Topic 1.3
Formal Reasoning

TMA1201 Discrete Structures & Probability
Faculty of Computing & Informatics
Multimedia University

What you will learn in this lecture:

- Arguments
- Premises
- Conclusions
- 8 Rules of Inference
  - Addition
  - Simplification
  - Modus Ponens
  - Modus Tollens
  - Hypothetical Syllogism
  - Disjunctive Syllogism
  - Resolution
  - Conjunction
- 4 Additional Rules of Inference for Predicates

What does it mean by a Formal Mathematical Reasoning?

Reasoning:
a cognitive process of looking for reasons, beliefs, and conclusions.

In mathematics, you have **_Sequences of Compound Propositions / Predicates_**

and you use Rules of Inference as Proper Reasoning Rules

to finally get a **_Valid Conclusion_**

Why do we need Formal Reasoning?

- Formal Reasoning is proving.
- Proof is a part of problem solving.
- Basically, if you think that you have found some hypotheses, and you think that these hypotheses are true, then you may wish to show and figure out why and how they are true.
- However, it is not necessary that these hypotheses will be true in all cases, sometimes they can be false.

Example 1

I am enrolled in one of the program in FCI, MMU. After talking with some seniors of the faculty, I have found the following facts:

- Students of the faculty study hard or don’t study hard.
- If the students study hard, they will succeed.
- If the students do not study hard, they will not succeed.

So, since you are also enrolled in this faculty, the seniors think that you will only succeed, if you study hard.

Are your seniors right?
Should you follow the advice?

To find out, you can use a formal reasoning technique.

Step 1: Formalize your arguments

An **argument** in formal reasoning is a sequence of compound propositions where all but the last one are called **premises** and the last one is called **conclusion**.

Define:

p: Students of the faculty study hard.

q: Students of the faculty will succeed.

Students of the faculty study hard or don't study hard.	$p \vee \neg p$
If the students study hard, they will succeed.	$p \rightarrow q$
If the students do not study hard, they will not succeed.	$\neg p \rightarrow \neg q$
You will only succeed, if you study hard.	$p \leftrightarrow q$

These four statements are known as ARGUMENTS.

These are known as the PREMISES.

Students of the faculty study hard or don't study hard. $p \vee \neg p$

If the students study hard, they will succeed. $p \rightarrow q$

If the students do not study hard, they will not succeed. $\neg p \rightarrow \neg q$

You will only succeed, if and only if you study hard. $p \leftrightarrow q$

This is known as the CONCLUSION.

Step 2: Connect each premise with a conjunctive ($\wedge$) operator

$(p \vee \neg p) \wedge (p \to q) \wedge (\neg p \to \neg q)$

Why conjunction $\wedge$?

Step 3: Link the premises to the conclusion with the logical consequence ($\Rightarrow$) operator

$$(p \vee \neg p) \wedge (p \to q) \wedge (\neg p \to \neg q) \Rightarrow (p \leftrightarrow q)$$

We are saying that all the PREMISES here imply the CONCLUSION.

Step 4: Show that the ARGUMENT is a TAUTOLOGY

There are 3 methods:

1. Truth Table
2. Logical Equivalences Simplification
3. Rules of Inference

Method 1: Truth Table

p	q	a			b		
		$p \vee \neg p$	$p \to q$	$\neg p \to \neg q$	$p \leftrightarrow q$	a	$a \to b$
T	T	T	T	T	T	T	T
T	F	T	F	T	F	F	T
F	T	T	T	F	F	F	T
F	F	T	T	T	T	T	T

This column shows that the ARGUMENT is VALID.

An argument is said to be VALID if whenever all the PREMISES are TRUE, then the CONCLUSION is also TRUE.

Method 2: Logical Equivalences Simplification

$$
\begin{aligned}
& (p \vee \neg p) \wedge (p \to q) \wedge (\neg p \to \neg q) \to (p \leftrightarrow q) \\
\equiv\; & T \wedge (p \to q) \wedge (\neg p \to \neg q) \to (p \leftrightarrow q) \\
\equiv\; & (p \to q) \wedge (\neg p \to \neg q) \to (p \leftrightarrow q) \\
\equiv\; & (p \to q) \wedge (q \to p) \to (p \leftrightarrow q) \\
\equiv\; & (p \leftrightarrow q) \to (p \leftrightarrow q) \\
\equiv\; & \neg(p \leftrightarrow q) \vee (p \leftrightarrow q) \\
\equiv\; & T
\end{aligned}
$$

This T shows that the ARGUMENT is VALID.

Method 3: Rules of Inference

Rules of Inference are common sense rules that provide the justification of the steps used to show that a CONCLUSION follows logically from a set of PREMISES.

There are ONLY 8 Rules to Remember!

Name	Tautological Arguments	Premises	Conclusion
Addition	$p \Rightarrow p \vee q$	1. p	$p \vee q$
Simplification	$(p \wedge q) \Rightarrow p$	1. p 2. q	p
Modus Ponens	$[p \wedge (p \to q)] \Rightarrow q$	1. p 2. $p \to q$	q
Modus Tollens	$[\neg q \wedge (p \to q)] \Rightarrow \neg p$	1. $\neg q$ 2. $p \to q$	$\neg p$

Name	**Tautological Arguments**	**Premises**	**Conclusion**
Hypothetical Syllogism	$[(p \to q) \wedge (q \to r)] \Rightarrow (p \to r)$	1. $p \to q$ 2. $q \to r$	$p \to r$
Disjunctive Syllogism	$[(p \vee q) \wedge \neg p] \Rightarrow q$	1. $p \vee q$ 2. $\neg p$	q
Resolution	$[(p \vee q) \wedge (\neg p \vee r)] \Rightarrow (q \vee r)$	1. $p \vee q$ 2. $\neg p \vee r$	$q \vee r$
Conjunction	$p \wedge q \Rightarrow (p \wedge q)$	1. p 2. q	$p \wedge q$

Rule #1: ADDITION

Given that p is true, then the **addition** of another premise q will still make the compound proposition true:

$$\begin{array}{l} 1.\, p \\ \hline \therefore p \vee q \end{array}$$

Example:

Suppose that the statement:
"It is sunny now." is true,
then the statement:
"It is either sunny or cloudy now." is also true.

Rule #2: SIMPLIFICATION

Given that the compound proposition *p* and *q* is true, removal of either one proposition and the remaining proposition is still true:

$$\begin{array}{l} 1.\ p \wedge q \\ \hline \therefore \quad p \end{array}$$

Example:

Suppose that the statement:
"It is cloudy and raining now." is true,
then the statement:
"It is cloudy now." or
"It is raining now." is also true.

Rule #3: MODUS PONENS

Given that both an implication and its hypothesis ($p \to q$ and p) are both true, then the conclusion of this implication (q) is true:

$$\begin{array}{l} 1.\ p \to q \\ \underline{2.\ p \qquad\quad} \\ \therefore \quad q \end{array}$$

Example:

Suppose that the statement:
"If it is sunny today, then we will go to the beach." is true.
The statement:
"It is sunny today." is also true.
Then the statement:
"We will go to the beach." must be true.

Rule #4: MODUS TOLLENS

Given that both an implication and the negation of its conclusion ($p \to q$ and $\neg q$) are true, then negation of its hypothesis $\neg p$ is also true:

1. $p \to q$
2. $\neg q$

$\therefore \quad \neg p$

Example:

Suppose that the statement:
"If I finish my homework before six, then I will go for a movie." is true.
The statement:
"I do not go for a movie." is also true.
Then the statement:
"I finish my homework before six." must NOT be true.

Rule #5: HYPOTHETICAL SYLLOGISM

Given that $p \to q$ and $q \to r$ are both true, then $p \to r$ is also true

$$
\begin{array}{ll}
1. & p \to q \\
2. & q \to r \\
\hline
\therefore & p \to r
\end{array}
$$

Example:

Suppose that the statement:
"If it is raining today, then we will stay at home." is true.
The statement:
"If we stay at home, we will watch TV." is also true.
Then the statement:
"If it is raining today, then we will watch TV." must be true.

Rule #6: DISJUNCTIVE SYLLOGISM

Given that $p \vee q$ and $\neg p$ are both true, then q is also true:

$$\begin{array}{l} 1.\ p \vee q \\ 2.\ \neg p \\ \hline \therefore \quad q \end{array}$$

Example:

Suppose that the statement:
"2 is either an odd number or an even number." is true.
The statement:
"2 is an odd number." is NOT true.
Then the statement:
"2 is an even number." must be true.

Rule #7: RESOLUTION

Given that $p \vee q$ and $\neg p \vee r$ are both true, then $q \vee r$ is also true:

1. $p \vee q$
2. $\neg p \vee r$

$\therefore \quad q \vee r$

Example:

Suppose that the statement:
"I am at home or it is sunny." is true.
The statement:
"I am not at home or I am watching TV." is also true.
Then the statement:
"It is sunny or I am watching TV." must be true.

Rule #8: CONJUNCTION

Given that p is true and q is true, then the conjunction $p \wedge q$ is also true:

$$\begin{array}{l} 1.\,P \\ 2.\,q \\ \hline \therefore p \wedge q \end{array}$$

Example:

Suppose that the statement:
"It is cloudy now." is true,
and the statement:
"It is raining now." is true.
Then, the statement:
"It is cloudy and raining now." is also true.

Abbreviations

P	Premise
CI	Conversion of Implication
ADD	Addition
SIMP	Simplification
CONJ	Conjunction
MP	Modus Ponens
MT	Modus Tollens
HS	Hypothetical Syllogism
DS	Disjunctive Syllogism
RES	Resolution

Let's continue to Method 3:

Recall:

1.	$p \vee \neg p$	: Students study hard or do not study hard.
2.	$p \rightarrow q$	: If the students study hard, they will succeed.
3.	$\neg p \rightarrow \neg q$	: If the students do not study hard, they will not succeed.
$\therefore$	$p \leftrightarrow \text{q}$	: The students will succeed if and only if they study hard.

We can give a formal proof like this:

1.	$p \vee \neg p$	Premise
2.	$p \rightarrow q$	Premise
3.	$\neg p \rightarrow \neg q$	Premise
4.	$\neg(\neg q) \rightarrow \neg(\neg p)$	Contra-positive of 3
5.	$q \rightarrow p$	4, Double Negation
6.	$(p \rightarrow q) \wedge (q \rightarrow p)$	2 and 5, CONJ
7.	$p \leftrightarrow q$	6, Equivalence law

Example 2:

Show that the following argument is valid:

If Susan gets the supervisor's position and works hard, then she will get a raise. If she gets the raise, she will buy a new car. She has not purchased a new car. Therefore, either Susan did not get the supervisor's position or she did not work hard.

Solution:

Let s: Susan gets the supervisor's position.

h: Susan works hard.

r: Susan gets a raise.

c: Susan buys a new car.

Premises: 1) $s \wedge h \to r$ 2) $r \to c$ 3) $\neg c$

Conclusion: $\neg s \vee \neg h$

Proof	Reason
1. $s \wedge h \to r$	P
2. $r \to c$	P
3. $\neg c$	P
4. $\neg r$	2 and 3, MT
5. $\neg(s \wedge h)$	1 and 4, MT
6. $\neg s \vee \neg h$	5, De Morgan's

Additional Rules of Inference for Predicates

Recall that predicates are statements with quantifiers.

For predicates, there are 4 additional Rules of Inference.

Universal Instantiation (UI)	**Universal Generalization (UG)**
$\underline{\forall x\, W(x)}$ $\therefore W(t)$ for an arbitrary t	$\underline{W(t) \text{ for an arbitrary } t}$ $\therefore \forall x\, W(x)$
Existential Instantiation (EI) $\underline{\exists x\, W(x)}$ $\therefore W(t)$ *for some element* t	**Existential Generalization (EG)** $\underline{W(t) \text{ for some element } t}$ $\therefore \exists x\, W(x)$

UOD = Universe of Discourse

Instantiation Rules

UI and EI are INSTANTIATION RULES.

With instantiation rules, you represent a case or an example for the statement, provided that the case or the example is true.

Example:
Every student has a student ID
$\therefore$ David has a student ID   (since David is a student)

$$\frac{\forall x\, W(x)}{\therefore W(t) \text{ for some element t}}$$

This is a UI Rule.

Generalization Rules

UG and EG are GENERALIZATION RULES.

With generalization rules, if you have a particular case or example and if this case or this example is true, you can generalize the case or the example to the whole universe of discourse.

Example:

1. Amy gets A+ in TMA1201.
2. Amy is a student of TC102.

$\therefore$ There is a student in TC102 who gets A+ in TMA1201.

$$\frac{W(t)}{\therefore \exists x\, W(x)}$$

This is an EG Rule.

A step-by-step example

Proof the following:

Every student who passed Discrete Structures worked hard.
Peter passed Discrete Structures.
Hence, there is one student who worked hard.

Step 1:
Define predicates that can be used to represent the sentences

DS(x) : x passed Discrete Structures
WH(x): x worked hard.

UOD = set of all students

A step-by-step example

Step 2:
Translate from English sentence to first order logic

Every student who passed Discrete Structures worked hard.
$\forall x\ (DS(x) \rightarrow WH(x))$

Peter passed Discrete Structures.
$DS(\text{Peter})$

Hence, there is one student who worked hard.
$\exists x\ WH(x)$

Remember?
$\forall$ expresses an implication
$\exists$ expresses a conjunction

Step 3:
Formalized the argument

$$\forall x\ (DS(x) \rightarrow WH(x)) \wedge DS(\text{Peter}) \Rightarrow \exists x\ WH(x)$$

A step-by-step example

Step 4:

Perform formal reasoning for $\forall x\,(DS(x) \rightarrow WH(x)) \wedge DS(\text{Peter}) \Rightarrow \exists x\, WH(x)$

Proof	Reason
1. $\forall x\,(DS(x) \rightarrow WH(x))$	P
2. $DS(\text{Peter})$	P
3. $DS(\text{Peter}) \rightarrow WH(\text{Peter})$	1, UI
4. $WH(\text{Peter})$	2 and 3, MP
5. $\exists x\, WH(x)$	4, EG

Notice that when doing formal reasoning involving predicates, the quantified statements have to be **instantiated using the EI then UI rules** before proceeding to apply the other 8 Rules of Inference.

Another example

Argue that the premises "All lions are fierce" and
"Some lions do not drink coffee" imply the conclusion
"Some fierce creatures do not drink coffee."

Step 1:
Define predicates that can be used to represent the sentences.
$P(x)$: x is a lion.
$Q(x)$: x is fierce
$R(x)$: x drinks coffee.
UOD: set of all animals

Step 2:
Translate from English sentence to first order logic
1. $\forall x\, (P(x) \rightarrow Q(x))$
2. $\exists x\, (P(x) \land \neg R(x))$

$\therefore \exists x\, (Q(x) \land \neg R(x))$

Another example

Step 3:

Formalized the argument

$(\forall x\,(P(x) \rightarrow Q(x))) \wedge (\exists x\,(P(x) \wedge \neg R(x))) \Rightarrow (\exists x\,(Q(x) \wedge \neg R(x)))$

Step 4:

Perform formal reasoning

Proof	Reason
1. $\forall x\,(P(x) \rightarrow Q(x))$	P
2. $\exists x\,(P(x) \wedge \neg R(x))$	P
3. $P(\mathrm{C}) \wedge \neg R(\mathrm{C})$	2, EI
4. $P(\mathrm{C})$	3, SIMP
5. $P(\mathrm{C}) \rightarrow Q(\mathrm{C})$	1, UI
6. $Q(\mathrm{C})$	4 and 5, MP
7. $\neg R(\mathrm{C})$	3, SIMP
8. $Q(\mathrm{C}) \wedge \neg R(\mathrm{C})$	6 and 7, CONJ
9. $\exists x\,(Q(x) \wedge \neg R(x))$	8, EG

Notice here that, ***EI is always evaluated before UI** since the existential statement has shown us that there already exists one example in the UOD .

*important

Sometimes you do not even need narratives to perform a Formal Proof!

Example

Proof	Reason
1. $p \vee q$	P
2. $p \vee r$	P
3. $\neg p$	P
4. q	1 and 3, DS
5. r	2 and 3, DS
6. $q \wedge r$	4 and 5, CONJ

Another Example without narratives

Show that $\forall x\,(P(x) \rightarrow Q(x)) \wedge (\forall x\, P(x)) \Rightarrow (\forall x\, Q(x))$

Proof	Reason
1. $\forall x\,(P(x) \rightarrow Q(x))$	P
2. $\forall x\, P(x)$	P
3. $P(\mathrm{T})$	2, UI
4. $P(\mathrm{T}) \rightarrow Q(\mathrm{T})$	1, UI
5. $Q(\mathrm{T})$	3 and 4, MP
6. $\forall x\, Q(x)$	5, UG

In Class Exercise

Given the following argument:

Jenny, a student in this class, knows how to write programs using C#. Everyone who knows how to write programs in C# can get a good salary job. Therefore someone in this class can get a good salary job.

a) Define 3 predicates from the above argument by assuming the *UOD is all students in MMU*.
   You must label the predicates as **A(*x*)**, **B(*x*) and C(*x*)**.

b) By using the predicates that you have defined in part (a) and logical connectives, **translate the argument into 3 premises and a conclusion**.

c) Show that the argument is valid with **formal reasoning**.

Summary

The following concepts related to formal reasoning were discussed:

- The meaning of formal reasoning
- The 4 steps to perform formal reasoning
- The 8 Rules of Inference
- The 4 additional Rules of Inference for Predicates

Proof Techniques (Self-reading)

- Direct Proof
- Indirect Proof (Proof by contrapositive & Proof by Contradiction)
- Proof by Cases

- https://www.whitman.edu/mathematics/higher_math_online/chapter02.html
- https://www.whitman.edu/mathematics/higher_math/higher_math.pdf
- https://sites.millersville.edu/bikenaga/math-proof/proof-by-cases/proof-by-cases.html

Exercise 1

Fill in the blanks with the arguments of the 8 inference rules.

1. Addition ______________________________
2. Simplification ______________________________
3. Modus Ponens ______________________________
4. Modus Tollens ______________________________
5. Hypothetical Syllogism ______________________________
6. Disjunctive Syllogism ______________________________
7. Resolution ______________________________
8. Conjunction ______________________________

Exercise 2

Verify whether the argument is valid

$$(p \to q) \wedge \neg(\neg p \vee r) \wedge p \Rightarrow q \wedge \neg r$$

a) by using a truth table,
b) by using formal reasoning.

Exercise 3

Show that $\exists x\big(P(x) \lor Q(x)\big) \equiv \exists x P(x) \lor \exists x Q(x)$

Exercise 4

Given the following argument:

If John passes his test and passes his final exam then he will pass this subject. If he is on time for the exam then he will answer all the questions. If he is late for the exam then he will fail this subject. John did not answer all the questions in the exam. Hence John fails his test or fails his final exam.

a) Define 5 propositions from the above argument. You must label them as **p**, **q**, **r**, **s**, **t**.
b) By using the propositions that you have defined in part (a) and logical connectives, translate the argument into 4 premises and a conclusion
c) Show that the argument is valid with formal reasoning.

Exercise 5: Where are your glasses?

Being in a hurry (because you didn't want to come late for TMA1201 class), you forgot your glasses. You remember the following facts:

1. If my glasses are on the kitchen table, then I saw them at breakfast.
2. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
3. If I was reading the newspaper in the living room, then my glasses are on the coffee table.
4. I did not see my glasses during breakfast.
5. If I was reading my book in bed, then my glasses are on the bed table.
6. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

First, find all propositions.
Second, give yourself suitable variables.
Third, write down clearly the argument.
Fourth, prove it. ***(Step by step, please!)***

Exercise 5: Where is the treasure?

In the back of an old cupboard, you discovered a note signed by a pirate famous for his bizarre sense of humor and love of logical puzzles. In the note he wrote that he had hidden treasure somewhere on the property. He listed five true statements:

a. If this house is next to a lake, then the treasure is not in the kitchen.
b. If the tree in the front yard is a pine, then the treasure is in the kitchen.
c. The house is next to a lake.
d. The tree in the front yard is a pine or the treasure is buried under the flagpole.
e. If the tree in the back yard is an oak, then the treasure is in the garage.

Can you guess where is the treasure? Verify your guess by formal reasoning.