Probability

Statistics

Experiments, Outcomes, Events and Sample Spaces

- **Experiment:** An experiment is any activity from which results are obtained. A random experiment is one in which the outcomes, or results, cannot be predicted with certainty.
- **Examples:**

1. Flip a coin
2. Flip a coin 3 times
3. Roll a die

- **Trial:** A physical action , the result of which cannot be predetermined
- **Basic Outcome:** A possible outcome of the experiment
- **Sample Space:** The set of all possible outcomes of an experiment

Examples

- **Example #1:** A company has offices in six cities, San Diego, Los Angeles, San Francisco, Denver, Paris, and London. A new employee will be randomly assigned to work in on of these offices.

- **Outcomes:** San Diego, Los Angeles, San Francisco, Denver, Paris, London.

- **Sample Space:**. San Diego, Los Angeles, San Francisco, Denver, Paris, London.

Examples

- **Example #2:** A random sample of size two is to be selected from the list of six cities, San Diego, Los Angeles, San Francisco, Denver, Paris, and London.
- **Outcomes:** San Diego (SD), Los Angeles (LA), San Francisco (SF), Denver (D), Paris (P), London (L).
- **Sample Space:** (SD, SD), (SD, LA), (SD, SF), (SD, D), (SD, P), (SD, L)
  (LA, SD), (LA, LA), (LA, SF), (LA, D), (LA, P), (LA, L)
  (SF, SD), (SF, LA), (SF, SF), (SF, D), (SF, P), (SF, L)
  (D, SD), (D, LA), (D, SF), (D, D), (D, P), (D, L)
  (P, SD), (P, LA), (P, SF), (P, D), (P, P), (P, L)
  (L, SD), (L, LA), (L, SF), (L, D), (L, P), (L, L)

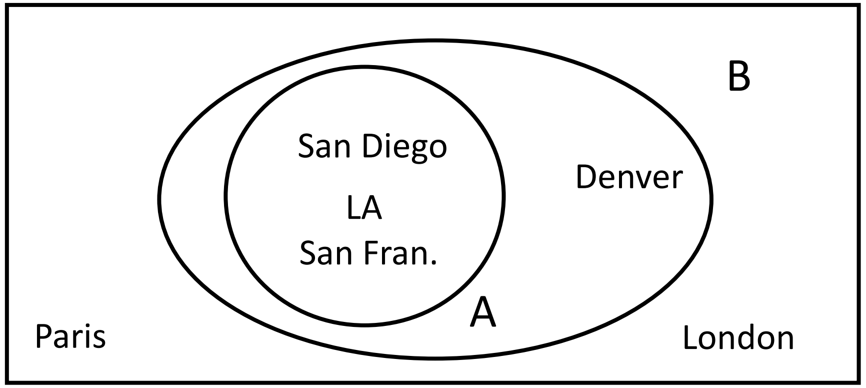
Events and Venn Diagrams

- **Events:** Collections of basic outcomes from the sample space. We say that an event occurs if any one of the basic outcomes in the event occurs.
- **Venn Diagram:** Graphical representation of sample space and events

- **Example #1** (cont.):

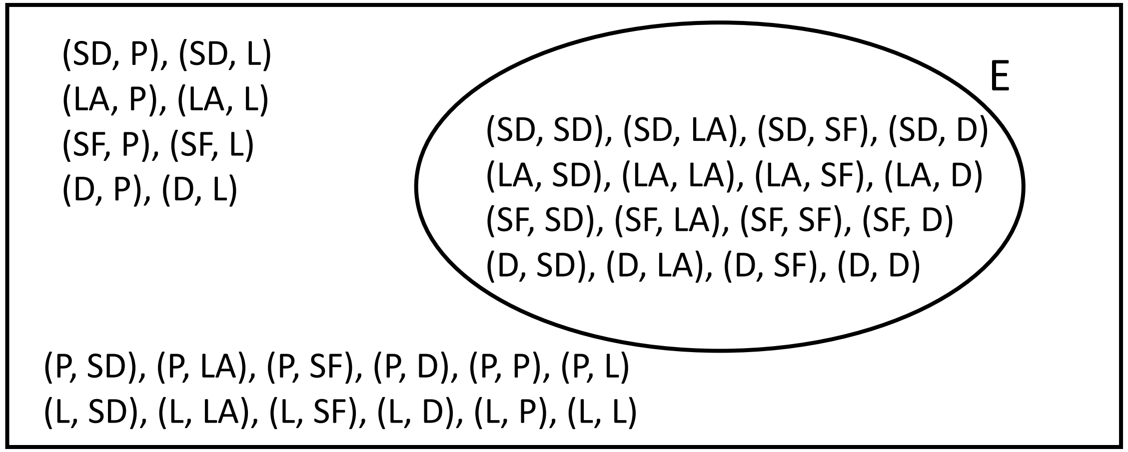
1. Let B be the event that the city selected is in the US

2. Let A be the event that the city selected is in California



- **Example #2:** A random sample of size two is to be selected from the list of six cities, San Diego, Los Angeles, San Francisco, Denver, Paris, and London.

- **Let E be the event that both cities selected are in the US**
- **E=** (SD, SD), (SD, LA), (SD, SF), (SD, D)
  (LA, SD), (LA, LA), (LA, SF), (LA, D)
  (SF, SD), (SF, LA), (SF, SF), (SF, D)
  (D, SD), (D, LA), (D, SF), (D, D)
- **Sample Space and Venn Diagram:**



Probabilities of an Event

Probability of an event P(E): "Chance" that an event will occur

- Must lie between 0 and 1
- "0" implies that the event will not occur
- "1" implies that the event will occur

- P(E) = $\frac{\text{Number of Outcomes in Event E}}{\text{Total Number of Possible Outcomes}}$

Types of Probability

- **Relative Frequency Approach:** Relative frequency of an event occurring in an infinitely large number of trials

Time Period	Number of Male Live Births	Total Number of Live Births	Relative Frequency of Live Male Birth
1965	1927.054	3760.358	0.51247
1965-1969	9219.202	17989.360	0.51248
1965-1974	17857.860	34832.050	0.51268

Types of Probability

- **Equally-likely Approach:** If an experiment must result in n equally likely outcomes, then each possible outcome must have probability 1/n of occurring.

Examples:

1. Roll a fair die
2. Select a simple random sample of size 2 from a population

- **Subjective Probability:** A number between 0 and 1 that reflects a person's degree of belief that an event will occur

- Example: Predictions for rain

Probability Questions

- **Problem 1:** The number on a license plate is any digit between 0 and 9. What is the probability that the first digit is a 3?

1/10

What is the probability that the first digit is less than 4?

4/10 = 2/5

- **Problem 2:** Two dice are rolled, find the probability that the sum is

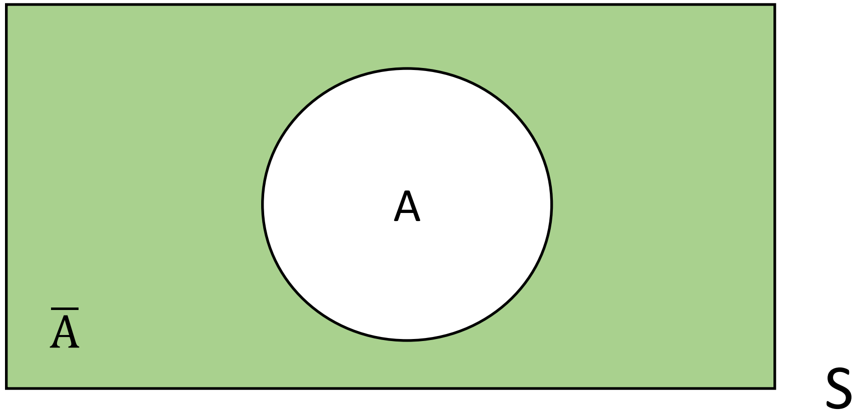
a) equal to 1 (0) (no outcomes are equal to 1, 0/36)

b) equal to 4 (3/36) (outcomes of event = (1,3), (2,2), (3,1))

c) less than 13 (1) (all outcomes are less than 13, 36/36)

Law of Complements:

"If A is an event, then the complement of A, denoted by $\bar{A}$ (or A') represents the event composed of all basic outcomes in S that do not belong to A."



$P(\bar{A}) = 1 - P(A)$

- **Example:** If the probability of getting a "working" computer is 0.9,

What is the probability of getting a defective computer?

1-0.9=0.1

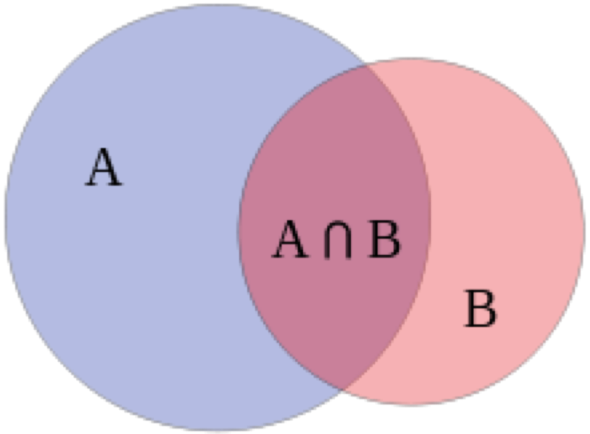
Unions and Intersections of Two Events

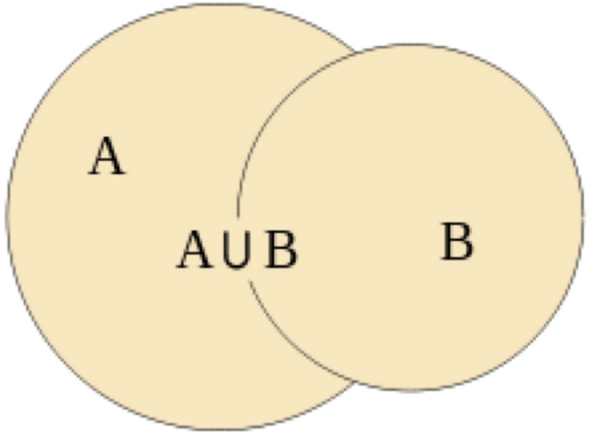
- **Unions of Two Events:**

“If A and B are events, then the union of A and B, denoted by A∪B, represents the event composed of all basic outcomes in A **or** B.”

- **Intersections of Two Events**

“If A and B are events, then the intersection of A and B, denoted by A∩B, represents the event composed of all basic outcomes in A **and** B.”

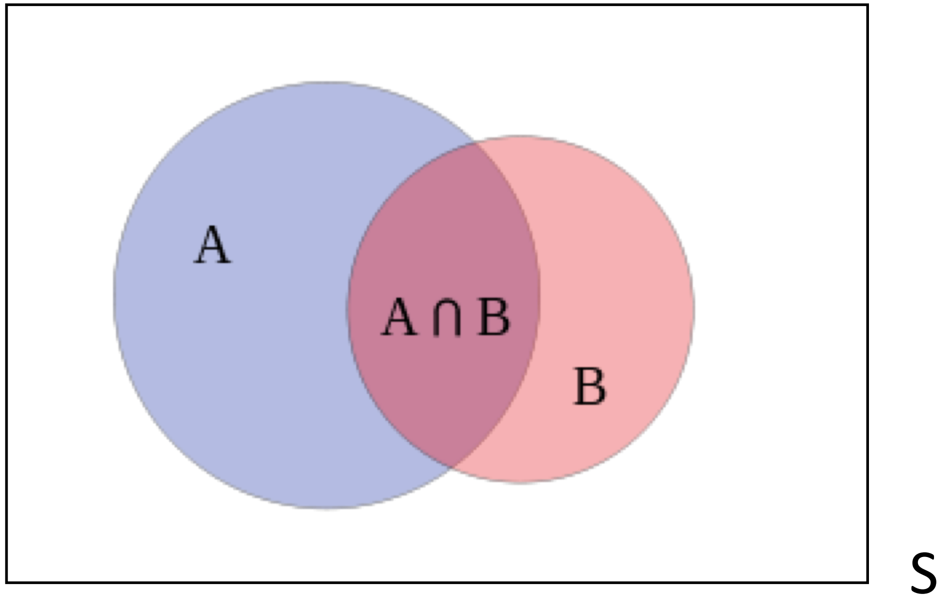




Additive Law of Probability

- Let A and B be two events in a sample space S. The probability of the union of A and B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Examples

- **Example 1:** At a university, all first-year science students must take chemistry and math. Suppose 15% fail chemistry, 12% fail math, and 5% fail both. If a first-year student is selected at random, what is the probability that student selected failed at least one of the courses?

  $P(C \cup M) = P(C) + P(M) – P(C \cap M) = 0.15+0.12-0.05 = 0.22$

- Example 2: If a fair dice is rolled, find the probability of that number that turns up is odd or divisible by 5?

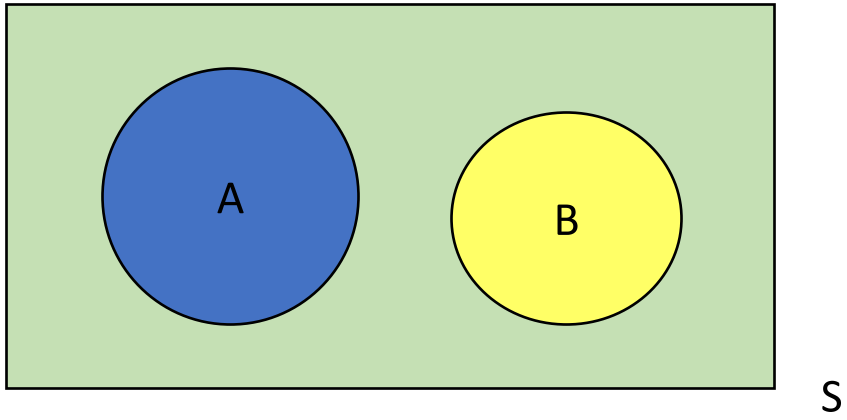
  $P(odd \cup 5) = P(odd) + P(5) – P(odd \cap 5) = 3/6 + 1/6 – 1/6 = 3/6 = 0.5$

Mutually Exclusive Events

- **Mutually Exclusive Events**: Events that have no basic outcomes in common, or equivalently, their intersection is the empty set.

- Let A and B be two events in a sample space S. The probability of the union of two mutually exclusive events A and B is

$$P(A \cup B) = P(A) + P(B)$$



Multiplication Rule and Independent Events

- Multiplication Rule for Independent Events: Let A and B be two independent events, then

$$P(A \cap B) = P(A) \times P(B)$$

- Therefore two events are independent if $P(A \cap B) = P(A) \times P(B)$

Examples:

1. Flip a coin twice. What is the probability of observing two heads? Are these events independent?

2. Flip a coin twice. What is the probability of getting a head and then a tail? A tail and then a head? One head?

3. Three computers are ordered. If the probability of getting a “working” computer is 0.9, what is the probability that all three are “working” ?

Examples:

1. Flip a coin twice. What is the probability of observing two heads? Are these events independent?

$P(1=H \text{ and } 2=H) = P(1=H \cap 2=H) = 1/4$

(sample space: (H,H), (H,T), (T,H), (T,T))

$P(1=H) = 2/4$, $P(2=H) = 2/4$,

Therefore: $P(1=H \cap 2=H) = P(1=H) \times P(2=H)$, events are independent.

2. Flip a coin twice. What is the probability of getting a head and then a tail? A tail and then a head? One head?

3. Three computers are ordered. If the probability of getting a "working" computer is 0.9, what is the probability that all three are "working" ?

Examples:

1. Flip a coin twice. What is the probability of observing two heads? Are these events independent?
2. Flip a coin twice. What is the probability of getting a head and then a tail? A tail and then a head? One head?

$P(1{=}H \text{ and } 2{=}T) = P(1{=}H \cap 2{=}T) = \frac{1}{4}$

(sample space: (H,H), (H,T), (T,H), (T,T))

$P(1{=}T \text{ and } 2{=}H) = P(1{=}T \cap 2{=}H) = \frac{1}{4}$

$P(\text{One head}) = P(1{=}H \cap 2{=}T) + P(1{=}T \cap 2{=}H) = 0.5$

3. Three computers are ordered. If the probability of getting a "working" computer is 0.9, what is the probability that all three are "working" ?

Examples:

1. Flip a coin twice. What is the probability of observing two heads? Are these events independent?

2. Flip a coin twice. What is the probability of getting a head and then a tail? A tail and then a head? One head?

3. Three computers are ordered. If the probability of getting a "working" computer is 0.9, what is the probability that all three are "working" ?

$P(3 \text{ working computers}) = P(C1=W \cap C2=W \cap C3=W)$

$= P(C1=W) \text{ and } P(C2=W) \text{ and } P(C3=W) = 0.9 \times 0.9 \times 0.9 = 0.729$

Conditional Probability

- For any two events A and B, where the $P(B) \neq 0$, the **conditional probability** of A given that B has occurred ("probability of A given B") is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example of Conditional Probability

- Roll two dice, so the sample space S has 36 elements or outcomes.

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

A = {sum of the results of the two dice in s is even}
B = {first die is even}
C = {second die is > 5}
Find $P(B|C)$? $P(A|B)$? $P(A|C)$?

Example of Conditional Probability

- Roll two dice, so the sample space S has 36 elements or outcomes.

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

A = {sum of the results of the two dice in s is even} => P(A) = 18/36
B = {first die is even} => P(B) = 18/36
C = {second die is > 5} => P(C) = 6/36
Find P(B|C)? P(A|B)? P(A|C)?

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{3/36}{6/36} = \frac{3}{6} = 0.5$$

Example of Conditional Probability

- Roll two dice, so the sample space S has 36 elements or outcomes.

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

A = {sum of the results of the two dice in s is even} => P(A) = 18/36
B = {first die is even} => P(B) = 18/36
C = {second die is > 5} => P(C) = 6/36
Find P(B|C)? P(A|B)? P(A|C)?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{9/36}{18/36} = \frac{9}{18} = 0.5$$

Example of Conditional Probability

- Roll two dice, so the sample space S has 36 elements or outcomes.

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
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(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

A = {sum of the results of the two dice in s is even} => P(A) = 18/36
B = {first die is even} => P(B) = 18/36
C = {second die is > 5} => P(C) = 6/36
Find P(B|C)? P(A|B)? P(A|C)?

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{3/36}{6/36} = \frac{3}{6} = 0.5$$

Example

A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease (H). Moreover, 312 of the 937 men had at least one parent who suffered from heart disease (D), and, of these 312 men, 102 died from causes related to heart disease.

$$P(H) = \frac{210}{937},\ P(D) = \frac{312}{937},\ P(H|D) = \frac{102}{312}$$

Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease. (i.e. $P(H|\overline{D})$)

- P(H) = $\frac{210}{937}$, P(D) = $\frac{312}{937}$, P(H|D) = $\frac{102}{312}$

P(H|$\overline{\mathrm{D}}$) $= \frac{\mathrm{P}(\mathrm{H}\cap\overline{\mathrm{D}})}{\mathrm{P}(\overline{\mathrm{D}})}$

P($\overline{\mathrm{D}}$) = 1 - $\frac{312}{937}$ = $\frac{625}{937}$

$\mathrm{P}(\mathrm{H} \cap \overline{\mathrm{D}})$=> 102 out of 312 died of heart disease and had parent(s) with heart disease

210-102 = 108 had died of heart disease and had no parents with heart disease

$\mathrm{P}(\mathrm{H} \cap \overline{\mathrm{D}})$= $\frac{108}{937}$

P(H|$\overline{\mathrm{D}}$) $= \frac{\mathrm{P}(\mathrm{H}\cap\overline{\mathrm{D}})}{\mathrm{P}(\overline{\mathrm{D}})}$

P(H|$\overline{\mathrm{D}}$) $= \frac{\frac{108}{937}}{\frac{625}{937}}$ = $\frac{108}{625}$

Law of Total Probability

- If B_1, B_2, B_3, … is a partition of the sample space S, then for any event A we have

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A \mid B_i)P(B_i)$$

Example:

Three bags that each contain 100 marbles:

- Bag 1 has 75 red and 25 blue marbles;
- Bag 2 has 60 red and 40 blue marbles;
- Bag 3 has 45 red and 55 blue marbles.

I choose one of the bags at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

$P(R|B1)=0.75$,

$P(R|B2)=0.60$,

$P(R|B3)=0.45$

$P(R)=P(R|B1)P(B1)+P(R|B2)P(B2)+P(R|B3)P(B3)$

$=(0.75)1/3+(0.60)1/3+(0.45)1/3$

$=0.60$

OR

$P(R\cap B1) =75/300$

$P(R\cap B2) =60/300$

$P(R\cap B3) =45/300$

$P(R) = P(R\cap B1) + P(R\cap B2) + P(R\cap B3)$

$= 75/300 + 60/300+45/300$

$=180/300$

$=0.6$

Bayes Theorem

- Bayes Theorem : $P(A|B) = \frac{P(B|A)}{P(B)} P(A)$

- **Example**: Interested in finding out a patient's probability of having liver disease if they are an alcoholic. Past data tells you that 10% of patients entering your clinic have liver disease. P(A) = 0.10. Five percent of the clinic's patients are alcoholics. You might also know that among those patients diagnosed with liver disease, 7% are alcoholics.

$P(A) = 0.10$, $P(B) = 0.05$, $P(B|A)=0.07$

$P(A|B) = \frac{P(B|A)}{P(B)} P(A) = (0.07 * 0.1)/0.05 = 0.14$

Example:

- A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease. Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

P(P|D) = 0.95, P(P| $\overline{\mathrm{D}}$) = 0.005, P(D) =0.01, want P(D|P)

P(D|P) = $\frac{\mathrm{P(P|D)}}{\mathrm{P(P)}}$ P(D)

P(P) = P(P|D)P(D) + P(P| $\overline{\mathrm{D}}$) P($\overline{\mathrm{D}}$)= 0.95$\times$0.01+0.005 $\times$ 0.99 = 0.01445

P(D|P) = $\frac{0.95}{0.01445}$ 0.01 = 0.6574394

P($\overline{\mathrm{D}}$|P) = 1 - 0.6574394 = 0.3425606 (= $\frac{0.005}{0.01445}$ 0.99)

Expected value

- The Expected Value of an event X is the sum of all the outcomes in the event multiplied by the probability of that outcome

$$E(X) = \sum_i x_i P(X = x_i)$$

- $E(X)$ is a weighted average of the values in the range of X.

- **Example:** What is the expected value of a dice?

E(X) = 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6)

= 3.5