6.1 EXPONENTIAL FUNCTIONS

(Adapted from "Precalculus" by Stewart et als.)

Exponential Functions _ Graphs of Exponential Functions _ Compound Interest
The Number ***e*** _ The Natural Exponential Function _

In this chapter we study a new class of functions called *exponential functions*.
Example: $f(x) = 2^x$ is an exponential function (with base 2).
Notice how quickly the values of this function increase:
$f(3) = 2^3 = 8$, $f(10) = 2^{10} = 1024$, $f(30) = 2^{30} = \mathbf{1{,}073{,}741{,}824}$

Compare this with the function $g(x) = x^2$, where $g(30) = 30^2 = 900$.
The point is that when the variable is in the exponent, even a small change in the variable can cause a dramatic change in the value of the function.

▼ Exponential Functions

To study exponential functions, we must first define what we mean by the exponential expression a^x when x is any real number.
In Section 1.2 we defined a^x for $a > 0$ and x a rational number, but we have not yet defined irrational powers. So what is meant by $5^{\sqrt{3}}$ or 2^{π}? To define a^x when x is irrational, we approximate x by rational numbers.
For example, since $\sqrt{3} = 1.73205...$ is an irrational number, we successively approximate $a^{\sqrt{3}}$ by the following rational powers:

$$a^{1.7}, a^{1.73}, a^{1.732}, a^{1.7320}, a^{1.73205}, \ldots$$

Intuitively, we can see that these rational powers of a are getting closer and closer to $a^{\sqrt{3}}$.
It can be shown by using advanced mathematics that there is exactly one number that these powers approach. We define $a^{\sqrt{3}}$ to be this number.

For example, using a calculator, we find

$$\begin{aligned} 5^{\sqrt{3}} &\approx 5^{1.732} \\ &\approx 16.2411\ldots \end{aligned}$$

The more decimal places of $\sqrt{3}$ we use in our calculation, the better our approximation of $5^{\sqrt{3}}$.
It can be proved that the *Laws of Exponents are still true when the exponents are real numbers*.

> **EXPONENTIAL FUNCTIONS**
>
> The exponential function with base a is defined for all real numbers x by
>
> $$f(x) = a^x$$
>
> where $a > 0$ and $a \neq 1$.

We assume that $a \neq 1$ because the function $f(x) = 1^x = 1$ is just a constant function.

Here are some examples of exponential functions:

$$f(x) = 2^x \qquad g(x) = 3^x \qquad h(x) = 10^x$$

Base 2 Base 3 Base 10

E X A M P L E 1 | Evaluating Exponential Functions

Let $f(x) = 3^x$. Using a calculator, we obtain

(a) $f(2) = 3^2 = 9$

(b) $f(-\frac{2}{3}) = 3^{-2/3} \approx 0.4807$

(c) $f(\pi) = 3^{\pi} \approx 31.544$

(d) $f(\sqrt{2}) = 3^{\sqrt{2}} \approx 4.7288$

NOW TRY:

Let $g(x) = 4^x$. Use a calculator to evaluate

$g(0.5)$, $g(\sqrt{2})$, $g(-\pi)$, $g(\frac{1}{3})$

▼ Graphs of Exponential Functions

We first graph exponential functions by plotting points. We will see that the graphs of such functions have an easily recognizable shape.

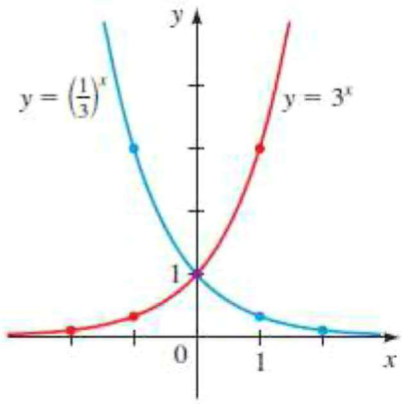
E X A M P L E 2 | Graphing Exponential Functions by Plotting Points

Draw the graph of each function.

(a) $f(x) = 3^x$ **(b)** $g(x) = (\frac{1}{3})^x$

S O LU T I O N We calculate values of $f(x)$ and $g(x)$ and plot points to sketch the graphs.

x	$f(x) = 3^x$	$g(x) = (\frac{1}{3})^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$



Notice that $g(x) = \left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x} = f(-x)$

so we could have obtained the graph of g from the graph of f by reflecting in the y-axis.

NOW TRY:

Graph the functions $f(x) = 2^x$ and $g(x) = 2^{-x}$ on one set of axes.

The figure on the right shows the graphs of the family of exponential functions $f(x) = a^x$ for various values of the base a.

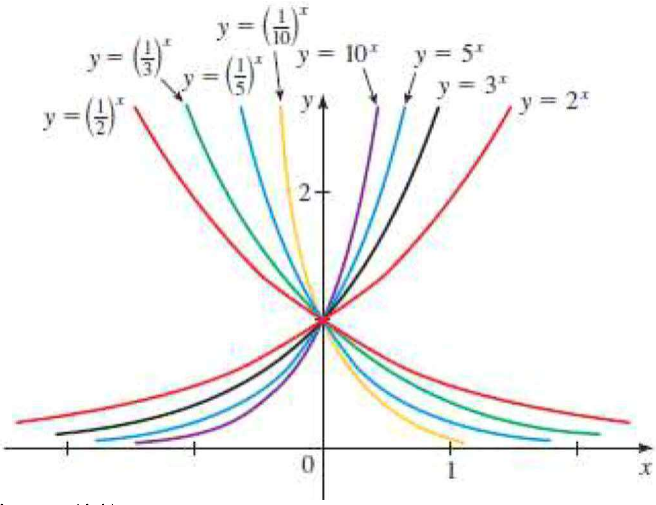


Figure (**)

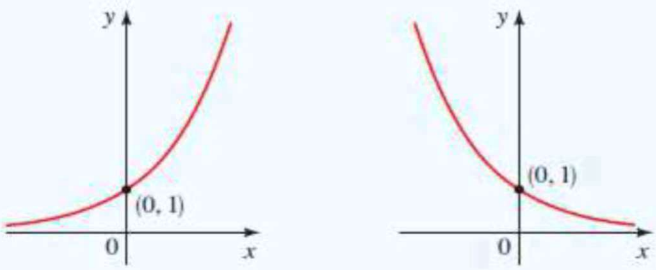
All of these graphs pass through the point $(0,1)$ because $a^0 = 1$ for $a \neq 0$.
You can see that there are two kinds of exponential functions
If $0 < a < 1$, the exponential function decreases rapidly.
If $a > 1$, the function increases rapidly.
The x-axis is a horizontal asymptote for the exponential function $f(x) = a^x$.
This is because when $a > 1$, we have $a^x \to 0$ as $x \to -\infty$,
and when $0 < a < 1$, we have $a^x \to 0$ as $x \to \infty$.
Also, $a^x > 0$ for all $x \in \mathbb{R}$, so the function $f(x) = a^x$ has domain $\mathbb{R}$ and range $(0,\infty)$.
These observations are summarized in the following box.

GRAPHS OF EXPONENTIAL FUNCTIONS

The exponential function

$$f(x) = a^x \qquad (a > 0, a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The line $y = 0$ (the x-axis) is a horizontal asymptote of f. The graph of f has one of the following shapes.

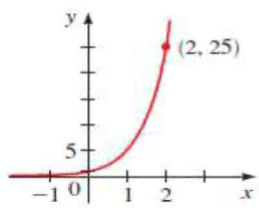


$f(x) = a^x$ for $a > 1$ $\qquad$ $f(x) = a^x$ for $0 < a < 1$

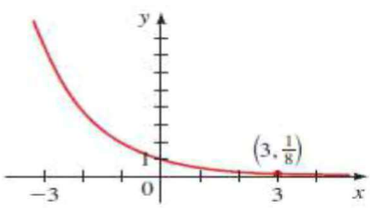
E X A M P L E 3 | Identifying Graphs of Exponential Functions

Find the exponential function $f(x) = a^x$ whose graph is given.

(a)



(b)



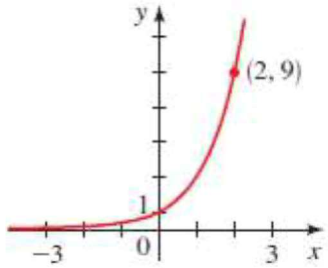
S O LU T I O N

(a) Since $f(2) = a^2 = 25$, we see that the base is $a = 5$. So $f(x) = 5^x$.

(b) Since $f(3) = a^3 = \frac{1}{8}$, we see that the base is $a = \frac{1}{2}$. So . $f(x) = \left(\frac{1}{2}\right)^x$

NOW TRY:

Find the exponential function $f(x) = a^x$ whose graph is given.



In the next example we see how to graph certain functions, not by plotting points, but by taking the basic graphs of the exponential functions in Figure(**) and applying shifting and reflecting transformations.

E X A M P L E 4 | Transformations of Exponential Functions

Use the graph of $f(x) = 2^x$ to sketch the graph of each function.

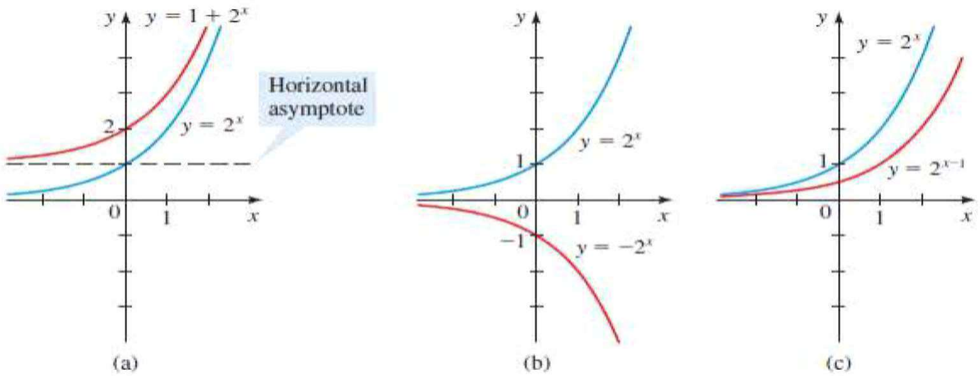
(a) $g(x) = 1 + 2^x$ **(b)** $h(x) = -2^x$ **(c)** $k(x) = 2^{x-1}$

S O LU T I O N

(a) To obtain the graph of $g(x) = 1 + 2^x$, we start with the graph of $f(x) = 2^x$ and shift it upward 1 unit. Notice from Figure (a) that the line $y = 1$ is now a horizontal asymptote.

(b) Again we start with the graph of $f(x) = 2^x$, but here we reflect in the x-axis to get the graph of $h(x) = -2^x$ shown in Figure (b).

(c) This time we start with the graph of $f(x) = 2^x$ and shift it to the right by 1 unit to get the graph of $k(x) = 2^{x-1}$ shown in Figure (c).



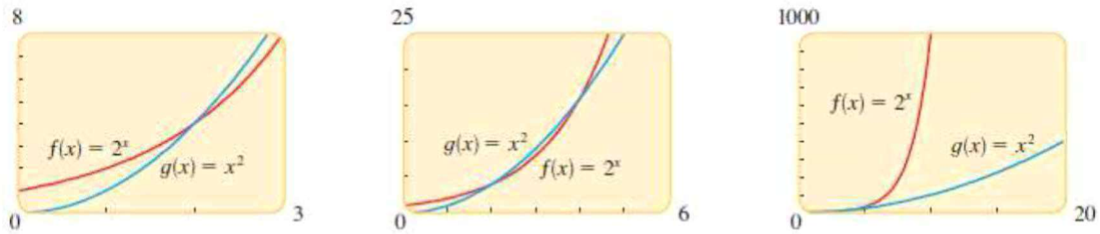
NOW TRY:

Graph the function, not by plotting points, but by starting from the graphs in Figure (**). State the domain, range, and asymptote.

(a) $f(x) = -3^x$ (b) $g(x) = 2^x - 3$ (c) $f(x) = 10^{x+3}$

E X A M P L E 5 | Comparing Exponential and Power Functions

Compare the rates of growth of the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$ by drawing the graphs of both functions in three viewing rectangles.



(a) (b) (c)

S O LU T I O N

(a) Figure (a) shows that the graph of $g(x) = x^2$ catches up with, and becomes higher than, the graph of $f(x) = 2^x$ at $x = 2$.

(b) The larger viewing rectangle in Figure (b) shows that the graph of $f(x) = 2^x$ overtakes that of $g(x) = x^2$ when $x = 4$.

(c) Figure (c) gives a more global view and shows that when x is large, $f(x) = 2^x$ is much larger than $g(x) = x^2$.

▼ The Number *e*

Any positive number can be used as a base for an exponential function. In this section we study the special base *e*, which is convenient for applications involving calculus.

The number *e* is defined as the value that $(1+\frac{1}{n})^n$ approaches as n becomes large. (In calculus this idea is made more precise through the concept of a limit.)
The table shows the values of the expression for increasingly large values of n.

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

It appears that, correct to five decimal places, $e \approx 2.71828$; in fact, the approximate value to 20 decimal places is $e \approx 2.71828182845904523536$
It can be shown that *e* is an irrational number, so we cannot write its exact value in decimal form.

▼ The Natural Exponential Function

The number e is the base for the natural exponential function. Why use such a strange base for an exponential function? It might seem at first that a base such as 10 is easier to work with. However, in certain applications the number e is the best possible base.

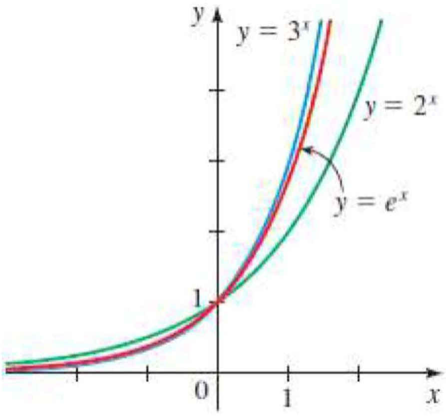
THE NATURAL EXPONENTIAL FUNCTION

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown below.



Graph of the natural exponential function

Scientific calculators have a special key for the function $f(x) = e^x$. We use this key in the next example.

E X A M P L E 6 | Evaluating the Exponential Function

Evaluate each expression rounded to five decimal places.

(a) e^3 (b) $e^{-0.53}$ (c) $e^{4.8}$

S O LU T I O N We use the $\boxed{e^x}$ key on a calculator to evaluate the exponential function.

(a) $e^3 = 20.08554$ (b) $2e^{0.53} = 1.17721$ (c) $e^{4.8} = 121.51042$

NOW TRY:

Use a calculator to evaluate the function $h(x) = e^x$ at the indicated values. Round your answers to three decimals.

$h(3)$, $h(0.23)$, $h(1)$, $h(-2)$

EXAMPLE 7| Transformations of the Exponential Function

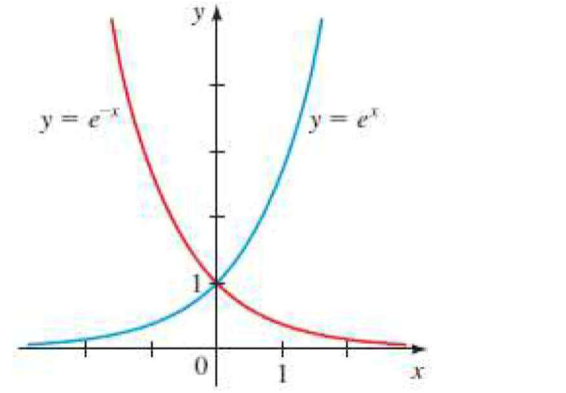
Sketch the graph of each function.

(a) $f(x) = e^{-x}$ **(b)** $g(x) = 3e^{0.5x}$

SOLUTION

(a)

We start with the graph of $y = e^x$ and reflect in the y-axis to obtain the graph of $y = e^{-x}$ as shown on the right.

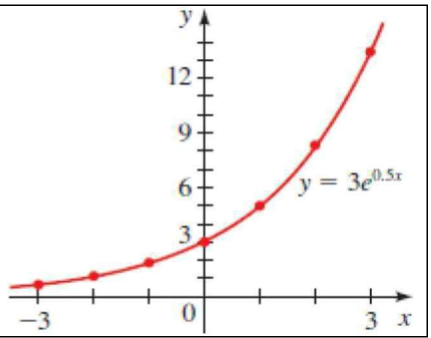


(b)

We calculate several values, plot the resulting points, then connect the points with a smooth curve.

The graph is shown on the right.

x	$f(x) = 3e^{0.5x}$
−3	0.67
−2	1.10
−1	1.82
0	3.00
1	4.95
2	8.15
3	13.45



NOW TRY:

(a) Complete the table of values, rounded to two decimal places, and sketch a graph of the function $f(x) = 3e^x$.

x	$f(x) = 3e^x$
−2	
−1	
−0.5	
0	
0.5	
1	
2	

(b) Graph the function $f(x) = -e^x$, not by plotting points, but by starting from the graph of $y = e^x$. State the domain, range, and asymptote.

EXAMPLE 8 | An Exponential Model for the Spread of a Virus

An infectious disease begins to spread in a small city of population 10,000. After t days, the number of people who have succumbed to the virus is modeled by the function

$$v(t) = \frac{10{,}000}{5 + 1245e^{-0.97t}}$$

(a) How many infected people are there initially (at time $t = 0$)?
(b) Find the number of infected people after one day, two days, and five days.
(c) Graph the function v, and describe its behavior.

S O LU T I O N

(a) Since $v(0) = \frac{10000}{5 + 1245e^{0}} = \frac{10000}{1250} = 8$, we conclude that 8 people initially have the disease.

(b) Using a calculator, we evaluate $v(1)$, $v(2)$, and $v(5)$ and then round off to obtain the following values.

Days	Infected people
1	21
2	54
5	678

(c) The graph in Figure 4 is called a *logistic curve* or a *logistic growth model*.
Curves like it occur frequently in the study of population growth.

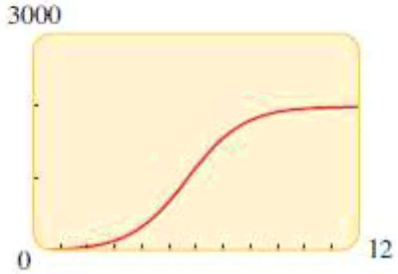


FIGURE 4

$$v(t) = \frac{10{,}000}{5 + 1245e^{-0.97t}}$$

Exercise

1–4 ■ Use a calculator to evaluate the function at the indicated values. Round your answers to three decimals.

1. $f(x) = 4^x$; $f(0.5), f(\sqrt{2}), f(\pi), f\left(\frac{1}{3}\right)$

2. $f(x) = 3^{x+1}$; $f(-1.5), f(\sqrt{3}), f(e), f\left(-\frac{5}{4}\right)$

3. $g(x) = \left(\frac{2}{3}\right)^{x-1}$; $g(1.3), g(\sqrt{5}), g(2\pi), g\left(-\frac{1}{2}\right)$

4. $g(x) = \left(\frac{3}{4}\right)^{2x}$; $g(0.7), g(\sqrt{7}/2), g(1/\pi), g\left(\frac{2}{3}\right)$

5–10 ■ Sketch the graph of the function by making a table of values. Use a calculator if necessary.

5. $f(x) = 2^x$

6. $g(x) = 8^x$

7. $f(x) = \left(\frac{1}{3}\right)^x$

8. $h(x) = (1.1)^x$

9. $g(x) = 3e^x$

10. $h(x) = 2e^{-0.5x}$

11–14 ■ Graph both functions on one set of axes.

11. $f(x) = 2^x$ and $g(x) = 2^{-x}$

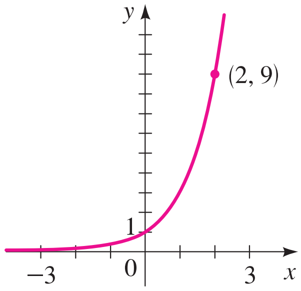
12. $f(x) = 3^{-x}$ and $g(x) = \left(\frac{1}{3}\right)^x$

13. $f(x) = 4^x$ and $g(x) = 7^x$

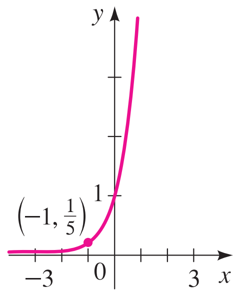
14. $f(x) = \left(\frac{2}{3}\right)^x$ and $g(x) = \left(\frac{4}{3}\right)^x$

15–18 ■ Find the exponential function $f(x) = a^x$ whose graph is given.

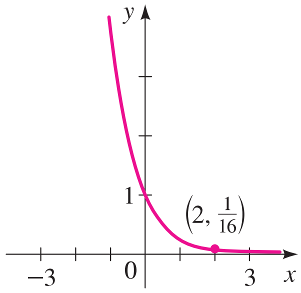
15.



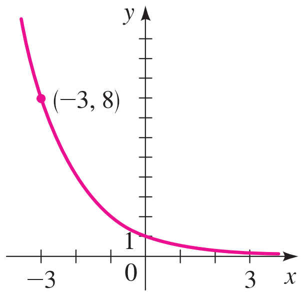
16.



17.



18.



19–24 ■ Match the exponential function with one of the graphs labeled I–VI.

19. $f(x) = 5^x$

20. $f(x) = -5^x$

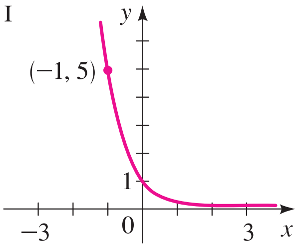
21. $f(x) = 5^{-x}$

22. $f(x) = 5^x + 3$

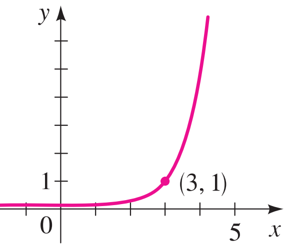
23. $f(x) = 5^{x-3}$

24. $f(x) = 5^{x+1} - 4$

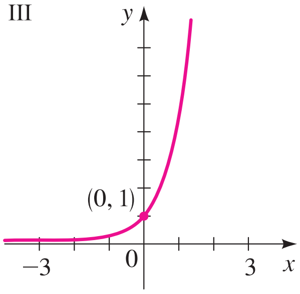
I



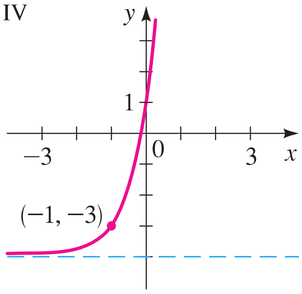
II



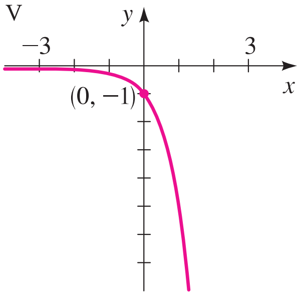
III



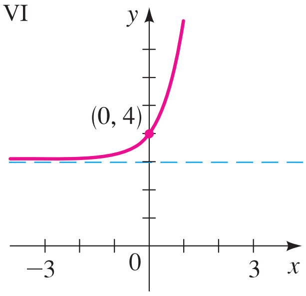
IV



V



VI



25–38 ■ Graph the function, not by plotting points, but by starting from the graphs in Figures 2 and 5. State the domain, range, and asymptote.

25. $f(x) = -3^x$

26. $f(x) = 10^{-x}$

27. $g(x) = 2^x - 3$

28. $g(x) = 2^{x-3}$

29. $h(x) = 4 + \left(\frac{1}{2}\right)^x$

30. $h(x) = 6 - 3^x$

31. $f(x) = 10^{x+3}$

32. $f(x) = -\left(\frac{1}{5}\right)^x$

33. $f(x) = -e^x$

34. $y = 1 - e^x$

35. $y = e^{-x} - 1$

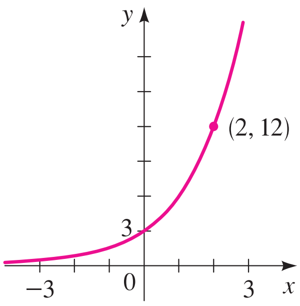
36. $f(x) = -e^{-x}$

37. $f(x) = e^{x-2}$

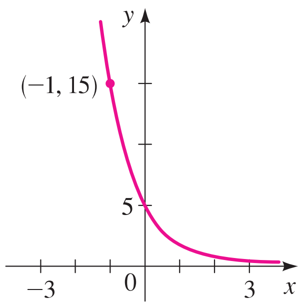
38. $y = e^{x-3} + 4$

39–40 ■ Find the function of the form $f(x) = Ca^x$ whose graph is given.

39.



40.



41. **(a)** Sketch the graphs of $f(x) = 2^x$ and $g(x) = 3(2^x)$.

(b) How are the graphs related?

42. **(a)** Sketch the graphs of $f(x) = 9^{x/2}$ and $g(x) = 3^x$.

(b) Use the Laws of Exponents to explain the relationship between these graphs.

43. If $f(x) = 10^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 10^x\left(\frac{10^h - 1}{h}\right)$$

44. Compare the functions $f(x) = x^3$ and $g(x) = 3^x$ by evaluating both of them for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15,$ and 20. Then draw the graphs of f and g on the same set of axes.

45. The *hyperbolic cosine function* is defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Sketch the graphs of the functions $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same axes and use graphical addition (see Section 2.7) to sketch the graph of $y = \cosh(x)$.

46. The *hyperbolic sine function* is defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Sketch the graph of this function using graphical addition as in Exercise 45.

47–50 ■ Use the definitions in Exercises 45 and 46 to prove the identity.

47. $\cosh(-x) = \cosh(x)$

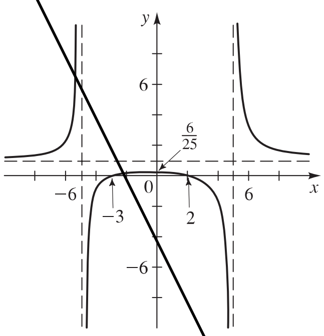
48. $\sinh(-x) = -\sinh(x)$

49. $[\cosh(x)]^2 - [\sinh(x)]^2 = 1$

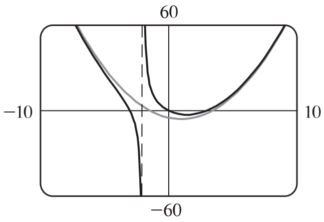
50. $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

Answer

(d)



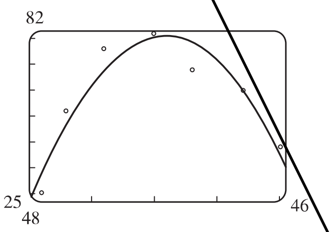
(e) $x^2 - 2x - 5$



Focus on Modeling ■ page 323

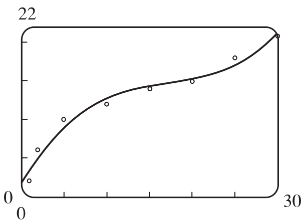
1. (a) $y = -0.275428x^2 + 19.7485x - 273.5523$

(b)



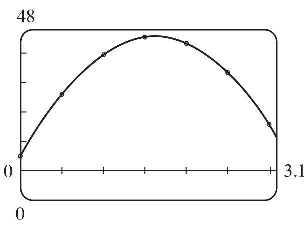
(c) 35.85 lb/in^2 **3. (a)** $y = 0.00203708x^3 - 0.104521x^2 + 1.966206x + 1.45576$

(b)



(c) 43 vegetables **(d)** 2.0 s **5. (a)** Degree 2

(b) $y = -16.0x^2 + 51.8429x + 4.20714$



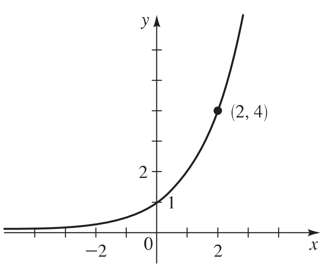
(c) 0.3 s and 2.9 s **(d)** 46.2 ft

Chapter 4

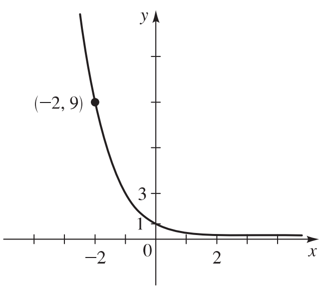
Section 4.1 ■ page 336

1. 2.000, 7.103, 77.880, 1.587 **3.** 0.885, 0.606, 0.117, 1.837

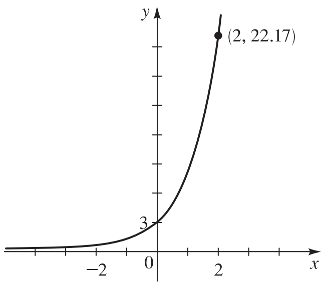
5.



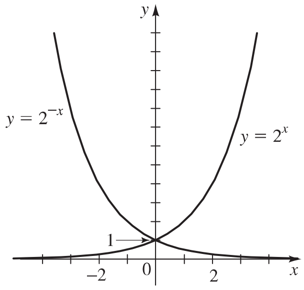
7.



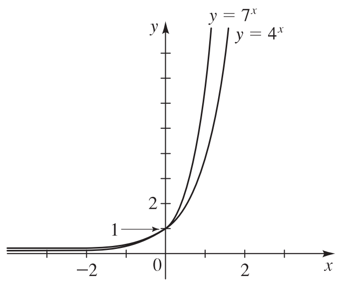
9.



11.

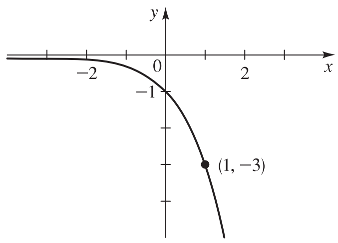


13.

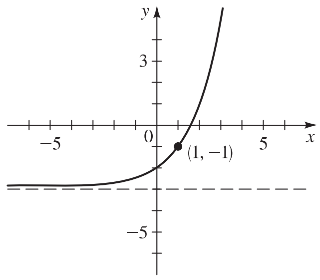


15. $f(x) = 3^x$ **17.** $f(x) = \left(\frac{1}{4}\right)^x$ **19.** III **21.** I **23.** II

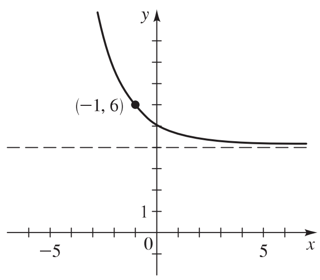
25. $\mathbb{R}$, $(-\infty, 0)$, $y = 0$



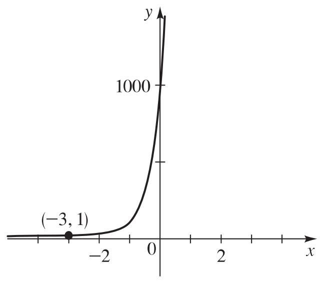
27. $\mathbb{R}$, $(-3, \infty)$, $y = -3$



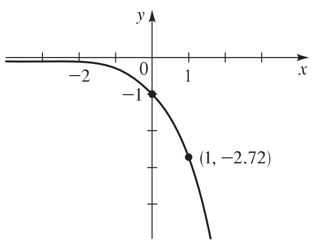
29. $\mathbb{R}$, $(4, \infty)$, $y = 4$



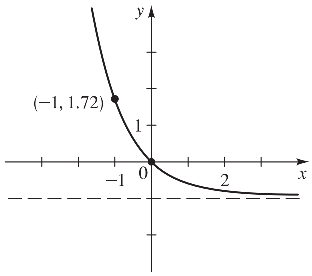
31. $\mathbb{R}$, $(0, \infty)$, $y = 0$



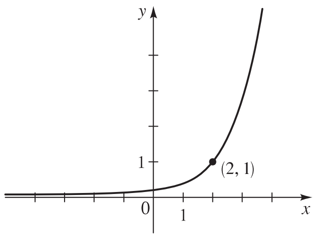
33. $\mathbb{R}$, $(-\infty, 0)$, $y = 0$



35. $\mathbb{R}$, $(-1, \infty)$, $y = -1$

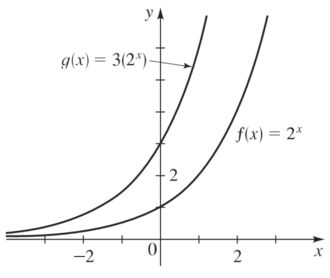


37. $\mathbb{R}, (0, \infty), y = 0$



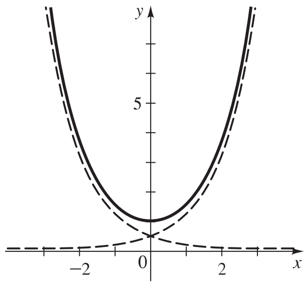
39. $y = 3(2^x)$

41. **(a)**



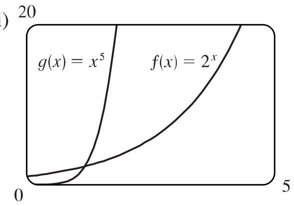
(b) The graph of g is steeper than that of f.

45.

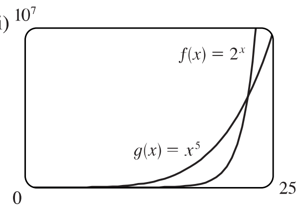


51. **(a)**

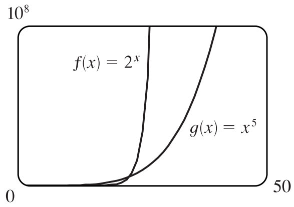
(i)



(ii)



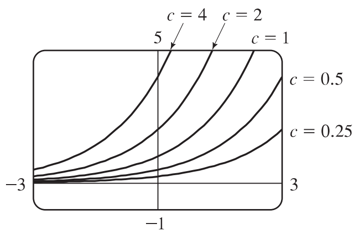
(iii)



The graph of f ultimately increases much more quickly than g.

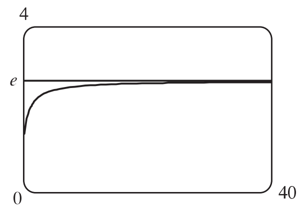
(b) 1.2, 22.4

53.

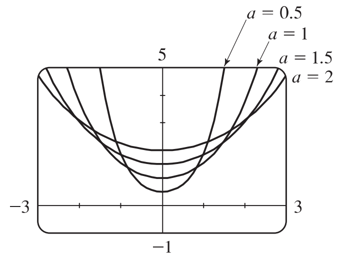


The larger the value of c, the more rapidly the graph increases.

55.

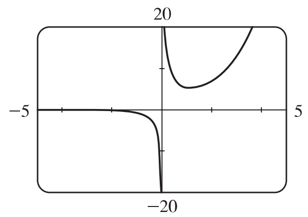


57. **(a)**



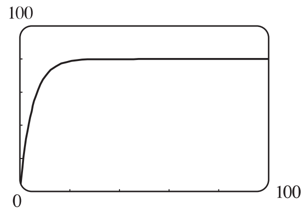
(b) The larger the value of a, the wider the graph.

59.



vertical asymptote $x = 0$
horizontal asymptote $y = 0$, left side only

61. Local minimum $\approx (0.27, 1.75)$ **63.** **(a)** Increasing on $(-\infty, 1.00]$, decreasing on $[1.00, \infty)$ **(b)** $(-\infty, 0.37]$
65. **(a)** 13 kg **(b)** 6.6 kg
67. **(a)** 0 **(b)** 50.6 ft/s, 69.2 ft/s
(c)



(d) 80 ft/s

69. **(a)** 100 **(b)** 482, 999, 1168 **(c)** 1200 **71.** 1.6 ft
73. \$5203.71, \$5415.71, \$5636.36, \$5865.99, \$6104.98, \$6353.71 **75.** **(a)** \$16,288.95 **(b)** \$26,532.98
(c) \$43,219.42 **77.** **(a)** \$4,615.87 **(b)** \$4,658.91
(c) \$4,697.04 **(d)** \$4,703.11 **(e)** \$4,704.68 **(f)** \$4,704.93
(g) \$4,704.94 **79.** (i) **81.** **(a)** \$7,678.96 **(b)** \$67,121.04

Section 4.2 ■ page 349

1.

Logarithmic form	Exponential form
$\log_8 8 = 1$	$8^1 = 8$
$\log_8 64 = 2$	$8^2 = 64$
$\log_8 4 = \frac{2}{3}$	$8^{2/3} = 4$
$\log_8 512 = 3$	$8^3 = 512$
$\log_8 \frac{1}{8} = -1$	$8^{-1} = \frac{1}{8}$
$\log_8 \frac{1}{64} = -2$	$8^{-2} = \frac{1}{64}$

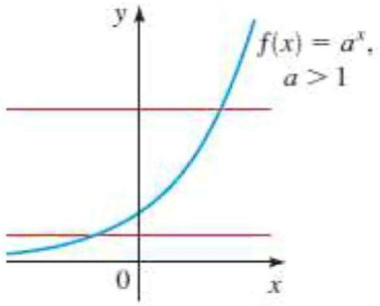
6.2 LOGARITHMIC FUNCTIONS
(Adapted from "Precalculus" by Stewart et als.)

Logarithmic Functions _ Graphs of Logarithmic Functions _ Common Logarithms _ Natural Logarithms

In this section we study the inverses of exponential functions.

▼ Logarithmic Functions

Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the Horizontal Line Test (see the figure on the right for the case $a > 1$) and therefore has an inverse function. The inverse function f^{-1} is called the *logarithmic function with base a* and is denoted by $\log_a$.



Recall that f^{-1} is defined by

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

This leads to the following definition of the logarithmic function.

DEFINITION OF THE LOGARITHMIC FUNCTION

Let a be a positive number with $a \neq 1$. The **logarithmic function with base a**, denoted by $\log_a$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

So $\log_a x$ is the *exponent* to which the base a must be raised to give x.

When we use the definition of logarithms to switch back and forth between the **logarithmic form** $\log_a x = y$ and the **exponential form** $a^y = x$, it is helpful to notice that, in both forms, the base is the same:

Logarithmic form	Exponential form
Exponent	Exponent
$\log_a x = y$	$a^y = x$
Base	Base

E X A M P L E 1 | Logarithmic and Exponential Forms

The logarithmic and exponential forms are equivalent equations: If one is true, then so is the other. So we can switch from one form to the other as in the following illustrations.

Logarithmic form	Exponential form
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2\left(\frac{1}{8}\right) = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 s = r$	$5^r = s$

NOW TRY:
Complete the table by finding the appropriate logarithmic or exponential form of the equation, as in Example 1.

Logarithmic form	Exponential form
$\log_8 8 = 1$	
$\log_8 64 = 2$	
	$8^{2/3} = 4$
	$8^3 = 512$
$\log_8\left(\frac{1}{8}\right) = -1$	
	$8^{-2} = \frac{1}{64}$

It is important to understand that $\log_a x$ is an *exponent*. For example, the numbers in the right column of the table are the logarithms (base 10) of the numbers in the left column.

x	$\log_{10} x$
10^4	4
10^3	3
10^2	2
10	1
1	0
10^{-1}	-1
10^{-2}	-2
10^{-3}	-3
10^{-4}	-4

This is the case for all bases, as the following example illustrates.

E X A M P L E 2 | Evaluating Logarithms

(a) $\log_{10} 1000 = 3$ because $10^3 = 1000$ **(b)** $\log_2 32 = 5$ because $2^5 = 32$

(c) $\log_{10} 0.1 = -1$ because $10^{-1} = 0.1$ **(d)** $\log_{16} 4 = \frac{1}{2}$ because $16^{\frac{1}{2}} = 4$

NOW TRY: Express the equation in exponential form.

(a) $\log_5 25 = 2$ (b) $\log_3 1 = 0$ (c) $\log_8 2 = \frac{1}{3}$ (d) $\log_2\left(\frac{1}{8}\right) = -3$

When we apply the Inverse Function Property to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we get

$$\log_a(a^x) = x, \qquad x \in \mathbb{R}$$

$$a^{\log_a x} = x, \qquad x > 0$$

We list these and other properties of logarithms discussed in this section.

PROPERTIES OF LOGARITHMS

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

E X A M P L E 3 | Applying Properties of Logarithms

We illustrate the properties of logarithms when the base is 5.

$\log_5 1 = 0$	Property 1	$\log_5 5 = 1$	Property 2
$\log_5 5^8 = 8$	Property 3	$5^{\log_5 12} = 12$	Property 4

NOW TRY:

Evaluate the expression.

(a) $\log_3 3$ **(b)** $\log_3 1$ **(c)** $\log_3 3^2$

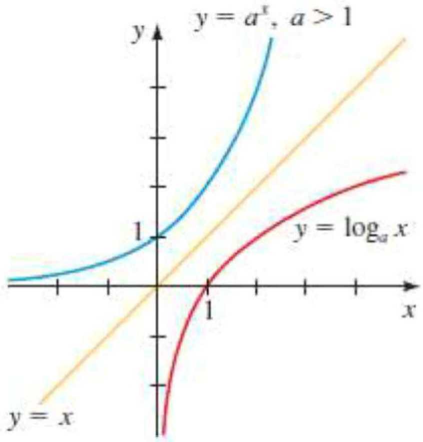
(a) $2^{\log_2 37}$ **(b)** $3^{\log_3 8}$ **(c)** $e^{\ln\sqrt{5}}$

▼ Graphs of Logarithmic Functions

Recall that if a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. Since the exponential function $f(x) = a^x$ with $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, we conclude that its inverse function, $f^{-1}(x) = \log_a x$, has domain $(0, \infty)$ and range $\mathbb{R}$.

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$.

The figure shows the case $a > 1$. The fact that $y = a^x$ (for $a > 1$) is a very rapidly increasing function for $x > 0$ implies that $y = \log_a x$ is a very slowly increasing function for $x > 1$.



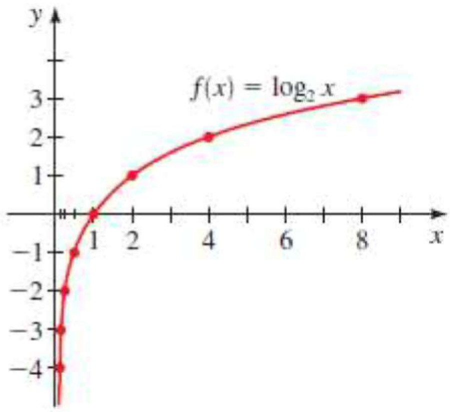
Since $\log_a 1 = 0$, the x-intercept of the function $y = \log_a x$ is 1. The y-axis is a vertical asymptote of $y = \log_a x$ because $\log_a x \to -\infty$, as $x \to 0^+$.

E X A M P L E 4 Graphing a Logarithmic Function by Plotting Points

Sketch the graph of $f(x) = \log_2 x$.

S O LU T I O N To make a table of values, we choose the *x*-values to be powers of 2 so that we can easily find their logarithms. We plot these points and connect them with a smooth curve.

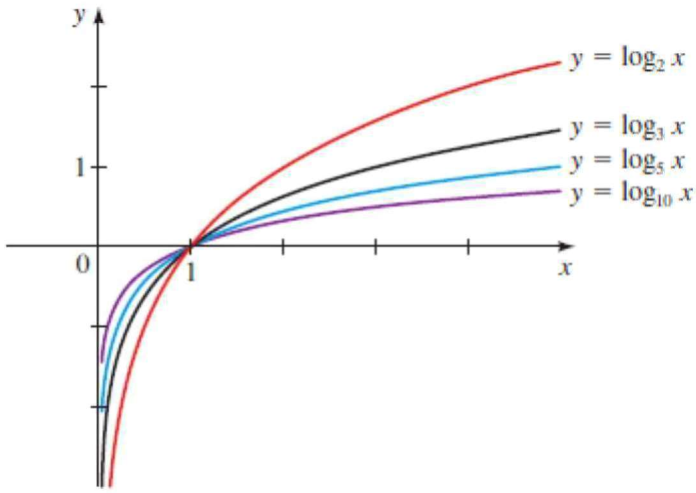
x	$\log_2 x$
2^3	3
2^2	2
2	1
1	0
2^{-1}	-1
2^{-2}	-2
2^{-3}	-3
2^{-4}	-4



NOW TRY:

Sketch the graph of $f(x) = \log_3 x$ by plotting points.

The figure on the right shows the graphs of the family of logarithmic functions with bases 2, 3, 5, and 10. These graphs are drawn by reflecting the graphs of $y = 2^x, y = 3^x, y = 5^x$ and $y = 10^x$ in the line $y = x$. We can also plot points as an aid to sketching these graphs, as illustrated in Example 4.



In the next two examples we graph logarithmic functions by starting with the basic graphs ($y = \log_a x$) and using transformations.

E X A M P L E 5 | Reflecting Graphs of Logarithmic Functions

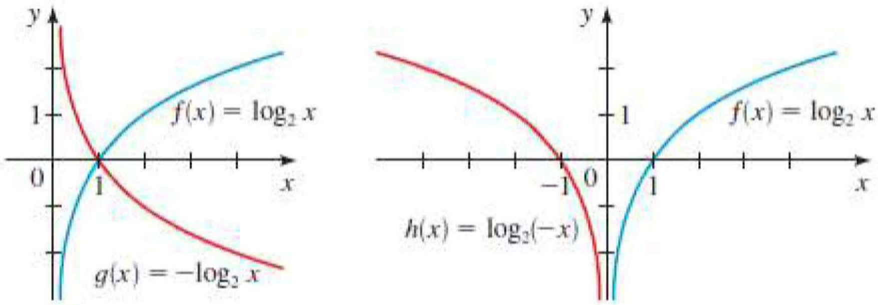
Sketch the graph of each function.

(a) $g(x) = -\log_2 x$ **(b)** $h(x) = \log_2(-x)$

S O LU T I O N

(a) We start with the graph of $f(x) = \log_2 x$ and reflect in the *x*-axis to get the graph of $g(x) = -\log_2 x$.

(b) We start with the graph of $f(x) = \log_2 x$ and reflect in the *y*-axis to get the graph of $h(x) = \log_2(-x)$.



(a) (b)

NOW TRY:

Graph the function $g(x) = \log_5(-x)$, not by plotting points, but by starting from the graphs $y = \log_5 x$. State the domain, range, and asymptote.

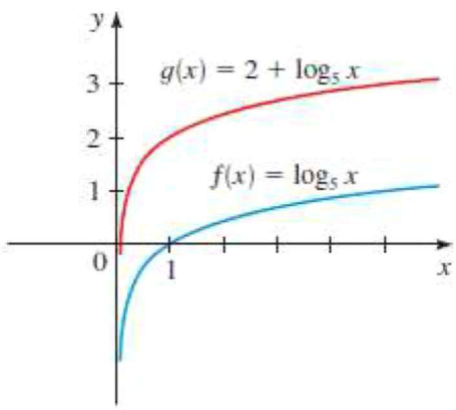
E X A M P L E 6 | Shifting Graphs of Logarithmic Functions

Find the domain of each function, and sketch the graph.

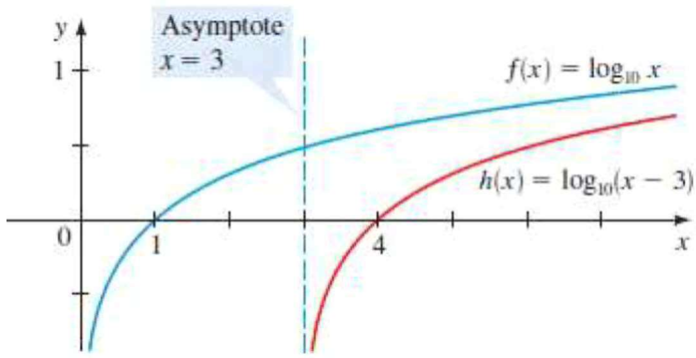
(a) $g(x) = 2 + \log_5 x$ **(b)** $h(x) = \log_{10}(x-3)$

S O LU T I O N

(a) The graph of g is obtained from the graph of $f(x) = \log_5 x$ by shifting upward 2 units.
The domain of f is $(0, \infty)$



(b) The graph of h is obtained from the graph of $f(x) = \log_{10} x$ by shifting to the right 3 units.
The line $x = 3$ is a vertical asymptote.
Since $\log_{10} x$ is defined only when $x > 0$, the domain of $h(x) = \log_{10}(x-3)$ is $\{x \mid x - 3 > 0\} = \{x \mid x > 3\} = (3, \infty)$



NOW TRY:

Graph the function, not by plotting points, but by starting from the graphs of the form $y = \log_a x$. State the domain, range, and asymptote.

(a) $f(x) = \log_2(x-4)$ (b) $y = 2 + \log_3 x$

▼ Common Logarithms

We now study logarithms with base 10.

> **COMMON LOGARITHM**
>
> The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:
>
> $$\log x = \log_{10} x$$

From the definition of logarithms we can easily find that $\log 10 = 1$ and $\log 100 = 2$. But how do we find log 50? We need to find the exponent y such that $10^y = 50$. Clearly, 1 is too small and 2 is too large. So $1 < \log 50 < 2$

To get a better approximation, we can experiment to find a power of 10 closer to 50. Fortunately,scientific calculators are equipped with a LOG key that directly gives values of common logarithms.

E X A M P L E 7 | Evaluating Common Logarithms

Use a calculator to find appropriate values of $y = \log x$ and use the values to sketch the graph.

S O LU T I O N We make a table of values, using a calculator to evaluate the function at those values of x that are not powers of 10. We plot those points and connect them by a smooth curve.in Figure 8.

x	$\log x$
0.01	−2
0.1	−1
0.5	−0.301
1	0
4	0.602
5	0.699
10	1

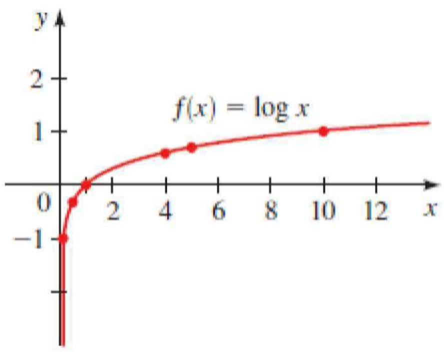


FIGURE 8

NOW TRY: Sketch the graph of the function $f(x) = 2\log x$ by plotting points.

E X A M P L E 8 | Common Logarithms and Sound

The perception of the loudness B (in decibels, dB) of a sound with physical intensity I (in W/m^2) is given by $B = 10\log\dfrac{I}{I_0}$ (as formulated by the psychologist Gustav Fechner) where I_0 is the physical intensity of a barely audible sound. Find the decibel level (loudness) of a sound whose physical intensity I is 100 times that of I_0.

S O LU T I O N We find the decibel level B by using the fact that $I = 100I_0$.

$$B = 10\log\frac{I}{I_0} = 10\log\frac{100I_0}{I_0} = 10\log 100 = 10\cdot 2 = 200$$

The loudness of the sound is 20 dB.

▼ Natural Logarithms

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of calculus is the number e, which we defined in Section 4.1.

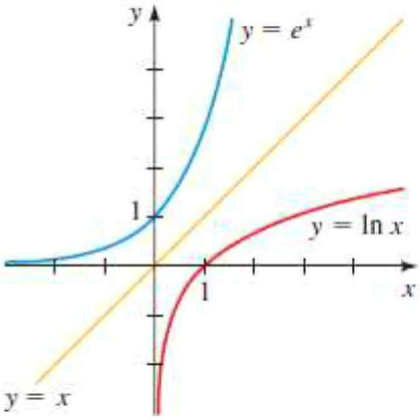
> **NATURAL LOGARITHM**
>
> The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:
>
> $$\ln x = \log_e x$$

The natural logarithmic function

$y = \ln x$

is the inverse function of the natural exponential function $y = e^x$.

Both functions are graphed here.



By the definition of inverse functions we have

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

If we substitute $a = e$ and write "ln" for "loge" in the properties of logarithms mentioned earlier, we obtain the following properties of natural logarithms.

PROPERTIES OF NATURAL LOGARITHMS

Property	Reason
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Calculators are equipped with an $\boxed{\text{LN}}$ key that directly gives the values of natural logarithms.

EXAMPLE 9 | Evaluating the Natural Logarithm Function

(a) $\ln e^8 = 8$ **(b)** $\ln\left(\dfrac{1}{e^2}\right) = \ln e^{-2} = -2$ **(c)** $\ln 5 \approx 1.609$ (Use $\boxed{\text{LN}}$ key on calculator)

NOW TRY: Use a calculator to evaluate $\ln 5$, correct to four decimal places.

EXAMPLE 10 | Finding the Domain of a Logarithmic Function

Find the domain of the function $f(x) = \ln(4 - x^2)$.

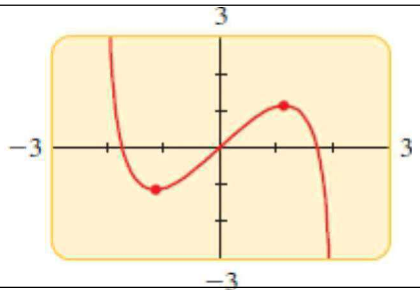
SOLUTION As with any logarithmic function, $\ln x$ is defined when $x > 0$. Thus, the domain of f is $\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = \{x \mid -2 < x < 2\}$

NOW TRY: Find the domain of the function $f(x) = \ln_{10}(x+3)$.

E X A M P L E 1 1 | Drawing the Graph of a Logarithmic Function

Draw the graph of the function $f(x) = x\ln(4 - x^2)$, and use it to find the asymptotes and local maximum and minimum values.

S O LU T I O N As in Example 10 the domain of this function is the interval $(-2, 2)$, so we choose the viewing rectangle [−3, 3] by [−3, 3]. The graph is shown here, and from it we see that the lines $x=-2$ and $x = 2$ are vertical asymptotes.



The function has a local maximum point to the right of $x = 1$ and a local minimum point to the left of $x=-1$. By zooming in and tracing along the graph with the cursor, we find that the local maximum value is approximately 1.13 and this occurs when $x \approx 1.15$. Similarly (or by noticing that the function is odd), we find that the local minimum value is about −1.13, and it occurs when $x \approx -1.15$.

Exercise

1–2 ■ Complete the table by finding the appropriate logarithmic or exponential form of the equation, as in Example 1.

1.

Logarithmic form	Exponential form
$\log_8 8 = 1$	
$\log_8 64 = 2$	
	$8^{2/3} = 4$
	$8^3 = 512$
$\log_8\left(\frac{1}{8}\right) = -1$	
	$8^{-2} = \frac{1}{64}$

2.

Logarithmic form	Exponential form
	$4^3 = 64$
$\log_4 2 = \frac{1}{2}$	
	$4^{3/2} = 8$
$\log_4\left(\frac{1}{16}\right) = -2$	
$\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$	
	$4^{-5/2} = \frac{1}{32}$

3–8 ■ Express the equation in exponential form.

3. **(a)** $\log_5 25 = 2$ **(b)** $\log_5 1 = 0$

4. **(a)** $\log_{10} 0.1 = -1$ **(b)** $\log_8 512 = 3$

5. **(a)** $\log_8 2 = \frac{1}{3}$ **(b)** $\log_2\left(\frac{1}{8}\right) = -3$

6. **(a)** $\log_3 81 = 4$ **(b)** $\log_8 4 = \frac{2}{3}$

7. **(a)** $\ln 5 = x$ **(b)** $\ln y = 5$

8. **(a)** $\ln(x + 1) = 2$ **(b)** $\ln(x - 1) = 4$

9–14 ■ Express the equation in logarithmic form.

9. **(a)** $5^3 = 125$ **(b)** $10^{-4} = 0.0001$

10. **(a)** $10^3 = 1000$ **(b)** $81^{1/2} = 9$

11. **(a)** $8^{-1} = \frac{1}{8}$ **(b)** $2^{-3} = \frac{1}{8}$

12. **(a)** $4^{-3/2} = 0.125$ **(b)** $7^3 = 343$

13. **(a)** $e^x = 2$ **(b)** $e^3 = y$

14. **(a)** $e^{x+1} = 0.5$ **(b)** $e^{0.5x} = t$

15–24 ■ Evaluate the expression.

15. **(a)** $\log_3 3$ **(b)** $\log_3 1$ **(c)** $\log_3 3^2$

16. **(a)** $\log_5 5^4$ **(b)** $\log_4 64$ **(c)** $\log_9 9$

17. **(a)** $\log_6 36$ **(b)** $\log_9 81$ **(c)** $\log_7 7^{10}$

18. **(a)** $\log_2 32$ **(b)** $\log_8 8^{17}$ **(c)** $\log_6 1$

19. **(a)** $\log_3\left(\frac{1}{27}\right)$ **(b)** $\log_{10}\sqrt{10}$ **(c)** $\log_5 0.2$

20. **(a)** $\log_5 125$ **(b)** $\log_{49} 7$ **(c)** $\log_9\sqrt{3}$

21. **(a)** $2^{\log_2 37}$ **(b)** $3^{\log_3 8}$ **(c)** $e^{\ln\sqrt{5}}$

22. **(a)** $e^{\ln \pi}$ **(b)** $10^{\log 5}$ **(c)** $10^{\log 87}$

23. **(a)** $\log_8 0.25$ **(b)** $\ln e^4$ **(c)** $\ln(1/e)$

24. **(a)** $\log_4\sqrt{2}$ **(b)** $\log_4\left(\frac{1}{2}\right)$ **(c)** $\log_4 8$

25–32 ■ Use the definition of the logarithmic function to find x.

25. **(a)** $\log_2 x = 5$ **(b)** $\log_2 16 = x$

26. **(a)** $\log_5 x = 4$ **(b)** $\log_{10} 0.1 = x$

27. **(a)** $\log_3 243 = x$ **(b)** $\log_3 x = 3$

28. **(a)** $\log_4 2 = x$ **(b)** $\log_4 x = 2$

29. **(a)** $\log_{10} x = 2$ **(b)** $\log_5 x = 2$

30. **(a)** $\log_x 1000 = 3$ **(b)** $\log_x 25 = 2$

31. **(a)** $\log_x 16 = 4$ **(b)** $\log_x 8 = \frac{3}{2}$

32. **(a)** $\log_x 6 = \frac{1}{2}$ **(b)** $\log_x 3 = \frac{1}{3}$

33–36 ■ Use a calculator to evaluate the expression, correct to four decimal places.

33. **(a)** $\log 2$ **(b)** $\log 35.2$ **(c)** $\log\left(\frac{2}{3}\right)$

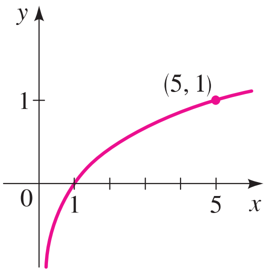
34. **(a)** $\log 50$ **(b)** $\log\sqrt{2}$ **(c)** $\log(3\sqrt{2})$

35. **(a)** $\ln 5$ **(b)** $\ln 25.3$ **(c)** $\ln(1 + \sqrt{3})$

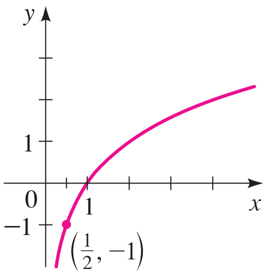
36. **(a)** $\ln 27$ **(b)** $\ln 7.39$ **(c)** $\ln 54.6$

37–40 ■ Find the function of the form $y = \log_a x$ whose graph is given.

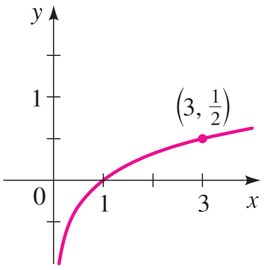
37.



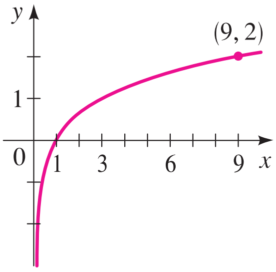
38.



39.



40.



41–46 ■ Match the logarithmic function with one of the graphs labeled I–VI.

41. $f(x) = -\ln x$

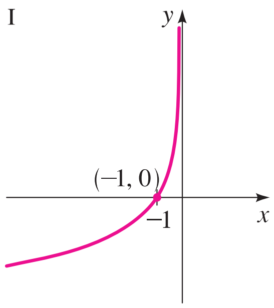
42. $f(x) = \ln(x - 2)$

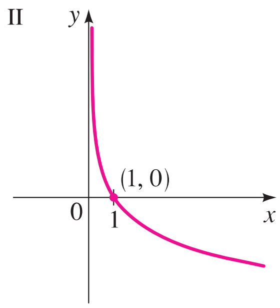
43. $f(x) = 2 + \ln x$

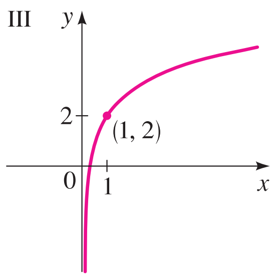
44. $f(x) = \ln(-x)$

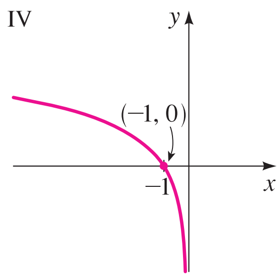
45. $f(x) = \ln(2 - x)$

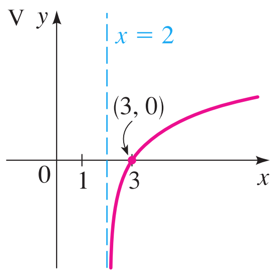
46. $f(x) = -\ln(-x)$

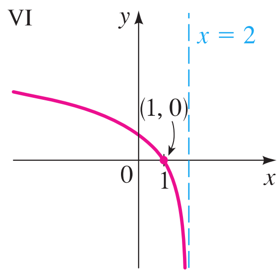












47. Draw the graph of $y = 4^x$, then use it to draw the graph of $y = \log_4 x$.

48. Draw the graph of $y = 3^x$, then use it to draw the graph of $y = \log_3 x$.

49–58 ■ Graph the function, not by plotting points, but by starting from the graphs in Figures 4 and 9. State the domain, range, and asymptote.

49. $f(x) = \log_2(x - 4)$

50. $f(x) = -\log_{10} x$

51. $g(x) = \log_5(-x)$

52. $g(x) = \ln(x + 2)$

53. $y = 2 + \log_3 x$

54. $y = \log_3(x - 1) - 2$

55. $y = 1 - \log_{10} x$

56. $y = 1 + \ln(-x)$

57. $y = |\ln x|$

58. $y = \ln|x|$

59–64 ■ Find the domain of the function.

59. $f(x) = \log_{10}(x + 3)$

60. $f(x) = \log_5(8 - 2x)$

61. $g(x) = \log_3(x^2 - 1)$

62. $g(x) = \ln(x - x^2)$

63. $h(x) = \ln x + \ln(2 - x)$

64. $h(x) = \sqrt{x - 2} - \log_5(10 - x)$

65–70 ■ Draw the graph of the function in a suitable viewing rectangle and use it to find the domain, the asymptotes, and the local maximum and minimum values.

65. $y = \log_{10}(1 - x^2)$

66. $y = \ln(x^2 - x)$

67. $y = x + \ln x$

68. $y = x(\ln x)^2$

69. $y = \dfrac{\ln x}{x}$

70. $y = x \log_{10}(x + 10)$

71. Compare the rates of growth of the functions $f(x) = \ln x$ and $g(x) = \sqrt{x}$ by drawing their graphs on a common screen using the viewing rectangle $[-1, 30]$ by $[-1, 6]$.

72. **(a)** By drawing the graphs of the functions

$$f(x) = 1 + \ln(1 + x) \qquad \text{and} \qquad g(x) = \sqrt{x}$$

in a suitable viewing rectangle, show that even when a logarithmic function starts out higher than a root function, it is ultimately overtaken by the root function.

(b) Find, correct to two decimal places, the solutions of the equation $\sqrt{x} = 1 + \ln(1 + x)$.

73–74 ■ A family of functions is given.

(a) Draw graphs of the family for $c = 1, 2, 3,$ and 4.

(b) How are the graphs in part (a) related?

73. $f(x) = \log(cx)$

74. $f(x) = c \log x$

75–76 ■ A function $f(x)$ is given.

(a) Find the domain of the function f.

(b) Find the inverse function of f.

75. $f(x) = \log_2(\log_{10} x)$

76. $f(x) = \ln(\ln(\ln x))$

77. **(a)** Find the inverse of the function $f(x) = \dfrac{2^x}{1 + 2^x}$.

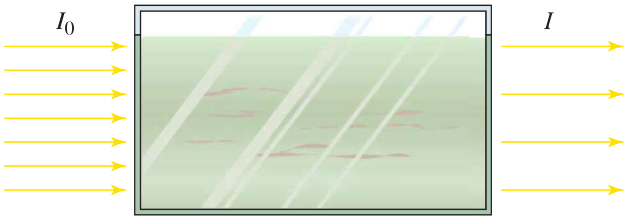
(b) What is the domain of the inverse function?

Applications

78. Absorption of Light A spectrophotometer measures the concentration of a sample dissolved in water by shining a light through it and recording the amount of light that emerges. In other words, if we know the amount of light absorbed, we can calculate the concentration of the sample. For a certain substance, the concentration (in moles/liter) is found using the formula

$$C = -2500 \ln\left(\frac{I}{I_0}\right)$$

where I_0 is the intensity of the incident light and I is the intensity of light that emerges. Find the concentration of the substance if the intensity I is 70% of I_0.



79. Carbon Dating The age of an ancient artifact can be determined by the amount of radioactive carbon-14 remaining in it. If D_0 is the original amount of carbon-14 and D is the amount remaining, then the artifact's age A (in years) is given by

$$A = -8267 \ln\left(\frac{D}{D_0}\right)$$

Find the age of an object if the amount D of carbon-14 that remains in the object is 73% of the original amount D_0.

80. Bacteria Colony A certain strain of bacteria divides every three hours. If a colony is started with 50 bacteria, then the time t (in hours) required for the colony to grow to N bacteria is given by

$$t = 3\frac{\log(N/50)}{\log 2}$$

Find the time required for the colony to grow to a million bacteria.

81. Investment The time required to double the amount of an investment at an interest rate r compounded continuously is given by

$$t = \frac{\ln 2}{r}$$

Find the time required to double an investment at 6%, 7%, and 8%.

82. Charging a Battery The rate at which a battery charges is slower the closer the battery is to its maximum charge C_0. The time (in hours) required to charge a fully discharged battery to a charge C is given by

$$t = -k \ln\left(1 - \frac{C}{C_0}\right)$$

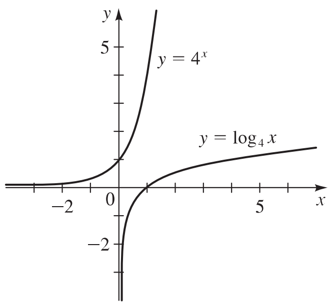
where k is a positive constant that depends on the battery. For a certain battery, $k = 0.25$. If this battery is fully discharged, how long will it take to charge to 90% of its maximum charge C_0?

1. Answer

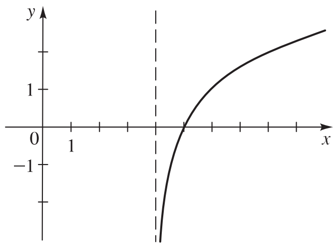
Logarithmic form	Exponential form
$\log_8 8 = 1$	$8^1 = 8$
$\log_8 64 = 2$	$8^2 = 64$
$\log_8 4 = \frac{2}{3}$	$8^{2/3} = 4$
$\log_8 512 = 3$	$8^3 = 512$
$\log_8 \frac{1}{8} = -1$	$8^{-1} = \frac{1}{8}$
$\log_8 \frac{1}{64} = -2$	$8^{-2} = \frac{1}{64}$

3. **(a)** $5^2 = 25$ **(b)** $5^0 = 1$ **5.** **(a)** $8^{1/3} = 2$ **(b)** $2^{-3} = \frac{1}{8}$
7. **(a)** $e^x = 5$ **(b)** $e^5 = y$ **9.** **(a)** $\log_5 125 = 3$
(b) $\log_{10} 0.0001 = -4$ **11.** **(a)** $\log_8 \frac{1}{8} = -1$
(b) $\log_2 \frac{1}{8} = -3$ **13.** **(a)** $\ln 2 = x$ **(b)** $\ln y = 3$
15. **(a)** 1 **(b)** 0 **(c)** 2 **17.** **(a)** 2 **(b)** 2 **(c)** 10
19. **(a)** -3 **(b)** $\frac{1}{2}$ **(c)** -1 **21.** **(a)** 37 **(b)** 8 **(c)** $\sqrt{5}$
23. **(a)** $-\frac{2}{3}$ **(b)** 4 **(c)** -1 **25.** **(a)** 32 **(b)** 4
27. **(a)** 5 **(b)** 27 **29.** **(a)** 100 **(b)** 25 **31.** **(a)** 2 **(b)** 4
33. **(a)** 0.3010 **(b)** 1.5465 **(c)** -0.1761 **35.** **(a)** 1.6094
(b) 3.2308 **(c)** 1.0051 **37.** $y = \log_5 x$ **39.** $y = \log_9 x$
41. II **43.** III **45.** VI

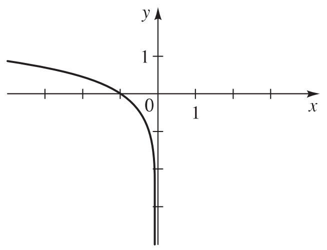
47.



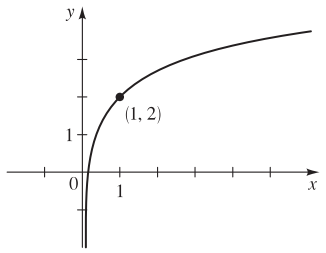
49. $(4, \infty)$, $\mathbb{R}$, $x = 4$



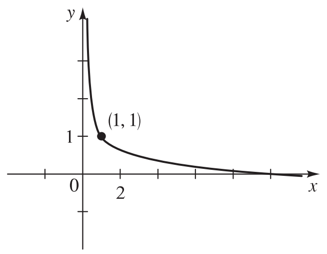
51. $(-\infty, 0)$, $\mathbb{R}$, $x = 0$



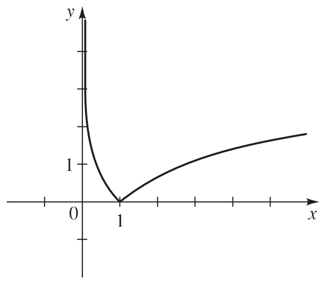
53. $(0, \infty)$, $\mathbb{R}$, $x = 0$



55. $(0, \infty)$, $\mathbb{R}$, $x = 0$

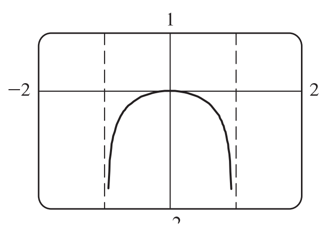


57. $(0, \infty)$, $[0, \infty)$, $x = 0$



59. $(-3, \infty)$ **61.** $(-\infty, -1) \cup (1, \infty)$ **63.** $(0, 2)$

65.



domain $(-1, 1)$
vertical asymptotes $x = 1$, $x = -1$
local maximum $(0, 0)$

6.3 LAWS OF LOGARITHMS

(Adapted from "Precalculus" by Stewart et als.)

Laws of Logarithms _ Expanding and Combining Logarithmic Expressions _ Change of Base Formula

Very briefly presented for exposure purpose

In this section we study properties of logarithms.

▼ Laws of Logarithms

Since logarithms are exponents, the Laws of Exponents give rise to the Laws of Logarithms.

LAWS OF LOGARITHMS

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C \log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

For proof, refer to the text or other similar books.

■

E X A M P L E 1 Using the Laws of Logarithms to Evaluate Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$ **(b)** $\log_2 80 - \log_2 5$ **(c)** $-\frac{1}{3}\log 8$

S O LU T I O N

(a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ Law 1

$= \log_4 64 = 3$ Because $64 = 4^3$

(b) $\log_2 80 - \log_2 5 = \log_2(\frac{80}{5})$ Law 2

$= \log_2 16 = 4$ Because $16 = 2^4$

(c) $-\frac{1}{3}\log 8 = \log 8^{-1/3}$ Law 3

$= \log(\frac{1}{2})$ Property of negative exponents

≈ -0.301 Calculator

NOW TRY: Evaluate the expression.

(a) $\log_3 \sqrt{27}$ (b) $\log 4 + \log 25$ (c) $\log_4 192 - \log_4 3$ ■

▼ Expanding and Combining Logarithmic Expressions

The Laws of Logarithms allow us to write the logarithm of a product or a quotient as the sum or difference of logarithms. This process, called ***expanding*** a logarithmic expression, is illustrated in the next example.

E X A M P L E 2 | Expanding Logarithmic Expressions
Use the Laws of Logarithms to expand each expression.

(a) $\log_2(6x)$ (b) $\log_5(x^3y^6)$ (c) $\ln\left(\frac{ab}{\sqrt[3]{c}}\right)$

S O LU T I O N

(a) $\log_2(6x) = \log_2 6 + \log_2 x$ (b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6 = 3\log_5 x + 6\log_5 y$

(c) $\ln\left(\frac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c} = \ln a + \ln b - \ln c^{\frac{1}{3}} = \ln a + \ln b - \frac{1}{3}\ln c$

NOW TRY: Use the Laws of Logarithms to expand the expression

(a) $\log_2(2x)$ (b) $\log_2(x(x-1))$ (c) $\log(\left(\frac{x^3y^4}{z^6}\right)$

The Laws of Logarithms also allow us to reverse the process of expanding that was done in Example 2. That is, we can write sums and differences of logarithms as a single logarithm. This process, called ***combining*** logarithmic expressions, is illustrated in the next example.

E X A M P L E 3 | Combining Logarithmic Expressions
Combine $3\log x + \frac{1}{2}\log(x+1)$ into a single logarithm.

S O LU T I O N

$$3\log x + \tfrac{1}{2}\log(x+1) = \log x^3 + \log(x+1)^{\frac{1}{2}} = \log\left(x^3(x+1)^{\frac{1}{2}}\right)$$

NOW TRY: Use the Laws of Logarithms to combine the expression

$$\log_2 A + \log_2 B - 2\log_2 C.$$

E X A M P L E 4 | Combining Logarithmic Expressions
Combine $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1)$ into a single logarithm.

S O LU T I O N

$$3\ln s + \tfrac{1}{2}\ln t - 4\ln(t^2+1) = \ln s^3 + \ln t^{\frac{1}{2}} - \ln(t^2+1)^4 = \ln(s^3t^{\frac{1}{2}}) - \ln(t^2+1)^4 = \ln\left(\frac{s^3t^{\frac{1}{2}}}{(t^2+1)^4}\right)$$

NOW TRY: Use the Laws of Logarithms to combine the expression
$4\log x - \frac{1}{3}\log(x^2+1) + 2\log(x-1)$

Warning Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, *there is no corresponding rule for the logarithm of a sum or a difference*. For instance,

$$\log_a(x + y) \neq \log_a x + \log_a y$$

In fact, we know that the right side is equal to $\log_a(xy)$. Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \neq \log\left(\frac{6}{2}\right) \qquad \text{and} \qquad (\log_2 x)^3 \neq 3\log_2 x$$

E X A M P L E 5 | The Law of Forgetting
(formulated by Hermann Ebbinghaus (1850–1909))

If a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies $\log P = \log P_0 - c\log(t+1)$

where c is a constant that depends on the type of task and t is measured in months.

(a) Solve for P.

(b) If your score on a history test is 90, what score would you expect to get on a similar test after two months? After a year? (Assume that $c = 0.2$.)

S O LU T I O N

(a) We first combine the right-hand side.

$\log P = \log P_0 - c\log(t+1)$ Given equation

$\log P = \log P_0 - \log(t+1)^c$ Law 3

$\log P = \log\dfrac{P_0}{(t+1)^c}$ Law 2

$P = \dfrac{P_0}{(t+1)^c}$ Because log is one-to-one

(b) Here $P_0 = 90$, $c = 0.2$, and t is measured in months.

In two months: $t = 2$ and $P = \dfrac{90}{(2+1)^{0.2}} \approx 72$

In one year: $t = 12$ and $P = \dfrac{90}{(12+1)^{0.2}} \approx 54$

Your expected scores after two months and one year are 72 and 54, respectively

▼ Change of Base Formula

For some purposes we find it useful to change from logarithms in one base to logarithms in another base. Suppose we are given $\log_a x$ and want to find $\log_b x$.

CHANGE OF BASE FORMULA

$$\log_b x = \frac{\log_a x}{\log_a b}$$

We may write the Change of Base Formula as

$$\log_b x = \left(\frac{1}{\log_a b}\right)\log_a x$$

So $\log_b x$ is just a constant multiple of $\log_a x$; the constant is $\dfrac{1}{\log_a b}$.

In particular,
if we put $x = a$,
then $\log_a a = 1$,

and this formula becomes

$$\log_b a = \frac{1}{\log_a b}$$

We can now evaluate a logarithm to *any* base by using the Change of Base Formula to express the logarithm in terms of common logarithms or natural logarithms and then using a calculator.

E X A M P L E 6 Evaluating Logarithms with the Change of Base Formula
Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, correct to five decimal places.
(a) $\log_8 5$ **(b)** $\log_9 20$

S O LU T I O N

(a) We use the Change of Base Formula with $b = 8$ and $a = 10$:

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} \approx 0.77398$$

(b) We use the Change of Base Formula with $b = 9$ and $a = e$:

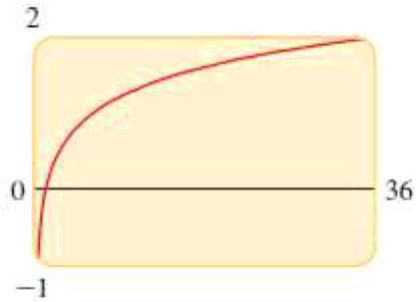
$$\log_9 20 = \frac{\ln 20}{\ln 9} \approx 1.36342$$

NOW TRY: Use the Change of Base Formula and a calculator to evaluate the logarithm, rounded to six decimal places. Use either natural or common logarithms.

(a) $\log_2 5$ (b) $\log_3 16$

E X A M P L E 7 Using the Change of Base Formula to Graph a Logarithmic Function
Use a graphing calculator to graph $f(x) = \log_6 x$.
S O LU T I O N Calculators don't have a key for $\log_6$, so we use the Change of Base Formula to write

$$f(x) = \log_6 x = \frac{\ln x}{\ln 6}$$



Since calculators do have an LN key, we can enter this new form of the function and graph it. The graph is shown in Figure 1.

NOW TRY: Use the Change of Base Formula to show that $\log_3 x = \dfrac{\ln x}{\ln 3}$.
Then use this fact to draw the graph of the function $f(x) = \log_3 x$

.

Exercise

1–12 ■ Evaluate the expression.

1. $\log_3 \sqrt{27}$

2. $\log_2 160 - \log_2 5$

3. $\log 4 + \log 25$

4. $\log \dfrac{1}{\sqrt{1000}}$

5. $\log_4 192 - \log_4 3$

6. $\log_{12} 9 + \log_{12} 16$

7. $\log_2 6 - \log_2 15 + \log_2 20$

8. $\log_3 100 - \log_3 18 - \log_3 50$

9. $\log_4 16^{100}$

10. $\log_2 8^{33}$

11. $\log(\log 10^{10,000})$

12. $\ln(\ln e^{e^{200}})$

13–38 ■ Use the Laws of Logarithms to expand the expression.

13. $\log_2(2x)$

14. $\log_3(5y)$

15. $\log_2(x(x - 1))$

16. $\log_5 \dfrac{x}{2}$

17. $\log 6^{10}$

18. $\ln \sqrt{z}$

19. $\log_2(AB^2)$

20. $\log_6 \sqrt[4]{17}$

21. $\log_3(x\sqrt{y})$

22. $\log_2(xy)^{10}$

23. $\log_5 \sqrt[3]{x^2 + 1}$

24. $\log_a\left(\dfrac{x^2}{yz^3}\right)$

25. $\ln \sqrt{ab}$

26. $\ln \sqrt[3]{3r^2 s}$

27. $\log\left(\dfrac{x^3y^4}{z^6}\right)$

28. $\log\left(\dfrac{a^2}{b^4\sqrt{c}}\right)$

29. $\log_2\left(\dfrac{x(x^2 + 1)}{\sqrt{x^2 - 1}}\right)$

30. $\log_5 \sqrt{\dfrac{x - 1}{x + 1}}$

31. $\ln\left(x\sqrt{\dfrac{y}{z}}\right)$

32. $\ln \dfrac{3x^2}{(x + 1)^{10}}$

33. $\log \sqrt[4]{x^2 + y^2}$

34. $\log\left(\dfrac{x}{\sqrt[3]{1 - x}}\right)$

35. $\log \sqrt{\dfrac{x^2 + 4}{(x^2 + 1)(x^3 - 7)^2}}$

36. $\log \sqrt{x\sqrt{y\sqrt{z}}}$

37. $\ln\left(\dfrac{x^3\sqrt{x - 1}}{3x + 4}\right)$

38. $\log\left(\dfrac{10^x}{x(x^2 + 1)(x^4 + 2)}\right)$

39–48 ■ Use the Laws of Logarithms to combine the expression.

39. $\log_3 5 + 5\log_3 2$

40. $\log 12 + \frac{1}{2}\log 7 - \log 2$

41. $\log_2 A + \log_2 B - 2\log_2 C$

42. $\log_5(x^2 - 1) - \log_5(x - 1)$

43. $4\log x - \frac{1}{3}\log(x^2 + 1) + 2\log(x - 1)$

44. $\ln(a + b) + \ln(a - b) - 2\ln c$

45. $\ln 5 + 2\ln x + 3\ln(x^2 + 5)$

46. $2(\log_5 x + 2\log_5 y - 3\log_5 z)$

47. $\frac{1}{3}\log(2x + 1) + \frac{1}{2}[\log(x - 4) - \log(x^4 - x^2 - 1)]$

48. $\log_a b + c\log_a d - r\log_a s$

49–56 ■ Use the Change of Base Formula and a calculator to evaluate the logarithm, correct to six decimal places. Use either natural or common logarithms.

49. $\log_2 5$

50. $\log_5 2$

51. $\log_3 16$

52. $\log_6 92$

53. $\log_7 2.61$

54. $\log_6 532$

55. $\log_4 125$

56. $\log_{12} 2.5$

57. Use the Change of Base Formula to show that

$$\log_3 x = \frac{\ln x}{\ln 3}$$

Then use this fact to draw the graph of the function $f(x) = \log_3 x$.

58. Draw graphs of the family of functions $y = \log_a x$ for $a = 2, e, 5$, and 10 on the same screen, using the viewing rectangle $[0, 5]$ by $[-3, 3]$. How are these graphs related?

59. Use the Change of Base Formula to show that

$$\log e = \frac{1}{\ln 10}$$

60. Simplify: $(\log_2 5)(\log_5 7)$

61. Show that $-\ln(x - \sqrt{x^2 - 1}) = \ln(x + \sqrt{x^2 - 1})$.

Applications

62. Forgetting Use the Ebbinghaus Forgetting Law (Example 5) to estimate a student's score on a biology test two years after he got a score of 80 on a test covering the same material. Assume $c = 0.3$ and t is measured in months.

63. Wealth Distribution Vilfredo Pareto (1848–1923) observed that most of the wealth of a country is owned by a few members of the population. **Pareto's Principle** is

$$\log P = \log c - k\log W$$

where W is the wealth level (how much money a person has) and P is the number of people in the population having that much money.

(a) Solve the equation for P.

(b) Assume $k = 2.1$, $c = 8000$, and W is measured in millions of dollars. Use part (a) to find the number of people who have \$2 million or more. How many people have \$10 million or more?

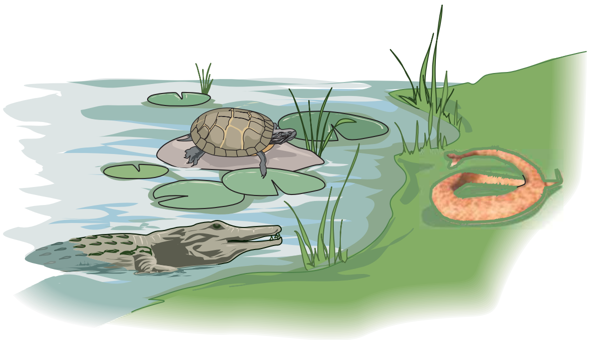
64. Biodiversity Some biologists model the number of species S in a fixed area A (such as an island) by the Species-Area relationship

$$\log S = \log c + k\log A$$

where c and k are positive constants that depend on the type of species and habitat.

(a) Solve the equation for S.

(b) Use part (a) to show that if $k = 3$ then doubling the area increases the number of species eightfold.



65. Magnitude of Stars The magnitude M of a star is a measure of how bright a star appears to the human eye. It is defined by

$$M = -2.5 \log\left(\frac{B}{B_0}\right)$$

where B is the actual brightness of the star and B_0 is a constant.

(a) Expand the right-hand side of the equation.

(b) Use part (a) to show that the brighter a star, the less its magnitude.

(c) Betelgeuse is about 100 times brighter than Albiero. Use part (a) to show that Betelgeuse is 5 magnitudes less than Albiero.

Discovery • Discussion

66. True or False? Discuss each equation and determine whether it is true for all possible values of the variables. (Ignore values of the variables for which any term is undefined.)

(a) $\log\left(\dfrac{x}{y}\right) = \dfrac{\log x}{\log y}$

(b) $\log_2(x - y) = \log_2 x - \log_2 y$

(c) $\log_5\left(\dfrac{a}{b^2}\right) = \log_5 a - 2\log_5 b$

(d) $\log 2^z = z \log 2$

(e) $(\log P)(\log Q) = \log P + \log Q$

(f) $\dfrac{\log a}{\log b} = \log a - \log b$

(g) $(\log_2 7)^x = x \log_2 7$

(h) $\log_a a^a = a$

(i) $\log(x - y) = \dfrac{\log x}{\log y}$

(j) $-\ln\left(\dfrac{1}{A}\right) = \ln A$

67. Find the Error What is wrong with the following argument?

$$\begin{aligned} \log 0.1 &< 2 \log 0.1 \\ &= \log(0.1)^2 \\ &= \log 0.01 \\ \log 0.1 &< \log 0.01 \\ 0.1 &< 0.01 \end{aligned}$$

68. Shifting, Shrinking, and Stretching Graphs of Functions Let $f(x) = x^2$. Show that $f(2x) = 4f(x)$, and explain how this shows that shrinking the graph of f horizontally has the same effect as stretching it vertically. Then use the identities $e^{2+x} = e^2 e^x$ and $\ln(2x) = \ln 2 + \ln x$ to show that for $g(x) = e^x$, a horizontal shift is the same as a vertical stretch and for $h(x) = \ln x$, a horizontal shrinking is the same as a vertical shift.

Answer

1. $\frac{3}{2}$ **3.** 2 **5.** 3 **7.** 3 **9.** 200 **11.** 4 **13.** $1 + \log_2 x$

15. $\log_2 x + \log_2(x - 1)$ **17.** $10 \log 6$ **19.** $\log_2 A + 2 \log_2 B$

21. $\log_3 x + \frac{1}{2}\log_3 y$ **23.** $\frac{1}{3}\log_5(x^2 + 1)$ **25.** $\frac{1}{2}(\ln a + \ln b)$

27. $3 \log x + 4 \log y - 6 \log z$

29. $\log_2 x + \log_2(x^2 + 1) - \frac{1}{2}\log_2(x^2 - 1)$

31. $\ln x + \frac{1}{2}(\ln y - \ln z)$ **33.** $\frac{1}{4}\log(x^2 + y^2)$

35. $\frac{1}{2}[\log(x^2 + 4) - \log(x^2 + 1) - 2\log(x^3 - 7)]$

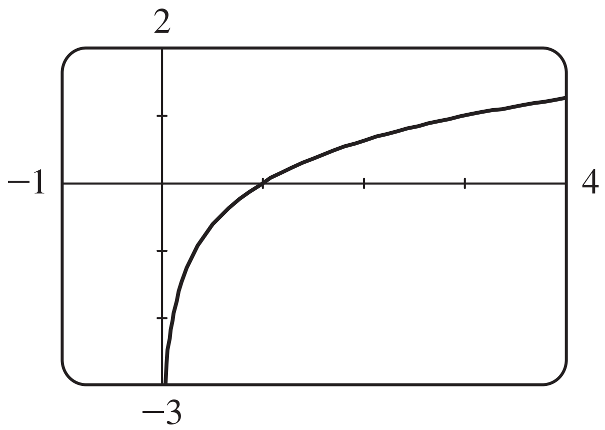
37. $3 \ln x + \frac{1}{2}\ln(x - 1) - \ln(3x + 4)$ **39.** $\log_3 160$

41. $\log_2(AB/C^2)$ **43.** $\log\left(\dfrac{x^4(x - 1)^2}{\sqrt[3]{x^2 + 1}}\right)$ **45.** $\ln(5x^2(x^2 + 5)^3)$

47. $\log\left(\sqrt[3]{2x + 1}\,\sqrt{(x - 4)/(x^4 - x^2 - 1)}\right)$

49. 2.321928 **51.** 2.523719 **53.** 0.493008 **55.** 3.482892

57.



63. **(a)** $P = c/W^k$ **(b)** 1866, 64

65. **(a)** $M = -2.5 \log B + 2.5 \log B_0$

4.4 EXPONENTIAL AND LOGARITHMIC EQUATIONS
(Adapted from "Precalculus" by Stewart et als.)

Exponential Equations _ Logarithmic Equations _

▼ Exponential Equations
An *exponential equation* is one in which the variable occurs in the exponent.
For example, $2^x = 7$
The variable x presents a difficulty because it is in the exponent. To deal with this difficulty, we take the logarithm of each side and then use the Laws of Logarithms to "bring down x" from the exponent.

$2^x = 7$	Given equation
$\ln 2^x = \ln 7$	Take ln of each side
$x \ln 2 = \ln 7$	Law 3 (bring down exponent)
$x = \dfrac{\ln 7}{\ln 2}$	Solve for x
≈ 2.807	Calculator

The method that we used to solve $2^x = 7$ is typical of how we solve exponential equations in general.

GUIDELINES FOR SOLVING EXPONENTIAL EQUATIONS

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

E X A M P L E 1 | Solving an Exponential Equation
Find the solution of the equation $3^{x+2} = 7$, rounded to six decimal places.

S O LU T I O N We take the common logarithm of each side and use Law 3.

$3^{x+2} = 7$ Given equation $\log 3^{x+2} = \log 7$ Take log of each side $(x+2)\log 3 = \log 7$ Law 3 (bring down exponent) $x+2 = \dfrac{\log 7}{\log 3}$ Divide by log 3	$x = \dfrac{\log 7}{\log 3} - 2$ Subtract 2 ≈ -0.228756 Calculator We could have used natural logarithms instead of common logarithms. In fact, using the same steps, we get $x = \dfrac{\ln 7}{\ln 3} - 2 \approx -0.228756$

CHECK YOUR ANSWER

Substituting $x = -0.228756$ into the original equation and using a calculator, we get

$$3^{(-0.228756)+2} \approx 7 \quad ✔$$

NOW TRY:
Find the solution of the exponential equation $2^{1-x} = 3$, rounded to four decimal places.

E X A M P L E 2 | Solving an Exponential Equation
Solve the equation $8e^{2x} = 20$

S O LU T I O N We first divide by 8 to isolate the exponential term on one side of the equation.

Given equation Divide by 8 Take ln of each side Property of ln $x = \frac{\ln 2.5}{2} \approx 0.458$ Divide by 2; calculator	**CHECK YOUR ANSWER** Substituting $x = 0.458$ into the original equation and using a calculator, we get $8e^{2(0.458)} \approx 20$ ✔

NOW TRY:
Find the solution of the exponential equation $3e^{x} = 10$, rounded to four decimal places.

E X A M P L E 3 Solving an Exponential Equation Algebraically and Graphically
Solve the equation $e^{3-2x} = 4$ algebraically and graphically.

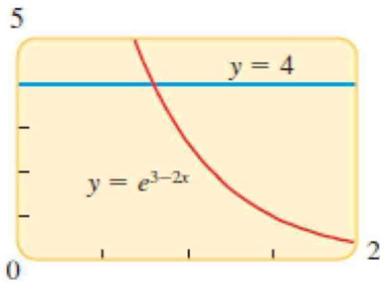
S O LU T I O N 1: Algebraic

Since the base of the exponential term is *e*, we use natural logarithms to solve this equation.

Given equation
Take ln of each side
Property of ln
Subtract 3
Multiply by $-\frac{1}{2}$

You should check that this answer satisfies the original equation.

S O LU T I O N 2: Graphical

We graph the equations $y = e^{3-2x}$ and $y = 4$ in the same viewing rectangle as in Figure 1. The solutions occur where the graphs intersect. Zooming in on the point of intersection of the two graphs, we see that $x = 0.81$.	 **FIGURE 1**

NOW TRY:
Find the solution of the exponential equation $e^{1-4x} = 2$, rounded to four decimal places.

E X A M P L E 4 | An Exponential Equation of Quadratic Type

Solve the equation . $e^{2x} - e^x - 6 = 0$

S O LU T I O N

To isolate the exponential term, we factor		Let $w = e^x$. We get the quadratic equation $w^2 - w - 6 = 0$ which factors as $(w-3)(w+2) = 0$
$e^{2x} - e^x - 6 = 0$	Given equation	
$(e^x)^2 - e^x - 6 = 0$	Law of Exponents	
$(e^x - 3)(e^x + 2) = 0$	Factor (a quadratic in e^x)	
$e^x - 3 = 0$ or $e^x + 2 = 0$	Zero-Product Property	
$e^x = 3$ $\quad$ $e^x = -2$		

The equation $e^x = 3$ leads to $x = \ln 3$.

But the equation $e^x = -2$ has no solution because $e^x > 0$ for all x.

Thus, $x = \ln 3 \approx 1.0986$ is the only solution.

[You should check that this answer satisfies the original equation.]

NOW TRY: Solve the equation $e^{2x} - 3e^x + 2 = 0$

E X A M P L E 5 | Solving an Exponential Equation

Solve the equation . $3xe^x + x^2e^x = 0$

S O LU T I O N

First we factor the left side of the equation.		CHECK YOUR ANSWER
$3xe^x + x^2e^x = 0$	Given equation	$x = 0$:
$x(3 + x)e^x = 0$	Factor out common factors	$3(0)e^0 + 0^2e^0 = 0$ ✔
$x(3 + x) = 0$	Divide by e^x (because $e^x \neq 0$)	$x = -3$:
$x = 0$ or $3 + x = 0$	Zero-Product Property	$3(-3)e^{-3} + (-3)^2e^{-3}$
Thus the solutions are $x = 0$ and $x = -3$.		$= -9e^{-3} + 9e^{-3} = 0$ ✔

NOW TRY: Solve the equation $x^2 2^x - 2^x = 0$

▼ Logarithmic Equations

A *logarithmic equation* is one in which a logarithm of the variable occurs.

For example, $\log_2(x+2) = 5$

To solve for x, we write the equation in exponential form.

$$x + 2 = 2^5 \qquad \text{Exponential form}$$

$$x = 32 - 2 = 30 \qquad \text{Solve for } x$$

Another way of looking at the first step is to raise the base, 2, to each side of the equation.

$$2^{\log_2(x+2)} = 2^5 \qquad \text{Raise 2 to each side}$$

$$x + 2 = 2^5 \qquad \text{Property of logarithms}$$

$$x = 32 - 2 = 30 \qquad \text{Solve for } x$$

The method used to solve this simple problem is typical. We summarize the steps as follows.

GUIDELINES FOR SOLVING LOGARITHMIC EQUATIONS

1. Isolate the logarithmic term on one side of the equation; you might first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.

E X A M P L E 6 | Solving Logarithmic Equations
Solve each equation for x.
(a) $\ln x = 8$ **(b)** $\log_2(25 - x) = 3$

S O LU T I O N
(a) $\ln x = 8$, $x = e^8 \approx 2981$

We can also solve this problem another way:
$\ln x = 8$, $e^{\ln x} = e^8$, $x = e^8 \approx 2981$

(b) The first step is to rewrite the equation in exponential form.
$\log_2(25 - x) = 3$, $25 - x = 2^3$, $25 - x = 8$, $x = 25 - 8 = 17$

NOW TRY:
Solve the logarithmic equation for x. (a) $\ln x = 10$ (b) $\log(3x + 5) = 2$

E X A M P L E 7 | Solving a Logarithmic Equation
Solve the equation . $4 + 3\log(2x) = 16$

S O LU T I O N We first isolate the logarithmic term. This allows us to write the equation in exponential form.
$4 + 3\log(2x) = 16$, $3\log(2x) = 12$, $\log(2x) = 4$, $2x = 10^4$, $x = 5000$

CHECK YOUR ANSWER

If $x = 5000$, we get

$$\begin{aligned} 4 + 3\log 2(5000) &= 4 + 3\log 10{,}000 \\ &= 4 + 3(4) \\ &= 16 \quad ✔ \end{aligned}$$

NOW TRY:
Solve the logarithmic equation $4 - \log(3 - x) = 3$ for x.

E X A M P L E 8 Solving a Logarithmic Equation Algebraically and Graphically
Solve the equation $\log(x + 2) + \log(x - 1) = 1$ algebraically and graphically.

SOLUTION 1: **Algebraic**

We first combine the logarithmic terms, using the Laws of Logarithms.

$\log(x+2)+\log(x-1)=1$

$(x+2)(x-1)=10$

$x^2+x-2=0$

$(x+4)(x-3)=0$

$x=-4$ or $x=3$

Checking these potential solutions in the original equation, we find that $x=-4$ is not a solution (because logarithms of negative numbers are undefined), but $x=3$ is a solution

CHECK YOUR ANSWER

$x=-4$:

$\log(-4+2)+\log(-4-1)$

$=\log(-2)+\log(-5)$

undefined ✗

$x=3$:

$\log(3+2)+\log(3-1)$

$=\log 5+\log 2=\log(5\cdot 2)$

$=\log 10=1$ ✓

SOLUTION 2: **Graphical**

We first move all terms to one side of the equation:

$\log(x+2)+\log(x-1)-1=0$

Then we graph $y=\log(x+2)+\log(x-1)-1$ as in Figure 2.

The solutions are the x-intercepts of the graph.

Thus, the only solution is $x = 3$.

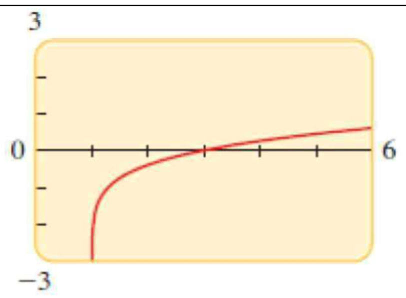


FIGURE 2

NOW TRY:

Solve the logarithmic equation $\log_5(x+1)-\log_5(x-1)=2$ for x.

EXAMPLE 9 | Solving a Logarithmic Equation Graphically

Solve the equation $x^2=2\ln(x+2)$

SOLUTION We first move all terms to one side of the equation

$$x^2-2\ln(x+2)=0$$

Then we graph $y=x^2-2\ln(x+2)$ as in Figure 3.

The solutions are the x-intercepts of the graph.

Zooming in on the x-intercepts, we see that there are two solutions:

$x\approx-0.71$ and $x\approx1.60$

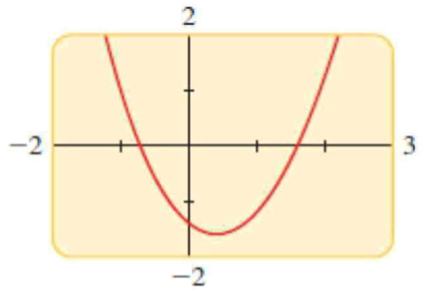


FIGURE 3

NOW TRY:

Use a graphing device to find all solutions of the equation $\ln x=3-x$, rounded to two decimal places.

Exercise

1–26 ▪ Find the solution of the exponential equation, correct to four decimal places.

1. $10^x = 25$

2. $10^{-x} = 4$

3. $e^{-2x} = 7$

4. $e^{3x} = 12$

5. $2^{1-x} = 3$

6. $3^{2x-1} = 5$

7. $3e^x = 10$

8. $2e^{12x} = 17$

9. $e^{1-4x} = 2$

10. $4(1 + 10^{5x}) = 9$

11. $4 + 3^{5x} = 8$

12. $2^{3x} = 34$

13. $8^{0.4x} = 5$

14. $3^{x/14} = 0.1$

15. $5^{-x/100} = 2$

16. $e^{3-5x} = 16$

17. $e^{2x+1} = 200$

18. $\left(\frac{1}{4}\right)^x = 75$

19. $5^x = 4^{x+1}$

20. $10^{1-x} = 6^x$

21. $2^{3x+1} = 3^{x-2}$

22. $7^{x/2} = 5^{1-x}$

23. $\dfrac{50}{1 + e^{-x}} = 4$

24. $\dfrac{10}{1 + e^{-x}} = 2$

25. $100(1.04)^{2t} = 300$

26. $(1.00625)^{12t} = 2$

27–34 ■ Solve the equation.

27. $x^2 2^x - 2^x = 0$

28. $x^2 10^x - x10^x = 2(10^x)$

29. $4x^3 e^{-3x} - 3x^4 e^{-3x} = 0$

30. $x^2 e^x + xe^x - e^x = 0$

31. $e^{2x} - 3e^x + 2 = 0$

32. $e^{2x} - e^x - 6 = 0$

33. $e^{4x} + 4e^{2x} - 21 = 0$

34. $e^x - 12e^{-x} - 1 = 0$

35–50 ■ Solve the logarithmic equation for x.

35. $\ln x = 10$

36. $\ln(2 + x) = 1$

37. $\log x = -2$

38. $\log(x - 4) = 3$

39. $\log(3x + 5) = 2$

40. $\log_3(2 - x) = 3$

41. $2 - \ln(3 - x) = 0$

42. $\log_2(x^2 - x - 2) = 2$

43. $\log_2 3 + \log_2 x = \log_2 5 + \log_2(x - 2)$

44. $2 \log x = \log 2 + \log(3x - 4)$

45. $\log x + \log(x - 1) = \log(4x)$

46. $\log_5 x + \log_5(x + 1) = \log_5 20$

47. $\log_5(x + 1) - \log_5(x - 1) = 2$

48. $\log x + \log(x - 3) = 1$

49. $\log_9(x - 5) + \log_9(x + 3) = 1$

50. $\ln(x - 1) + \ln(x + 2) = 1$

51. For what value of x is the following true?

$$\log(x + 3) = \log x + \log 3$$

52. For what value of x is it true that $(\log x)^3 = 3\log x$?

53. Solve for x: $2^{2/\log_5 x} = \frac{1}{16}$

54. Solve for x: $\log_2(\log_3 x) = 4$

55–62 ■ Use a graphing device to find all solutions of the equation, correct to two decimal places.

55. $\ln x = 3 - x$

56. $\log x = x^2 - 2$

57. $x^3 - x = \log(x + 1)$

58. $x = \ln(4 - x^2)$

59. $e^x = -x$

60. $2^{-x} = x - 1$

61. $4^{-x} = \sqrt{x}$

62. $e^{x^2} - 2 = x^3 - x$

63–66 ■ Solve the inequality.

63. $\log(x - 2) + \log(9 - x) < 1$

64. $3 \le \log_2 x \le 4$

65. $2 < 10^x < 5$

66. $x^2 e^x - 2e^x < 0$

Applications

67. Compound Interest A man invests \$5000 in an account that pays 8.5% interest per year, compounded quarterly.

(a) Find the amount after 3 years.

(b) How long will it take for the investment to double?

68. Compound Interest A man invests \$6500 in an account that pays 6% interest per year, compounded continuously.

(a) What is the amount after 2 years?

(b) How long will it take for the amount to be \$8000?

69. Compound Interest Find the time required for an investment of \$5000 to grow to \$8000 at an interest rate of 7.5% per year, compounded quarterly.

70. Compound Interest Nancy wants to invest \$4000 in saving certificates that bear an interest rate of 9.75% per year, compounded semiannually. How long a time period should she choose in order to save an amount of \$5000?

71. Doubling an Investment How long will it take for an investment of \$1000 to double in value if the interest rate is 8.5% per year, compounded continuously?

72. Interest Rate A sum of \$1000 was invested for 4 years, and the interest was compounded semiannually. If this sum amounted to \$1435.77 in the given time, what was the interest rate?

73. Annual Percentage Yield Find the annual percentage yield for an investment that earns 8% per year, compounded monthly.

74. Annual Percentage Yield Find the annual percentage yield for an investment that earns $5\frac{1}{2}\%$ per year, compounded continuously.

75. Radioactive Decay A 15-g sample of radioactive iodine decays in such a way that the mass remaining after t days is given by $m(t) = 15e^{-0.087t}$ where $m(t)$ is measured in grams. After how many days is there only 5 g remaining?

76. Skydiving The velocity of a sky diver t seconds after jumping is given by $v(t) = 80(1 - e^{-0.2t})$. After how many seconds is the velocity 70 ft/s?

77. Fish Population A small lake is stocked with a certain species of fish. The fish population is modeled by the function

$$P = \frac{10}{1 + 4e^{-0.8t}}$$

where P is the number of fish in thousands and t is measured in years since the lake was stocked.

(a) Find the fish population after 3 years.

1. 1.3979 **3.** -0.9730 **5.** -0.5850 **7.** 1.2040
9. 0.0767 **11.** 0.2524 **13.** 1.9349 **15.** -43.0677
17. 2.1492 **19.** 6.2126 **21.** -2.9469 **23.** -2.4423
25. 14.0055 **27.** ± 1 **29.** $0, \frac{4}{3}$ **31.** $\ln 2 \approx 0.6931, 0$
33. $\frac{1}{2} \ln 3 \approx 0.5493$ **35.** $e^{10} \approx 22026$ **37.** 0.01
39. $\frac{95}{3}$ **41.** $3 - e^2 \approx -4.3891$ **43.** 5 **45.** 5
47. $\frac{13}{12}$ **49.** 6 **51.** $\frac{3}{2}$ **53.** $1/\sqrt{5} \approx 0.4472$ **55.** 2.21
57. 0.00, 1.14 **59.** -0.57 **61.** 0.36
63. $2 < x < 4$ or $7 < x < 9$ **65.** $\log 2 < x < \log 5$
67. **(a)** \$6435.09 **(b)** 8.24 yr **69.** 6.33 yr **71.** 8.15 yr
73. 8.30% **75.** 13 days **77.** **(a)** 7337 **(b)** 1.73 yr
79. **(a)** $P = P_0 e^{-kh}$ **(b)** 56.47 kPa
81. **(a)** $t = -\frac{5}{13} \ln(1 - \frac{13}{60} I)$ **(b)** 0.218 s