Topic 1.2
Predicate Logic

TMA1201 Discrete Structures & Probability
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What you will learn in this lecture:

- Predicates
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Translating from English into Logical Expressions
- Logical Equivalences Involving Quantifiers
- De Morgan’s Law for Quantified Statements
- Nested Quantifiers Expression

What is Predicate Logic?

Predicate logic is also known as first order logic.

Recall that in English, when you have a sentence, it can be divided into two components: a subject and a predicate, where the predicate describes the subject.

In predicate logic, the predicate can be used to express the meaning and relationship of mathematical/computer science statements.

Quantification phrases are involved in a predicate for it to have a truth value.

The word *some*, *all*, *every*, *each one* are examples of quantification phrases in English.

The Quantifiers

Universal Quantifier

$\forall$

Existential Quantifier

$\exists$

Universal Quantifiers, $\forall$

For any predicate, a universal quantifier expresses the English word "for every", "for all", or "for any".

Example:
$\forall x\, Q(x)$ means "for all x , $Q(x)$ is true" or "for every x , $Q(x)$ is true"
$\forall x\, Q(x)$ is the same as $Q(x_1) \wedge Q(x_2) \wedge Q(x_3) \wedge Q(x_4) \wedge \ldots \wedge Q(x_n)$
(if the **domain of discourse** contains only $x_1, x_2, \ldots., x_n$)
If <u>an x</u> is causing $Q(x)$ to be false, then the statement $\forall x\, Q(x)$ is false.

The symbol $\forall$ is the universal quantifier.
$Q(x)$ is the predicate.
x is the variable (subject).
Both alphabets of Q and x can be changed.

A **domain of discourse**
is the universal set
where the variable x lies, which
is also known as **universe**
of discourse (UOD).

Existential Quantifiers, $\exists$

For any predicate, an existential quantifier expresses the English word "for some", "there exist at least" , "for at least one", or "there is".

Example:
$\exists x\, Q(x)$ means "for some x , $Q(x)$ is true" or "there is at least one x such that $Q(x)$ is true"
$\exists x\, Q(x)$ is the same as $Q(x_1) \vee Q(x_2) \vee Q(x_3) \vee Q(x_4) \vee \ldots \vee Q(x_n)$
(if the **domain of discourse** contains only $x_1, x_2, \ldots., x_n$)
If at least one x is causing $Q(x)$ to be true, then $\exists x\, Q(x)$ is true.

The symbol $\exists$ is the existential quantifier.
$Q(x)$ is the predicate .
x is the variable (subject)
Both alphabets of Q and x can be changed.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.
- For example, $\forall x P(x) \lor Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. In other words, it means $(\forall x P(x)) \lor Q(x)$ rather than $\forall x (P(x) \lor Q(x))$.

Binding Variables

- When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**.
- An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.

Example 1

a) In the statement: $\exists x(x + y = 1)$,

x and y are bounded variable and free variable, respectively.

b) In the statement $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$,

all variables are bound. Furthermore the existential quantifier $\exists x$ binds the variable x in $P(x) \wedge Q(x)$ and the universal quantifier $\forall x$ binds the variable x in $R(x)$. Which also means the statement equal to $\exists x(P(x) \wedge Q(x)) \vee \forall y R(y)$.

Truth Value

As with propositions, the truth value for the predicates can be evaluated.

To conclude that a statement of the form $\exists x\, Q(x)$ **is true**, we need to only find one value of x in the universe of discourse for which $Q(x)$ is **true**.

To conclude that a statement of the form $\forall x\, P(x)$ **is false**, we need to only find one value of x in the universe of discourse for which $P(x)$ is **false**. Such a value of x is called the **counter example** to the statement $\forall x\, P(x)$.

Example 2

Given a predicate,

$$Q(x): x^2 > 10.$$

What is the truth value of $\exists x\, Q(x)$, if the domain of discourse consists the numbers of 0, 1, 2, 3, and 4?

The universe of discourse contains the number 4. Since $Q(4)$, which is the statement "$16 > 10$" is true, therefore $\exists x\, Q(x)$ is true.

This is an evaluation for truth value of an existential statement.
So <u>find one x value</u> that makes the predicate statement to be true to show that the quantified statement as a whole to be true.

How about the statement $\forall x\, Q(x)$?
Is it True or False?
If False, what is the counter example?

Example 3

Given that the domain of discourse consists of all integers, what is the truth value of $\forall x$ (x is even)?

If your answer is false, give a counter example.

Statement is false.
Counter example: $x = 1$.

Can you think of other counterexamples?

Example 4

Determine the truth value of the following statements.
If the statement is false, explain why.
Assume that the domain of discourse for x is :

(i) the set of real numbers
(ii) the set of all positive real numbers
(iii) the set of integers
(iv) the set of positive integers

(a) $\forall x\,(2x \geq x)$ (b) $\exists x\,(x^3 = -1)$ (c) $\exists x\,(x^2 = -1)$

Solution:

(a) The statement is true for (ii) and (iv), false for (i) and (iii).
Counterexample: $x = -1$, $2x = -2 < -1 = x$.
(b) The statement is true for (i) and (iii), false for (ii) and (iv), since the only solution (in real numbers) to the equation $x^3 = -1$ is $x = -1$ (integers are real numbers too).
(c) The statement is always false, since for all real number s or integers, x, $x^2 \geq 0$ (we can never get a negative value).

Translate quantified English statements

Quantified English statements can be translated into predicates with quantifiers $\forall$, $\exists$ and with logical connectives such as $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$.

There can be more than one variable for a predicate, such as $P(x)$, Q(x,y) or $R(x, y, z, \ldots)$ depending on the English statements.

<u>Example</u>

$P(x, y)$: x eats y

What is the truth value for *P*(cow, grass)? How about *P*(cow, cat)?

Quantified statements for predicates with more that one variable may involve more that one quantifiers (more to come).

Example 5

Express the statement "**Every student in this class has studied calculus**" using predicates and quantifiers.

1) Rewrite the statement if necessary:

   "For every student in this class, that student has studied calculus."

2) Let the variable x be students in the class. We have

   "For every student x in this class, x has studied calculus."

3) Introduce predicate(s). Let $C(x)$: x has studied calculus, we thus have

$$\forall x \; C(x)$$

Note: This statement however is **NOT GENERALLY TRUE** if the domain of x contains any student NOT in this class.

Example 5 (Cont.)

In order to have correct statement for people other than those in this class, we rewrite the statement as

> "For every person x, if person x is a student in this class then x has studied calculus."

3) Let S(x): x in this class and C(x): x has studied calculus. We thus have

$$\forall x(S(x) \rightarrow C(x)).$$

Caution! The statement cannot be expressed as $\forall x\big(S(x) \wedge C(x)\big)$. This will means "all people are students in this class and have studied calculus!"

Example 6

Express the statement "**Some students in this class have studied calculus**" using predicates and quantifiers.

1) Rewrite the statement

   "There is a person x having the property that x is a student in this class and x has studied calculus."

2) Let $S(x)$: x in this class and $C(x)$: x has studied calculus. We thus have

   $$\exists x(S(x) \wedge C(x))$$

Caution! Our statement cannot be expressed as ∃x(S(x) → C(x)), which is true when there is someone not in the class because, in that case, for such a person x, $S(x) \rightarrow C(x)$ becomes either $F \rightarrow T$ or $F \rightarrow F$, both of which are true.

Example 7

Let $S(x)$: x is a subject in MMU and $T(x)$: x is a tough subject.

Translation:

Some subjects in MMU is tough.	$\exists x\,(S(x) \wedge T(x))$
No subject in MMU is tough.	$\neg\,\exists x\,(S(x) \wedge T(x))$ OR $\forall x\,(S(x) \rightarrow \neg\,T(x))$
All subjects in MMU are tough.	$\forall x\,(S(x) \rightarrow T(x))$
Not all subjects in MMU are tough.	$\neg\,\forall x\,(S(x) \rightarrow T(x))$ OR $\exists x\,(S(x) \wedge \neg\,T(x))$

Logically equivalent

Definition: **Logically equivalent**

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.

We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example 8

Assume that the domain is nonempty. Show that

$\forall x\big(P(x) \land Q(x)\big) \equiv \forall x P(x) \land \forall x Q(x)$ and

$\exists x\big(P(x) \lor Q(x)\big) \equiv \exists x P(x) \lor \exists x Q(x)$.

However

$\forall x(P(x) \lor Q(x)) \not\equiv \forall x P(x) \lor \forall x Q(x)$ and

$\exists x(P(x) \land Q(x)) \not\equiv \exists x P(x) \land \exists x Q(x)$

Can you find some counter examples?

Negating quantified statements using De Morgan's Law

Whenever negation (¬) occurs before a quantified statement, the **De Morgan's Law for Quantifiers** can be used to write an equivalent statement.

Negation	Equivalent Statement
$\neg\exists x\, P(x)$	$\forall x\, \neg P(x)$
$\neg\forall x\, P(x)$	$\exists x\, \neg P(x)$

RECALL:

No subject in MMU is tough. $\neg\exists x\, (S(x) \wedge T(x))$
$\equiv \forall x\, (S(x) \rightarrow \neg T(x))$

Not all subjects in MMU are tough. $\neg\forall x\, (S(x) \rightarrow T(x))$
$\equiv \exists x\, (S(x) \wedge \neg T(x))$

Example 9

$P(x)$: x is a student of TMA1201 from MMU's Fosee.

"Every student of TMA1201 is from MMU's Fosee." $\forall x P(x)$

The negation of the statement is
"Not every student of TMA1201 is from MMU's Fosee.", $\neg\forall x P(x)$

which is the same as
"There is a student of TMA1201 that is not from MMU's Fosee". $\exists x \neg P(x)$

Example 10

P(*x*): *x* is a man taller than 3 meters.

"There is a man taller than 3 meters." $\exists x P(x)$

The negation of the statement is
"There does not exist a man taller than 3 meters.", $\neg \exists x P(x)$

which is the same as
"Every man is not taller than 3 meters.". $\forall x \neg P(x)$

Nested Quantifiers

For a predicate with two or more variables, we can have two or more quantifiers to describe the quantity of these variables. There can be 6 possible combinations as below:

Statement	Meaning
$\forall x \, \forall y \, P(x, y)$ $\forall y \, \forall x \, P(x, y)$	$P(x, y)$ is true for every pair of x, y.
$\forall x \, \exists y \, P(x, y)$	For every x there is a y for $P(x, y)$ to be true.
$\exists x \, \forall y \, P(x, y)$	There is an x for every y for $P(x, y)$ to be true.
$\exists x \, \exists y \, P(x, y)$ $\exists y \, \exists x \, P(x, y)$	There exist at least a pair of x, y for $P(x, y)$ to be true.

Order of Quantifiers

The order of these quantifiers gives different meaning to the statement.

Given that $F(x, y)$: x and y are friends.
The domain of discourse is all the students in MMU.

Translate the following statements into English.

a) $\forall x\, \exists y\, F(x, y)$

b) $\exists x\, \forall y\, F(x, y)$

c) $\exists x\, \forall y\, \forall z\, (F(x, y) \wedge F(x, z) \wedge (y \neq z) \rightarrow \neg F(y, z))$

d) $\forall x\, \exists y\, \exists z\, (F(x, y) \wedge F(x, z) \wedge (y \neq z) \wedge \neg F(y, z))$

Solution:

Given that $F(x, y)$*: x* and *y* are friends.
The domain of discourse is all the students in MMU.

Translate the following statements into English.

a) $\forall x\, \exists y\, F(x, y)$
Every students in MMU has some friends.

b) $\exists x\, \forall y\, F(x, y)$
There is a student in MMU who is a friend to all students in MMU.

c) $\exists x\, \forall y\, \forall z\, (F(x, y) \land F(x, z) \land (y \neq z) \to \neg F(y, z))$
There is a student in MMU where none of his/her friends are friends to each other.

d) $\forall x\, \exists y\, \exists z\, (F(x, y) \land F(x, z) \land (y \neq z) \land \neg F(y, z))$
Every students in MMU has at least two friends where these two friends are not friends to each other.

The last two are trickier, **but the first two you should be able to answer correctly.**

Order of Quantifiers and Truth Value

The order of these quantifiers whenever they differ gives different truth values.

Example

Translate the following two statements into English

a) $\forall x \, \exists y \, (x + y = 0)$

b) $\exists y \, \forall x \, (x + y = 0)$

Do they express the same thing? Here the domain of discourse is the set of real numbers.

a) For every real number *x*, there exists a real number *y* such that their sum $x + y$ is zero.

b) There exists a real number *y*, such that its sum with any real number *x* is zero.

The two statements do not express the same thing. In fact, the first one is true, and the second one is false.

Example 11

If the domain of discourse is the set of real numbers, determine the truth value of the following statements:

a) $\forall x \, \exists y \; x^2 = y$

b) $\exists x \, \forall y \, (xy = 0)$

c) $\forall x \, \exists y \, (x + 2y = 3)$

a) True. Why?

b) True. $x = 0$.

d) True. $y = (3-x)/2$.

In Class Exercise

Given that Q(x, y) : $x = y^2$ where the domain of discourse is the set of integer, determine the truth value of the following statements and Justify each of your answer.

a) $\forall y\ Q(3, y)$

b) $\exists y\ Q(3, y)$

c) $\forall y\ Q(3, y) \to \exists y\ Q(3, y)$

d) $\forall x\ Q(x, 3)$

e) $\exists x\ Q(x, 3)$

f) $\forall x\ Q(x, 3) \to \exists x\ Q(x, 3)$

g) $\exists x\ Q(x, 3) \to \forall x\ Q(x, 3)$

h) $\forall x\ \exists y\ Q(x, y)$

i) $\exists x\ \forall y\ Q(x, y)$

Summary

We have discussed the following concepts related to predicate logic:

- Meaning of predicates with universal quantifiers and existential quantifiers
- Evaluating the truth values of statements with predicates.
- Translation of quantified English statements into predicates with quantifiers and logical connectives.
- Conversion of a negated predicate statements by De Morgan's Law.
- Order of quantifiers are important whenever two or more quantifiers are involved.

Exercise 1

Lets define the following predicate on the set of integer,

$P(x)$: "x % 4 == 0"

Evaluate

(a) $P(8)$

(b) $P(9)$

(c) $P(123464)$

(d) $\forall x\, P(x)$

(e) $\exists x\, P(x)$

a % b is the remainder when a is divided by b

Exercise 2

Determine the truth value of each of the following statements.

If it is false, find a counterexample. Consider the case where the domain of

discourse is

(i)the set of real numbers

(ii)the set of integers for

(a) $\forall x\ (x^2 \neq x)$ (b) $\forall x\ (x^2 \neq 2)$ (c) $\forall x\ (|x| > 0)$.

Exercise 3

Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(A \rightarrow P(x)) \equiv A \rightarrow \forall x P(x)$

b) $\forall x(P(x) \rightarrow A) \equiv \exists x P(x) \rightarrow A$

Exercise 4

Use the following predicates

$S(x)$: "x is student"

$P(x)$: "x is always punctual"

$L(x, y)$: "x likes y"

to express these sentences in first order logic.

a) There is a student who is always punctual.
b) All students are always punctual.
c) Some students are not always punctual.
d) There is a student who does not like punctual students.

The last one is trickier, but the first three you should be able to answer correctly.
Remember the clear rule about $\exists$ and $\forall$.

Exercise 5

Translate each of the following statements into logical expressions using predicates, quantifiers, and logical connectives. The domain of discourse is all the people in this world

(a) No one is perfect.

(b) Not everyone is perfect.

(c) All your friends are perfect.

(d) One of your friends is perfect.

(e) Everyone is your friend and is perfect.

(f) Not everyone is your friend or someone is not perfect.