Topic 9: Ordinary Differential Equation

9.1 Basic Concepts and Ideas

Definition:

A ***differential equation*** **(DE)** is an equation involving an unknown function and its derivatives.

Differential equations are classified according to type, order, and linearity.

Classification of differential equation

An equation containing only ordinary derivatives, with respect to a ***single*** *independent variable*, is said to be an ***ordinary*** *differential equation*.

The following are differential equations involving the unknown function *y*.

Example 1: (i) $\frac{dy}{dx} = \cos x$ or $y' = \cos x$ or $dy = \cos x dx$

(ii) $\frac{dy}{dx} = -\frac{x}{y}$ or $y' = -\frac{x}{y}$ or $dy = -\frac{x}{y}dx$

(iii) $x\frac{dy}{dx} - 4y = x^6 e^x$. or $xy' - 4y = x^6 e^x$

(iv) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ or $y'' - 5y' + 6y = 0$

A **partial differential equation** (or briefly a **PDE**) is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the independent variables.

Example 2: Here $u = u(t,x)$ is the unknown function with two independent variables *t* and *x*.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{(heat equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{(Laplace's equation)}$$

Classification by Order

The order of the highest-order derivative in a differential equation is called the ***order*** of the equation.

Example 3:

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$ ***second-order*** ordinary differential equation.

$$a^2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$ ***fourth-order*** partial differential equation.

Classification as Linear or Nonlinear

An ordinary differential equation is said to be linear if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

It is characterized by two properties:

(i) The dependent variable y and all its derivatives are of the ***first degree***; that is, the power of each term involving y is 1.

(ii) Each coefficient depends on only the independent variable x.

An equation that is not linear is said to be ***nonlinear***.

Example 4:

$xdy + ydx = 0$ Linear first-order ordinary differential equation

$y'' - 2y' + y = 0$ Linear second-order ordinary differential equation

$x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = e^x$ Linear third-order ordinary differential equation

$yy'' - 2y' = x$ Nonlinear second-order ordinary differential equation because it involves the product of y *and* y''.

$\frac{d^3y}{dx^3} - y^2 = 0$ Nonlinear third-order ordinary differential equation

Concept of Solution

Definition: Any function f defined on some interval I, which when substituted into a differential equation reduces the equation to an identity, is said to be a ***solution*** of the equation on the interval.

Example 5:

Verify that $y = x^2$ is a solution of the differential equation (DE) $xy' = 2y$ for all x.

Solution:
To show that $y = x^2$ is a solution of the DE, we have to show that the LHS of the DE is equal to the RHS. Differentiating $y = x^2$ with respect to x *and* substituting $y' = 2x$ into the LHS of the DE ,we obtain

$$\text{LHS} = xy' = x(2x) = 2x^2$$
$$\text{RHS} = 2y = 2x^2$$

We have an identity in x *because LHS=RHS*. Therefore $y = x^2$ is a solution of the DE.

Remark: Verifying that $y = \boldsymbol{f}(x)$ is a solution of a DE is usually relatively easy as it involves differentiation. Solving a DE is much more difficult as it involves finding the unknown function $y = f(x)$.

<u>Explicit and Implicit Solutions</u>

A solution of an ordinary differential equation that can be written in the form $y = f(x)$ is said to be an ***explicit solution***. It is also a solution in which the dependent variable is expressed solely in terms of the independent variable and constant.

A relation $\boldsymbol{G}(x, y) = 0$ is said to be an ***implicit solution*** of an ordinary differential equation on an interval I provided it determines implicitly a differentiable function $y = f(x)$ that satisfies the differential equation on I.

Example 6:
For $-1 < x < 1$, show that the relation $x^2 + y^2 - 1 = 0$ is an implicit solution of the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

Solution: We are going to show by differentiating $x^2 + y^2 - 1 = 0$ with respect to x, we arrive at the DE $\dfrac{dy}{dx} = -\dfrac{x}{y}$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(1) = 0$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}.$$

Example 7:
Show that the function $y = 3xe^x$ is a solution of the linear (differential) equation

$$y'' - 2y' + y = 0$$

Solution: We find $\quad y' = 3xe^x + 3e^x$

$$y'' = 3xe^x + 3e^x + 3e^x$$
$$= 3xe^x + 6e^x$$

Therefore

$$y'' - 2y' + y = (3xe^x + 6e^x) - 2(3xe^x + 3e^x) + 3xe^x = 0$$

Hence $y = 3xe^x$ is a solution of the DE

In general, it can be shown that $y = Axe^x$, where A is an arbitrary constant, is a solution of the differential equation $y'' - 2y' + y = 0$.

Hence this is known as the ***general solution*** of the differential equation while $y = 3xe^x$ is a ***particular solution***.

The most general function that will satisfy the differential equation contains one or more arbitrary constants; it is known as the ***general solution*** of the differential equation. Giving particular numerical values to one or more of the constants in the general solution results in a ***particular solution*** of the equation.

Example 8:

Solve $y' = \cos x$.

Solution:

$$y = \sin x + c \quad \text{with arbitrary } c.$$

Figure 1 shows some of the solutions, for c = -3, -2, -1, 0, 1, 2, 3, 4.

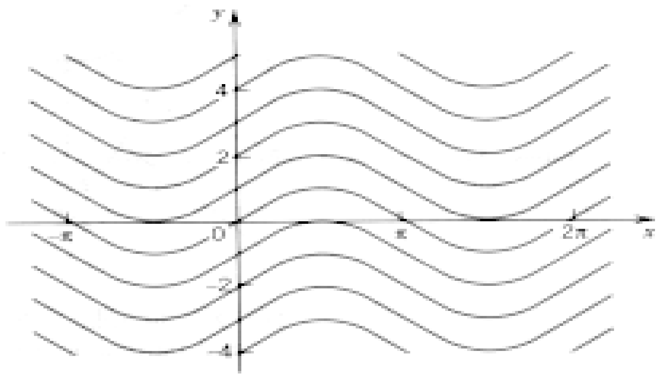

Fig. 1. Solutions of $y' = \cos x$

Initial-Value Problem

An **initial value problem** is an ordinary differential equation

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0 \quad \text{(which is an } \textbf{\textit{nth-order}} \text{ differential equation)}$$

together with the initial condition

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \ \ldots, \ y^{(n-1)}(x_0) = y_{n-1},$$

where $x_0, y_0, y_1, \ldots, y_{n-1}$ are arbitrary constants.

Example 9 :

1. The initial value problem. $y'(x) = y; \quad y(0) = 3$
2. The initial value problem $\dfrac{d^2 y}{dx^2} + y = 0; \ y(0) = -1, \ y'(0) = 1.$

9.2 Separable Differential Equations

Definition:

A first-order differential equation that can be expressed in the form

$$g(y)\frac{dy}{dx} = f(x) \qquad \text{or} \qquad g(y)dy = f(x)dx \qquad (1)$$

is said to be ***separable*** or to have ***separable variables*** where *f(x)* is a function that depends only on *x* and *g(y)* is a function that depends only on *y*.

Example 10: Show that $\frac{dy}{dx} = xe^{(x+2y)}$ is separable.

Solution:

$$\frac{dy}{dx} = xe^x e^{2y} \ ; \qquad dy = xe^x e^{2y} dx \ ;$$

$$e^{-2y} dy = xe^x dx$$

which is of the form $g(y)dy = f(x)dx$

Example 10a: The differential equation $\frac{dy}{dx} = 3x - y$ is not separable because it cannot be expressed in the form $g(y)dy = f(x)dx$

Method of Solution : Separable equation

To solve a separable DE $g(y)\frac{dy}{dx} = f(x)$ we integrate on both sides with respect to *x*, obtaining

$$\int g(y)\frac{dy}{dx}dx = \int f(x)dx + c.$$

$$\int g(y)dy = \int f(x)dx + c.$$

Example 11:

Solve the differential equation $\frac{dy}{dx} = 1 + y$

Solution: We note that the DE is separable because it can be expressed in the form $g(y)dy = f(x)dx$

$$\frac{1}{1+y}dy = dx$$

$$\int \frac{1}{1+y}dy = \int dx$$

$$\ln|1+y| = x + c$$

This is an implicit solution of the DE. It can be converted into an explicit solution of the form $y = f(x)$. How?

Example 12:

Solve the differential equation $9yy' + 4x = 0$.

Solution:

$$\frac{dy}{dx} = \frac{-4x}{9y}$$

$$\int 9y\,dy = -\int 4x\,dx$$

$$\frac{9}{2}y^2 = -2x^2 + c^*$$

$$\frac{x^2}{9} + \frac{y^2}{4} = c$$

The solution represents a family of ellipses.

9.3 Linear Differential Equations

Definition:

A differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$ is said to be a ***first-order linear equation***.

For example,

$$x\frac{dy}{dx} - 4y = x^6e^x$$

is a first order linear DE.

Here $a_1(x) = x, \quad a_0(x) = -4, \quad and \quad g(x) = x^6e^x$

Method of solution :First Order Linear Differential equation

1. Make the coefficient of $\frac{dy}{dx}$ unity. i.e.

$$\frac{dy}{dx} + P(x)y = r(x)$$

For homogeneous equation, $r(x) = 0$,

$$\frac{dy}{dx} + P(x)y = 0$$ is a separable equation.

2. Identify $p(x)$ and find the integrating factor

$$\mu(x) = e^{\int P(x)dx}.$$

3. Multiply the equation obtained in step (1) by the integrating factor:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = e^{\int P(x)dx}r(x).$$

4. The left side of the equation in step (3) is the derivative of the product of the integrating factor and the dependent variable y; that is,

$$\frac{d}{dx}[e^{\int P(x)dx}y] = e^{\int P(x)dx}r(x).$$

5. Integrate both sides of the equation found in step (4).

Example 13:

Solve $x\dfrac{dy}{dx} - 4y = x^6 e^x$.

Solution:

1. Rewrite the DE as $\dfrac{dy}{dx} - \dfrac{4}{x}y = x^5 e^x$.

2 We then note that $P(x) = -\dfrac{4}{x}$. Hence, the integrating factor is given by

$$\mu(x) = e^{\int\left(-\frac{4}{x}\right)dx} = e^{-4\ln x} = e^{\ln(x^{-4})} = \tfrac{1}{x^4} \quad \text{because} \quad e^{\ln f(x)} = f(x)$$

3. $$\therefore \frac{1}{x^4}\frac{dy}{dx} - \frac{4}{x^5}y = \frac{1}{x^4}(x^5 e^x)$$

4. $$\frac{d}{dx}\left(\frac{1}{x^4}y\right) = xe^x$$

5. $$\frac{1}{x^4}y = \int xe^x dx = xe^x - e^x + c$$

$$y = x^5 e^x - x^4 e^x + cx^4$$

Example 14:

Solve the initial value problem: $y' + 2xy = x, \quad y(0) = 1.$

Solution:

Here $P(x) = 2x,$

Integrating factor, $\mu(x) = e^{\int P(x)dx} = e^{\int 2x dx} = e^{x^2}$.

Multiplying into the equation, $$e^{x^2}\left(\frac{dy}{dx} + 2xy\right) = xe^{x^2}$$

$$\frac{d}{dx}\left(e^{x^2}y\right) = xe^{x^2}$$

$$e^{x^2}y = \int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c$$

$$\therefore \quad y(x) = \frac{1}{2} + ce^{-x^2}.$$

From the initial condition, when $x = 0, \quad y = 1$

$$\therefore \quad 1 = \frac{1}{2} + c \qquad \text{Hence, } c = \frac{1}{2}$$

The solution of our initial value problem is $y(x) = \dfrac{1}{2} + \dfrac{1}{2}e^{-x^2}$.

9.4 Exact Differential Equations

Revision on Partial Differentiation (Topic 8)

Example:

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x, y) = x^2 + 3xy + y - 1$.

Solution: Regarding y as a constant and differentiating f(x,y) with respect to x, we obtain

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 3xy + y - 1\right) = 2x + 3y$$

Regarding x as a constant and differentiating f(x,y) with respect to y, we obtain

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3xy + y - 1\right) = 3x + 1$$

Example:

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x, y) = y \sin xy$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y \sin xy) = y\frac{\partial}{\partial x}(\sin xy) = y^2 \cos xy$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y \sin xy\right) = y\frac{\partial}{\partial y}\left(\sin xy\right) + (\sin xy)\frac{\partial}{\partial y}(y)$$

$$= y \cos xy \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy$$

Definition of Total Differential

If $f = f(x, y)$ then the differential of f, denoted df, is defined by

$$df = f_x(x,y)dx + f_y(x,y)dy \quad \text{or} \quad df = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy$$

df is also called the total differential of f.

Example: Let $F = F(x, y) = \frac{1}{3}x^3y^3$. Then

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = \frac{\partial}{\partial x}(\frac{1}{3}x^3y^3)dx + \frac{\partial}{\partial y}(\frac{1}{3}x^3y^3)dy$$

$$dF = x^2y^3dx + x^3y^2dy$$

Definition of Exact Differential Equations

A differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be ***exact*** in a region ***R*** of the *xy*-plane if there is a function *F(x, y)* such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y).$$

That is, the total differential of *F* satisfies

$$dF(x,y) = M(x,y)dx + N(x,y)dy.$$

Example 15:

1. Show that the differential equation $x^2y^3dx + x^3y^2dy = 0$ is exact.

Solution: To show that the DE is exact we have to find a function $F(x,y)$ such that its differential

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = x^2y^3dx + x^3y^2dy$$

We claim that $F(x,y) = \frac{1}{3}x^3y^3$ is such a function because $\frac{\partial F}{\partial x} = x^2y^3$ and $\frac{\partial F}{\partial y} = x^3y^2$

$$dF = x^2y^3dx + x^3y^2dy.$$

Remark : In practice, producing such a function $F(x,y)$ to show that the DE is exact is not that easy. In fact if we can produce such a function, then the solution of the DE is given implicitly by $F(x,y) = c$. Later we will give an easier criterion for testing whether a given DE is exact or not.

Example 15a: Solve $\dfrac{dy}{dx} = \dfrac{\sin y}{2y - x\cos y}$.

Solution: The above d.e. in differential form can be written as

$$\sin y\, dx + (x\cos y - 2y)dy = 0$$

To solve the DE we would have to produce a function $F(x,y)$ such that the LHS of the above DE is $dF(x,y)$, *the total differential of* $F(x,y)$. We can verify that such a function is

$$F(x,y) = x\sin y - y^2.$$

Therefore

$$d(x\sin y - y^2) = 0$$

Hence $x\sin y - y^2 = c$ *is the solution of the DE.*

Theorem (Criterion for an Exact Differential)

Let $\boldsymbol{M}(x, y)$ and $\boldsymbol{N}(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region $\boldsymbol{R}$. Then a necessary and sufficient condition that

$$M(x,y)dx + N(x,y)dy$$

be an exact differential is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Method of solution : Exact equation

1. If $Mdx + Ndy = 0$ is exact, then $\dfrac{\partial F}{\partial x} = M$. Integrate this last equation with respect to x to get $F(x,y) = \int M(x,y)dx + g(y)$. (2)

2. To determine g(y), take the partial derivative with respect to y of both sides of equation (2) and substitute N for $\dfrac{\partial F}{\partial y}$. We can now solve for g'(y).

3. Integrate g'(y) to obtain g(y) up to a numerical constant. Substituting g(y) into equation (2) gives $F(x, y)$.

4. The solution to $Mdx + Ndy = 0$ is given implicitly by $F(x, y) = C$.
(Alternatively, starting with $\dfrac{\partial F}{\partial y} = N$, the implicit solution can be found by first integrating with respect to y)

Example 16:

Solve $(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$.

Solution:

Here $M(x,y) = (e^{2y} - y\cos xy)$ and $N(x,y) = (2xe^{2y} - x\cos xy + 2y)$.

Therefore $\dfrac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, a function $F(x,y)$ exists for which $M(x,y) = \dfrac{\partial F}{\partial x}$ and $N(x,y) = \dfrac{\partial F}{\partial y}$.

$$\therefore \frac{\partial F}{\partial x} = e^{2y} - y\cos xy$$

$$F(x,y) = \int e^{2y}dx - y\int \cos xy dx = xe^{2y} - \sin xy + g(y)$$

$$\frac{\partial F}{\partial y} = 2xe^{2y} - x\cos xy + g'(y) = N = 2xe^{2y} - x\cos xy + 2y$$

so that $g'(y) = 2y$ and $g(y) = y^2 + c$.

Hence, a one parameter family of solutions is given by

$$xe^{2y} - \sin xy + y^2 + C = 0.$$

THE END
(nby, July 2016)