Topic 2
Set Theory

TMA1201 Discrete Structures & Probability
Faculty of Computing & Informatics
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What you will learn in this lecture:

- Sets and its Structures
- Set Elements
- Cardinality of a Set
- Infinite Sets
- Equality of a Set
- Subset of a Set
- Power Set of a Set
- Operations on Sets
- Set Identities

Definition of a Set

A set is an ***unordered*** collection of objects/elements/members.

Sets are usually denoted as follows

$S = \{a, b, c\}$

$T = \{x \mid P(x)\}$

Set S is defined by listing its elements.
Set T is defined by property of its element *x*.

This is read as "*x* such that P(*x*)"

$x \in S$ informs us that object *x* is an element (a member) of the set S.

This is read as "*x* is an element of S"

$N = \{0, 1, 2, 3, 4, \ldots\}$, so $3 \in N$

$T = \{x \mid x$ is a letter of the English alphabets$\}$, so $a \in T$

Definition of a Set

Sets are ***unordered***.

Thus {a, b, c} = {a, c, b} = {b, a, c} = {b, c, a} = {c, a, b} = {c, b, a}.

All of those sets above are equal!

All elements are ***distinct*** (unequal); multiple listings make no difference!

Suppose a = b, then

T = {a, b, c} = {a, c} = {b, c} ={a, a, b, a, b, c, c, c, c}.

If *x* is NOT an object in a set, then we use the symbol, $\notin$

$x \notin S$ informs us that object *x* is NOT in S.

Cardinality of a Set

Cardinality: is defined as the number of ***distinct elements*** in a set.

$|M|$ refers to the cardinality of set M.

These two lines are used to refer to cardinality.

If $|T| = 0$, then we also write $T = \varnothing$ or $T = \{\ \}$.

These are known as ***empty set*** or ***null set***.

If $|S| = \mathbf{A}$, where **A** is some number, then we say S is ***finite*** or *countable*.

Otherwise, we say S is ***infinite*** or *uncountable.*

Note:

$\{\varnothing\} \neq \{\}$

$\{\} \neq \varnothing$

Example 1: TRUE or FALSE?

- $e \in \{x \mid x \text{ is a vowel}\}$ T/F
- cat $\in \{x \mid x \text{ is an animal}\}$ T/F
- $\{3\} \in \{x \mid x \text{ is an odd number}\}$ T/F
- Peter $\in \{x \mid x \text{ is a male}\}$ T/F

Example 2

Determine the **cardinality** of the following:

$|\{\text{cat, rabbit, parrot}\}|$ = _3___

$|\{a, b, c, a, c\}|$ = ___

$|\{\{1,2,3\},\{4,5\}\}|$ = _____

$|\{\, x \mid x \text{ is non negative even and } x < 11\}|$ = ____

Common Infinite Sets to Know

N = {0, 1, 2, 3, 4, 5, …} — Set of **natural** numbers

Also known as Counting Number or Nonnegative Integer

Z = {…, –2, –1, 0, 1, 2, …} — Set of **integers**

$\mathbf{Z}^+$ = {1, 2, 3, …} — Set of **positive integers**

$\mathbf{Z}^-$ = {-1, -2, -3, …} — Set of **negative integers**

Q = $\{p/q \mid p \in Z, q \in Z, q \neq 0\}$ — Set of **rational** numbers

Another name for Rational Numbers is Fractions.

R = {All real numbers} — Set of **real** numbers

Examples of real numbers:
-12.66547, 100000000.02, 244.0, $\sqrt{2}$

Equality of a Set

Two sets are ***equal*** if and only if they have the same members.

It does not matter *how the set is defined* or *denoted!*

Example 3

Given

$M_1 = \{x | x$ is an integer where $x>0$ and $x<5$ $\}$

$M_2 = \{x | x$ is a positive integer whose square is >0 and $<25\}$

$M_3 = \{x | x$ is an integer whose square is >0 and $<25\}$

Determine if

$M_1 = M_2$?

$M_1 = M_3$?

$M_2 = M_3$?

Subsets and Proper Subsets

A set ***S*** is said to be a ***subset*** of another set ***T*** if and only if every element of set ***S*** is contained in set ***T***.

The notation for subset is $\subseteq$

$$S \subseteq T$$

Note that:
Subsets of set T includes the set itself.
That is, $T \subseteq T$.

A set ***S*** is said to be a ***proper subset*** of another set ***T*** if and only if set ***S*** is a subset of set ***T*** but set ***S*** is not equal to set ***T***.

The notation for proper subset is $\subset$

$$S \subset T : S \subseteq T \text{ and } S \neq T$$

This means:
A proper subset for set T is ALL subsets of set T
EXCEPT **the set itself.**

Example 4: True or False?

- $4 \subseteq \{x|x \text{ is an even number}\}$ T/F
- $\{5/8\} \subseteq \{x|x \text{ is a fraction}\}$ T/F
- $\{\text{rice, ice-cream}\} \subseteq \{x|x \text{ is food}\}$ T/F
- $\{\text{rice, ice-cream}\} \subset \{x|x \text{ is food}\}$ T/F
- $\{1, 2, 3\} \subseteq \{1, 2, 3\}$ T/F
- $\{1, 2, 3\} \subseteq \{1, 2, 2, 3\}$ T/F
- $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ T/F
- $\{1, 2, 3\} \subset \{1, 2, 3, 4\}$ T/F
- $\{1, 2, 3\} \subset \{1, 2, 2, 3\}$ T/F

Power Sets

A ***power set*** P(S) of a set S is the set of all subsets of S.

$$P(S) = \{x \mid x \subseteq S\}$$

Example: If S = {a,b}, then P(S) = {∅, {a}, {b}, {a,b}}.

Cardinality of a Power Set

If S is finite then $|P(S)| = 2^{|S|}$.

$|P(N)| > |N|$.

Example:

A={0, 2, b}

|A|=3

P(A) = {∅, {0}, {2}, {b}, {0,2}, {0,b}, {2,b}, {0,2,b} }.

$|P(A)| = 2^{|A|} = 2^3 = 8$

Example 5

Let A = {1, 2, 3}

1) Write out P(A).

2) What is |P(A)|?

The Universal Set

The idea of a "set of all sets" leads to logical difficulties.

Difficulties are avoided by always working within a local "universal set" which includes only those objects under consideration.

For example, when discussing arithmetic it might be sufficient to work just with the numbers . . . , -3, -2, -1, 0, 1, 2, 3, . . .
Our universal set could then be taken as Z and other sets of interest, such as $\{x: x \text{ is prime}\}$, are parts of Z.

The **Universal set notation** is U.

Operations on Sets

Operation	**Name**
$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$	Union
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$	Intersection
$A - B = \{x \mid x \in A \text{ and } x \notin B\}$	Difference
$\overline{A} = A^c = U - A$ where U is the universal set	Complement of A
$A \triangle B = (A - B) \cup (B - A)$ $= (A \cup B) - (A \cap B)$	Symmetric difference of two sets A and B

Example 6

Determine the elements resulting from the set operations of the following:

- $\{a,b,c\} \cup \{2,3\} =$ ______________
- $\{2,3,5\} \cup \{3,5,7\} =$ ______________
- $\{a,b,c\} \cap \{2,3\} =$ ______________
- $\{2,4,6\} \cap \{3,4,5\} =$ ______________
- $\{1,2,3,4,5,6\} - \{2,3,5,7,9,11\} =$ ______________
- $\{3,4,5,6\} \triangle \{5,7,9\} =$ ______________
- $Z - N = \{\ldots, -1, 0, 1, 2, \ldots\} - \{0, 1, \ldots\} =$ ______________
- Given $A=\{2, 3, 4, 5, 6, 7, 8\}$ and $U=\{ x \in Z^+ \mid x<10 \}$.
  $\overline{A} =$ ________________

Set Identities/Laws

Identity:	$A \cup \varnothing = A$ $A \cap U = A$
Domination:	$A \cup U = U$ $A \cap \varnothing = \varnothing$
Idempotent:	$A \cup A = A = A \cap A$
Double complement:	$(A^c)^c = \overline{(\overline{A})} = A$
Commutative:	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative:	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$

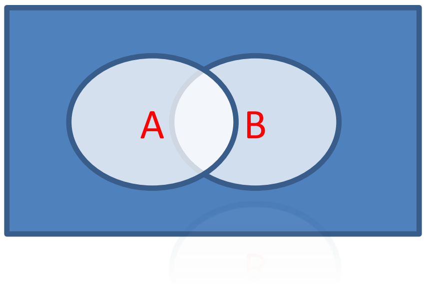
Set Identities/Laws

Distributive:	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
De Morgan's:	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$
Absorption:	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Complement:	$A \cup A^c = U$ $A \cap A^c = \varnothing$

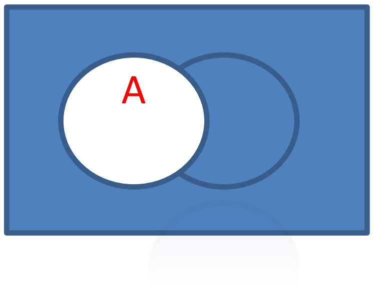
To understand all these identities, you can draw a **Venn Diagram** but a Venn Diagram *does not prove correctness*!

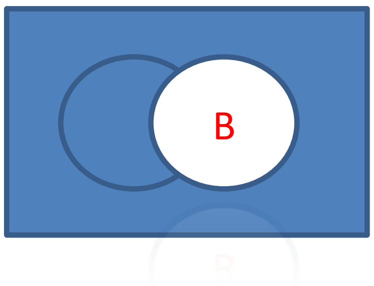
A Venn diagram can be used to illustrate the set identities
e.g. De Morgan's: $(A \cup B)' = A' \cap B'$

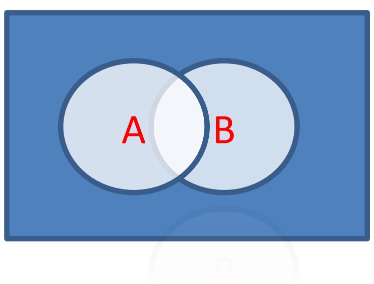
Left side of the equation



Right side of the equation







Summary

We have learnt the following concepts related to sets:

- Notation for sets,
- $\in$ relational operator,
- cardinality $|S|$, the empty set $\varnothing$ and infinite sets N, Z, Q, R,
- equal sets, subsets and proper subsets $=$, $\subseteq$, $\subset$,
- power sets P(S),
- set operations $\cup$, $\cap$, $-$, S^c, $\triangle$
- set identities.

Exercise 1

Let $A = \{x \mid x \in N \wedge x \text{ divides } 24 \ \wedge \ x < 10\}$ and
$B = \{x \mid x \in N \wedge x \text{ is prime number } \wedge \ x < 16\}$

a) List all elements in *A* and *B.*

b) Find

i) $A \cap B =$

ii) $A \cup B =$

iii) $A - B =$

iv) $A \, \Delta \, B =$

v) $P(A \cap B) =$

vi) $|P(A \cap B)| =$

Exercise 2

Given the following sets

$M_1 = \{1, 3, 5, 7\}$
$M_2 = \{\varnothing, \{1, K\}\}$

Give

$P(M_1) =$

$P(M_2) =$

Exercise 3

Given the following sets

M_1 = { x | x $\in$ N and x + 1 is even}
M_2 = { x | x $\in$ N and x has 1 as its first digit}
M_3 = { x | x $\in$ N and x is divisible by 3}

Give the first 10 elements of the following sets and describe them informally in English:

$M_1 \cup M_3 =$
$M_2 - M_1 =$
$M_1 \cap M_2 =$
$M_1 \cap M_3 =$