2.1 EQUATIONS (Adapted from "Precalculus" by Stewart et als.)

Solving Linear Equations _ Solving Quadratic Equations _ Other Types of Equations

An equation is a statement that two mathematical expressions are equal.
For example, $3+4=7$ is an equation involving numbers.
Equations may contain variables, which are symbols (usually letters) that stand for numbers.
In the equation $3x+4=17$ the letter x is the variable. We think of x as the "unknown" in the equation, and our goal is to find the value of x that makes the equation true.

The values of the unknown that make the equation true are called the **solutions** or **roots** of the equation, and the process of finding the solutions is called **solving the equation**.

Two equations with exactly the same solutions are called **equivalent equations**.
To solve an equation, we try to find a simpler, equivalent equation in which the variable stands alone on one side of the "equal" sign.

Here are the properties that we use to solve an equation.
(In these properties, A, B, and C stand for any algebraic expressions, and the symbol $\Leftrightarrow$ means "**is equivalent to**.")

PROPERTIES OF EQUALITY

Property	Description
1. $A = B \quad \Leftrightarrow \quad A + C = B + C$	Adding the same quantity to both sides of an equation gives an equivalent equation.
2. $A = B \quad \Leftrightarrow \quad CA = CB \quad (C \neq 0)$	Multiplying both sides of an equation by the same nonzero quantity gives an equivalent equation.

▼ Solving Linear Equations

The simplest type of equation is a ***linear equation***, or **first-degree equation**, which is an equation in which each term is either a constant or a nonzero multiple of the variable.

LINEAR EQUATIONS

A **linear equation** in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

Here are some examples that illustrate the difference between linear and nonlinear equations.

Linear equations	Nonlinear equations	
$4x - 5 = 3$	$x^2 + 2x = 8$	Not linear; contains the square of the variable
$2x = \frac{1}{2}x - 7$	$\sqrt{x} - 6x = 0$	Not linear; contains the square root of the variable
$x - 6 = \frac{x}{3}$	$\frac{3}{x} - 2x = 1$	Not linear; contains the reciprocal of the variable

E X A M P L E 1 Solving a Linear Equation
Solve the equation 7x – 4 = 3x + 8.

S O LU T I O N We solve this by changing it to an equivalent equation with all terms that have the variable *x* on one side and all constant terms on the other.

$$\begin{aligned} 7x - 4 &= 3x + 8 && \text{Given equation} \\ (7x - 4) + 4 &= (3x + 8) + 4 && \text{Add 4} \\ 7x &= 3x + 12 && \text{Simplify} \\ 7x - 3x &= (3x + 12) - 3x && \text{Subtract } 3x \\ 4x &= 12 && \text{Simplify} \\ \tfrac{1}{4} \cdot 4x &= \tfrac{1}{4} \cdot 12 && \text{Multiply by } \tfrac{1}{4} \\ x &= 3 && \text{Simplify} \end{aligned}$$

CHECK YOUR ANSWER

$x = 3$:

$x = 3$ → LHS $= 7(3) - 4 = 17$

$x = 3$ → RHS $= 3(3) + 8 = 17$

LHS = RHS ✔

$7x-4=3x+8$ $7x=3x+8+4$ $7x=3x+12$ $7x-3x=12$ $4x=12$ $x=\frac{12}{4}=3$	OR	$7x-4=3x+8$ $7x=3x+8+4$ $7x=3x+12$ $7x-3x=12$ $4x=12$ $\frac{1}{4}\cdot 4x=\frac{1}{4}\cdot 12$ $x=3$	OR	$7x-4=3x+8$ $7x-3x=8+4$ $4x=12$ $x=\frac{12}{4}=3$

NOW TRY: Solve the equation $-7w = 15 - 2w$

EXAMPLE 2 Solving for One Variable in Terms of Others

Solve for the variable M in the equation $F = G\dfrac{mM}{r^2}$

S O LU T I O N Although this equation involves more than one variable, we solve it as usual by isolating M on one side and treating the other variables as we would numbers.

$$F = \left(\frac{Gm}{r^2}\right)M$$

$$\left(\frac{r^2}{Gm}\right)F = \left(\frac{r^2}{Gm}\right)\left(\frac{Gm}{r^2}\right)M$$

$$\frac{r^2F}{Gm} = M$$

The solution is $M = \dfrac{r^2F}{Gm}$.

NOW TRY: Solve the equation $PV = nRT$ for R.

E X A M P L E 3 Solving for One Variable in Terms of Others

The surface area A of a closed rectangular box with length l, width w, and the height h is given by $A = 2lw + 2wh + 2lh$.

Solve for w in terms of the other variables in this equation.

S O LU T I O N Although this equation involves more than one variable, we solve it as usual by isolating w on one side, treating the other variables as we would numbers.

The solution is ????

NOW TRY: Solve the equation $P = 2l + 2w$ for w.

▼ Solving Quadratic Equations

Quadratic equations are **second-degree** equations like $x^2 + 2x - 3 = 0$ or $2x^2 + 3 = 5x$.

QUADRATIC EQUATIONS

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$.

Some quadratic equations can be solved by factoring and using the following basic property of real numbers.

ZERO-PRODUCT PROPERTY

$AB = 0$ if and only if $A = 0$ or $B = 0$

This means that if we can factor the left-hand side of a quadratic (or other) equation, then we can solve it by setting each factor equal to 0 in turn. This method works only when the right-hand side of the equation is 0.

E X A M P L E 4 Solving a Quadratic Equation by Factoring

Solve the equation $x^2 + 5x = 24$.

S O LU T I O N We must first rewrite the equation so that the right-hand side is 0.

Subtract 24 ???

Factor ???

Zero-Product Property $x-3=0$ **or** $x+8=0$

Solve ???

The solutions are $x = 3$ and $x = -8$.

NOW TRY: Solve $x^2 + x - 12 = 0$ by factoring.

Do you see why one side of the equation _must be_ 0 in Example 4? Factoring the equation as $x(x+5)=24$ does not help us find the solutions, since 24 can be factored in infinitely many ways, such as $6 \cdot 4, \frac{1}{2} \cdot 48, \left(-\frac{2}{5}\right) \cdot (-60)$, and so on.

SOLVING A SIMPLE QUADRATIC EQUATION

The solutions of the equation $x^2 = c$ are $x = \sqrt{c}$ and $x = -\sqrt{c}$.

E X A M P L E 5 Solving Simple Quadratics

Solve each equation.

(a) $x^2 = 5$ (b) $(x - 4)^2 = 5$

S O LU T I O N

(a) From the principle in the preceding box, we get $x = \pm\sqrt{5}$.

(b) We can take the square root of each side of this equation as well.

$$(x-4)^2 = 5$$

$$x - 4 = \pm\sqrt{5} \qquad \text{Take the square root}$$

$$x = 4 \pm \sqrt{5} \qquad \text{Add 4}$$

The solutions are $x = 4+\sqrt{5}$ and $x = 4-\sqrt{5}$.

NOW TRY: Solve (a) $2x^2 = 8$ (b) $(3x+2)^2 = 10$

As we saw in Example 5, if a quadratic equation is of the form $(x \pm a)^2 = c$, then we can solve it by taking the square root of each side. In an equation of this form the left-hand side is a *perfect square*: the square of a linear expression in x. So if a quadratic equation does not factor readily, then we can solve it using the technique of **completing the square**. This means that we add a constant to an expression to make it a perfect square. For example, to make $x^2 - 6x$ a perfect square, we must add 9, since $x^2 - 6x + 9 = (x-3)^2$.

COMPLETING THE SQUARE

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$, the square of half the coefficient of x. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

EXAMPLE 6 Solving Quadratic Equations by Completing the Square

Solve each equation.

(a) $x^2 - 8x + 13 = 0$ **(b)** $3x^2 - 12x + 6 = 0$

SOLUTION

(a) $x^2 - 8x + 13 = 0$ Given equation

$x^2 - 8x = -13$ Subtract 13

$x^2 - 8x + 16 = -13 + 16$ Complete the square: add $\left(\frac{-8}{2}\right)^2 = 16$

$(x-4)^2 = 3$ Perfect square

$x - 4 = \pm\sqrt{3}$ Take square root

$x = 4 \pm \sqrt{3}$ Add 4

(b) After subtracting 6 from each side of the equation, we must factor the coefficient of x^2 (the 3) from the left side to put the equation in the correct form for completing the square.

$3x^2 - 12x + 6 = 0$ Given equation

$3x^2 - 12x = -6$ Subtract 6

$x^2 - 4x = -2$ *Divide by 3 throughout*

Complete the square: add ?

Perfect square

Add 2

Take square root

Add 2

NOW TRY: Solve by completing the square (a) $x^2 + 2x - 5 = 0$ (b) $2x^2 + 8x + 1 = 0$

We can use the technique of completing the square to derive a formula for the roots of the general quadratic equation $ax^2+bx+c=0$.

THE QUADRATIC FORMULA

The roots of the quadratic equation $ax^2+bx+c=0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

PROOF First, we divide each side of the equation by a and move the constant to the right side, giving

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Divide by } a$$

To complete the square, ADD ??? (***Make sure you know how to continue***.)

Perfect square
Take square root
Subtract

■

The Quadratic Formula could be used to solve the equations in Examples 4 and 6. You should carry out the details of these calculations.

E X A M P L E 7 Using the Quadratic Formula

Find all solutions of each equation.

(a) $3x^2-5x-1=0$ (b) $4x^2+12x+9=0$ (c) $x^2+2x+2=0$

S O LU T I O N

(a) In this quadratic equation a = 3, b = –5, and c = –1.

By the Quadratic Formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2-4(3)(-1)}}{2(3)} = \frac{5 \pm \sqrt{37}}{6}$$

If approximations are desired, we can use a calculator to obtain 1.8471 and –0.1805 [*This last line is not needed for our course*.]

(b) Using the Quadratic Formula with a = ?, b = ?, and c = ? gives

???

This equation has only one solution, x = …….. .

(c) Using the Quadratic Formula, x = ……..

Since the square of any real number is nonnegative, $\sqrt{-1}$ is undefined in the real number system. **The equation has no real solution**.

[In the complex number system, the square roots of negative numbers do exist. The equation in Example 7(c) does have solutions in the complex number system.]

See next sub-section.

NOW TRY: Find all real solutions of the equation.

(a) $x^2 - 7x + 10 = 0$ (b) $3x^2 + 6x - 5 = 0$ (c) $w^2 = 3(w-1)$

The quantity $b^2 - 4ac$ that appears under the square root sign in the quadratic formula is called the ***discriminant*** of the equation $ax^2 + bx + c = 0$ and is given the symbol D.
If $D < 0$, then $\sqrt{b^2 - 4ac}$ is undefined in the real number system, and the quadratic equation has no real solution, as in Example 7(c).
If $D = 0$, then the equation has only one real solution, as in Example 7(b).
Finally, if $D > 0$, then the equation has two distinct real solutions, as in Example 7(a).

THE DISCRIMINANT

The **discriminant** of the general quadratic $ax^2 + bx + c = 0$ $(a \neq 0)$ is $D = b^2 - 4ac$.

1. If $D > 0$, then the equation has two distinct real solutions.
2. If $D = 0$, then the equation has exactly one real solution.
3. If $D < 0$, then the equation has no real solution.

E X A M P L E 8 Using the Discriminant
Use the discriminant to determine how many real solutions each equation has.
(a) $x^2 + 4x - 1 = 0$ (b) $4x^2 - 12x + 9 = 0$ (c) $x^2 - 2x + 4 = 0$

S O LU T I O N
(a) The discriminant is D = 42 – 4(1)(–1) = 20 > 0, so the equation has two distinct real solutions.

(b)

(c)

NOW TRY: Use the discriminant to determine the number of real solutions of the equation.
(a) $x^2 - 6x + 1 = 0$ **(b)** $x^2 + 2.20x + 1.21 = 0$ **(c)** $4x^2 + 5x + \frac{13}{8} = 0$

▼ Complex Solutions of Quadratic Equations

We have already seen that if $a \neq 0$, then the solutions of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$, then the equation has no real solution. But in the complex number system, this equation will always have solutions, because negative numbers have square roots in this expanded setting.

E X A M P L E C1 | Quadratic Equations with Complex Solutions

Solve each equation.

(a) $x^2 + 9 = 0$ **(b)** $x^2 + 4x + 5 = 0$

S O LU T I O N

(a) $x^2 + 4 = 0$

$x^2 = -4$

So $x = \pm\sqrt{-9} = \pm i\sqrt{9} = \pm 3i$

The solutions are therefore $3i$ and $-3i$.

(b) By the Quadratic Formula we have

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = \frac{2(-2 \pm i)}{2} = -2 \pm i$$

So the solutions are $-2 + i$ and $-2 - i$.

NOW TRY: Find all solutions of the equation and express them in the form $a + bi$.

(a) $x^2 + 4 = 0$ (b) $x^2 - 4x + 5 = 0$

E X A M P L E C2 | Complex Conjugates as Solutions of a Quadratic

$4x^2 - 24x + 37 = 0$

We use the Quadratic Formula to get the solutions.

$$x = \frac{24 \pm \sqrt{(24)^2 - 4(4)(37)}}{2(4)}$$

$$= \frac{24 \pm \sqrt{-16}}{8} = \frac{24 \pm 4i}{8} = 3 \pm \frac{1}{2}i$$

So the solutions are $3 + \frac{1}{2}i$ and $3 - \frac{1}{2}i$, and these are complex conjugates.

NOW TRY: Find all solutions of the equation and express them in the form $a + bi$.

$2x^2 - 2x + 1 = 0$

▼ Other Types of Equations

Now we study other types of equations, including those that involve higher powers, fractional expressions, radicals, and absolute values.

E X A M P L E 1 0 An Equation Involving ***Fractional Expressions***

Solve the equation $\dfrac{3}{x} + \dfrac{5}{x+2} = 2.$

S O LU T I O N We eliminate the denominators by multiplying each side by the lowest common denominator.

$$\left(\frac{3}{x}+\frac{5}{x+2}\right)x(x+2) = 2x(x+2)$$ Multiply by LCD $x(x+2)$

$$3(x+2)+5x = 2x^2+4x$$ Expand

$$8x+6 = 2x^2+4x$$ Expand LHS

$$0 = 2x^2-4x-6$$ Subtract $8x+6$

$$0 = x^2-2x-3$$ Divide both sides by 2

$$0 = (x-3)(x+1)$$ Factor

$$x-3=0 \quad \text{or} \quad x+1=0$$ Zero-Product Property

$$x=3 \qquad\qquad x=-1$$ Solve

We must check our answers because multiplying by an expression that contains the variable can introduce extraneous solutions.

Checking Answers

$x=3$: LHS $=\frac{3}{3}+\frac{5}{3+2}$ $=1+1=2$ RHS $=2$ LHS $=$ RHS ✔	$x=-1$: LHS $=\frac{3}{-1}+\frac{5}{-1+2}$ $=-3+5=2$ RHS $=2$ LHS $=$ RHS ✔

From "*Checking Answers*" we see that the solutions are $x=3$ and -1.

NOW TRY: Find all real solutions of the equation.

$$\frac{1}{x-1}+\frac{1}{x+2}=\frac{5}{4}$$

When you solve an equation that involves radicals, you must be especially careful to check your final answers. The next example demonstrates why.

E X A M P L E 1 1 An Equation Involving a ***Radical***

Solve the equation $2x = 1-\sqrt{2-x}$.

S O LU T I O N To eliminate the square root, we first isolate it on one side of the equal sign, then square:

Subtract 1

Square each side

Expand LHS

Add $-2+x$

Factor

Zero-Product Property

Solve

The values $x=-\frac{1}{4}$ and $x=1$ are only potential solutions.

(**We must check them to see if they satisfy the original equation**.)

Checking Answers

$x=-\frac{1}{4}$: $\quad \text{LHS} = 2(-\frac{1}{4}) = -\frac{1}{2}$ $\text{RHS} = 1-\sqrt{2-(-\frac{1}{4})}$ $= 1-\sqrt{\frac{9}{4}}$ $= 1-\frac{3}{2} = -\frac{1}{2}$ LHS = RHS ✔	$x=1$: $\quad \text{LHS} = 2(1) = 2$ $\text{RHS} = 1-\sqrt{2-1}$ $= 1-1 = 0$ LHS ≠ RHS ✗

From *Checking Answers* we see that $x=-\dfrac{1}{4}$ is a solution but $x = 1$ is not.

The only solution is $x=-\dfrac{1}{4}$.

NOW TRY: Find all real solutions of the equation.

$$\sqrt{2x+1}+1=x$$

When we solve an equation, we may end up with one or more **extraneous solutions**, that is, potential solutions that do not satisfy the original equation.

In Example 11 the value $x = 1$ is an extraneous solution. Extraneous solutions may be introduced when we square each side of an equation because the operation of squaring can turn a false equation into a true one.

For example, $-1 \neq 1$, but $(-1)^2 = 1^2$.

Thus, the squared equation may be true for more values of the variable than the original equation.

That is why you must always check your answers to make sure that each satisfies the original equation.

An equation of the form $aW^2 + bW + c = 0$, where W is an algebraic expression, is an equation of **quadratic type**. We solve equations of quadratic type by substituting for the algebraic expression, as we see in the next two examples.

E X A M P L E 1 2 A Fourth-Degree Equation of Quadratic Type

Find all solutions of the equation $x^4 - 8x^2 + 8 = 0$.

SOLUTION If we set $W = x^2$, then we get a quadratic equation in the new variable W:

$$(x^2)^2 - 8x^2 + 8 = 0 \qquad \text{Write } x^4 \text{ as } (x^2)^2$$

$$W^2 - 8W + 8 = 0 \qquad \text{Let } W = x^2$$

$$W = \frac{-(-8) \pm \sqrt{(-8)^2 - 4\cdot 8}}{2} = 4 \pm 2\sqrt{2} \qquad \text{Quadratic Formula}$$

$$x^2 = 4 \pm 2\sqrt{2} \qquad W = x^2$$

Take square roots $\quad x = \pm\sqrt{4 \pm 2\sqrt{2}}$

So, there are four solutions: $\sqrt{4+2\sqrt{2}}, \sqrt{4-2\sqrt{2}}, -\sqrt{4+2\sqrt{2}}, -\sqrt{4-2\sqrt{2}}$

NOW TRY: Find all real solutions of the equation.

$$x^4 - 13x^2 + 40 = 0$$

E X A M P L E 1 3 An Equation Involving Fractional Powers

Find all solutions of the equation $x^{\frac{1}{3}} + x^{\frac{1}{6}} - 2 = 0$.

S O LU T I O N This equation is of quadratic type because if we let $W = x^{\frac{1}{6}}$, then $W^2 = (x^{\frac{1}{6}})^2 = x^{\frac{1}{3}}$. The equation becomes $W^2 + W - 2 = 0$.

Factor

Zero-Product Property

Solve

$$W = x^{\frac{1}{6}}$$

Take the 6th power

Checking Answers

$x = 1$:	LHS $= 1^{1/3} + 1^{1/6} - 2 = 0$ RHS $= 0$ LHS $=$ RHS ✔	$x = 64$:	LHS $= 64^{1/3} + 64^{1/6} - 2$ $= 4 + 2 - 2 = 4$ RHS $= 0$ LHS $\neq$ RHS ✗

From *Checking Answers* we see that $x = 1$ is a solution but $x = 64$ is not. The only solution is $x = 1$.

NOW TRY: Find all real solutions of the equation. $x^{\frac{4}{3}} - 5x^{\frac{2}{3}} + 6 = 0$

When solving equations that involve absolute values, we usually take cases.

E X A M P L E 1 4 An **Absolute Value** Equation

Solve the equation $|2x - 5| = 3$.

SOLUTION By the definition of absolute value, $|2x - 5| = 3$ is equivalent to

$$\begin{aligned} 2x - 5 &= 3 & \quad\text{or}\quad 2x - 5 &= -3 \\ 2x &= 8 & 2x &= 2 \\ x &= 4 & x &= 1 \end{aligned}$$

The solutions are $x = 1$, $x = 4$.

NOW TRY: Find all real solutions of the equation. $|3x + 5| = 1$

(nby, Mar 2016)

Exercise

1–4 ■ Determine whether the given value is a solution of the equation.

1. $4x + 7 = 9x - 3$
(a) $x = -2$ (b) $x = 2$

2. $1 - [2 - (3 - x)] = 4x - (6 + x)$
(a) $x = 2$ (b) $x = 4$

3. $\dfrac{1}{x} - \dfrac{1}{x-4} = 1$
(a) $x = 2$ (b) $x = 4$

4. $\dfrac{x^{3/2}}{x-6} = x - 8$
(a) $x = 4$ (b) $x = 8$

5–22 ■ The given equation is either linear or equivalent to a linear equation. Solve the equation.

5. $2x + 7 = 31$

6. $5x - 3 = 4$

7. $\frac{1}{2}x - 8 = 1$

8. $3 + \frac{1}{3}x = 5$

9. $-7w = 15 - 2w$

10. $5t - 13 = 12 - 5t$

11. $\frac{1}{2}y - 2 = \frac{1}{3}y$

12. $\dfrac{z}{5} = \dfrac{3}{10}z + 7$

13. $2(1 - x) = 3(1 + 2x) + 5$

14. $\dfrac{2}{3}y + \dfrac{1}{2}(y - 3) = \dfrac{y+1}{4}$

15. $x - \frac{1}{3}x - \frac{1}{2}x - 5 = 0$

16. $2x - \dfrac{x}{2} + \dfrac{x+1}{4} = 6x$

17. $\dfrac{1}{x} = \dfrac{4}{3x} + 1$

18. $\dfrac{2x-1}{x+2} = \dfrac{4}{5}$

19. $\dfrac{3}{x+1} - \dfrac{1}{2} = \dfrac{1}{3x+3}$

20. $\dfrac{4}{x-1} + \dfrac{2}{x+1} = \dfrac{35}{x^2-1}$

21. $(t - 4)^2 = (t + 4)^2 + 32$

22. $\sqrt{3}x + \sqrt{12} = \dfrac{x+5}{\sqrt{3}}$

23–36 ■ Solve the equation for the indicated variable.

23. $PV = nRT$; for R

24. $F = G\dfrac{mM}{r^2}$; for m

25. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$; for R_1

26. $P = 2l + 2w$; for w

27. $\dfrac{ax+b}{cx+d} = 2$; for x

28. $a - 2[b - 3(c - x)] = 6$; for x

29. $a^2x + (a - 1) = (a + 1)x$; for x

30. $\dfrac{a+1}{b} = \dfrac{a-1}{b} + \dfrac{b+1}{a}$; for a

31. $V = \frac{1}{3}\pi r^2 h$; for r

32. $F = G\dfrac{mM}{r^2}$; for r

33. $a^2 + b^2 = c^2$; for b

34. $A = P\left(1 + \dfrac{i}{100}\right)^2$; for i

35. $h = \frac{1}{2}gt^2 + v_0 t$; for t

36. $S = \dfrac{n(n+1)}{2}$; for n

37–44 ■ Solve the equation by factoring.

37. $x^2 + x - 12 = 0$

38. $x^2 + 3x - 4 = 0$

39. $x^2 - 7x + 12 = 0$

40. $x^2 + 8x + 12 = 0$

41. $4x^2 - 4x - 15 = 0$

42. $2y^2 + 7y + 3 = 0$

43. $3x^2 + 5x = 2$

44. $6x(x - 1) = 21 - x$

45–52 ■ Solve the equation by completing the square.

45. $x^2 + 2x - 5 = 0$

46. $x^2 - 4x + 2 = 0$

47. $x^2 + 3x - \frac{7}{4} = 0$

48. $x^2 = \frac{3}{4}x - \frac{1}{8}$

49. $2x^2 + 8x + 1 = 0$

50. $3x^2 - 6x - 1 = 0$

51. $4x^2 - x = 0$

52. $-2x^2 + 6x + 3 = 0$

53–68 ■ Find all real solutions of the quadratic equation.

53. $x^2 - 2x - 15 = 0$

54. $x^2 + 30x + 200 = 0$

55. $x^2 + 3x + 1 = 0$

56. $x^2 - 6x + 1 = 0$

57. $2x^2 + x - 3 = 0$

58. $3x^2 + 7x + 4 = 0$

59. $2y^2 - y - \frac{1}{2} = 0$

60. $\theta^2 - \frac{3}{2}\theta + \frac{9}{16} = 0$

61. $4x^2 + 16x - 9 = 0$

62. $w^2 = 3(w - 1)$

63. $3 + 5z + z^2 = 0$

64. $x^2 - \sqrt{5}x + 1 = 0$

65. $\sqrt{6}x^2 + 2x - \sqrt{3/2} = 0$

66. $3x^2 + 2x + 2 = 0$

67. $25x^2 + 70x + 49 = 0$

68. $5x^2 - 7x + 5 = 0$

69–74 ■ Use the discriminant to determine the number of real solutions of the equation. Do not solve the equation.

69. $x^2 - 6x + 1 = 0$

70. $3x^2 = 6x - 9$

71. $x^2 + 2.20x + 1.21 = 0$

72. $x^2 + 2.21x + 1.21 = 0$

73. $4x^2 + 5x + \frac{13}{8} = 0$

74. $x^2 + rx - s = 0 \quad (s > 0)$

75–98 ■ Find all real solutions of the equation.

75. $\dfrac{1}{x-1} + \dfrac{1}{x+2} = \dfrac{5}{4}$

76. $\dfrac{10}{x} - \dfrac{12}{x-3} + 4 = 0$

77. $\dfrac{x^2}{x+100} = 50$

78. $\dfrac{1}{x-1} - \dfrac{2}{x^2} = 0$

79. $\dfrac{x+5}{x-2} = \dfrac{5}{x+2} + \dfrac{28}{x^2-4}$ **80.** $\dfrac{x}{2x+7} - \dfrac{x+1}{x+3} = 1$

81. $\sqrt{2x+1} + 1 = x$ **82.** $\sqrt{5-x} + 1 = x - 2$

83. $2x + \sqrt{x+1} = 8$ **84.** $\sqrt{\sqrt{x-5} + x} = 5$

85. $x^4 - 13x^2 + 40 = 0$ **86.** $x^4 - 5x^2 + 4 = 0$

87. $2x^4 + 4x^2 + 1 = 0$ **88.** $x^6 - 2x^3 - 3 = 0$

89. $x^{4/3} - 5x^{2/3} + 6 = 0$ **90.** $\sqrt{x} - 3\sqrt[4]{x} - 4 = 0$

91. $4(x+1)^{1/2} - 5(x+1)^{3/2} + (x+1)^{5/2} = 0$

92. $x^{1/2} + 3x^{-1/2} = 10x^{-3/2}$

93. $x^{1/2} - 3x^{1/3} = 3x^{1/6} - 9$ **94.** $x - 5\sqrt{x} + 6 = 0$

95. $|2x| = 3$ **96.** $|3x + 5| = 1$

97. $|x - 4| = 0.01$ **98.** $|x - 6| = -1$

Applications

99–100 ■ Falling-Body Problems Suppose an object is dropped from a height h_0 above the ground. Then its height after t seconds is given by $h = -16t^2 + h_0$, where h is measured in feet. Use this information to solve the problem.

99. If a ball is dropped from 288 ft above the ground, how long does it take to reach ground level?

100. A ball is dropped from the top of a building 96 ft tall.

(a) How long will it take to fall half the distance to ground level?

(b) How long will it take to fall to ground level?

101–102 ■ Falling-Body Problems Use the formula $h = -16t^2 + v_0 t$ discussed in Example 9.

101. A ball is thrown straight upward at an initial speed of $v_0 = 40$ ft/s.

(a) When does the ball reach a height of 24 ft?

(b) When does it reach a height of 48 ft?

(c) What is the greatest height reached by the ball?

(d) When does the ball reach the highest point of its path?

(e) When does the ball hit the ground?

102. How fast would a ball have to be thrown upward to reach a maximum height of 100 ft? [*Hint:* Use the discriminant of the equation $16t^2 - v_0 t + h = 0$.]

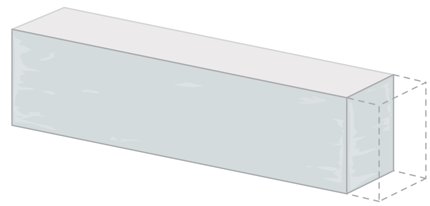
103. Shrinkage in Concrete Beams As concrete dries, it shrinks—the higher the water content, the greater the shrinkage. If a concrete beam has a water content of w kg/m^3, then it will shrink by a factor

$$S = \frac{0.032w - 2.5}{10{,}000}$$

where S is the fraction of the original beam length that disappears due to shrinkage.

(a) A beam 12.025 m long is cast in concrete that contains 250 kg/m^3 water. What is the shrinkage factor S? How long will the beam be when it has dried?

(b) A beam is 10.014 m long when wet. We want it to shrink to 10.009 m, so the shrinkage factor should be $S = 0.00050$. What water content will provide this amount of shrinkage?



104. The Lens Equation If F is the focal length of a convex lens and an object is placed at a distance x from the lens, then its image will be at a distance y from the lens, where F, x, and y are related by the *lens equation*

$$\frac{1}{F} = \frac{1}{x} + \frac{1}{y}$$

Suppose that a lens has a focal length of 4.8 cm, and that the image of an object is 4 cm closer to the lens than the object itself. How far from the lens is the object?

105. Fish Population The fish population in a certain lake rises and falls according to the formula

$$F = 1000(30 + 17t - t^2)$$

Here F is the number of fish at time t, where t is measured in years since January 1, 2002, when the fish population was first estimated.

(a) On what date will the fish population again be the same as on January 1, 2002?

(b) By what date will all the fish in the lake have died?

106. Fish Population A large pond is stocked with fish. The fish population P is modeled by the formula $P = 3t + 10\sqrt{t} + 140$, where t is the number of days since the fish were first introduced into the pond. How many days will it take for the fish population to reach 500?

107. Profit A small-appliance manufacturer finds that the profit P (in dollars) generated by producing x microwave ovens per week is given by the formula $P = \frac{1}{10}x(300 - x)$ provided that $0 \le x \le 200$. How many ovens must be manufactured in a given week to generate a profit of \$1250?

108. Gravity If an imaginary line segment is drawn between the centers of the earth and the moon, then the net

Answer

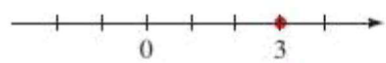
1. **(a)** No **(b)** Yes **3.** **(a)** Yes **(b)** No **5.** 12 **7.** 18
9. -3 **11.** 12 **13.** $-\frac{3}{4}$ **15.** 30 **17.** $-\frac{1}{3}$ **19.** $\frac{13}{3}$ **21.** -2
23. $R = \dfrac{PV}{nT}$ **25.** $R_1 = \dfrac{RR_2}{R_2 - R}$ **27.** $x = \dfrac{2d - b}{a - 2c}$
29. $x = \dfrac{1 - a}{a^2 - a - 1}$ **31.** $r = \pm\sqrt{\dfrac{3V}{\pi h}}$
33. $b = \pm\sqrt{c^2 - a^2}$ **35.** $t = \dfrac{-V_0 \pm \sqrt{V_0^2 + 2gh}}{g}$
37. $-4, 3$ **39.** $3, 4$ **41.** $-\frac{3}{2}, \frac{5}{2}$ **43.** $-2, \frac{1}{3}$ **45.** $-1 \pm \sqrt{6}$
47. $-\frac{7}{2}, \frac{1}{2}$ **49.** $-2 \pm \dfrac{\sqrt{14}}{2}$ **51.** $0, \frac{1}{4}$ **53.** $-3, 5$
55. $\dfrac{-3 \pm \sqrt{5}}{2}$ **57.** $-\frac{3}{2}, 1$ **59.** $\dfrac{1 \pm \sqrt{5}}{4}$ **61.** $-\frac{9}{2}, \frac{1}{2}$
63. $\dfrac{-5 \pm \sqrt{13}}{2}$ **65.** $-\dfrac{\sqrt{6}}{2}, \dfrac{\sqrt{6}}{6}$ **67.** $-\frac{7}{5}$
69. 2 **71.** 1 **73.** No real solution
75. $-\frac{7}{5}, 2$ **77.** $-50, 100$ **79.** -4 **81.** 4 **83.** 3
85. $\pm 2\sqrt{2}, \pm\sqrt{5}$ **87.** No real solution
89. $\pm 3\sqrt{3}, \pm 2\sqrt{2}$ **91.** $-1, 0, 3$ **93.** 27, 729 **95.** $-\frac{3}{2}, \frac{3}{2}$
97. 3.99, 4.01 **99.** 4.24 s **101.** **(a)** After 1 s and $1\frac{1}{2}$ s
(b) Never **(c)** 25 ft **(d)** After $1\frac{1}{4}$ s **(e)** After $2\frac{1}{2}$ s
103. **(a)** 0.00055, 12.018 m **(b)** 234.375 kg/m^3
105. **(a)** After 17 yr, on Jan. 1, 2019 **(b)** After 18.621 yr, on Aug. 12, 2020 **107.** 50 **109.** 132.6 ft

2.2 INEQUALITIES (Adapted from "Precalculus" by Stewart et als.)

Solving Linear Inequalities   Solving Nonlinear Inequalities
Absolute Value Inequalities

Some problems in algebra lead to **inequalities** instead of equations. An inequality looks just like an equation, except that in the place of the equal sign is one of the symbols, $<, >, \leq$ or $\geq$. Here is an example of an inequality: $4x - 7 \leq 19$

To **solve** an inequality that contains a variable means to find all values of the variable that make the inequality true.
Unlike an equation, an inequality generally has infinitely many solutions, which form an interval or a union of intervals on the real line.
The following illustration shows how an inequality differs from its corresponding equation:

		Solution	Graph
Equation:	$4x + 7 = 19$	$x = 3$	
Inequality:	$4x + 7 \leq 19$	$x \leq 3$	

To solve inequalities, we use the following rules to isolate the variable on one side of the inequality sign. These rules tell us when two inequalities are *equivalent* (the symbol $\Leftrightarrow$ means "is equivalent to"). In these rules the symbols A, B, and C stand for real numbers or algebraic expressions. Here we state the rules for inequalities involving the symbol $\leq$, but they apply to all four inequality symbols.

RULES FOR INEQUALITIES

Rule	Description
1. $A \leq B \quad \Leftrightarrow \quad A + C \leq B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \leq B \quad \Leftrightarrow \quad A - C \leq B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \leq B \quad \Leftrightarrow \quad CA \leq CB$	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \leq B \quad \Leftrightarrow \quad CA \geq CB$	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \leq B \quad \Leftrightarrow \quad \frac{1}{A} \geq \frac{1}{B}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$	Inequalities can be added.

Pay special attention to Rules 3 and 4. Rule 3 says that we can multiply (or divide) each side of an inequality by a *positive* number, but Rule 4 says that if we multiply each side of an inequality by a *negative* number, then we reverse the direction of the inequality.
For example, if we start with the inequality $3 < 5$ and multiply by 2, we get $6 < 10$ but if we multiply by -2, we get $-6 > -10$.

▼ Solving Linear Inequalities

An inequality is **linear** if each term is constant or a multiple of the variable. To solve a linear inequality, we isolate the variable on one side of the inequality sign.

EXAMPLE 1 Solving a Linear Inequality

Solve the inequality $3x < 9x - 4$ and sketch the solution set.

SOLUTION

$3x < 9x + 4$	Given inequality
$3x - 9x < 9x + 4 - 9x$	Subtract $9x$
$-6x < 4$	Simplify
$(-\frac{1}{6})(-6x) > (-\frac{1}{6})(4)$	Multiply by $-\frac{1}{6}$ and reverse inequality
$x > -\frac{2}{3}$	Simplify

The solution set is $\left(-\frac{2}{3}, \infty\right)$ interval notation

or $\{x \mid x > -\frac{2}{3}\}$ or $\{x \mid -\frac{2}{3} < x\}$ set builder notation

The solution set shown graphically :

[Three forms given here; you must master all THREE forms.]

NOW TRY: Solve the linear inequality. Express the solution using interval notation and graph the solution set.

$$\tfrac{1}{2}x - \tfrac{2}{3} > 2$$

EXAMPLE 2 Solving a Pair of Simultaneous Inequalities

Solve the inequalities $4 \leq 3x - 2 < 13$.

SOLUTION The solution set consists of all values of x that satisfy both of the inequalities $4 \leq 3x - 2$ and $3x - 2 < 13$.

$4 \leq 3x - 2 < 13$	Given inequality
$6 \leq 3x < 15$	Add 2
$2 \leq x < 5$	Divide by 3

Therefore, the solution set is $[2,5)$, as shown here graphically.

NOW TRY: Solve the linear inequality. Express the solution using interval notation and graph the solution set.

$$-2 < 8 - 2x \leq -1$$

▼ Solving Nonlinear Inequalities

To solve inequalities involving squares and other powers of the variable, we use factoring, together with the following principle.

THE SIGN OF A PRODUCT OR QUOTIENT

If a product or a quotient has an *even* number of *negative* factors, then its value is *positive*.

If a product or a quotient has an *odd* number of *negative* factors, then its value is *negative*.

For example, to solve the inequality $x^2 - 5x \le -6$, we first move all terms to the left-hand side and factor to get $(x-2)(x-3) \le 0$.

This form of the inequality says that the product must be negative or zero, so to solve the inequality, we must determine where each factor is negative or positive (because the sign of a product depends on the sign of the factors).

The details are explained in Example 3, in which we use the following guidelines.

GUIDELINES FOR SOLVING NONLINEAR INEQUALITIES

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals that are determined by these numbers.
4. **Make a Table or Diagram.** Use test values to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Determine the solution of the inequality from the last row of the sign table. Be sure to check whether the inequality is satisfied by some or all of the endpoints of the intervals. (This may happen if the inequality involves $\le$ or $\ge$.)

The factoring technique that is described in these guidelines works only if all nonzero terms appear on one side of the inequality symbol. If the inequality is not written in this form, first rewrite it, as indicated in Step 1.

E X A M P L E 3 | Solving a Quadratic Inequality

Solve the inequality $x^2 \le 5x - 6$.

S O LU T I O N We will follow the guidelines given above.

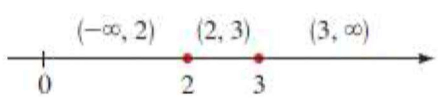
Given inequality $\qquad x^2 \le 5x - 6$

1. Move all terms to one side

Subtract $5x$, add 6

2. Factor

3. Find the intervals

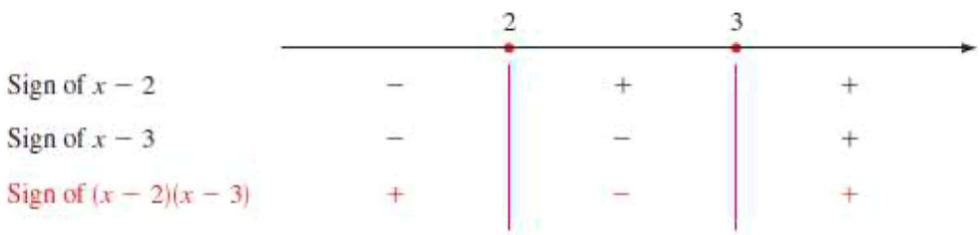


$(-\infty, 2), (2, 3), (3, \infty)$

4. Make a table or diagram

Interval	$(-\infty, 2)$	$(2, 3)$	$(3, \infty)$
Sign of $x-2$ Sign of $x-3$	$-$ $-$	$+$ $-$	$+$ $+$
Sign of $(x-2)(x-3)$	$+$	$-$	$+$

or



5. Solve We read from the table or the diagram that $(x-2)(x-3)$ is negative on the interval (2, 3). [*You must check if any of the endpoints must be included in the solution*.]
Thus, the solution of the inequality $(x-2)(x-3) \le 0$ is [2, 3].

[Two other forms: $\{x \mid 2 \le x \le 3\}$ or ]

[The endpoints 2 and 3 are included because we seek values of x such that the product is either less than *or equal to* zero. The solution is illustrated here.
]

After the factoring step, there are other ways of presenting the answer in addition to the two ways shown here. We may discuss a third way.

NOW TRY: Solve the nonlinear inequality. Express the solution using interval notation and graph the solution set. $2x^2 + x \ge 1$

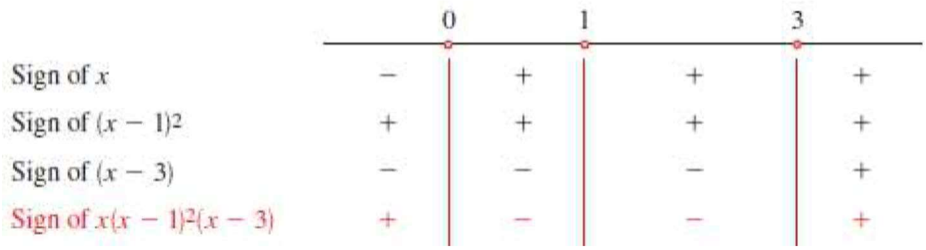
E X A M P L E 4 | Solving an Inequality with Repeated Factors
Solve the inequality $x(x-1)^2(x-3) < 0$.

S O LU T I O N All nonzero terms are already on one side of the inequality, and the nonzero side of the inequality is already factored. So we begin by finding the intervals for this inequality.

Find the intervals. The factors of the left-hand side are x, $(x-1)^2$, and $x - 3$. These are zero when $x = 0, 1, 3$. These numbers divide the real line into the intervals

$$(-\infty,0), (0,1), (1,3), (3,\infty)$$

Make a diagram. (*Test points could be used to determine the sign of each factor in each interval.*)



Solve. From the diagram we see that $x(x-1)^2(x-3)<0$ for x in the interval (0, 1) or for x in (1, 3). So the solution set is the union of these two intervals $(0,1)\cup(1,3)$

The solution set is graphed here.

[*Check endpoints*?]

NOW TRY: Solve the nonlinear inequality. Express the solution using interval notation and graph the solution set. $(x-2)^2(x-3)(x+1)\le 0$

E X A M P L E 5 Solving an Inequality Involving a Quotient

Solve the inequality $\dfrac{1+x}{1-x}\ge 1$.

S O LU T I O N

Move all terms to one side.

(We move the terms to the left-hand side and simplify using a common denominator.)

Given inequality

Subtract 1

Common denominator 1 – x

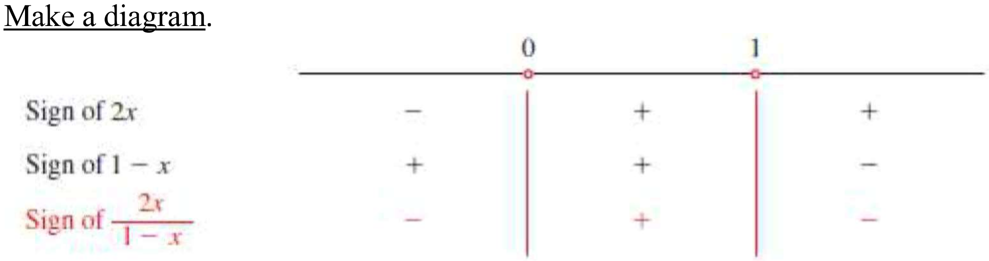
Combine the fractions

Simplify

Find the intervals. The factors of the left-hand side are $2x$ and $1-x$. These are zero when x is 0 and 1. These numbers divide the real line into the intervals

$$(-\infty,0),\ (0,1),\ (1,\infty)$$

Make a diagram.



Solve. From the diagram, the solution set is the interval [0, 1)

[*Check endpoints*?]

[We see that for x in the interval (0, 1). We include the endpoint 0 because the original inequality requires that the quotient be greater than *or equal to* 1. However, we do not include the other endpoint 1 because the quotient in the inequality is not defined at 1.]

The solution set is graphed here.

NOW TRY: Solve the nonlinear inequality $\dfrac{4x}{2x+3} > 2$.

Express the solution using interval notation and graph the solution set.

Example 5 shows that we should always check the endpoints of the solution set to see whether they satisfy the original inequality.

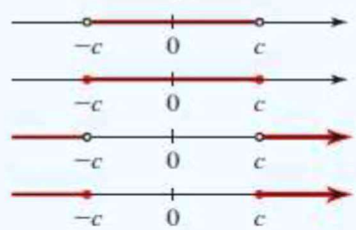
▼ Absolute Value Inequalities

We use the following properties to solve inequalities that involve absolute value.

PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

Inequality	Equivalent form
1. $\lvert x\rvert < c$	$-c < x < c$
2. $\lvert x\rvert \le c$	$-c \le x \le c$
3. $\lvert x\rvert > c$	$x < -c$ or $c < x$
4. $\lvert x\rvert \ge c$	$x \le -c$ or $c \le x$

Graph



These properties can be proved using the definition of absolute value. To prove Property 1, for example, note that the inequality $|x| < c$ says that the distance from x to 0 is less than c, and this is true if and only if x is between $-c$ and c.

E X A M P L E 6 Solving an Absolute Value Inequality

Solve the inequality $|x-5| < 2$.

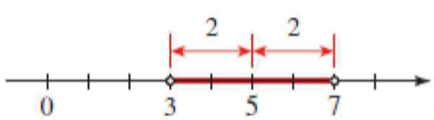
S O LU T I O N 1 The inequality $|x-5| < 2$ is equivalent to

$$-2 < x - 5 < 2 \qquad \text{Property 1}$$

$$3 < x < 7 \qquad \text{Add 5}$$

The solution set is the open interval $(3, 7)$.

S O LUT I O N 2 Geometrically, the solution set consists of all numbers x whose distance from 5 is less than 2. From



we see that this is the interval $(3,7)$.

NOW TRY: Solve the absolute value inequality $|2x-3| \le 0.4$

. Express the answer using interval notation and graph the solution set.

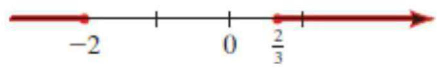
E X A M P L E 7 Solving an Absolute Value Inequality

Solve the inequality . $|3x+2| \ge 4$

S O LU T I O N The inequality is equivalent to

$$\begin{aligned} 3x + 2 &\geq 4 \quad &\text{or} \quad 3x + 2 &\leq -4 & \\ 3x &\geq 2 & 3x &\leq -6 & \quad \text{Subtract 2} \\ x &\geq \tfrac{2}{3} & x &\leq -2 & \quad \text{Divide by 3} \end{aligned}$$

So the solution set is $\{x \mid x \leq -2 \quad \text{or} \quad x \geq \tfrac{2}{3}\} = (-\infty, -2] \cup [\tfrac{2}{3}, \infty)$
The set is graphed here.



NOW TRY: Solve the absolute value inequality $\left|\dfrac{x-2}{3}\right| < 2$.
Express the answer using interval notation and graph the solution set.

(nby, Mar 2016)

1–6 ■ Let $S = \{-2, -1, 0, \frac{1}{2}, 1, \sqrt{2}, 2, 4\}$. Determine which elements of S satisfy the inequality.

1. $3 - 2x \le \frac{1}{2}$

2. $2x - 1 \ge x$

3. $1 < 2x - 4 \le 7$

4. $-2 \le 3 - x < 2$

5. $\dfrac{1}{x} \le \dfrac{1}{2}$

6. $x^2 + 2 < 4$

7–28 ■ Solve the linear inequality. Express the solution using interval notation and graph the solution set.

7. $2x - 5 > 3$

8. $3x + 11 < 5$

9. $7 - x \ge 5$

10. $5 - 3x \le -16$

11. $2x + 1 < 0$

12. $0 < 5 - 2x$

13. $3x + 11 \le 6x + 8$

14. $6 - x \ge 2x + 9$

15. $\frac{1}{2}x - \frac{2}{3} > 2$

16. $\frac{2}{5}x + 1 < \frac{1}{5} - 2x$

17. $\frac{1}{3}x + 2 < \frac{1}{6}x - 1$

18. $\frac{2}{3} - \frac{1}{2}x \ge \frac{1}{6} + x$

19. $4 - 3x \le -(1 + 8x)$

20. $2(7x - 3) \le 12x + 16$

21. $2 \le x + 5 < 4$

22. $5 \le 3x - 4 \le 14$

23. $-1 < 2x - 5 < 7$

24. $1 < 3x + 4 \le 16$

25. $-2 < 8 - 2x \le -1$

26. $-3 \le 3x + 7 \le \frac{1}{2}$

27. $\dfrac{1}{6} < \dfrac{2x - 13}{12} \le \dfrac{2}{3}$

28. $-\dfrac{1}{2} \le \dfrac{4 - 3x}{5} \le \dfrac{1}{4}$

29–62 ■ Solve the nonlinear inequality. Express the solution using interval notation and graph the solution set.

29. $(x + 2)(x - 3) < 0$

30. $(x - 5)(x + 4) \ge 0$

31. $x(2x + 7) \ge 0$

32. $x(2 - 3x) \le 0$

33. $x^2 - 3x - 18 \le 0$

34. $x^2 + 5x + 6 > 0$

35. $2x^2 + x \ge 1$

36. $x^2 < x + 2$

37. $3x^2 - 3x < 2x^2 + 4$

38. $5x^2 + 3x \ge 3x^2 + 2$

39. $x^2 > 3(x + 6)$

40. $x^2 + 2x > 3$

41. $x^2 < 4$

42. $x^2 \ge 9$

43. $-2x^2 \le 4$

44. $(x + 2)(x - 1)(x - 3) \le 0$

45. $x^3 - 4x > 0$

46. $16x \le x^3$

47. $\dfrac{x - 3}{x + 1} \ge 0$

48. $\dfrac{2x + 6}{x - 2} < 0$

49. $\dfrac{4x}{2x + 3} > 2$

50. $-2 < \dfrac{x + 1}{x - 3}$

51. $\dfrac{2x+1}{x-5} \le 3$

52. $\dfrac{3+x}{3-x} \ge 1$

53. $\dfrac{4}{x} < x$

54. $\dfrac{x}{x+1} > 3x$

55. $1 + \dfrac{2}{x+1} \le \dfrac{2}{x}$

56. $\dfrac{3}{x-1} - \dfrac{4}{x} \ge 1$

57. $\dfrac{6}{x-1} - \dfrac{6}{x} \ge 1$

58. $\dfrac{x}{2} \ge \dfrac{5}{x+1} + 4$

59. $\dfrac{x+2}{x+3} < \dfrac{x-1}{x-2}$

60. $\dfrac{1}{x+1} + \dfrac{1}{x+2} \le 0$

61. $x^4 > x^2$

62. $x^5 > x^2$

63–76 ■ Solve the absolute value inequality. Express the answer using interval notation and graph the solution set.

63. $|x| \le 4$

64. $|3x| < 15$

65. $|2x| > 7$

66. $\frac{1}{2}|x| \ge 1$

67. $|x-5| \le 3$

68. $|x+1| \ge 1$

69. $|2x-3| \le 0.4$

70. $|5x-2| < 6$

71. $\left|\dfrac{x-2}{3}\right| < 2$

72. $\left|\dfrac{x+1}{2}\right| \ge 4$

73. $|x+6| < 0.001$

74. $3 - |2x+4| \le 1$

75. $8 - |2x-1| \ge 6$

76. $7|x+2| + 5 > 4$

77–80 ■ A phrase describing a set of real numbers is given. Express the phrase as an inequality involving an absolute value.

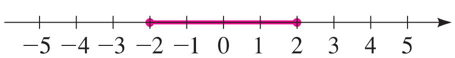
77. All real numbers x less than 3 units from 0

78. All real numbers x more than 2 units from 0

79. All real numbers x at least 5 units from 7

80. All real numbers x at most 4 units from 2

81–86 ■ A set of real numbers is graphed. Find an inequality involving an absolute value that describes the set.

81.


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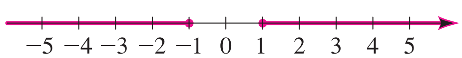

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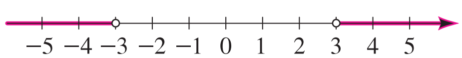

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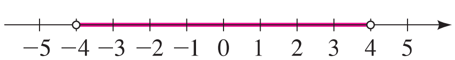

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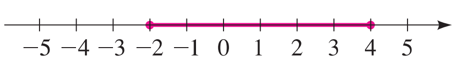

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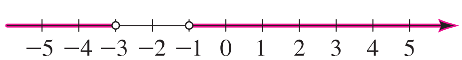


87–90 ■ Determine the values of the variable for which the expression is defined as a real number.

87. $\sqrt{16 - 9x^2}$

88. $\sqrt{3x^2 - 5x + 2}$

89. $\left(\dfrac{1}{x^2 - 5x - 14}\right)^{1/2}$

90. $\sqrt[4]{\dfrac{1-x}{2+x}}$

91. Solve the inequality for x, assuming that a, b, and c are positive constants.

(a) $a(bx - c) \ge bc$ **(b)** $a \le bx + c < 2a$

92. Suppose that a, b, c, and d are positive numbers such that

$$\frac{a}{b} < \frac{c}{d}$$

Show that $\dfrac{a}{b} < \dfrac{a+c}{b+d} < \dfrac{c}{d}$

Applications

93. **Temperature Scales** Use the relationship between C and F given in Example 9 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \le C \le 30$.

94. **Temperature Scales** What interval on the Celsius scale corresponds to the temperature range $50 \le F \le 95$?

95. **Car Rental Cost** A car rental company offers two plans for renting a car.

Plan A: \$30 per day and 10¢ per mile
Plan B: \$50 per day with free unlimited mileage

For what range of miles will plan B save you money?

96. **Long-Distance Cost** A telephone company offers two long-distance plans.

Plan A: \$25 per month and 5¢ per minute
Plan B: \$5 per month and 12¢ per minute

For how many minutes of long-distance calls would plan B be financially advantageous?

97. **Driving Cost** It is estimated that the annual cost of driving a certain new car is given by the formula

$$C = 0.35m + 2200$$

where m represents the number of miles driven per year and C is the cost in dollars. Jane has purchased such a car, and decides to budget between \$6400 and \$7100 for next year's driving costs. What is the corresponding range of miles that she can drive her new car?

98. **Gas Mileage** The gas mileage g (measured in mi/gal) for a particular vehicle, driven at v mi/h, is given by the formula $g = 10 + 0.9v - 0.01v^2$, as long as v is between 10 mi/h and 75 mi/h. For what range of speeds is the vehicle's mileage 30 mi/gal or better?

99. Gravity The gravitational force F exerted by the earth on an object having a mass of 100 kg is given by the equation

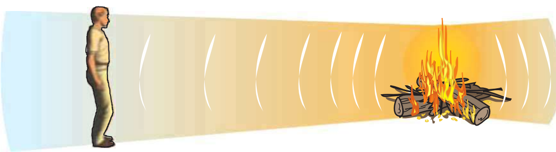
$$F = \frac{4{,}000{,}000}{d^2}$$

where d is the distance (in km) of the object from the center of the earth, and the force F is measured in newtons (N). For what distances will the gravitational force exerted by the earth on this object be between 0.0004 N and 0.01 N?

100. Bonfire Temperature In the vicinity of a bonfire, the temperature T in °C at a distance of x meters from the center of the fire was given by

$$T = \frac{600{,}000}{x^2 + 300}$$

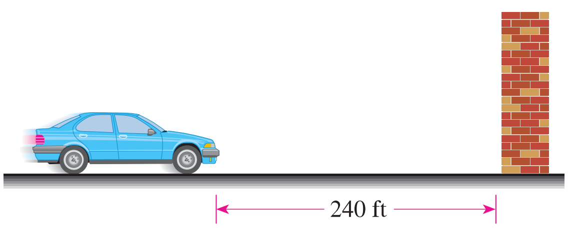
At what range of distances from the fire's center was the temperature less than 500°C?



101. Stopping Distance For a certain model of car the distance d required to stop the vehicle if it is traveling at v mi/h is given by the formula

$$d = v + \frac{v^2}{20}$$

where d is measured in feet. Kerry wants her stopping distance not to exceed 240 ft. At what range of speeds can she travel?



102. Manufacturer's Profit If a manufacturer sells x units of a certain product, his revenue R and cost C (in dollars) are given by:

$$R = 20x$$
$$C = 2000 + 8x + 0.0025x^2$$

Use the fact that

$$\text{profit} = \text{revenue} - \text{cost}$$

to determine how many units he should sell to enjoy a profit of at least \$2400.

103. Air Temperature As dry air moves upward, it expands and in so doing cools at a rate of about 1 °C for each 100-meter rise, up to about 12 km.

(a) If the ground temperature is 20 °C, write a formula for the temperature at height h.

(b) What range of temperatures can be expected if a plane takes off and reaches a maximum height of 5 km?

104. Airline Ticket Price A charter airline finds that on its Saturday flights from Philadelphia to London, all 120 seats will be sold if the ticket price is \$200. However, for each \$3 increase in ticket price, the number of seats sold decreases by one.

(a) Find a formula for the number of seats sold if the ticket price is P dollars.

(b) Over a certain period, the number of seats sold for this flight ranged between 90 and 115. What was the corresponding range of ticket prices?

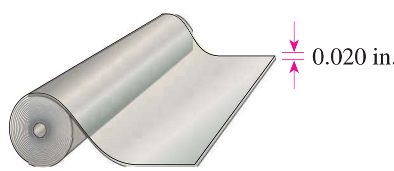
105. Theater Tour Cost A riverboat theater offers bus tours to groups on the following basis. Hiring the bus costs the group \$360, to be shared equally by the group members. Theater tickets, normally \$30 each, are discounted by 25¢ times the number of people in the group. How many members must be in the group so that the cost of the theater tour (bus fare plus theater ticket) is less than \$39 per person?

106. Fencing a Garden A determined gardener has 120 ft of deer-resistant fence. She wants to enclose a rectangular vegetable garden in her backyard, and she wants the area enclosed to be at least 800 ft^2. What range of values is possible for the length of her garden?

107. Thickness of a Laminate A company manufactures industrial laminates (thin nylon-based sheets) of thickness 0.020 in, with a tolerance of 0.003 in.

(a) Find an inequality involving absolute values that describes the range of possible thickness for the laminate.

(b) Solve the inequality you found in part (a).



108. Range of Height The average height of adult males is 68.2 in, and 95% of adult males have height h that satisfies the inequality

$$\left| \frac{h - 68.2}{2.9} \right| \le 2$$

Solve the inequality to find the range of heights.

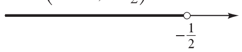
Section 1.7 ■ page 84

1. $\{\sqrt{2}, 2, 4\}$ **3.** $\{4\}$ **5.** $\{-2, -1, 2, 4\}$

7. $(4, \infty)$

9. $(-\infty, 2]$

11. $\left(-\infty, -\frac{1}{2}\right)$

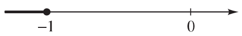


13. $[1, \infty)$

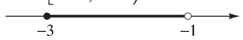
15. $\left(\frac{16}{3}, \infty\right)$

17. $(-\infty, -18)$

19. $(-\infty, -1]$



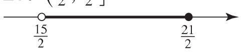
21. $[-3, -1)$



23. $(2, 6)$

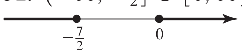
25. $\left[\frac{9}{2}, 5\right)$

27. $\left(\frac{15}{2}, \frac{21}{2}\right]$

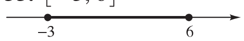


29. $(-2, 3)$

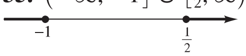
31. $\left(-\infty, -\frac{7}{2}\right] \cup [0, \infty)$



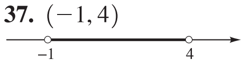
33. $[-3, 6]$



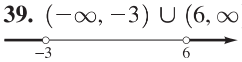
35. $(-\infty, -1] \cup \left[\frac{1}{2}, \infty\right)$



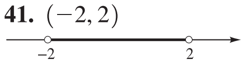
37. $(-1, 4)$



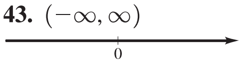
39. $(-\infty, -3) \cup (6, \infty)$



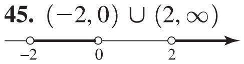
41. $(-2, 2)$



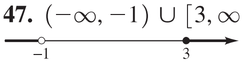
43. $(-\infty, \infty)$



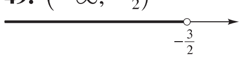
45. $(-2, 0) \cup (2, \infty)$



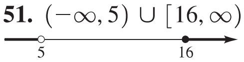
47. $(-\infty, -1) \cup [3, \infty)$



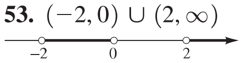
49. $\left(-\infty, -\frac{3}{2}\right)$



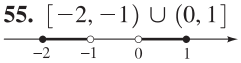
51. $(-\infty, 5) \cup [16, \infty)$



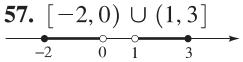
53. $(-2, 0) \cup (2, \infty)$



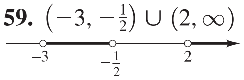
55. $[-2, -1) \cup (0, 1]$



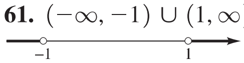
57. $[-2, 0) \cup (1, 3]$



59. $\left(-3, -\frac{1}{2}\right) \cup (2, \infty)$

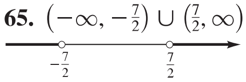


61. $(-\infty, -1) \cup (1, \infty)$



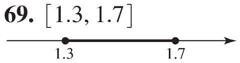
63. $[-4, 4]$

65. $\left(-\infty, -\frac{7}{2}\right) \cup \left(\frac{7}{2}, \infty\right)$

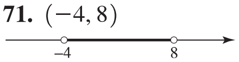


67. $[2, 8]$

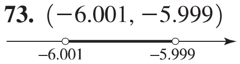
69. $[1.3, 1.7]$



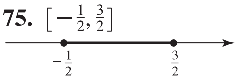
71. $(-4, 8)$



73. $(-6.001, -5.999)$



75. $\left[-\frac{1}{2}, \frac{3}{2}\right]$



77. $|x| < 3$

79. $|x - 7| \geq 5$ **81.** $|x| \leq 2$ **83.** $|x| > 3$

85. $|x - 1| \leq 3$ **87.** $-\frac{4}{3} \leq x \leq \frac{4}{3}$ **89.** $x < -2$ or $x > 7$