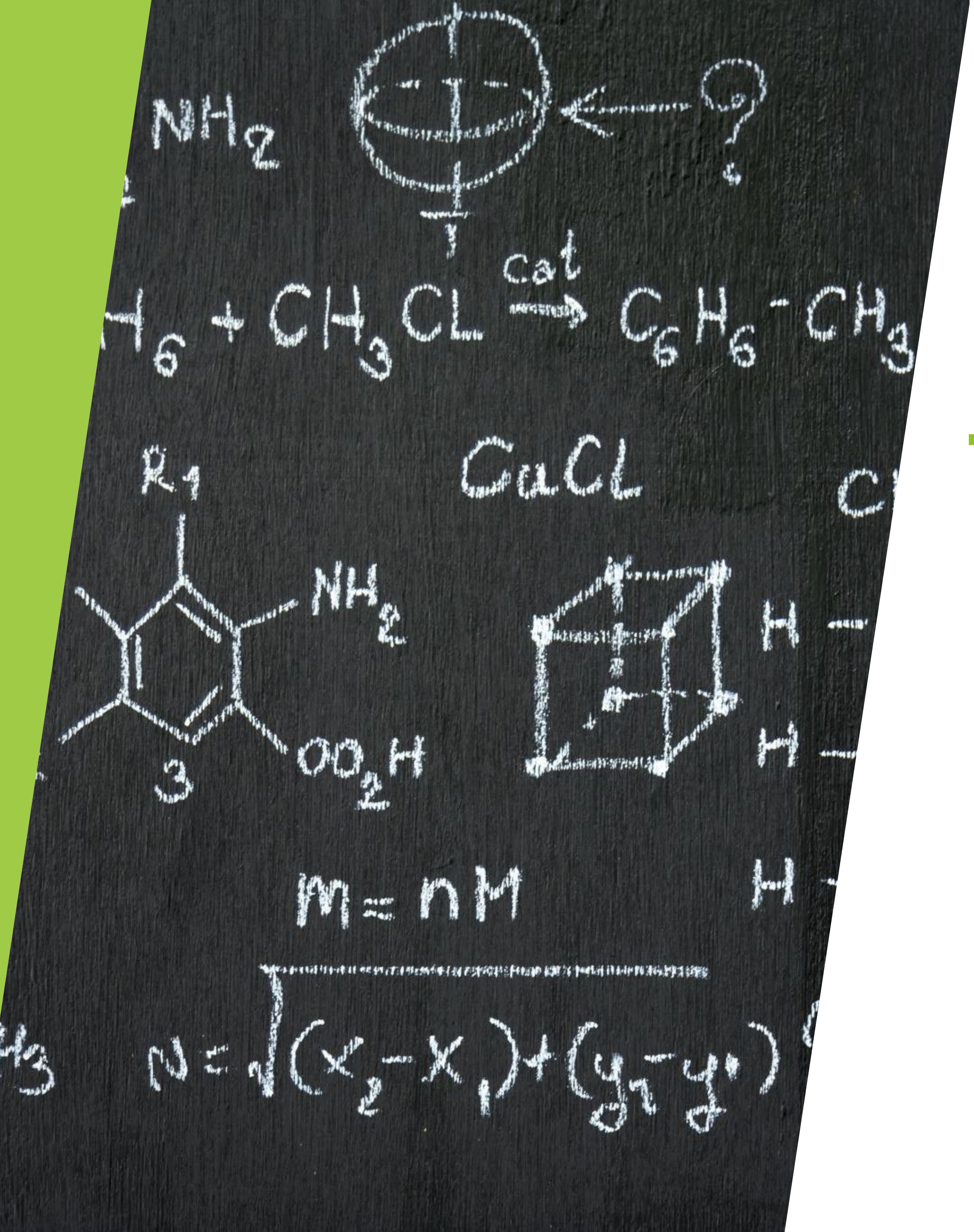


Chapter 1 Trigonometry (Part 2)

LECTURE 05 – 16.11.2022

2.3 Law or Sine and Law of Cosine

Objectives:

- Law of Sines and the Law of Cosines.
- Area of a triangle.

Chapter Outline

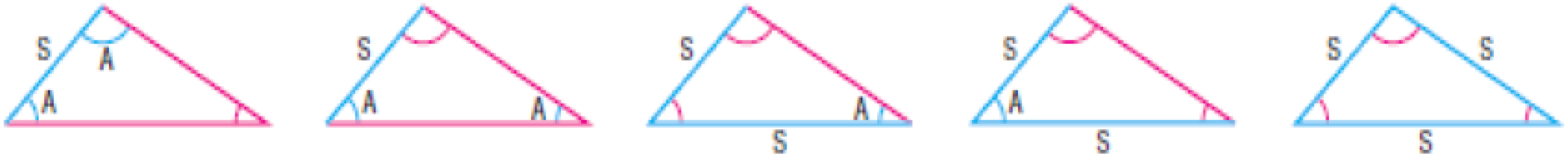
2.3.1 The Law of Sines

Definition:

A triangle is called ***oblique***, if none of the angles of the triangle is a right angle.

To solve an oblique triangle means to find the lengths of its sides and the measurements of it angles. Four possibilities to consider:

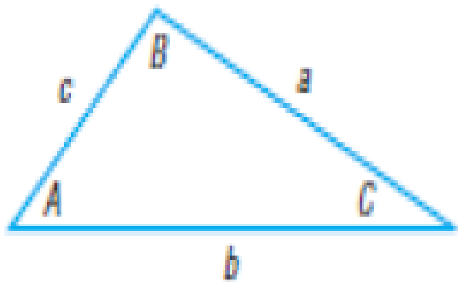
Case 1: One side and two angles are known (ASA or SAA).
Case 2: Two sides and the angle opposite one of them are known (SSA).
Case 3: Two sides and the included angle are known (SAS).
Case 4: Three sides are known (SSS).



Case 1: ASA | Case 1: SAA | Case 2: SSA | Case 3: SAS | Case 4: SSS

The first two cases require the Law of Sines and the last two requires the Law of Cosines.

Law of Sines:



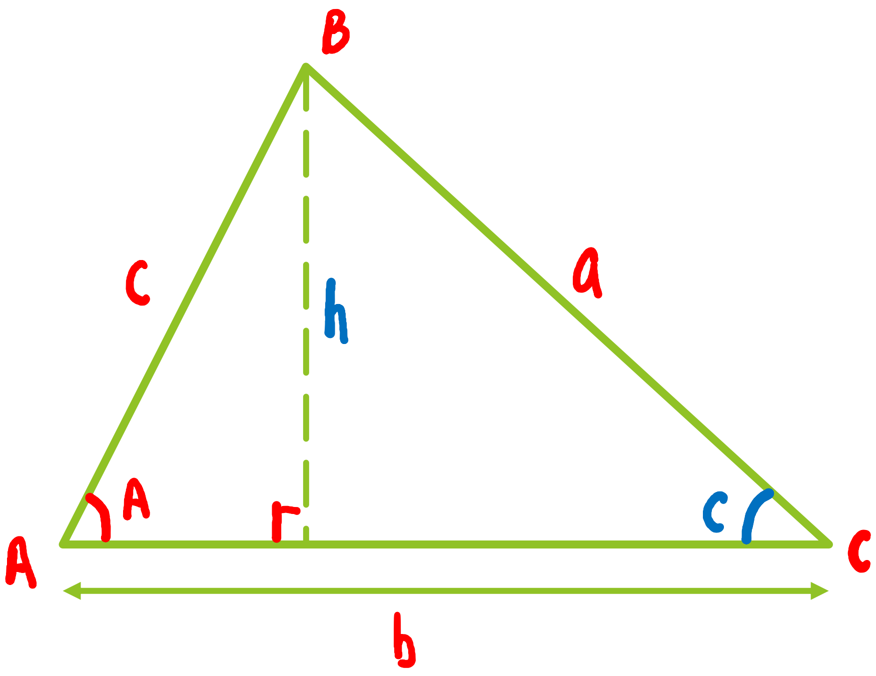
For a triangle with sides a, b, c and opposite angle A, B, C, respectively,

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

- Alternatively:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

How is SINE law derived?



$$\sin A = \frac{h}{c} \qquad \sin C = \frac{h}{a}$$

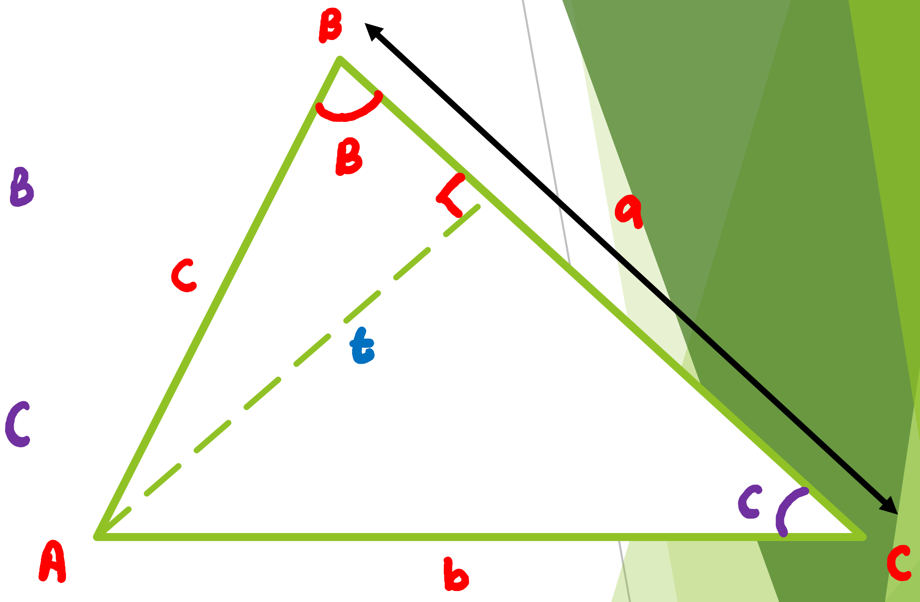
$$h = c \sin A \longleftrightarrow h = a \sin C$$

$$\sin B = \frac{t}{c}$$

$$t = c \sin B$$

$$\sin C = \frac{t}{b}$$

$$t = b \sin C$$



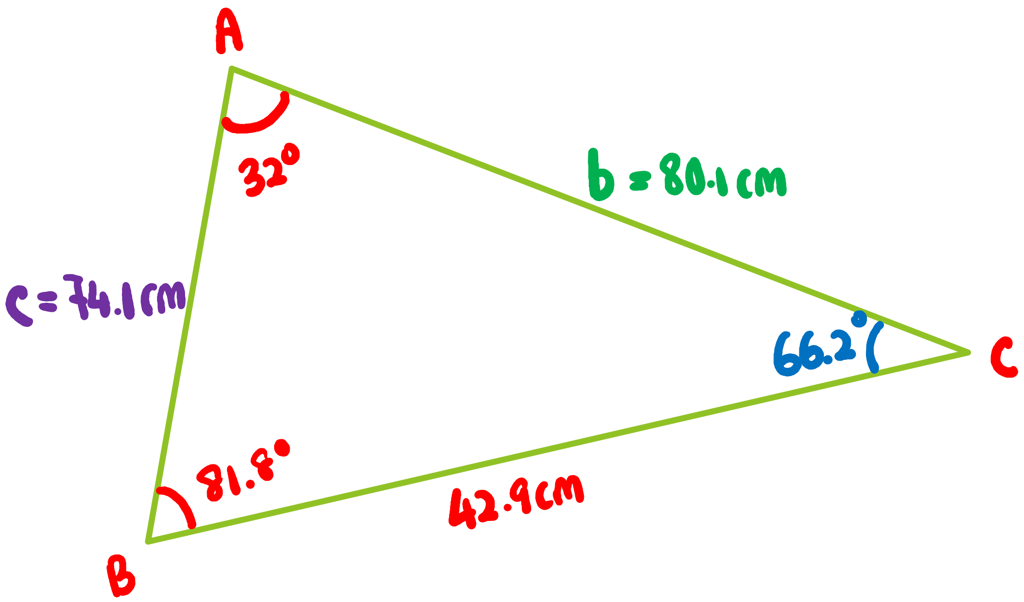
$$b \sin C = c \sin B$$

$$\Rightarrow \frac{\sin C}{c} = \frac{\sin B}{b}$$

$$c \sin A = a \sin C \Rightarrow \frac{\sin A}{a} = \frac{\sin C}{c} = \frac{\sin B}{b}$$

Example: Solve the triangle with $A = 32°, B = 81.8°$ and $a = 42.9$ cm.

$\angle C = 180° - 81.8° - 32° = 66.2°$ #



$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{b}{\sin 81.8°} = \frac{42.9\text{cm}}{\sin 32°}$$

$$\therefore b = \frac{42.9\text{cm}}{\sin 32°} \times \sin 81.8°$$

$$= 80.1\text{cm} \ \#$$

$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{c}{\sin 66.2°} = \frac{42.9\text{cm}}{\sin 32°}$$

$$\therefore c = \frac{42.9\text{cm}}{\sin 32°} \times \sin 66.2° = 74.1\text{cm} \ \#$$

Example

A satellite orbiting the earth passes directly overhead at observation stations in Phoenix and Los Angeles, 340 mi apart. At an instant when the satellite is between these two stations, its angle of elevation is simultaneously observed to be 60° at Phoenix and 75° at Los Angeles. How far is the satellite from Los Angeles?

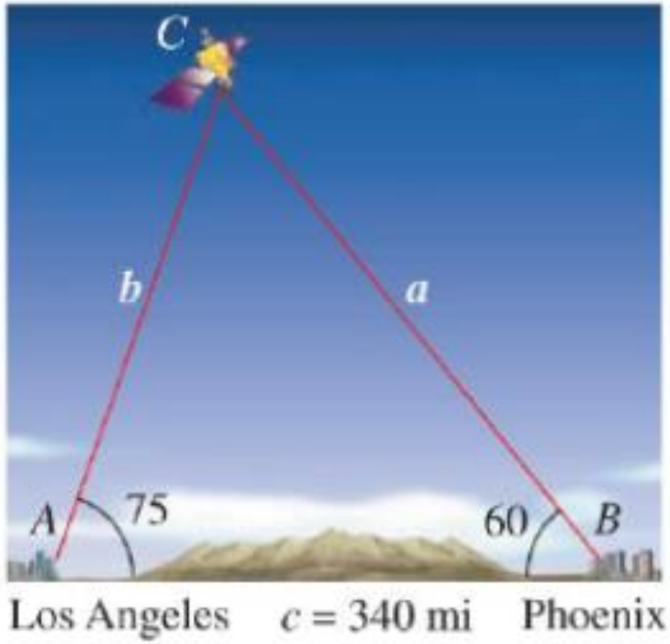
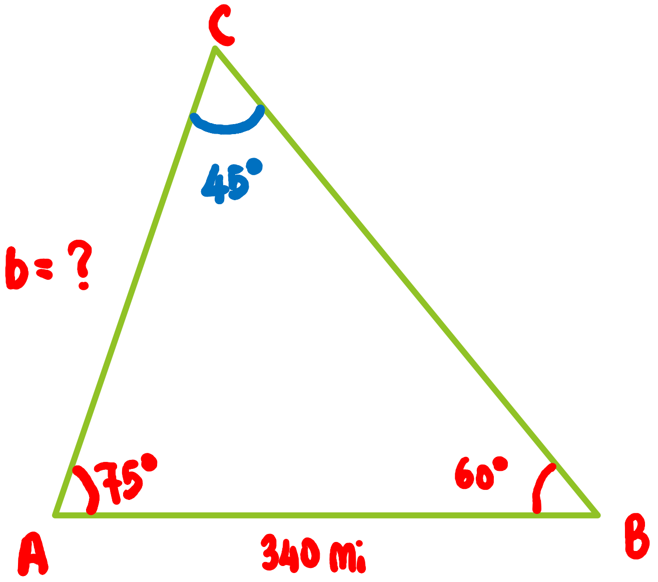


FIGURE 4



$$\measuredangle C = 180° - 75° - 60°$$

$$= 45°$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

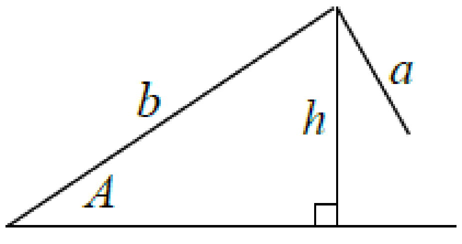
$$\frac{b}{\sin 60°} = \frac{340 \text{ mi}}{\sin 45°}$$

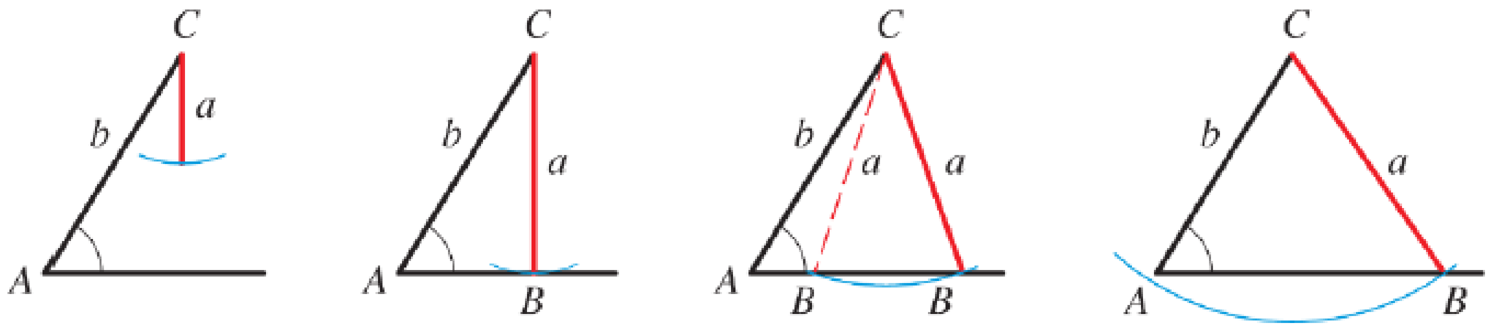
$$\therefore b = \frac{340 \text{ mi}}{\sin 45°} \times \sin 60°$$

$$= 416 \text{ miles}$$

The Ambiguous Case

When solving triangle where two sides and the angle opposite one of them are known, the result will lead to one triangle, two triangles or no triangle at all. (*a, b* are sides of the triangle and *h* is the height).

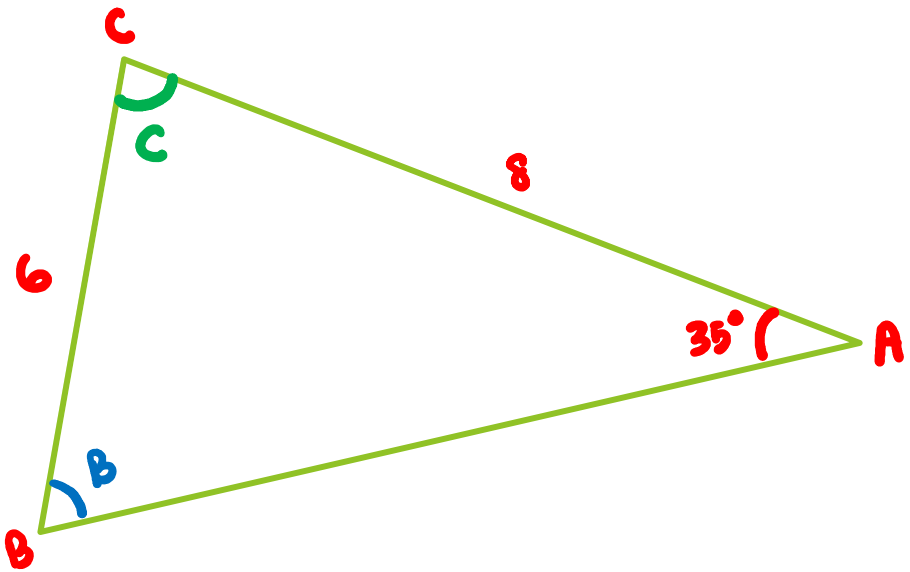
 $\sin A = \frac{h}{b}$	If $a < h = b\sin A$, then clearly side a is not sufficiently long to form a triangle	No Triangle
	If $a = h = b\sin A$, then side a is just long enough to form a right triangle	One Right Triangle
	If $a < b$ and $h = b \sin A < a$, then two distinct triangles can be formed.	Two Triangles
	If $a \geq b$, then only one triangle can be formed	One Triangle.



(a) (b) (c) (d)

Example:

Solve the triangle: $a = 6$, $b = 8$, $A = 35^\circ$



$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin B}{8} = \frac{\sin 35^\circ}{6}$$

$$\sin B = \frac{\sin 35^\circ}{6} \times 8 = 0.7648$$

$$\therefore B = \sin^{-1}(0.7648) = 49.9^\circ \;\#$$

$$= 130.1^\circ \;\#$$

- Since sin function is positive in Quadrant 1 and 2, i.e.

$$0^\circ \leq \theta \leq 180^\circ$$

For $B = 49.9^\circ$, $\angle C = 180^\circ - 35^\circ - 49.9^\circ$

$= 95.1^\circ \;\#$

$\rightarrow \frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{c}{\sin 95.1^\circ} = \frac{6}{\sin 35^\circ}$

$\therefore c = \frac{6}{\sin 35^\circ} \times \sin 95.1^\circ = 10.4 \;\#$

For $B = 130.1^\circ$, $\angle C = 180^\circ - 35^\circ - 130.1^\circ$

$= 14.9^\circ \;\#$

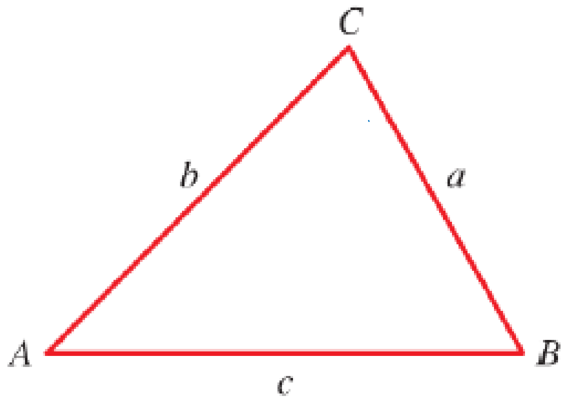
$\rightarrow \frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{c}{\sin 14.9^\circ} = \frac{6}{\sin 35^\circ}$

$\therefore c = \frac{6}{\sin 35^\circ} \times \sin 14.9^\circ = 2.7 \;\#$

2.3.2 The Law of Cosines

For a triangle with sides a, b, c and opposite angles A, B, C respectively,

$$\boxed{\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ b^2 &= a^2 + c^2 - 2ab\cos B \\ a^2 &= b^2 + c^2 - 2ab\cos A \end{aligned}}$$



- The Cosine Law is useful especially when we have SAS case.
- For example (refer to the triangle ABC at the side):

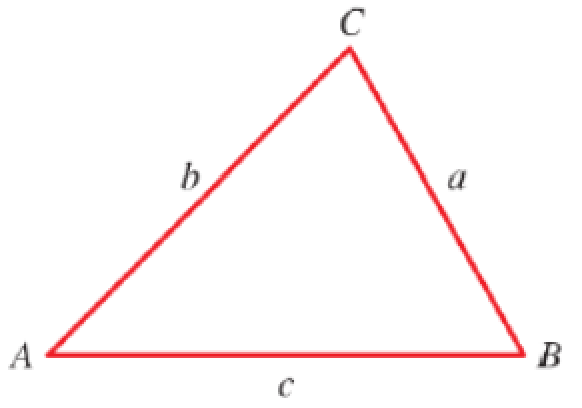
1) Length a and c, angle B are known.
2) Length b and c, angle A are known.

- If we want to use the COSINE law to find the length of the triangle sides, we can simply rearrange the formula.
- This is especially when you have SSS case, that is, all the length of the triangle sides are given.
- For example, given length a, b, and c, we can choose to compute any angle of A, B, or C using Cosine Law:

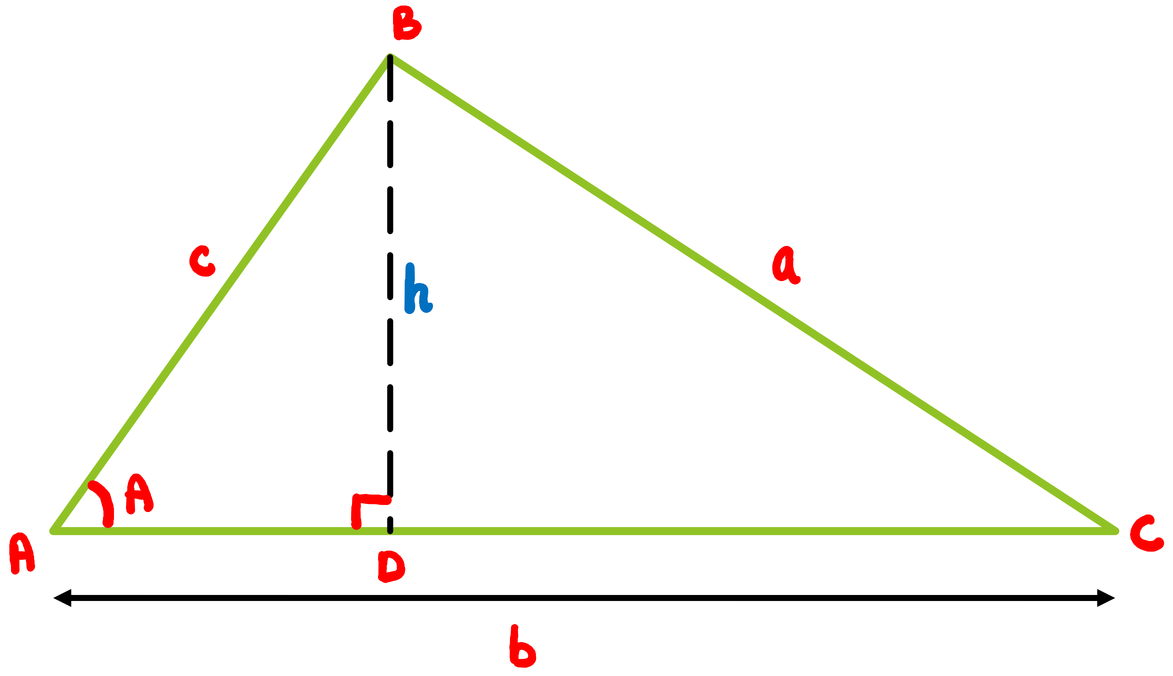
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \rightarrow \quad A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$\cos B = \frac{a + c^2 - b^2}{2ac} \quad \rightarrow \quad B = \cos^{-1}\left(\frac{a + c^2 - b^2}{2ac}\right)$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad \rightarrow \quad C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$



How is COSINE law derived?



Pythagoras Theorem:
$a^2 = b^2 + c^2$

$$a^2 = h^2 + DC^2$$
$$= (c \sin A)^2 + (b - c\cos A)^2$$
$$= c^2 \sin^2 A + (b^2 - 2bc\cos A + c^2\cos^2 A)$$
$$= c^2 \sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A$$
$$= c^2(\sin^2 A + \cos^2 A) + b^2 - 2bc\cos A$$
$$= c^2 + b^2 - 2bc\cos A$$

Trigonometric Identity:
$\sin^2 A + \cos^2 A = 1$

$\sin A = \dfrac{h}{c}$

$\therefore h = c \sin A$

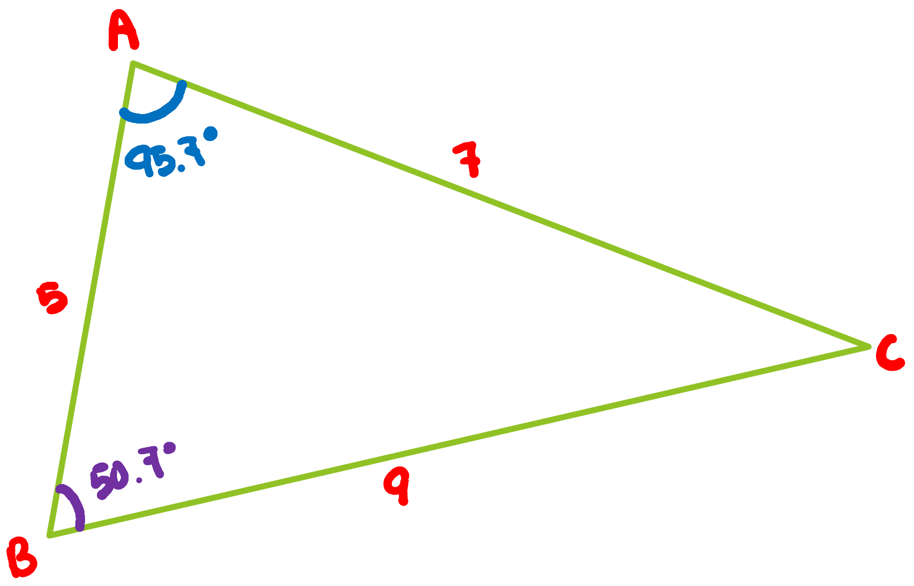
$\cos A = \dfrac{AD}{c}$

$\therefore AD = c\cos A$

length $DC = b - AD$

$= b - c\cos A$

Example: Solve the triangle if $a = 9$, $b = 7$ and $c = 5$.



$a^2 = b^2 + c^2 - 2bc\cos A$

$9^2 = 5^2 + 7^2 - 2(5)(7)\cos A$

$\therefore \cos A = \dfrac{5^2+7^2-9^2}{2(5)(7)} = -0.1$

$\therefore A = \cos^{-1}(-0.1) = 95.7°$ #

$\dfrac{\sin B}{b} = \dfrac{\sin A}{a} \Rightarrow \dfrac{\sin B}{7} = \dfrac{\sin 95.7°}{9}$

$\therefore \sin B = \dfrac{\sin 95.7°}{9} \times 7 = 0.7739$

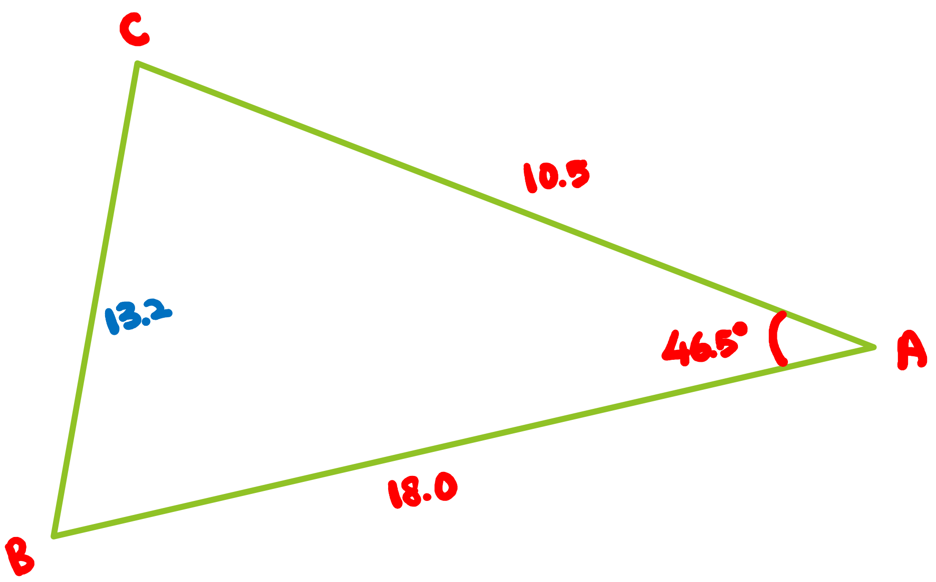
$\therefore B = \sin^{-1}(0.7739) = 50.7°$ #

$\therefore \angle C = 180° - 95.7° - 50.7°$

$= 33.6°$ #

Example SAS, the law of Cosines

Solve triangle ABC where $\angle A = 46.5^\circ$ $b = 10.5$ and $c = 18.0$



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$= 10.5^2 + 18.0^2 - 2(10.5)(18.0)\cos 46.5^\circ$$

$$= 174.1$$

$$\therefore a = \sqrt{174.1} = 13.2$$

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{10.5} = \frac{\sin 46.5^\circ}{13.2}$$

$$\sin B = \frac{\sin 46.5^\circ}{13.2} \times 10.5 = 0.5770$$

$$\therefore B = \sin^{-1}(0.5770) = 35.2^\circ$$

$$= 144.8^\circ$$

For $\angle B = 35.2^\circ$

$$\therefore \angle C = 180^\circ - 46.5^\circ - 35.2^\circ$$

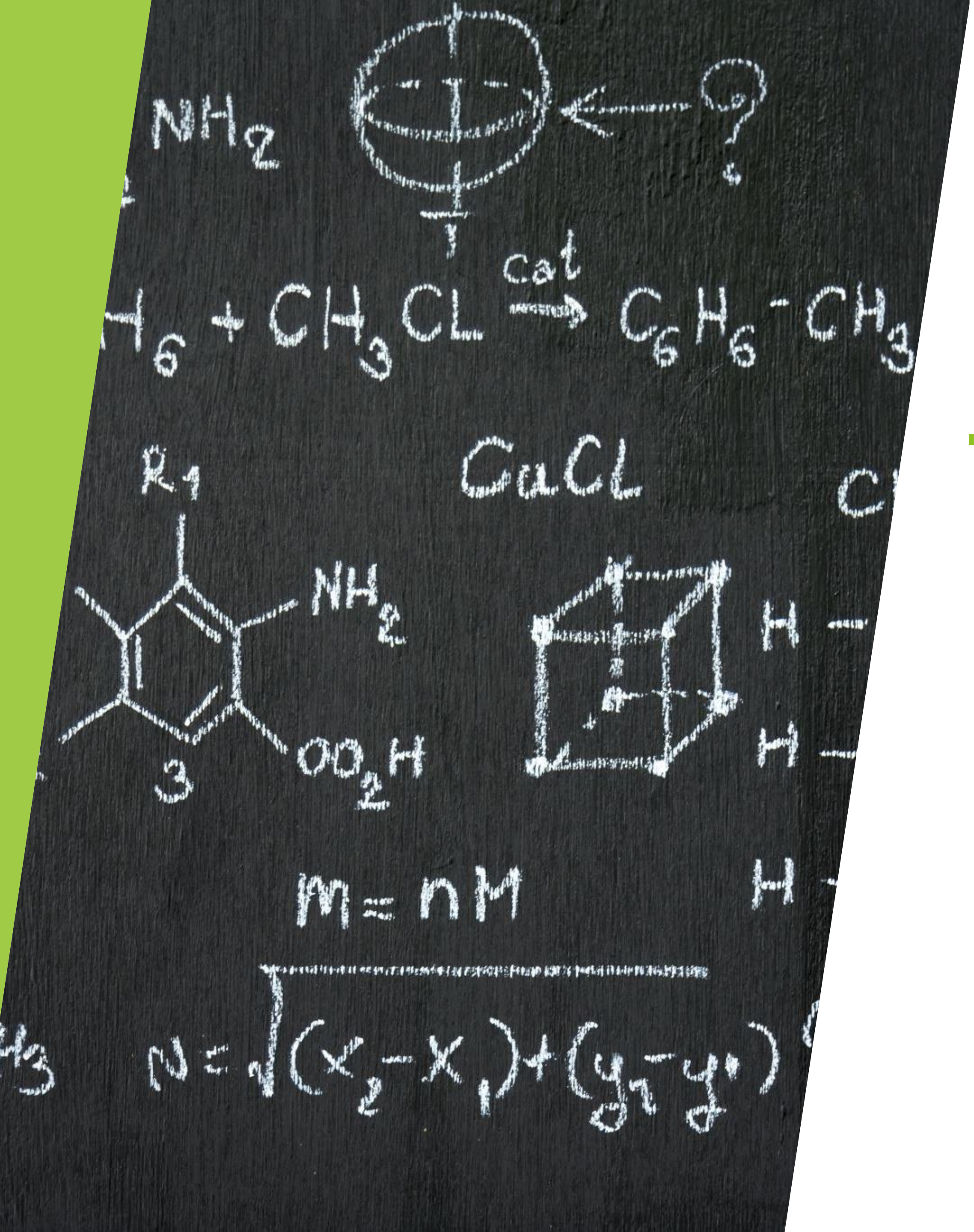
$$= 98.3^\circ$$

For $\angle B = 144.8^\circ$

$$\therefore \angle C = 180^\circ - 46.5^\circ - 144.8^\circ$$

$$= -11.3^\circ$$

(NOT VALID!!!)



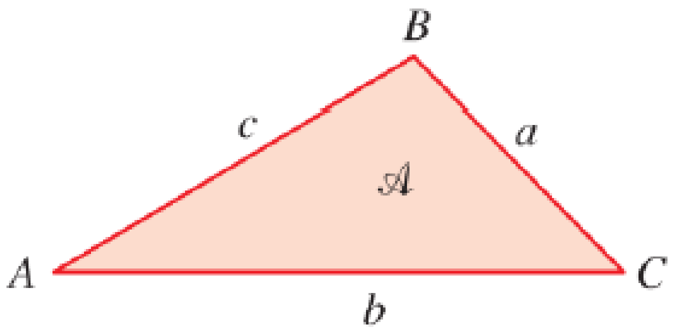
Chapter 1 Trigonometry (Part 2)

LECTURE 06 – 23.11.2022

2.3.3 Area of A Triangle

All students are familiar with the formula area $\mathfrak{A} = \frac{1}{2}bh$ (b is the base and h is the height) to get the area of a triangle. Other forms of formulas used to get the area of triangles are:

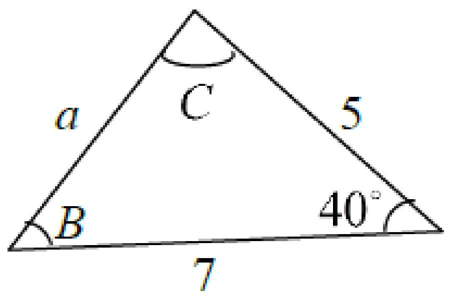
$$\boxed{\begin{aligned} \mathfrak{A} &= \frac{1}{2}ab\sin C \\ \mathfrak{A} &= \frac{1}{2}bc\sin A \\ \mathfrak{A} &= \frac{1}{2}ac\sin B \end{aligned}}$$



or

The area of a triangle equals one-half the product of two sides times the sine of their included angle.

Example:



Find the area $\mathfrak{A}$ of the triangle for which $b = 5$ and $c = 7$, and $A = 40^\circ$.

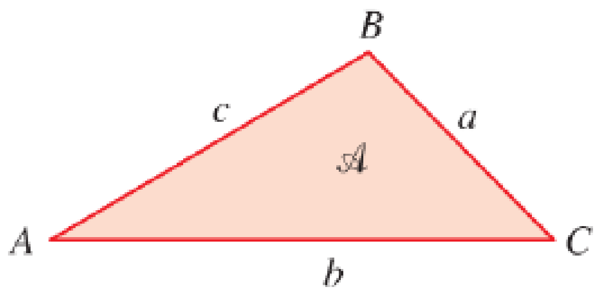
Area $= \frac{1}{2}bc \sin A$

$= \frac{1}{2}(5)(7) \sin 40^\circ$

$= 11.25$ unit^2

Heron's Formula

Given a triangle with the length of the sides are a, b and c respectively



HERON'S FORMULA

The area $\mathscr{A}$ of triangle ABC is given by

$$\mathscr{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$ is the **semiperimeter** of the triangle; that is, s is half the perimeter.

Example: Find the area of a triangle whose sides are 8, 4, and 5.

semiperimeter, $s = \frac{1}{2}(8+4+5) = 8.5$

$\therefore \text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{8.5(8.5-8)(8.5-4)(8.5-5)}$

$= \sqrt{8.5(0.5)(4.5)(3.5)}$

$= \sqrt{66.9375}$

$= 8.18 \text{ unit}^2$ #

Types of Triangles

1. Equilateral Triangle – all angles are equal of $60°$.
2. Isosceles Triangle – two sides have the same length, hence equal angle.
3. Right-angle Triangle – one angle of $90°$, satisfies Pythagorean Theorem.
4. Oblique Triangle – all sides and angles are different.

☺ ~ THE END ~ ☺