Topic 1.1
Propositional Logic

TMA1201 Discrete Structures & Probability
Faculty of Computing & Informatics
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What you will learn in this lecture:

- Propositions
- Truth Values
- Compound Propositions
- Logical Connectives
- Inverse, Converse, Contrapositive Statements
- Tautology, Contradiction, Contingency
- Logical Equivalence

Definition of a proposition:

A proposition is a ***declarative statement*** that is either ***true*** or ***false,*** but not ***both***.

Examples of sentences that are propositions:

1. **Kuala Lumpur is the capital of Malaysia.**
2. **Superman can fly.**
3. **1+5=9.**
4. **Tuesday is the day after Wednesday.**

Examples of sentences that are NOT propositions:

1. **Please sit down.**
2. **x+1=2.**
3. **What time is it?**

Truth value of a proposition:

A proposition can be either TRUE or FALSE.

The TRUE or FALSE is known as the truth value for the proposition.

TRUE is abbreviated as **T** and FALSE is abbreviated as **F**.

The notation [[ ]] is used to indicate finding the truth value of a proposition.

Examples:

1. [[Kuala Lumpur is the capital of Malaysia]] = T
2. [[1+6=9]] = F
3. [[Tuesday is the day after Wednesday]] = ?
4. [[Superman can fly]] = ?

Representation of a proposition:

Since propositions are naturally long, they are normally represented as, p, q, r, s or some other alphabets.

Examples:

p : TMA1201 is a compulsory subject for students who major in Finance at MMU.

q : You are sitting in this class to learn mathematical logic.

$[[p]] =$

$[[q]] =$

Compound Propositions via Logical Connectives

Compound proposition is a term to describe two or more propositions combined by logical connectives.

The four basic logical connectives are:

- Negation ($\neg$)
- Conjunction ($\wedge$)
- Disjunction ($\vee$)
- Implication ($\rightarrow$)

A TRUTH TABLE can display the relationships between the truth values of the propositions when connected with these logical connectives.

Negation (¬)

Negation is a connective “not” in English.

It is symbolized by “¬”.

The truth table of a negation operator for a proposition *p* is:

p	**¬*p***
T	F
F	T

Example:
p: It is sunny.
¬*p*: It is not sunny.
The statement “It is not sunny.” is true only when the statement “It is sunny.” is false.

Conjunction ($\wedge$)

Conjunction is a connective "and" in English.

It is symbolized by "$\wedge$".

The truth table of a conjunction operator for propositions *p* and *q* is:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example:
p: Today is Tuesday.
q: It is raining today.
$p \wedge q$: Today is Tuesday and it is raining today.
The statement "Today is Tuesday and it is raining today." is true only when both the statements "Today is Tuesday." and "It is raining today." are true.

Disjunction ($\vee$)

Disjunction is a connective "or" in English.

It is symbolized by "$\vee$".

The truth table of a disjunction operator for propositions *p* and *q* is:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:
p: Mary likes jogging.
q: Mary likes dancing.
$p \vee q$: Mary likes jogging or dancing.
The statement "Mary likes jogging or dancing" is true if Mary likes either jogging or dancing. It is false only when Mary likes neither.

Implication ($\rightarrow$)

Implication is a connective "if … then" in English.

It is symbolized by "$\rightarrow$".

The truth table of an implication operator for propositions *p* and *q* is:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In English, this compound proposition can have these meaning:

"*p* implies *q*"

"*p* only if *q*"

"*p* is sufficient for *q*"

"*q* is necessary for *p*"

"*q* if *p*"

"*q* when *p*"

"*q* whenever *p*"

Example:

p: It is sunny.

q: We go to the beach.

$p \rightarrow q$: If it is sunny, we go to the beach.

The statement "If it is sunny, we go to the beach" is false only when it is sunny, but we do not go to the beach.

Other Connectives

Two other connectives related to "$\rightarrow$" are
"$\leftrightarrow$" (if and only if) and "$\oplus$" (exclusive or)

The truth values of these two operators for propositions *p* and *q* are:

p	q	$p \leftrightarrow q$	$p \oplus q$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

In English, the compound proposition $p \leftrightarrow q$ can have these meaning:
"*p* if and only if *q*"
"*p* is a necessary and sufficient condition for *q*"

$p \leftrightarrow q$ is defined to be $(p \rightarrow q) \wedge (q \rightarrow p)$, it is True when p, q are both true or both false.

$p \oplus q$ is the negation of $p \leftrightarrow q$, it is True when exactly one of p or q is true (but not both).

Example 1

Given

p : It is raining.

q : The sun is shining.

r : There are clouds in the sky.

Write the following propositions using p, q, r and logical operators:

a) If it is raining, then the sun is not shining and there are clouds in the sky.
b) If the sun is shining or there are no clouds in the sky, then it is not raining.
c) The sun is shining if and only if it is not raining.
d) If there are no clouds in the sky, then it is not raining and the sun is shining.

Solution:

a) $p \rightarrow (\neg q \wedge r)$

b) $(q \vee \neg r) \rightarrow \neg p$

c) $q \leftrightarrow \neg p$

d) $\neg r \rightarrow (\neg p \wedge q)$

Example 2

Given

p : Today is Monday.

q : It is raining.

r : It is hot.

Express each of the following propositions as an English sentence.

a) $\neg p \rightarrow q \vee r$

b) $\neg (p \vee q) \leftrightarrow r$

c) $p \wedge (q \vee r) \rightarrow r \vee (q \vee p)$

Solution:

a) If today is not Monday, then it is raining or it is hot.

b) It is not the case that today is Monday or it is raining if and only if it is hot.

c) Today is Monday and either it is raining or it is hot imply that it is hot or it is raining or today is Monday.

Constructing The Truth Table for a Compound Proposition

$(p \to q) \land (r \lor \neg q)$

p	q	r	$(p \to q)$	$\neg q$	$(r \lor \neg q)$	$(p \to q) \land (r \lor \neg q)$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

Converse, Inverse, Contrapositive

For a compound proposition involving the implication connective ($p \to q$), we can write the **converse, inverse, and contrapositive** statement of the implication.

Mathematically,

the **converse** of $p \to q$ is $q \to p$
the **inverse** of $p \to q$ is $\neg p \to \neg q$
the **contrapositive** of $p \to q$ is $\neg q \to \neg p$

Example 3

p : It is sunny.
q : We go to the beach.
$p \to q$: If it is sunny, then we go to the beach.

Its **converse**, $q \to p$, is "If we go to the beach, then it is sunny."

Its **inverse**, $\neg p \to \neg q$, is "If it is not sunny, then we do not go to the beach."

Its **contrapositive**, $\neg q \to \neg p$, is "If we do not go to the beach, it is not sunny."

Let’s Look at Its Truth Table

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

The truth values for the **implication** and its **contrapositive** are similar in every row, BUT not with its **inverse** and **converse**.

So we said that the implication and its contrapositive are **logically equivalent** .

Classification of Compound Propositions

A compound proposition is called

- a **tautology** if it is **always true**, no matter what the truth values of the propositions that occur in it.
- a **contradiction** if it is **always false**, no matter what the truth values of the propositions that occur in it.
- a **contingency** if it is neither a tautology nor a contradiction.

Example: Given

p: Today is raining.

q: I am going out.

$p \vee \neg p$ is a tautology.

Today is raining or today is not raining.

$p \wedge \neg p$ is a contradiction.

Today is raining and today is not raining.

$p \vee q$ is a contingency.

Today is raining or I am going out.

Let's Look at Its Truth Table

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Tautology

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Contradiction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Contingency

Precedence of Logical Operators

Logical operator	Precedence
( )	1
$\neg$	2
$\wedge$	3
$\vee$	4
$\rightarrow$	5
$\leftrightarrow$	6

Note:

- $\neg p \vee q \wedge r$ is equivalent to $(\neg p) \vee (q \wedge r)$
- $p \rightarrow q \wedge q \vee r$ is equivalent to $p \rightarrow ((q \wedge q) \vee r)$

Exercise:

Add parentheses to the following to show how operator precedence groups operands:

$p \leftrightarrow q \wedge \neg r \vee s$

is equivalent to ____________________

Logical Equivalence

A compound proposition containing propositions, p and q are said to be **logically equivalence** when $p \leftrightarrow q$ is a TAUTOLOGY.

Logical equivalence can also be denoted as follows:

$$p \equiv q$$

$$p \Leftrightarrow q$$

When using the truth table to show logical equivalence, basically the columns of giving the final truth values agreed.

Example 4

Verify that the statements, $(p \vee q) \wedge q$ and q, are logically equivalent.

p	q	$p \vee q$	$(p \vee q) \wedge q$	$(p \vee q) \wedge q \leftrightarrow q$
T	T	T	T	T
T	F	T	F	T
F	T	T	T	T
F	F	F	F	T

These columns are equivalent!
$(p \vee q) \wedge q \Leftrightarrow q$

This column shows that
$(p \vee q) \wedge q \leftrightarrow q$ is a tautology!

$(p \vee q) \wedge q \Leftrightarrow q$ **if and only if** $(p \vee q) \wedge q \leftrightarrow q \Leftrightarrow \mathrm{T}$

Logical Equivalence by Identities and Laws

Other than using the Truth Table method, Equivalences' Laws can be used to show logical equivalence.

Example

Show that $\neg(p \vee q) \vee (\neg p \wedge q)$ is logically equivalent to $\neg p$.

$$
\begin{aligned}
& \neg(p \vee q) \vee (\neg p \wedge q) & & \\
\Leftrightarrow\ & (\neg p \wedge \neg q) \vee (\neg p \wedge q) & & \text{De Morgan's Law} \\
\Leftrightarrow\ & \neg p \wedge (\neg q \vee q) & & \text{Distributive Law} \\
\Leftrightarrow\ & \neg p \wedge \mathrm{T} & & \text{Negation Law} \\
\Leftrightarrow\ & \neg p & & \text{Identity Law}
\end{aligned}
$$

These are some of the Equivalences' Laws.

Equivalences' Laws

Identity laws $p \wedge \text{T} \Leftrightarrow p$ $p \vee \text{F} \Leftrightarrow p$	**Associative laws** $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	**Conversion of Implication** $p \rightarrow q \Leftrightarrow \neg p \vee q$ $p \vee q \Leftrightarrow \neg p \rightarrow q$
Domination laws $p \vee \text{T} \Leftrightarrow \text{T}$ $p \wedge \text{F} \Leftrightarrow \text{F}$	**Absorption Laws** $p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$	**Contrapositive** $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$ $p \leftrightarrow q \Leftrightarrow \neg q \leftrightarrow \neg p$
Idempotent laws $p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$	**Commutative laws** $p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	**Conversion of Equivalence** $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
Negation laws $p \vee \neg p \Leftrightarrow \text{T}$ $p \wedge \neg p \Leftrightarrow \text{F}$ **Double Negation laws** $\neg(\neg p) \Leftrightarrow p$	**Distributive laws** $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$ $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$	**De Morgan's laws** $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

Example 5

Show that $p \rightarrow (q \rightarrow r)$ is logically equivalent to $q \rightarrow (p \rightarrow r)$

	$p \rightarrow (q \rightarrow r)$	
$\Leftrightarrow$	$p \rightarrow (\neg q \vee r)$	Conversion of implication
$\Leftrightarrow$	$\neg p \vee (\neg q \vee r)$	Conversion of implication
$\Leftrightarrow$	$(\neg p \vee \neg q) \vee r$	Associative law
$\Leftrightarrow$	$(\neg q \vee \neg p) \vee r$	Commutative law
$\Leftrightarrow$	$\neg q \vee (\neg p \vee r)$	Associative law
$\Leftrightarrow$	$q \rightarrow (\neg p \vee r)$	Conversion of implication
$\Leftrightarrow$	$q \rightarrow (p \rightarrow r)$	Conversion of implication

Convert the expression based on equivalence laws.

Example 6

Show that $p \wedge (\neg p \vee q) \leftrightarrow (p \wedge q)$ is a tautology.

	$p \wedge (\neg p \vee q) \leftrightarrow (p \wedge q)$	
$\Leftrightarrow$	$(p \wedge \neg p) \vee (p \wedge q) \leftrightarrow (p \wedge q)$	Distributive law
$\Leftrightarrow$	$\mathrm{F} \vee (p \wedge \mathrm{q}) \leftrightarrow (p \wedge q)$	Negation law
$\Leftrightarrow$	$(p \wedge q) \leftrightarrow (p \wedge q)$	Identity law
$\Leftrightarrow$	T	

Remember a TAUTOLOGY means equal to TRUE or T.

Summary

We have learnt the following concepts, terms and notation related to mathematical logic:

- Definition of a proposition and compound propositions
- Logical connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\oplus$
- Truth values and truth table
- Rephrasing into a Converse, Inverse, Contrapositive Statements
- Classification into Tautologies, Contingency, Contradiction Statements
- Precedence of Logical Operators
- The meaning of Logical Equivalence

Exercise 1

Which statements are syntactically incorrect? Why?

1. $(p \rightarrow q)\ (\rightarrow r \vee \neg q)$
2. $\neg\neg(p \wedge \neg q)$
3. $(p \wedge p \wedge p)$
4. $\neg(\vee\ p)$
5. $p \rightarrow (\neg p \rightarrow (p \rightarrow \neg p))$

Exercise 2

Given that [[p]] = T, [[q]] = F, [[r]] = T, evaluate:

1. $[[p \wedge (q \vee r)]]$
2. $[[\neg(p \oplus q) \rightarrow (p \leftrightarrow q)]]$
3. $[[\neg p \wedge q]]$

Exercise 3

Create the truth table for the compound proposition:

$$p \to q \wedge r \vee \neg q$$

Exercise 4

Let

h: John is healthy, *w*: John is wealthy, and *s*: John is wise

A) Express the following propositions using *h*, *w*, s and logical operators.

1. John is healthy and wise but not wealthy.
2. John is neither wealthy nor wise, but he is healthy.

B) Express each of the following propositions as an English sentence.

1. $w \to \neg(h \lor s)$
2. $s \land h \to w$

C) Write the converse, inverse, and contrapositive statements for (B)

Exercise 5

Characterize each of the following formulae as a tautology, a contingency, or a contradiction.

1) $p \rightarrow \neg(q \vee p)$
2) $\neg p \rightarrow (q \vee \neg p)$
3) $(q \vee \neg q) \rightarrow (p \wedge \neg p)$
4) $\neg(p \wedge q) \vee (q \wedge \neg p)$

Exercise 6

Fill in the blanks with the correct equivalences.

$p \to q$	$\Leftrightarrow$	__________________	Conversion of Implication
$p \vee q$	$\Leftrightarrow$	__________________	De Morgan
$p \wedge q$	$\Leftrightarrow$	__________________	De Morgan
$p \leftrightarrow q$	$\Leftrightarrow$	__________________	Conversion of Equivalences
$(q \wedge r) \vee p$	$\Leftrightarrow$	__________________	Distributive law
$p \to q$	$\Leftrightarrow$	__________________	Contra-positive
$p \wedge (q \vee r)$	$\Leftrightarrow$	__________________	Distributive law

Exercise 7

Prove that $[\neg(\neg p \rightarrow q) \vee p] \Leftrightarrow p \vee \neg q$ by using

a) Truth table

b) Logical equivalence laws