Topic 8 Introduction to Combinatory

TMA1201 Discrete Structures & Probability
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What you will learn in this lecture:

- Basic counting principles
- The principle of inclusion-exclusion
- Pigeonhole principle
- Permutations
- Combinations
- Generalized permutations
- Generalized combinations

The Product Rule

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Example 1

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution: The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees. ◀

Example 2

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution: The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600. ◀

Example 3

There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

Solution: The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the microcomputer and 24 ways to choose the port no matter which microcomputer has been selected, the product rule shows that there are $32 \cdot 24 = 768$ ports. ◀

Example 4

Counting Functions How many functions are there from a set with m elements to a set with n elements?

Solution: A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. Hence, by the product rule there are $n \cdot n \cdot \cdots \cdot n = n^m$ functions from a set with m elements to one with n elements. For example, there are $5^3 = 125$ different functions from a set with three elements to a set with five elements. ◀

Example 5

Counting One-to-One Functions How many one-to-one functions are there from a set with m elements to one with n elements?

Solution: First note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.

Now let $m \leq n$. Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of the function at a_1. Because the function is one-to-one, the value of the function at a_2 can be picked in $n - 1$ ways (because the value used for a_1 cannot be used again). In general, the value of the function at a_k can be chosen in $n - k + 1$ ways. By the product rule, there are $n(n-1)(n-2)\cdots(n-m+1)$ one-to-one functions from a set with m elements to one with n elements.

For example, there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions from a set with three elements to a set with five elements. ◀

Sum Rule

THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example 6

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects respectively. How many possible projects are there to choose from?

(Solution) The student can choose a project from the first list in 23 ways, from the second list in 15 ways, and from the third list in 19 ways. Hence, there are 23 + 15 + 19 = 57 projects to choose from.

Example 7

A swimming team has 4 athletes who swim backstroke, 3 who swim breaststroke, 3 who swim butterfly, and 5 who swim freestyle. A medley relay consists of one swimmer from each of the four strokes.

(i) How many choices does the coach have for a medley relay?

(ii) How many choices does the coach have for a medley relay if one of the 3 athletes who swim butterfly is also one of the 5 athletes who swim freestyle?

- **(i) No. of choices = 4 x 3 x 3 x 5 = 180 (product rule)**

  (ii) We break the problem into 3 disjoint cases and use the sum rule. Let *V* denotes the versatile swimmer.

  - **Case 1: *V* swims butterfly: $4 \times 3 \times 1 \times 4 = 48$ (product rule)**
  - **Case 2: *V* swims freestyle: $4 \times 3 \times 2 \times 1 = 24$ (…)**
  - **Case 3: *V* swims neither: $4 \times 3 \times 2 \times 4 = 96$ (…)**

  Hence, the number of choices is 48 + 24 + 96 = 168.

The Principle of Inclusion-Exclusion

- Suppose that a task can be done in one of two ways, but some of the ways to do it are common to both ways. In this situation, we cannot use the sum rule to count the number of ways to do the task. If we add the number of ways to do the tasks in these two ways, we get an overcount of the total number of ways to do it, because the ways to do the task that are common to the two ways are counted twice. To correctly count the number of ways to do the two tasks, we must subtract the number of ways that are counted twice. This leads us to an important counting rule.

The Subtraction Rule

THE SUBTRACTION RULE If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

- Suppose that A_1 and A_2 are sets. Then, there are $|A_1|$ ways to select an element from A_1 and $|A_2|$ ways to select an element from A_2. The number of ways to select an element from A_1 or from A_2, that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element from A_1 and the number of ways to select an element from A_2, minus the number of ways to select an element that is in both A_1 and A_2. Because there are $|A_1 \cup A_2|$ ways to select an element in either A_1 or in A_2, and $|A_1 \cap A_2|$ ways to select an element common to both sets, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Example 8

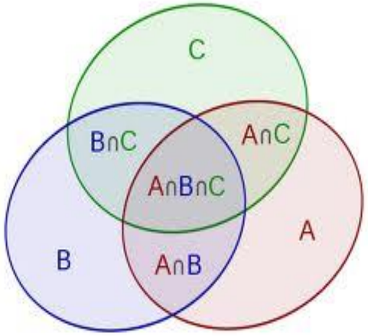
How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

- **The first task, constructing a bit string of length eight that begins with a 1, can be done in $2^7 = 128$ ways.**
- **The second task, constructing a bit string of length eight that ends with the two bits 00, can be done in $2^6 = 64$ ways.**
- **However, both tasks include those bit strings that begin with a 1 and end with 00. Thus, such bit strings are counted twice, and we must deduct them to eliminate over counting.**
- **The number of ways to construct a bit string that begins with a 1 and ends with 00 is $2^5 = 32$ ways.**
- **Consequently, the number of bit strings of length eight that begin with a 1 or end with 00, which equals to the number of ways to do either the first task or the second task, will be 128 + 64 – 32 = 160.**

The Subtraction Rule

- Suppose we have a task can be done in 3 ways. Let A, B and C denote sets containing the ways to do the task in first, second and third ways, respectively. The total number of ways to do the task is the number of elements in the union of three sets *A*, *B* and *C*.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Example 10

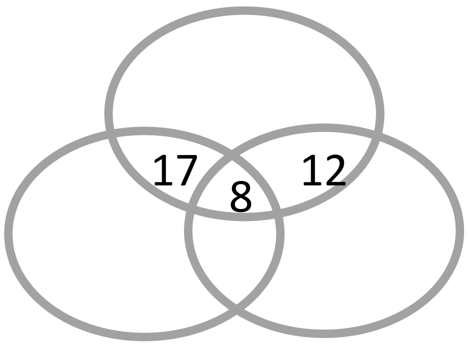
Find the number of students at a university taking at least one of the languages Malay, Mandarin, and Tamil, knowing the following information:

65 study Malay	20 study Malay and Mandarin
45 study Mandarin	25 study Malay and Tamil
42 study Tamil	15 study Mandarin and Tamil
8 study all three languages	

- **We want to find $|M \cup D \cup T|$ where *M*, *D*, and *T* denote the sets of students studying Malay, Mandarin, and Tamil, respectively.**

Solution

- **By Vann-Diagram:**



$|M \cup D \cup T| =$

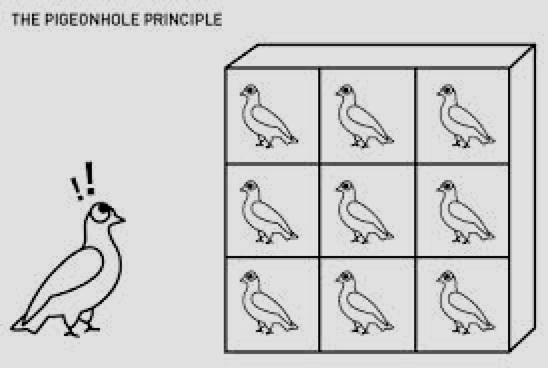
- **By the principle of inclusion-exclusion,**

$$|M \cup D \cup T| = |M| + |D| + |T| - |M \cap D| - |M \cap T| - |D \cap T| + |M \cap D \cap T|$$

$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

The Pigeonhole Principle

- Suppose there are n pigeons, k pigeonholes, and $n > k$. If these n pigeons fly into these k pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.



Example 11

1) A function f from a set with k + 1 or more elements to a set with k elements is not one-to-one.

Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain of f such that f(x)= y. Because the domain contains k + 1 or more elements and the codomain contains only k elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain. This means that f cannot be one-to-one.

Example 11

2) Use Pigeonhole principle to show that in any group of 27 English words, there must be at least two words that begin with the same letter.

Suppose for each English letter, a box is associated with it and word that begin with that letter will be assigned to the box. As such we have 27 words to be assigned to 26 boxes. By Pigeonhole Principle, there will be at least a box containing at least 2 words. Hence this show that there must be at least two words that begin with the same letter.

The Generalized Pigeonhole Principle

- Suppose there are n pigeons, k pigeonholes, $n > k$, and $m = \lceil n / k \rceil$. If these n pigeons fly into these k pigeonholes, then there is at least one pigeonhole containing at least m pigeons.

Example 12

Among 100 people at least how many people were born in the same month?

- Suppose we have a box for each month and people that were born with same month are assigned to the same box.

  There are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example 13

What is the minimum number of students required in a discrete mathematics class to be sure that at least seven will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solution:

Suppose a box is associated to each grade and students having the same grade are assigned to it. As such we have 5 boxes and is denoted as k= 5. Let N be minimum number of students that at least 7 student having the same grade. By Generalized Pigeonhole Principle:

$$\lceil N/k \rceil = 7$$

$$N = (7-1) \cdot k + 1 = 6*5+1 = 31$$

If the number of students N=30, it is possible for there to be six who have received each grade so that no seven students have received the same grade. Hence the minimum number of students needed to ensure that at least seven students receive the same grade is 31.

Try This

a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

b) How many must be selected to guarantee that at least three hearts are selected?

n-Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.

- Let S be a set with n distinct elements, where $n > 0$. An n-permutation of S (or just termed as permutation of S) is the number of ordered arrangements of the n distinct elements, which is equal to:

$$\boldsymbol{n}! = \boldsymbol{n} \times (\boldsymbol{n} - \mathbf{1}) \times (\boldsymbol{n} - \mathbf{2}) \times \cdots \times \mathbf{2} \times \mathbf{1}$$

(can you prove it?)

Example 14

- Let S = {1, 2, 3}. The arrangement 1, 2, 3 is a permutation of S. The arrangement 3, 1, 2 is another permutation of S. Altogether there are

  $3! = 6$ permutations of S.

- To arrange four persons in a line for a photo session, there are

  $4! = 24$ ways of arrangement.

r-Permutations

- We can also find the ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an r-permutation.

- Let S be a set with n distinct elements, where $n > 0$. Let $0 < r \leq n$. An r-permutation of S is

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

(can you prove it?)

which can also be written as

$$P(n, r) = {}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Example 15

A president and a treasurer are to be chosen from a student club consisting of 5 people, {*A*, *B*, *C*, *D*, *E*}. How many different choices of officers are possible if

(i) there are no restrictions.

(ii) *A* will serve only if he is a president.

(iii) *B* and *C* will serve together or not at all.

(iv) *D* and *E* will not serve together.

- **Let *n* be the number of ways,**

(i) There are no restrictions.

$\boldsymbol{n = {}^{5}P_{2} = 20}$

(ii) *A* will serve only if he is a president.

$\boldsymbol{n = {}^{4}P_{1} + {}^{4}P_{2} = 16}$

(iii) *B* and *C* will serve together or not at all.

$\boldsymbol{n = 2! + {}^{3}P_{2} = 8}$

(iv) *D* and *E* will not serve together.

$\boldsymbol{n = 20 - 2! = 18}$

Combinations

- A selection of objects without regard to order is called a combination.

- An *r*-combination of elements of a set is an unordered selection of *r* elements from the set.

- Let *S* be a set with *n* distinct elements, *n*> 0. Let $0 < r \leq n$. An *r*-combination of *S* is

$$\boldsymbol{C(n,r) = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}}$$

(can you prove it?)

Examples 16

1. How many ways are there to select four players from a 10-member tennis team to make a trip to a match at another school?

- **$C(10, 4) = 10!/(4!6!) = 210$ ways.**

2. In how many ways we can select a committee of two women and three men from a group of five women and six men?

- **The committee can be formed in two successive steps:**

  Select the women, then select the men.

  To select two women from a group of five: $C(5, 2) = 10$ ways

  To select three men from a group of six: $C(6, 3) = 20$ ways

  Thus, by the multiplication principle, the total number of ways

  is $10 \times 20 = 200$.

Generalized Permutations and Combinations

Permutations with Repetition

The number of r-permutations of a set of n objects with repetition allowed is n^r.

Example 17

How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly, we see that there are 26^r strings of uppercase English letters of length r.

Combinations with Repetition

Example 18

How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution: To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

4 apples	4 oranges	4 pears
3 apples,1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	
2 pears, 1 apple, 1 orange		

The solution is the number of 4-combinations with repetition allowed from a three-element set, {apple, orange, pear}.

Combinations with Repetition

There are $C(n + r - 1, r) = C(n + r - 1, n - 1)$ r-combinations from a set with n elements when repetition of elements is allowed.

From Example 18

Imagine that we have 3 containers which are used to keep the selected fruits apple, orange and pear separately. Let "*" denotes a fruit and "|" represent the separator between the containers. In this case 2 separators is required to separate the 3 containers. The ways to select four pieces of fruits from a bowl containing apples, oranges, and pears (where repetition is allowed) can now be represented by a sequence of four "*"s and two "|"s.

For example:

4 apples	****| |
3 oranges,1 pear	|***|*
2 pears, 1 apple, 1 orange	*|*|**

…

This also means that the total number of ways to select fruits mentioned above is the same as numbers of ways to select four "*" in a row with a total of 6 positions which gives

$$c(6,4) = \frac{6!}{4!\cdot(6-4)!} = \frac{6\cdot 5\cdot 4!}{4!2!} = \frac{6\cdot 5}{1\cdot 2} = 15$$

From Example 18 (Cont.)

Or by using the Theorem

The ways to select four pieces of fruits (r=4) from a bowl containing apples, oranges and pears (n=3) (where repetition is allowed) is

$$C(n+r-1,r) = C(6,4) = \ldots = 15$$

Example 19

How many ways are there to select five bills from a cash box containing $1 bills, $2 bills, $5 bills, $10 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.

(Try to solve the problem without using the formula.)

Examples 20

Find the number of integer solutions to the equation

$$x_1 + x_2 = 3, \text{ where } x_1 \geq 0 \text{ and } x_2 \geq 0.$$

(Try to solve the problem without using the formula.)

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Indistinguishable Objects

- Suppose that a sequence S of n items has

  n_1 identical objects of type 1

  n_2 identical objects of type 2

  :

  :

  n_k identical objects of type k

Then, the number of orderings of S is

$$\frac{n!}{n_1!\,n_2!\,\ldots n_{k-1}!\,n_k!}$$

Example 21

How many different strings can be made by reordering the letters of the word SUCCESS?

Solution: Because some of the letters of SUCCESS are the same, the answer is not given by the number of permutations of seven letters. This word contains three Ss, two Cs, one U, and one E. To determine the number of different strings that can be made by reordering the letters, first note that the three Ss can be placed among the seven positions in C(7,3) different ways, leaving four positions free. Then the two Cs can be placed in C(4,2)ways, leaving two free positions. The U can be placed in C(2,1)ways, leaving just one position free. Hence E can be placed in C(1,1) way. Consequently, from the product rule, the number of different strings that can be made is

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!\cdot 4!}\cdot\frac{4!}{2!\cdot 2!}\cdot\frac{2!}{1!\cdot 1!}\cdot\frac{1!}{1!\cdot 0!}$$

$$= \frac{7!}{3!\cdot 2!\cdot 1!\cdot 1!} = 420.$$

Example 22

How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Solution: We will use the product rule to solve this problem. To begin, note that the first player can be dealt 5 cards in C(52,5)ways. The second player can be dealt 5 cards in C(47,5)ways, because only 47 cards are left. The third player can be dealt 5 cards in C(42,5) ways. Finally, the fourth player can be dealt 5 cards in C(37,5)ways. Hence, the total number of ways to deal four players 5 cards each is

$$\begin{aligned} C(52,5)C(47,5)C(42,5)C(37,5) &= \frac{52!}{47!\,5!}\cdot\frac{47!}{42!\,5!}\cdot\frac{42!}{37!\,5!}\cdot\frac{37!}{32!\,5!} \\ &= \frac{52!}{5!\,5!\,5!\,5!\,32!}. \end{aligned}$$

Summary

Materials covered in this lecture:

- The addition principle
- The multiplication principle
- The principle of inclusion-exclusion
- Pigeonhole principle
- Generalized pigeonhole principle
- *n*-Permutations
- *r*-Permutations
- Combinations
- Generalized permutations
- Generalized combinations

Exercise 1

There are 6 flavors of ice-cream, and 3 different cones. How many different single-scoop ice-creams you could order?

Exercise 2

A student can choose a computer project from one of three lists. The three lists contain 23, 15 and 19 possible projects, respectively. How many possible projects are there to choose from?

Exercise 3

How many integers from 1 to 100 are either multiples of 3 or multiples of 5?

Exercise 4

Among 18 students in a room, 7 study mathematics, 10 study science, and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many of these students study none of the three subjects?

Exercise 5

How many students in a class must there be to ensure that there are at least 6 students get the same grade (A, B, C, D, or E)?

Exercise 6

In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom if the bride must be next to the groom?

Exercise 7

3 marbles are drawn at random from a bag containing 3 red and 5 white marbles. How many different draws would contain 1 red and 2 white marbles?

Exercise 8

Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.