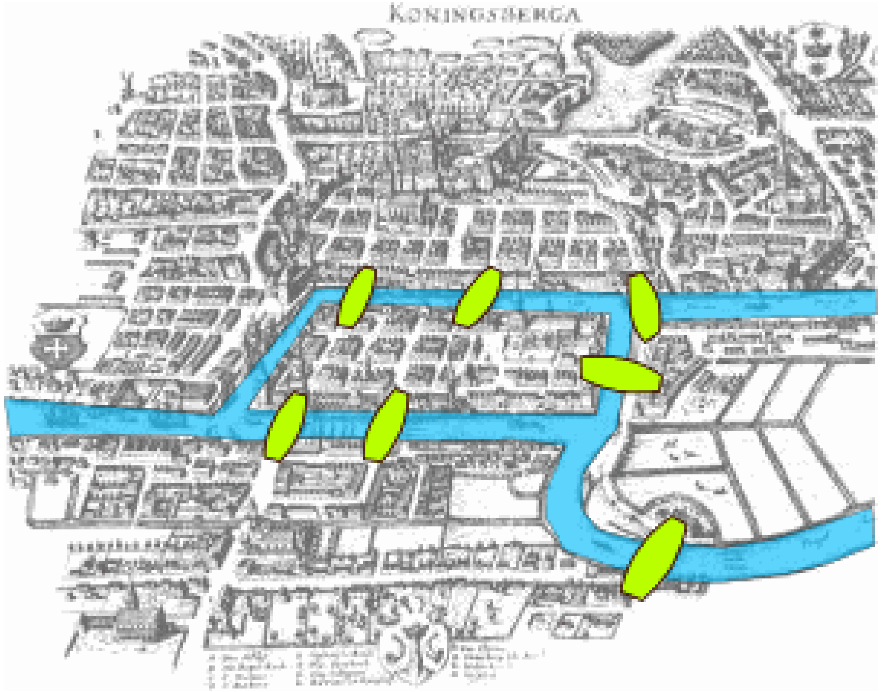
Topic 6.2
Trail, Path and Circuit

TMA1201 Discrete Structures & Probability
Faculty of Computing & Informatics
Multimedia University

What you will learn in this lecture:

- Walk, trail, path, closed walk, circuit, simple circuit
- Euler trail
- Euler circuit
- Hamiltonian path
- Hamiltonian circuit

The seven bridges of Königsberg



Question:
Is there a way for a citizen of Königsberg to find a route which crosses each bridge exactly once?

Some definitions

Let G be a graph, and let v and w be vertices in G.

A **walk from *v* to *w*** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for all $i = 1, 2, \ldots n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from *v* to *v*** consists of the single vertex v.

A **trail from *v* to *w*** is a walk from v to w that does not contain a repeated edge.

A **path from *v* to *w*** is a trail that does not contain a repeated vertex.

A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

Some definitions

- The definition are summarized as in the table:

	Repeated Edge?	Repeated Vertex?	Starts and Ends at Same Point?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple circuit	no	first and last only	yes	yes

Some definitions

Example 1:

- A **walk** from v2 to v6

  v2 e2 v3 e5 v5 e6 v4 e6 v5 e8 v6

- A **trail** from v2 to v6

  v2 e2 v3 e5 v5 e6 v4 e4 v3 e7 v6

- A **path** from v2 to v6

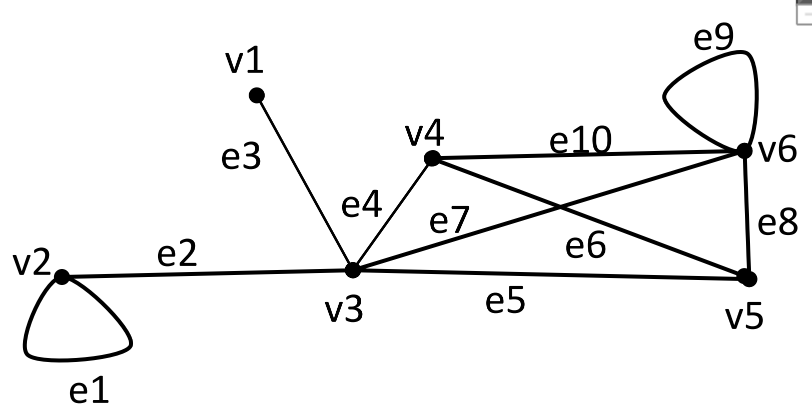
  v2 e2 v3 e5 v5 e8 v6

- A **closed walk**

  v2 e2 v3 e5 v5 e6 v4 e4 v3 e2 v2

- A **circuit** which is also a **simple circuit**

  v3 e5 v5 e8 v6 e7 v3



Some definitions

<u>Example 2:</u>

Determine which of the following walks are trails, paths, circuits, or simple circuits.

a) $v_1\, e_1\, v_2\, e_3\, v_3\, e_4\, v_3\, e_5\, v_4$

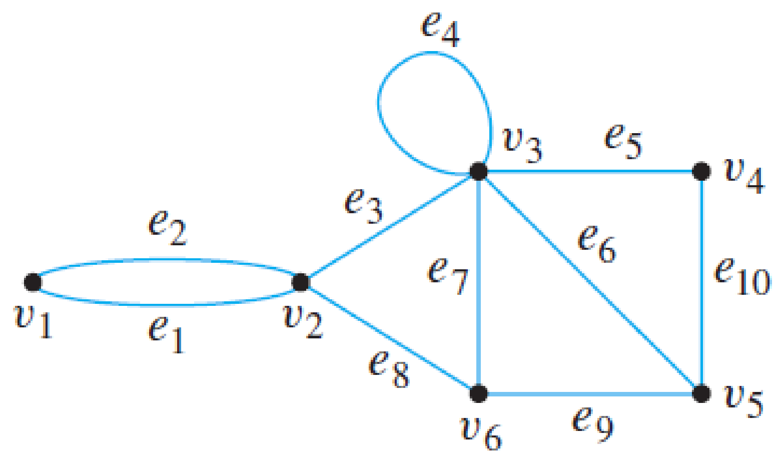
b) $e_1\, e_3\, e_5\, e_5\, e_6$

c) $v_2\, v_3\, v_4\, v_5\, v_3\, v_6\, v_2$

d) $v_2\, v_3\, v_4\, v_5\, v_6\, v_2$

e) $v_1\, e_1\, v_2\, e_1\, v_1$

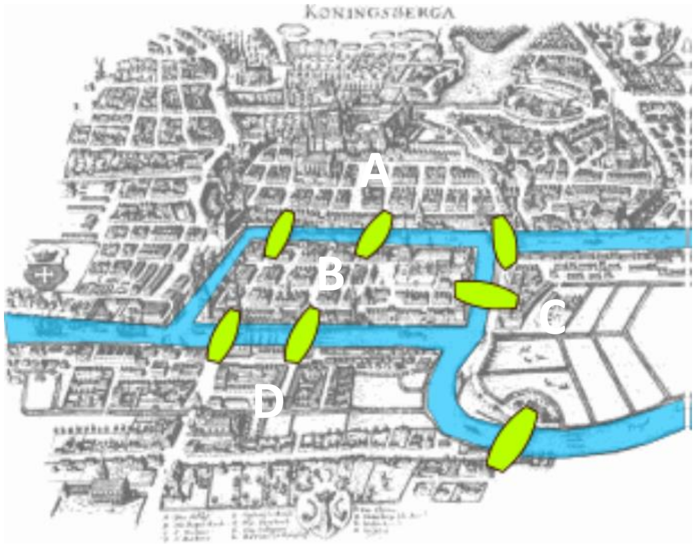
f) v_1

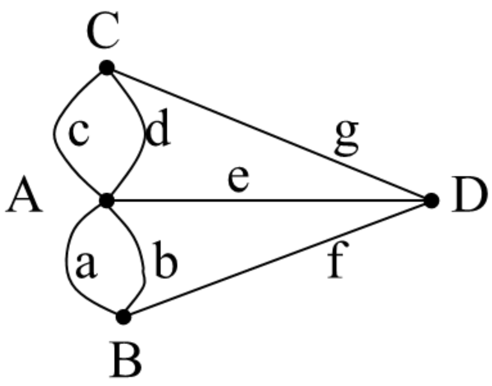


<u>Solution:</u>

The seven bridges of Königsberg

- The bridges of Königsberg problem is often regarded as the birthplace of graph theory. Leonhard Euler published a paper in 1736 giving a solution to the above puzzle. He translated the puzzle (on the left) into a graph theory problem (on the right):





Is there a simple circuit that contains every edge in the graph?

Euler trail and Euler circuit

Let G be a graph

- An **Euler trail** for G is a walk of G that contains every edge of G exactly once and every vertex of G at least once.
- An **Euler circuit** is an Euler trail that starts and ends at the same vertex.

Theorem (on the Existence of Euler Circuit)

- A connected graph *G* has an Euler circuit if and only if every vertex of *G* has a positive even degree.

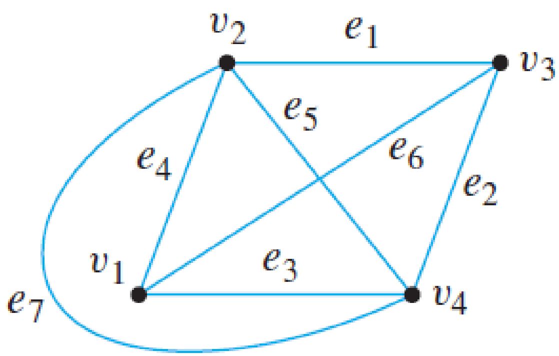
Corollary (on the Existence of Euler Trail)

- A connected graph has an Euler trail if and only if it has at most two vertices with odd degrees.

Euler trail and Euler circuit

Example 3:

Show that the graph below does not have an Euler circuit.

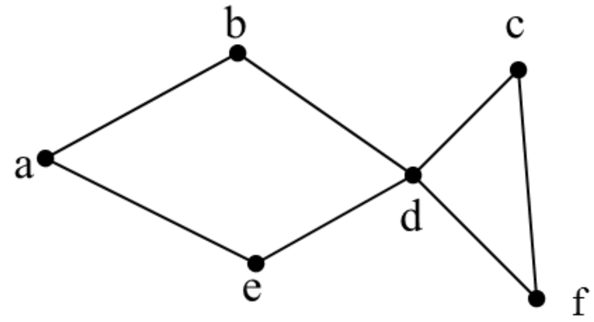


Solution:

Euler trail and Euler circuit

Example 4:

Does the graph below has an Euler circuit and/or an Euler trail?



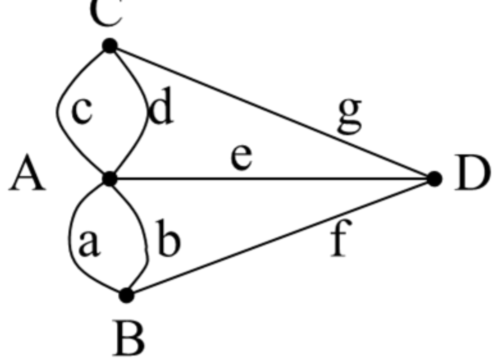
deg(a) = 2, deg(b) = 2, deg(c) = 2, deg(d) = 4, deg(e) = 2, deg(f) = 2

All vertices are even number degrees, the graph may have an Euler circuit.

Euler trail and Euler circuit

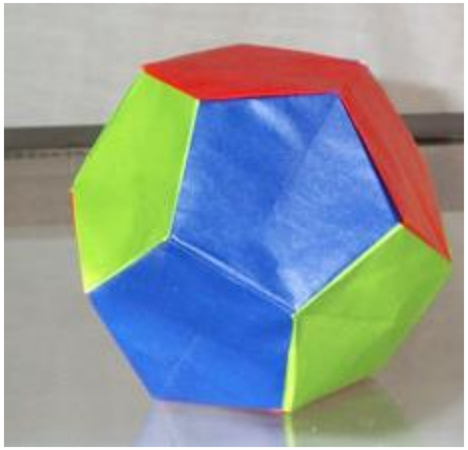
Example 5:

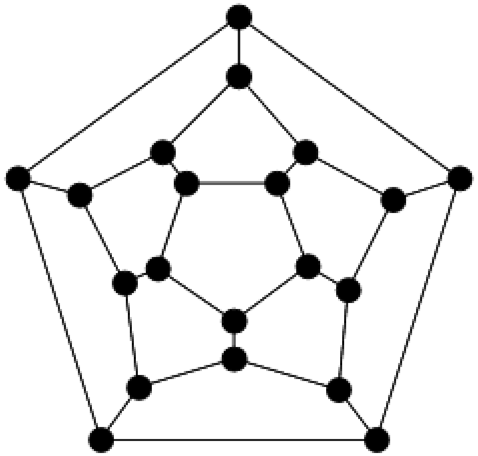
Euler proved that it is impossible to cross all the 7 bridges exactly once. Why?



The Dodecahedron

- Sir William Rowan Hamilton (1805 - 1865) invented a game called Icosian puzzle, which consisted of a wooden dodecahedron (a polyhedron with 12 regular pentagons as faces).

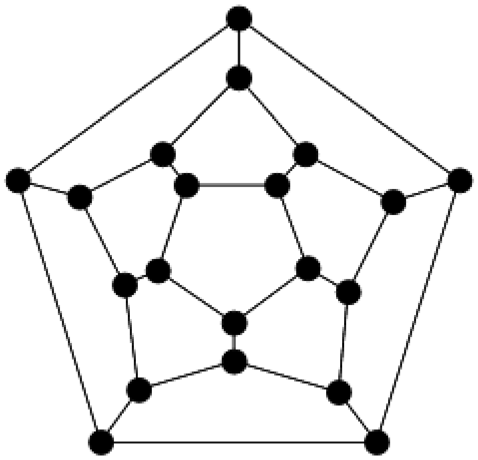




- The 20 vertices were labeled with different cities in the world. The objective of the puzzle was to start at a city and travel along the edges of the dodecahedron, visiting each of the other 19 cities exactly once, and end back at the starting city.

The Dodecahedron

- So the question is:
  Is it possible to visit all vertices in the graph exactly once and come back to the starting vertex again?



Hamiltonian path and Hamiltonian circuit

Given a graph G,

- A **Hamiltonian path** of G is a path that passes every vertex of G exactly once.
- A **Hamiltonian circuit** of G is a simple circuit that includes every vertex of *G*.

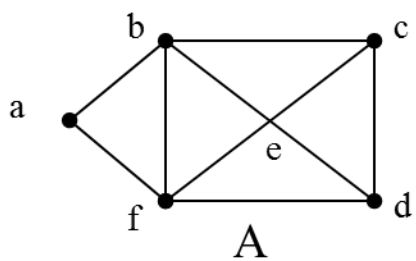
There are no known simple necessary and sufficient criteria for the existence of Hamiltonian circuits.

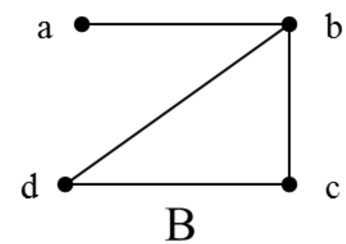
Certain properties can be used to show that a graph *does not have* a Hamiltonian circuit, for example, a graph with a vertex of degree one cannot have a Hamiltonian circuit.

Hamiltonian circuit

<u>Example 6:</u>

Do the graphs below have Hamiltonian circuits?





<u>Solution:</u>

Graph A has a Hamiltonian circuit, namely *abcdefa*

Graph B does not have a Hamiltonian circuit, note that any circuit containing every vertex must contain the edge *ab* twice.

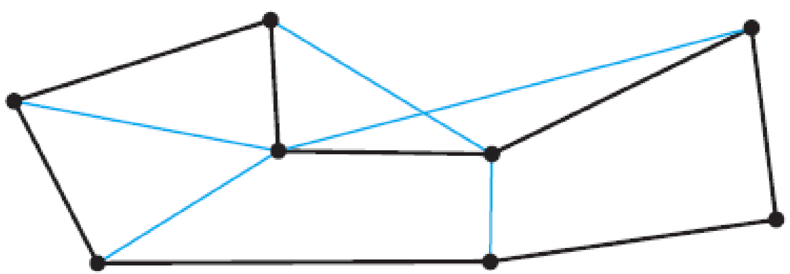
Hamiltonian circuit

Proposition:

If a graph *G* has a Hamiltonian circuit, then *G* has a subgraph *H* with the following properties:

1. *H* contains every vertex of *G*.
2. *H* is connected.
3. *H* has the same number of edges as vertices.
4. Every vertex of *H* has degree 2.

An example of such an *H* is shown below:

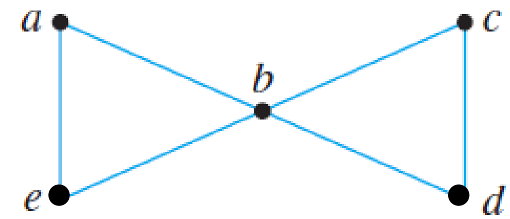


H is indicated by the black lines.

Hamiltonian circuit

<u>**Example 7:**</u>

Show that the graph G does not have a Hamiltonian circuit.



<u>Solution:</u>

If G has a Hamiltonian circuit, then by the Proposition, G has a subgraph H that

(1) contains every vertex of G *(vertices {a,b,c,d,e})*

(2) is connected,

(3) has the same number of edges as vertices *(five edges)*

(4) is such that every vertex has degree 2.

If H exists, then two edges incident on b must be removed from G to create H. Edge $\{a, b\}$ cannot be removed because if it were, vertex a would have degree less than 2 in H. Edges $\{e, b\}$, $\{b, a\}$, and $\{b, d\}$ cannot be removed either with similar reason.

This contradicts the condition that every vertex in H has degree 2 in H.

Summary

Materials covered in this lecture?

- Definition of walk, trail, path.
- Definition of closed walk, circuit, simple circuit.
- Definition of Euler trail and Euler circuit.
- How to determine the existence of Euler trail and Euler circuit.
- Definition of Hamiltonian path and Hamiltonian circuit.
- How to determine the existence of Hamiltonian circuit.

Exercise 1

Give an example of a graph in which every vertex has degree 2, but has no Euler trail.

Exercise 2

Determine whether the graphs have Euler circuits. If the graph does not have an Euler circuit, explain why not. If it does have an Euler circuit, describe one.