Topic 3.1
Relation

TMA1201 Discrete Structures & Probability
Faculty of Computing & Informatics
Multimedia University

What you will learn in this lecture:

- Cartesian products
- Binary Relations
- Inverse of Relations
- Composition of Relations
- Equivalence Relation
- Partial Ordering Relation

Introduction

- Relationship between elements of sets occur in many contexts. For example a relationship of a person and their country origin which can be described as "x is a citizen of y", where x is from the set of people and y is from the set of countries.

- ***Relationships between elements of sets*** are represented using the structure called a ***relation***, which is just a ***subset*** of the ***Cartesian product of the sets***.

Cartesian products

Definition: **Ordered *n*-tuples**

The ordered *n*-tuple (a_1, a_2,..., a_n) is the ordered collection that has a_1 as its first element, a_2 as its second element,...,and a_n as its *n*th element.

Example of ordered 2-tuples which also called ordered pairs:

(1, 2), (c, d)

Note:

- The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.
- Hence (a, b) and (b, a) are not equal unless $a = b$.

Cartesian products

Definition: **Cartesian product of *A* and *B***

- Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, $$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Example 1

What is the Cartesian product of $A = \{1,2\}$ and $B = \{a, b, c\}$

Solution

The Cartesian product $A \times B$ is

$$A \times B =$$

Cartesian products

Definition: **Cartesian product of the sets $\boldsymbol{A_1, A_2, \cdots, A_n}$**

The Cartesian product of the sets $A_1, A_2, \cdots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered *n*-tuples $(a_1, a_2, \cdots, a_n)$, where a_i belongs to A_i for *i* = 1, 2,...,*n*. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) \big| a_i \in A_i \text{ for } i = 1,2, \cdots n\}$$

Example 2

What is the Cartesian product of $A \times B \times C$, where *A*={Ali, Sam}, *B* = {TMA1201, TMA1101}, and *C* = {P, F}

$A \times B \times C =$

Representation of Relation

- A relation *R* is a subset of Cartesian product.
- A relation that expresses relationship between two sets are known as binary relation.

Definition: **Binary Relation**

Let *A* and *B* be sets. A binary relation from *A* to *B* is a subset of *A* × *B*.

- For $x \in A, y \in B$, we use notation xRy to denotes $(x, y) \in R$ and $x \not R y$ denotes $(x, y) \notin R$.
- For example A= {a, b, c} and B = {1, 2} and R = {(a, 1), (b, 2), (c, 1)}
- Hence $aR1$ and $a \not R 2$

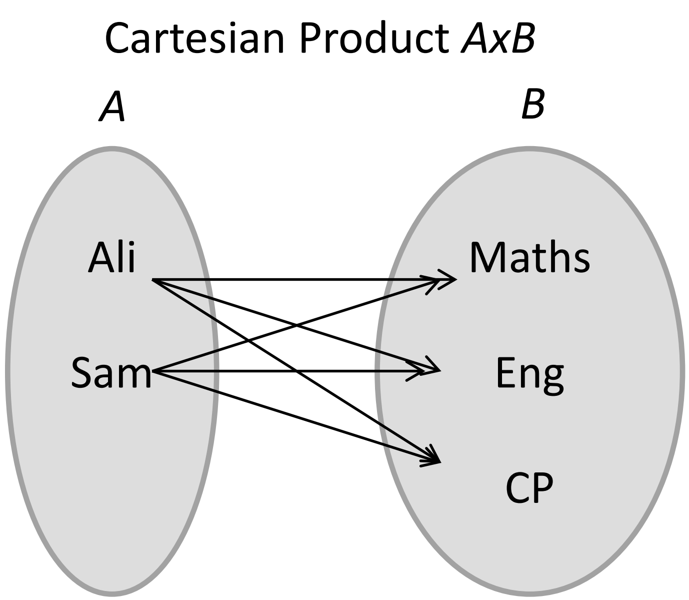
Arrow Diagram

Suppose R is a relation from a set A to a set B.
An arrow diagram can be used to illustrate R.

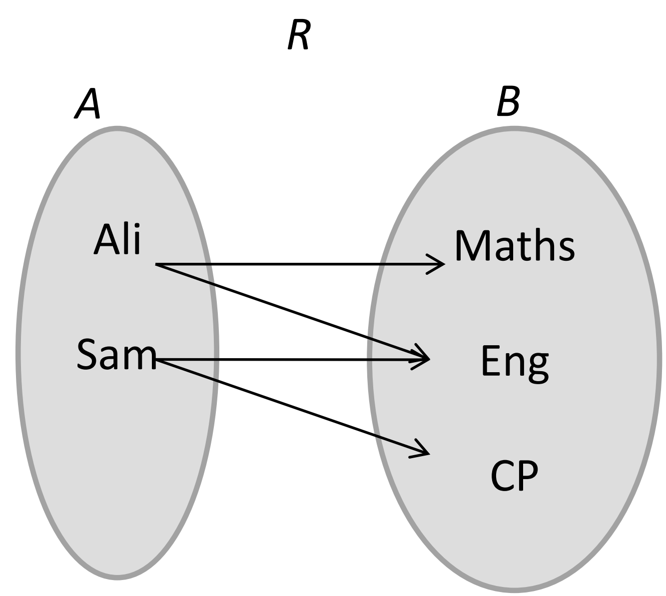
Example 3

Suppose A be set of students, B be subjects offered by FCI and R denotes the enrollment of students from A to subjects B. For simplicity let A = {Ali, Sam}, B = {Maths, Eng, CP} and

R={(Ali, Maths), (Ali Eng), (Sam, Eng), (Sam, CP)}.

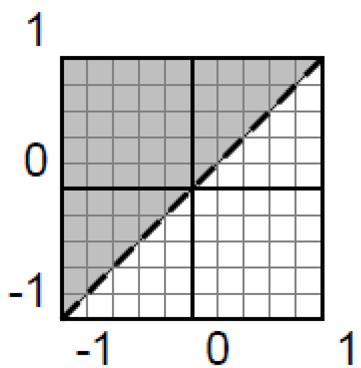


$\supseteq$



Example 4

Let S be the relation "<" on the real number set.
The set of S thus consists of all the pairs (x, y) with $x < y$. It is the following shaded subset of the *xy* plane.



(The dashed line indicates that the points where $x = y$ are omitted.)

Note: order is important in this relation, e.g., 1S2 but $2 \not S 1$

An Example of *n*-ary Relation:

Assuming that MMU students' attendance database contains the following:

A_1 be a set of positive integers for student ID

A_2 a set of alphabetic character strings for name

A_3 a set of numeric character strings for date of attendance for a subject

A_4 a set of logical character T for "attended" and F for "absence" for indicating attendance

Assume that a student, "Tanpa Nama" wishes to retrieve his attendance detail for a particular date. The data that will be retrieved can be in the following form:

(100000001, Tanpa Nama, 250612, T)

Student ID

Name

Date

Attendance Indicator

For this particular example, the relation is known as a quaternary relation R on the sets A_1, A_2, A_3, $A_{4,}$ where:

$(a_1, a_2, a_3, a_4) \in R$
$\Leftrightarrow$ a student with student ID number a_1, named a_2, date of attendance a_3, with attendance info a_4.

The elements of set R above may contain those of below:

{(100000011, No Name, 250612, F),
(100000111, Namaewa Iie, 250612, T), …}

Inverse of a Relation

If R is a relation from set A to set B, the inverse of a relation denoted as R^{-1} can be defined to be a relation from set B to set A.

R^{-1} can be found by interchanging the elements of all ordered pairs.

Its mathematical definition is as below:

$$\boxed{\text{For all } x \in A \text{ and } y \in B, \quad (y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R.}$$

Note: A and B can be a similar set or a different set

Example 5

Let A ={2,3,4} and B ={2,6,8} and let the relation

R= {(x, y) ∈ $A \times B$ | y is divisible by x}.

a) State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}.

b) Describe R^{-1} in words.

Example 6

List the elements of below:

Let R_2 be the relation define on the set of integers and $R_2 = \{(n, m) \mid n < m\}$

List down the elements of R_2

$R_2 =$

List down the elements of R_2^{-1}

$R_2^{-1} =$

Write the definition of R_2^{-1}

Composition of Relations

- Objects in a set can comprise of more than one relation.
- The composition of relations is denoted by the symbol $\circ$
- $S \circ R = \{(a, c) \mid$ there exist $b \in B$ where aRb and $bSc\}$ is called the composite of $R \subseteq A \times B$ and $S \subseteq B \times C$.

Example 7

Let R and S be relations on the set {0,1,2,3,4} and

$$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$$

$$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$$

Find $R \circ S$ and $S \circ R$.

Solution:

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$$

$$R \circ S = \{(3, 1), (3, 4), (3, 3), (4, 1), (4, 4)\}$$

How will these looks like in the form of arrow diagram?

Example 8

Given R = { (1, 3), (2, 6), (3, 2), (2,7)} on the set A={1, 2, 3, 4, 5, 6, 7}.

Find R^2, R^3, R^4, R^5, and R^6.

<u>Solution</u>:

$R^2 = R \circ R = \{(1, 2), (3, 6), (3, 7)\}$

$R^3 = R^2 \circ R =$

$R^4 = R^3 \circ R =$

$R^5 = R^4 \circ R =$

$R^6 = R^5 \circ R =$

Relations on One Set

Definition: **A relation on one set**

A relation on a set A is a relation from A to A.

(relationships between elements of a single set)

Example 9

Let A be the set {0, 1, 2, 3, 4}. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b \land a, b \in A\}$?

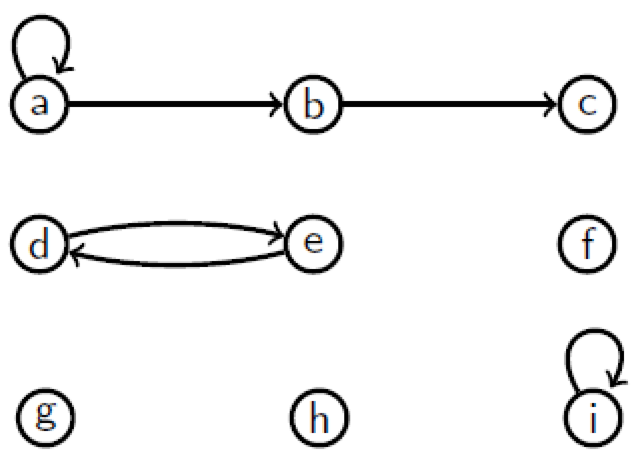
$R =$

Relations on One Set

Relation on one set can be represented by **directed graphs**, or **digraphs**.

Example 10

Let R be the relation on set A={a, b, c, d, e, f, g, h, i} and relation R = {(a, a), (a, b), (b, c), (d, e), (e, d), (i, i)}. The arrow diagram of R that indicates transitions between elements in A is as the following:



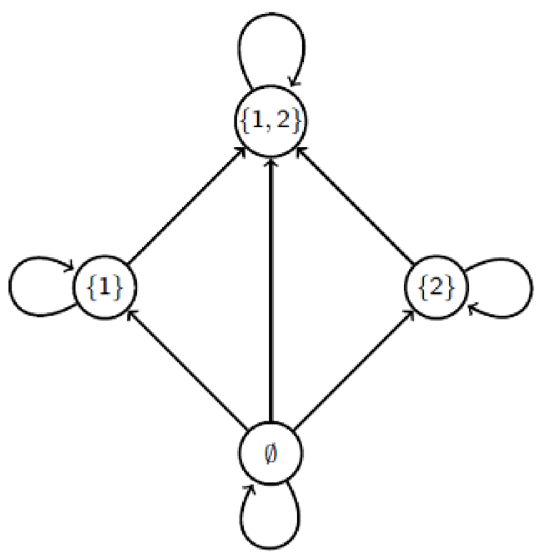
Example 11

Write the set of ordered pairs for the relation "$\subseteq$" on P({1, 2}) and draw an arrow diagram for it.

<u>Solution</u>:

P({1, 2}) = {∅, {1}, {2}, {1, 2}}.

R= { (∅, ∅), (∅, {1}), (∅, {2}), (∅, {1, 2}), ({1}, {1}), ({1}, {1, 2}), ({2}, {2}), ({2}, {1, 2}), ({1, 2}, {1, 2}) }



Properties of relation

A relation R defined on a set A has these properties,

REFLEXIVE

if and only if, for all $a \in A$, $(a, a) \in R$

SYMMETRIC

if and only if, for all $a, b \in A$, if $(a, b) \in R$ then $(b, a) \in R$

ANTISYMMETRIC

if and only if, for all $a, b \in A$ and $a \neq b$, if $(a, b) \in R$ then $(b, a) \notin R$

TRANSITIVE

if and only if, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Properties of relation

A relation *R* defined on a set A has these properties,

REFLEXIVE

$\leftrightarrow\ \forall a \in A, (a, a) \in R$

SYMMETRIC

$\leftrightarrow\ \forall a, b \in A, \qquad (a, b) \in R \rightarrow (b, a) \in R$

ANTISYMMETRIC

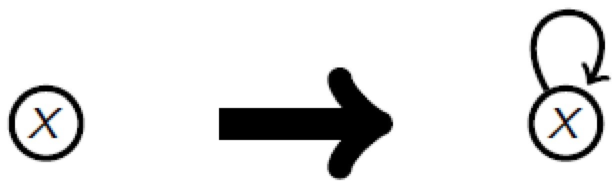
$\leftrightarrow\ \forall a, b \in A$ and $a \neq b, \qquad (a, b) \in R \rightarrow (b, a) \notin R$

TRANSITIVE

$\leftrightarrow\ \forall a, b, c \in A, \qquad (a, b) \in R$ and $(b, c) \in R \rightarrow (a, c) \in R$

Reflexive (For a binary relation R on a set A)

Think about it as...



To prove R is reflexive, show that “For all $x \in$ A, xRx”.

To prove R is not reflexive, show that “There is an $x \in$ A such that $x \not R x$.

Example 12

Consider the following relations on A ={1, 2, 3}

$R_1 = \{(1,1),(2,2),(3,3)\}$

$R_2 = \{(1,1)\ ,(1,2),(2,1),(1,3)\}$

$R_3 = \{(1,1),(1,2),(1,3),(2,2),(2,1),(3,3),(3,1)\}$

$R_4 = \{(1,1),(1,2),(2,2)\ ,(2,3),(3,3)\}$

$R_5 = \{(1,1),(1,2),(2,2)\ ,(2,3),(3,1)\}$

$R_6 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

Which of these relations are reflexive? Explain your answer.

<u>Sample solution:</u>

R_1, R_3, R_4 and R_6 are **reflexive** because, $\forall a \in A$, they contain all pairs in the form (a,a), namely (1,1), (2,2),(3,3).

R_2 is **irreflexive** because $2 \in A$, but $(2,2) \notin R_2$ (counter example).

R_3 is **irreflexive** because_____

Example 13

Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of these relations are reflexive?

Sample solution:

R_1 is **reflexive** because, $\forall y \in Z, y \leq y$ is always true, i.e., $(y, y) \in R_1$.

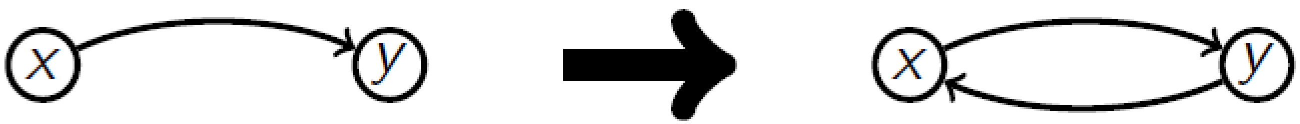
R_3 is **reflexive** because

R_4 is **reflexive** because

R_2 is **irreflexive** because, 2>2 is false (counter example), hence $(2,2) \notin R_2$, i.e., $\forall y \in Z, y > y \text{ is false}$.

Symmetric (For a binary relation R on a set A)

Think about it as...



To prove R is symmetric, show that “For all *x*, *y* $\in$ A, if *x*R*y* then *y*R*x*”.

To prove R is not symmetric, show that “There are some *x*, *y* $\in$ A such that *x*R*y* but $y \not R x$.

Example 14

Consider the following relations on A ={1, 2, 3}

$R_1 = \{(1,1), (2,2), (3,3)\}$

$R_2 = \{(1,1)\ , (1,2), (2,1), (1,3)\}$

$R_3 = \{(1,1), (1,2), (1,3), (2,2), (2,1), (3,3), (3,1)\}$

$R_4 = \{(1,1), (1,2), (2,2)\ , (2,3), (3,3)\}$

$R_5 = \{(1,1), (1,2), (2,2)\ , (2,3), (3,1)\}$

$R_6 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Which of these relations are symmetric? Explain your answer.

Sample solution:

R_1, R_3 and R_6 are **symmetric** because, $\forall x, y \in A$, if xRy then yRx.

R_2 is **not symmetric** because, (1,3) $\in R_2$ but (3,1) $\notin R_2$ (counter example).

R_4 is **not symmetric** because…

R_5 is **not symmetric** because...

Example 15

Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of the relations are symmetric?

Sample solution:

R_3 is **symmetric** because, $\forall x, y \in Z$, if $x = y$ then $y = x$ is true, or, if $x = -y$ then $y = -x$ is true.

R_4 is **symmetric** because…

R_6 is **symmetric** because…

R_1 is not **symmetric** because $2 \leq 3$ is true, $i.e.$ $(2,3) \in R_1$, but $3 \leq 2$ is false, $i.e.$ $(2,3) \notin R_1$

R_2 is not **symmetric** because …

R_5 is not **symmetric** because …

Antisymmetric
(For a binary relation R on a set A)

Think about it as...

To prove R is antisymmetric, show that "For all x, $y \in$ A, if xRy and yRx then $x = y$".

To prove R is not antisymmetric, show that "There are some x, $y \in$ A such that $x \neq y$, xRy and yRx.
and

Example 16

Consider the following relations on A ={1, 2, 3}

$R_1 = \{(1,1), (2,2), (3,3)\}$

$R_2 = \{(1,1)\ , (1,2), (2,1), (1,3)\}$

$R_3 = \{(1,1), (1,2), (1,3), (2,2), (2,1), (3,3), (3,1)\}$

$R_4 = \{(1,1), (1,2), (2,2)\ , (2,3), (3,3)\}$

$R_5 = \{(1,1), (1,2), (2,2)\ , (2,3), (3,1)\}$

$R_6 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Which of these relations are antisymmetric? Explain your answer.

Sample solution:

R_1, R_4 **and** R_5 are **antisymmetric** because, $\forall x, y \in A,$ if xRy and yRx then $y = x$.

R_2 is not **antisymmetric** because $1 \neq 2$ and $(1,2) \in R_2$, but $(2,1) \in R_2$.

R_3 is not **antisymmetric** because …

R_6 is not **antisymmetric** because …

Example 17

Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of the relations are antisymmetric?

<u>Sample solution:</u>

R_1 is **antisymmetric** because….

R_2 is **antisymmetric** because…

R_4 is **antisymmetric** because…

R_5 is **antisymmetric** because …

R_3 is not **antisymmetric** because …

R_6 is not **antisymmetric** because …

Transitive (For a binary relation R on a set A)

Think about it as...

To prove R is transitive, show that "For all x, y, $z \in$ A, if xRy and yRz then xRz".

To prove R is not transitive, show that "There are some x, y, $z \in$ A such that xRy and yRz but $x \not\mathrel{R} z$".

Example 18

Consider the following relations on A ={1, 2, 3}

$R_1 = \{(1,1), (2,2), (3,3)\}$

$R_2 = \{(1,1)\ , (1,2), (2,1), (1,3)\}$

$R_3 = \{(1,1), (1,2), (1,3), (2,2), (2,1), (3,3), (3,1)\}$

$R_4 = \{(1,1), (1,2), (2,2)\ , (2,3), (3,3)\}$

$R_5 = \{(1,1), (1,2), (2,2)\ , (2,3), (3,1)\}$

$R_6 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Which of these relations are transitive? Explain your answer.

Sample solution:

R_1 and R_6 are **transitive** because, $\forall x, y, z \in A$, if xRy and yRz then xRz.

R_2 is **not transitive**because...

R_3 is **not transitive** because...

R_4 is **not transitive** because ...

R_5 is **not transitive** because (1,2) , (2,3) ∈ R_5, but (1,3) $\notin R_5$.

Example 19

Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of the relations are transitive?

Sample solution:

R_1 is **transitive** because, $\forall x, y, z \in Z$, if $x \leq y$ and $y \leq z$ then $x \leq z$, i.e., $(x, y), (y, z) \in R_1 \rightarrow (x, z) \in R_1$

R_2 is **transitive**because…

R_3 is **transitive** because…

R_4 is **transitive** because …

R_5 is **not transitive** because

R_6 is **not transitive** because 3+0 $\leq$3 and 0+2 $\leq$3 but (counter example) 3+2 $\not\leq$ 3,
i.e., (3,0),(0,2) $\in R_6$, but (3,2) $\notin R_6$.

Equivalence Relation

A relation is said to be an equivalence relation if it has the properties of,

reflexive, symmetric, and transitive.

Example 20:

Let *R* be the relation on Z^+ such that *aRb* if and only if *a* divides *b. I*s *R* reflexive, symmetric, and/or transitive? Is *R* an equivalence relation?

<u>Solution</u>:

Reflexive? Yes -> *a* divides *a* for all $a \in Z^+$.

Symmetric? No -> 3 divides 6 but 6 does not divide 3, so 3R6 but $6 \not R 3$.

Transitive? Yes -> if *a* divides *b* and *b* divides *c*, then *a* divides *c* for all *a*, *b*, *c* $\in Z^+$.

R is not equivalent relation since it is not symmetric

Example 21

Is the relation “5 divides $(x – y)$” an equivalence relation on Z?

Equivalence Class

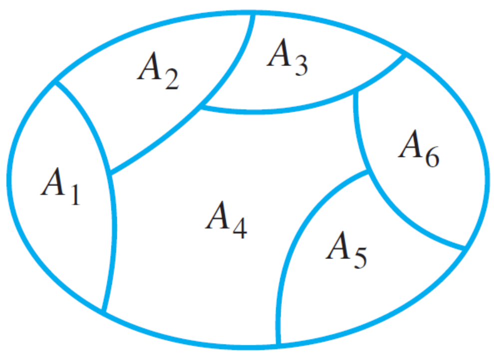
Giving an equivalence relation on a nonempty set is equivalent to partitioning the set into many subsets $\{A_1, A_2, \ldots\}$ with the following requirements:

$A_i \neq \varnothing$ for each i

A_i's are mutually disjoint, i.e. $A_i \cap A_j = \varnothing$ for $i \neq j$, and

$A = \cup A_n$.

Example 22:



$A_i \cap A_j = \emptyset$, whenever $i \neq j$
$A_i \cup A_2 \cup \cdots \cup A_6 = A$

$A_1, A_2, A_3, A_4, A_5, A_6$ are called the equivalence classes.

Equivalence Class

There are 5 equivalence classes for the equivalence relation 5 divides $x - y$ on the set Z:

{..., -10, -5, 0, 5, 10, ...}

{..., -9, -4, 1, 6, 11, ...}

{..., -8, -3, 2, 7, 12, ...}

{..., -7, -2, 3, 8, 13, ...}

{..., -6, -1, 4, 9, 14, ...}

Partial Ordering Relation

A relation is said to be a partial order if it has the properties of,

reflexive, anti-symmetric, and transitive.

The ordered pair objects in *R* is called a **poset** .

A partial order set can indicate that for certain pairs of elements in the set, one of the element is of higher order than the other.

An example is the pre-requisite subjects that is imposed on a student's study plan at a university.

Example 23

Given,

$R = \{(i, j) \mid i \text{ divides } j\}$,

where $i, j \in Z^+$

Determine if R is a partial order.

Solution:

Check the followings:

is it reflexive?

is it antisymmetric?

is it transitive?

Make sure that you are able to explain the answer based on the definition of the properties!

Summary

We have learnt the following concepts related to relations:

- Relations occur in a set and may involve two or more sets.
- Relations are a subset of Cartesian products.
- Relations have inverse and can form composition.
- Relations may have none/any/all of the following properties: reflexive, symmetric, antisymmetric, transitive.
- An equivalence relation has the properties reflexive, symmetric and transitive.
- A partial order has the properties reflexive, antisymmetric and transitive.

Exercise 1

What is the difference between {a, b, c} and (a, b, c)? Explain.

Exercise 2

Give the set of ordered pairs for the relation ≤ on the set {1, 2, 3, 4, 5, 6}. Draw an arrow diagram to illustrate the relationship between elements in the set.

Exercise 3

Given,

R_1 = {(x, y) | x and y are human beings and x is taller than y}

R_2 = {(x, y) | x and y are human beings, and both have the same height}

$R_3 = \{(a, b) \mid |a - b| \leq 4\}$, where $a, b \in Z$

$R_4 = \{(a, b) \mid (a - b)$ is a multiple of $7\}$, where $a, b \in Z$

1) Determine if each of these are equivalence relation. Justify your answer.
2) Determine if each of these are partial order. Justify your answer.