Tutorial 9 - Ordinary Differential Equations

1. State the **order** and **linearity** of each differential equation and verify that the given function is a solution.

 a) $y''+9y=0;\qquad y=A\cos(3x)+B\sin(3x)$

 b) $y''+2y'+y=0;\qquad y=2e^{-x}+xe^{-x}$

 c) $2yy'=9\sin(2x);\qquad y_1(x)=\sin x,\qquad y_2(x)=3\sin x$

 d) $y''-xy'+y=0\qquad y=x$

First-Order Linear Differential Equations

2. Solve the following differential equations **by separation of variables**.
[The solution you obtain, if correct, may not be exactly the same as what is given here, but would be equivalent to it.]

 a) $\dfrac{dy}{dx}+3y=0$ [$y=Ce^{-3x}$]

 b) $\dfrac{dy}{dx}-4xy=0$ [$y=Ce^{2x^2}$]

 c) $y'=xe^{x^2}$ (Hint: substitution) [$y=\dfrac{1}{2}e^{x^2}+C$]

 d) $y'=\dfrac{x\cos x}{6y^5-1}$ (Hint: by parts) [$y^6-y=x\sin x+\cos x+c$]

 e) $e^x y\dfrac{dy}{dx}=e^{-y}+e^{-2x-y}$ (Hint: by parts) [$ye^y-e^y+e^{-x}+\dfrac{1}{3}e^{-3x}=c$]

 f) $x^2\dfrac{dy}{dx}=y-xy\qquad y(1)=1$ [$xy=e^{1-\frac{1}{x}}$]

 g) $y'=y^2+y-6;\ y(5)=10$ (Hint: partial fractions) [$\dfrac{y-2}{y+3}=\dfrac{8}{13}\exp(5(x-5))$]

3. Solve the following **first-order linear differential equations**.
[The solution you obtain, if correct, may not be exactly the same as what is given here, but would be equivalent to it.]

 a) $y'+4y=e^{-3x}$ [$y=e^{-3x}+Ce^{-4x}$]

 b) $y'=\dfrac{2y}{x}+x^2e^x$ (Hint: $e^{\ln x}=x$) [$y=x^2e^x+cx^2$]

 c) $y'+2xy=x$ [$y=\dfrac{1}{2}+Ce^{-x^2}$]

 d) $xy'+4y=x^5$ [$y=\dfrac{C}{x^4}+\dfrac{1}{9}x^5$]

 e) $y'-2y=2\cos(2x)+4,\ y(0)=-\dfrac{5}{4}$ [$y=\dfrac{1}{2}(\sin 2x-\cos 2x)-2+\dfrac{5}{4}e^{2x}$]

 f) $(1+x)y'-xy=x+x^2$ [$(x+1)e^{-x}y=-x(x+1)e^{-x}-2x-3+c$]

g) $xy' + y = e^x$ $\quad y(1) = 2$ $\quad [\, y = \frac{1}{x}e^x + \frac{2-e}{x} \,]$

h) $(x+1)\frac{dy}{dx} + y = \ln x$ $\quad y(1) = 10$ $\quad [\,(x+1)y = x\ln x - x + 21\,]$

4. Determine whether the given differential equation is **exact.** If it is exact, solve it.

a) $\frac{dy}{dx} = \left(\frac{x^2+y^2}{y-2xy}\right)$ $\quad [\, \frac{x^3}{3} + xy^2 - \frac{y^2}{2} = c \,]$

b) $\left(2y - \frac{1}{x} + \cos 3x\right)\frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$

c) $2x^3(y-1)dy + 3x^2(y-1)^2 dx = 0;$ $\quad y(-2) = 5/2$

$[\, x^3(y-1)^2 + 18 = 0 \,]$

d) $\left(\frac{3y^2 - t^2}{y^5}\right)\frac{dy}{dt} + \frac{t}{2y^4} = 0$ $\quad y(1) = 1$ $\quad [\, \frac{t^2}{4y^4} - \frac{3}{2y^2} = -\frac{5}{4} \,]$

e) $(2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0$ $\quad [\, x^2 y - \tan x + y^2 = C \,]$

<u>Second-Order Linear Differential Equations</u>

5. Find a general solution for each of the following second-order homogeneous linear differential equations.

(a) $y'' - 2y' - 8y = 0$ (b) $y'' - 3y' = 0$ (c) $y'' - 4y = 0$

(d) $y'' - 6y' + 25y = 0$ (e) $y'' + 4y = 0$ (f) $y'' + 6y' + 9y = 0$

(g) $\frac{d^2y}{dx^2} - 20\frac{dy}{dx} + 64y = 0$ (h) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$

6. Solve the following initial-value problems.

(a) $y'' - 4y' + 13y = 0;\ y(0) = 5,\ y'(0) = -2$

(b) $y'' + 16y = 0;\ y(0) = 2,\ y'(0) = -2$

7. Solve the following second-order non-homogeneous linear differential equations using the method of undetermined coefficients.

(a) $y'' - 2y' - 8y = 4x - 3e^x$ (b) $y'' - 2y' - 8y = \sin x$

(c) $y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x$ (d) $y'' + 3y' = 4x + 5$

(e) $y'' + 6y' + 8y = 3e^{-2x} + 2x$ (f) $y'' - 2y' + y = xe^x$

(g) $y'' + 10y' + 25y = e^{-5x}$ (h) $y'' - 5y' = 2;\ y(0) = 2,\ y'(0) = 2$