Chapter 2:
Analytic Trigonometry and Polar Coordinates (Part 2)

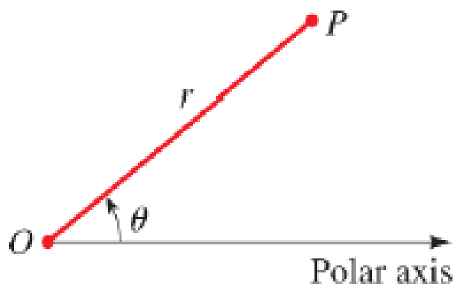
Lecture 09 – 2.12.2022

- Polar coordinates
- Complex Number & De Moivre's Theorem

2.4.1 Polar Coordinates

The **polar coordinate system** uses distances and directions to specify the location of a point in the plane.

To set up this system, we choose a fixed point O in the plane called the **pole** (or **origin**) and draw from O a ray (half-line) called the **polar axis** as in Figure below.



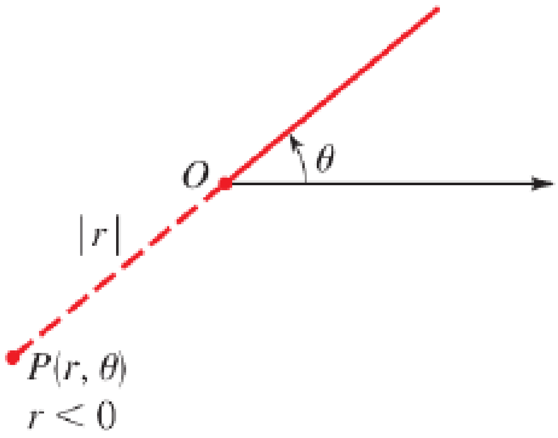
Then each point P can be assigned polar coordinates $P(r, \theta)$ where

r is the *distance* from O to P

θ is the *angle* between the polar axis and the segment $\overline{OP}$

We use the convention that θ is positive if measured in a counterclockwise direction from the polar axis or negative if measured in a clockwise direction.

If r is negative, then $P(r, \theta)$ is defined to be the point that lies $|r|$ units from the pole in the direction opposite to that given by θ (see Figure below).

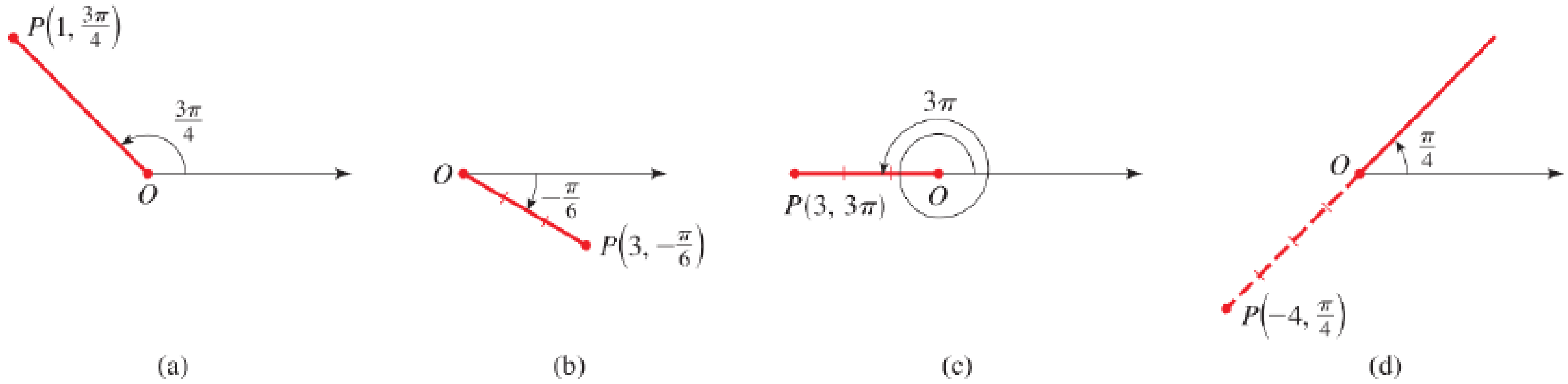


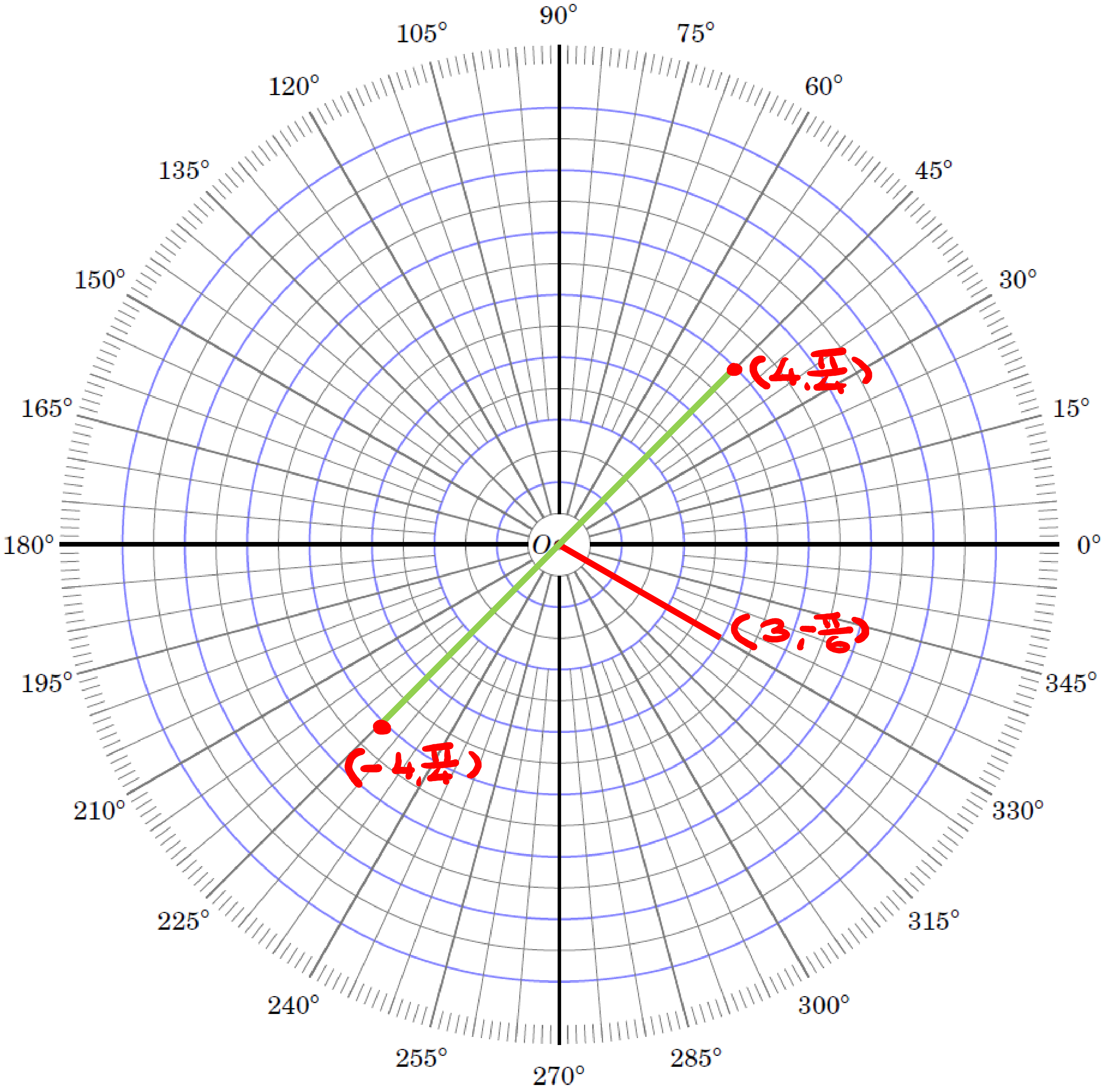
Example

Plot the points whose polar coordinates are given.

(a) $(1, 3\pi/4)$ **(b)** $(3, -\pi/6)$ **(c)** $(3, 3\pi)$ **(d)** $(-4, \pi/4)$

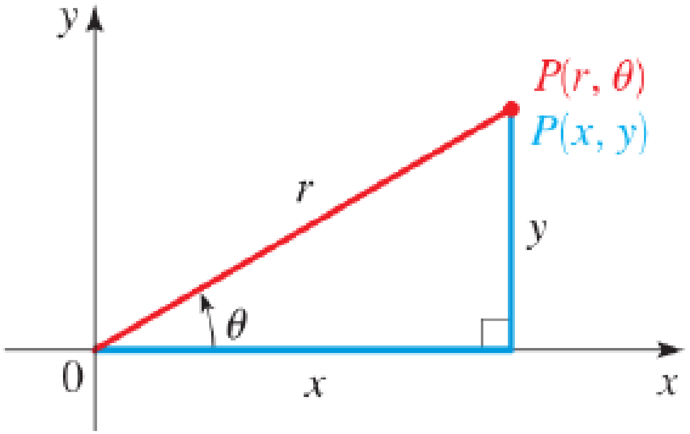
SOLUTION The points are plotted in Figure 3. Note that the point in part (d) lies 4 units from the origin along the angle $5\pi/4$, because the given value of r is negative.




90°
105°
75°
120°
60°
135°
45°
150°
30°
165°
15°
180°
O
0°
195°
345°
210°
330°
225°
315°
240°
300°
255°
285°
270°
$(4, \frac{\pi}{4})$
$(3, -\frac{\pi}{6})$
$(-4, \frac{\pi}{4})$

2.4.1.1 Relationship between Polar and Rectangular Coordinates

The connection between the two systems is illustrated in Figure below, where the polar axis coincides with the positive x-axis.



The formulas in the following box are obtained from the figure using the definitions of the trigonometric functions and the Pythagorean Theorem.

RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES

1. To change from polar to rectangular coordinates, use the formulas

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

2. To change from rectangular to polar coordinates, use the formulas

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

Example

Find rectangular coordinates for the point that has polar coordinates $(4, 2\pi/3)$.

$$x = r\cos\theta$$
$$= 4\cos\left(\frac{2\pi}{3}\right)$$
$$= 4\left(-\frac{1}{2}\right) = -2$$

$$y = r\sin\theta$$
$$= 4\sin\left(\frac{2\pi}{3}\right)$$
$$= 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

$\therefore$ rectangular coordinate $= (-2, 2\sqrt{3})$ #

Example

Find polar coordinates for the point that has rectangular coordinates (2, -2)

$$r^2 = x^2 + y^2$$
$$= 2^2 + (-2)^2$$
$$= 8$$
$$\therefore r = \pm\sqrt{8}$$

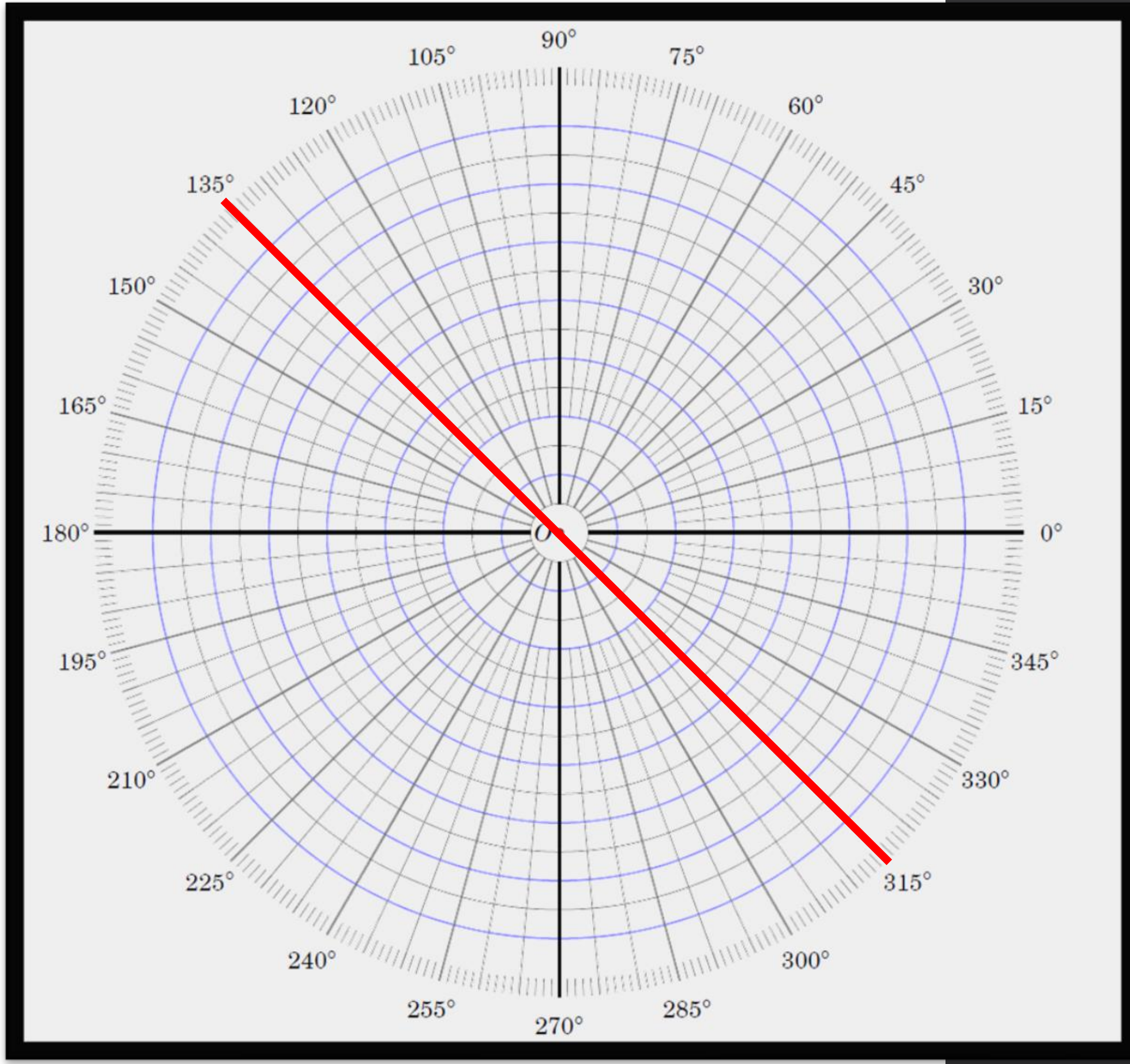
$$\tan\theta = -\frac{2}{2}$$

$$\therefore \theta = \tan^{-1}(-1)$$
$$= 315^\circ$$

$\therefore$ polar coordinates $= (\sqrt{8}, 315^\circ) = (\sqrt{8}, -45^\circ)$

$= (-\sqrt{8}, 315^\circ)$

$= (-\sqrt{8}, 135^\circ)$



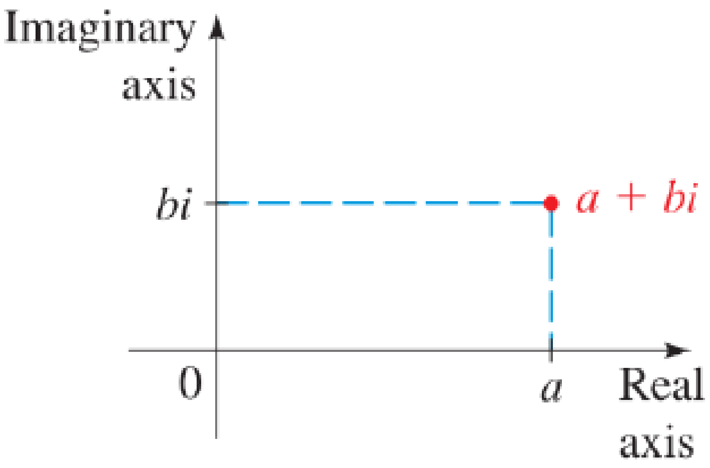
Note that the equations relating polar and rectangular coordinates do not uniquely determine r or θ. When we use these equations to find the polar coordinates of a point, we must be careful that the values we choose for r and θ give us a point in the correct quadrant.

2.4.2 Complex Numbers In Polar Form.

2.4.2.1 Complex Number

Complex Number z is a number that can be expressed as $z = a + bi$ where $a, b \in \mathbb{R}$ and i denotes the imaginary unit satisfy $i^2 = -1$.

To graph the complex number $z = a + bi$, we plot the ordered pair of numbers (a, b) in the plane:

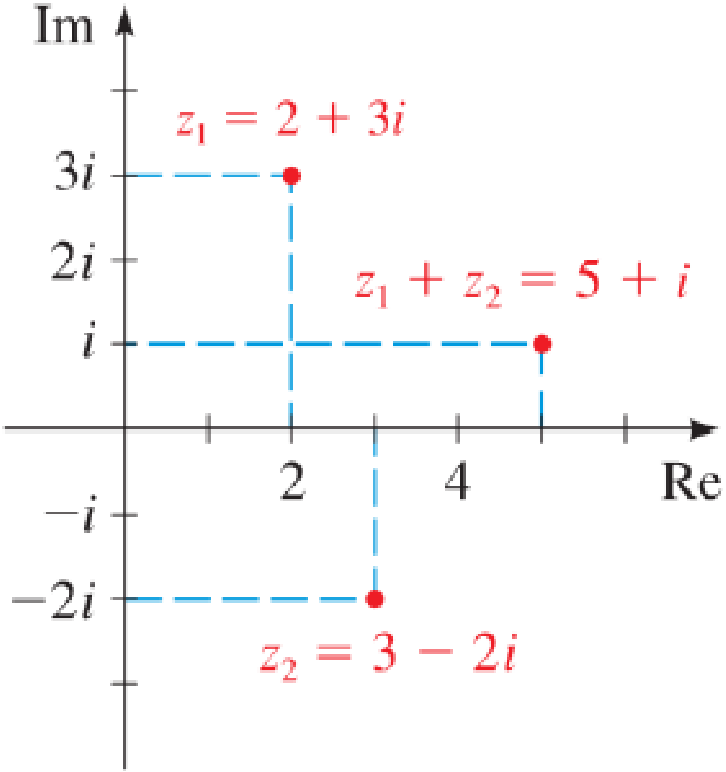


Example

Graph the complex number $z_1 = 2+3i$, $z_2 = 3-2i$, and $z_1 + z_2$

Solution

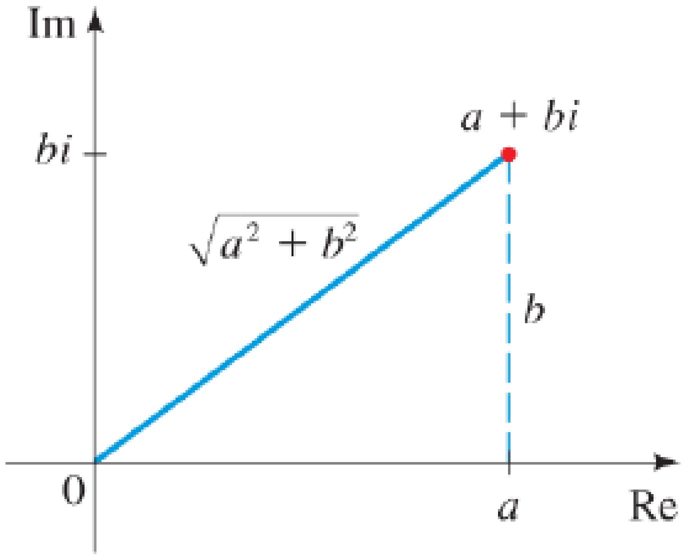
$$z_1 + z_2 = (2+3i)+(3-2i) = 5+i$$



MODULUS OF A COMPLEX NUMBER

The **modulus** (or **absolute value**) of the complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}$$



Example

Find the moduli of the complex numbers $3+4i$ and $8-5i$

a) $Z_1 = 3+4i$

$\therefore |Z_1| = \sqrt{3^2+4^2}$

$= \sqrt{9+16}$

$= \sqrt{25}$

$= 5$ #

b) $Z_2 = 8-5i$

$|Z_2| = \sqrt{8^2+(-5)^2}$

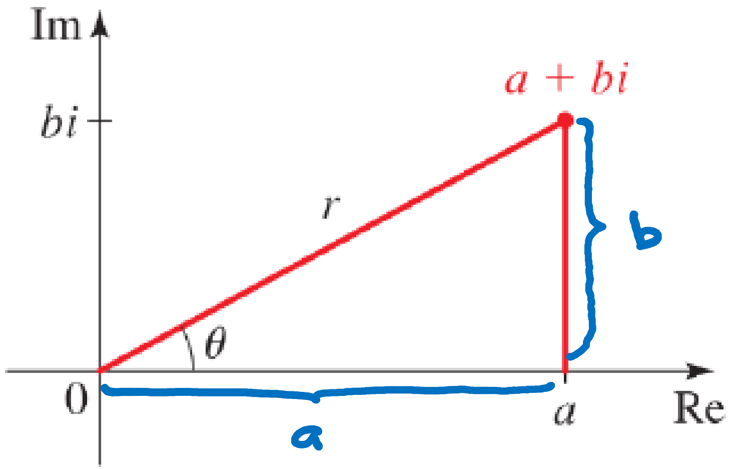
$= \sqrt{64+25}$

$= \sqrt{89}$ #

2.4.2.2 Polar Form of Complex Number

Let $z = a + bi$ be a complex number, and in the complex plane let's draw the line segment joining the origin to the point $a + bi$ (see figure below).

The length of this line segment is $r = |z| = \sqrt{a^2 + b^2}$.



If θ is an angle in standard position whose terminal side coincides with this line segment, then by the definitions of sine and cosine

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

so

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

$\cos\theta = \frac{a}{r} \Rightarrow a = r\cos\theta$

$\sin\theta = \frac{b}{r} \Rightarrow b = r\sin\theta$

Some properties about imaginary number i:

1. $i = \sqrt{-1}$
2. $i^2 = -1$
3. $i^3 = -i$
4. $i^4 = 1$

POLAR FORM OF COMPLEX NUMBERS

A complex number $z = \overset{\text{Re}}{a} + \overset{\text{Im}}{b}i$ has the **polar form** (or **trigonometric form**)

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is an **argument** of z.

θ = argument of z

$= \arg(z)$

Example

Write each complex number in polar form

(a) $1+i$ (b) $-1+\sqrt{3}i$

a) $z_1 = 1 + i$

$r = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

$\theta = \arg(z_1)$

$= \tan^{-1}\left(\frac{1}{1}\right)$

$= 45°$

$\therefore z_1 = \sqrt{2}\,(\cos 45° + i \sin 45°)$

b) $z_2 = -1 + \sqrt{3}i$

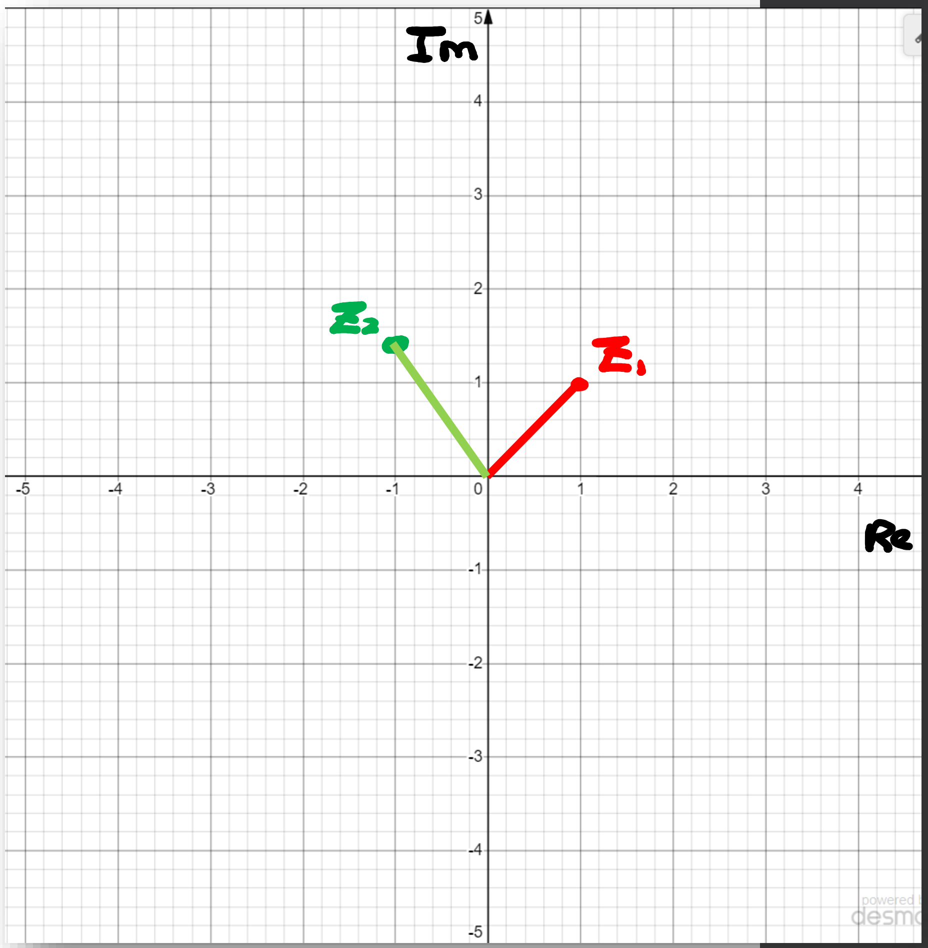
$r = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= \sqrt{1+3} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$

$= 120°$

$\therefore z_2 = 2\,(\cos 120° + i \sin 120°)$



Example

Write each complex number in polar form

(c) $-4\sqrt{3}-4i$ (d) $3+4i$

$z_3 = -4\sqrt{3} - 4i$

$r = \sqrt{(-4\sqrt{3})^2 + (-4)^2}$

$= \sqrt{48+16}$

$= \sqrt{64} = 8$

$\theta = \tan^{-1}\left(\frac{-4}{-4\sqrt{3}}\right)$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 210°$

$\therefore z_3 = 8(\cos 210° + i \sin 210°)$ #

$z_4 = 3 + 4i$

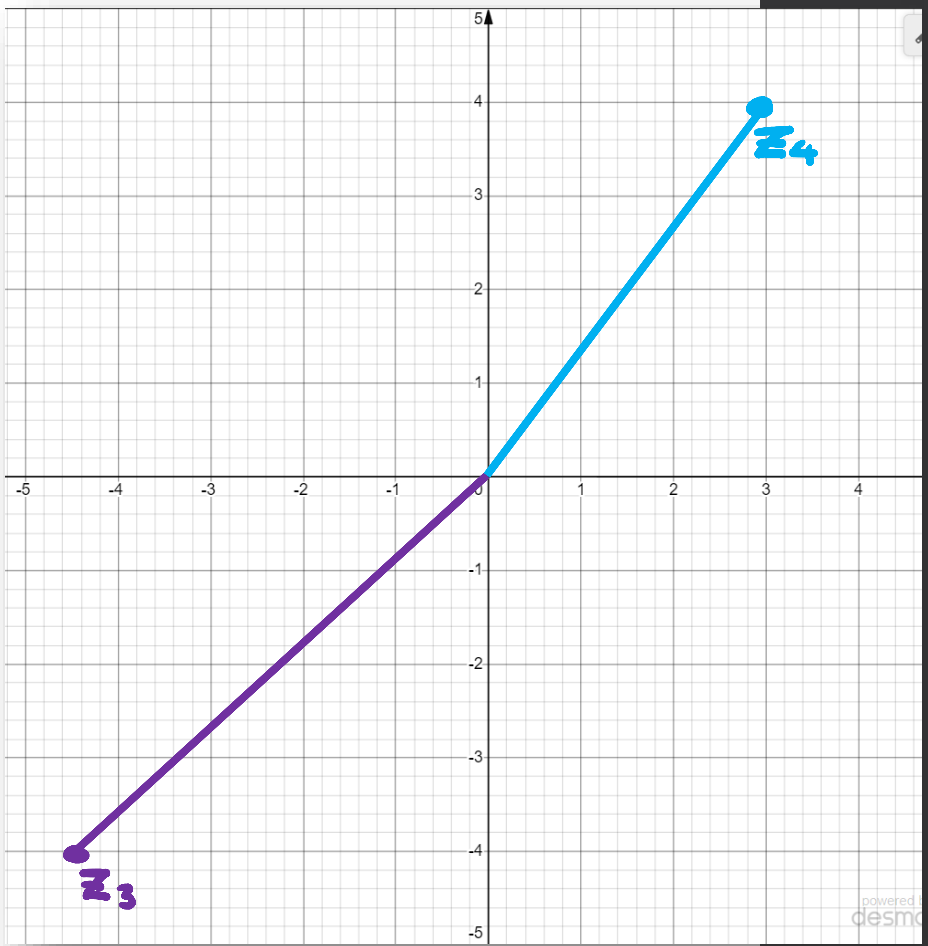
$r = \sqrt{3^2 + 4^2}$

$= \sqrt{9+16} = \sqrt{25} = 5$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$= 53.1°$

$\therefore z_4 = 5(\cos 53.1° + i \sin 53.1°)$ #



2.4.2.3 Multiplication and Division of Complex Numbers

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS

If the two complex numbers z_1 and z_2 have the polar forms

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{Multiplication}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad (z_2 \neq 0) \qquad \text{Division}$$

Tips:

- Given any complex number in the form of $z = a + ib$, always change it into **Polar form** of $z = r(\cos\theta + i\sin\theta)$.

- Proving $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Let $z_1 = r(\cos\theta + i\sin\theta)$ and $z_2 = s(\cos\alpha + i\sin\alpha)$

$$z_1z_2 = [r(\cos\theta + i\sin\theta)][s(\cos\alpha + i\sin\alpha)]$$

$i^2 = -1$

$$= rs[\cos\theta\cos\alpha + i\cos\theta\sin\alpha + i\sin\theta\cos\alpha + \underbrace{i^2}_{-1}\sin\theta\sin\alpha]$$

$$= rs[\cos\theta\cos\alpha - \sin\theta\sin\alpha + i(\sin\theta\cos\alpha + \cos\theta\sin\alpha)]$$

$$= rs[\cos(\theta+\alpha) + i\sin(\theta+\alpha)]$$ #

Compound (addition) Angle Formula

- Proving $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad (z_2 \neq 0)$

Let $z_1 = r(\cos\theta + i\sin\theta)$ and $z_2 = s(\cos\alpha + i\sin\alpha)$

$$\frac{z_1}{z_2} = \frac{r(\cos\theta + i\sin\theta)}{s(\cos\alpha + i\sin\alpha)} \times \frac{\cos\alpha - i\sin\alpha}{\cos\alpha - i\sin\alpha}$$

$+i\sin\alpha \quad -i\sin\alpha$

conjugate

$$= \frac{r(\cos\theta + i\sin\theta)(\cos\alpha - i\sin\alpha)}{s(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$-i^2 = -(-1)$
$= +1$

$$= \frac{r[\cos\theta\cos\alpha - i\cos\theta\sin\alpha + i\sin\theta\cos\alpha \underbrace{- i^2}_{+1}\sin\theta\sin\alpha]}{s[\cos^2\alpha \cancel{- i\cos\alpha\sin\alpha} \cancel{+ i\sin\alpha\cos\alpha} \underbrace{- i^2}_{+1}\sin^2\alpha]}$$

$\leftarrow \cos^2\alpha + \sin^2\alpha = 1$

$$= \frac{r}{s}[\cos\theta\cos\alpha + \sin\theta\sin\alpha + i(\sin\theta\cos\alpha - \cos\theta\sin\alpha)]$$

$$= \frac{r}{s}[\cos(\theta - \alpha) + i\sin(\theta - \alpha)]$$

Compound angle formula
(addition)

Chapter 2:
Analytic Trigonometry and Polar Coordinates (Part 2)

Lecture 10 – 7.12.2022

- Polar coordinates
- Complex Number & De Moivre's Theorem

Example

Let

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad \text{and} \qquad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Find **(a)** z_1z_2 and **(b)** z_1/z_2.

a) $z_1z_2 = 2(5)\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\right]$

$= 10\left[\cos\frac{7}{12}\pi + i\sin\frac{7}{12}\pi\right]$ #

$= 10\,(-0.2588 + 0.9659i)$

$= -2.588 + 9.659i$ #

Example

Let

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{and} \quad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Find **(a)** $z_1 z_2$ and **(b)** z_1/z_2.

$$\text{b) } \frac{z_1}{z_2} = \frac{2}{5}\left[\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\right]$$

$$= \frac{2}{5}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right]$$

$$= \frac{2}{5}\left[\cos\frac{23}{12}\pi + i\sin\frac{23}{12}\pi\right] \#$$

If you want to remain the answer in the **Polar Form**, convert the θ in to its **positive angle**.

$$= \frac{2}{5}(0.9659 - 0.2588i)$$

$$= 0.3864 - 0.1035i \#$$

If you want answer it in complex form $a + bi$, then you can directly evaluate the negative angle $-\theta$.

2.4.2.4 De Moivre's Theorem

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$, then for any integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

This theorem says: *To take the nth power of a complex number, we take the nth power of the modulus and multiply the argument by n*

Tips:

- Given any complex number in the form of $z = a + ib$, always change it into **Polar form** of $z = r(\cos\theta + i\sin\theta)$.

Example

Find $\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}$

Let $z = \frac{1}{2} + \frac{1}{2} i$

$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \sqrt{\frac{1}{4}+\frac{1}{4}}$

$= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$\theta = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right)$

$= \tan^{-1}(1)$

$= 45^\circ$

$z = \frac{1}{\sqrt{2}}(\cos 45^\circ + i \sin 45^\circ)$

$\therefore z^{10} = \left(\frac{1}{\sqrt{2}}\right)^{10}\left[\cos 45(10) + i \sin 45(10)\right]$

$= \frac{1}{32}(\cos 450^\circ + i \sin 450^\circ)$

$= \frac{1}{32}(0 + i)$

$= \frac{i}{32}$ #

2.4.2.5 *n*th Roots of Complex Numbers

An ***n*th root** of a complex number z is any complex number w such that $w^n = z$.

De Moivre's Theorem gives us a method for calculating the nth roots of any complex number.

nth ROOTS OF COMPLEX NUMBERS

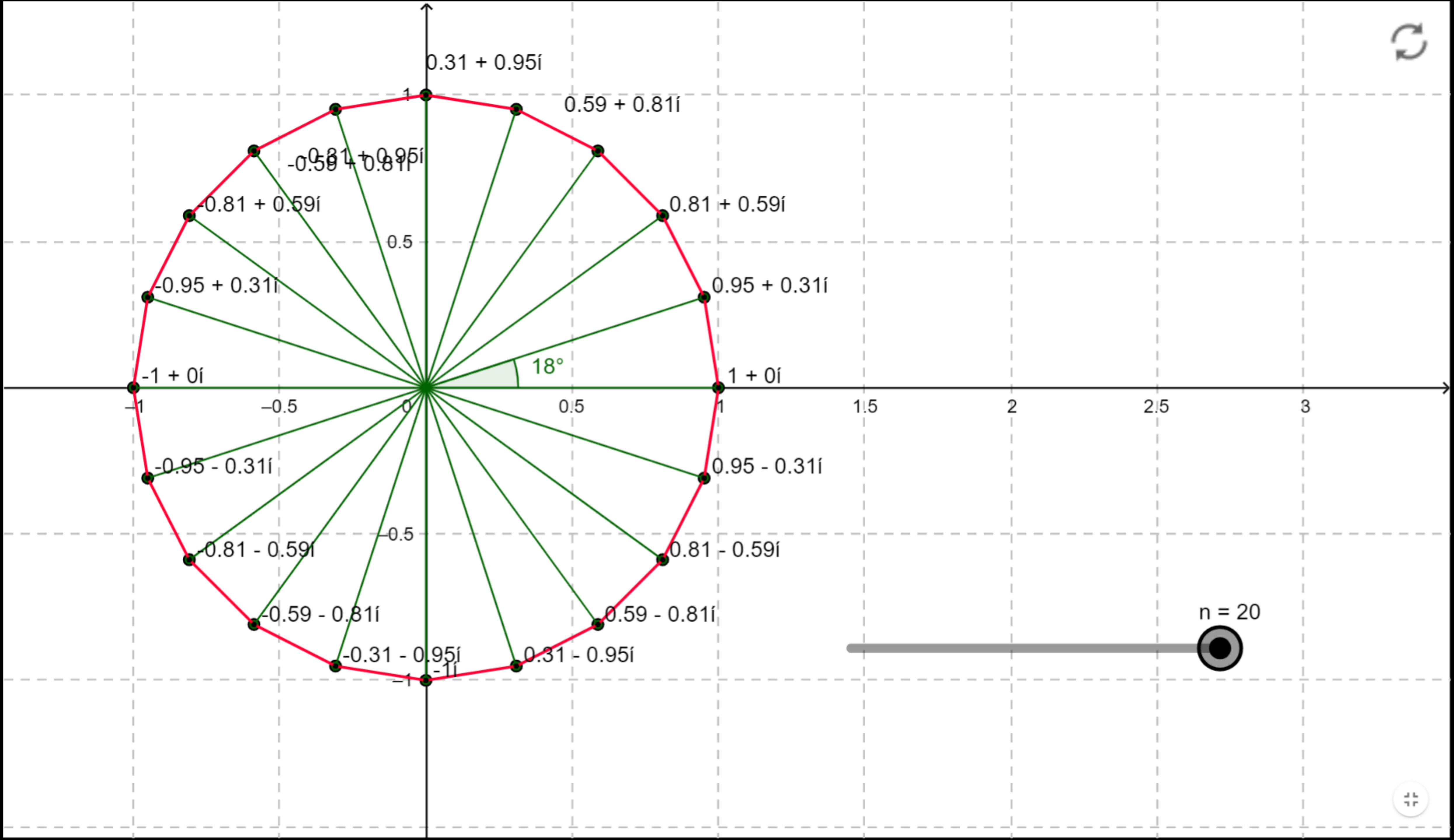
If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n - 1$.

Note:

The n^{th}-root of complex number is also known as the **n^{th}-root of unity**.



0.31 + 0.95i
0.59 + 0.81i
-0.31 + 0.95i
-0.59 + 0.81i
-0.81 + 0.59i
0.81 + 0.59i
-0.95 + 0.31i
0.95 + 0.31i
18°
-1 + 0i
1 + 0i
-0.95 - 0.31i
0.95 - 0.31i
-0.81 - 0.59i
0.81 - 0.59i
-0.59 - 0.81i
0.59 - 0.81i
-0.31 - 0.95i
0.31 - 0.95i
-1i
n = 20

FINDING THE nth ROOTS OF $z = r(\cos\theta + i\sin\theta)$

1. The modulus of each nth root is $r^{1/n}$.
2. The argument of the first root is θ/n.
3. We repeatedly add $2\pi/n$ to get the argument of each successive root.

Tips:

- Given any complex number in the form of $z = a + ib$, always change it into **Polar form** of $z = r(\cos\theta + i\sin\theta)$.
- Any **real number** is a subset of **complex number**. For example $z = 3$ is also a complex number because $z = 3 + 0i$.

Example

Find the six sixth roots of z = –64, and graph these roots in the complex plane.

$Z = -64 + 0i$ ← between Q2 and Q3

$r = \sqrt{(-64)^2 + 0^2}$

$= 64$

$\theta = \tan^{-1}\left(\frac{0}{-64}\right)$

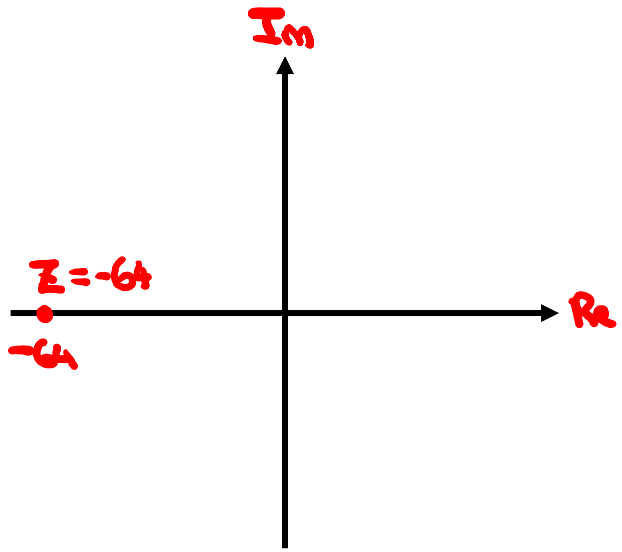
$= \tan^{-1}(0)$

$= 180°$

$\therefore Z = 64(\cos 180° + i\sin 180°)$

$\therefore w_k = Z^{\frac{1}{6}} = 64^{\frac{1}{6}}\left[\cos\left(\frac{180° + 360°k}{6}\right) + i\sin\left(\frac{180° + 360°k}{6}\right)\right]$

for $k = 0, 1, 2, 3, 4, 5$



$$\therefore w_k = z^{\frac{1}{6}} = 64^{\frac{1}{6}}\left[\cos\left(\frac{180^\circ+360^\circ k}{6}\right) + i\sin\left(\frac{180^\circ+360^\circ k}{6}\right)\right]$$

$$w_0 = 64^{\frac{1}{6}}\left(\cos\frac{180^\circ}{6} + i\sin\frac{180^\circ}{6}\right)$$

$$= 2(\cos 30^\circ + i\sin 30^\circ)$$

$$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i \;\#$$

$$w_1 = 2(\cos 90^\circ + i\sin 90^\circ)$$

$$= 2(0 + i)$$

$$= 2i \;\#$$

$$w_2 = 2(\cos 150^\circ + i\sin 150^\circ)$$

$$= 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= -\sqrt{3} + i \;\#$$

$$w_3 = 2(\cos 210^\circ + i\sin 210^\circ)$$

$$= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= -\sqrt{3} - i \;\#$$

$$w_4 = 2(\cos 270^\circ + i\sin 270^\circ)$$

$$= 2(0 - i)$$

$$= -2i \;\#$$

$$w_5 = 2(\cos 330^\circ + i\sin 330^\circ)$$

$$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= \sqrt{3} - i \;\#$$

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