \$100 par value bond with semiannual coupons of 8% is priced to yield 8% convertible semiannually. The duration of the bond is 1.887545517. Estimate the price using the duration if the yield changes to 7.8%.

The price at 8% is \$1000 because coupon rate equals yield rate and it redeems at par.

We could convert the rates to annual effective rates, but the formula really just demands the annual accumulation factors, so we'll use those:

$$P_{i_1} \approx P_{i_0} \left(\frac{1+i_0}{1+i_1} \right)^{\text{MacD}} = 1000 \left(\frac{(1.04)^2}{(1.039)^2} \right)^{1.887545517} \approx 1003.638244$$

The actual new price is

$$40a_{\overline{4}|0.039} + 1000v_{0.039}^4 = 1003.638466$$

Earlier, we saw this same example, with the price estimated using the ModD approximation. Repeating that calculation using a less-rounded value for ModD (1.814947612) gives an estimated price of 1003.629895.

- The relative error for using MacD here is 0.000022%
- The relative error for using ModD here is 0.000854%

The change in the yield rate was small, allowing both approximations to do a good job. MacD has appeared to have done a bit better, however.

Comparison of Methods

Approximating using MacD using ModD:

- The MacD approximation will do better than the ModD approximation as long as the future cashflows are all positive (if some are negative, then this doesn't always apply)
- With all future cashflows positive, the relative error of the MacD approximation will be roughly of **one order** of magnitude (10^{-1}) better than the error of the ModD approximation
- The situation where there is only one future cashflow is different: in this special case, the MacD approximation will give the *exact* new price $-$ no error at all

Example 7.21. A five-year bond with annual coupons of \$100 and a redemption amount of \$1500 is priced at \$1608.24 to yield 5%. The duration of this bond is 4.435. Estimate the percent change in price that would occur if the yield rose to 5.5%.

$$\%\Delta P = \left(\frac{1.05}{1.055} \right)^{4.435} - 1 \approx \boxed{-0.02085}$$

An alternative solution is to use the formula for the MacD approximation of the **price**. We can use that formula to estimate the new price, and then compute the percent change in price manually.

$$P_{i_1} = P_{i_0} \left(\frac{1+i_0}{1+i_1} \right)^{\text{MacD}} = 1608.24 \left(\frac{1.05}{1.055} \right)^{4.435} \approx 1574.41$$

$$\%\Delta P = \frac{P_{i_i} - P_{i_0}}{P_{i_0}} = \frac{1574.71 - 1608.24}{1608.24} \approx \boxed{-0.02085}$$

Example 7.22. A four-year bond with semiannual coupons is priced at \$1179.25 to yield 5% convertible semiannually.

Suppose the bond's price rises to \$1199.30, implying a new nominal yield of 4.5%. Estimate the MacD of the bond prior to the interest rate change.

$$P_{i_1} \approx P_{i_0} \left(\frac{1+i_0}{1+i_1} \right)^{\text{MacD}}$$

$$1199.30 \approx 1179.25 \left(\frac{(1.025)^2}{(1.0225)^2} \right)^{\text{MacD}}$$

$$\ln \left(\frac{1199.30}{1179.25} \right) \approx \text{MacD} \ln \left(\frac{(1.025)^2}{(1.0225)^2} \right)$$

$$\boxed{\text{MacD} \approx 3.452}$$

8 Convexity and Immunization

We will use convexity and immunization to further refine approximation on pricing and protecting portfolio duration.

8.1 Convexity

We've seen that we can calculate price sensitivity from its duration

$$\%\Delta P = \frac{P_{i_1} - P_{i_0}}{P_{i_0}}$$

or we can estimate it using modified duration

$$\%\Delta P \approx -\Delta i \cdot \text{ModD}$$

The estimate formula came from using a linear approximation to the price vs. yield curve. We can improve our approximation by using a quadratic curve instead of a line. Mathematically speaking, this is the "second-order Taylor approximation."

A linear approximation at $i = i_0$ is the tangent line.

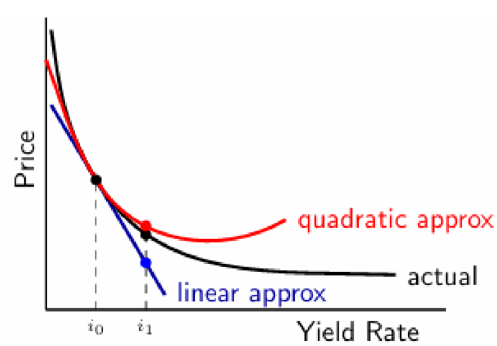
$$P_i - P_{i_0} = P'_{i_0}(i - i_0)$$

Geometrically, this line matches the original curve by having the same value (P_{i_0}) at $i = i_0$ and the same slope (i.e. first derivative) at $i = i_0$.

A quadratic approximation builds on the linear approximation by adding a quadratic term

$$y - y_0 = P'(i_0)(i - i_0) + \frac{1}{2}P''(i_0)(i - i_0)^2$$

Now, it matches the value, first derivative **and** second derivative of the original curve at $i = i_0$. This lets it curve more like the real thing, resulting in a better approximation.



For the red curve:

$$P_i - P_{i_0} = \left(\frac{d}{dt}P_{i_0}\right)(i - i_0) + \frac{1}{2}\left(\frac{d^2}{di^2}P_{i_0}\right)(i - i_0)^2$$

The estimated value of P_{i_1} is

$$P_{i_1} \approx P_{i_0} + \left(\frac{d}{dt}P_{i_0}\right)\Delta i + \frac{1}{2}\left(\frac{d^2}{di^2}P_{i_0}\right)(\Delta i)^2$$

The % change in price is:

$$\begin{aligned}\%\Delta P = \frac{P_{i_1} - P_{i_0}}{P_{i_0}} &\approx \frac{\left(\frac{d}{dt}P_{i_0}\right)\Delta i + \frac{1}{2}\left(\frac{d^2}{di^2}P_{i_0}\right)(\Delta i)^2}{P_{i_0}} \\ &= -\Delta i\left(-\frac{\frac{d}{di}P_{i_0}}{P_{i_0}}\right) + \frac{(\Delta i)^2}{2}\left(\frac{\frac{d^2}{di^2}P_{i_0}}{P_{i_0}}\right)\end{aligned}$$

We recognize the first term in parentheses as the modified duration. As before, we assign a name to the other term in parentheses.

$$\boxed{\text{Convexity} = \frac{\frac{d^2}{di^2}P_{i_0}}{P_{i_0}}}$$

Rewriting our approximations using this new name

$$\boxed{\%\Delta P \approx -\Delta i \cdot \text{ModD} + \frac{(\Delta i)^2}{2} \cdot \text{Convexity}}$$

Alternatively, a formula for the change in price is

$$\boxed{\Delta P \approx -P_{i_0} \cdot \Delta i \cdot \text{ModD} + P_{i_0} \cdot \frac{(\Delta i)^2}{2} \cdot \text{Convexity}}$$

Calculating Convexity

Let i be the nominal annual rate convertible m-thly. Then

$$P_i = \sum_{\text{all } t} A_t = \sum_{\text{all}} tCF_t\left(1 + \frac{i}{m}\right)^{-mt}$$

Earlier we showed

$$\frac{d}{di}P_i = \sum_{\text{all } t} -mtCF_t\left(1 + \frac{i}{m}\right)^{-mt-1} \cdot \frac{1}{m}$$

It follows that

$$\frac{d^2}{di^2}P_i = \sum_{\text{all } t} t\left(t + \frac{1}{m}\right) \cdot A_t \cdot v^2$$

Therefore,

$$\boxed{\text{Convexity} = \frac{\sum_{\text{all } t} t\left(t + \frac{1}{m}\right) \cdot A_t \cdot v^2}{\sum_{\text{all } t} A_t}}$$

where t is in years and v is the per-period discount factor.

Example 8.1. A three-year bond with 8% annual coupons has a redemption value of 105% of the par value and is priced to yield at an annual effective rate of 6%. Calculate the convexity.

For convenience, fix a par value of \$100.

$$\sum_{\text{all } t} t(t+1)A_t v^2 = 16(1.06)^{-3} + 48(1.06)^{-4} + 12(113)(1.06)^{-5} \approx 1064.75$$

$$\sum_{\text{all } t} A_t = 8(1.06)^{-1} + 8(1.06)^{-2} + 113(1.06)^{-3} \approx 109.54$$

$$\text{Convexity } = \frac{1064.75}{109.54} \approx \boxed{9.72}$$

Example 8.2. A two-year \$1000 par value bond with semiannual coupons of 8% is priced to yield 8% convertible semiannually. The modified duration of the bond is 1.814948. The convexity is 4.277335. Estimate the price if the yield changes to 7.8% using the modified duration and convexity.

$$\%\Delta P \approx -\Delta i \cdot \text{ModD} + \frac{(\Delta i)^2}{2} \cdot \text{Convexity}$$

$$= -(-0.002)(1.814948) + \frac{(-0.002)^2}{2}(4.277335) \approx 0.003638451$$

The price at 8% is 1000 because coupon rate equals yield rate. The estimated new price is

$$1000(1.003638451) \approx \boxed{1003.638451}$$

The actual new price is

$$40a_{\overline{4}|0.039} + 1000v_{0.039}^4 \approx 1003.638466$$

offering an even more precise estimation compared to our first-order approximation.

8.2 Macaulay Convexity and Portfolios

We've seen modified duration and convexity naturally occur together. Macaulay duration also has a partner: Macaulay convexity.

Recall when interest is compounded continuously, ModD matches MacD. This led to an alternative expression for MacD:

$$\lim_{m\to\infty} \text{ModD} = \lim_{m\to\infty} -\frac{\frac{d}{di}P_i}{P_i} = -\frac{\frac{d}{\delta}P_\delta}{P_\delta} = \text{MacD}$$

This MacD form can be used even when interest isn't continuously compounded $-$ just use the δ equivalent to $i^{(m)}$.

By analogy, define MacC:

$$\lim_{m\to\infty} \text{Convexity} = \; = \lim_{m\to\infty} \frac{\frac{d^2}{di^2}P_i}{P_i} = \frac{\frac{d^2}{d\delta^2}P_\delta}{P_\delta} = \text{MacC}$$

That definition gives us two ways to arrive at a useful form for MacC. The first of which is to start with convexity and take the limit as the number of compounding periods to infinity:

$$\text{MacC} = \lim_{m\to\infty} \text{Convexity} = \lim_{m\to\infty} \frac{\sum t\left(t+\frac{1}{m}\right)A_t v^2}{\sum A_t} = \frac{\sum t^2 A_t}{A_t}$$

The second approach is to simply do the derivatives:

$$P_\delta = \sum A_t = \sum CF_t e^{-\delta t}$$

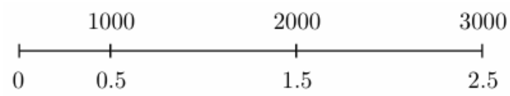
$$\frac{d}{d\delta}P_\delta = \sum -t\cdot CF_t e^{-\delta t}$$

$$\frac{d^2}{d\delta^2}P_\delta = \sum(-t)(-t)CF_t e^{-\delta t} = \sum t^2 A_t$$

$$\boxed{\text{MacC} = \frac{\sum_{\text{all } t} t^2 A_t}{\sum_{\text{all } t} A_t} \quad \text{where } t \text{ is in years}}$$

Note: Convexity can not be easily "modified" from Macaulay convexity. Thus we generally refer to it as convexity instead of modified convexity.

Example 8.3. Find the Macaulay convexity of a 3-year increasing annuity with annual payments. The first payment of \$1000 is due in six months and each subsequent payment increases by \$1000. The effective annual interest rate is 4%.



$$\sum t^2 A_t = (0.5)^2(1000)(1.04)^{-0.5} + (1.5)^2(2000)(1.04)^{-1.5} + (2.5)^2(3000)(1.04)^{-2.5} \approx 21486.83$$

$$\sum A_t = 1000(1.04)^{-0.5} + 2000(1.04)^{-1.05} + 3000(1.04)^{-2.5} \approx 5586.12$$

$$\text{MacC} = \frac{21486.83}{5586.12} \approx \boxed{3.846}$$

Convexity of a Portfolio

Use the same methods we use to find duration of a portfolio, with convexity or MacC instead of ModD or MacD.

1. Determine the total cash flow of the entire portfolio at each time t, and then use Convexity or MacC summation formulas.
2. Compute the weighted (by price) average of the Convexity or MacC values of the portfolio components

Example 8.4 (Convexity of a Portfolio). A portfolio contains three bonds, all of which have par value \$1000. The first has a two-year term and no coupons, the second is a three-year bond with no coupons, and the third is a two-year 6% bond with semiannual coupons. If each bond is purchased to yield 5% convertible semiannually, find the convexity of the portfolio.

Use the weighted average of convexities by price approach. The prices were already computed in a previous example (7.18):

$$P_1 = 905.95 \qquad P_2 = 862.30 \qquad P_3 = 1018.81$$

$$\text{Conv}_1 = 2(2.5)\left(\frac{P_1 v^2}{P_1}\right) \approx 4.759 \qquad \text{Conv}_2 = 3(3.5)v^2 \approx 9.994$$

$$\text{Conv}_3 = \frac{0.5(1)(30v^3) + 1(1.5)(30v^4) + 1.5(2)(30v^5) + 2(2.5)(1030)v^6}{1018.81} \approx 4.491$$

$$\text{Conv}_P = \frac{4.759P_1 + 9.994P_2 + 4.491P_3}{P_1 + P_2 + P_3} \approx \boxed{6.281}$$

Example 8.5. A three-year bond with 8% annual coupons has a redemption value of 105% of the par value and is priced to yield at an annual effective rate of 6%. Calculate the Macaulay convexity.

For convenience, pick a par value of \$100

$$\sum t^2 A_t = 8(1.06)^{-1} + 32(1.06)^{-2} + 9(113)(1.06)^{-3} \approx 889.92$$

$$\sum A_t = 8(1.06)^{-1} + 8(1.06)^{-2} + 113(1.06)^{-3} \approx 109.54$$

$$\text{MacD} = \frac{889.92}{109.54} \approx \boxed{8.124}$$

Example 8.6. We construct a portfolio of zero-coupon bonds

Term (years)	1	2	3
Price of 100 par value zero-coupon bond	94.12	88.17	82.55

Our portfolio has \$1000 par value of 1-year bonds, \$2500 par value of 2-year bonds, and \$2000 par value of 3-year bonds. Find the Macaulay convexity of the portfolio.

Use the weighted (by price) average of the Macaulay convexities method:

$$P_1 = 10(94.12) = 941.20 \qquad P_2 = 25(88.17) = 2204.25 \qquad P_3 = 20(82.55) = 1651.00$$

$$\text{MacC}_1 = 1 \qquad \text{MacC}_2 = 4 \qquad \text{MacC}_3 = 9$$

$$\text{MacC}_P = \frac{941.20 + 4(2204.25) + 9(1651.00)}{941.20 + 2204.25 + 1651.00} \approx \boxed{5.132}$$

8.3 Redington Immunization

Some definitions before proceeding:

Liability cash flows: payments that a company is required to make

Asset cash flows: payments that a company will receive from its investments

Surplus: $S = P_A - P_L$

- If interest rates fall, P_A may increase by less than P_L
- If interest rates rise, P_A may fall by more than P_L

In both cases, S declines!

Immunization: the act of protecting a surplus position from changes in the rate of interest.

Redington Immunization

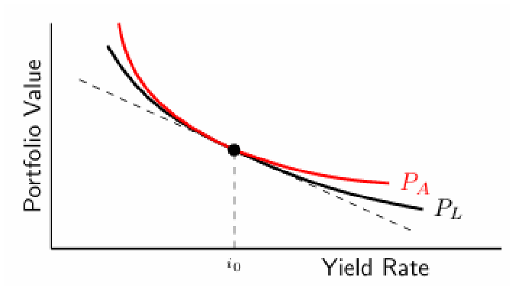
The two main assumptions for Redington Immunization to apply are (1) a flat yield curve (i.e. all spot rates are the same) and (2) there are only parallel shifts in the yield curve. If both apply, then **Redington Immunization** will protect the surplus from *small changes* in the interest rate.

Conditions for Redington Immunization

- $P_A = P_L$
- $\text{ModD}_A = \text{ModD}_L$ or equivalently $\text{MacD}_A = \text{MacD}_L$
- $\text{Convexity}_A > \text{Convexity}_L$ or equivalently $\text{MacC}_A > \text{MacC}_L$

Note: Depending on what is given, it could save time to compute MacD rather than ModD (or vice versa) or MacC rather than Convexity (or vice versa). Also, you can mix-and-match: for example using ModD for condition 2 and MacC for condition 3.

The three conditions for Redington immunization correspond to the picture



At the current yield rate i_0:

- Condition 1: The present values match (black dot)
- Condition 2: The slopes match (same tangent line)
- Condition 3: P_A curves up more quickly to either side of i_0

Alternate Redington Conditions

As implied by the picture, we can use derivatives to express the conditions:

Alternative for condition 2: $\frac{d}{di}P_a = \frac{d}{di}P_L$

Alternative for condition 3: $\frac{d^2}{di^2}P_A > \frac{d^2}{di^2}P_L$

Using the alternate conditions and moving everything to one side, we get the Redington Conditions in terms of the surplus S:

1. $S(i) = 0$
2. $S'(i) = 0$
3. $S''(i) > 0$

Mathematically, this says the surplus is at a local minimum!

Example 8.7. A company has a single liability of \$10,000 due in 5 years. The company can buy 3-year and 7-year bonds to cover this liability. The 3-year bonds have an annual coupons of 5% and the 7-year bonds are zero-coupon bonds. Both bonds have a par value of \$100 and are priced to yield 5%. How much of each bond should the company buy in order to meet the first two conditions of Redington immunization? Does this allocation meet condition 3?

1. Flat yield curve assumption: 5-year bond also yields 5%.

$$P_L = \frac{10000}{(1.05)^5} \approx 7835.26 \implies P_A = P_3 + P_7 = 7835.26$$

2. $$\text{MacD}_L = 5 \implies \text{MacD}_A = \frac{\text{MacD}_3 P_3 + 7P_7}{P_3 + P_7} = 5$$

Condition 1 rearranges to: $P_3 = 7835.26 - P_7$. Using our shortcut, $\text{MacD}_3 = \ddot{a}_{\overline{3}|0.05} \approx 2.8594$

$$\frac{(2.8594)(7835.26 - P_7) + 7P_7}{7835.26} = 5 \qquad \implies \qquad P_7 \approx 4050.66, P_3 \approx 3784.60$$

Now, we check if condition 3 is satisfied ($\text{MacC}_A > \text{MacC}_L$)

$$\text{MacC}_L = 5^2 = 25 \qquad \text{MacC}_7 = 7^2 = 49$$

$$\text{MacC}_3 = \frac{1^2 \cdot 5v + 2^2 \cdot 5v^2 + 3^2 \cdot 105v^3}{100} \approx 8.3923$$

$$\text{MacC}_A = \frac{8.3923(3784.60) + 49(4050.66)}{3784.60 + 4050.66} \approx 29.386$$

$$\boxed{29.386 > 25}$$

which verifies that Condition 3 is indeed satisfied.

Example 8.8. Infinite Life buys a 5-year zero-coupon bond that will mature for \$1000 to immunize two liability cash flows of X and Y. X occurs in 3 years and Y occurs in 10 years. Find X and Y such that conditions 1 and 2 of Redington immunization are satisfied if the effective annual rate of interest is 6%.

1. $$1000v^5 = P_A = P_L = Xv^3 + Yv^{10}$$

2. $$5 = \text{MacD}_A = \text{MacD}_L = \frac{3Xv^3 + 10Yv^{10}}{Xv^3 + Yv^{10}}$$
$$5(Xv^3 + Yv^{10}) = 3Xv^3 + 10Yv^{10} \qquad \Longrightarrow \qquad 2Xv^3 = 5Yv^{10}$$

   Divide both sides by 2 and substitute into Condition 1:

$$1000v^5 = \left(\frac{5}{2}Yv^{10}\right) + Yv^{10}$$

$$\boxed{Y \approx 382.35, X \approx 635.71}$$

If Infinite Life buys the bond and guarantees the payments of X and Y, has it achieved Redington Immunization?

Check if Condition 3 holds. We know $\text{MacC}_A = 5^2 = 25$.

$$\text{MacC}_L = \frac{3^2(635.71)v^3 + 10^2(382.35)v^{10}}{635.71v^3 + 382.35v^{10}} \approx 35$$

Since $\text{MacC}_A < \text{MacC}_L$, Redington Immunization has **not** been achieved.

8.4 Full Immunization and Dedication

Under the same assumptions (flat yield curve/constant spot rates, only parallel shifts in the yield curve), **Full Immunization** will protect the surplus from *any size of changes* in the interest rate.

The first two conditions of Redington Immunization are also required for Full Immunization. The third condition, however, requires that asset cash flows must occur both before **and** after liability cash flows.

More importantly, if a portfolio achieves full immunization, it also achieves Redington immunization. However, the converse does not hold.

Refer to Example 8.7.

1. The allocation satisfies the first two conditions of Full Immunization
2. Because asset cash flows occur before the 5-year liability (3-year coupon cash flows) and after (7-year redemption), condition 3 of Full Immunization is satisfied
3. Therefore, this allocation also reaches Redington Immunization

Rebalancing

Recall that duration changes as time passes, and it can change differently for each asset and each liability. Therefore, immunization may not last without adjustments.

Rebalancing: adjusting assets to maintain immunization.

- Rebalancing carries costs
- More frequent rebalancing vs. less useful immunization

Other weaknesses for immunization are as follows:

- We assumed a flat yield curve, which is not generally the case
- We assumed that when the yield changes, it has the same change in yield for all terms (i.e. a "parallel shift). Again, generally not true
- Liability cash flows may not be completely known − the amount or the timing or both may be unknown (ex. insurance payout)
- Assets with the needed maturities may not be available, particularly for very long terms

Dedication

Dedication is immunization by **exact matching**. More precisely, each asset cash flow is "dedicated" to a single liability. When an asset cash flow is received, it immediately pays the liability cash flow.

If interest rate changes, any effect on CF_A is the same as effect on CF_L since they occur at the same time. Thus, they are immunized against any change in interest rate and we can ignore assumptions about a flat yield curve or parallel shifts.

In theory, this is a great idea and fixes a main concern of immunization, but there are still some issues:

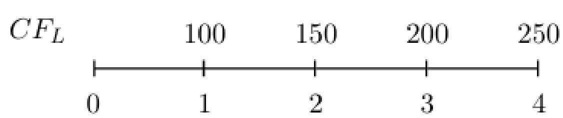
- Timing or size of liability CFs may be uncertain
- Assets to exactly match liabilities may not be available

- Might ignore higher yield opportunity
- Asset CFs may not be exactly predictable (ex. callable bonds, early repayment of loans)

Example 8.9 (Dedication Example). A company's only liability is a four-year annuity starting at \$100 one year from now and increasing by \$50 each year. The company wants to use exact matching to immunize the portfolio. The following bonds are available for purchase:

Term	1 year	2 years	3 years	4 years
Coupon Rate	5%	0%	7%	8%

All bonds pay annual coupons, are redeemable at par and can be bought in any quantity. How much par value of each bond should the company buy?



Start with the 4-year bond, since it affects all 4 cash flow times.

$$CF_4 = 250 = W + 0.08W \implies W \approx 231.48$$

The 4-year bond contributes $0.08W = 18.52$ to CF_3, CF_2, and CF_1.

$$CF_3 = 200 = 18.52 + X + 0.07X \implies X \approx 169.61$$

The 3-year bond contributes $0.07X = 11.87$ to CF_2 and CF_1

$$CF_2 = 150 = 18.52 + 11.87 + Y \implies Y \approx 119.61$$

The 2-year bond is zero-coupon, so it contributes nothing to CF_1

$$CF_1 = 100 = 18.52 + 11.87 + Z + 0.05Z \implies Z \approx 66.30$$

Example 8.10. A life insurance company has written a policy that pays \$100,000 one year after the death of the policyholder.

What makes this liability difficult to immunize?

The timing of the liability is uncertain.

- This foils exact matching, since we don't know when the asset cash flow should arrive

- The unknown timing also makes it difficult to satisfy the second condition of Full/Redington Immunization
- The term of the liability could be very long − it could be decades before the policyholder dies. It may be difficult to find assets with decades-long terms

Example 8.11. A company must pay L one year from now and $2L$ two years from now. The company buys two bonds to match the liability cash flows. The first bond is a \$1000 one-year bond. The second bond is a \$2500 two-year bond. Both bonds pay annual coupons at a rate r. Calculate L.

The two-year bond pays $2500 + 2500r$ at time 2.

$$2L = 2500 + 2500r$$

The one-year bond pays $1000 + 1000r$ at time 1, and the two-year bond pays $2500r$ at time 1.

$$L = 1000 + 3500r$$

Substitute for L in our first equation:

$$2(1000 + 3500r) = 2500 + 2500r \quad \Longrightarrow \quad r = \frac{1}{9}$$

$$L = 1000 + 3500\left(\frac{1}{9}\right) \approx \boxed{1388.89}$$

9 Important Concepts and Formulas

9.1 Measurement of Interest

Definition 9.1 (Accumulation Function). The **accumulation function** is a special case of the amount function where the original investment is one unit. $a(t)$ is the AV at time t of an original investment of k, thus

$$A(t) = k \cdot a(t)$$

Definition 9.2 (Effective Rate of Interest). The **effective rate of interest**, i, is the amount of money that one unit invested at the beginning of a period will earn during the period, where interest is paid at the end of the period.

If I_1 is the amount of interest and $A(0)$ is the principal investment, then

$$i = \frac{I_1}{A(0)}$$

Definition 9.3 (Discount Functions and Factors). Given an accumulation function $a(t)$, the **discount function**

$$a^{-1}(t) = \frac{1}{a(t)}$$

reveals the discounted cash flows from time t to time 0.

The **discount factor** d discounts the value of a cash flow from time t to time $t-1$:

$$v_t = \frac{a(t-1)}{a(t)}$$

Additionally, if $i_t = \frac{a(t)-a(t-1)}{a(t-1)}$ is the effective rate of interest, then

$$d_t = \frac{a(t)-a(t-1)}{a(t)}$$

is the **effective rate of discount**. Consequentially, $v_t = 1 - d_t$.

Corollary 9.4 (Relationship between i_t and d_t). *Let i_t and d_t be the effective rates of interest and discount, respectively. Then*

$$i_t = \frac{d_t}{1-d_t}, \quad d_t = \frac{i_t}{1+i_t}$$

Definition 9.5 (Simple Interest). As opposed to applying interest from the previous period, **simple interest** adds interest based on the interest earned in the first period. The accumulated value at time t, given interest rate i, is

$$a(t) = 1 + it$$

Definition 9.6 (Nominal Annual Rate of Interest). **Nominal rates** are interest rates in name only. The **nominal annual rate (NAR)** $i^{(m)}$ is the basic interest rate before compounding or inflating. The effective rate of interest per m-th of a year is

$$\frac{i^{(m)}}{m}$$

Definition 9.7 (Nominal Annual Rate of Discount). We say $d^{(m)}$ is the nominal rate of discount convertible m times per year and

$$\frac{d^{(m)}}{m}$$

is the effective rate of discount per m-th of a year.

Corollary 9.8 (Converting between two nominal annual rates of interest and discount).

$$i^{(n)} = n\left[\left(1+\frac{i^{(m)}}{m}\right)^{\frac{m}{n}} - 1\right]$$

$$d^{(m)} = n\left[1-\left(1-\frac{d^{(m)}}{m}\right)^{\frac{m}{n}}\right]$$

Corollary 9.9 (Converting annual effective rate to m-thly). *Let i be the annual effective rate, d be the annual effective discount rate, and m be the number of calculations per year. Then*

$$i_m = (1+i)^{\frac{1}{m}} - 1$$

$$d_m = 1 - (1-d)^{\frac{1}{m}}$$

represent the effective rate per $\frac{1}{m}$ years.

Corollary 9.10 (Equating nominal and annual effective rate of interest).

$$i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]$$

$$d^{(m)} = m\left[1 - (1-d)^{\frac{1}{m}}\right]$$

Corollary 9.11 (Equating nominal interest and discount).

$$\left(1 + \frac{i^{(n)}}{n}\right)^n = \left(1 - \frac{d^{(m)}}{m}\right)^{-m}$$

Both measure the accumulated value after n or m periods

Definition 9.12 (Force of Interest). The **force of interest** at time t, denoted as δ_t, is a measure of the intensity of interest at time t. The measurement is expressed as a rate per measure period (typically one year). If $a(t)$ is the accumulation function, then

$$\delta_t = \frac{a'(t)}{a(t)}$$

where $a(0) = 1$

Definition 9.13 (Accumulation for Force of Interest). For a constant force of interest δ

$$a(t) = e^{\delta t} \quad a(t_1, t_2) = e^{\delta(t_2 - t_1)} \quad a^{-1}(t) = e^{-\delta(t_2 - t_1)}$$

Corollary 9.14 (Converting Force of Interest to Nominal Interest Rate).

$$i^{(m)} = \frac{1}{m}\left[(1 + e^{\delta})\right]^{\frac{1}{m}} - 1$$

Corollary 9.15 (Relationship between i and δ).

$$\delta = \ln(1+i) \qquad e^{\delta} = 1+i \qquad i = e^{\delta} - 1$$

9.2 Annuities

Corollary 9.16 (Sum of Geometric Series). *Let $S = a + ar + ar^2 + \cdots + ar^{n-1}$. Then*

$$S = \frac{a - ar^n}{1-r}$$

Suppose S is an infinite series, where $n \to \infty$. Then

$$S = \frac{a}{1-r}$$

Corollary 9.17 (Annuity-immediate: Present and Accumulated value).

$$a_{\overline{n}|i} = \frac{1-v^n}{i} \qquad s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

Or,

$$s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|}$$

Corollary 9.18 (Annuity-due: Present and Accumulated value).

$$\ddot{a}_{\overline{n}|i} = \frac{1-v^n}{d} \qquad \ddot{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{d}$$

Corollary 9.19 (Relationship between Annuity-immediate and Annuity-due).

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$$

Corollary 9.20 (Present value: Perpetuity-immediate and perpetuity-due).

$$a_{\overline{\infty}|} = \frac{1}{i} \qquad \ddot{a}_{\overline{\infty}|} = \frac{1}{d}$$

Definition 9.21 (Deferred Annuity). A **deferred annuity** has payments that do not start in the first period. We use

$$\text{“}k|\text{”}$$

to defer the start of the annuity for k periods.

$$_{m|}a_{\overline{n}|} = v^m a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$$

$$_{m|}\ddot{a}_{\overline{n}|} = v^m \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|}$$

Accumulated value is expressed as

$$_{m|}s_{\overline{n}|}(1+i)^m \qquad _{m|}\ddot{s}_{\overline{n}|}(1+i)^{m-1}$$

Corollary 9.22 (Grouping Payments).

$$a_{\overline{2n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|}, \quad a_{\overline{3n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} + v^{2n} a_{\overline{n}|}$$

$$s_{\overline{3n}|} = s_{\overline{n}|}(1+i)^{2n} + s_{\overline{n}|}(1+i)^n + s_{\overline{n}|}$$

Definition 9.23 (Fission Method). If annuities are paid between time periods of equal length t, $\sum_{k=1}^{n} d^{kt}$ represents the present value of annuity payments. This geometric series is equal to

$$\frac{a_{\overline{nt}|}}{s_{\overline{t}|}}$$

Corollary 9.24 (Present Value Annuities Payable m-thly).

$$a^{(m)}_{\overline{n}|i} = \frac{1}{m} a_{\overline{m\times n}|\frac{i^{(m)}}{m}}$$

Corollary 9.25 (Fusion Method). *Serves as an alternative to annuity payable m-thly problems:*

$$a^{(m)}_{\overline{n}|i} = s_{\overline{m}|j} a_{\overline{i}|}$$

where $j = (1+i)^{\frac{1}{m}} - 1$ is the effective rate per m-th of a year and i is the effective annual rate.

Corollary 9.26 (Arithmetic Increasing Annuity-immediates). *Suppose an annuity-immediate has payments equal to its corresponding time period, then*

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} = \frac{(1+i)a_{\overline{n}|} - nv^n}{i}$$

$$(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

Corollary 9.27 (Arithmetic Increasing Annuity-dues).

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} = \frac{(1+i)a_{\overline{n}|} - nv^n}{d}$$

$$(I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d}$$

Corollary 9.28 (Arithmetic Increasing Perpetuities).

$$(Ia)_{\overline{\infty}|} = \frac{1}{id} \qquad (I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$$

Corollary 9.29 (Arithmetic Decreasing Annuity-immediates).

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

$$(Ds)_{\overline{n}|} = (Da)_{\overline{n}|}(1+i)^n$$

Corollary 9.30 (Arithmetic Decreasing Annuity-dues).

$$(D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d}$$

$$(D\ddot{s})_{\overline{n}|} = (D\ddot{a})_{\overline{n}|}(1+i)$$

Corollary 9.31 (Geometric Annuity-Immediates and Annuity-Dues).

$$(Ga)_{\overline{n}|i,r} = \frac{1 - \left(\frac{1+r}{1+i}\right)^n}{i - r}$$

$$(G\ddot{a})_{\overline{n}|i,r} = (1+i)(Ga)_{\overline{n}|i,r} = \ddot{a}_{\overline{n}|j}$$

where $j = \frac{i-r}{1+r}$

Corollary 9.32 (Accumulated Value of Geometric Annuities). *For a geometric annuity-immediate, the accumulated value over n time periods is*

$$(Gs)_{\overline{n}|i,r} = \frac{(1+i)^n - (1+r)^n}{i - r}$$

For a geometric annuity-due,

$$(G\ddot{s})_{\overline{n}|i,r} = (Ga)_{\overline{n}|i,r}(1+i)^{n+1}$$

Corollary 9.33 (Present value of Geometric Perpetuities).

$$(Ga)_{\overline{\infty}|i,r} = \begin{cases} \frac{1}{i-r} & r < i \\ \text{undefined} & r \geq i \end{cases}$$

$$(G\ddot{a})_{\overline{\infty}|i,r} = \begin{cases} \frac{1+i}{i-r} & r < i \\ \text{undefined} & r \geq i \end{cases}$$

Corollary 9.34 (Present value of geometric annuities with $i = r$). *In the special case where $r = i$, then*

1. $(Ga)_{\overline{n}|r,r} = nv_r$.
2. $(Gs)_{\overline{n}|r,r} = n(1+r)^{n-1}$.
3. $(G\ddot{a})_{\overline{n}|r,r} = n$.
4. $(G\ddot{s})_{\overline{n}|r,r} = n(1+r)^n$.

Corollary 9.35 (Present Value of Continuous Annuities).

$$\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta} \qquad \bar{a}_{\overline{\infty}|} = \frac{1}{\delta}$$

Corollary 9.36 (Accumulated Value of Continuous Annuities).

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

Corollary 9.37 (Present and accumulated value of level continuous annuities with varying rate and force of interest). *Consider an n-year continuous annuity with, at time t, a rate of payment $f(t)$ and force of interest δ_t. Then the amount of payment between time t and $t + dt = f(t)dt$. The present value and accumulated value become:*

$$PV = \int_0^n \frac{f(t)}{a(t)} dt = \int_0^n f(t) \exp\left(-\int_0^t \delta_s ds\right) dt$$

$$AV = \int_0^n f(t) a(t,n) dt = \int_0^n f(t) \exp\left(\int_t^n \delta_s ds\right) dt$$

Corollary 9.38 (Continuously Increasing Annuities). *Consider a continuously increasing annuity of n years with continuous payments. The rate of payment at time t is t. If the force of interest is δ, then the present value and accumulated value are given by*

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} \qquad (\bar{I}\bar{s})_{\overline{n}|} = \frac{\bar{s}_{\overline{n}|} - n}{\delta}$$

Corollary 9.39 (Continuously Decreasing Annuities). *Consider a continuous annuity with a rate of payment that starts at n and linearly decreases to 0 by time n. Then the rate of payment at time t is $n - t$. If the force of interest is δ, the present value and accumulated value are given by*

$$(\overline{D}\bar{a})_{\overline{n}|} = \frac{n - \bar{a}_n}{\delta} \qquad (\overline{D}\bar{s})_{\overline{n}|} = (\overline{D}\bar{a})_{\overline{n}|}(1+i)^n$$

Corollary 9.40 (Continuously Increasing Perpetuities). *Consider a continuously increasing perpetuity with continuous payments. The rate of payment at time t is t. If the force of interest is δ, then the present value is given by*

$$(\bar{I}\bar{a})_{\overline{\infty}|} = \frac{1}{\delta^2}$$

Corollary 9.41 (Double Dots Cancel). *If you have an annuity-due on both sides on both sides, then you can replace those annuity-dues with annuity-immediates*

$$X\ddot{a}_{\overline{n}|} = Y\ddot{a}_{\overline{m}|} \quad \Longrightarrow \quad Xa_{\overline{n}|} = Ya_{\overline{m}|}$$

Corollary 9.42 (Upper (m)'s Cancel). *If you have m-thly annuities on both sides of an equation, then you can replace those annuities with annual annuities:*

$$Xa^{(m)}_{\overline{p}|} = Ys^{(m)}_{\overline{q}|}$$

Corollary 9.43 ($a_{\overline{2n}|}/a_{\overline{n}|}$ Trick).

$$\frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$$

Corollary 9.44 (Pyramid Annuities).

$$PV_{pyramid\text{-}imm} = a_{\overline{n}|}\ddot{a}_{\overline{n}|} \qquad PV_{pyramid\text{-}due} = \ddot{a}_{\overline{n}|}\ddot{a}_{\overline{n}|}$$

9.3 Yield Rates and Amortization

Definition 9.45 (Net Present Value). For an n-period project and a series of cash flows (CFs) at times $0 \le t \le n$, the net present value is

$$\text{NPV} = \sum_{i=0}^{n} \frac{(\text{net CF})_t}{(1+r)^t}$$

where r is the required return, cost of capital, opportunity cost of capital or benchmark interest rate. If NPV > 0, then the company should do the project. If choosing between projects, do the project with the greatest NPV.

Definition 9.46 (Internal Rate of Return). **Internal Rate of Return (IRR)**: The rate such that the present value of the cash inflows is equal to the present value of the cash outflows. In other words, the rate such that the NPV is 0.

If IRR $>$ r, then do the project. If choosing between projects, do the project with the greatest IRR.

Corollary 9.47 (Reinvestment Rates). *If interest payments are reinvested at a different rate (rate j) than interest is earned (rate i),*

$$\text{total balance} = \underbrace{nX}_{i\ \text{fund}} + \underbrace{(iX)(Is)_{\overline{n}|j}}_{j\ \text{fund}}$$

Definition 9.48 (Yield Rates). Let y represent the yield rate:

$$\sum_{i=0}^{n} \frac{(\text{cash inflow})_t}{(1+y)^t} = \sum_{i=0}^{n} \frac{(\text{cash outflow})_t}{(1+y)^t}$$

In other words

What I have = What I demand

Synonymous with Internal Rate of Return

Definition 9.49 (Loan Terms).

- t: time (in years unless otherwise stated)
- R_t: payment amount at time t
- I_t: amount of interest paid in R_t
- P_t: amount of principal paid in R_t
- OB_t: outstanding loan balance **after** payment at time t
- L: original loan amount

Corollary 9.50 (Amortization Schedule).

1. $L = \sum_{\text{all } t} R_t v^t$
2. $I_t = OB_{t-1} \cdot i$
3. $P_t = R_t - I_t$
4. $OB_t = OB_{t-1} - P_t$

Corollary 9.51 (Premium and Discount Factors). *For a loan with level payments of R,*

$$P_t = Rv^{n-t-1}$$

Corollary 9.52 (Prospective vs. Retrospective Method For Level Payments). *There are two main ways to measure the outstanding balance of a loan with level payments. The prospective method states*

$$OB_t = Ra_{\overline{n-t|}}$$

and the retrospective method states

$$OB_t = L(1+i)^t - Rs_{\overline{t|}i}$$

Corollary 9.53 (Relationship between principal payments at different time periods). *Let t and k be two time periods in a loan payment schedule. Then*

$$P_{t+k} = P_t(1+i)^k$$

Corollary 9.54 (Original Loan Amount). *We can write the original loan amount in terms of the first principal payment.*

$$L = P_1 s_{\overline{n|}}$$

Definition 9.55 (Drop and Balloon Payments). A **drop payment** is a payment at maturity that is less than the coupon payments. Suppose Y is a drop payment, then

$$Y = OB_{n-1}(1+i)$$

A **balloon payment** is a payment at maturity that is greater than the coupon payments. Suppose X is a balloon payment, then

$$X = OB_n + L$$

9.4 Bonds

Corollary 9.56 (T-bill price).

$$\textit{Price} = \textit{Face Amount}\left[1 - \textit{ discount yield } \cdot \frac{n}{360}\right]$$

Definition 9.57 (Bond Notation).

- P: price of the bond
- F: face amount or par value. It is the unit in which the bond is issue (e.g. 100, 1000, 10000) and is used in determining the coupon payment
- r: coupon rates per payment period. They are typically payable semiannually. If given the annual nominal rate, divide by 2 for semiannual, 4 for quarterly, or 12 for monthly
- Fr: amount of each coupon payment. For instance, a 1000 par value bond with 6% semiannual coupons, then

$$Fr = 1000(0.03) = 30 \text{ every six months}$$

- C: redemption value of the bond. Unless otherwise stated, assume $C = F$
- g: modified coupon rate. It follows that

$$Fr = Cg \quad \Rightarrow \quad g = \frac{Fr}{C}$$

- i: yield rate of the bond. Also called yield-to-maturity. **Always expressed as effective rate per coupon payment period**
- n: number of coupon payments. For example, a 30-year bond with semiannual coupons implies 60 coupon payments
- K: present value of the redemption value

$$K = Cv_i^n$$

Corollary 9.58 (Bond Pricing Formulas).

1. *Basic/Frank's Formula:*

$$P = Fra_{\overline{n}|i} + Cv^n = Fra_{\overline{n}|i} + K$$

2. *Premium/Discount Formula:*

$$P = C + (Fr - Ci)a_{\overline{n}|i} = C + (Cg - Ci)a_{\overline{n}|i}$$

3. *Makeham's Formula:*

$$P = K + \frac{g}{i}(C - K)$$

Corollary 9.59 (Premiums vs. Discounts (P and C)). *If $P > C$, then bond is said to be bought at a premium. If $P < C$, then bond is said to be bought at a discount. If $P = C$, then the bought is said to be bought at par.*

Corollary 9.60 (Premiums vs. Discounts (Fr and Ci)). *If $Fr > Ci$, then the bond is bought at a premium*

$$P - C = (Fr - Ci)a_{\overline{n}|} = \text{ premium}$$

If $Fr < Ci$, then the bond is bought at a discount

$$C - P = (Ci - Fr)a_{\overline{n}|} = \text{ discount}$$

If $Fr = Ci$, then bond is bought at par

Corollary 9.61 (Premiums vs. Discounts (i and g)). *If $i < g$, then bond is said to be bought at a premium. If $g < i$, then bond is said to be bought at a discount. If $g = i$, then the bought is said to be bought at par.*

Corollary 9.62 (Bond Amortization).

- *Book Value:*

$$BV_t = Fra_{\overline{n-t}|i} + Cv_i^{n-t} \text{ (prospective)}$$
$$= P(1+i)^t - Frs_{\overline{t}|i} \text{ (retrospective)}$$
$$= BV_{t-1} - P_t$$

- *Interest:*

$$I_t = BV_{t-1} \cdot i$$

- *Write-up/Write-down:*

$$P_t = Fr - I_t$$
$$= (Fr - Ci)v^{n-t+1}$$

Just like for loans, each principal payment grows by a factor of $1+i$

$$P_{t+k} = P_t(1+i)^k$$

Definition 9.63 (Write-ups and Write-downs). If the redemption value is greater than the book value, then the principal payments are **write-downs**. If the redemption value is less than the book value, then the principal payments are **write-ups**.

Corollary 9.64 (Yield of a Callable Bond).

- *Yield rate depends on when the bond is redeemed*
- *Bond issuer may call bond at worst time for investor*
- *For a callable bond with a constant redemption amount, the worst possible yield for investor occurs at the earliest call date for a premium bond, and at the latest call date for a discount bond*

Corollary 9.65 (Price of a Callable Bond). *If the bond is bought at a discount, then the bond is priced using the yield rate at maturity. If the bond is bought at a discount, then the bond is priced using the yield rate at the earliest callable time period.*

9.5 Interest Rate Behavior

Definition 9.66 (Yield Curves). Depicts how current interest rates vary depending on the term of the investment

Definition 9.67 (Spot Rates). The t-year **spot rate**, s_t, is the yield rate for a zero-coupon bond with maturity in t years. s_t is the rate of interest between time 0 and t. The present value of CF_t is

$$\frac{CF_t}{(1+s_t)^t}$$

Definition 9.68 (Forward Rates). **Forward rates** are rates of interest that can be earned on an investment at a future point in time. Denoted as $f_{t,t+m}$

Corollary 9.69 (Relationship between Spot and Forward Rates).

$$(1+s_t)^t(1+f_{t,t+m})^m = (1+s_{t+m})^{t+m}$$

$$f_{t,t+m} = \left[\frac{(1+s_{t+m})^{t+m}}{(1+s_t)^t}\right]^{\frac{1}{m}} - 1$$

$$s_t = \left[\prod_{t=0}^{t-1}(1+f_{t,t+1})\right]^{\frac{1}{t}} - 1$$

Definition 9.70 (Real Rate of Interest). Known as the inflation-adjusted return. If r, i, and i' are the inflation rate, annual rate of interest, and real rate of return, then

$$(1+i) = (1+r)(1+i')$$

$$i' = \frac{i-r}{1+r}$$

9.6 Duration, Convexity, and Immunization

Corollary 9.71 (Price as a Function of Yield).

$$P = \sum_{\text{all } t} CF_t(1+i)^{-t}$$

where CF_t is the cash flow at time t

Definition 9.72 (Price Sensitivity). **Price sensitivity** is the percentage change in the price of an asset from a shift in the yield curve.

$$\text{price sensitivity } = \frac{P_{i_1} - P_{i_0}}{P_{i_0}} = \%\Delta P$$

Cash flows further into the future are more sensitive to changes in interest rate, and price change will be greater for a yield shift in smaller rates (ex. 2% to 3%) than it will be for a yield shift in larger rates (10% to 11%).

Definition 9.73 ((Macaulay) Duration). The **(Macaulay) duration** of a single cash flow is the length of time, in years, until the cash flow occurs. If there are multiple cash flows, then the Macaulay duration is the weighted average of times cash flows occur, weighted by their present value/price.

$$\text{MacD} = \frac{\sum t A_t}{P} = -\frac{\frac{d}{d\delta} P_\delta}{P_\delta}$$

Corollary 9.74 (Relationship between Macaulay Duration, coupon payments, and yield).

1. *A bond with higher coupons has a shorter MacD*
2. *A bond with a higher yield has a shorter MacD*

Corollary 9.75 (Macaulay Duration of a Par Bond).

$$\textit{MacD of a Par Bond } = \ddot{a}^{(m)}_{\overline{i}|} = \frac{1}{m}\ddot{a}_{\overline{n\times m}|j}$$

where $j = \frac{i^{(m)}}{m}$ and $m =$ number of coupon payments per year. Par value does not matter; choose freely if not given.

Corollary 9.76 (Macaulay Duration of a Level Annuity-Immediate). *The duration of a n-year level annuity with m payments per year is*

$$\frac{(Ia)_{\overline{m\times n}|}}{ma_{\overline{m\times n}|}}$$

Corollary 9.77 (Macaulay Duration of a Zero-Coupon Bond). *The Macaulay Duration of a zero-coupon bond is the time to maturity (time length of the bond).*

Corollary 9.78 (Duration of a Portfolio).

$$MacD_P = \frac{P_1 MacD_1 + P_2 MacD_2 + \cdots + P_n MacD_n}{P_1 + P_2 + \cdots + P_n}$$

Definition 9.79 (Passage of Time). Duration decreases over time with jumps up after each cash flow.

Definition 9.80 (Modified Duration). Measures the average cash-weighted term to maturity of a bond.

$$\text{ModD} = \frac{v \sum t A_t}{P}$$

where v is the discount factor for one period. The estimated price is always smaller than the actual price using modified duration.

Corollary 9.81 (Measuring Price and Sensitivity with Modified Duration).

$$\%\Delta P \approx -(\Delta i) \cdot ModD$$

The change in price approximation can be thought of as an order-one Taylor Series

$$\Delta P \approx -(\Delta i) \cdot ModD \cdot P_{i_0}$$

$$P_{i_1} \approx P_{i_0} - (\Delta i) \cdot ModD \cdot P_{i_0}$$

Corollary 9.82 (Relationship between Modified and Macaulay Duration).

$$ModD = vMacD$$

where v = discount factor for one period. As the number of payments approach infinity, these two quantities are equivalent:

$$MacD = \lim_{m \to \infty} ModD$$

Corollary 9.83 (Approximating Price using Duration).

$$P_{i_1} \approx P_{i_0} \left(\frac{1+i_0}{1+i_1} \right)^{MacD}$$

$$\% \Delta P \approx \left(\frac{1+i_0}{1+i_1} \right)^{MacD} - 1$$

where

- P_{i_0} *is the price at the original effective annual yield rate of* i_0
- P_{i_1} *is the price at the new effective annual yield rate of* i_1

Interest rates must be in the form of effective annual rates since MacD is in years. If all future cashflows are positive, approximating the new price using MacD will do a better job than using ModD

Definition 9.84 (Convexity).

$$\text{Convexity } = \frac{\frac{d^2}{di^2} P_{i_0}}{P_{i_0}} = \frac{\sum t \left(t + \frac{1}{m} \right) A_t v^2}{P}$$

with t in years and v as the per-period discount factor.

Corollary 9.85 (Approximating Price using Convexity). *The change in price approximation can be thought of as an order-two Taylor Series*

$$\%\Delta P \approx -(\Delta i) ModD + \frac{(\Delta i)^2}{2} \cdot \; Convexity$$

$$\Delta P \approx -P_{i_0}(\Delta i) ModD + \frac{P_{i_0}(\Delta i)^2}{2} \cdot \; Convexity$$

Definition 9.86 (Macaulay Convexity).

$$\text{MacC} = \lim_{m\to\infty} \text{Convexity} = \frac{\sum t^2 A_t}{P}$$

Corollary 9.87 (Convexity of a Portfolio).

$$Conv_P = \frac{P_1 Conv_1 + P_2 Conv_2 + \cdots + P_n Conv_n}{P_1 + P_2 + \cdots + P_n}$$

Definition 9.88 (Redington Immunization). **Redington Immunization** protects the surplus from *small changes* in the interest rate used to discount assets and liabilities. Assumes that yields do not vary by maturity and there are only parallel shifts in the yield curve (typically these aren't true). The following conditions must be met to achieve Redington Immunization:

1. $PV_A = PV_L$ or $P_A = P_L$ or $S(i) = 0$
2. $\text{ModD}_A = \text{ModD}_L$ or $P'_A = P'_L$ or $S'(i) = 0$
3. $\text{Conv}_A > \text{Conv}_L$ or $P''_A = P''_L$ or $S''(i) > 0$

Can use MacD, MacC in (2) or (3) as convenient

Definition 9.89 (Full Immunization).

- **Full Immunization** protects the surplus for interest rate changes of any size.
- Follows the same assumptions as Redington Immunization (flat yield curve and only parallel shifts)
- The first two conditions are the same as Redington Immunization
- The third condition states that an asset cash flow must occur before and after each liability cash flow

Corollary 9.90 (Immunization Weaknesses).

- *We assumed a flat yield curve, which is not generally the case*
- *We assumed that when the yield changes, it has the same change in yield for all terms (i.e. a "parallel shift). Again, generally not true*
- *Liability cash flows may not be completely known – the amount or the timing or both may be unknown (ex. insurance payout)*
- *Assets with the needed maturities may not be available, particularly for very long terms*

Definition 9.91 (Dedication). **Dedication** is immunization by **exact matching**. More precisely, each asset cash flow is "dedicated" to a single liability. When an asset cash flow is received, it immediately pays the liability cash flow. This way, surplus is not affected by changes in interest rates. However, there are still weaknesses to this approach:

- Timing or size of liability CFs may be uncertain
- Assets to exactly match liabilities may not be available
- Might ignore higher yield opportunity
- Asset CFs may not be exactly predictable (ex. callable bonds, early repayment of loans)

10 References

Most of the material and diagrams were taken from
https://www.theinfiniteactuary.com/