1 Introduction to Options and Stocks

At every moment, stocks are always susceptible to change. While these fluctuations are often subtle, most quantitative traders cannot predict any outcome, even dramatic changes in stock prices. Here, we only provide a simple heuristic for thinking about option pricing, which hopefully gets us one foot in the door into optimally managing stocks. We begin with some definitions that underline the relationship between consumers and sellers in the market:

Definition 2.1: Options

Options are financial derivatives that give the buyer the *right* the right to buy or sell an underlying asset at a pre-agreed price at a future date. Options are purchased at the price the shorter sets, or the *premium*. One cannot use the option until the premium is paid in full.

1. *Longing* an option gives you the right to buy the asset and *shorting*, or selling, an option means you are selling rights to the counterparty with an obligation to fulfill.
2. *Call* options allows the buyer the right to *purchase* the asset at the pre-agreed price any time before the expiration date; *put* options allows the buyer the right to sell the asset at the pre-agreed price any time before the expiration date.

The pre-agreed price is referred to as the *strike price.*

Keep in mind that when we long an option, we are not obligated to buy or sell the asset at any point. In fact, if the price of the stock changes such that we are at a loss by buying or selling, it is recommended to not take action and let the option go through. When this happens, we say the option *expires worthless*.

Combining the notion of longing/shorting and calls/puts gives us four different types of option pricing:

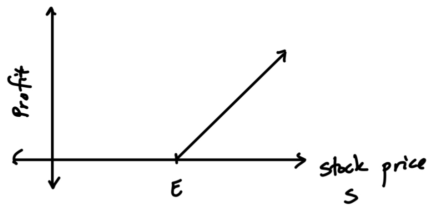
Definition 2.2: Long Calls

Longing a call is the right to *exercise* a purchase of the stock at price E. Only exercise your right if the stock price exceeds the strike price ($S > E$). The maximum possible loss is the premium.

$$C_E(S,t) = \max(0, S - E).$$

Generally, long calls are the safest option to invest in as your possible returns are boundless. If the stock price falls below the strike price, you let the option expire worthless and only lose the premium you paid for. We can construct a graph (porfolio value vs. stock price) to better understand the relationship.

Here the x-axis and y-axis depict the price of the stock price and portfolio value at the expiration date, respectively. So long as $S > E$, you are making profit. If $E < S$, the option expires worthless. The graph assumes there is no premium.



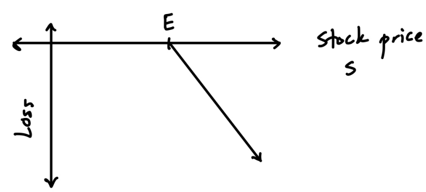
Definition 2.3: Short Calls

Shorting a call is selling the call option to a buyer. You are obligated to purchase the price at the current stock price if the buyer exercises their right to do so.

$$-C_E(S,t) = -\max(0, S-E).$$

People who short calls are typically very knowledgeable in quantitative trading as the potential for loss is significant. They hope that the stock price decreases so that they can repurchase the stock at a lower price (since the option will expire worthless). Otherwise, if it goes above E, the buyer can choose to buy the stock S at price E and they must buy at the current price S. This can yield unlimited loss for the seller. The maximum profit is the premium.

One can easily observe that the portfolio value for a short call is the reflection of a long call along the x-axis.



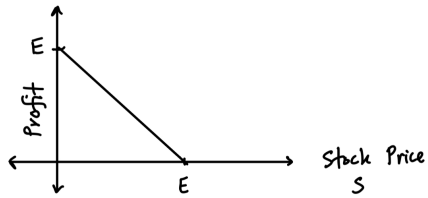
Definition 2.4: Long Puts

Longing a put option means you have the obligation to sell the stock at the strike price.

$$P_E = \max(0, E-S).$$

One expects the stock price to go down and profit when the stock price is less than the stirke price (because you own the right to sell a cheaper stock for a greater price). The maximum profit is the strike price minus the premium and the maximum loss is the premium.

Judging by the graph, we want to let the option expire worthless if $S \geq E$. For $S < E$, we can sell the stock at the strike price = profit!



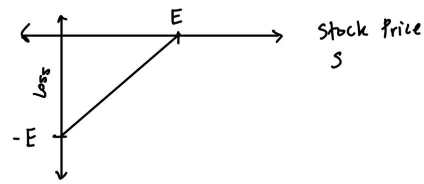
Definition 2.5: Short Puts

Shorting a put means you are obligated to buy the stock at the strike price if the buyer exercises that right.

$$-P_E = -\max(0, E - S)$$

You are expecting the stock price to go up and profit when the stock price is greater than the strike price (so the option expires worthless). On the other hand, if it is less tan the strike price, then the buyer will exercise the option and you must buy the stock at the underlying strike price. The maximum loss is the cost of the strike price and maximum profit is the premium.

As with long and short calls, the graph of a short put is the reflection of a long put along the y-axis.

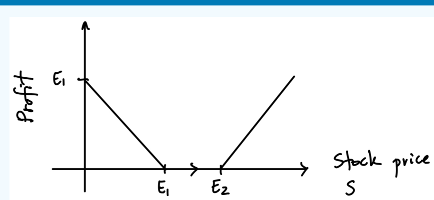


We have overviewed the four main types of options pricing. In general, we can express any portfolio as a linear combination of these. Before we proceed, let us go over general notation. For puts and calls, we express them as C_E, P_E, where E is the underlying strike price of the option. We often define portfolios for an asset as one with calls and puts for multiple strike prices. The next example will highlight this type of portfolio:

Definition 2.6: Long Strangles

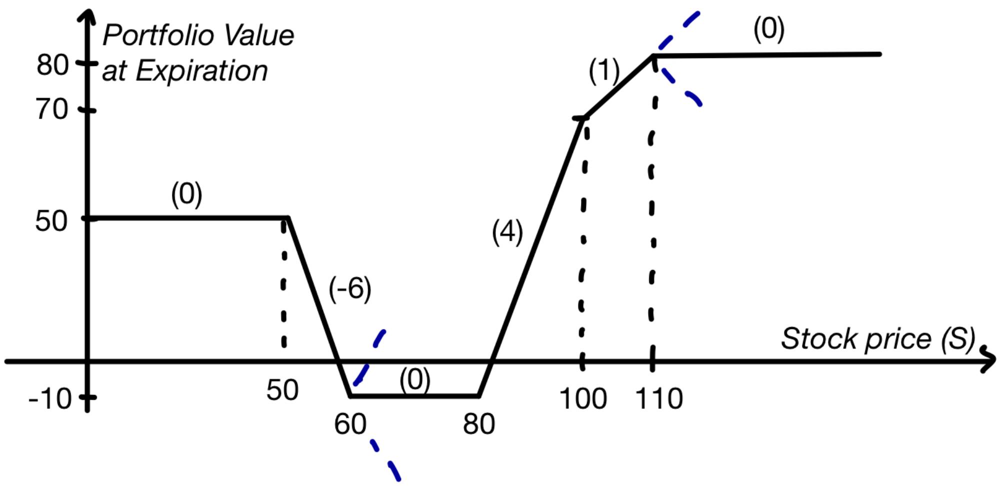
Long strangles is the linear combination of longing a put at strike price E_1 and longing a call at a different strike price E_2.

$$\text{Portfolio Value} = P_{E_1} + C_{E_2}.$$



Long Strangles are another safe portfolio; our loss is restricted to the two premiums we paid for. Investors profit from long strangles when the stock falls below E_1 or above E_2.

We continue with a more complicated example. Below is a portfolio with multiple strike prices and the corresponding portfolio value at expiration:



First, we will describe the portfolio using only call options. The numbers in parentheses indicate the slopes of each line in our piecewise-defined graph. This is important in keeping track of the changes in slope as the stock price increases (or decreases).

1. From $0 \leq S \leq 50$, the slope is zero. No call options are bought in this interval.
2. From $0 \leq S \leq 60$, the slope is -6. At this point the graph resembles that of shorting 6 calls with a strike price of \$50. For every dollar the stock rises above \$50, we are losing \$6. If our portfolio ended here, we could potentially suffer an infinite loss, as seen from the blue dashed line pointing downward.
3. A wise investor would not let this happen. Therefore, they buying 6 call options at the strike price \$60. While we are still losing money from shorting C_{50}, we are gaining money from C_{60} if the stock price rises above \$60. Hence, the \$6 lost for every dollar increase in stock from shorting the previous option is offset by the \$6 gained for every dollar increase from calling this new option. The slopes cancel out as seen by the blue dashed lines.
4. Here, the portfolio value goes up, which is only achieved by longing 4 call options, this time at strike price \$80. We earn \$4 for each dollar increase in stock.
5. From $100 \leq S \leq 110$., the slope starts to flatten, which implies that we are shorting some call options. Since the slope decreases from 4 to 1, we are shorting 3 call options at the strike price \$100.
6. The slope returns to zero, so we are shorting a call option at strike price \$110. Notice again how the slopes cancel out. This concludes the portfolio.

To describe this using notation, one would write

$$\text{Portfolio Value} = -6C_{50} + 6C_{60} + 4C_{80} - 3C_{100} - C_{110}.$$

Similarly, we can describe the same portfolio using only puts. However, we work right to left as puts derive value when stock prices go down. After going through each step, we actually find that the two portfolios look very similar. We describe the portfolio with only puts as

$$\text{Portfolio Value} = -P_{110} - 3P_{100} + 4P_{80} + 6P_{60} - 6P_{50}.$$

The similarity is heavily tied to a principle called *Put-Call Parity*, which we will define later.

While we can describe portfolios with only calls and only puts, using a combination of calls and puts will *optimize* the portfolio, or be the cheapest. First, we look at the following definition:

Definition 2.7: Intrinsic Value of Options

The intrinsic value of an option is measured relative to underlying stock price S at any time. We say an option is

- *Out of the money* if the option is worth something. This is the case if $S > E$ for calls and $S < E$ for puts.
- *At the money* $S = E$ for both calls and puts.
- *In the money* if it is rendered worthless. This is the case if $S < E$ for calls and $S > E$ for puts.

When optimizing our portfolio, we obviously want only out of the money options. For example, if today's stock is \$70, we only use puts for options with strike prices less than \$70 and only calls for options with strike prices greater than \$70. Hence, we have the optimized portfolio

$$\text{Portfolio Value} = -6P_{50} + 6P_{60} + 4C_{80} - 3C_{100} - C_{110}.$$

A word of caution: The portfolio we analyzed is by no means a representation of those that we observe in the real world. In fact, they are far more complicated and composed of many more options and stocks are always changing; they hardly ever remain flat.

2 Formulas

Expected Value and Variance of a Continuous Random Variable

Let $X(x)$ be a random payoff variable with $f(x)$ as the PDF. Then,

$$E[X] = \int_{-\infty}^{\infty} X(x)f(x)dx, \qquad \text{Var}(X) = \int_{-\infty}^{\infty} (X(x))^2 f(x)dx - (E[X])^2 .$$

If X follows a normal distribution $N \sim (\mu, \sigma^2)$, then it has mean $\mu = E[X]$ and variance $\sigma^2 = \text{Var}(X)$.

Recall that $E[X]$ is the weight average of all possible values of X, weighted by their probability density $f(x)$. Variance is the spread of the random variable around its mean.

Put-Call Parity

Let S be the price of the stock, E be the strike price of the put and call options $P_E(S,t), C_E(S,t)$. If no arbitrage exists, then

$$S + P_E(S,t) = C_E(S,t) + Ee^{-r(T-t)}.$$

$T - t$ is the time remaining until expiration.

Arbitrage exists if = is replaced with an inequality.

If LHS > RHS, then you sell $S + P$, buy C, and deposit $Ee^{-r(T-t)}$ in the bank.

If LHS < RHS, then you buy $S + P$, sell C, and borrow $Ee^{-r(T-t)}$ from the bank.

Then, pocket the remaining sum as an arbitrage profit.