Tutorial 3 (Limits & Continuity)

(You should practise writing proper steps.)

A) Limits

1. Evaluate the following limits:

(a) $\lim_{x\to 2}(-x^2+x-2)$ -4 (b) $\lim_{x\to 4} 2(x-3)(x-5)$ -2 (c) $\lim_{x\to 2}\dfrac{x^2-2}{x-1}$ 2 (d) $\lim_{x\to -1}\dfrac{2x^2-x-3}{x+1}$ -5 (2x-3)(x+1)

7/2 x-3 x+4 / 2(x-3) (e) $\lim_{x\to 3}\dfrac{x^2+x-12}{2x-6}$ (f) $\lim_{x\to -1}\dfrac{x^3+1}{x+1}$ 1 0 [Do you know how to factorise x^3+1?] (x+1)(x2-x+1) x2-2x+1

(g) $\lim_{x\to 4}\dfrac{\sqrt{x}-1}{x-1}$ 1/3 (h) $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}$ DNE 1/2 (1/(/x + 1)

(i) $\lim_{x\to 0}\dfrac{\sqrt{3+x}-\sqrt{3}}{x}$ [The expression $\sqrt{3+x}+\sqrt{3}$ may be useful.] 1/ (/3 + /3) 1/ 2(/3)

-1/4 (j) $\lim_{x\to 0}\dfrac{\frac{1}{2+x}-\frac{1}{2}}{x}$ [Simplify the expression first.] (k) $\lim_{x\to 3}\left[\dfrac{x^2-9}{x-3}-\dfrac{x^2-1}{x+1}\right]$ 4

1/(cos x) * sin x/ x = 1 (l) $\lim_{x\to 0}\dfrac{\tan x}{x}$ [Hint: $\tan x=\dfrac{\sin x}{\cos x}$] (m) $\lim_{x\to 2}\dfrac{|x-1|}{x-1}$ 1 (n) $\lim_{x\to 2.5}\lfloor x\rfloor$ 2

2. (a) Given that $\lim_{x\to 0}\dfrac{x}{\sin x}=L$, what is the value of L? 1

(b) Evaluate the following limits:

(i) $\lim_{u\to 0}\dfrac{\sin u}{u}$ 1 (ii) $\lim_{x\to 0}\dfrac{\sin 3x}{x}$ 3 (iii) $\lim_{x\to 0}\dfrac{\sin x^3}{x^3}$ 1

2x(x+1) / sin3x * sin3x/3x =2/3(x+1) = 2/3 (iv) $\lim_{x\to 0}\dfrac{2x(x+1)}{\sin 3x}$ [The expression $\dfrac{3x}{\sin 3x}$ may be useful.] (v) $\lim_{x\to 0} x^3\sin\dfrac{1}{x^3}$ -x3 < x3 sin1/x3 < x3 0

3. (a) Determine $\lim_{x\to 3^+}\lceil x\rceil$ 3 4 and $\lim_{x\to 3^-}\lceil x\rceil$ 3. Does $\lim_{x\to 3}\lceil x\rceil$ 3 exist? Why?

(b) Determine $\lim_{x\to 2^+}\dfrac{|x-2|}{x-2}$ 3, 1 and $\lim_{x\to 2^-}\dfrac{|x-2|}{x-2}$ 1, -1. Does $\lim_{x\to 2}\dfrac{|x-2|}{x-2}$ exist? Why? DNE, not same Undefined, both undefined

4. (a) Given that $1-\dfrac{x^2}{4}\le f(x)\le 1+\dfrac{x^2}{2}$ for all $x\ne 0$, find $\lim_{x\to 0} f(x)$. sandwich

[Which theorem do you use?]

(b) Prove that $\lim_{x\to 0} x\cos\dfrac{1}{x}=0$. [Try to sandwich $x\cos\dfrac{1}{x}$ between two appropriate expressions.]

(c) Given that $3x-5\le f(x)\le x^2-3x+4$ for $x\ge 0$, find $\lim_{x\to 3} f(x)$ 4 4 4

(d) Find $\lim_{x\to 0} x^4\sin\dfrac{1}{x^3}$. 0

5. Determine whether the limit exists by considering the corresponding one-sided limits. Give the value of the limit if it exists.

(a) $\lim_{x->2} f(x)$, $f(x)=\begin{cases} \dfrac{x+2}{2}, & x<2 \\ \dfrac{12-2x}{3}, & x\geq 2 \end{cases}$ 1, 3/2 3, 2

DNE

(c) $\lim_{x\to 2} f(x), f(x)=\begin{cases} 3-x & ,x<2 \\ 2 & ,x=2 \\ \dfrac{x}{2} & ,x>2 \end{cases}$ 1, 2 2, 1 2,1 4, 2

2

(b) $\lim_{x->2} f(x), f(x)=\begin{cases} 3x-2 & x<2 \\ \dfrac{6}{x}+1 & x\geq 2 \end{cases}$ 1, 1 2, 4 3, 3 2, 4

DNE

4, cont

For each of the above, is f continuous at 2? c

6. Determine the value of each of the following limits if it exists. If it does not exist, explain why.

(x-2)(x+2) : 4
-(x-2)(x+2) : -4
DNE, not same

(a) $\lim_{x\to 2}\dfrac{|x^2-4|}{x-2}$ *Hint*: Use $|x^2-4|=\begin{cases} -(\ ?\ ), & \text{if } x<2 \\ x^2-4, & \text{if } x\geq 2 \end{cases}$ or try another way.

(b) $\lim_{x\to 2}\left\lfloor \dfrac{x+1}{2} \right\rfloor$ 1 [What are the values of $\left\lfloor \dfrac{x+1}{2} \right\rfloor$ for values of x near 2, say for $1<x<3$?]

B) Continuity

1. Determine whether the following functions are continuous at $x=3$.

(a) $f(x)=\begin{cases} 2x^2-4 & (x>3) \\ x+11 & (x\leq 3) \end{cases}$ Yes

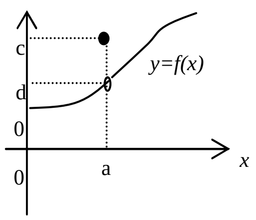
(b) $f(x)=\dfrac{2x}{3x^2-9x}$ No

(c) $f(x)=\begin{cases} 3x-2 & x<2 \\ \dfrac{6}{x}+1 & x\geq 2 \end{cases}$ No

2. ? For each of the functions graphed below, explain why the function is not continuous at $x=a$.

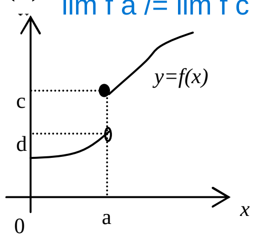
(a) f a /= limit f(x) -> a

f a DNE



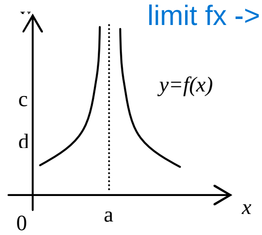
(b) limit fx -> a DNE

lim f a /= lim f c



(c) -> a inf

limit fx -> a DNE



3. (a) If you look at the graph of $y=\tan x$, where are the points of discontinuity?

x=n pi /2 (2n+1)pi/2, n (Z

(2n+1) pi/6 + 2/3

Find the discontinuities of $f(x) = \tan(3x-2)$.

(b) Determine the discontinuities, if any, of the following function.

$$f(x) = \begin{cases} 2x+1, & x \le 0 \\ 1, & 0 < x \le 1 \\ x^2+1, & x > 1 \end{cases}$$

0- = 1
0+ = 1
1- = 1
1+ = 2
discontinuous at x>1, limit x-> 1 DNE

4. Evaluate each of the following limits by observing that each expression involved is a composition of continuous functions.

(a) $\lim_{x \to \pi} \cos(x + \sin x)$ -1 (b) $\lim_{x \to 2} e^{x^2-1}$ e^3

5. At what points are the functions continuous?

(a) $f(x) = \dfrac{x+1}{x^2-3x+2}$ (b) $g(x) = \begin{cases} 3-x & if\ x<3 \\ 2x+1 & if\ x \ge 3 \end{cases}$

x /= 2 , 1, ~~x (R~~

6. (a) Explain why $g(x) = \begin{cases} 3-x & if\ x<3 \\ 2x+1 & if\ x \ge 3 \end{cases}$ is a continuous function on the interval $(0,3)$ but not a continuous function on the interval $[0,3]$

(b) Explain why $g(x) = \dfrac{x}{x-2}$ is not a continuous function on the interval $[0,3]$

7. (a) State the intermediate value theorem (i.e. the full statement including the hypothesis and the conclusion).

(b) Show that there is a root of the equation $x - \cos x = 0$ in the interval $[0, \frac{\pi}{2}]$.

-1 < 0 < pi/2 f(0) < 0 < f(pi/2)

[Let $f(x) = x - \cos x$. Then apply Intermediate Value Theorem.]

8. (a) If $f(x) = x^3 - 8x + 10$, show that there is a value of c in the interval $(0,1)$ for which $f(c) = \pi$. f(1) < f(c) < f(0)

(b) Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 3 = 0$ in the interval $[1,2]$.

-2 < 0 < 9

C) Limits involving infinity

1. Show that $\lim\limits_{x\to 0^+}\left(\dfrac{1}{|x|}-\dfrac{1}{x}\right)$ exists but $\lim\limits_{x\to 0^-}\left(\dfrac{1}{|x|}-\dfrac{1}{x}\right)$ does not.

?

inf - inf=0   - inf - inf   inf -(-inf) = inf

What can you conclude about $\lim\limits_{x\to 0}\left(\dfrac{1}{|x|}-\dfrac{1}{x}\right)$? [Reminder: $|x|=\begin{cases} x & if\ x\geq 0 \\ -x & if\ x<0\end{cases}$]

2. (a) Determine the vertical and/or horizontal asymptote(s) for the graph of each function defined as follows.

[Consider $\lim\limits_{x\to\infty} f(x)$, $\lim\limits_{x\to-\infty} f(x)$, $\lim\limits_{x\to a^-} f(x)$ and/or $\lim\limits_{x\to a^+} f(x)$ for appropriate a.]

(i) $f(x)=\dfrac{2x+1}{2x^2-5x+2}$

(ii) $f(x)=\dfrac{3x^2-2x+4}{2x^2-5x+2}$ [Factoring $2x^2-5x+2$ may help.]

(iii) $f(x)=\dfrac{x+3}{\sqrt{x^2+2x-8}}$

(b) The graph of $f(x)=\dfrac{4x+8}{x^2-4}$ has a horizontal asymptote. Give the equation of this asymptote.

Show that $x=-2$ is NOT a vertical asymptote for the graph of $f(x)=\dfrac{4x+8}{x^2-4}$.

3. For each of the following limits, determine if it exists. If it does not exist, could you write as $\lim\limits_{x\to a} f(x)=\infty$ or $\lim\limits_{x\to a} f(x)=-\infty$. Show steps to justify your answers.

(a) $\lim\limits_{x\to+\infty}\dfrac{2x^3-4x}{5x^3+2}$ (b) $\lim\limits_{x\to+\infty}\dfrac{5x^5-3}{3x^3-2}$ (c) $\lim\limits_{x\to\infty}\left(\dfrac{8x^2+7}{2x^2}-\dfrac{9x^3+27}{3x^3-3}\right)$

(d) $\lim\limits_{x\to\infty}\left(\sqrt{x(x+2)}-x\right)$ (e) $\lim\limits_{x\to 2}\dfrac{2x+3}{3x^2-4x-4}$ (f) $\lim\limits_{x\to\infty}\dfrac{2x+3}{3x^2-4x-4}$

(g) $\lim\limits_{x\to 0}\dfrac{\cos x}{x^4}$ (h) $\lim\limits_{x\to 0}\dfrac{\sin x}{x^4}$

(i) $\lim\limits_{x\to 0}\dfrac{x^2+1}{x^4}$ (j) $\lim\limits_{x\to\infty}\dfrac{x^2+1}{x^4}$

(nby, Jun 2017)