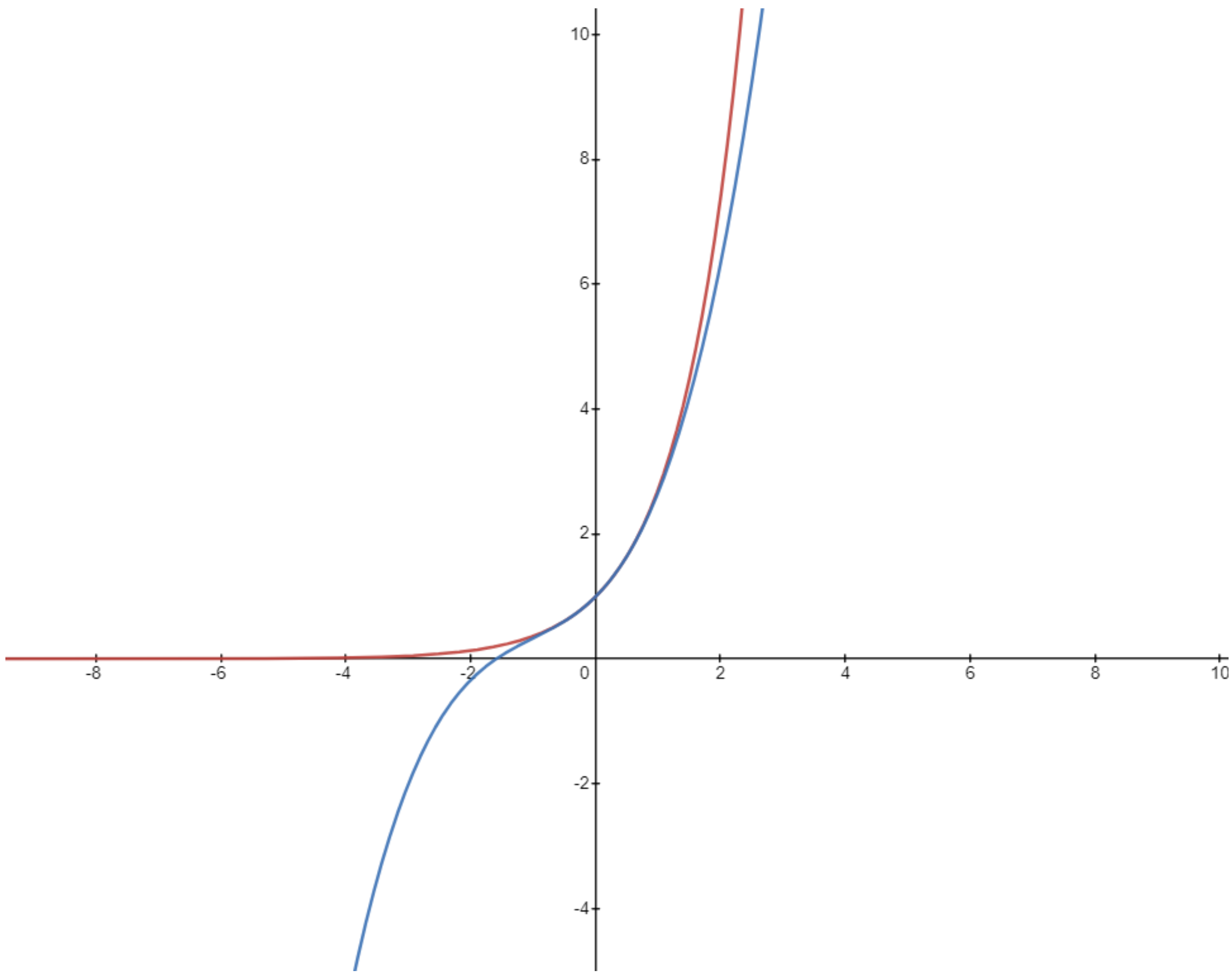
Proof of Euler's Formula and Identity



Ryan Hausner

Euler's Formula

Euler's Formula is given as: $e^{ix} = \cos x + i \sin x$

Euler's Identity is given as: $e^{i\pi} + 1 = 0$

In this paper we are going to prove both Euler's Formula and Identity using Maclaurin Series.

The Maclaurin Series is a way to approximate transcendental functions using polynomials.

Maclaurin Series

The Maclaurin Series is defined as:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For Maclaurin Series, a is equal to 0 because the approximation is centered at 0. We can now rewrite the formula for the Maclaurin Series as:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

The three series we need for this proof are for e^x, $\sin x$, and $\cos x$.

e^x:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$\sin x$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$\cos x$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Using this information, we can now write our power series for e^{ix}.

e^{ix}:

$$\sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

Now, we can write out the first few terms of this power series:

$$\sum_{n=0}^{\infty}\frac{(ix)^n}{n!} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

Substituting: $i^2 = -1$:

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} + \frac{x^6}{6!} + \cdots$$

Factoring out i:

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots)$$

Here you'll notice the first group of terms without i is the power series for $\cos x$ and the second set of terms with i is the power series for $\sin x$.

Therefore,

$$e^{ix} = \cos x + i \sin x$$

Now, if we plug in π:

$e^{i\pi} = \cos(\pi) + i \sin(\pi)$

$e^{i\pi} = -1 + 0$

$e^{i\pi} + 1 = 0$