Topic 9.2 Discrete Random Variables and Distributions

TMA1201 Discrete Structures & Probability
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What you will learn in this lecture:

- What is a random variable?
- What is a discrete random variable?
- What is a continuous random variable?
- Probability distribution function of discrete random variable
- Cumulative distribution function of discrete random variable
- Expected value of discrete random variable
- Variance of discrete random variable

Random Variables

- A random variable (rv) is a function from sample space of an experiment to the set of real numbers. That is a random variable assigns a real numbers to each possible outcome.

Note:

- Random variable is a function. It is not a variable and it is not random.

- It can be categorized into:
  - Discrete random variable and
    Continuous random variable

Definition of Discrete and Continuous Random Variables

- A discrete random variable is an rv whose possible values either constitute a finite set or else can be listed in an infinite sequence in which there is a first element, a second element, and so on ("countably" infinite).
- A random variable is continuous if both of the following apply:
  1. Its set of possible values consists either of all numbers in a single interval on the number line (possibly infinite in extent, e.g., from $-\infty$ to ∞) or all numbers in a disjoint union of such intervals (e.g., $[0, 10] \cup [20, 30]$).
  2. No possible value of the variable has positive probability, that is, for any possible value c. $P(X = c) = 0$ for any possible value c.

Example 1

- Suppose that a coin is flipped three times. Let X(t) be the random variable that equals to the number of heads appear when t is the outcome.
- The experiment has the sample space

  S={HHH, HHT, HTH, THH, HTT, HTH, THH, TTT} and
- X(t) is the function that maps each of the outcome to a numeric value as follows:

  X(HHH) = 3

  X(HHT)=X(HTH)=X(THH) = 2

  X(HTT)=X(THT)=X(TTH) = 1

  X(TTT) = 0

Random Variables

Question	Random Variable x	Type
Family size	x = Number of dependents in family reported on tax return	Discrete
Distance from home to store	x = Distance in miles from home to the store site	Continuous
Own dog or cat	x = 1 if own no pet; = 2 if own dog(s) only; = 3 if own cat(s) only; = 4 if own dog(s) and cat(s)	Discrete

Discrete Random Variables

- *Probabilities assigned to various outcomes in S in turn determine probabilities associated with the values of any particular rv X. The probability distribution of X illustrates how the total probability of 1 is distributed among (allocated to) the various possible X values.*

- *The* ***probability distribution*** *or* ***probability mass function*** *(pmf) of a discrete rv is defined for every number* x *by*

$$p(x) = P(X = x) = P(all\ s \in S\colon X(s) = x).$$

Example 1 (Continue)

- From Example 1 suppose that the coin used in the experiment is a fair coin. As X(t) denotes the rv of number of head occurred in flipping of the coin three times, we have the probability mass function p(x) :

$$p(0) = P(X=0)=P(\{TTT\}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$p(1) = P(X=1)=P(\{HTT, THT, TTH\}) = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$p(2) = P(X=2)=P(\{HHT, HTH, THH\}) = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$p(3) = P(X=3)=P(\{HHH\}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Example 2

- Tossing a balance die

Example of a Discrete Probability Distribution

X=outcome from tossing a die

x	P(X=x)
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Cumulative Distribution Function

- The cumulative distribution function (cdf), $F(x)$ of a discrete random variable X with probability mass function p(x) is defined by

$$F(x) = P(X \leq x) = \sum_{y:y\leq x} p(y)$$

and $$p(x_i) = F(x_i) - F(x_{i-1})$$

Example 3

- Consider a discrete random variable *X* with the following pmf. Find the corresponding cdf.

x	0	1	2	3
$p(x)$ or $f(x)$	1/8	3/8	3/8	1/8
$F(x)$	1/8	4/8	7/8	1

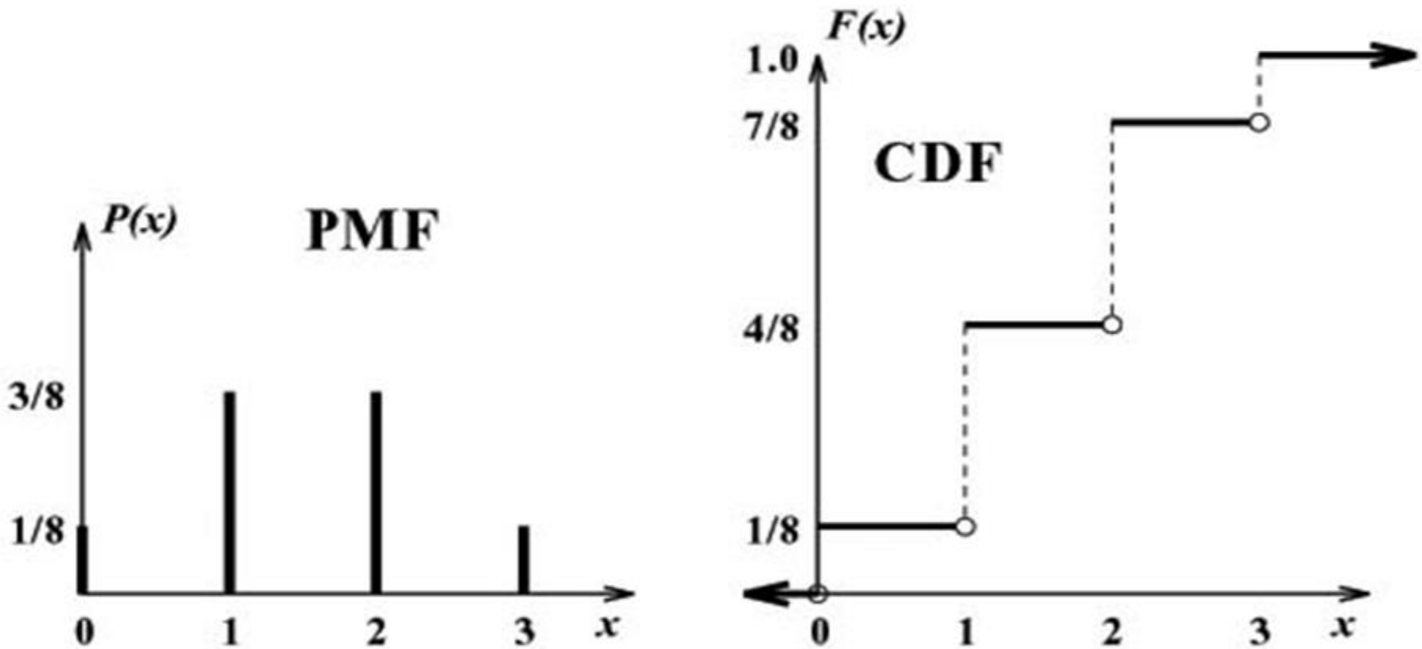
Example 3

x	0	1	2	3
$p(x)$ or $f(x)$	1/8	3/8	3/8	1/8
$F(x)$	1/8	4/8	7/8	1

$$F(x)=\begin{cases} 0 & ; \quad x<0 \\ 1/8 & ; \quad 0\le x<1 \\ 4/8 & ; \quad 1\le x<2 \\ 7/8 & ; \quad 2\le x<3 \\ 1 & ; \quad 3\le x \end{cases}$$

Example3

The pmf and cdf for the previous random variable are:



Example 4

From a box containing four 10 cents and two 5 cents, three coins are selected at random without replacement. Let X be random variable indicate the sum of three selected coins. Find the probability distribution and the cumulative distribution of X.

x	20	25	30
$p(x)$ or $f(x)$	1/5	3/5	1/5
$F(x)$	1/5	4/5	1

Expected Value of a Discrete Random Variable

Let X be a discrete random variable with the set of possible values x and probability mass function $p(x)$.

- The expected value (mean) of X denoted by $E(X)$ or μ_X is

$$\boldsymbol{E(X) = \mu_X = \sum_{x} x_i \cdot p(x_i)}$$

Example 5

Find the expected value for the following random variable.

x	-1	1	2	3	4
$f(x)$	1/10	3/10	3/10	2/10	1/10

Solution:

$$E(X)$$

$$= (-1)\left(\frac{1}{10}\right) + (1)\left(\frac{3}{10}\right) + (2)\left(\frac{3}{10}\right) + (3)\left(\frac{2}{10}\right) + (4)\left(\frac{1}{10}\right)$$

$$=$$

Expected Value of a Function of Discrete Random Variable

Let X be a discrete random variable with the set of possible values x and pmf $p(x)$, the expected value of any function of X, $h(X)$, is given by

$$\boldsymbol{E[h(X)] = \sum_{x} h(x_i) \cdot p(x_i)}$$

Example 6

Let X denotes the number of computer sold per day in an IT shop, and suppose that $p(x)$ is the pmf:

$p(0) = 0.1$, $p(1) = 0.2$, $p(2) = 0.3$, $p(3) = 0.4$

If the profit function is $h(X)=800X - 900$.

$$E[h(X)] = \sum_x h(x_i) \cdot p(x_i)$$

The mean profit is

$E[h(X)] = h(0)p(0) + h(1)p(1) + h(2)p(2) + h(3)p(3)$

$= (-900)(0.1) + (-100)(0.2) + (700)(0.3) + (1500)(0.4)$

$= \$700$

Properties of Expected Value

1. $E(a) = a$, where a is any constant
2. $E(aX) = aE(X)$, where a is any constant
3. $E(aX + b) = aE(X) + b$, where a, b are any constants

Example 7:

Given $E(X) = 5$

a) $E(3X) = 3E(X) = 15$

b) $E(X - 6) = E(X) - E(6)$

$= 5 - 6 = -1$

Variance of a Discrete Random Variable

Let X be a discrete random variable with the set of possible values x and probability mass function $p(x)$

- The variance of X, denoted by $V(X)$ or σ_X^2, or just σ^2, is

$$V(X) = \sigma^2 = \sum_x (x_i - \mu)^2 \cdot p(x_i)$$

$$= E\left[(X - \mu)^2\right] = E\left(X^2\right) - [E(X)]^2$$

Example 8

Find the variance for the following random variable.

x	-1	1	2	3	4
$p(x)$	1/10	3/10	3/10	2/10	1/10

Solution:

We know that $E(X)$ = __________ from Example 4, slide #16. Thus,

$V(X) = \sum_x (x_i - \mu)^2 \cdot p(x_i)$ =

OR

$$V(X) = E(X^2) - [E(X)]^2$$
$$= \sum_x x_i^2 p(x_i) - [E(X)]^2 =$$

Properties of Variance

1. $V(a) = 0$, where a is any constant
2. $V(aX) = a^2V(X)$, where a is any constant
3. $V(aX + b) = a^2V(X) + 0$, where a, b are any constants

Example 9:

Given $V(X) = 2$

a) $V(3X) = 3^2V(X) = 18$

b) $V(3X - 4) = 9V(X)$

$= 18$

Summary

Materials covered in this lecture?

- Random variables (discrete/continuous)
- Probability distribution function (probability mass function) of a discrete random variable
- Cumulative distribution function of a discrete random variable
- Expected value (mean) of a discrete random variable
- Properties of expected value
- Variance of a discrete random variable
- Properties of variance

Exercise 1

Let X be the outcome of rolling a die where the odd numbers has twice the probability to appear compared to the even numbers.

1) Construct the pmf and cdf of X.

2) Find the expected value of X.

3) Find the variance of X.

4) Find $E(-2X + 4)$ and $V(-2X + 4)$.

Exercise 2

A certain gas station has six pumps. Let X denote the number of pumps that are In use at a particular time of day. Suppose that the probability distribution of X is as given in the following table; the first row of the table lists the possible X values and the second row gives the probability of each such value.

x	0	1	2	3	4	5	6
$P(x)$	0.05	0.10	0.15	a	0.20	0.15	0.10

Find

(a) the value of a.

(b) the cumulative distribution function, $F(x)$.

(c) the probability that at most 2 pumps are in use.

(d) the expected value of X.

(e) the variance value of X.

Exercise 3

Suppose a bookstore purchases ten copies of a book at RM6 each to sell at RM12 with the understanding that at the end of a 3-month period any unsold copies can be redeemed for RM2. Let X = the number of copies sold, find the expected net revenue, $h(x)$.