

## Tutorial 7: Sequences and Series

### 7.1 Sequences

1. Find a formula for the general term  $a_n$  of the sequence, assuming that the pattern of the first few terms continues.

(a)  $\{2, 5, 8, 11, \dots\}$

(b)  $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right\}$

(c)  $\left\{-1, -\frac{1}{3}, \frac{3}{5}, -\frac{5}{7}, \dots\right\}$

(d)  $\{1, 0, 1, 0, 1, \dots\}$

(e)  $\left\{0, \frac{1}{2}, 0, \frac{1}{2}, 0, \dots\right\}$

2. Does the sequence  $\left\{\frac{\cos n}{n}\right\}$  converge or diverge? Find the limit if it is a convergent sequence.

3. Determine whether the given sequence is convergent. If the sequence is convergent, find its limit.

(a)  $\left\{\frac{3n^4}{2n^4 + 1}\right\}$

(b)  $\left\{\left(\frac{1}{2}\right)^n + \frac{1}{\sqrt{3^n}}\right\}$

(c)  $\left\{\frac{n^2 + n + 8}{4n^3 + n^2}\right\}$

(d)  $\left\{\frac{1 + n^4}{n^3 + 3000}\right\}$

(e)  $\left\{\frac{\ln n}{n^{\frac{1}{n}}}\right\}$

(f)  $\left\{\sin\left(\frac{\pi}{2} + \frac{1}{n}\right)\right\}$

(g)  $\left\{\frac{\sin^2 n}{\sqrt{n}}\right\}$

(h)  $\left\{\sqrt[n]{10n}\right\}$

(i)  $\left\{\frac{n!}{(n+2)!}\right\}$

(j)  $\left\{\left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}}\right\}$

(k)  $\left\{n - \sqrt{n^2 - n}\right\}$

(l)  $\left\{\frac{n}{5 + \sqrt{n}}\right\}$

[Note: (i) Knowing  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$  may be quite useful.

(ii) The following property may be useful for (f):

If  $f(x)$  is a continuous function and the sequence  $a_n \rightarrow L$ , then  $f(a_n) \rightarrow f(L)$ .]

## 7.2 Series

1. Write down the first four terms of the following series. Does the series converge or diverge? Find the sum if it converges. Explain.

$$\sum_{n=1}^{\infty} \frac{2}{10^n}$$

2. The series  $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \dots$  can be written in the form  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  for a suitable value of  $p$ . What is the value of  $p$  here? Is this series convergent or divergent?

3. Determine whether the given series is convergent or divergent. (Try to use the properties of geometric series or p-series, or the divergence test.) For series that converges, find its sum.

(a)  $\frac{1}{4} + \frac{1}{2} + 1 + 2 + \dots$

(b)  $3 + 0.3 + 0.03 + 0.003 + \dots$

(c)  $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \dots$

(d)  $1 + \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{3}} + \frac{4}{\sqrt{4}} + \dots$

(e)  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$

(f)  $\sum_{n=1}^{\infty} \frac{2}{10^n}$

(g)  $\sum_{n=0}^{\infty} \left( \frac{2^n - 1}{3^n} \right)$

(h)  $\sum_{n=1}^{\infty} \frac{3n^2}{5n^2 + 1}$

(i)  $\sum_{n=1}^{\infty} (-1)^n$

(j)  $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)(n+3)}$

(k)  $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$

(l)  $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$

(m)  $\sum_{n=1}^{\infty} e^{-n}$

4. Determine whether the given series is convergent or divergent using Comparison Test.

(a)  $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

(b)  $\sum_{n=1}^{\infty} \frac{n^2}{n^4 + 1}$

(c)  $\sum_{n=1}^{\infty} \frac{1}{1000n + 1}$

(d)  $\sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-n}$

(e)  $\sum_{n=1}^{\infty} \frac{n+2}{n\sqrt{n}}$

(f)  $\sum_{n=1}^{\infty} \frac{n+2}{n^2\sqrt{n}}$

$$\begin{array}{ll} \text{(g)} & \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}} \\ \text{(i)} & \sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n} \end{array} \quad \begin{array}{ll} \text{(h)} & \sum_{n=1}^{\infty} \frac{3-(-1)^n}{n\sqrt{n}} \\ \text{(j)} & \sum_{n=1}^{\infty} \frac{1+5^n}{1+3^n} \end{array}$$

5. Use the ratio test or the root test to decide on the convergence of the series  $\sum_{n=0}^{\infty} a_n$ .

$$\begin{array}{lll} \text{(a)} & \sum_{n=1}^{\infty} \frac{n^2}{3^n} & \text{(b)} \quad \sum_{n=1}^{\infty} n \left( \frac{3}{4} \right)^n \\ \text{(d)} & \sum_{n=1}^{\infty} \frac{10^n}{n^n} & \text{(e)} \quad \sum_{n=1}^{\infty} \frac{10^n}{n!} \\ \text{(g)} & \sum_{n=1}^{\infty} \frac{10^n}{n \cdot 3^{2n}} & \text{(h)} \quad \sum_{n=1}^{\infty} \left( \frac{2n+5}{4n+1} \right)^n \end{array} \quad \begin{array}{l} \text{(c)} \quad \sum_{k=1}^{\infty} \frac{k!}{k^k} \\ \text{(f)} \quad \sum_{n=1}^{\infty} \frac{10^n}{n \cdot 4^{2n}} \\ \text{(i)} \quad \sum_{n=1}^{\infty} \left( \frac{4n+1}{2n+9} \right)^n \end{array}$$

### 7.3 Taylor Series and Maclaurin series

1. Find the Taylor series for  $f(x)$  at the given value of  $a$ .

$$\begin{array}{ll} \text{(a)} & f(x) = 3x^2 + 2x + 1, \quad a = 3 \\ \text{(b)} & f(x) = e^{2x} \sin x, \quad a = \frac{\pi}{2} \\ \text{(c)} & f(x) = \frac{1}{x}, \quad a = 1 \\ \text{(d)} & f(x) = \frac{1}{3-x}, \quad a = 2 \end{array}$$

2. Find the Maclaurin series for  $f(x)$ .

$$\begin{array}{ll} \text{(a)} & f(x) = \frac{1}{1-x} \\ \text{(b)} & f(x) = \cos 3x \\ \text{(c)} & f(x) = xe^x \end{array}$$

## 7.4 Fourier Series

In each of the following, a periodic function of period  $2\pi$  is specified over one period.

(i) Sketch a graph of the function for  $-2\pi < x < 2\pi$ ,  $-3\pi < x < 3\pi$  and  $-4\pi < x < 4\pi$ .

(ii) Obtain a Fourier series representation for the function.

1. 
$$f(x) = \begin{cases} 1 & \text{if } -\pi \leq x < 0 \\ -1 & \text{if } 0 \leq x < \pi \end{cases}$$

2. 
$$f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ x & \text{if } 0 \leq x < \pi \end{cases}$$

3. 
$$f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ \cos x & \text{if } 0 \leq x < \pi \end{cases}$$

4. 
$$f(x) = \begin{cases} -1 & \text{if } -\pi \leq x < -\frac{\pi}{2} \\ 1 & \text{if } -\frac{\pi}{2} \leq x < 0 \\ 0 & \text{if } 0 \leq x < \pi \end{cases}$$

5. 
$$f(x) = x \text{ for } -\pi \leq x < \pi$$

6. 
$$f(x) = x^2 \text{ for } -\pi \leq x < \pi$$