4.0 RELATIONS AND FUNCTIONS

Definition of a Relation
A **relation** is any set of ordered pairs. The set of all first components of the ordered pairs is called the **domain** of the relation and the set of all second components is called the **range** of the relation.

E X A M P L E 1 | Finding the Domain and Range of a Relation
Find the domain and range of the relation:
{(Ali, 15), (Bala, 20), (Chong, 20), (Daniel, 35), (Evelyn, 31)}.
S O LU T I O N
The domain is the set of all first components.
Thus, the domain is {Ali, Bala, Chong, Daniel, Evelyn}.
The range is the set of all second components.
Thus, the range is {15, 20, 35, 31}.

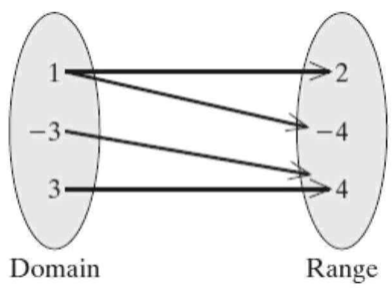
Note: Although Bala and Chong are 20 years old, it is not necessary to list 20 twice.

We would be more interested in looking at ordered pairs of numbers.

E X A M P L E 2 | Different Representations for a Relation
Look at this relation.
{(1, 2), (–3, 4), (1, –4), (3, 4)}

Three other ways of representing this relation:

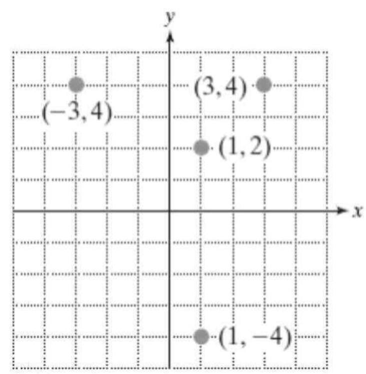
Arrow/Mapping Diagram



Table

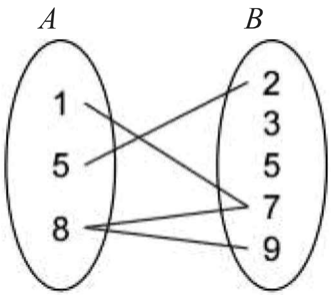
x	y
1	2
1	–4
–3	4
3	4

Graph in a Rectangular Coordinate System



A **relation** can also be seen as a correspondence between two sets where each element in the first set corresponds to *at least* one element in the second set.

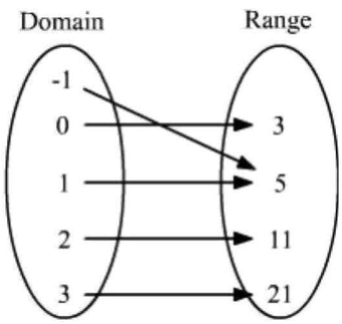
E X A M P L E 3



Here, the domain is the set $A = \{1, 5, 8\}$
while the range is the set $\{2, 7, 9\}$.
The set B is known as the **codomain**.

In general, the range is contained in the codomain; there could be some elements of the codomain that are not in the range.

E X A M P L E 4

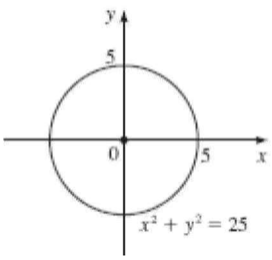


When the set of ordered pairs is infinite, arrow/mapping diagrams or tables may not be suitable.

E X A M P L E 5

A relation may be expressed by an equation such as $x^2 + y^2 = 25$, with x as the first component and y as the second.

The solutions to this equation define an infinite set of ordered pairs of the form
$\{(x, y) \mid x^2 + y^2 = 25\}$
The solutions can also be represented by a graph in a rectangular coordinate system.

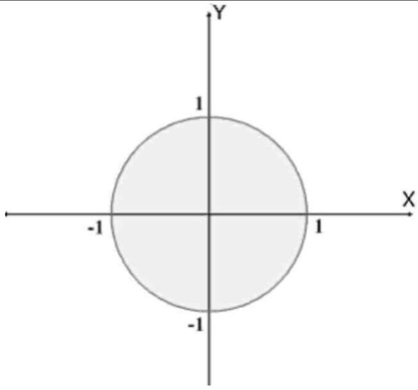


Graph of the relation $x^2 + y^2 = 25$

E X A M P L E 6

The graph of the relation $x^2 + y^2 = 1$ is a circle with centre at the origin and radius equal to one unit.

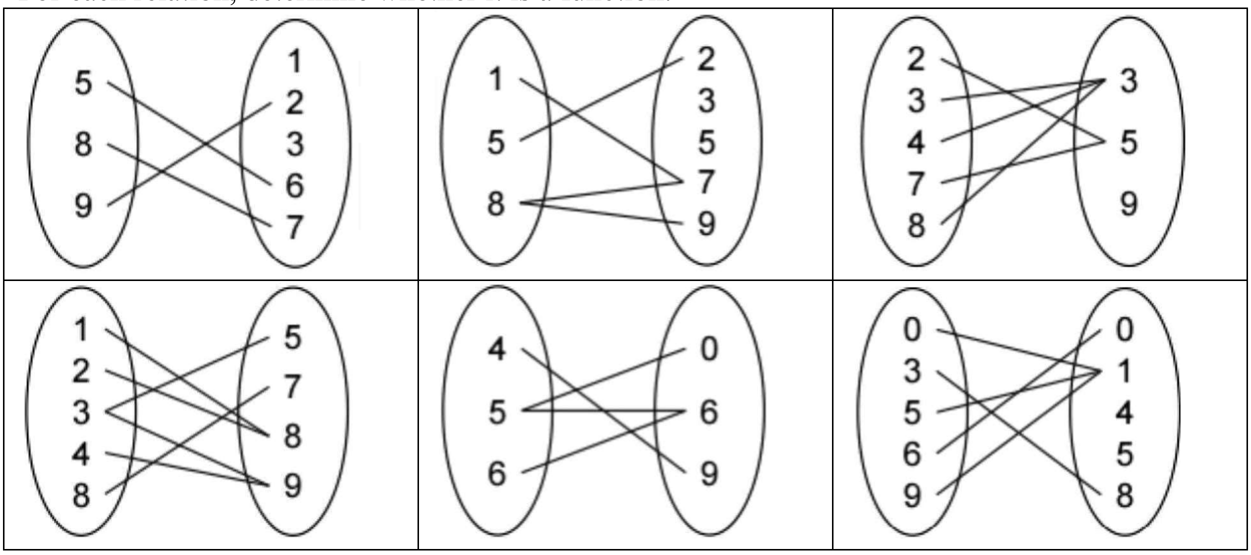
The graph of the relation $x^2 + y^2 \leq 1$ is the circular region including the boundary which is the circle with centre at the origin and radius equal to one unit.

	Domain of the relation $x^2 + y^2 = 1$? Range of the relation $x^2 + y^2 = 1$? Domain of the relation $x^2 + y^2 \leq 1$? Range of the relation $x^2 + y^2 \leq 1$? (*Use interval notation.*)

A relation in which each element in the domain corresponds to exactly one element in the codomain is a **function**.

E X A M P L E 7

The following diagrams represent relations.
For each relation, determine whether it is a function.

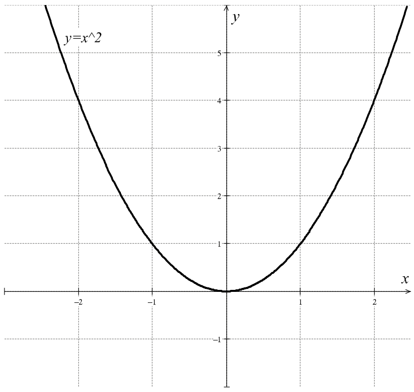


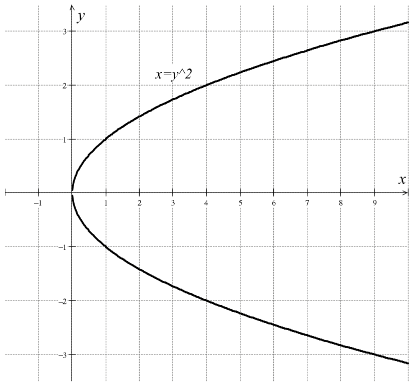
E X A M P L E 8

The graphs of two relations

R1 $= \{ (x, y) \mid y = x^2 \}$

R2 $= \{ (x, y) \mid x = y^2 \}$





Equation for Graph of R1: $y = x^2$

Equation for Graph of R2: $x = y^2$

Can you suggest two ordered pairs in R2 with the same first component, but different second components?

Is it possible to do so with R1?

Based on this, which of R1 and R2 is a function?

In the sub-sections to follow, we shall focus on functions.

4.1 WHAT IS A FUNCTION? (Adapted from "Precalculus" by Stewart et als.)

Functions All Around Us ┐ Definition of Function ┐ Evaluating a Function
┐ Domain of a Function ┐ Four Ways to Represent a Function

▼ Functions All Around Us

In nearly every physical phenomenon we observe that one quantity depends on another.

For example, your height depends on your age, the temperature depends on the date, the cost of mailing a package depends on its weight.

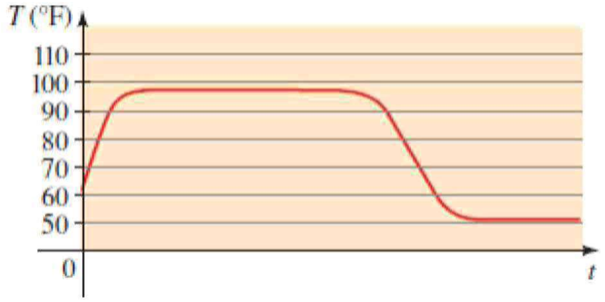
We use the term ***function*** to describe this dependence of one quantity on another.
That is, we say the following:

■ Height is a function of age.
■ Temperature is a function of date.
■ Cost of mailing a package is a function of weight.

■ The area of a circle is a function of its radius.
■ The number of bacteria in a culture is a function of time.
■ The weight of an astronaut is a function of her elevation.
■ The price of a commodity is a function of the demand for that commodity.

The rule that describes how the area A of a circle depends on its radius r is given by the formula $A = \pi r^2$.
Even when a precise rule or formula describing a function is not available, we can still describe the function by a graph.

■ The temperature of water from the faucet is a function of time.



Graph of water temperature T as a function of time t (the time that has elapsed since the faucet was turned on)

▼ Definition of Function

A function is a rule. To talk about a function, we need to give it a name. We will use letters such as *f*, *g*, *h*, . . . to represent functions.

For example, we can use the letter f to represent a rule as follows:

"f" is the rule "square the number"

When we write $f(2)$, we mean "apply the rule f to the number 2." Applying the rule gives $f(2)=2^2=4$. Similarly, $f(3)=3^2=9$, $f(4)=4^2=16$, and in general $f(x)=x^2$.

DEFINITION OF A FUNCTION

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

We usually consider functions for which the sets A and B are sets of real numbers. The symbol $f(x)$ is read "f of x" or "f at x" and is called the **value of *f* at *x***, or the **image of *x* under** f. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain, that is,

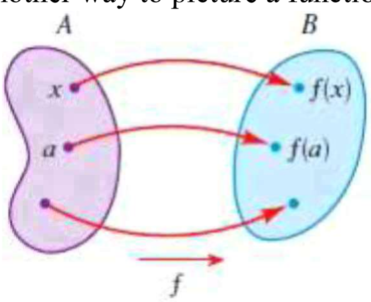
$$\text{range of } f = \{f(x) \mid x \in A\}$$

The symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**. The symbol that represents a number in the range of f is called a **dependent variable**. So if we write $y = f(x)$, then x is the independent variable and y is the dependent variable.

It is helpful to think of a function as a **machine**.

If x is in the domain of the function f, then when x enters the machine, it is accepted as an **input** and the machine produces an **output** $f(x)$ according to the rule of the function.
Thus, we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

Another way to picture a function is by an **arrow diagram** as shown here.



Each arrow connects an element of A to an element of B. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

EXAMPLE 1 | Analyzing a Function

A function f is defined by the formula $f(x) = x^2 + 4$

(a) Express in words how f acts on the input x to produce the output $f(x)$.

(b) Evaluate $f(3)$, $f(-2)$, and $f(\sqrt{5})$.

(c) Find the domain and range of f.

(d) Draw a machine diagram for f.

SOLUTION

(a) The formula tells us that f first squares the input x and then adds 4 to the result.
So f is the function: "square, then add 4"

(b) The values of f are found by substituting for x in the formula $f(x) = x^2 + 4$. ???

Replace x by 3

Replace x by -2

Replace x by $\sqrt{5}$

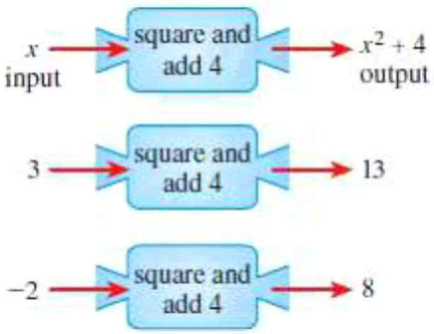
(c) The domain of f consists of all possible inputs for f. Since we can evaluate the formula $f(x) = x^2 + 4$ for every real number x, the domain of f is the set $\mathbb{R}$ of all real numbers.

The range of f consists of all possible outputs of f.

Because $x^2 \geq 0$ for all real numbers x, we have $x^2 + 4 \geq 4$, so for every output of f we have $f(x) \geq 4$.

Thus, the range of f is $\{y \mid y \geq 4\} = [4, \infty)$.

(d) A machine diagram for f:



NOW TRY: For each of the functions

(i) $f(x) = x^2 + 2$ (ii) $f(x) = \sqrt{x-1}$ (iii) $f(x) = x^2 - 6$ (iv) $f(x) = 2x$. Do the following.

(a) Express in words how f acts on the input x to produce the output $f(x)$.

(b) Evaluate $f(3)$, $f(10)$, and $f(0)$. [*If the value does not exist, say that it is "undefined"*.]

(c) Find the domain and range of f.

(d) Draw a machine diagram for f.

▼ Evaluating a Function

In the definition of a function the independent variable x plays the role of a placeholder.
For example, the function $f(x) = 3x^2 + x - 5$ can be thought of as

$$f(\square) = 3\cdot\square^2 + \square - 5$$

To evaluate f at a number, we substitute the number for the placeholder.

E X A M P L E 2 | **Evaluating a Function**

Let $f(x) = 3x^2 + x - 5$. Evaluate each function value.

(a) $f(-2)$ **(b)** $f(0)$ **(c)** $f(4)$ **(d)** $f(\frac{1}{2})$

S O LU T I O N To evaluate *f* at a number, we substitute the number for *x* in the definition of *f*. ....???.

NOW TRY: Evaluate the function at the indicated values.

$f(x) = 2x + 1$; $f(1), f(-2), f(\frac{1}{2}), f(a), f(-a), f(a+b)$

E X A M P L E 3 | **A Piecewise Defined Function**

A cell phone plan costs $39 a month. The plan includes 400 free minutes and charges 20¢ for each additional minute of usage. The monthly charges are a function of the number of minutes used, given by

$$C(x) = \begin{cases} 39 & \text{if } 0 \le x \le 400 \\ 39 + 0.20(x - 400) & \text{if } x > 400 \end{cases}$$

Find $C(100)$, $C(400)$, and $C(480)$.

S O LU T I O N ....???.....

Thus, the plan charges $39 for 100 minutes, $39 for 400 minutes, and $55 for 480 minutes.

NOW TRY: Evaluate the piecewise defined function at the indicated values.

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x + 1 & \text{if } x \ge 0 \end{cases}; \quad f(-2), f(-1), f(0), f(1), f(2)$$

E X A M P L E 4 | **Evaluating a Function**

If $f(x) = 2x^2 + 3x - 1$, evaluate the following.

(a) $f(a)$ **(b)** $f(-a)$ **(c)** $f(a+h)$ **(d)** $\dfrac{f(a+h) - f(a)}{h}$, $h \ne 0$

S O LU T I O N ....???.....

NOW TRY: For the function $f(x) = 3x + 2$, find $f(a), f(a+h)$, and the difference quotient $\dfrac{f(a+h) - f(a)}{h}$, where $h \ne 0$,

▼ The Domain of a Function

Recall that the *domain* of a function is the set of all inputs for the function. The domain of a function may be stated explicitly.

For example, if we write $f(x) = x^2 \quad 0 \le x \le 5$

then the domain is the set $\{x \in \Re \mid 0 \le x \le 5\}$, i.e., the interval $[0,5]$.

If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention *the domain of the function is the domain of the algebraic expression—that is, the set* *of all real numbers for which the expression is defined as a real number.*

For example, consider the functions $f(x)=\dfrac{1}{x-4}$, $g(x)=\sqrt{x}$

The function f is not defined at $x = 4$, so its domain is $\{x \mid x \neq 4\}$. Interval notation?

The function g is not defined for negative x, so its domain is $\{x \mid x \geq 0\}$. Interval notation?

E X A M P L E 6 | **Finding Domains of Functions**

Find the domain of each function.

(a) $f(x)=\dfrac{1}{x^2-x}$ **(b)** $g(x)=\sqrt{9-x^2}$ **(c)** $h(t)=\dfrac{t}{\sqrt{t+1}}$

S O LU T I O N

(a) [A rational expression is not defined when the denominator is 0.]

$f(x)$ is not defined when $x^2-x=0$, i.e., $x(x-1)=0$.

So, $f(x)$ is not defined when $x = 0$ or $x = 1$.

Thus, the domain of f is ??? $\{x \mid x \neq 0, x \neq 1\}$

The domain may also be written in interval notation as ??? [*Do you know the interval notation*?]

(b) We can't take the square root of a negative number, so we must have $9-x^2 \geq 0$.

To solve the inequality $9-x^2 \geq 0$, we obtain $-3 \leq x \leq 3$

Thus, the domain of g is ???

(c) We can't take the square root of a negative number, and we can't divide by 0, so we must have $t + 1 > 0$, that is, $t > -1$. So the domain of h is $\{t \mid$........??? $\}$ = (interval notation?)

NOW TRY: Find the domain of the function. [Make sure you know how to give your answer using interval notation, in addition to the set builder notation]

(a) $f(x)=\dfrac{1}{x-3}$ (b) $g(x)=\sqrt{x-5}$

▼ Four Ways to Represent a Function

To help us understand what a function is, we have used machine and arrow diagrams. We can describe a specific function in the following four ways:

- verbally (by a description in words)
- algebraically (by an explicit formula)
- visually (by a graph)
- numerically (by a table of values)

FOUR WAYS TO REPRESENT A FUNCTION

Verbal

Using words:

"To convert from Celsius to Fahrenheit, multiply the Celsius temperature by $\frac{9}{5}$, then add 32."

Relation between Celsius and Fahrenheit temperature scales

Algebraic

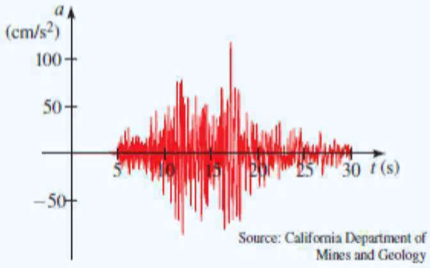
Using a formula:

$$A(r) = \pi r^2$$

Area of a circle

Visual

Using a graph:



Vertical acceleration during an earthquake

Numerical

Using a table of values:

w (ounces)	$C(w)$ (dollars)
$0 < w \le 1$	1.22
$1 < w \le 2$	1.39
$2 < w \le 3$	1.56
$3 < w \le 4$	1.73
$4 < w \le 5$	1.90
⋮	⋮

Cost of mailing a first-class parcel

E X A M P L E 7

Representing a Function Verbally, Algebraically, Numerically, and Graphically

Let $F(C)$ be the Fahrenheit temperature corresponding to the Celsius temperature C. (Thus, F is the function that converts Celsius inputs to Fahrenheit outputs.) The box above gives a verbal description of this function. Find ways to represent this function

(a) Algebraically (using a formula)

(b) Numerically (using a table of values)

(c) Visually (using a graph)

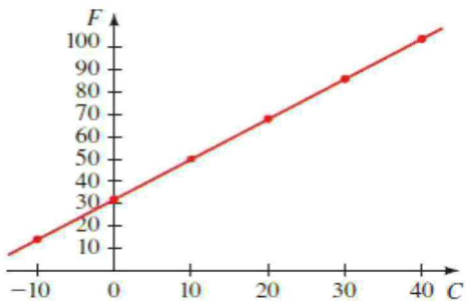
S O LU T I O N

(a) The verbal description tells us that we should first multiply the input C by and then add 32 to the result. So we get $F(C) = \frac{9}{5}C + 32$

(b) We use the algebraic formula for F that we found in part (a) to construct a table of values:

C (Celsius)	F (Fahrenheit)
−10	14
0	32
10	50
20	68
30	86
40	104

(c) We use the points tabulated in part (b) to help us draw the graph of this function.



Exercise

1–4 ■ Express the rule in function notation. (For example, the rule "square, then subtract 5" is expressed as the function $f(x) = x^2 - 5$.)

1. Add 3, then multiply by 2
2. Divide by 7, then subtract 4
3. Subtract 5, then square
4. Take the square root, add 8, then multiply by $\frac{1}{3}$

5–8 ■ Express the function (or rule) in words.

5. $f(x) = \dfrac{x-4}{3}$
6. $g(x) = \dfrac{x}{3} - 4$
7. $h(x) = x^2 + 2$
8. $k(x) = \sqrt{x+2}$

9–10 ■ Draw a machine diagram for the function.

9. $f(x) = \sqrt{x-1}$
10. $f(x) = \dfrac{3}{x-2}$

11–12 ■ Complete the table.

11. $f(x) = 2(x-1)^2$

x	$f(x)$
-1	
0	
1	
2	
3	

12. $g(x) = |2x + 3|$

x	$g(x)$
-3	
-2	
0	
1	
3	

13–20 ■ Evaluate the function at the indicated values.

13. $f(x) = 2x + 1$;
$f(1), f(-2), f(\frac{1}{2}), f(a), f(-a), f(a+b)$

14. $f(x) = x^2 + 2x$;
$f(0), f(3), f(-3), f(a), f(-x), f\left(\dfrac{1}{a}\right)$

15. $g(x) = \dfrac{1-x}{1+x}$;
$g(2), g(-2), g(\frac{1}{2}), g(a), g(a-1), g(-1)$

16. $h(t) = t + \dfrac{1}{t}$;
$h(1), h(-1), h(2), h(\frac{1}{2}), h(x), h\left(\dfrac{1}{x}\right)$

17. $f(x) = 2x^2 + 3x - 4$;
$f(0), f(2), f(-2), f(\sqrt{2}), f(x+1), f(-x)$

18. $f(x) = x^3 - 4x^2$;
$f(0), f(1), f(-1), f(\frac{3}{2}), f\left(\dfrac{x}{2}\right), f(x^2)$

19. $f(x) = 2|x - 1|$;
$f(-2), f(0), f(\frac{1}{2}), f(2), f(x+1), f(x^2+2)$

20. $f(x) = \dfrac{|x|}{x}$;
$f(-2), f(-1), f(0), f(5), f(x^2), f\left(\dfrac{1}{x}\right)$

21–24 ■ Evaluate the piecewise defined function at the indicated values.

21. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$
$f(-2), f(-1), f(0), f(1), f(2)$

22. $f(x) = \begin{cases} 5 & \text{if } x \leq 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$
$f(-3), f(0), f(2), f(3), f(5)$

23. $f(x) = \begin{cases} x^2 + 2x & \text{if } x \leq -1 \\ x & \text{if } -1 < x \leq 1 \\ -1 & \text{if } x > 1 \end{cases}$
$f(-4), f(-\frac{3}{2}), f(-1), f(0), f(25)$

24. $f(x) = \begin{cases} 3x & \text{if } x < 0 \\ x + 1 & \text{if } 0 \leq x \leq 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$
$f(-5), f(0), f(1), f(2), f(5)$

25–28 ■ Use the function to evaluate the indicated expressions and simplify.

25. $f(x) = x^2 + 1$; $f(x+2), f(x) + f(2)$
26. $f(x) = 3x - 1$; $f(2x), 2f(x)$
27. $f(x) = x + 4$; $f(x^2), (f(x))^2$
28. $f(x) = 6x - 18$; $f\left(\dfrac{x}{3}\right), \dfrac{f(x)}{3}$

29–36 ■ Find $f(a)$, $f(a+h)$, and the difference quotient $\dfrac{f(a+h) - f(a)}{h}$, where $h \neq 0$.

29. $f(x) = 3x + 2$
30. $f(x) = x^2 + 1$

31. $f(x) = 5$

32. $f(x) = \dfrac{1}{x+1}$

33. $f(x) = \dfrac{x}{x+1}$

34. $f(x) = \dfrac{2x}{x-1}$

35. $f(x) = 3 - 5x + 4x^2$

36. $f(x) = x^3$

37–58 ■ Find the domain of the function.

37. $f(x) = 2x$

38. $f(x) = x^2 + 1$

39. $f(x) = 2x, \quad -1 \le x \le 5$

40. $f(x) = x^2 + 1, \quad 0 \le x \le 5$

41. $f(x) = \dfrac{1}{x-3}$

42. $f(x) = \dfrac{1}{3x-6}$

43. $f(x) = \dfrac{x+2}{x^2-1}$

44. $f(x) = \dfrac{x^4}{x^2+x-6}$

45. $f(x) = \sqrt{x-5}$

46. $f(x) = \sqrt[4]{x+9}$

47. $f(t) = \sqrt[3]{t-1}$

48. $g(x) = \sqrt{7-3x}$

49. $h(x) = \sqrt{2x-5}$

50. $G(x) = \sqrt{x^2-9}$

51. $g(x) = \dfrac{\sqrt{2+x}}{3-x}$

52. $g(x) = \dfrac{\sqrt{x}}{2x^2+x-1}$

53. $g(x) = \sqrt[4]{x^2-6x}$

54. $g(x) = \sqrt{x^2-2x-8}$

55. $f(x) = \dfrac{3}{\sqrt{x-4}}$

56. $f(x) = \dfrac{x^2}{\sqrt{6-x}}$

57. $f(x) = \dfrac{(x+1)^2}{\sqrt{2x-1}}$

58. $f(x) = \dfrac{x}{\sqrt[4]{9-x^2}}$

Applications

59. Production Cost The cost C in dollars of producing x yards of a certain fabric is given by the function

$$C(x) = 1500 + 3x + 0.02x^2 + 0.0001x^3$$

(a) Find $C(10)$ and $C(100)$.

(b) What do your answers in part (a) represent?

(c) Find $C(0)$. (This number represents the *fixed costs*.)

60. Area of a Sphere The surface area S of a sphere is a function of its radius r given by

$$S(r) = 4\pi r^2$$

(a) Find $S(2)$ and $S(3)$.

(b) What do your answers in part (a) represent?

61. How Far Can You See? Due to the curvature of the earth, the maximum distance D that you can see from the top of a tall building or from an airplane at height h is given by the function

$$D(h) = \sqrt{2rh + h^2}$$

where $r = 3960$ mi is the radius of the earth and D and h are measured in miles.

(a) Find $D(0.1)$ and $D(0.2)$.

(b) How far can you see from the observation deck of Toronto's CN Tower, 1135 ft above the ground?

(c) Commercial aircraft fly at an altitude of about 7 mi. How far can the pilot see?

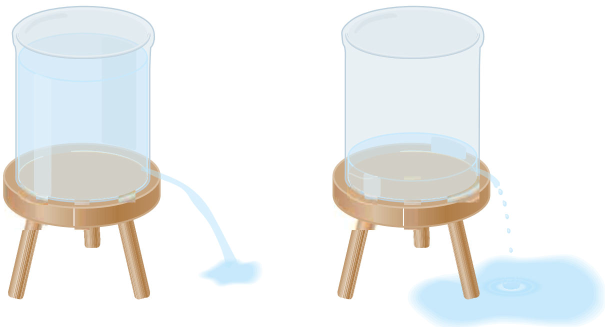
62. Torricelli's Law A tank holds 50 gallons of water, which drains from a leak at the bottom, causing the tank to empty in 20 minutes. The tank drains faster when it is nearly full because the pressure on the leak is greater. **Torricelli's Law** gives the volume of water remaining in the tank after t minutes as

$$V(t) = 50\left(1 - \frac{t}{20}\right)^2 \qquad 0 \le t \le 20$$

(a) Find $V(0)$ and $V(20)$.

(b) What do your answers to part (a) represent?

(c) Make a table of values of $V(t)$ for $t = 0, 5, 10, 15, 20$.



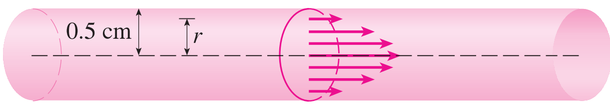
63. Blood Flow As blood moves through a vein or an artery, its velocity v is greatest along the central axis and decreases as the distance r from the central axis increases (see the figure). The formula that gives v as a function of r is called the **law of laminar flow**. For an artery with radius 0.5 cm, we have

$$v(r) = 18{,}500(0.25 - r^2) \qquad 0 \le r \le 0.5$$

(a) Find $v(0.1)$ and $v(0.4)$.

(b) What do your answers to part (a) tell you about the flow of blood in this artery?

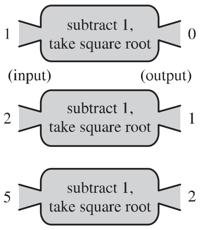
(c) Make a table of values of $v(r)$ for $r = 0, 0.1, 0.2, 0.3, 0.4, 0.5$.



Answer

1. $f(x) = 2(x + 3)$
3. $f(x) = (x - 5)^2$
5. Subtract 4, then divide by 3
7. Square, then add 2

9.



11.

x	$f(x)$
-1	8
0	2
1	0
2	2
3	8

13. $3, -3, 2, 2a+1, -2a+1, 2a+2b+1$

15. $-\frac{1}{3}, -3, \frac{1}{3}, \frac{1-a}{1+a}, \frac{2-a}{a}$, undefined

17. $-4, 10, -2, 3\sqrt{2}, 2x^2+7x+1, 2x^2-3x-4$

19. $6, 2, 1, 2, 2|x|, 2(x^2+1)$ **21.** $4, 1, 1, 2, 3$

23. $8, -\frac{3}{4}, -1, 0, -1$ **25.** x^2+4x+5, x^2+6

27. $x^2+4, x^2+8x+16$ **29.** $3a+2, 3(a+h)+2, 3$

31. $5, 5, 0$ **33.** $\frac{a}{a+1}, \frac{a+h}{a+h+1}, \frac{1}{(a+h+1)(a+1)}$

35. $3-5a+4a^2, 3-5a-5h+4a^2+8ah+4h^2,$ $-5+8a+4h$

37. $(-\infty, \infty)$ **39.** $[-1, 5]$ **41.** $\{x \mid x \neq 3\}$

43. $\{x \mid x \neq \pm 1\}$ **45.** $[5, \infty)$ **47.** $(-\infty, \infty)$ **49.** $[\frac{5}{2}, \infty)$

51. $[-2, 3) \cup (3, \infty)$ **53.** $(-\infty, 0] \cup [6, \infty)$ **55.** $(4, \infty)$

57. $(\frac{1}{2}, \infty)$ **59.** **(a)** $C(10) = 1532.1, C(100) = 2100$
(b) The cost of producing 10 yd and 100 yd
(c) $C(0) = 1500$

61. **(a)** $D(0.1) = 28.1, D(0.2) = 39.8$ **(b)** 41.3 mi
(c) 235.6 mi **63.** **(a)** $v(0.1) = 4440, v(0.4) = 1665$
(b) Flow is faster near central axis.
(c)

r	$v(r)$
0	4625
0.1	4440
0.2	3885
0.3	2960
0.5	0

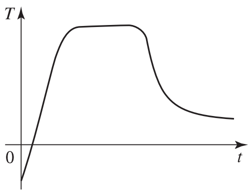
65. **(a)** 8.66 m, 6.61 m, 4.36 m
(b) It will appear to get shorter.

67. **(a)** \$90, \$105, \$100, \$105 **(b)** Total cost of an order, including shipping

69. **(a)** $F(x) = \begin{cases} 15(40-x) & \text{if } 0 < x < 40 \\ 0 & \text{if } 40 \le x \le 65 \\ 15(x-65) & \text{if } x > 65 \end{cases}$

(b) \$150, \$0, \$150 **(c)** Fines for violating the speed limits

71.



4.2 GRAPHS OF FUNCTIONS (Adapted from "Precalculus" by Stewart et als.)

Graphing Functions by Plotting Points ┐ Graphing Functions with a Graphing Calculator ┐ Graphing Piecewise Defined Functions ┐ The Vertical Line Test ┐ Equations That Define Functions

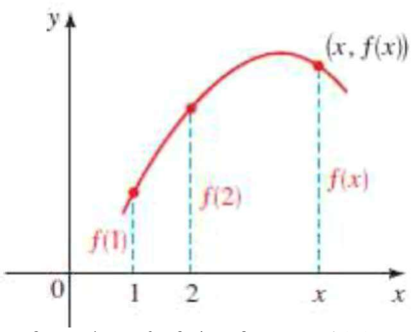
The most important way to visualize a function is through its graph. In this section we investigate in more detail the concept of graphing functions.

▼ Graphing Functions by Plotting Points

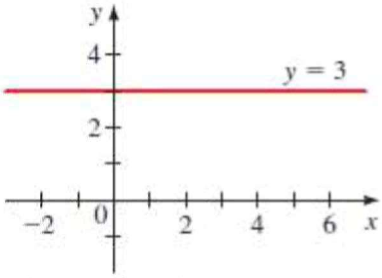
To graph a function *f*, we plot the points $(x, f(x))$ in a coordinate plane. In other words, we plot the points (x, y) whose *x*-coordinate is an input and whose *y*-coordinate is the corresponding output of the function.

> **THE GRAPH OF A FUNCTION**
>
> If f is a function with domain A, then the **graph** of f is the set of ordered pairs
>
> $$\{(x, f(x)) \mid x \in A\}$$
>
> plotted in a coordinate plane. In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.

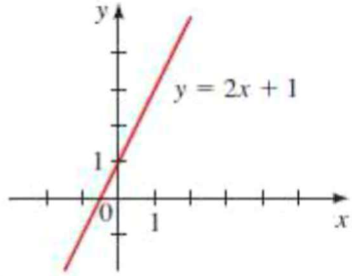
The graph of a function *f* gives a picture of the behavior or "life history" of the function. We can read the value of $f(x)$ from the graph as being the height of the graph above the point *x*.



A function *f* of the form $f(x) = mx + b$ is called a **linear function** because its graph is the graph of the equation $y = mx + b$, which represents a line with slope m and y-intercept b. A special case of a linear function occurs when the slope is $m = 0$. The function $f(x) = b$, where b is a given number, is called a **constant function** because all its values are the same number, namely, b. Its graph is the horizontal line $y = b$.



The constant function $f(x) = 3$



The linear function $f(x) = 2x + 1$

E X A M P L E 1 | Graphing Functions by Plotting Points
Sketch graphs of the following functions.
(a) $f(x) = x^2$ **(b)** $g(x) = x^3$ **(c)** $h(x) = \sqrt{x}$

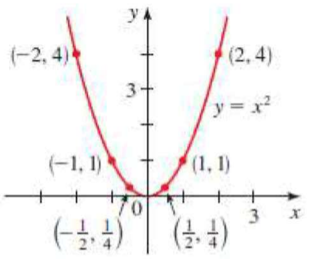
S O LU T I O N We first make a table of values.

x	$f(x) = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

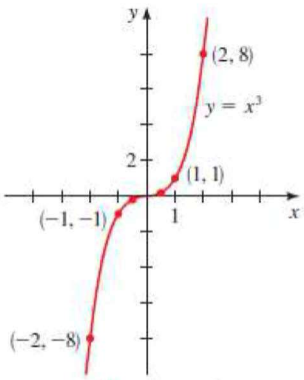
x	$g(x) = x^3$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
$-\frac{1}{2}$	$-\frac{1}{8}$
-1	-1
-2	-8

x	$h(x) = \sqrt{x}$
0	0
1	1
2	$\sqrt{2}$
3	$\sqrt{3}$
4	2
5	$\sqrt{5}$

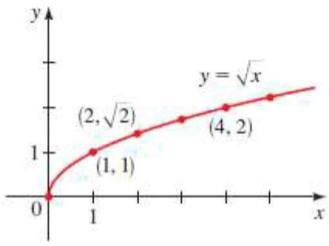
Then we plot the points given by the table and join them by a smooth curve to obtain the graph. The graphs are sketched here.



(a) $f(x) = x^2$



(b) $g(x) = x^3$



(c) $h(x) = \sqrt{x}$

NOW TRY: Sketch the graph of the function by first making a table of values.
(a) $f(x) = -x^2$ (b) $f(x) = x^3 - 8$ (c) $f(x) = 1 + \sqrt{x}$

▼ Graphing Functions with a Graphing Calculator

A convenient way to graph a function is to use a graphing calculator.

E X A M P L E 2 | A Family of Power Functions

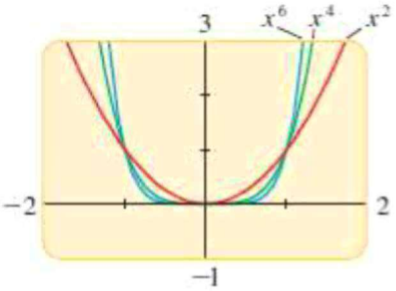
(a) Graph the functions $f(x) = x^n$ for $n = 2$, 4, and 6 in the viewing rectangle [–2, 2] by [–1, 3].

(b) Graph the functions $f(x) = x^n$ for $n = 1$, 3, and 5 in the viewing rectangle [–2, 2] by [–2, 2].

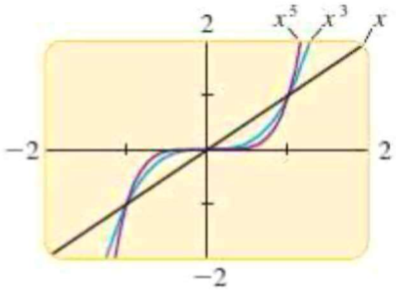
(c) What conclusions can you draw from these graphs?

S O LU T I O N

To graph the function $f(x) = x^n$, we graph the equation $y = x^n$. The graphs for parts (a) and (b) are shown here.



(a) Even powers of x



(b) Odd powers of x

(c) We see that the general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.

If n is even, the graph of is similar to the parabola $y = x^2$.

If n is odd, the graph of is similar to that of $y = x^3$.

Notice from the graphs that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $x > 1$. When $0 < x < 1$, the lower powers of x are the "bigger" functions. But when $x > 1$, the higher powers of x are the dominant functions.

▼ Graphing Piecewise Defined Functions

A piecewise defined function is defined by different formulas on different parts of its domain., The graph of such a function consists of separate pieces.

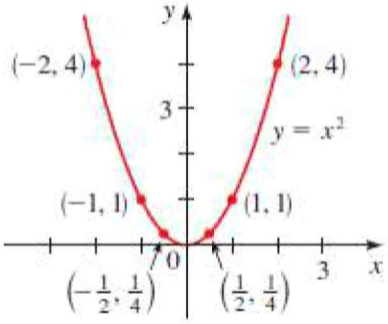
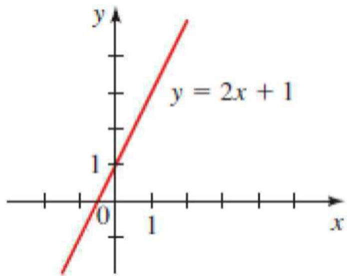
E X A M P L E 5 | Graph of a Piecewise Defined Function

Sketch the graph of the function.

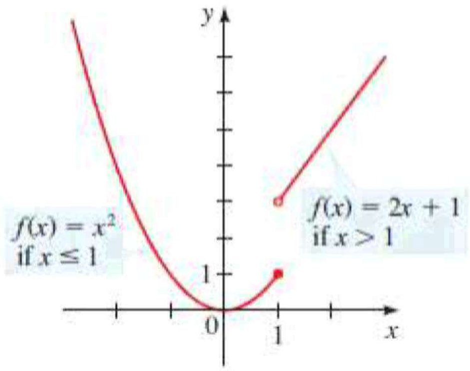
$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$$

S O LU T I O N

If $x \le 1$, then $f(x) = x^2$, so the part of the graph to the left of $x = 1$ coincides with the graph of $y = x^2$.	If $x > 1$, then $f(x) = 2x - 1$, so the part of the graph to the right of $x = 1$ coincides with the line $y = 2x + 1$.

 Graph of $y = x^2$	 Graph of $y = 2x + 1$

This enables us to sketch the graph of $f(x)=\begin{cases} x^2 & \text{if } x \le 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$



The solid dot at (1, 1) indicates that this point is included in the graph; the open dot at (1, 3) indicates that this point is excluded from the graph.

NOW TRY: Sketch the graph of the piecewise defined function.

$$f(x)=\begin{cases} 3 & \text{if } x \le 2 \\ x-1 & \text{if } x > 2 \end{cases}$$

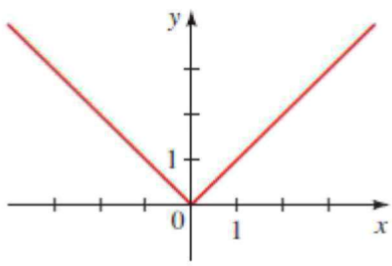
E X A M P L E 6 | Graph of the Absolute Value Function

Sketch a graph of the absolute value function $f(x) = |x|$.

S O LU T I O N

Recall that

$$|x| = \begin{cases} x & \text{if } x \le 0 \\ -x & \text{if } x > 0 \end{cases}$$



NOW TRY Sketch the graph of the function by first making a table of values.

$H(x) = |2x|$

The **greatest integer function** is defined by

$$[\![x]\!] = \text{greatest integer less than or equal to } x$$

For example, $[\![2]\!] = 2$, $[\![2.3]\!] = 2$, $[\![1.999]\!] = 1$, $[\![0.002]\!] = 0$, $[\![-3.5]\!] = -4$, and $[\![-0.5]\!] = -1$.

It is also called the **floor function**.

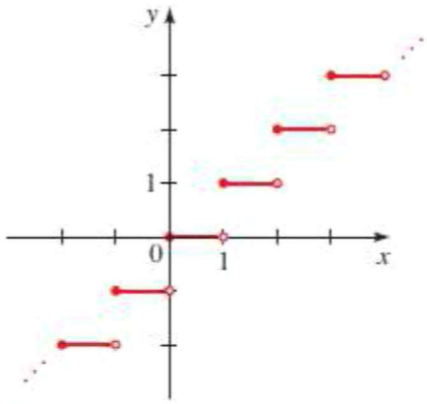
.

E X A M P L E 7 | Graph of the Greatest Integer Function

Sketch a graph of $f(x) = [\![x]\!]$.

SOLUTION The table shows the values of f for some values of x. Note that $f(x)$ is constant between consecutive integers, so the graph between integers is a horizontal line segment, as shown in Figure 8.

x	$[\![x]\!]$
$\vdots$	$\vdots$
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2
$\vdots$	$\vdots$



The greatest integer function is an example of a **step function**. The next example gives a real-world example of a step function.

E X A M P L E 8 | The Cost Function for Long-Distance Phone Calls

The cost of a long-distance daytime phone call from Toronto, Canada, to Mumbai, India, is 69 cents for the first minute and 58 cents for each additional minute (or part of a minute). Draw the graph of the cost C (in dollars) of the phone call as a function of time t (in minutes).

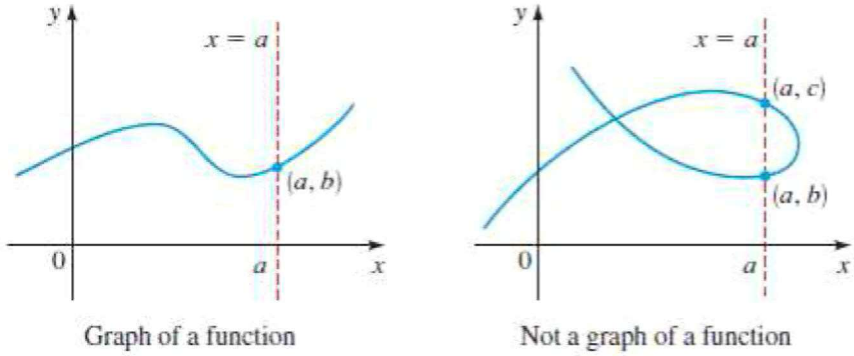
A function is called **continuous** if its graph has no "breaks" or "holes." The functions in Examples 1, 2, 3, and 5 are continuous; the functions in Examples 4, 6, and 7 are not continuous.

▼ The Vertical Line Test

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by the following test.

THE VERTICAL LINE TEST

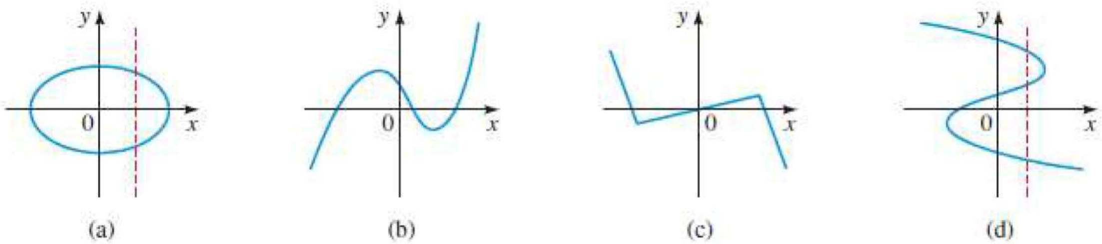
A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.



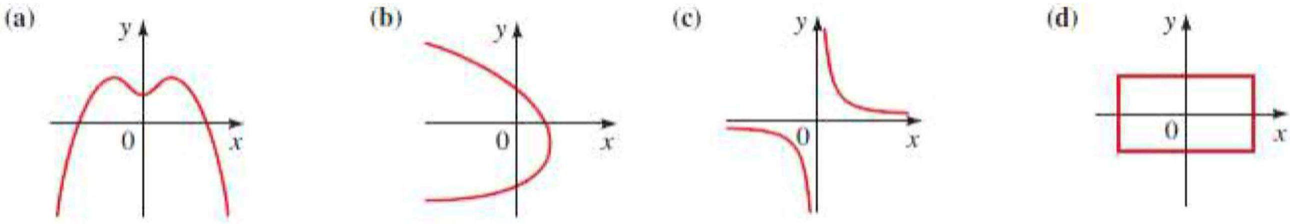
If each vertical line $x = a$ intersects a curve only once at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and at (a, c), then the curve cannot represent a function because a function cannot assign two different values to a.

E X A M P L E 9 | Using the Vertical Line Test

Using the Vertical Line Test, we see that the curves in parts (b) and (c) of Figure 11 represent functions, whereas those in parts (a) and (d) do not.



NOW TRY: Using the Vertical Line Test to determine whether the curve is the graph of a function of x.



▼ Equations That Define Functions

Any equation in the variables x and y defines a **relationship** between these variables. For example, the equation

$y - x^2 = 0$ defines a relationship between y and x.

Does this equation define y as a ***function*** of x?
To find out, we solve for y and get $y = x^2$
We see that the equation defines a rule, or function, that gives one value of y for each value of x. We can express this rule in function notation as $f(x) = x^2$.
But not every equation defines y as a function of x, as the following example shows.

E X A M P L E 10 | Equations That Define Functions
Does the equation define y as a function of x?
(a) $y - x^2 = 2$ **(b)** $x^2 + y^2 = 4$
S O LU T I O N
(a) Solving for y in terms of x gives ???
Add x^2
The last equation is a rule that gives one value of y for each value of x, so it defines y as a function of x. We can write the function as $f(x) =$????.
(b) We try to solve for y in terms of x:

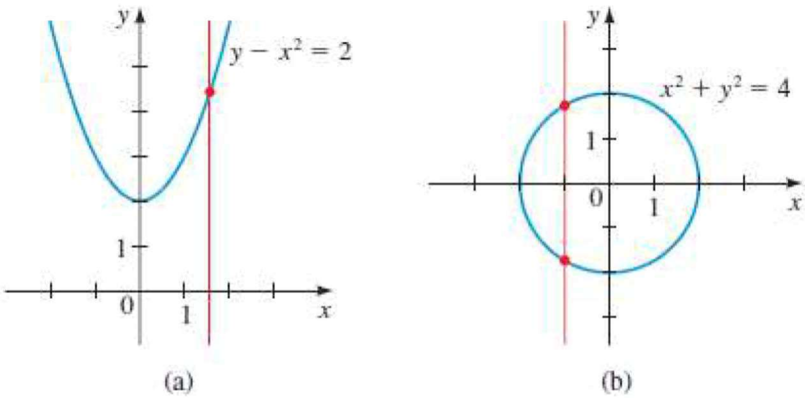
$$x^2 + y^2 = 4$$

$$y^2 = 4 - x^2 \qquad \text{Subtract } x^2$$

$$y = \pm\sqrt{4 - x^2} \qquad \text{Take square roots}$$

The last equation gives two values of y for a given value of x. Thus, the equation does not define y as a function of x.

The graphs of the two equations in Example 9 are shown here. The Vertical Line Test shows graphically that the equation in Example 9(a) defines a function but the equation in Example 9(b) does not.

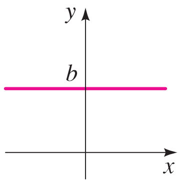


NOW TRY: Determine whether the equation defines y as a function of x.
(a) $x^2 + 2y = 4$ (b) $x + y^2 = 9$

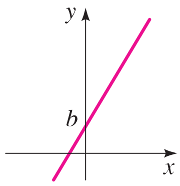
The following table shows the graphs of some functions that you will see frequently in this book.

Some Functions and Their Graphs

Linear functions
$f(x) = mx + b$

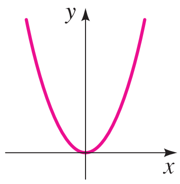


$f(x) = b$

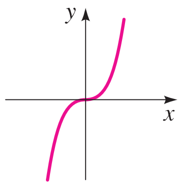


$f(x) = mx + b$

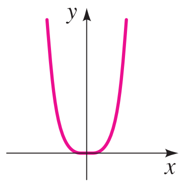
Power functions
$f(x) = x^n$



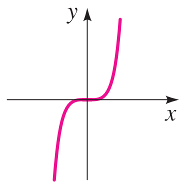
$f(x) = x^2$



$f(x) = x^3$

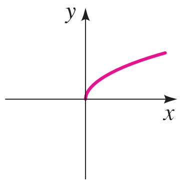


$f(x) = x^4$

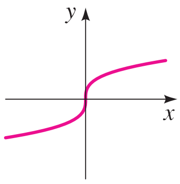


$f(x) = x^5$

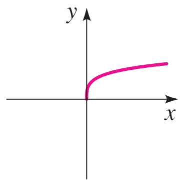
Root functions
$f(x) = \sqrt[n]{x}$



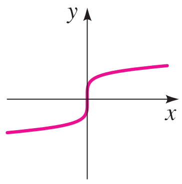
$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$

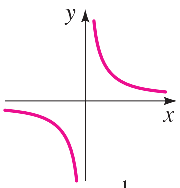


$f(x) = \sqrt[4]{x}$

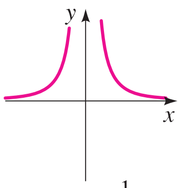


$f(x) = \sqrt[5]{x}$

Reciprocal functions
$f(x) = 1/x^n$

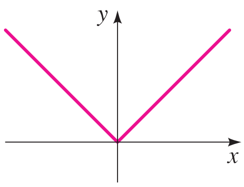


$f(x) = \dfrac{1}{x}$



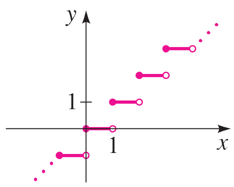
$f(x) = \dfrac{1}{x^2}$

Absolute value function
$f(x) = |x|$



$f(x) = |x|$

Greatest integer function
$f(x) = [\![x]\!]$



$f(x) = [\![x]\!]$

Exercise

1–22 ■ Sketch the graph of the function by first making a table of values.

1. $f(x) = 2$

2. $f(x) = -3$

3. $f(x) = 2x - 4$

4. $f(x) = 6 - 3x$

5. $f(x) = -x + 3, \quad -3 \le x \le 3$

6. $f(x) = \dfrac{x-3}{2}, \quad 0 \le x \le 5$

7. $f(x) = -x^2$

8. $f(x) = x^2 - 4$

9. $g(x) = x^3 - 8$

10. $g(x) = 4x^2 - x^4$

11. $g(x) = \sqrt{x + 4}$

12. $g(x) = \sqrt{-x}$

13. $F(x) = \dfrac{1}{x}$

14. $F(x) = \dfrac{1}{x + 4}$

15. $H(x) = |2x|$

16. $H(x) = |x + 1|$

17. $G(x) = |x| + x$

18. $G(x) = |x| - x$

19. $f(x) = |2x - 2|$

20. $f(x) = \dfrac{x}{|x|}$

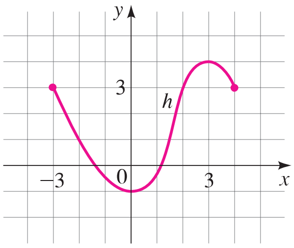
21. $g(x) = \dfrac{2}{x^2}$

22. $g(x) = \dfrac{|x|}{x^2}$

23. The graph of a function h is given.

(a) Find $h(-2)$, $h(0)$, $h(2)$, and $h(3)$.

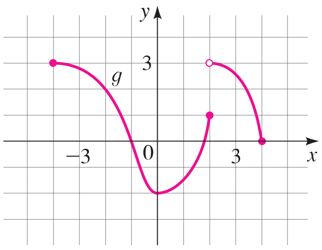
(b) Find the domain and range of h.



24. The graph of a function g is given.

(a) Find $g(-4)$, $g(-2)$, $g(0)$, $g(2)$, and $g(4)$.

(b) Find the domain and range of g.

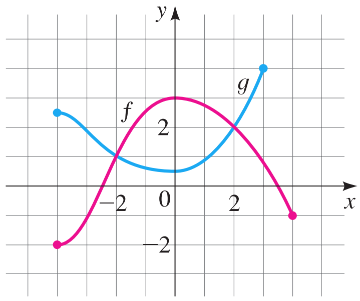


25. Graphs of the functions f and g are given.

(a) Which is larger, $f(0)$ or $g(0)$?

(b) Which is larger, $f(-3)$ or $g(-3)$?

(c) For which values of x is $f(x) = g(x)$?

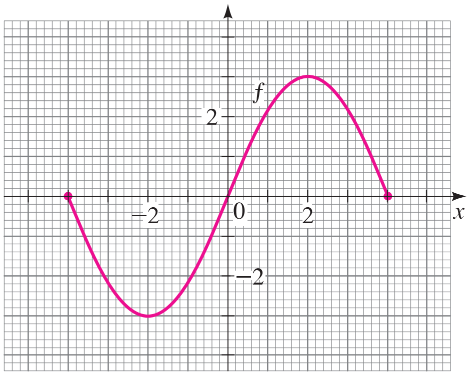


26. The graph of a function f is given.

(a) Estimate $f(0.5)$ to the nearest tenth.

(b) Estimate $f(3)$ to the nearest tenth.

(c) Find all the numbers x in the domain of f for which $f(x) = 1$.



27–36 ■ A function f is given.

(a) Use a graphing calculator to draw the graph of f.

(b) Find the domain and range of f from the graph.

27. $f(x) = x - 1$

28. $f(x) = 2(x + 1)$

29. $f(x) = 4$

30. $f(x) = -x^2$

31. $f(x) = 4 - x^2$

32. $f(x) = x^2 + 4$

33. $f(x) = \sqrt{16 - x^2}$

34. $f(x) = -\sqrt{25 - x^2}$

35. $f(x) = \sqrt{x - 1}$

36. $f(x) = \sqrt{x + 2}$

37–50 ■ Sketch the graph of the piecewise defined function.

37. $f(x) = \begin{cases} 0 & \text{if } x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

38. $f(x) = \begin{cases} 1 & \text{if } x \le 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

39. $f(x) = \begin{cases} 3 & \text{if } x < 2 \\ x - 1 & \text{if } x \ge 2 \end{cases}$

40. $f(x) = \begin{cases} 1 - x & \text{if } x < -2 \\ 5 & \text{if } x \ge -2 \end{cases}$

41. $f(x) = \begin{cases} x & \text{if } x \le 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

42. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \ge -1 \end{cases}$

43. $f(x) = \begin{cases} -1 & \text{if } x < -1 \\ 1 & \text{if } -1 \le x \le 1 \\ -1 & \text{if } x > 1 \end{cases}$

44. $f(x) = \begin{cases} -1 & \text{if } x < -1 \\ x & \text{if } -1 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

45. $f(x) = \begin{cases} 2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

46. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \le 2 \\ x & \text{if } x > 2 \end{cases}$

47. $f(x) = \begin{cases} 0 & \text{if } |x| \le 2 \\ 3 & \text{if } |x| > 2 \end{cases}$

48. $f(x) = \begin{cases} x^2 & \text{if } |x| \le 1 \\ 1 & \text{if } |x| > 1 \end{cases}$

49. $f(x) = \begin{cases} 4 & \text{if } x < -2 \\ x^2 & \text{if } -2 \le x \le 2 \\ -x + 6 & \text{if } x > 2 \end{cases}$

50. $f(x) = \begin{cases} -x & \text{if } x \le 0 \\ 9 - x^2 & \text{if } 0 < x \le 3 \\ x - 3 & \text{if } x > 3 \end{cases}$

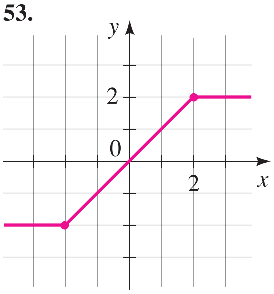
51–52 ■ Use a graphing device to draw the graph of the piecewise defined function. (See the margin note on page 162.)

51. $f(x) = \begin{cases} x + 2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

52. $f(x) = \begin{cases} 2x - x^2 & \text{if } x > 1 \\ (x - 1)^3 & \text{if } x \le 1 \end{cases}$

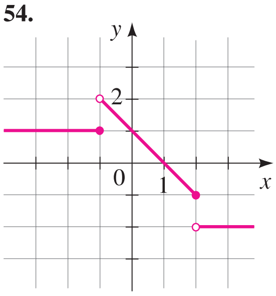
53–54 ■ The graph of a piecewise defined function is given. Find a formula for the function in the indicated form.

53.



$f(x) = \begin{cases} \quad & \text{if } x < -2 \\ \quad & \text{if } -2 \le x \le 2 \\ \quad & \text{if } x > 2 \end{cases}$

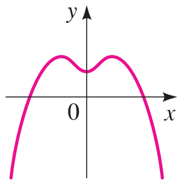
54.



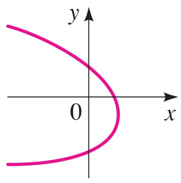
$f(x) = \begin{cases} \quad & \text{if } x \le -1 \\ \quad & \text{if } -1 < x \le 2 \\ \quad & \text{if } x > 2 \end{cases}$

55–56 ■ Determine whether the curve is the graph of a function of x.

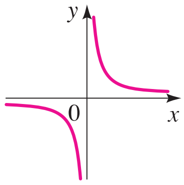
55. **(a)**



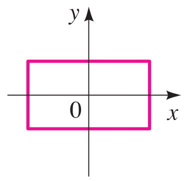
(b)



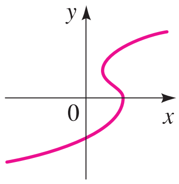
(c)



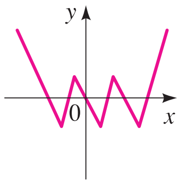
(d)



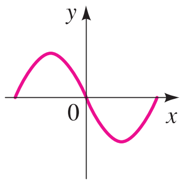
56. **(a)**



(b)



(c)



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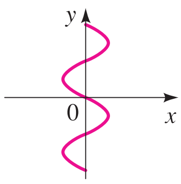
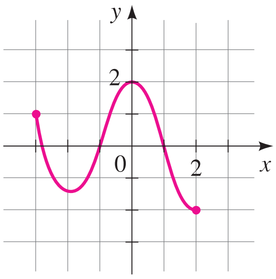


57–60 ■ Determine whether the curve is the graph of a function x. If it is, state the domain and range of the function.

57.



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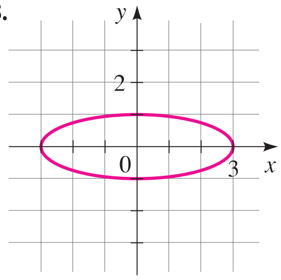

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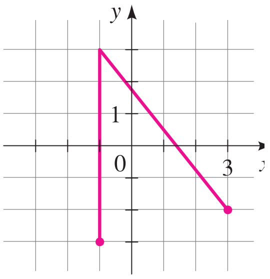

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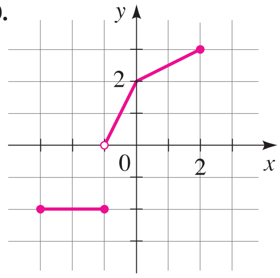


61–72 ■ Determine whether the equation defines y as a function of x. (See Example 10.)

61. $x^2 + 2y = 4$

62. $3x + 7y = 21$

63. $x = y^2$

64. $x^2 + (y - 1)^2 = 4$

65. $x + y^2 = 9$

66. $x^2 + y = 9$

67. $x^2y + y = 1$

68. $\sqrt{x} + y = 12$

69. $2|x| + y = 0$

70. $2x + |y| = 0$

71. $x = y^3$

72. $x = y^4$

73–78 ■ A family of functions is given. In parts (a) and (b) graph all the given members of the family in the viewing rectangle indicated. In part (c) state the conclusions you can make from your graphs.

73. $f(x) = x^2 + c$

(a) $c = 0, 2, 4, 6;\quad [-5, 5]$ by $[-10, 10]$

(b) $c = 0, -2, -4, -6;\quad [-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

74. $f(x) = (x - c)^2$

(a) $c = 0, 1, 2, 3;\quad [-5, 5]$ by $[-10, 10]$

(b) $c = 0, -1, -2, -3;\quad [-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

75. $f(x) = (x - c)^3$

(a) $c = 0, 2, 4, 6;\quad [-10, 10]$ by $[-10, 10]$

(b) $c = 0, -2, -4, -6;\quad [-10, 10]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

76. $f(x) = cx^2$

(a) $c = 1, \frac{1}{2}, 2, 4;\quad [-5, 5]$ by $[-10, 10]$

(b) $c = 1, -1, -\frac{1}{2}, -2;\quad [-5, 5]$ by $[-10, 10]$

(c) How does the value of c affect the graph?

77. $f(x) = x^c$

(a) $c = \frac{1}{2}, \frac{1}{4}, \frac{1}{6};\quad [-1, 4]$ by $[-1, 3]$

(b) $c = 1, \frac{1}{3}, \frac{1}{5};\quad [-3, 3]$ by $[-2, 2]$

(c) How does the value of c affect the graph?

78. $f(x) = 1/x^n$

(a) $n = 1, 3;\quad [-3, 3]$ by $[-3, 3]$

(b) $n = 2, 4;\quad [-3, 3]$ by $[-3, 3]$

(c) How does the value of n affect the graph?

79–82 ■ Find a function whose graph is the given curve.

79. The line segment joining the points $(-2, 1)$ and $(4, -6)$

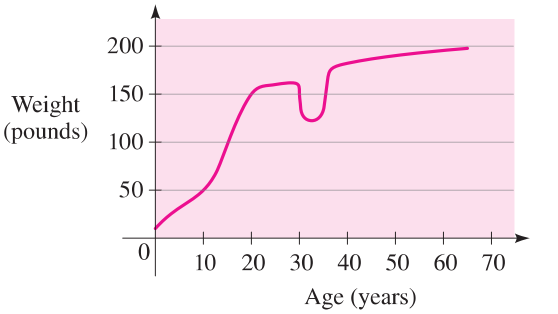
80. The line segment joining the points $(-3, -2)$ and $(6, 3)$

81. The top half of the circle $x^2 + y^2 = 9$

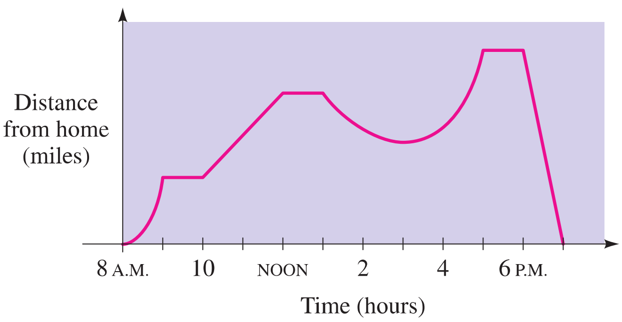
82. The bottom half of the circle $x^2 + y^2 = 9$

Applications

83. Weight Function The graph gives the weight of a certain person as a function of age. Describe in words how this person's weight has varied over time. What do you think happened when this person was 30 years old?



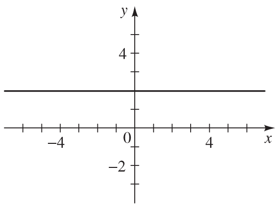
84. Distance Function The graph gives a salesman's distance from his home as a function of time on a certain day. Describe in words what the graph indicates about his travels on this day.



Answer

Section 2.2 ■ page 167

1.



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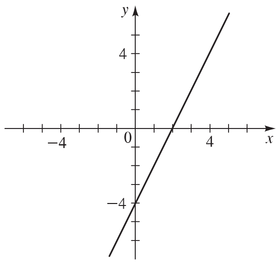

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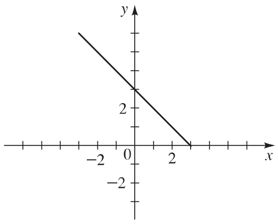

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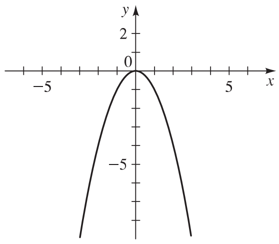

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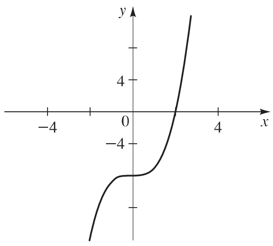

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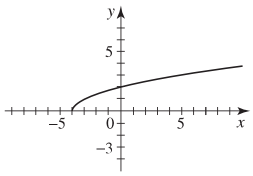

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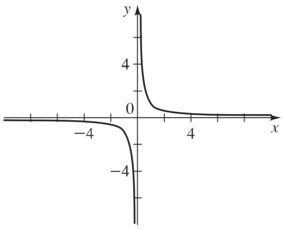

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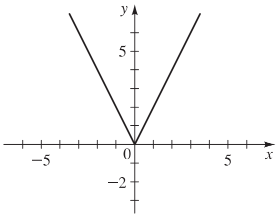

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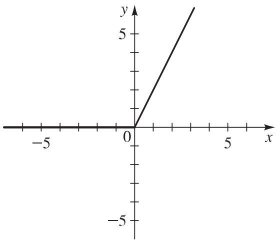

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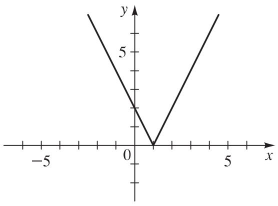

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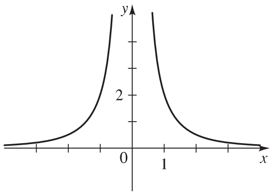
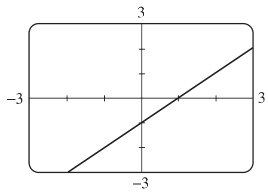


23. **(a)** $1, -1, 3, 4$ **(b)** Domain $[-3, 4]$, range $[-1, 4]$

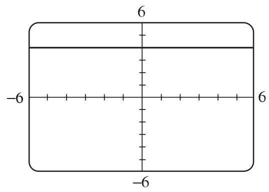
25. **(a)** $f(0)$ **(b)** $g(-3)$ **(c)** $-2, 2$

27. **(a)**



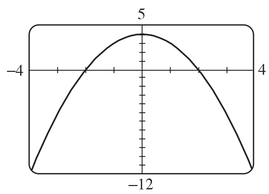
(b) Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

29. **(a)**



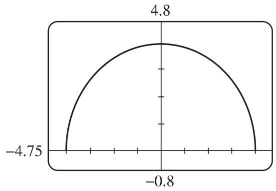
(b) Domain $(-\infty, \infty)$, range $\{4\}$

31. **(a)**



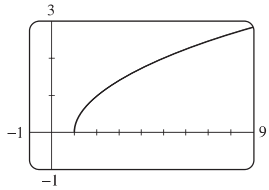
(b) Domain $(-\infty, \infty)$, range $(-\infty, 4]$

33. **(a)**



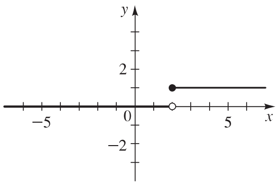
(b) Domain $[-4, 4]$, range $[0, 4]$

35. **(a)**



(b) Domain $[1, \infty)$, range $[0, \infty)$

37.



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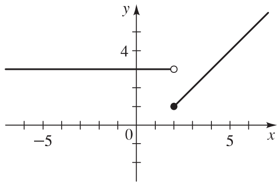

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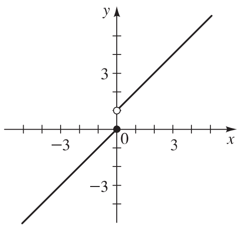

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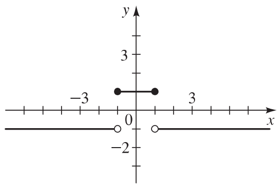

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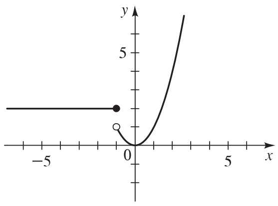

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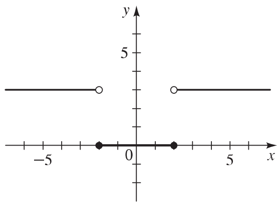

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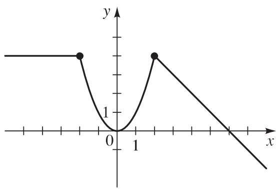

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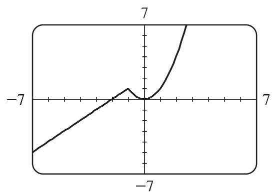


53. $f(x) = \begin{cases} -2 & \text{if } x < -2 \\ x & \text{if } -2 \le x \le 2 \\ 2 & \text{if } x > 2 \end{cases}$

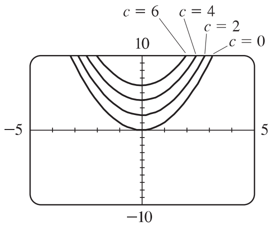
55. **(a)** Yes **(b)** No **(c)** Yes **(d)** No

57. Function, domain $[-3, 2]$, range $[-2, 2]$

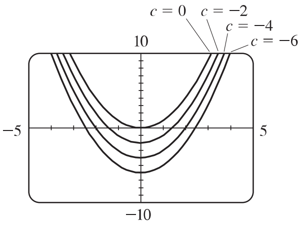
59. Not a function **61.** Yes **63.** No **65.** No

67. Yes **69.** Yes **71.** Yes

73. **(a)**

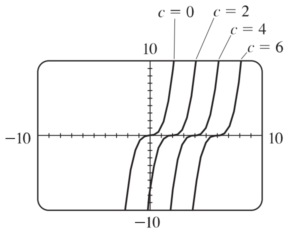


(b)

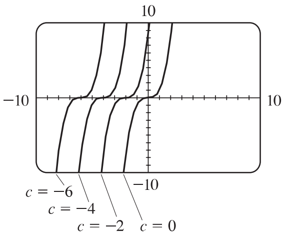


(c) If $c > 0$, then the graph of $f(x) = x^2 + c$ is the same as the graph of $y = x^2$ shifted upward c units. If $c < 0$, then the graph of $f(x) = x^2 + c$ is the same as the graph of $y = x^2$ shifted downward c units.

75. **(a)**

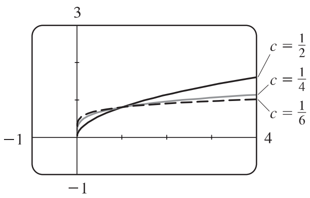


(b)

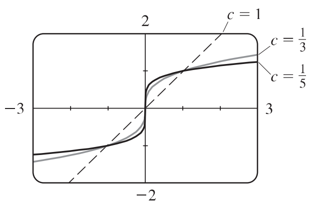


(c) If $c > 0$, then the graph of $f(x) = (x - c)^3$ is the same as the graph of $y = x^3$ shifted right c units. If $c < 0$, then the graph of $f(x) = (x - c)^3$ is the same as the graph of $y = x^3$ shifted left c units.

77. **(a)**



(b)



(c) Graphs of even roots are similar to $\sqrt{x}$; graphs of odd roots are similar to $\sqrt[3]{x}$. As c increases, the graph of $y = \sqrt[c]{x}$ becomes steeper near 0 and flatter when $x > 1$.

79. $f(x) = -\frac{7}{6}x - \frac{4}{3}, -2 \le x \le 4$

81. $f(x) = \sqrt{9 - x^2}, -3 \le x \le 3$

83. This person's weight increases as he grows, then continues to increase; the person then goes on a crash diet (possibly) at age 30, then gains weight again, the weight gain eventually leveling off.

4.3 GETTING INFORMATION FROM THE GRAPH OF A FUNCTION
(Adapted from "Precalculus" by Stewart et als.)

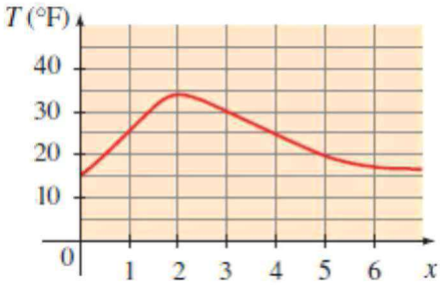
Values of a Function; Domain and Range ┐ Increasing and Decreasing Functions ┐
Local Maximum and Minimum Values of a Function

Many properties of a function are more easily obtained from a graph than from the rule that describes the function. We will see in this section how a graph tells us whether the values of a function are increasing or decreasing and also where the maximum and minimum values of a function are.

▼ Values of a Function; Domain and Range (refer to Section 2.1)

A complete graph of a function contains all the information about a function, because the graph tells us which input values correspond to which output values. To analyze the graph of a function, we must keep in mind that *the height of the graph is the value of the function*. So we can read off the values of a function from its graph.

E X A M P L E 1 | Finding the Values of a Function from a Graph
The function T, with graph shown here, gives the temperature between noon and 6:00 p.m. at a certain weather station.



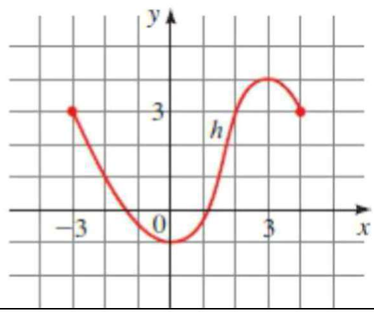
Temperature function

(a) Find $T(1)$, $T(3)$, and $T(5)$.
(b) Which is larger, $T(2)$ or $T(4)$?
(c) Find the value(s) of x for which $T(x) = 25$.
(d) Find the value(s) of x for which $T(x) \geq 25$.

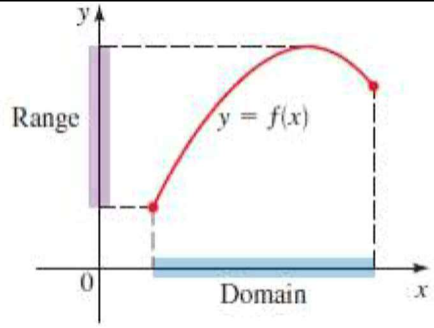
S O LU T I O N
(a) $T(1)$ is the temperature at 1:00 P.M. It is represented by the height of the graph above the x-axis at $x = 1$. Thus, . Similarly, $T(3) = 30$ and $T(5) = 20$.
(b) Since the graph is higher at $x = 2$ than at $x = 4$, it follows that $T(2)$ is larger than $T(4)$.
(c) The height of the graph is 25 when x is 1 and when x is 4. In other words, the temperature is 25 at 1:00 P.M. and 4:00 P.M.
(d) The graph is higher than 25 for x between 1 and 4. In other words, the temperature was 25 or greater between 1:00 P.M. and 4:00 P.M.

NOW TRY:

The graph of a function h is given.
(a) Find $h(-2)$, $h(0)$, $h(2)$, and $h(3)$.
(b) Find the domain and range of h.
(c) Find the values of x for which $h(x) = 3$.
(d) Find the values of x for which $h(x) \leq 3$.



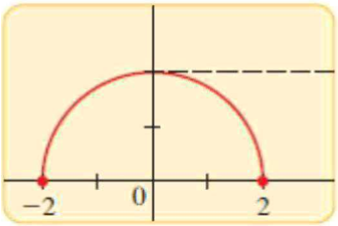
The graph of a function helps us to picture the domain and range of the function on the x-axis and y-axis, as shown here.



Domain and range of f

E X A M P L E 2 | Finding the Domain and Range from a Graph

The graph of $f(x) = \sqrt{4 - x^2}$.

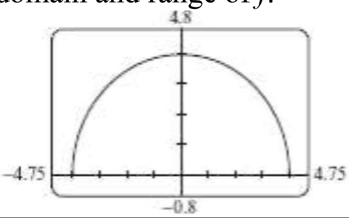
i. 

Find the domain and range of f.

S O LU T I O N

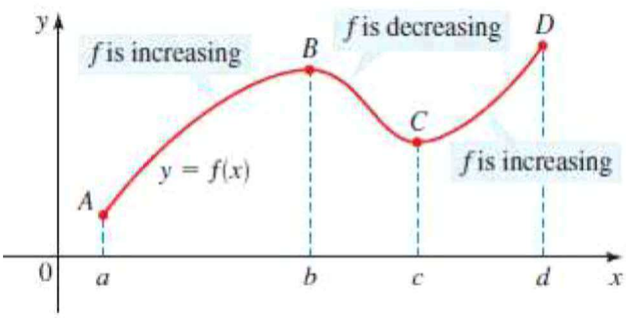
From the graph we see that the domain is [–2, 2] and the range is [0, 2].

NOW TRY: From the graph of a function f as shown on a graphing calculator, find the domain and range of f.



▼ Increasing and Decreasing Functions

It is very useful to know where the graph of a function rises and where it falls. The graph shown here rises, falls, then rises again as we move from left to right: It rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be *increasing* when its graph rises and *decreasing* when its graph falls.



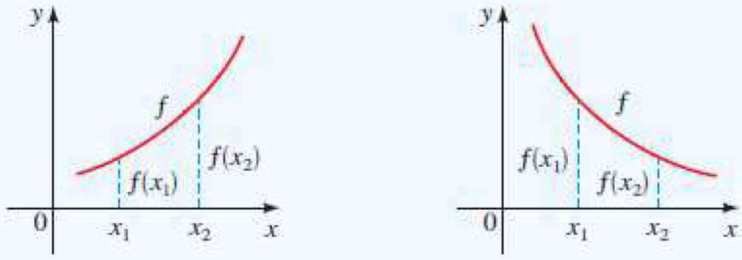
f is increasing on $[a, b]$ and $[c, d]$.
f is decreasing on $[b, c]$.

We have the following definition.

DEFINITION OF INCREASING AND DECREASING FUNCTIONS

f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

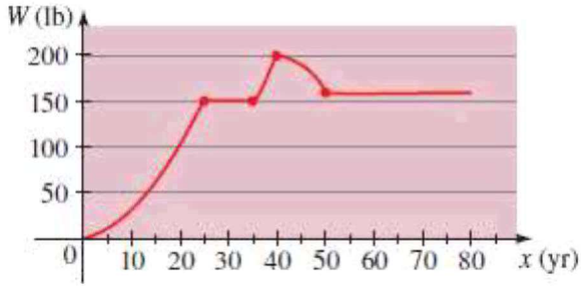


f is increasing

f is decreasing

E X A M P L E 3 Intervals on Which a Function Increases and Decreases

The graph gives the weight W of a person at age x.
Determine the intervals on which the function W is increasing and on which it is decreasing.

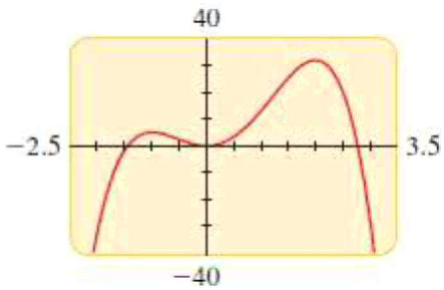


Weight as a function of age

S O LU T I O N The function W is increasing on [0, 25] and [35, 40]. It is decreasing on [40, 50]. The function W is constant (neither increasing nor decreasing) on [25, 35] and [50, 80]. This means that the person gained weight until age 25, then gained weight again between ages 35 and 40. He lost weight between ages 40 and 50.

E X A M P L E 4 Finding Intervals Where a Function Increases and Decreases (Using a graphing calculator)

(a) A graph of the function $f(x) = 12x^2 + 4x^3 - 3x^4$



i.

(b) Find the domain and range of f.

(c) Find the intervals on which f increases and decreases.

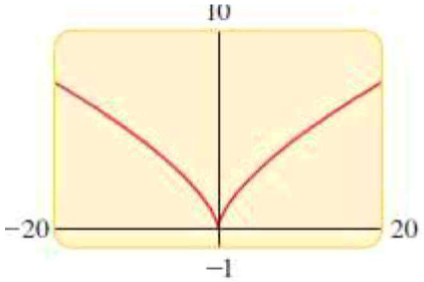
S O LU T I O N

(b) The domain of f is $\mathbb{R}$ because f is defined for all real numbers. Using the $\boxed{\text{TRACE}}$ feature on the calculator, we find that the highest value is $f(2) = 32$.
So the range of f is $(-\infty,32]$

(c) From the graph we see that f is increasing on the intervals $(-\infty,-1]$ and $[0, 2]$ and is decreasing on $[-1, 0]$ and $[2,\infty)$.

E X A M P L E 5 Finding Intervals Where a Function Increases and Decreases

(a) A graph of the function $f(x) = x^{\frac{2}{3}}$



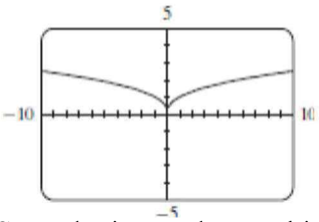
(b) Find the domain and range of the function.
(c) Find the intervals on which f increases and decreases.

S O LU T I O N

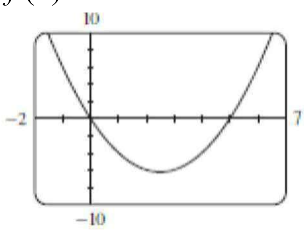
(b) From the graph we observe that the domain of f is $\mathbb{R}$ and the range is $[0,\infty)$.
(c) From the graph we see that f is decreasing on $(-\infty,0]$ and increasing on .$[0,\infty)$.

NOW TRY: A graph of the given function is shown.

(a) $f(x) = x^{\frac{2}{5}}$



(b) $f(x) = x^2 - 5x$



State the intervals on which f is increasing and on which f is decreasing.

▼ Local Maximum and Minimum Values of a Function

Finding the largest or smallest values of a function is important in many applications. For example, if a function represents revenue or profit, then we are interested in its maximum value. For a function that represents cost, we would want to find its minimum value.

We can easily find these values from the graph of a function. We first define what we mean by a local maximum or minimum.

LOCAL MAXIMA AND MINIMA OF A FUNCTION

1. The function value $f(a)$ is a **local maximum value** of f if

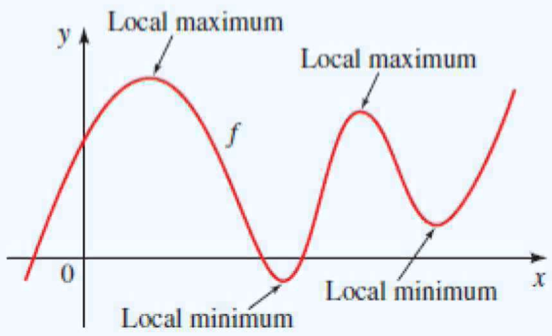
$$f(a) \geq f(x) \quad \text{when } x \text{ is near } a$$

(This means that $f(a) \geq f(x)$ for all x in some open interval containing a.) In this case we say that f has a **local maximum** at $x = a$.

2. The function value $f(a)$ is a **local minimum** of f if

$$f(a) \leq f(x) \quad \text{when } x \text{ is near } a$$

(This means that $f(a) \leq f(x)$ for all x in some open interval containing a.) In this case we say that f has a **local minimum** at $x = a$.

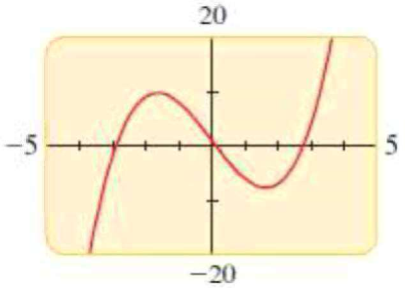


•

E X A M P L E 6 | Finding Local Maxima and Minima from a Graph

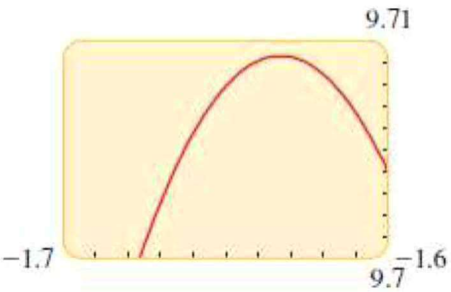
Find the local maximum and minimum values of the function $f(x) = x^3 - 8x + 1$, correct to three decimal places. (Using a graphing calculator)

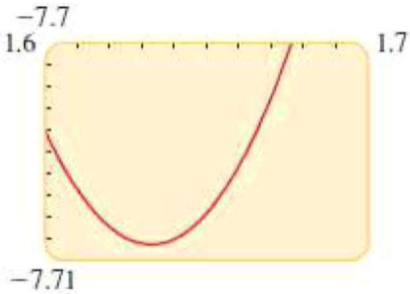
S O LU T I O N A graph of f is shown here.



There appears to be one local maximum between $x = -2$ and $x = -1$, and one local minimum between $x = 1$ and $x = 2$.

Let's find the coordinates of the local maximum point first. We use the "zoom" feature on the calculator to enlarge the area near this point, and move the cursor along the curve and observe how the y-coordinates change. The local maximum value of y is 9.709, and this value occurs when x is 1.633, correct to three decimal places.





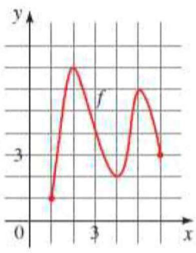
We locate the minimum value in a similar fashion. We find that the local minimum value is about 7.709, and this value occurs when $x = 1.633$.

E X A M P L E 7 | A Model for the Food Price Index (Section 2.5 in 5th edition)
(nby, June 2015)

Exercises 4.3

CONCEPTS

1–4 ■ These exercises refer to the graph of the function f shown below.



1. To find a function value $f(a)$ from the graph of f, we find the height of the graph above the x-axis at $x =$ ________. From the graph of f we see that $f(3) =$ ________.

2. The domain of the function f is all the ________-values of the points on the graph, and the range is all the corresponding ________-values. From the graph of f we see that the domain of f is the interval ________ and the range of f is the interval ________.

3. (a) If f is increasing on an interval, then the y-values of the points on the graph ________ as the x-values increase. From the graph of f we see that f is increasing on the intervals ________ and ________.

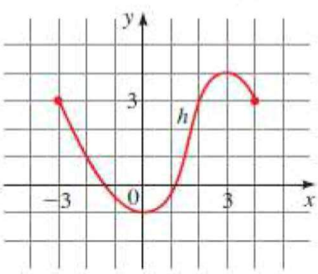
(b) If f is decreasing on an interval, then y-values of the points on the graph ________ as the x-values increase. From the graph of f we see that f is decreasing on the intervals ________ and ________.

4. (a) A function value $f(a)$ is a local maximum value of f if $f(a)$ is the ________ value of f on some interval containing a. From the graph of f we see that one local maximum value of f is ________ and that this value occurs when x is ________.

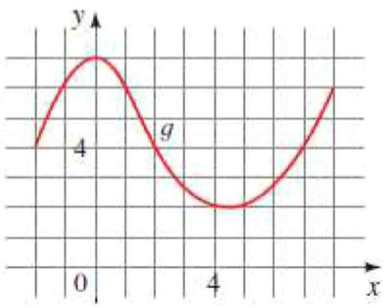
(b) The function value $f(a)$ is a local minimum value of f if $f(a)$ is the ________ value of f on some interval containing a. From the graph of f we see that one local minimum value of f is ________ and that this value occurs when x is ________.

SKILLS

5. The graph of a function h is given.
(a) Find $h(-2)$, $h(0)$, $h(2)$, and $h(3)$.
(b) Find the domain and range of h.
(c) Find the values of x for which $h(x) = 3$.
(d) Find the values of x for which $h(x) \le 3$.



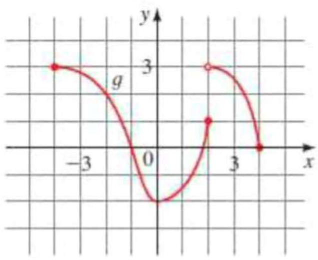
6. The graph of a function g is given.
(a) Find $g(-2)$, $g(0)$, and $g(7)$.
(b) Find the domain and range of g.
(c) Find the values of x for which $g(x) = 4$.
(d) Find the values of x for which $g(x) > 4$.



7. The graph of a function g is given.

(a) Find $g(-4)$, $g(-2)$, $g(0)$, $g(2)$, and $g(4)$.

(b) Find the domain and range of g.

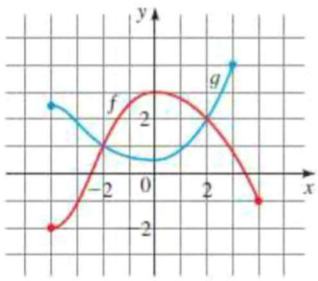


8. Graphs of the functions f and g are given.

(a) Which is larger, $f(0)$ or $g(0)$?

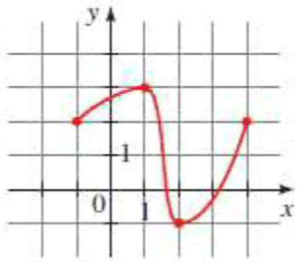
(b) Which is larger, $f(-3)$ or $g(-3)$?

(c) For which values of x is $f(x) = g(x)$?

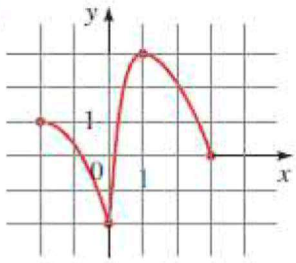


19–22 ■ The graph of a function is given. Determine the intervals on which the function is (a) increasing and (b) decreasing.

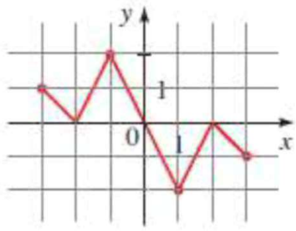
19.



20.



21.



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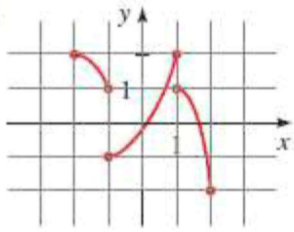
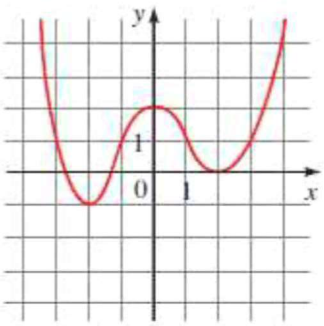
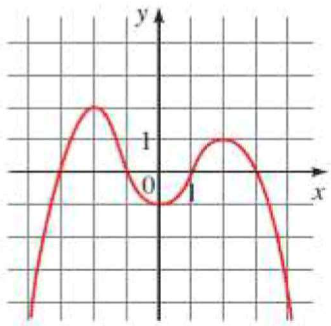


31–34 ■ The graph of a function is given. (a) Find all the local maximum and minimum values of the function and the value of x at which each occurs. (b) Find the intervals on which the function is increasing and on which the function is decreasing.

31.



32.



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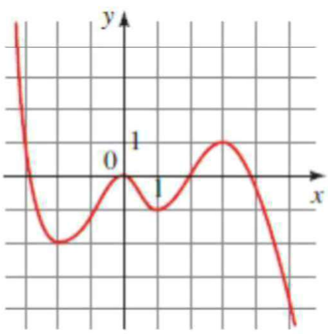
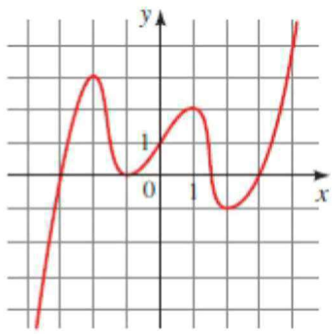


34.



4.4 TRANSFORMATIONS OF FUNCTIONS
(Adapted from "Precalculus" by Stewart et als.)

Vertical Shifting ┐ Horizontal Shifting ┐ Reflecting Graphs ┐
Vertical Stretching and Shrinking ┐ Horizontal Stretching and Shrinking ┐
Even and Odd Functions

In this section we study how certain transformations of a function affect its graph. This will give us a better understanding of how to graph functions. The transformations that we study are shifting, reflecting, and stretching.

▼ Vertical Shifting

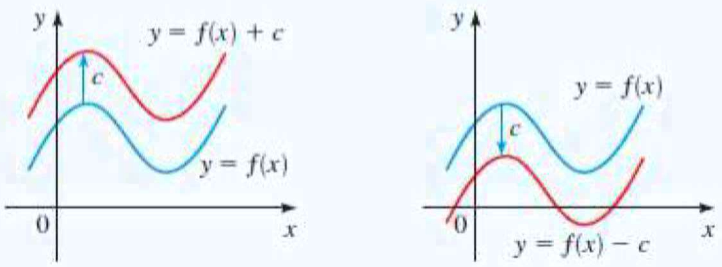
Adding a constant to a function shifts its graph vertically:
upward if the constant is positive and downward if it is negative.

In general, suppose we know the graph of $y = f(x)$.
How do we obtain from it the graphs of $y = f(x) + c$ and $y = f(x) - c$ $(c > 0)$
The y-coordinate of each point on the graph of $y = f(x) + c$ is c units above the y-coordinate of the corresponding point on the graph of $y = f(x)$.
So we obtain the graph of $y = f(x) + c$ simply by shifting the graph of $y = f(x)$ upward c units.
Similarly, we obtain the graph of $y = f(x) - c$ by shifting the graph of $y = f(x)$ downward c units

VERTICAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.
To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



E X A M P L E 1 | Vertical Shifts of Graphs

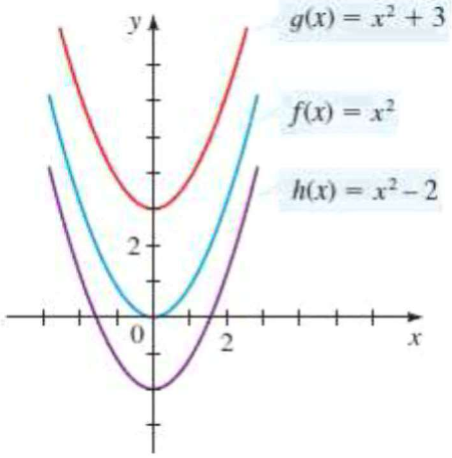
Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = x^2 + 3$ **(b)** $h(x) = x^2 - 2$

S O LU T I O N

(a) Observe that $g(x) = x^2 + 3 = f(x) + 3$

So the y-coordinate of each point on the graph of g is 3 units above the corresponding point on the graph of f.
This means that to graph g, we shift the graph of f upward 3 units.
(b) Similarly, to graph h, we shift the graph of f downward 2 units.



NOW TRY: Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

(a) $f(x) = x^2 - 1$ (b) $f(x) = \sqrt{x} + 1$

▼ Horizontal Shifting

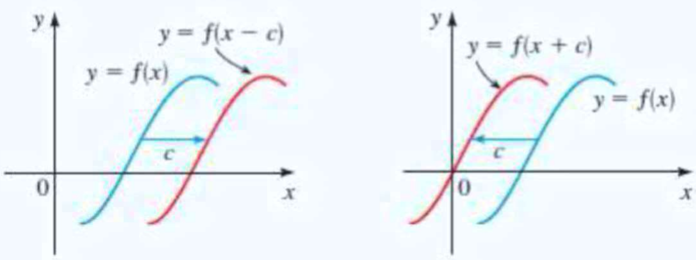
Suppose that we know the graph of $y = f(x)$.
How do we use it to obtain the graphs of $y = f(x+c)$ and $y = f(x-c)$ $(c > 0)$

The value of $f(x-c)$ at x is the same as the value of $f(x)$ at $x-c$. Since $x-c$ is c units to the left of x, it follows that the graph of $y = f(x-c)$ is just the graph of $y = f(x)$ shifted to the right c units.
Similar reasoning shows that the graph of $y = f(x+c)$ is the graph of $y = f(x)$ shifted to the left c units. The following box summarizes these facts.

HORIZONTAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.
To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.



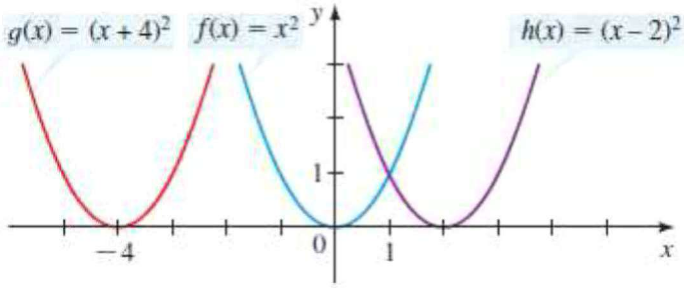
EXAMPLE 2 | Horizontal Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = (x+4)^2$ **(b)** $h(x) = (x-2)^2$

SOLUTION

(a) To graph g, we shift the graph of f to the left 4 units.
(b) To graph h, we shift the graph of f to the right 2 units.
The graphs of g and h are sketched here.



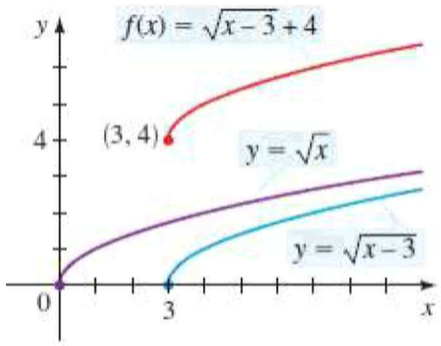
NOW TRY: Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

(a) $f(x) = (x-5)^2$ (b) $f(x) = \sqrt{x+4}$

EXAMPLE 3 | Combining Horizontal and Vertical Shifts

Sketch the graph of $f(x) = \sqrt{x-3} + 4$.

SOLUTION We start with the graph of $y = \sqrt{x}$ and shift it to the right 3 units to obtain the graph of $y = \sqrt{x-3}$. Then we shift the resulting graph upward 4 units to obtain the graph of $f(x) = \sqrt{x-3} + 4$ shown here.



NOW TRY: Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

$f(x) = (x-3)^2 + 5$

▼ Reflecting Graphs

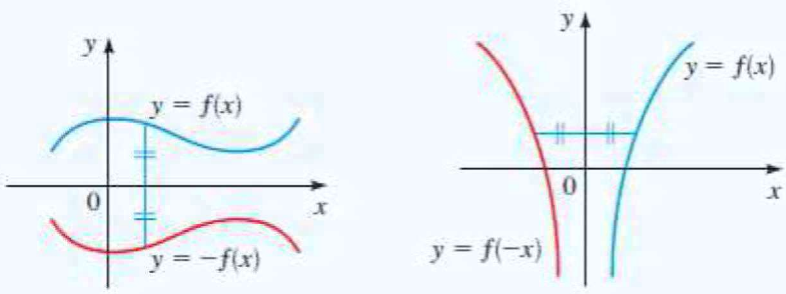
Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graphs of $y = -f(x)$ and $y = f(-x)$?

The y-coordinate of each point on the graph of $y=-f(x)$ is simply the negative of the y-coordinate of the corresponding point on the graph of $y=f(x)$. So the desired graph is the reflection of the graph of $y=f(x)$ in the x-axis.

On the other hand, the value of $y=f(-x)$ at x is the same as the value of $y=f(x)$ at $-x$, so the desired graph here is the reflection of the graph of $y=f(x)$ in the y-axis. The following box summarizes these observations.

REFLECTING GRAPHS

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.
To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.

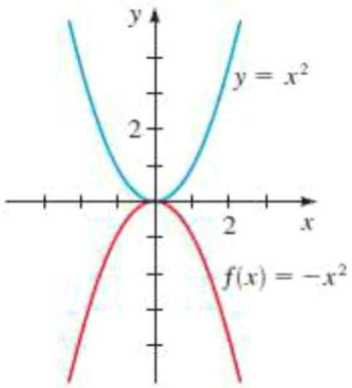


E X A M P L E 4 | Reflecting Graphs
Sketch the graph of each function.

(a) $f(x)=-x^2$ **(b)** $g(x)=\sqrt{-x}$

S O LU T I O N

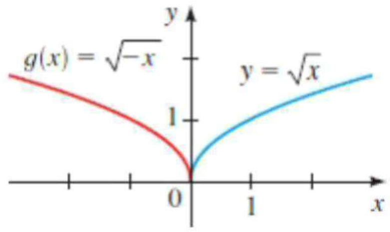
(a) We start with the graph of $y=x^2$. The graph of $f(x)=-x^2$ is the graph of $y=x^2$ reflected in the x-axis.



(b) We start with the graph of $y=\sqrt{x}$

The graph of $g(x)=\sqrt{-x}$ is the graph of $y=\sqrt{x}$ reflected in the y-axis.

Note that the domain of the function $g(x)=\sqrt{-x}$ is $\{x \mid x \le 0\}$.



NOW TRY: Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

(a) $f(x) = -x^3$ (b) $f(x) = \sqrt[4]{-x}$

▼ Vertical Stretching and Shrinking

Suppose we know the graph of $y = f(x)$.

How do we use it to obtain the graph of $y = cf(x)$?

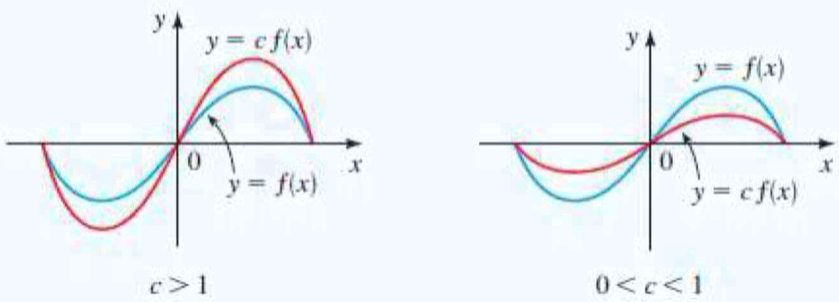
The y-coordinate of $y = cf(x)$ at x is the same as the corresponding y-coordinate of $y = f(x)$ multiplied by c. Multiplying the y-coordinates by c has the effect of vertically stretching or shrinking the graph by a factor of c.

VERTICAL STRETCHING AND SHRINKING OF GRAPHS

To graph $y = cf(x)$:

If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.



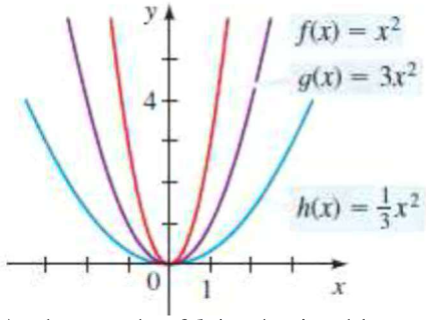
EXAMPLE 5 | Vertical Stretching and Shrinking of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = 3x^2$ **(b)** $h(x) = \frac{1}{3}x^2$

SOLUTION

(a) The graph of g is obtained by multiplying the y-coordinate of each point on the graph of f by 3. That is, to obtain the graph of g, we stretch the graph of f vertically by a factor of 3. The result is the narrower parabola as shown below.



(b) The graph of h is obtained by multiplying the y-coordinate of each point on the graph of f by $\frac{1}{3}$. That is, to obtain the graph of h, we shrink the graph of f vertically by a factor of $\frac{1}{3}$. The result is the wider parabola as shown above.

NOW TRY: Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

(a) $f(x) = \frac{1}{4}x^2$ (b) $f(x) = 3|x|$

We illustrate the effect of combining shifts, reflections, and stretching in the following example.

E X A M P L E 6 | Combining Shifting, Stretching, and Reflecting

Sketch the graph of the function $f(x) = 1 - 2(x-3)^2$.

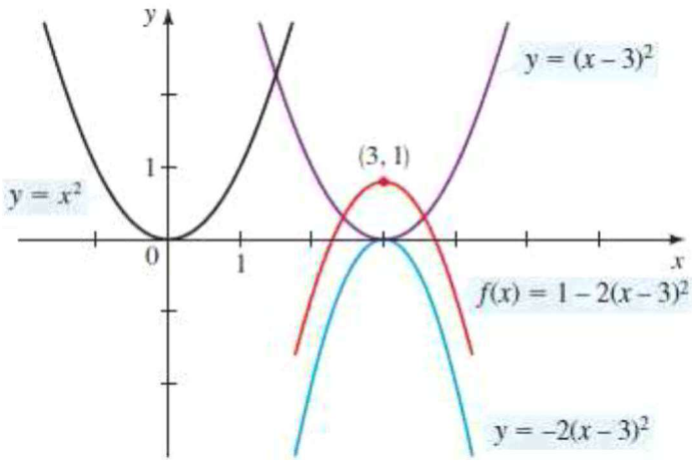
S O LU T I O N Starting with the graph of $y = x^2$,

we first shift to the right 3 units to get the graph of $y = (x-3)^2$.

Then we reflect in the x-axis and stretch by a factor of 2 to get the graph of

$$y = -2(x-3)^2.$$

Finally, we shift upward 1 unit to get the graph of $f(x) = 1 - 2(x-3)^2$ as shown here.



NOW TRY: Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

$f(x) = 3 - \frac{1}{2}(x-1)^2$

Horizontal Stretching and Shrinking

Now we consider horizontal shrinking and stretching of graphs.
If we know the graph of $y \quad f(x)$, then how is the graph of related to it?

The y-coordinate of $y = f(cx)$ at x is the same as the y-coordinate of $y = f(x)$ at cx. Thus, the x-coordinates in the graph of $y = f(x)$ correspond to the x-coordinates in the graph of $y = f(cx)$ multiplied by c.

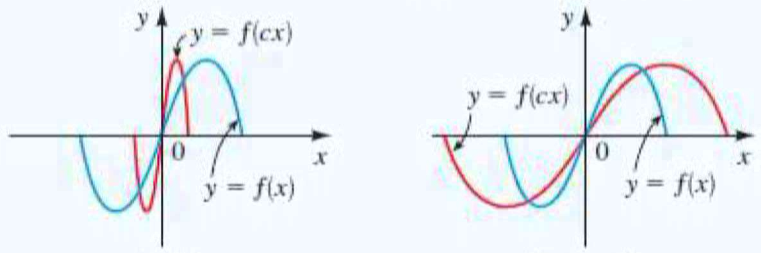
Looking at this the other way around, we see that the x-coordinates in the graph of $y = f(cx)$ are the x-coordinates in the graph of $y = f(x)$ multiplied by $1/c$. In other words, to change the graph of $y = f(x)$ to the graph of $y = f(cx)$, we must shrink (or stretch) the graph horizontally by a factor of $1/c$, as summarized in the following box.

HORIZONTAL SHRINKING AND STRETCHING OF GRAPHS

To graph $y = f(cx)$:

If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

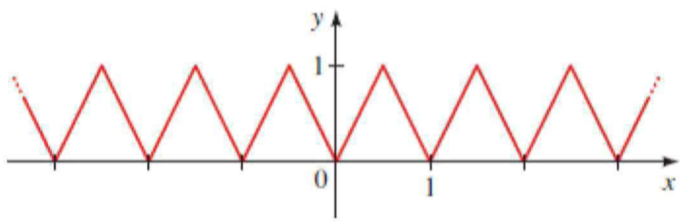
If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.



$c > 1$ $\qquad$ $0 < c < 1$

E X A M P L E 7 | Horizontal Stretching and Shrinking of Graphs

The graph of $y = f(x)$ is shown here.

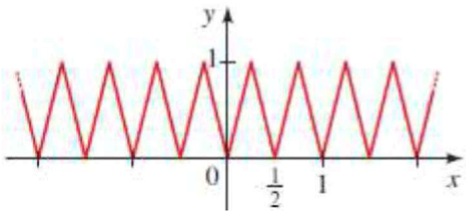


$y = f(x)$

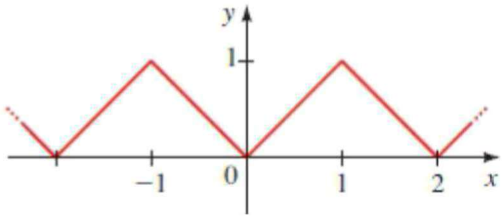
Sketch the graph of each function.

(a) $y = f(2x)$ $\qquad$ **(b)** $y = f(\frac{1}{2}x)$

S O LU T I O N

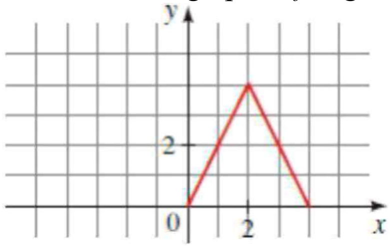


$y = f(2x)$

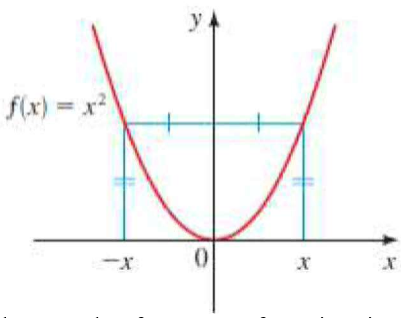
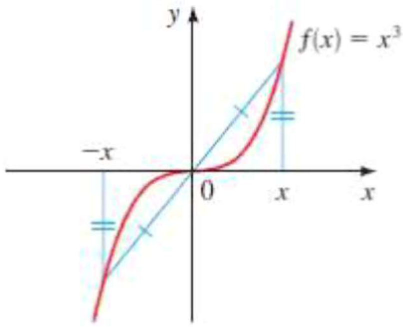


$y = f(\frac{1}{2}x)$

NOW TRY: The graph of *f* is given. Sketch the graphs of the following functions.



▼ Even and Odd Functions

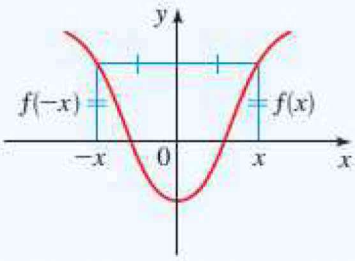
If a function *f* satisfies $f(-x) = f(x)$ for every number x in its domain, then *f* is called an **even function**	If *f* satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**.
For instance, the function $f(x) = x^2$ is even because $f(-x) = (-x)^2 = (-1)^2(-x)^2 = x^2 = f(x)$  The graph of an even function is symmetric with respect to the y-axis This means that if we have plotted the graph of *f* for $x \geq 0$, then we can obtain the entire graph simply by reflecting this portion in the y-axis.	For example, the function $f(x) = x^2$ is odd because $f(-x) = (-x)^3 = (-1)^3(-x)^3 = -x^3 = -f(x)$  The graph of an odd function is symmetric about the origin. If we have plotted the graph of *f* for $x \geq 0$, then we can obtain the entire graph by rotating this portion through $180°$ about the origin. (This is equivalent to reflecting first in the x-axis and then in the y-axis.)

EVEN AND ODD FUNCTIONS

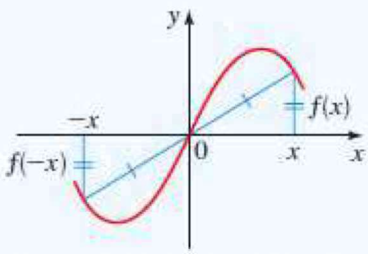
Let f be a function.

f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.



The graph of an even function is symmetric with respect to the y-axis.



The graph of an odd function is symmetric with respect to the origin.

EXAMPLE 8 | Even and Odd Functions

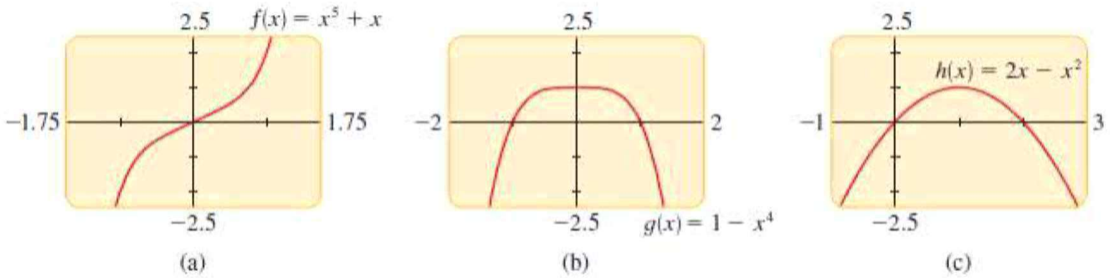
Determine whether the functions are even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ **(b)** $g(x) = 1 - x^4$ **(c)** $h(x) = 2x - x^2$

SOLUTION

(a) $f(-x) =$ **(b)** $g(-x) =$ **(c)** $h(-x) =$

The graphs of the functions in Example 8 are shown here.
The graph of f is symmetric about the origin,
and the graph of g is symmetric about the y-axis.
The graph of h is not symmetric either about the y-axis or the origin.



NOW TRY: Determine whether the function f is even, odd, or neither. If f is even or odd, use symmetry to sketch its graph.

(a) $f(x)$—x^4 **(b)** $g(x)$—$x^2 + x$ **(c)** $h(x) = x^3 - x$

(b)

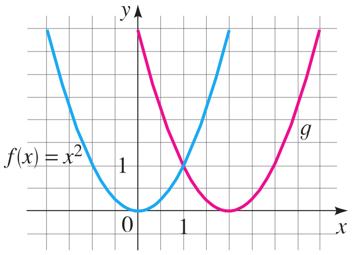
Exercise

1–10 ■ Suppose the graph of f is given. Describe how the graph of each function can be obtained from the graph of f.

1. **(a)** $y = f(x) - 5$ **(b)** $y = f(x - 5)$
2. **(a)** $y = f(x + 7)$ **(b)** $y = f(x) + 7$
3. **(a)** $y = f(x + \frac{1}{2})$ **(b)** $y = f(x) + \frac{1}{2}$
4. **(a)** $y = -f(x)$ **(b)** $y = f(-x)$
5. **(a)** $y = -2f(x)$ **(b)** $y = -\frac{1}{2}f(x)$
6. **(a)** $y = -f(x) + 5$ **(b)** $y = 3f(x) - 5$
7. **(a)** $y = f(x - 4) + \frac{3}{4}$ **(b)** $y = f(x + 4) - \frac{3}{4}$
8. **(a)** $y = 2f(x + 2) - 2$ **(b)** $y = 2f(x - 2) + 2$
9. **(a)** $y = f(4x)$ **(b)** $y = f(\frac{1}{4}x)$
10. **(a)** $y = -f(2x)$ **(b)** $y = f(2x) - 1$

11–16 ■ The graphs of f and g are given. Find a formula for the function g.

11.



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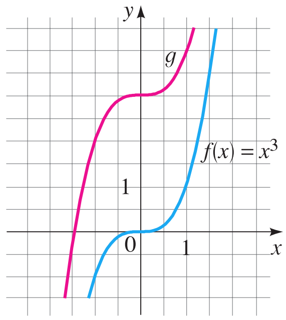

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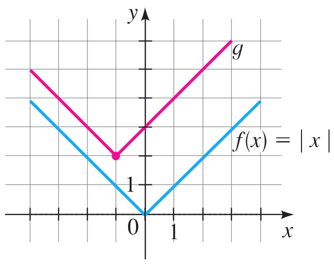

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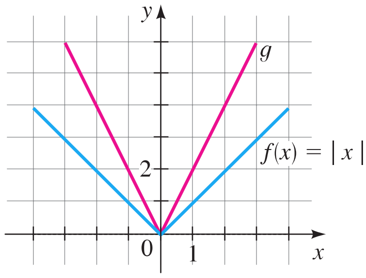

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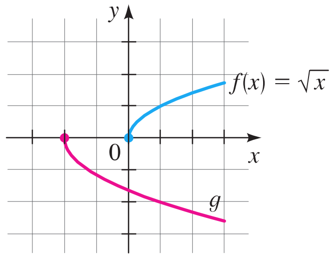

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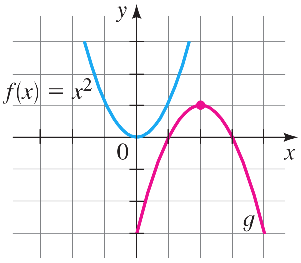
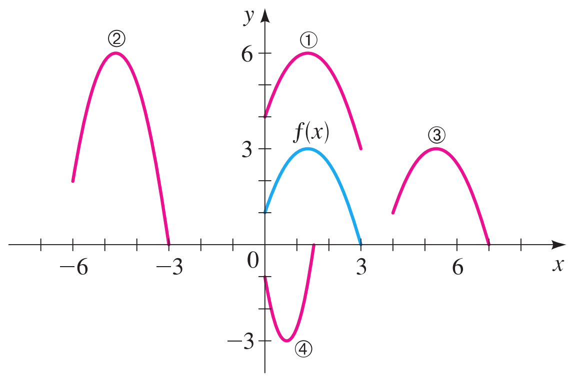


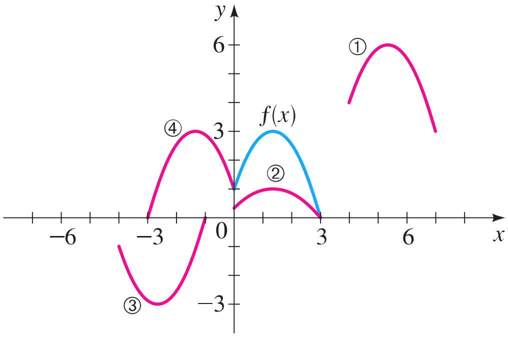
17–18 ■ The graph of $y = f(x)$ is given. Match each equation with its graph.

17. **(a)** $y = f(x - 4)$ **(b)** $y = f(x) + 3$
 (c) $y = 2f(x + 6)$ **(d)** $y = -f(2x)$



18. **(a)** $y = \frac{1}{3}f(x)$ **(b)** $y = -f(x + 4)$

(c) $y = f(x - 4) + 3$ **(d)** $y = f(-x)$

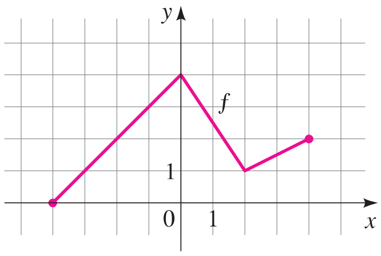


19. The graph of f is given. Sketch the graphs of the following functions.

(a) $y = f(x - 2)$ **(b)** $y = f(x) - 2$

(c) $y = 2f(x)$ **(d)** $y = -f(x) + 3$

(e) $y = f(-x)$ **(f)** $y = \frac{1}{2}f(x - 1)$

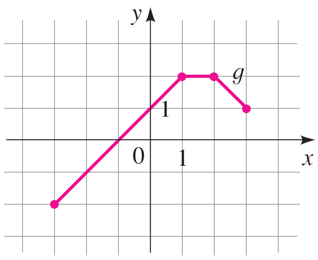


20. The graph of g is given. Sketch the graphs of the following functions.

(a) $y = g(x + 1)$ **(b)** $y = -g(x + 1)$

(c) $y = g(x - 2)$ **(d)** $y = g(x) - 2$

(e) $y = -g(x) + 2$ **(f)** $y = 2g(x)$



21. **(a)** Sketch the graph of $f(x) = \dfrac{1}{x}$ by plotting points.

(b) Use the graph of f to sketch the graphs of the following functions.

(i) $y = -\dfrac{1}{x}$ (ii) $y = \dfrac{1}{x - 1}$

(iii) $y = \dfrac{2}{x + 2}$ (iv) $y = 1 + \dfrac{1}{x - 3}$

22. **(a)** Sketch the graph of $g(x) = \sqrt[3]{x}$ by plotting points.

(b) Use the graph of g to sketch the graphs of the following functions.

(i) $y = \sqrt[3]{x - 2}$ (ii) $y = \sqrt[3]{x + 2} + 2$

(iii) $y = 1 - \sqrt[3]{x}$ (iv) $y = 2\sqrt[3]{x}$

23–26 ■ Explain how the graph of g is obtained from the graph of f.

23. **(a)** $f(x) = x^2$, $g(x) = (x + 2)^2$

(b) $f(x) = x^2$, $g(x) = x^2 + 2$

24. **(a)** $f(x) = x^3$, $g(x) = (x - 4)^3$

(b) $f(x) = x^3$, $g(x) = x^3 - 4$

25. **(a)** $f(x) = \sqrt{x}$, $g(x) = 2\sqrt{x}$

(b) $f(x) = \sqrt{x}$, $g(x) = \frac{1}{2}\sqrt{x - 2}$

26. **(a)** $f(x) = |x|$, $g(x) = 3|x| + 1$

(b) $f(x) = |x|$, $g(x) = -|x + 1|$

27–32 ■ A function f is given, and the indicated transformations are applied to its graph (in the given order). Write the equation for the final transformed graph.

27. $f(x) = x^2$; shift upward 3 units and shift 2 units to the right

28. $f(x) = x^3$; shift downward 1 unit and shift 4 units to the left

29. $f(x) = \sqrt{x}$; shift 3 units to the left, stretch vertically by a factor of 5, and reflect in the x-axis

30. $f(x) = \sqrt[3]{x}$; reflect in the y-axis, shrink vertically by a factor of $\frac{1}{2}$, and shift upward $\frac{3}{5}$ unit

31. $f(x) = |x|$; shift to the right $\frac{1}{2}$ unit, shrink vertically by a factor of 0.1, and shift downward 2 units

32. $f(x) = |x|$; shift to the left 1 unit, stretch vertically by a factor of 3, and shift upward 10 units

33–48 ■ Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

33. $f(x) = (x - 2)^2$ **34.** $f(x) = (x + 7)^2$

35. $f(x) = -(x + 1)^2$ **36.** $f(x) = 1 - x^2$

37. $f(x) = x^3 + 2$ **38.** $f(x) = -x^3$

39. $y = 1 + \sqrt{x}$ **40.** $y = 2 - \sqrt{x + 1}$

41. $y = \frac{1}{2}\sqrt{x + 4} - 3$ **42.** $y = 3 - 2(x - 1)^2$

43. $y = 5 + (x + 3)^2$ **44.** $y = \frac{1}{3}x^3 - 1$

45. $y = |x| - 1$ **46.** $y = |x - 1|$

47. $y = |x + 2| + 2$ **48.** $y = 2 - |x|$

49–52 ■ Graph the functions on the same screen using the given viewing rectangle. How is each graph related to the graph in part (a)?

49. Viewing rectangle $[-8, 8]$ by $[-2, 8]$

(a) $y = \sqrt[4]{x}$ **(b)** $y = \sqrt[4]{x + 5}$

(c) $y = 2\sqrt[4]{x + 5}$ **(d)** $y = 4 + 2\sqrt[4]{x + 5}$

50. Viewing rectangle $[-8, 8]$ by $[-6, 6]$

(a) $y = |x|$ **(b)** $y = -|x|$

(c) $y = -3|x|$ **(d)** $y = -3|x - 5|$

51. Viewing rectangle $[-4, 6]$ by $[-4, 4]$

(a) $y = x^6$ **(b)** $y = \frac{1}{3}x^6$

(c) $y = -\frac{1}{3}x^6$ **(d)** $y = -\frac{1}{3}(x - 4)^6$

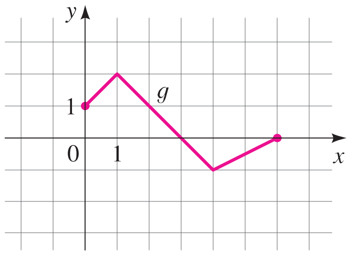
52. Viewing rectangle $[-6, 6]$ by $[-4, 4]$

(a) $y = \dfrac{1}{\sqrt{x}}$ **(b)** $y = \dfrac{1}{\sqrt{x + 3}}$

(c) $y = \dfrac{1}{2\sqrt{x + 3}}$ **(d)** $y = \dfrac{1}{2\sqrt{x + 3}} - 3$

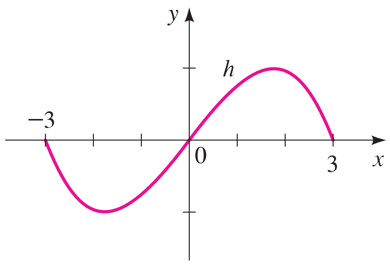
53. The graph of g is given. Use it to graph each of the following functions.

(a) $y = g(2x)$ **(b)** $y = g(\frac{1}{2}x)$



54. The graph of h is given. Use it to graph each of the following functions.

(a) $y = h(3x)$ **(b)** $y = h(\frac{1}{3}x)$

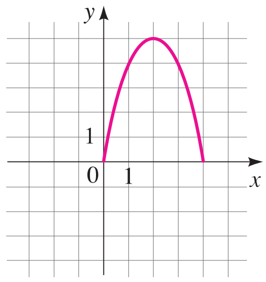


55–56 ■ The graph of a function defined for $x \geq 0$ is given. Complete the graph for $x < 0$ to make

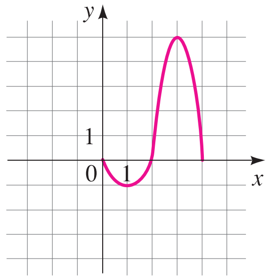
(a) an even function

(b) an odd function

55.



56.



57–58 ■ Use the graph of $f(x) = [\![x]\!]$ described on pages 162–163 to graph the indicated function.

57. $y = [\![2x]\!]$ **58.** $y = [\![\frac{1}{4}x]\!]$

59. If $f(x) = \sqrt{2x - x^2}$, graph the following functions in the viewing rectangle $[-5, 5]$ by $[-4, 4]$. How is each graph related to the graph in part (a)?

(a) $y = f(x)$ **(b)** $y = f(2x)$ **(c)** $y = f(\frac{1}{2}x)$

60. If $f(x) = \sqrt{2x - x^2}$, graph the following functions in the viewing rectangle $[-5, 5]$ by $[-4, 4]$. How is each graph related to the graph in part (a)?

(a) $y = f(x)$ **(b)** $y = f(-x)$ **(c)** $y = -f(-x)$

(d) $y = f(-2x)$ **(e)** $y = f(-\frac{1}{2}x)$

61–68 ■ Determine whether the function f is even, odd, or neither. If f is even or odd, use symmetry to sketch its graph.

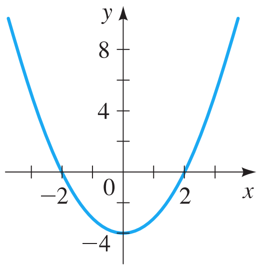
61. $f(x) = x^{-2}$ **62.** $f(x) = x^{-3}$

63. $f(x) = x^2 + x$ **64.** $f(x) = x^4 - 4x^2$

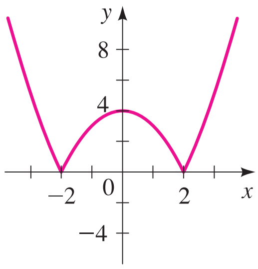
65. $f(x) = x^3 - x$ **66.** $f(x) = 3x^3 + 2x^2 + 1$

67. $f(x) = 1 - \sqrt[3]{x}$ **68.** $f(x) = x + \dfrac{1}{x}$

69. The graphs of $f(x) = x^2 - 4$ and $g(x) = |x^2 - 4|$ are shown. Explain how the graph of g is obtained from the graph of f.



$f(x) = x^2 - 4$

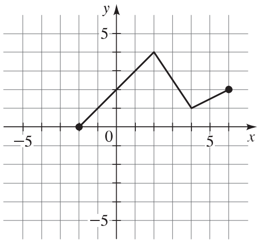


$g(x) = |x^2 - 4|$

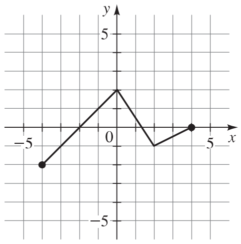
Section 2.4 ■ page 190

1. **(a)** Shift downward 5 units **(b)** Shift right 5 units
3. **(a)** Shift left $\frac{1}{2}$ unit **(b)** Shift up $\frac{1}{2}$ unit **5.** **(a)** Reflect in the x-axis and stretch vertically by a factor of 2 **(b)** Reflect in the x-axis and shrink vertically by a factor of $\frac{1}{2}$ **7.** **(a)** Shift right 4 units and upward $\frac{3}{4}$ unit **(b)** Shift left 4 units and downward $\frac{3}{4}$ unit **9.** **(a)** Shrink horizontally by a factor of $\frac{1}{4}$

(b) Stretch horizontally by a factor of 4 **11.** $g(x) = (x - 2)^2$
13. $g(x) = |x + 1| + 2$ **15.** $g(x) = -\sqrt{x + 2}$
17. (a) 3 **(b)** 1 **(c)** 2 **(d)** 4
19. (a)



(b)



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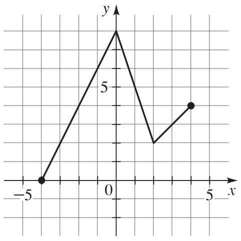
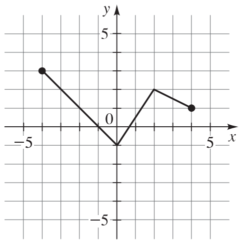
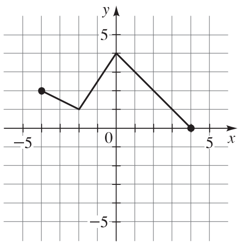


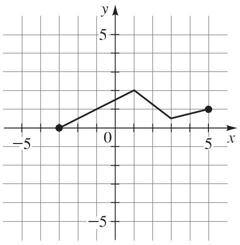
(d)



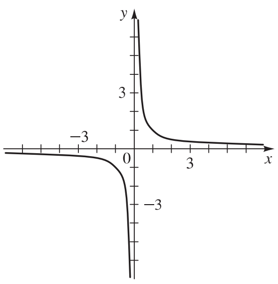
(e)



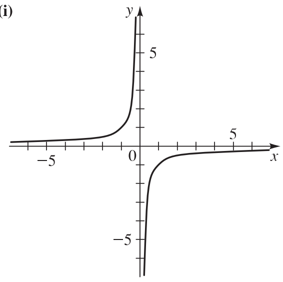
(f)



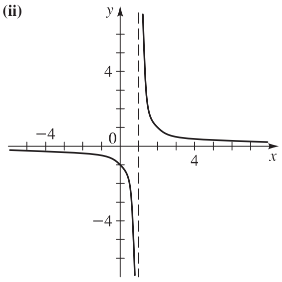
21. (a)



(b) (i)



(ii)



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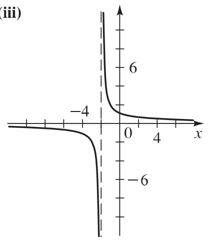
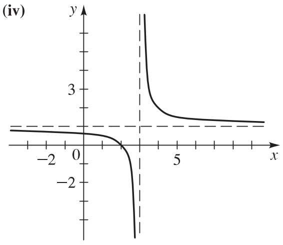
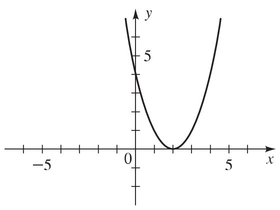


(iv)



23. (a) Shift left 2 units **(b)** Shift up 2 units
25. (a) Stretch vertically by a factor of 2
(b) Shift right 2 units, then shrink vertically by a factor of $\frac{1}{2}$
27. $g(x) = (x - 2)^2 + 3$ **29.** $g(x) = -5\sqrt{x + 3}$
31. $g(x) = 0.1|x - \frac{1}{2}| - 2$
33.



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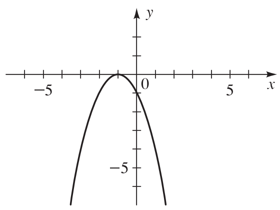

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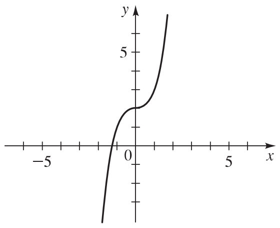

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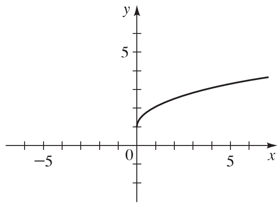

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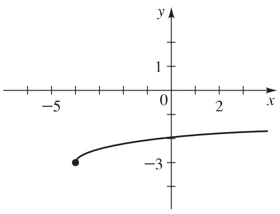

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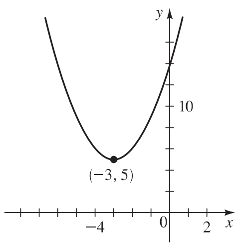

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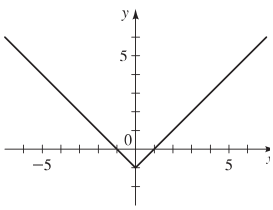

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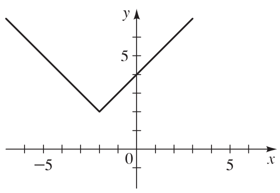
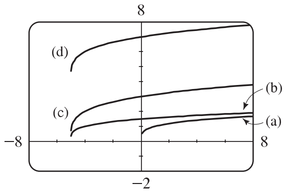
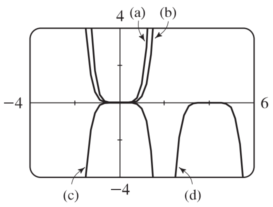


49.



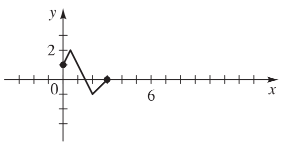
For part (b) shift the graph in (a) left 5 units; for part (c) shift the graph in (a) left 5 units and stretch vertically by a factor of 2; for part (d) shift the graph in (a) left 5 units, stretch vertically by a factor of 2, and then shift upward 4 units.

51.

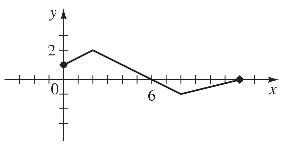


For part (b) shrink the graph in (a) vertically by a factor of $\frac{1}{3}$; for part (c) shrink the graph in (a) vertically by a factor of $\frac{1}{3}$ and reflect in the x-axis; for part (d) shift the graph in (a) right 4 units, shrink vertically by a factor of $\frac{1}{3}$, and then reflect in the x-axis.

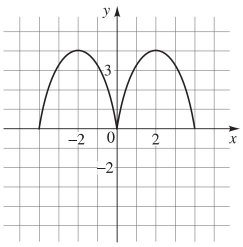
53. (a)



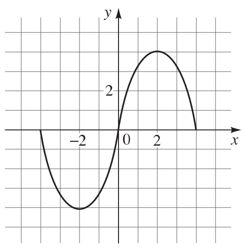
(b)



55. (a)



(b)



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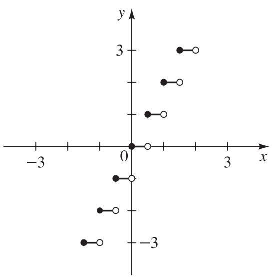

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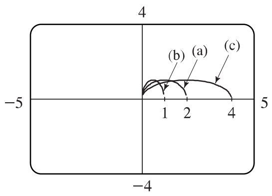
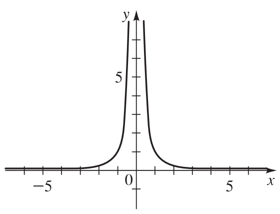


61. Even



63. Neither

4.5 COMBINING FUNCTIONS
(Adapted from "Precalculus" by Stewart et als.)

Sums, Differences, Products, and Quotients ┐ Composition of Functions

In this section we study different ways to combine functions to make new functions.

▼ Sums, Differences, Products, and Quotients

Two functions f and g can be combined to form new functions $f+g, f-g, fg$, and $\frac{f}{g}$ in a manner similar to the way we add, subtract, multiply, and divide real numbers.

For example, we define the function $f+g$ by $(f+g)(x)=f(x)+g(x)$

The new function is called the **sum** of the functions f and g; its value at x is $f(x)+g(x)$. Of course, the sum on the right-hand side makes sense only if both $f(x)$ and $g(x)$ are defined, that is, if x belongs to the domain of f and also to the domain of g.
So if the domain of f is A and the domain of g is B, then the domain of $f+g$ is the intersection of these domains, that is, $A \cap B$.

Similarly, we can define the **difference** $f-g$, the **product** fg, and the **quotient** $\frac{f}{g}$ of the functions f and g. Their domains are $A \cap B$, but in the case of the quotient we must remember not to divide by 0.

ALGEBRA OF FUNCTIONS

Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows.

$(f+g)(x)=f(x)+g(x)$	Domain $A \cap B$
$(f-g)(x)=f(x)-g(x)$	Domain $A \cap B$
$(fg)(x)=f(x)g(x)$	Domain $A \cap B$
$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

EXAMPLE 1 | Combinations of Functions and Their Domains

Let $f(x)=\frac{1}{x-2}$ and . $g(x)=\sqrt{x}$

(a) Find the functions $f+g, f-g, fg$, and $\frac{f}{g}$ and their domains.

(b) Find $(f+g)(4), (f-g)(4), (fg)(4)$, and $\left(\frac{f}{g}\right)(4)$.

S O LU T I O N

(a) The domain of f is $\{x \mid x \neq 2\}$, and the domain of g is $\{x \mid x \geq 0\}$.

The intersection of the domains of f and g is $\{x \mid x \geq 0 \text{ and } x \neq 2\} = [0, 2) \cup (2, \infty)$

Thus, we have

$$(f+g)(x) = f(x) + g(x) = \frac{1}{x-2} + \sqrt{x} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$(f-g)(x) = f(x) - g(x) = \frac{1}{x-2} - \sqrt{x} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$(fg)(x) = f(x)g(x) = \frac{\sqrt{x}}{x-2} \qquad \text{Domain } \{x \mid x \geq 0 \text{ and } x \neq 2\}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{(x-2)\sqrt{x}} \qquad \text{Domain } \{x \mid x > 0 \text{ and } x \neq 2\}$$

Note that in the domain of $\frac{f}{g}$ we exclude 0 because $g(0) = 0$.

(b) Each of these values exist because $x = 4$ is in the domain of each function.

$(f+g)(4) =$??? $(f-g)(4) =$??? $(fg)(4) =$??? $\left(\frac{f}{g}\right)(4) =$???

NOW TRY: Find $f+g, f-g, fg$, and $\frac{f}{g}$ and their domains.

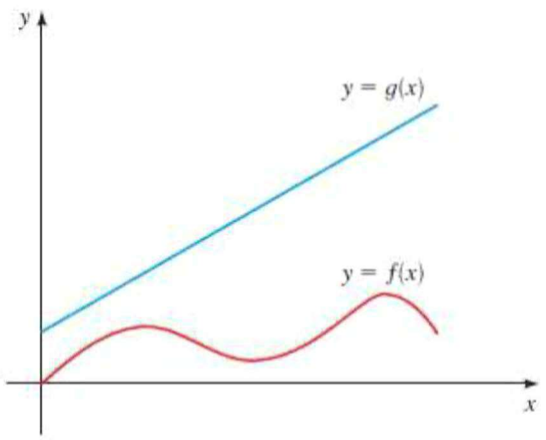
$f(x) = x - 3$ and . $g(x) = x^2$

The graph of the function $f+g$ can be obtained from the graphs of f and g by **graphical addition**. This means that we add corresponding y-coordinates, as illustrated in the next example.

E X A M P L E 2 | Using Graphical Addition

The graphs of f and g are shown here. Use graphical addition to graph the function $f+g$.

S O LU T I O N

<table>
<tr>
<td>

Graphs of f and g.</td>
<td><u>Graphical addition</u>:
We obtain the graph of $f+g$ by "graphically adding" the value of $f(x)$ to $g(x)$.
This is implemented by copying the line segment PQ on top of PR to obtain the point S on the graph of $f+g$.</td>
</tr>
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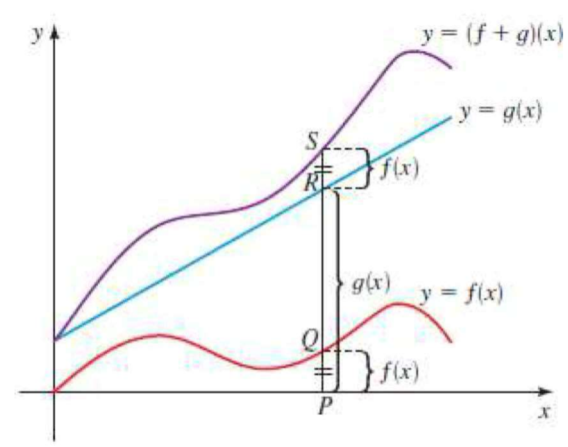
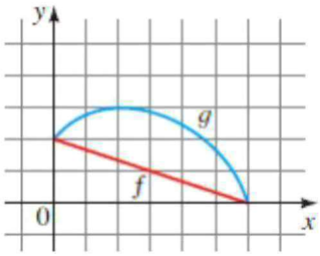


NOW TRY: Use graphical addition to sketch the graph of $f + g$.



▼ Composition of Functions

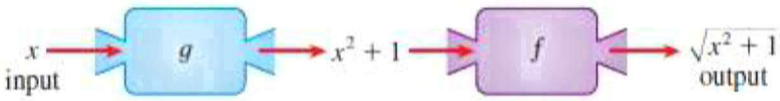
Now let's consider a very important way of combining two functions to get a new function.

Suppose $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$. We may define a new function h as

$$h(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The function h is made up of the functions f and g in an interesting way:
Given a number x, we first apply the function g to it, then apply f to the result. In this case, f is the rule "take the square root," g is the rule "square, then add 1," and h is the rule "square, then add 1, then take the square root." In other words, we get the rule h by applying the rule g and then the rule f.

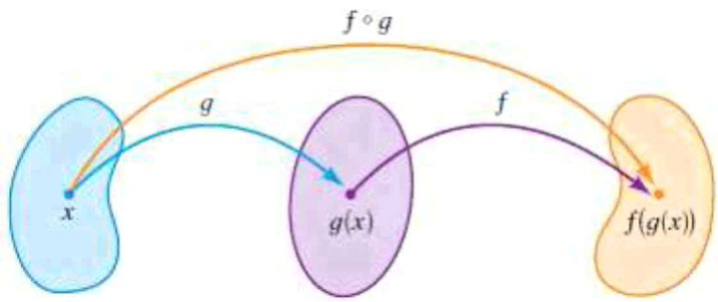
Machine diagram for *h*



The h machine is composed of the g machine (first) and then the f machine.
In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number is in the domain of f, we can then calculate the value of $f(g(x))$. The result is a new function $h(x) = f(g(x))$ that is obtained by substituting g into f. It s called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ ("f composed with g").

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

<u>Arrow diagram for</u> $f \circ g$



E X A M P L E 3 | Finding the Composition of Functions

Let $f(x) = x^2$ and $g(x) = x - 3$.

(a) Find the functions $f \circ g$ and $g \circ f$ and their domains.

(b) Find $(f \circ g)(5)$ and $(g \circ f)(7)$.

S O LU T I O N

(a)

$(f \circ g)(x) = f(g(x))$	Definition of $f \circ g$	$(g \circ f)(x) = g(f(x))$	Definition of $g \circ f$
$= f(x-3)$	Definition of g	$= g(x^2)$	Definition of f
$= (x-3)^2$	Definition of f	$= x^2 - 3$	Definition of g

The domains of both $f \circ g$ and $g \circ f$ are R.

(b)

$$(f \circ g)(5) = f(g(5)) = f(2) = 2^2 = 4$$

$$(g \circ f)(7) = g(f(7)) = g(49) = 49 - 3 = 46$$

NOW TRY:

(a) Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate $f(g(0))$ and $g(f(0))$

(b) Find the functions $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ and their domains.

$f(x) = x^2$ and $g(x) = x + 1$.

You can see from Example 3 that, in general, $f \circ g \neq g \circ f$. Remember that the notation $f \circ g$ means that the function g is applied first and then f is applied second.

E X A M P L E 4 | Finding the Composition of Functions

If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{2-x}$, find the following functions and their domains.

(a) $f\circ g$ **(b)** $g\circ f$ **(c)** $f\circ f$ **(d)** $g\circ g$

S O LU T I O N

(a)

$$\begin{aligned}(f\circ g)(x) &= f(g(x)) && \text{Definition of } f\circ g\\ &= f(\sqrt{2-x}) && \text{Definition of } g\\ &= \sqrt{\sqrt{2-x}} && \text{Definition of } f\\ &= \sqrt[4]{2-x}\end{aligned}$$

The domain of $f\circ g$ is $\{x\,|\,2-x\ge 0\}=\{x\,|\,x\le 2\}=(-\infty,2]$.

(b)

For $\sqrt{x}$ to be defined, we must have $x\ge 0$. For $\sqrt{2-\sqrt{x}}$ to be defined, we must have $2-\sqrt{x}\ge 0$, that is, $\sqrt{x}\le 2$, or $x\le 4$. Thus, we have $0\le x\le 4$, so the domain of $g\circ f$ is the closed interval $[0, 4]$.

(c)

$$\begin{aligned}(f\circ f)(x) &= f(f(x)) && \text{Definition of } f\circ f\\ &= f(\sqrt{x}) && \text{Definition of } f\\ &= \sqrt{\sqrt{x}} && \text{Definition of } f\\ &= \sqrt[4]{x}\end{aligned}$$

The domain of $f\circ f$ is $[0,\infty)$.

(d)

$$\begin{aligned}(g\circ g)(x) &= g(g(x)) && \text{Definition of } g\circ g\\ &= g(\sqrt{2-x}) && \text{Definition of } g\\ &= \sqrt{2-\sqrt{2-x}} && \text{Definition of } g\end{aligned}$$

This expression is defined when both $2-x\ge 0$ and $2-\sqrt{2-x}\ge 0$. The first inequality means $x\le 2$, and the second is equivalent to $\sqrt{2-x}\le 2$, or $2-x\le 4$, or $x\ge -2$. Thus, $-2\le x\le 2$, so the domain of $g\circ g$ is $[-2, 2]$.

NOW TRY:

Find the functions $f\circ g$, $g\circ f$, $f\circ f$ and $g\circ g$ and their domains.

$$f(x)=\frac{x}{x+1} \quad\text{and}\quad g(x)=2x-1.$$

It is possible to take the composition of three or more functions. For instance, the composite function $f\circ g\circ h$ is found by first applying h, then g, and then f as follows:

$$(f\circ g\circ h)(x)=f(g(h(x)))$$

E X A M P L E 5 | A Composition of Three Functions

Find $f \circ g \circ h$ if $f(x) = \dfrac{x}{x+1}$, $g(x) = x^{10}$, and $h(x) = x + 3$.

S O LU T I O N $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+9)) = \ldots$

NOW TRY: Find $f \circ g \circ h$ if $f(x) = x - 1$, $g(x) = \sqrt{x}$, and $h(x) = x - 3$.

So far, we have used composition to build complicated functions from simpler ones. But in calculus it is useful to be able to "decompose" a complicated function into simpler ones, as shown in the following example.

E X A M P L E 6 | Recognizing a Composition of Functions

Given $F(x) = \sqrt[4]{x+9}$, find functions f and g such that $F = f \circ g$.

S O LU T I O N Since the formula for F says to first add 9 and then take the fourth root, we let $f(x) = \sqrt[4]{x}$ and $g(x) = x + 9$.

Then $(f \circ g)(x) = f(g(x)) = f(x+9) = \ldots$

NOW TRY : Express the function in the form $f \circ g$. $F(x) = (x-9)^5$

Exercise

1–6 ■ Find $f + g$, $f - g$, fg, and f/g and their domains.

1. $f(x) = x - 3, \quad g(x) = x^2$

2. $f(x) = x^2 + 2x, \quad g(x) = 3x^2 - 1$

3. $f(x) = \sqrt{4 - x^2}, \quad g(x) = \sqrt{1 + x}$

4. $f(x) = \sqrt{9 - x^2}, \quad g(x) = \sqrt{x^2 - 4}$

5. $f(x) = \dfrac{2}{x}, \quad g(x) = \dfrac{4}{x + 4}$

6. $f(x) = \dfrac{2}{x + 1}, \quad g(x) = \dfrac{x}{x + 1}$

7–10 ■ Find the domain of the function.

7. $f(x) = \sqrt{x} + \sqrt{1 - x}$

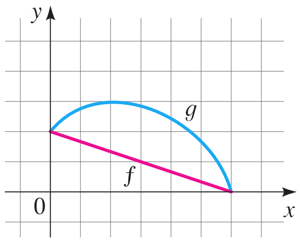
8. $g(x) = \sqrt{x + 1} - \dfrac{1}{x}$

9. $h(x) = (x - 3)^{-1/4}$

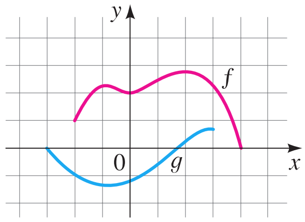
10. $k(x) = \dfrac{\sqrt{x + 3}}{x - 1}$

11–12 ■ Use graphical addition to sketch the graph of $f + g$.

11.



12.



13–16 ■ Draw the graphs of f, g, and $f + g$ on a common screen to illustrate graphical addition.

13. $f(x) = \sqrt{1 + x}, \quad g(x) = \sqrt{1 - x}$

14. $f(x) = x^2, \quad g(x) = \sqrt{x}$

15. $f(x) = x^2, \quad g(x) = \frac{1}{3}x^3$

16. $f(x) = \sqrt[4]{1 - x}, \quad g(x) = \sqrt{1 - \dfrac{x^2}{9}}$

17–22 ■ Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate the expression.

17. **(a)** $f(g(0))$ **(b)** $g(f(0))$

18. **(a)** $f(f(4))$ **(b)** $g(g(3))$

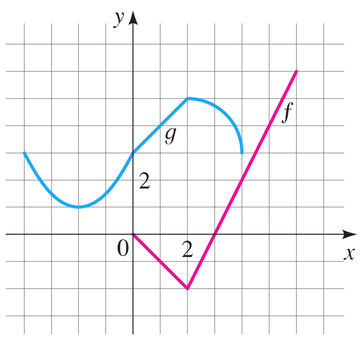
19. **(a)** $(f \circ g)(-2)$ **(b)** $(g \circ f)(-2)$

20. **(a)** $(f \circ f)(-1)$ **(b)** $(g \circ g)(2)$

21. **(a)** $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$

22. **(a)** $(f \circ f)(x)$ **(b)** $(g \circ g)(x)$

23–28 ■ Use the given graphs of f and g to evaluate the expression.



23. $f(g(2))$

24. $g(f(0))$

25. $(g \circ f)(4)$

26. $(f \circ g)(0)$

27. $(g \circ g)(-2)$

28. $(f \circ f)(4)$

29–40 ■ Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

29. $f(x) = 2x + 3, \quad g(x) = 4x - 1$

30. $f(x) = 6x - 5, \quad g(x) = \dfrac{x}{2}$

31. $f(x) = x^2, \quad g(x) = x + 1$

32. $f(x) = x^3 + 2, \quad g(x) = \sqrt[3]{x}$

33. $f(x) = \dfrac{1}{x}, \quad g(x) = 2x + 4$

34. $f(x) = x^2, \quad g(x) = \sqrt{x - 3}$

35. $f(x) = |x|, \quad g(x) = 2x + 3$

36. $f(x) = x - 4, \quad g(x) = |x + 4|$

37. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = 2x - 1$

38. $f(x) = \dfrac{1}{\sqrt{x}}, \quad g(x) = x^2 - 4x$

39. $f(x) = \sqrt[3]{x}, \quad g(x) = \sqrt[4]{x}$

40. $f(x) = \dfrac{2}{x}, \quad g(x) = \dfrac{x}{x + 2}$

41–44 ■ Find $f \circ g \circ h$.

41. $f(x) = x - 1, \quad g(x) = \sqrt{x}, \quad h(x) = x - 1$

42. $f(x) = \dfrac{1}{x}, \quad g(x) = x^3, \quad h(x) = x^2 + 2$

43. $f(x) = x^4 + 1, \quad g(x) = x - 5, \quad h(x) = \sqrt{x}$

44. $f(x) = \sqrt{x}, \quad g(x) = \dfrac{x}{x - 1}, \quad h(x) = \sqrt[3]{x}$

45–50 ■ Express the function in the form $f \circ g$.

45. $F(x) = (x - 9)^5$

46. $F(x) = \sqrt{x} + 1$

Answer

local minimum ≈ -0.38 when $x \approx 1.73$ **59.** 25 ft
61. \$4,000, 100 units **63.** 30 times **65.** 50 trees per acre
67. 20 mi/h **69.** $r \approx 0.67$ cm

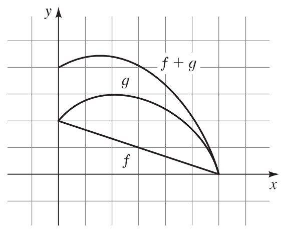
Section 2.6 ■ page 210

1. $A(w) = 3w^2, w > 0$ **3.** $V(w) = \frac{1}{2}w^3, w > 0$
5. $A(x) = 10x - x^2, 0 < x < 10$
7. $A(x) = (\sqrt{3}/4)x^2, x > 0$ **9.** $r(A) = \sqrt{A/\pi}, A > 0$
11. $S(x) = 2x^2 + 240/x, x > 0$ **13.** $D(t) = 25t, t \geq 0$
15. $A(b) = b\sqrt{4 - b}, 0 < b < 4$
17. $A(h) = 2h\sqrt{100 - h^2}, 0 < h < 10$
19. (b) $p(x) = x(19 - x)$ **(c)** 9.5, 9.5 **21.** $-12, -12$
23. (b) $A(x) = x(2400 - 2x)$ **(c)** 600 ft by 1200 ft
25. (a) $f(w) = 8w + 7200/w$ **(b)** Width along road is 30 ft, length is 40 ft **(c)** 15 ft to 60 ft
27. (a) $R(p) = -3000p^2 + 57{,}000p$ **(b)** \$19 **(c)** \$9.50
29. (a) $A(x) = 15x - \left(\dfrac{\pi + 4}{8}\right)x^2$ **(b)** Width $\approx$ 8.40 ft, height of rectangular part $\approx$ 4.20 ft
31. (a) $A(x) = x^2 + 48/x$ **(b)** Height $\approx$ 1.44 ft, width $\approx$ 2.88 ft **33. (a)** $A(x) = 2x + 200/x$
(b) 10 m by 10 m
35. (a) $E(x) = 14\sqrt{25 + x^2} + 10(12 - x)$
(b) To point C, 5.1 mi from point B

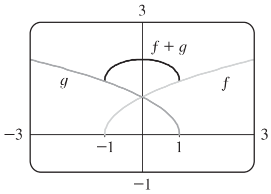
Section 2.7 ■ page 219

1. $(f + g)(x) = x^2 + x - 3, (-\infty, \infty)$;
$(f - g)(x) = -x^2 + x - 3, (-\infty, \infty)$;
$(fg)(x) = x^3 - 3x^2, (-\infty, \infty)$;
$\left(\dfrac{f}{g}\right)(x) = \dfrac{x - 3}{x^2}, (-\infty, 0) \cup (0, \infty)$
3. $(f + g)(x) = \sqrt{4 - x^2} + \sqrt{1 + x}, [-1, 2]$;
$(f - g)(x) = \sqrt{4 - x^2} - \sqrt{1 + x}, [-1, 2]$;
$(fg)(x) = \sqrt{-x^3 - x^2 + 4x + 4}, [-1, 2]$;
$\left(\dfrac{f}{g}\right)(x) = \sqrt{\dfrac{4 - x^2}{1 + x}}, (-1, 2]$
5. $(f + g)(x) = \dfrac{6x + 8}{x^2 + 4x}, x \neq -4, x \neq 0$;
$(f - g)(x) = \dfrac{-2x + 8}{x^2 + 4x}, x \neq -4, x \neq 0$;
$(fg)(x) = \dfrac{8}{x^2 + 4x}, x \neq -4, x \neq 0$;
$\left(\dfrac{f}{g}\right)(x) = \dfrac{x + 4}{2x}, x \neq -4, x \neq 0$
7. $[0, 1]$ **9.** $(3, \infty)$

11.



13.



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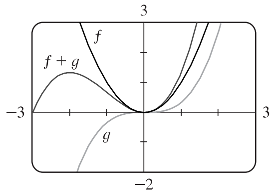


17. (a) 1 **(b)** -23 **19. (a)** -11 **(b)** -119
21. (a) $-3x^2 + 1$ **(b)** $-9x^2 + 30x - 23$
23. 4 **25.** 5 **27.** 4
29. $(f \circ g)(x) = 8x + 1, (-\infty, \infty)$;
$(g \circ f)(x) = 8x + 11, (-\infty, \infty)$;
$(f \circ f)(x) = 4x + 9, (-\infty, \infty)$;
$(g \circ g)(x) = 16x - 5, (-\infty, \infty)$
31. $(f \circ g)(x) = (x + 1)^2, (-\infty, \infty)$;
$(g \circ f)(x) = x^2 + 1, (-\infty, \infty)$; $(f \circ f)(x) = x^4, (-\infty, \infty)$;
$(g \circ g)(x) = x + 2, (-\infty, \infty)$
33. $(f \circ g)(x) = \dfrac{1}{2x + 4}, x \neq -2$; $(g \circ f)(x) = \dfrac{2}{x} + 4, x \neq 0$;
$(f \circ f)(x) = x, x \neq 0$, $(g \circ g)(x) = 4x + 12, (-\infty, \infty)$
35. $(f \circ g)(x) = |2x + 3|, (-\infty, \infty)$;
$(g \circ f)(x) = 2|x| + 3, (-\infty, \infty)$; $(f \circ f)(x) = |x|, (-\infty, \infty)$;
$(g \circ g)(x) = 4x + 9, (-\infty, \infty)$
37. $(f \circ g)(x) = \dfrac{2x - 1}{2x}, x \neq 0$;
$(g \circ f)(x) = \dfrac{2x}{x + 1} - 1, x \neq -1$;
$(f \circ f)(x) = \dfrac{x}{2x + 1}, x \neq -1, x \neq -\frac{1}{2}$;
$(g \circ g)(x) = 4x - 3, (-\infty, \infty)$
39. $(f \circ g)(x) = \sqrt[12]{x}, [0, \infty)$; $(g \circ f)(x) = \sqrt[12]{x}, [0, \infty)$;
$(f \circ f)(x) = \sqrt[9]{x}, (-\infty, \infty)$; $(g \circ g)(x) = \sqrt[16]{x}, [0, \infty)$
41. $(f \circ g \circ h)(x) = \sqrt{x - 1} - 1$
43. $(f \circ g \circ h)(x) = (\sqrt{x} - 5)^4 + 1$
45. $g(x) = x - 9, f(x) = x^5$
47. $g(x) = x^2, f(x) = x/(x + 4)$
49. $g(x) = 1 - x^3, f(x) = |x|$
51. $h(x) = x^2, g(x) = x + 1, f(x) = 1/x$
53. $h(x) = \sqrt[3]{x}, g(x) = 4 + x, f(x) = x^9$
55. $R(x) = 0.15x - 0.000002x^2$

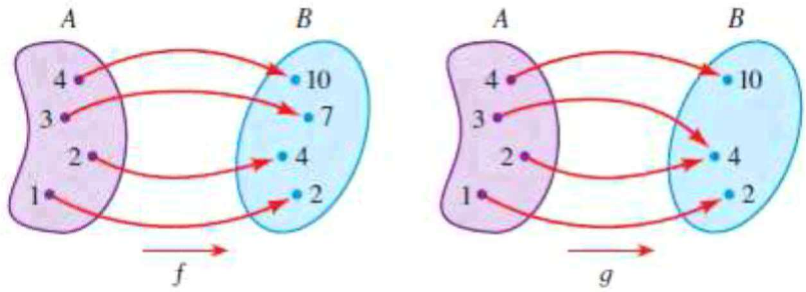
4.6 ONE-TO-ONE FUNCTIONS AND THEIR INVERSES
(Adapted from "Precalculus" by Stewart et als.)

One-to-One Functions ┐ The Inverse of a Function ┐ Graphing the Inverse of a Function

The ***inverse*** of a function is a rule that acts on the output of the function and produces the corresponding input. So the inverse "undoes" or reverses what the function has done. Not all functions have inverses; those that do are called ***one-to-one***.

▼ One-to-One Functions

Let's compare the functions *f* and *g* whose arrow diagrams are shown here.



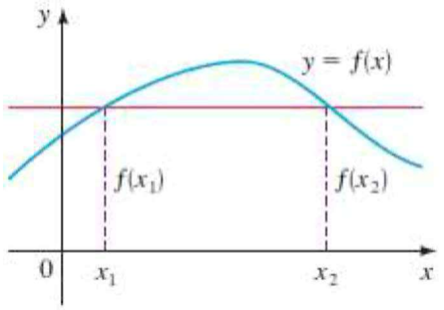
f is one-to-one $\qquad$ *g* is not one-to-one

Note that *f* never takes on the same value twice (any two numbers in *A* have different images), whereas *g* does take on the same value twice (both 2 and 3 have the same image, 4). In symbols, $g(2) = g(3)$ but $f(x_1) \neq f(x_2)$ .whenever $x_1 \neq x_2$.
Functions that have this latter property are called ***one-to-one***.

> **DEFINITION OF A ONE-TO-ONE FUNCTION**
>
> A function with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,
>
> $$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

An equivalent way of writing the condition for a one-to-one function is this:

If $f(x_1) = f(x_2)$, then $x_1 = x_2$.

	If a horizontal line intersects the graph of f at more than one point, then we see from the figure that there are numbers $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. This function is not one-to-one because $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Therefore, we have the following geometric method for determining whether a function is one-to-one.

HORIZONTAL LINE TEST

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

E X A M P L E 1 | Deciding Whether a Function Is One-to-One

Is the function $f(x) = x^3$ one-to-one?

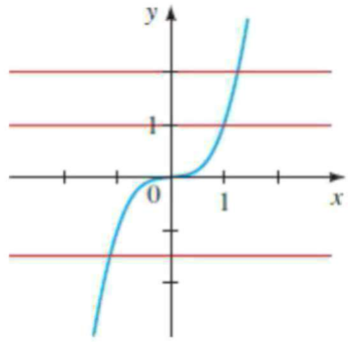
S O LU T I O N 1 (Algebraically)

If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers cannot have the same cube).

Therefore, $f(x) = x^3$ is one-to-one.

[nby: ***The solution here*** is based on the condition "if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$" in the definition; the equivalent condition (the contrapositive) "if $f(x_1) = f(x_2)$, then $x_1 = x_2$" is usually used as seen in Example 3 below.]

S O LU T I O N 2 (From graph)



No horizontal line intersects the graph of $f(x) = x^3$ more than once.

Therefore, by the Horizontal Line Test, f is one-to-one.

***Note*: This example serves to help students understand one-to-one concept; it is based on the graph.**

NOW TRY: Determine whether the function is one-to-one.

$g(x) = \sqrt{x}$

Notice that the function f of Example 1 is increasing and is also one-to-one. In fact, it can be proved that *every increasing function and every decreasing function is one-to-one*.

E X A M P L E 2 | Deciding Whether a Function Is One-to-One

Is the function $g(x) = x^2$ one-to-one?

S O LU T I O N 1 (Algebraically)

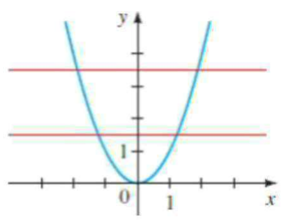
This function is not one-to-one because, for instance,

$g(1) = 1$ and

$g(-1) = 1$

so 1 and -1 have the same image.

S O LU T I O N 2 (From graph)



There are horizontal lines that intersect the graph of g more than once.
Therefore, by the Horizontal Line Test, g is not one-to-one.

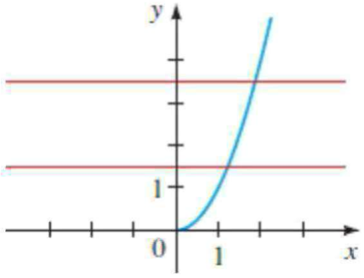
NOW TRY: Determine whether the function is one-to-one.

$h(x) = x^2 - 2x$

Although the function g in Example 2 is not one-to-one, it is possible to restrict its domain so that the resulting function is one-to-one. In fact, if we define

$$h(x) = x^2 \qquad x \geq 0$$

then h is one-to-one, as you can see from the graph and the Horizontal Line Test.



$f(x) = x^2\ (x \geq 0)$ is one-to-one.

[***It's important*** to know what the domain is; you cannot just look at the expression only.]

EXAMPLE 3 | Showing That a Function Is One-to-One

Show that the function $f(x) = 3x + 4$ is one-to-one.

SOLUTION Suppose $f(x_1) = f(x_2)$.

Then $3x_1 + 4 = 3x_2 + 4$

Subtract 4 $\quad 3x_1 = 3x_2$

Divide by 3 $\quad x_1 = x_2$

Therefore, f is one-to-one.

NOW TRY: Determine whether the function is one-to-one.

$f(x) = 2x + 4$

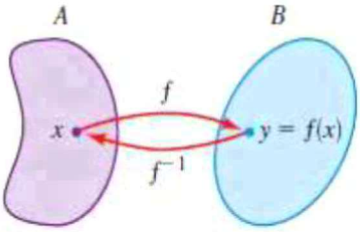
▼ The Inverse of a Function

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

> **DEFINITION OF THE INVERSE OF A FUNCTION**
>
> Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by
>
> $$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$
>
> for any y in B.

This definition says that if f takes x to y, then f^{-1} takes y back to x.

(If f were not one-to-one, then f^{-1} would not be defined uniquely.)

The arrow diagram 

indicates that f^{-1} reverses the effect of f.

From the definition we have

$$\text{domain of } f^{-1} = \text{range of } f$$

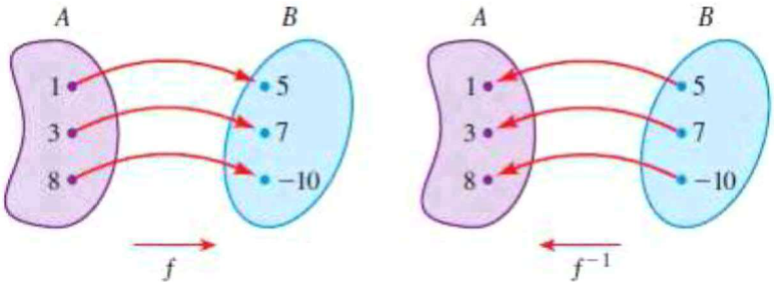
$$\text{range of } f^{-1} = \text{domain of } f$$

Don't mistake the -1 in f^{-1} for an exponent.

$f^{-1}(x)$ *does not mean* $\dfrac{1}{f(x)}$ The reciprocal $\dfrac{1}{f(x)}$ is written as $(f(x))^{-1}$

E X A M P L E 4 | Finding f^{-1} for Specific Values f^{-1}

If $f(1)=5$, $f(3)=7$, and $f(8)=-10$, find $f^{-1}(5)$, $f^{-1}(7)$, and $f^{-1}(10)$.



How f^{-1} reverses the effect of f

S O LU T I O N From the definition of f^{-1} we have

$$f^{-1}(5)=1 \quad \text{because} \quad f(1)=5$$

NOW TRY: Assume that f is a one-to-one function.

(a) If $f(2)=7$, find $f^{-1}(7)$. **(b)** If $f^{-1}(3)=-1$, find $f(-1)$.

By definition the inverse function f^{-1} undoes what f does: If we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started. Similarly, f undoes what f^{-1} does. In general, any function that reverses the effect of f in this way must be the inverse of f. These observations are expressed precisely as follows.

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

These properties indicate that f is the inverse function of f^{-1},
so we say that f and f^{-1} are *inverses of each other*.

EXAMPLE 5 | Verifying That Two Functions Are Inverses

Show that $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$ are inverses of each other.

SOLUTION Note that the domain and range of both f and g is R. We have

$$g(f(x)) = g(x^3) = (x^3)^{\frac{1}{3}} = x$$
$$f(g(x)) = f(x^{\frac{1}{3}}) = (x^{\frac{1}{3}})^3 = x$$

So by the Property of Inverse Functions, f and g are inverses of each other. These equations simply say that the cube function and the cube root function, when composed, cancel each other.

NOW TRY: Use the Inverse Function Property to show that f and g are inverses of each other. $\quad f(x) = 2x - 5;\ \ g(x) = \dfrac{x+5}{2}$

Now let's examine how we compute inverse functions. We first observe from the definition of f^{-1} that

$$y = f(x) \quad \Leftrightarrow \quad f^{-1}(y) = x$$

So if $y = f(x)$ and if we are able to solve this equation for x in terms of y, then we must have $x = f^{-1}(y)$. If we then interchange x and y, we have $y = f^{-1}(x)$, which is the desired equation.

HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

1. Write $y = f(x)$. 2. Solve this equation for x in terms of y (if possible). 3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$	With step 2, you have $x = f^{-1}(y)$

Note that Steps 2 and 3 can be reversed. In other words, we can interchange x and y first and then solve for y in terms of x.

E X A M P L E 6 | Finding the Inverse of a Function

Find the inverse of the function $f(x) = 3x - 2$.

S O L U T I O N

First we write . $y = f(x)$

$y = 3x - 2$

Then we solve this equation for x (in terms of y).

$$y = 3x - 2$$
$$3x = y + 2$$
$$x = \frac{y+2}{3}$$

← *This step gives you* $x = f^{-1}(y)$

Finally, we interchange x and y.

$$y = \frac{x+2}{3}$$

← *You now have* $y = f^{-1}(x)$

Therefore, the inverse function is .

$$f^{-1}(x) = \frac{x+2}{3}$$

CHECK YOUR ANSWER

We use the Inverse Function Property.

$$f^{-1}(f(x)) = f^{-1}(3x - 2) = \frac{(3x-2)+2}{3} = \frac{3x}{3} = x$$

$$f(f^{-1}(x)) = f\left(\frac{x+2}{3}\right) = 3\left(\frac{x+2}{3}\right) - 2 = x + 2 - 2 = x \checkmark$$

NOW TRY: Find the inverse function of f. $f(x) = 2x + 5$

E X A M P L E 7 | Finding the Inverse of a Function

Find the inverse of the function $f(x) = \dfrac{x^5 - 3}{2}$.

S O L U T I O N

We first write $y = \dfrac{x^5 - 3}{2}$ and solve for x.

$$x^5 = 2y + 3$$
$$x = (2y+3)^{\frac{1}{5}}$$

Then we interchange x and y to get

$y = (2x+3)^{\frac{1}{5}}$.

Therefore, the inverse function is

$$f^{-1}(x) = (2x+3)^{\frac{1}{5}}$$

CHECK YOUR ANSWER

We use the Inverse Function Property.

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x^5-3}{2}\right) = \left[2\left(\frac{x^5-3}{2}\right) + 3\right]^{1/5} = (x^5 - 3 + 3)^{1/5} = (x^5)^{1/5} = x$$

$$f(f^{-1}(x)) = f((2x+3)^{1/5}) = \frac{[(2x+3)^{1/5}]^5 - 3}{2} = \frac{2x + 3 - 3}{2} = \frac{2x}{2} = x \checkmark$$

NOW TRY: Find the inverse function of f. $f(x) = 4 - x^2, \quad x \geq 0$

A **rational function** is a function defined by a rational expression. In the next example we find the inverse of a rational function.

E X A M P L E 8 | Finding the Inverse of a Rational Function

Find the inverse of the function $f(x)=\dfrac{2x+3}{x-1}$.

S O LU T I O N We first write $y=\dfrac{2x+3}{x-1}$ and solve for x.

$$y = \frac{2x+3}{x-1} \qquad \text{Equation defining function}$$

$$y(x-1) = 2x+3 \qquad \text{Multiply by } x-1$$

$$yx - y = 2x+3 \qquad \text{Expand}$$

$$yx - 2x = y+3 \qquad \text{Bring } x\text{-terms to LHS}$$

$$x(y-2) = y+3 \qquad \text{Factor } x$$

$$x = \frac{y+3}{y-2} \qquad \text{Divide by } y-2$$

(Having solved for x in terms of y, interchange x and y. The interchange step is skipped here.)

Therefore the inverse function is $f^{-1}(x)=\dfrac{x+3}{x-2}$.

NOW TRY: Find the inverse function of f. $f(x)=\dfrac{x}{x+4}$

▼ Graphing the Inverse of a Function

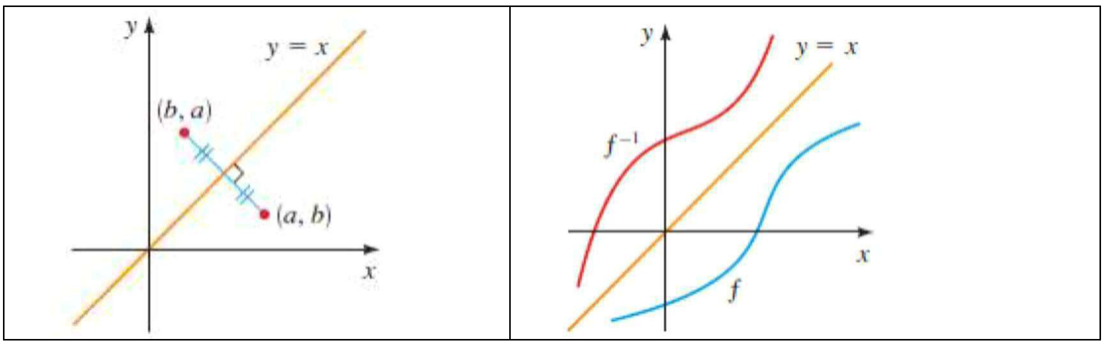
The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f.

If $f(a)=b$, then $f^{-1}(b)=a$.

Thus, the point (a,b) is on the graph of f

if and only if the point (b,a) is on the graph of f^{-1}.

But we get the point (b,a) from the point (a,b) by reflecting in the line $y=x$.



The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

E X A M P L E 9 | Graphing the Inverse of a Function

(a) Sketch the graph of $f(x) = \sqrt{x-2}$.

(b) Use the graph of *f* to sketch the graph of f^{-1}.

(c) Find an equation for f^{-1}.

S O LU T I O N

(a) (Using transformation)

We sketch the graph of $y = \sqrt{x-2}$ by plotting(*or* sketching) the graph of the function $y = \sqrt{x}$ and moving it to the right 2 units

(c) Solve

$y = \sqrt{x-2}$ for *x*, noting that $y \geq 0$.

$$\sqrt{x-2} = y$$

Square each side $\quad x - 2 = y^2$

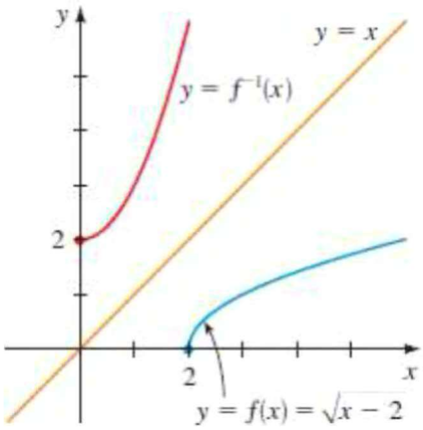
Add 2 $\quad x = y^2 + 2 \quad y \geq 0$

Interchange *x* and *y*: $\quad y = x^2 + 2 \quad x \geq 0$

Thus $\quad f^{-1}(x) = x^2 + 2 \quad x \geq 0$

This expression shows that the graph of f^{-1} is the right half of the parabola $y = x^2 + 2$, and from the graph shown in (b), this seems reasonable.

(b) The graph of f^{-1} is obtained from the graph of *f* in part (a) by reflecting it in the line $y = x$.



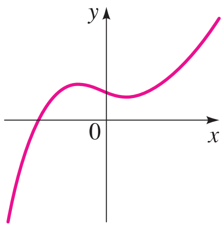
NOW TRY A function f is given. **(a)** Sketch the graph of *f*. **(b)** Use the graph of *f* to sketch the graph of f^{-1}. **(c)** Find f^{-1}

$$f(x) = \sqrt{x+1}$$

Exercise

1–6 ■ The graph of a function f is given. Determine whether f is one-to-one.

1.



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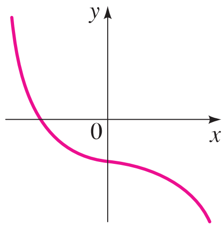

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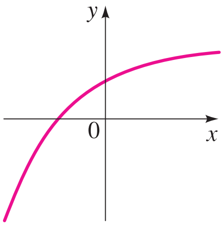

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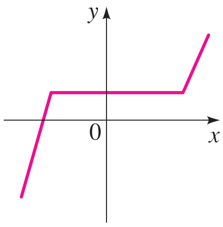

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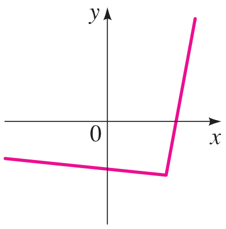
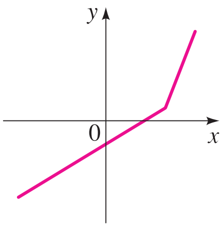


6.



7–16 ■ Determine whether the function is one-to-one.

7. $f(x) = -2x + 4$

8. $f(x) = 3x - 2$

9. $g(x) = \sqrt{x}$

10. $g(x) = |x|$

11. $h(x) = x^2 - 2x$

12. $h(x) = x^3 + 8$

13. $f(x) = x^4 + 5$

14. $f(x) = x^4 + 5, \quad 0 \le x \le 2$

15. $f(x) = \dfrac{1}{x^2}$

16. $f(x) = \dfrac{1}{x}$

17–18 ■ Assume f is a one-to-one function.

17. **(a)** If $f(2) = 7$, find $f^{-1}(7)$.
(b) If $f^{-1}(3) = -1$, find $f(-1)$.

18. **(a)** If $f(5) = 18$, find $f^{-1}(18)$.
(b) If $f^{-1}(4) = 2$, find $f(2)$.

19. If $f(x) = 5 - 2x$, find $f^{-1}(3)$.

20. If $g(x) = x^2 + 4x$ with $x \ge -2$, find $g^{-1}(5)$.

21–30 ■ Use the Inverse Function Property to show that f and g are inverses of each other.

21. $f(x) = x - 6, \quad g(x) = x + 6$

22. $f(x) = 3x, \quad g(x) = \dfrac{x}{3}$

23. $f(x) = 2x - 5; \quad g(x) = \dfrac{x + 5}{2}$

24. $f(x) = \dfrac{3 - x}{4}; \quad g(x) = 3 - 4x$

25. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

26. $f(x) = x^5, \quad g(x) = \sqrt[5]{x}$

27. $f(x) = x^2 - 4, \quad x \ge 0;$
$g(x) = \sqrt{x + 4}, \quad x \ge -4$

28. $f(x) = x^3 + 1; \quad g(x) = (x - 1)^{1/3}$

29. $f(x) = \dfrac{1}{x - 1}, \quad x \neq 1;$

$g(x) = \dfrac{1}{x} + 1, \quad x \neq 0$

30. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2;$

$g(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

31–50 ■ Find the inverse function of f.

31. $f(x) = 2x + 1$

32. $f(x) = 6 - x$

33. $f(x) = 4x + 7$

34. $f(x) = 3 - 5x$

35. $f(x) = \dfrac{x}{2}$

36. $f(x) = \dfrac{1}{x^2}, \quad x > 0$

37. $f(x) = \dfrac{1}{x + 2}$

38. $f(x) = \dfrac{x - 2}{x + 2}$

39. $f(x) = \dfrac{1 + 3x}{5 - 2x}$

40. $f(x) = 5 - 4x^3$

41. $f(x) = \sqrt{2 + 5x}$

42. $f(x) = x^2 + x, \quad x \ge -\frac{1}{2}$

43. $f(x) = 4 - x^2, \quad x \ge 0$

44. $f(x) = \sqrt{2x - 1}$

45. $f(x) = 4 + \sqrt[3]{x}$

46. $f(x) = (2 - x^3)^5$

47. $f(x) = 1 + \sqrt{1 + x}$

48. $f(x) = \sqrt{9 - x^2}, \quad 0 \le x \le 3$

49. $f(x) = x^4, \quad x \ge 0$

50. $f(x) = 1 - x^3$

51–54 ■ A function f is given.

(a) Sketch the graph of f.

(b) Use the graph of f to sketch the graph of f^{-1}.

(c) Find f^{-1}.

51. $f(x) = 3x - 6$

52. $f(x) = 16 - x^2, \quad x \ge 0$

53. $f(x) = \sqrt{x + 1}$

54. $f(x) = x^3 - 1$

55–60 ■ Draw the graph of f and use it to determine whether the function is one-to-one.

55. $f(x) = x^3 - x$

56. $f(x) = x^3 + x$

57. $f(x) = \dfrac{x + 12}{x - 6}$

58. $f(x) = \sqrt{x^3 - 4x + 1}$

59. $f(x) = |x| - |x - 6|$

60. $f(x) = x \cdot |x|$

61–64 ■ A one-to-one function is given.

(a) Find the inverse of the function.

(b) Graph both the function and its inverse on the same screen to verify that the graphs are reflections of each other in the line $y = x$.

61. $f(x) = 2 + x$

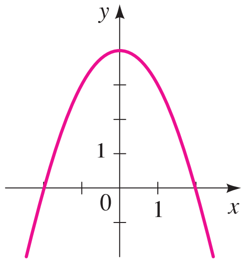
62. $f(x) = 2 - \frac{1}{2}x$

63. $g(x) = \sqrt{x + 3}$

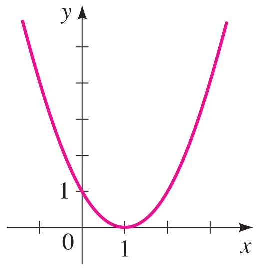
64. $g(x) = x^2 + 1, \quad x \ge 0$

65–68 ■ The given function is not one-to-one. Restrict its domain so that the resulting function *is* one-to-one. Find the inverse of the function with the restricted domain. (There is more than one correct answer.)

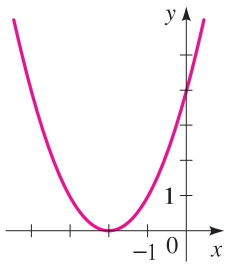
65. $f(x) = 4 - x^2$



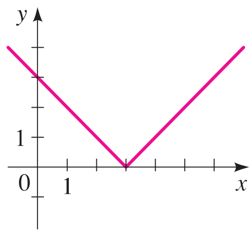
66. $g(x) = (x - 1)^2$



67. $h(x) = (x + 2)^2$

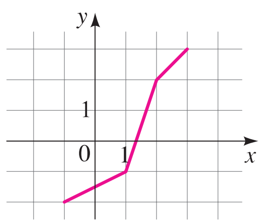


68. $k(x) = |x - 3|$

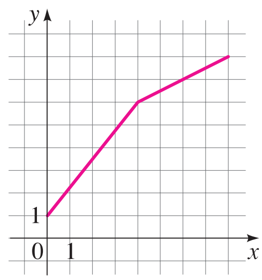


69–70 ■ Use the graph of f to sketch the graph of f^{-1}.

69.



70.



Applications

71. Fee for Service For his services, a private investigator requires a $500 retention fee plus $80 per hour. Let x represent the number of hours the investigator spends working on a case.

(a) Find a function f that models the investigator's fee as a function of x.

(b) Find f^{-1}. What does f^{-1} represent?

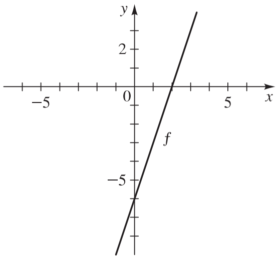
(c) Find $f^{-1}(1220)$. What does your answer represent?

Answer

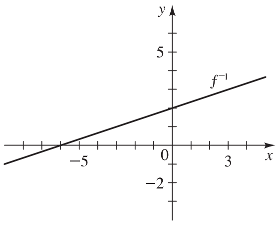
57. (a) $g(t) = 60t$ **(b)** $f(r) = \pi r^2$
(c) $(f \circ g)(t) = 3600\pi t^2$ **59.** $A(t) = 16\pi t^2$
61. (a) $f(x) = 0.9x$ **(b)** $g(x) = x - 100$
(c) $f \circ g(x) = 0.9x - 90$, $g \circ f(x) = 0.9x - 100$, $f \circ g$: first rebate, then discount, $g \circ f$: first discount, then rebate, $g \circ f$ is the better deal

Section 2.8 ■ page 230

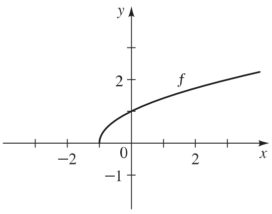
1. No **3.** Yes **5.** No **7.** Yes **9.** Yes **11.** No
13. No **15.** No **17. (a)** 2 **(b)** 3 **19.** 1
31. $f^{-1}(x) = \frac{1}{2}(x - 1)$ **33.** $f^{-1}(x) = \frac{1}{4}(x - 7)$
35. $f^{-1}(x) = 2x$ **37.** $f^{-1}(x) = (1/x) - 2$
39. $f^{-1}(x) = (5x - 1)/(2x + 3)$
41. $f^{-1}(x) = \frac{1}{5}(x^2 - 2), x \geq 0$
43. $f^{-1}(x) = \sqrt{4 - x}, x \leq 4$ **45.** $f^{-1}(x) = (x - 4)^3$
47. $f^{-1}(x) = x^2 - 2x, x \geq 1$ **49.** $f^{-1}(x) = \sqrt[4]{x}$
51. (a)



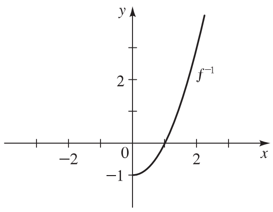
(b)



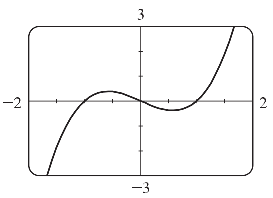
(c) $f^{-1}(x) = \frac{1}{3}(x + 6)$
53. (a)



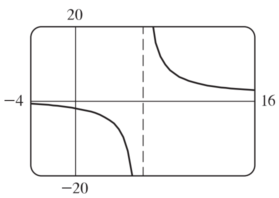
(b)



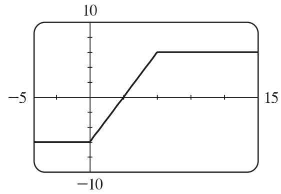
(c) $f^{-1}(x) = x^2 - 1, x \geq 0$
55. Not one-to-one



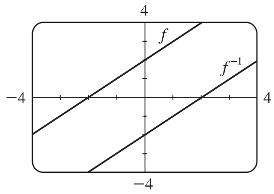
57. One-to-one



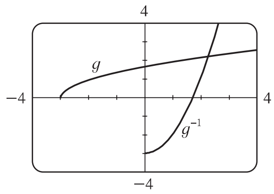
59. Not one-to-one



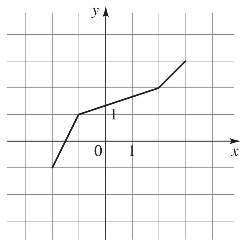
61. (a) $f^{-1}(x) = x - 2$
(b)



63. (a) $g^{-1}(x) = x^2 - 3, x \geq 0$
(b)



65. $x \geq 0$, $f^{-1}(x) = \sqrt{4 - x}$
67. $x \geq -2$, $h^{-1}(x) = \sqrt{x} - 2$
69.



71. (a) $f(x) = 500 + 80x$ **(b)** $f^{-1}(x) = \frac{1}{80}(x - 500)$, the number of hours worked as a function of the fee **(c)** 9; if he charges \$1220, he worked 9 h
73. (a) $v^{-1}(t) = \sqrt{0.25 - \dfrac{t}{18{,}500}}$ **(b)** 0.498; at a distance 0.498 from the central axis, the velocity is 30
75. (a) $F^{-1}(x) = \frac{5}{9}(x - 32)$; the Celsius temperature when the Fahrenheit temperature is x **(b)** $F^{-1}(86) = 30$; when the temperature is 86°F, it is 30°C
77. (a) $f(x) = \begin{cases} 0.1x & \text{if } 0 \leq x \leq 20{,}000 \\ 2000 + 0.2(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$

(b) $f^{-1}(x) = \begin{cases} 10x & \text{if } 0 \leq x \leq 2000 \\ 10{,}000 + 5x & \text{if } x > 2000 \end{cases}$
If you pay x euros in taxes, your income is $f^{-1}(x)$.
(c) $f^{-1}(10{,}000) = 60{,}000$ **79.** $f^{-1}(x) = \frac{1}{2}(x - 7)$. A pizza costing x dollars has $f^{-1}(x)$ toppings.