CHAPTER 5

INTEGRATION

LECTURE 19 – 13.1.2023

- ü Indefinite Integral
- ü Integration Rules
- ü Technique of Integration
- ü Definite Integral
- ü Application: Areas & Volumes

A. Indefinite Integral

Antiderivative

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

The process of finding antiderivatives is called **integration** where we denote the integration process by $\int f(x)dx = F(x) + C$

$\int$ is called integral sign

$f(x)$ is called the integrand

$F(x)$ is the antiderivative of f

C is a constant of integration.

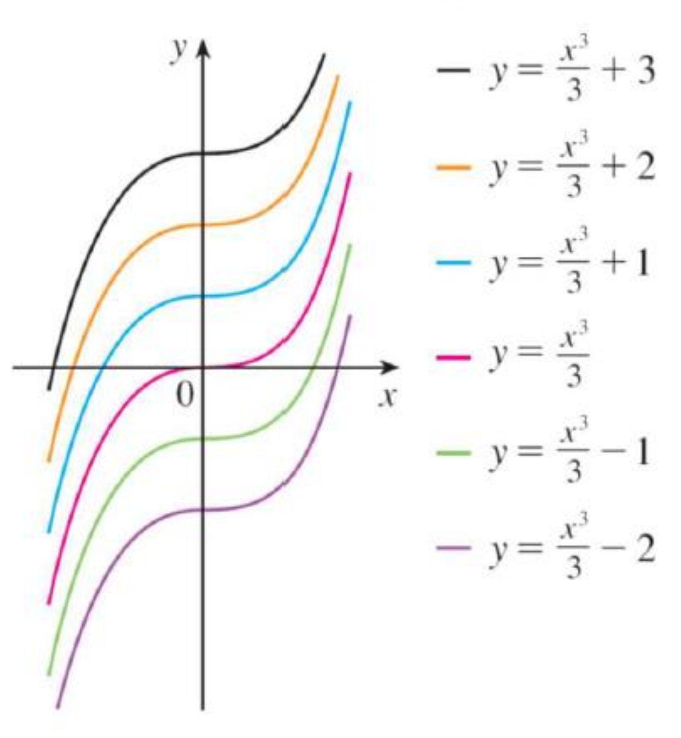
Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

For example, we can write $\int x^2\,dx = \frac{x^3}{3} + C$ because $\frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$

$\int \sec^2 x\,dx = \tan x + C$ because $\frac{d}{dx}(\tan x + C) = \sec^2 x$



The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Note: The Fundamental Theorem of Calculus (Part 2) will be discussed in Definite Integrals.

$$g(x) = \int_a^x f(t)\,dt$$
$$= \Big[F(t) + c\Big]_a^x$$
$$= [F(x)+c] - [F(a)+c]$$
$$= F(x) - F(a)$$
$$\therefore g'(x) = F'(x) - F'(a)$$
$$= f(x) - 0$$
$$= f(x)$$

Example: The Fundamental Theorem of Calculus (Part 1)

Find the derivative of the functions.

	SOLUTION
(a) $g(x) = \int_0^x (1+t^2)\,dt$	$g'(x) = \dfrac{d}{dx}\int_0^x (1+t^2)\,dt = 1 + x^2$
(b) $g(x) = \int_4^x te^t\,dt$	$g'(x) = \dfrac{d}{dx}\int_4^x te^t\,dt = xe^x$
(c) $g(x) = \int_1^{x^4} \sin t\,dt$	Let $u = x^4$, then $g'(x) = \dfrac{d}{dx}\int_1^u \sin t\,dt$ $= \dfrac{d}{du}\left[\int_1^u \sin t\,dt\right]\dfrac{du}{dx} = \sin u\,\dfrac{du}{dx} = \sin(u)4x^3 = 4x^3\sin(x^4)$

B. Integration Rules and Formula

Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively, and k is an arbitrary number. Then:

1. Constant Multiple Rule:	$\int kf(x)dx = k\int f(x)dx = kF(x) + C$
2. Rule for Negatives:	$\int -f(x)dx = -\int f(x)dx = -F(x) + C$
3. Sum and Difference Rule:	$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx = F(x) \pm G(x) + C$

The formulas of integration come from basic differentiation.

	Differentiation Formulas	*Integration Formulas*
1.	$\frac{d}{dx}[x^n] = nx^{n-1}, \quad n \in \Re$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
2.	$\frac{d}{dx}[kx] = k$	$\int k.dx = kx + C$
3.	$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\, dx = \sin x + C$
4.	$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x.dx = -\cos x + C$
5.	$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$

6.	$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x \; dx = \sec x + C$
7	$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x \; dx = -\cot x + C$
8.	$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x \; dx = -\csc x + C$
9.	$\frac{d}{dx}[e^x] = e^x$	$\int e^x \, dx = e^x + C$
10.	$\frac{d}{dx}[\ln x] = \frac{1}{x}; \quad x > 0 \quad ;$ $\frac{d}{dx} \ln\lvert x\rvert = \frac{1}{x}, \qquad x \neq 0$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$

NOTE Note that the Power Rule for Integration has the restriction that $n \neq -1$.

Example: Rewriting the Integrals Before Integrating

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
$\int \sqrt{x}\,dx$	$\int x^{1/2}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
$\int 2\sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2\cos x + C$

Example: Use the basic integration formulas to integrate the following functions.

Function	*Integration* *Rewrite the Integrals Before Integrating*
$f(x) = x^8 + 2$	$\int (x^8 + 2)\,dx = \frac{x^9}{9} + 2x + c$
$f(x) = x^{-2}$	$\int x^{-2}\,dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$
$f(x) = 2x^{2/3}$	$\int 2x^{2/3}\,dx = 2\left(\frac{x^{5/3}}{\frac{5}{3}}\right) + c = \frac{6}{5}x^{5/3} + c$
* $f(x) = \frac{1}{2x^2}$	$\int \frac{1}{2}x^{-2}\,dx = \frac{1}{2}\left(-\frac{1}{x}\right) + C = -\frac{1}{2x} + C$
* $f(x) = \sqrt[3]{x}$	$\int x^{\frac{1}{3}}\,dx = \frac{3}{4}x^{\frac{4}{3}} + C$
* $f(x) = x^3\sqrt{x}$	$\int x^3 \cdot x^{\frac{1}{2}}\,dx = \int x^{\frac{7}{2}}\,dx = \frac{2}{9}x^{\frac{9}{2}} + C$
$f(x) = 1$	$\int 1\,dx = x + C$
$f(x) = 3x^3 + 2x - 50$	$\int 3x^3 + 2x - 50\,dx = \frac{3}{4}x^4 + x^2 - 50x + C$

* $f(x)=\dfrac{x^5+2x^2-1}{x^4}$	$\int x+2x^{-2}-x^{-4}\,dx = \frac{1}{2}x^2-\frac{2}{x}+\frac{1}{3x^3}+C$
$f(x)=-7\sin x+2\cos x$	$\int -7\sin x+2\cos x\,dx = 7\cos x+2\sin x+C$
$f(x)=10e^x+x$	$\int 10e^x+x\,dx = 10e^x+\frac{1}{2}x^2+C$
$f(x)=\dfrac{3}{x}$	$\int \frac{3}{x}\,dx = 3\ln\lvert x\rvert+C$

C. Techniques of Integration

In this section, we will discuss three techniques of integration: (i) Integration by Substitution (ii) Integration by Parts, and (iii) Integration by Partial Fractions

<u>INTEGRATION BY SUBSTITUTION</u>

Integration by Substitution is used to transform complicated integration problems into simpler ones that is easier to integrate.

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then $\int \underbrace{f(g(x))}_{f(u)} \, . \underbrace{g'(x)dx}_{du} = \int f(u)du$

Example: Find $\int 3x^2(x^3-5)^7\,dx$.

SOLUTION

Let $u = x^3 - 5$, $\rightarrow \dfrac{du}{dx} = 3x^2$, or $du = 3x^2dx$

Using the Substitution Rule, we have

$$\int \underbrace{(x^3-5)^7}_{u^7}\ \underbrace{3x^2\,dx}_{du} = \int u^7\,du$$

Integral in terms of u

These steps are required

$$= \frac{u^8}{8} + C$$

Antiderivative in terms of u

$$= \frac{\left(x^3-5\right)^8}{8} + C$$

Antiderivative in terms of x

Example: Find $\int x\sqrt{x^2+4}\,dx$

SOLUTION

Let $u = x^2 + 4$, $\rightarrow \dfrac{du}{dx} = 2x$, or $xdx = \dfrac{du}{2}$

$$\int x\sqrt{x^2+4}\,dx = \int \underbrace{(x^2+4)^{1/2}}_{u^{1/2}} \underbrace{xdx}_{du/2}$$

*These steps are required

$$= \frac{1}{2}\int u^{1/2}\,du$$

Integral in terms of u

$$= \frac{1}{2}\left(\frac{2}{3}\right)u^{3/2} + c$$

Antiderivative in terms of u

$$= \frac{1}{3}\left(x^2+4\right)^{3/2} + c$$

Simplify.

Antiderivative in terms of x

Example: Find the integrals.

(a) $\int e^x\left(e^x+1\right)^{-2}dx$

SOLUTION

Let $u=e^x+1$, $\frac{du}{dx}=e^x$ or $du=e^x dx$

$$\int \underbrace{(1+e^x)^{-2}}_{u^{-2}}\ \underbrace{e^x dx}_{\text{du}} = \int u^{-2}\ du$$

$$= \frac{u^{-1}}{-1}+c$$

$$= -\frac{1}{u}+c$$

$$= -\frac{1}{e^x+1}+c$$

(b) $\int e^{3x}dx$

SOLUTION

Let $u=3x$, $\rightarrow \frac{du}{dx}=3$ or $\frac{du}{3}=dx$

$$\int e^{3x}dx = \int e^u\left(\frac{du}{3}\right)$$

$$= \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u+c$$

$$= \frac{1}{3}e^{3x}+c$$

(c) $\int \sin 4x \, dx$.

SOLUTION

Let $u = 4x$, $du = 4dx$ or $dx = \frac{1}{4} du$

$$\int \sin(\underbrace{4x}_{u}) \, dx = \int \sin u \left(\frac{du}{4}\right)$$

$$= \frac{1}{4} \int \sin u \, du$$

$$= -\frac{1}{4} \cos u + C$$

$$= -\frac{1}{4} \cos 4x + c$$

(d) $\int \cos 5x \, dx$.

SOLUTION Let $u = 5x$ $\frac{du}{dx} = 5 \Rightarrow \frac{du}{5} = dx$

$$\int \cos 5x \, dx$$

$$= \int \cos u \left(\frac{du}{5}\right)$$

$$= \frac{1}{5} \int \cos u \, du = \frac{1}{5} \sin u + C$$

$$= \frac{1}{5} \sin 5x + c$$

****Note*: *For integrals in the form* $\int e^{kx}\,dx$, $\int \cos kx\,dx$, $\int \sin kx\,dx$ *and* $\int \sec^2 kx\,dx$, *we can write down the final answer fast without showing the detail steps. For e.g.,*

$$\int \sin 2x\,dx = -\frac{\cos 2x}{2} + c,\quad \int \cos\left(\frac{1}{2}x\right)dx = 2\sin\left(\frac{1}{2}x\right) + c,\quad \int \sec^2 2x\,dx = \frac{\tan 2x}{2} + c$$

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} + c,\quad \int e^{\frac{1}{2}x}\,dx = 2e^{\frac{1}{2}x} + c,\quad \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + c$$

Example: Evaluate the indefinite integral by using The Substitution Rule.

(a) $\int (x+3)\left(\frac{1}{2}x^2+3x\right)^5 dx$

Let $u = \frac{1}{2}x^2 + 3x$

$\frac{du}{dx} = x+3 \Rightarrow du = (x+3)dx$

$= \int u^5 \, du$

$= \frac{u^6}{6} + C$

$= \frac{1}{6}\left(\frac{1}{2}x^2 + 3x\right)^6 + C$

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Example: Evaluate the indefinite integral by using The Substitution Rule.

(b) $\int (2x-1)^{-3}\,dx$

Let $u = 2x-1$

$\frac{du}{dx} = 2 \Rightarrow \frac{du}{2} = dx$

$= \int u^{-3}\left(\frac{du}{2}\right)$

$= \frac{1}{2}\int u^{-3}\,du$

$= \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) + C$

$= -\frac{1}{4}u^{-2} + C = -\frac{1}{4(2x-1)^2} + C$ #

Example: Evaluate the indefinite integral by using The Substitution Rule.

(c) $\int 8x^3(2x^4-1)^{1/2}\,dx$

Let $u = 2x^4 - 1$

$\frac{du}{dx} = 8x^3 \Rightarrow du = 8x^3\,dx$

$= \int u^{\frac{1}{2}}\,du$

$= \frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C$

$= \frac{2}{3}(2x^4-1)^{\frac{3}{2}} + C$

Example: Evaluate the indefinite integral by using The Substitution Rule.

(d) $\int e^x \sqrt{e^x + 4}\, dx = \int e^x \left(e^x + 4\right)^{1/2} dx$

Let $u = e^x + 4$

$\frac{du}{dx} = e^x \Rightarrow du = e^x\, dx$

$= \int u^{\frac{1}{2}}\, du$

$= \frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C$

$= \frac{2}{3}\left(e^x+4\right)^{\frac{3}{2}} + C$ #

Example: Evaluate the indefinite integral by using The Substitution Rule.

(e) $\int \frac{1}{x+3} dx$ $= \int \frac{1}{u} dx$

Let $u = x+3$

$\frac{du}{dx} = 1 \Rightarrow du = dx$

$= \ln|u| + C$

$= \ln|x+3| + C$ #

Example: Evaluate the indefinite integral by using The Substitution Rule.

(f) $\int \frac{1}{(x+3)^2}dx = \int \frac{1}{u^2}\,du$

$= \int u^{-2}\,du$

$= \frac{u^{-1}}{-1} + C$

$= -\frac{1}{x+3} + C$ #

Let $u = x+3$

$\frac{du}{dx} = 1 \Rightarrow du = dx$

Example: Find $\int \sin x \cos x dx$.

SOLUTION

Let $u = \sin x$, then we have $du = \cos x \; dx$

$$\int \underbrace{\sin x}_{u} \; \underbrace{\cos x dx}_{du} = \int u du \qquad \text{Integral in terms of } u$$

$$= \frac{u^2}{2} + c \qquad \text{Antiderivative in terms of } u$$

$$= \frac{(\sin x)^2}{2} + c = \frac{\sin^2 x}{2} + c \qquad \text{Antiderivative in terms of } x$$

Try: For the same integral above, $\int \sin x \cos x dx$, use substitution technique with $u = \cos x$. Can you get the same answer as given above?

Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$\Rightarrow \frac{du}{-1} = \sin x \, dx$

$\int \sin x \cos x \, dx = \int u \, (-du)$

$= -\int u \, du$

$= -\frac{u^2}{2} + C = -\frac{\cos^2 x}{2} + C$ #

By trigonometric identity

$\sin^2 x + \cos^2 x = 1$

$\therefore \sin^2 x - 1 = -\cos^2 x$

$-\frac{\cos^2 x}{2} + C$

$= \frac{\sin^2 x - 1}{2} + C$

$= \frac{\sin^2 x}{2} \underbrace{- \frac{1}{2} + C}$

$= \frac{\sin^2 x}{2} + C$

- **What about this example ?**

Let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$\frac{du}{-1} = \sin x \, dx$$

Can you solve this ?

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

$$= \int \frac{1}{u} (-du)$$

$$= -\int \frac{1}{u} du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C \quad \#$$

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INTEGRATION BY PARTS

If function $u(x)$ and $v(x)$ are differentiable functions, then we have

$$\begin{aligned}\frac{d}{dx}[uv] &= u\frac{dv}{dx} + v\frac{du}{dx} = uv' + vu' \\ uv &= \int uv'\,dx + \int vu'\,dx \\ &= \int u\,dv + \int v\,du.\end{aligned}$$

By rewriting this equation, we obtain the following theorem.

THEOREM Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du.$$

Guidelines for Integration by Parts

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
2. Try letting u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.

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Example: Evaluate the integrals

(a) $\int x \ln x \, dx = \frac{x^2}{2}\ln x - \int \left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right) dx$

Let $u = \ln x$ $\quad dv = x$

$du = \frac{1}{x}$ $\quad v = \frac{x^2}{2}$

$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\, dx$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

Example: Evaluate the integrals

(b) $\int xe^x dx$. $= xe^x - \int 1.e^x dx$

$= xe^x - \int e^x dx$

$= xe^x - e^x + C$

Let $u = x$ $\quad dv = e^x$

$du = 1$ $\quad v = e^x$

Example: Evaluate the integrals

Let $u = x$ $\quad dv = e^{2x}$

$du = 1$ $\quad v = \frac{1}{2}e^{2x}$

(c) $\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \int 1.\left(\frac{1}{2}e^{2x}\right)dx$

$= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$

$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$ #

$= \frac{1}{4}e^{2x}(2x-1) + C$ #

Example: Evaluate the integrals

Let $u = x^2$ $dv = e^x$

$du = 2x$ $v = e^x$

(d) $\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$

$= x^2 e^x - \left[2x e^x - \int 2e^x dx \right]$

Let $u = 2x$ $dv = e^x$

$du = 2$ $v = e^x$

$= x^2 e^x - 2xe^x + 2e^x + C$ #

Example: Evaluate the integrals

(e) $\int 2x\cos x\, dx$.

Let $u = 2x$ $\quad dv = \cos x$

$du = x$ $\quad v = \sin x$

$$= 2x \sin x - \int x \sin x \, dx$$

Let $u = x$ $\quad dv = \sin x$

$du = 1$ $\quad v = -\cos x$

$$= 2x \sin x - \left[-x \cos x - \int -\cos x \, dx \right]$$

$$= 2x \sin x + x \cos x - \sin x + C$$

Example: Evaluate the integrals

(f) $\int x\sin(4x)\,dx$

Let $u = x$ $\quad dv = \sin 4x$

$du = 1$ $\quad v = -\frac{\cos 4x}{4}$

$$= -\frac{1}{4}x\cos 4x - \int -\frac{\cos 4x}{4}\,dx$$

$$= -\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C \;\#$$

- **Another example:**

Let $u = \ln x \qquad dv = 1$

$du = \frac{1}{x} \qquad v = x$

$$\int \ln x \, dx = \int 1 . \ln x \, dx$$

$$= x \ln x - \int x \left(\frac{1}{x}\right) dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

- **Another example:**

$$\int e^x \sin x \, dx$$

Let $u = \sin x$ $\quad dv = e^x$

$du = \cos x$ $\quad v = e^x$

$= e^x \sin x - \int e^x \cos x \, dx$

Let $u = \cos x$ $\quad dv = e^x$

$du = -\sin x$ $\quad v = e^x$

$= e^x \sin x - \left[e^x \cos x - \int -e^x \sin x \, dx \right]$

$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right]$

$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

$\therefore \int e^x \sin x \, dx + \int e^x \sin x \, dx = e^x (\sin x - \cos x)$

$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x)$

$\therefore \int e^x \sin x \, dx = \frac{e^x}{2} (\sin x - \cos x) + C$ #

INTEGRATION BY PARTIAL FRACTIONS

For a rational function, which is ratio of two polynomials, we can decompose the function into a sum of simple rational expressions.

For example: $$\frac{5x-10}{(x-4)(x+4)} = \boxed{\frac{2}{x-4}+\frac{3}{x+1}}$$ *sum of simpler rational expressions*

Given any rational function $P(x)/Q(x)$, to decompose it into partial fractions, the denominator need to be completely factorize into linear and/or irreducible quadratic factors.

If degree of P<degree of Q then it is called <u>proper rational</u> function.
If degree of $P \geq$ degree of Q then it is called <u>improper rational</u> function.

Types of Partial Fractions

1. Denominator with **Linear Factor**

$$\frac{A}{ax+b}$$

2. Denominator with **Quadratic Factor**

$$\frac{Ax+B}{ax^2+bx+c}$$

3. Denominator with **Repeated Linear Factors**

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

***For this course, we shall consider only the case where f is a __proper rational function__ and Q(x) is a __product of distinct linear factors__*

Example: Evaluate the integral $\int \frac{1}{x^2-4}\,dx$

SOLUTION

First, write as a sum of rational expressions: $\frac{1}{x^2-4} = \frac{1}{(x+2)(x-2)}$

$$\frac{1}{(x+2)(x-2)} = \frac{A}{x-2} + \frac{B}{x+2},$$ *then multiply both sides by (x+2)(x-2)*

$$\therefore 1 = A(x+2) + B(x-2)$$

Find the values of A and B:

Method 1: *(substitute different values of x)*	Method 2: *(by equating coefficients)*
$1 = A(x+2) + B(x-2)$ when x=2: $1 = A(2+2) + B(2-2) = 4A$ $A = 1/4$ when x= -2: $1 = A(x+2) + B(x-2)$ $1 = A(-2+2) + B(-2-2) = -4B$ $B = -1/4$	$1 = A(x+2) + B(x-2)$ $1 = Ax + 2A + Bx - 2B$ $1 = x(A+B) + 2A - 2B$ Equating coefficients: $\left.\begin{array}{r} 2A-2B=1 \\ A+B=0 \end{array}\right\}$ --> $A = \frac{1}{4}$, $B = -\frac{1}{4}$

We have shown that $\dfrac{1}{x^2-4} = \dfrac{A}{x-2} + \dfrac{B}{x+2} = \dfrac{1/4}{x-2} + \dfrac{-1/4}{x+2}$,

Therefore, $$\int \frac{1}{x^2-4}\,dx = \frac{1}{4}\int \frac{1}{x-2}dx - \frac{1}{4}\int \frac{1}{x+2}dx$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

Note: Use Integration by Substitution

Example: Evaluate the integral $\int \frac{3x+11}{(x^2-x-6)}\,dx$

SOLUTION

$$\frac{3x+11}{(x^2-x-6)} = \frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$3x+11 = A(x+2) + B(x-3)$$

Find the values of A and B:

when $x = 3$,

$3(3)+11 = 5A$

$20 = 5A$

$\therefore A = 4$

when $x = -2$

$3(-2)+11 = -5B$

$5 = -5B$

$\therefore B = -1$

$$\therefore \frac{3x+11}{x^2-x-6} = \frac{4}{x-3} - \frac{1}{x+2}$$

$$\int \frac{3x+11}{x^2-x-6}\,dx = \int \frac{4}{x-3} - \frac{1}{x+2}\,dx$$

$$= 4\int \frac{1}{x-3}\,dx - \int \frac{1}{x+2}\,dx$$

$$= 4\ln|x-3| - \ln|x+2| + C$$

- **Another example:**

$$\frac{7x-4}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$= \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)}$$

when $x = 0$; $-4 = A(-1)(2) = -2A$

$\therefore A = 2$

when $x = 1$; $3 = B(1)(3) = 3B$

$\therefore B = 1$

when $x = -2$; $-18 = C(-2)(-3) = 6C$

$\therefore C = -3$

$$\int \frac{7x-4}{x(x-1)(x+2)}\,dx = \int \frac{2}{x} + \frac{1}{x-1} - \frac{3}{x+2}\,dx$$

$$= \int \frac{2}{x}\,dx + \int \frac{1}{x-1}\,dx - \int \frac{3}{x+2}\,dx$$

$$= 2\ln|x| + \ln|x-1| - 3\ln|x+2| + C$$

CHAPTER 5

INTEGRATION

LECTURE 22 – 25.1.2023

- ü Indefinite Integral
- ü Integration Rules
- ü Technique of Integration
- ü Definite Integral
- ü Application: Areas & Volumes

D. Definite Integrals

<u>**Definition of Definite Integral**</u>

Definition of a Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is **integrable** on $[a, b]$ and the limit is denoted by

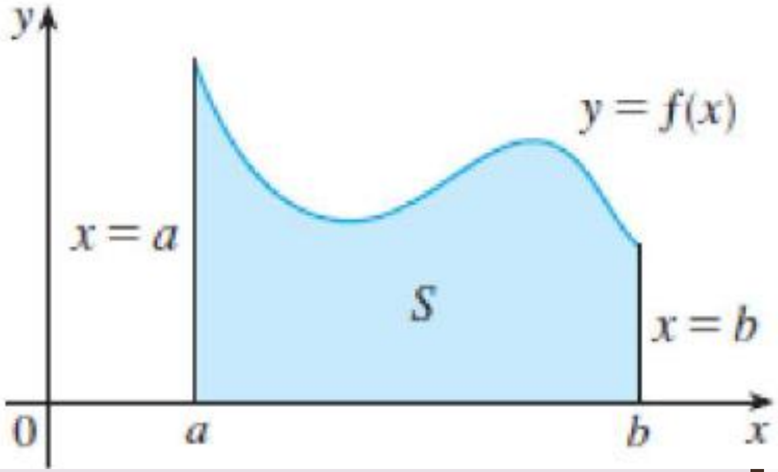
$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

THEOREM Continuity Implies Integrability

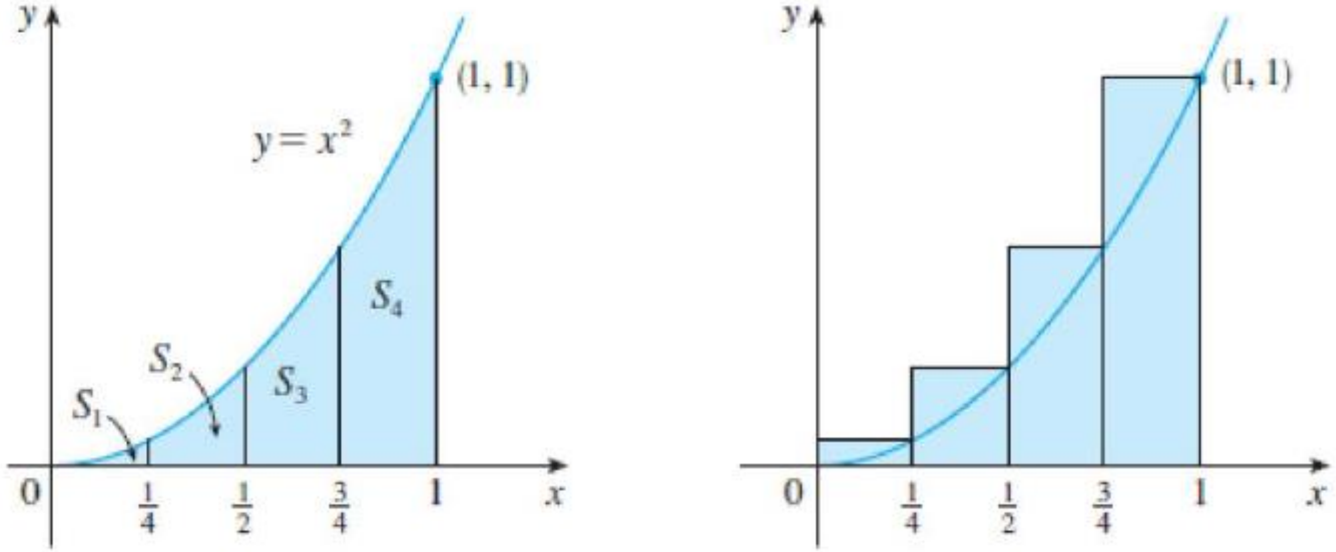
If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.

The Area Problem



Notice that region S is bounded by the graph of a continuous function f vertical line $x=a$ and $x=b$ and the x-axis. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

Consider the area S under the parabola $y=x^2$ from 0 to 1. Suppose we divide S into n=four strips. We could use the rectangles to estimate the area and obtain better estimates by increasing the number of n subintervals.



It appears that, as n increases, the approximations to the area of S become better and better. Therefore we define the area A to be the limit of the sums of the areas of the approximating rectangles.

THEOREM The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

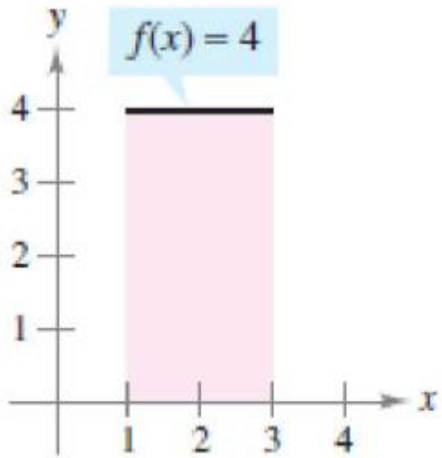
$$\text{Area} = \int_a^b f(x)\,dx.$$

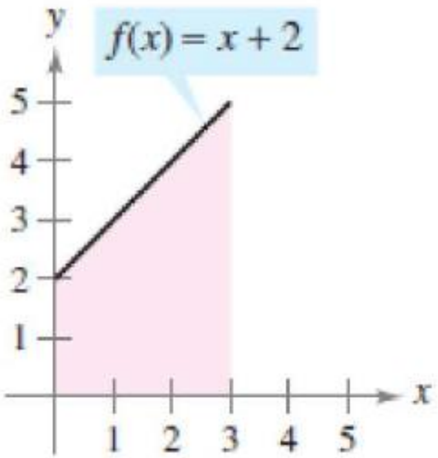
Example:

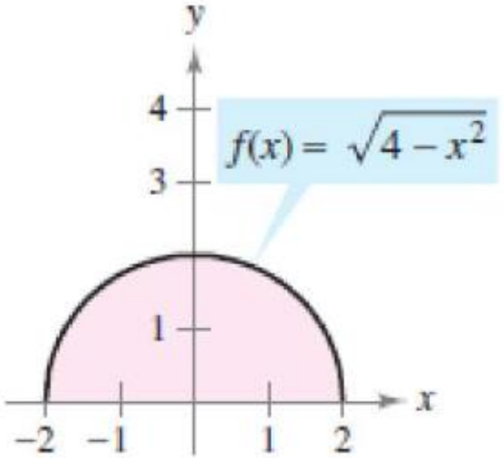
Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a. $\int_1^3 4\,dx$ b. $\int_0^3 (x+2)\,dx$ c. $\int_{-2}^2 \sqrt{4-x^2}\,dx$

SOLUTION







Properties of the Definite Integral

1. $$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

2. $$\int_a^a f(x)dx = 0$$

3. $$\int_a^b c.dx = c(b-a)$$

4. $$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

5. $$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

6. $$\int_a^b cf(x)dx = c\int_a^b f(x)dx, \qquad c \text{ is any constant}$$

7. $$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

Example: Use the Properties of the Definite Integral

Suppose that f and h such that $\int_1^9 f(x)dx = 10$, $\int_7^9 f(x)dx = 5$, $\int_7^9 h(x)dx = 4$

Definite Integral	SOLUTION
$\int_1^9 -2f(x)\ dx$	$= -2\int_1^9 f(x)\ dx = -2(10) = -20$
$\int_7^9 [f(x)+h(x)]\ dx$	$= \int_7^9 f(x)dx + \int_7^9 h(x)dx = 5+4 = 9$
$\int_7^9 [2f(x)-3h(x)]\ dx$	$= \int_7^9 2f(x)\,dx - \int_7^9 3h(x)\,dx = 2(5)-3(4) = -2$
$\int_9^1 f(x)\ dx$	$= -\int_1^9 f(x)\,dx = -10$
$\int_1^7 f(x)\ dx$	$= \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = 10-5 = 5$
$\int_9^7 [h(x)-f(x)]\ dx$	$= -\int_7^9 h(x)\,dx + \int_7^9 f(x)\,dx = -4+5 = 1$

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Guidelines for Using the Fundamental Theorem of Calculus

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b$$
$$= [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a).$$

Example: Evaluate each definite integral.

a. $\displaystyle\int_1^2 (x^2 - 3)\,dx$ b. $\displaystyle\int_1^4 3\sqrt{x}\,dx$ c. $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx$

$$= \left[\frac{x^3}{3} - 3x\right]_1^2$$

$$= \left[\frac{2^3}{3} - 3(2)\right] - \left[\frac{1^3}{3} - 3(1)\right]$$

$$= \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right)$$

$$= -\frac{10}{3} - \left(-\frac{8}{3}\right)$$

$$= -\frac{2}{3} \;\#$$

$$= \int_1^4 3x^{\frac{1}{2}}\,dx$$

$$= \left[3\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)\right]_1^4$$

$$= \left[2x^{\frac{3}{2}}\right]_1^4$$

$$= \left[2(4)^{\frac{3}{2}}\right] - \left[2(1)^{\frac{3}{2}}\right]$$

$$= 8 - 2$$

$$= 6 \;\#$$

$$= \left[\tan x\right]_0^{\frac{\pi}{4}}$$

$$= \left[\tan\frac{\pi}{4}\right] - \left[\tan 0\right]$$

$$= 1 - 0$$

$$= 1 \;\#$$

Example: Integration by Substitution in definite integral

Evaluate $\int_0^1 x(x^2+1)^3\,dx.$

Let $u = x^2+1$

$\frac{du}{dx} = 2x \Rightarrow \frac{du}{2} = x\,dx$

$= \int_0^1 u^3\left(\frac{du}{2}\right)$

$= \int_0^1 \frac{u^3}{2}\,du$

$* = \left[\frac{u^4}{8}\right]_0^1$

$= \left[\frac{(x^2+1)^4}{8}\right]_0^1$

$= \left[\frac{(1^2+1)^4}{8}\right] - \left[\frac{(0^2+1)^4}{8}\right]$

$= \left[\frac{16}{8}\right] - \left[\frac{1}{8}\right]$

$= \frac{15}{8}$ #

Alternatively,

$u = x^2+1$

when $x = 0,\ u = 1$

$x = 1,\ u = 2$

From (*)

$\left[\frac{u^4}{8}\right]_1^2 = \left[\frac{2^4}{8}\right] - \left[\frac{1^4}{8}\right]$

$= \frac{16}{8} - \frac{1}{8}$

$= \frac{15}{8}$ #

CHAPTER 5

INTEGRATION

LECTURE 23 – 27.1.2023

- ü Indefinite Integral
- ü Integration Rules
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- ü Definite Integral
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Example: Integration by Parts in definite integral

Evaluate $\int_0^2 xe^x dx$

Let $u = x$ $\quad dv = e^x$

$du = 1$ $\quad v = e^x$

$= [xe^x]_0^2 - \int_0^2 e^x dx$

$= [2e^2 - 0e^0] - [e^x]_0^2$

$= 2e^2 - [e^2 - e^0]$

$= 2e^2 - e^2 + 1$

$= e^2 + 1$ #

Example: Integration by Partial Fraction in definite integral

Evaluate $\int_0^1 \frac{1}{x^2-4}\,dx$

$$= \int_0^1 \frac{1}{4}\left(\frac{1}{x-2}\right) - \frac{1}{4}\left(\frac{1}{x+2}\right) dx$$

$$= \left[\frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2|\right]_0^1$$

$$= \left[\cancel{\frac{1}{4}\ln 1}^{\,0} - \frac{1}{4}\ln 3\right] - \left[\cancel{\frac{1}{4}\ln 2} - \cancel{\frac{1}{4}\ln 2}\right]$$

$$= -\frac{1}{4}\ln 3 \;\#$$

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-2)$$

when $x = -2$

$1 = -4B$

$\therefore B = -\frac{1}{4}$

when $x = 2$

$1 = 4A$

$\therefore A = \frac{1}{4}$

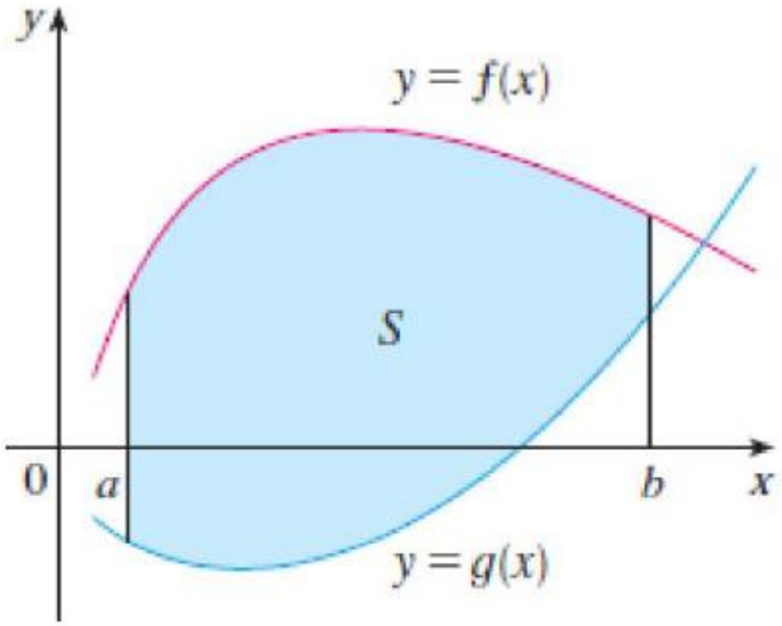
$$\therefore \frac{1}{x^2-4} = \frac{1}{4}\left(\frac{1}{x-2}\right) - \frac{1}{4}\left(\frac{1}{x+2}\right)$$

E. Application of Integration: Area and Volume

Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

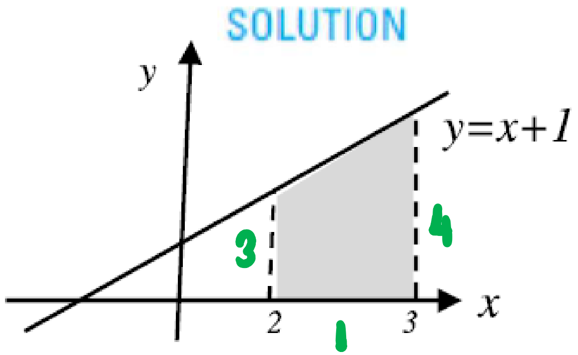


**<u>Note:</u> Observe that f(x) above g(x)*

Example: Region between a curve and x-axis

Sketch the region corresponding to each definite integral. Then evaluate each integral.

(a) $\int_2^3 (x+1)\,dx$



$$\int_2^3 (x+1)\,dx = \left[\frac{x^2}{2} + x\right]_2^3$$

$$= \left[\frac{3^2}{2} + 3\right] - \left[\frac{2^2}{2} + 2\right]$$

$$= \left[\frac{9}{2} + 3\right] - \left[\frac{4}{2} + 2\right]$$

$$= \frac{15}{2} - 4$$

$$= \frac{7}{2} \text{ unit}^2$$

$y = x+1$

at $x = 2$, $y = 3$

at $x = 3$, $y = 4$

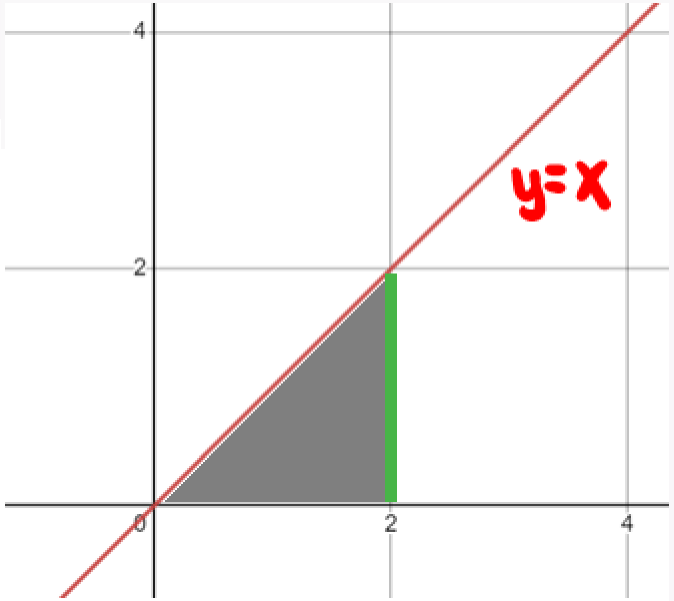
Area of Trapezium $= \frac{1}{2}(4+3)(1)$

$= \frac{7}{2}$ unit^2

(b) $\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2$

$= \left[\frac{2^2}{2}\right] - \left[\frac{0^2}{2}\right]$

$= 2 - 0 = 2$ unit^2

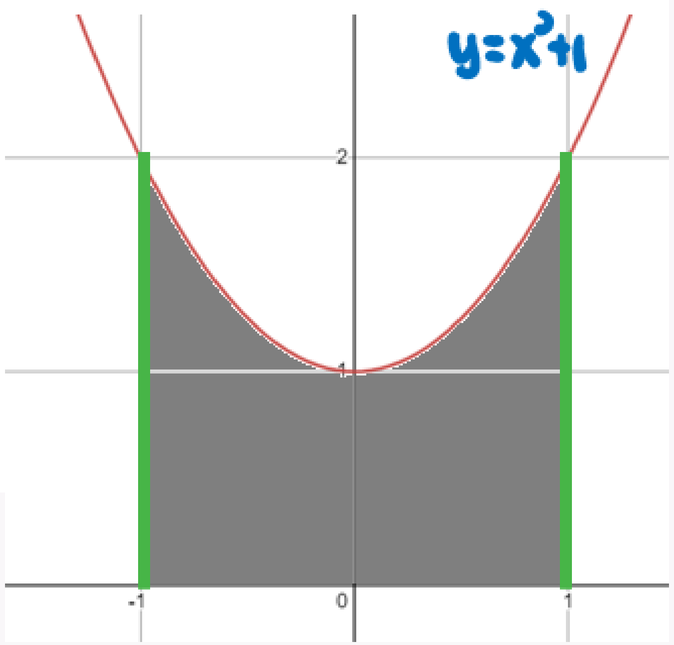


(c) $\int_{-1}^1 x^2 + 1\,dx = \left[\frac{x^3}{3} + x\right]_{-1}^1$

$= \left[\frac{1^3}{3} + 1\right] - \left[\frac{(-1)^3}{3} + (-1)\right]$

$= \frac{4}{3} - \left(-\frac{1}{3} - 1\right)$

$= \frac{8}{3}$ unit^2



Example: Region Between Two Curves

Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$, and $x = 1$.

$$\text{Bounded Area} = \int_0^1 (x^2+2)-(-x)\,dx$$

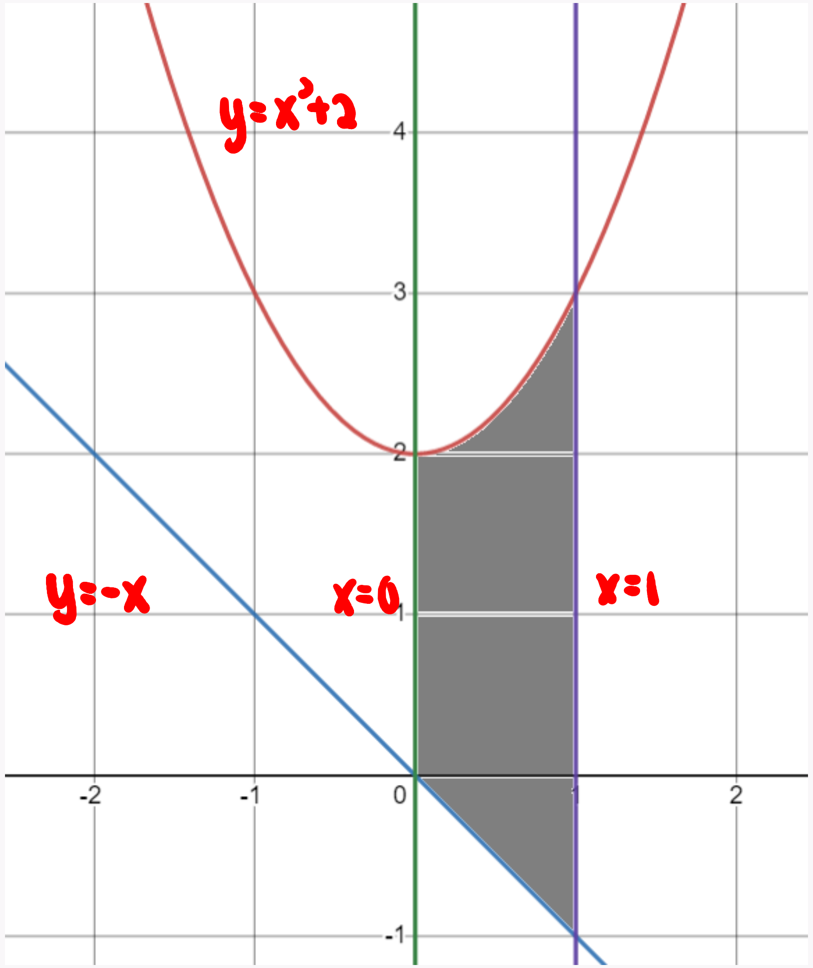
$$= \int_0^1 x^2+2+x\,dx$$

$$= \left[\frac{x^3}{3} + 2x + \frac{x^2}{2}\right]_0^1$$

$$= \left[\frac{1^3}{3} + 2(1) + \frac{1^2}{2}\right] - \left[\frac{0^3}{3} + 2(0) + \frac{0^2}{2}\right]$$

$$= \frac{17}{6} - 0$$

$$= \frac{17}{6} \text{ unit}^2$$



Example: Region Lying Between Two Intersecting Curves

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

$$\text{Bounded Area} = \int_{-2}^{1} (2-x^2) - x \, dx$$

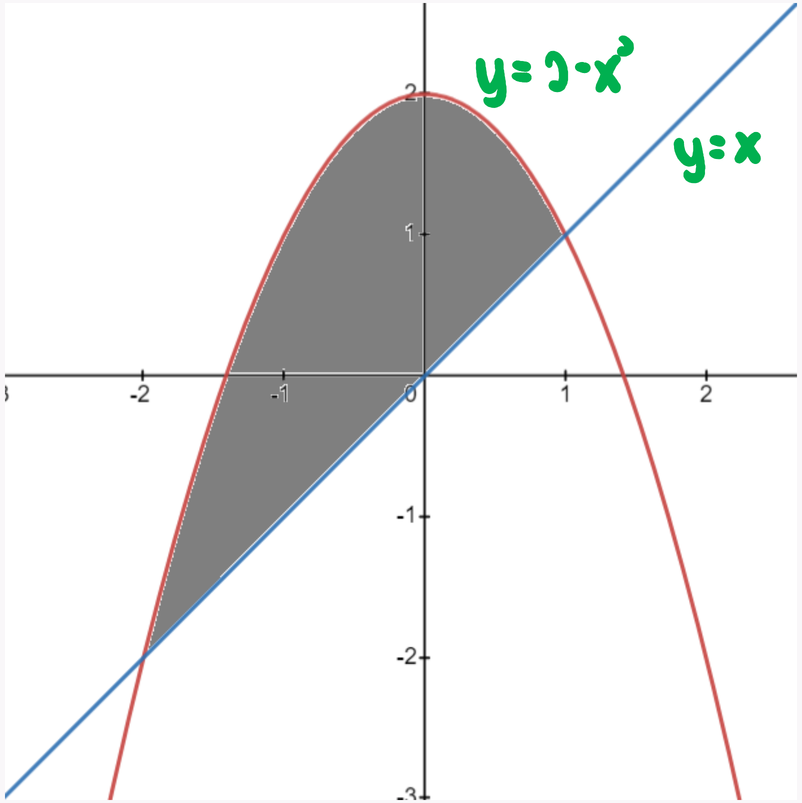
$$= \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1}$$

$$= \left[2(1) - \frac{1^3}{3} - \frac{1^2}{2}\right] - \left[2(-2) - \frac{(-2)^3}{3} - \frac{(-2)^2}{2}\right]$$

$$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - \frac{4}{2}\right)$$

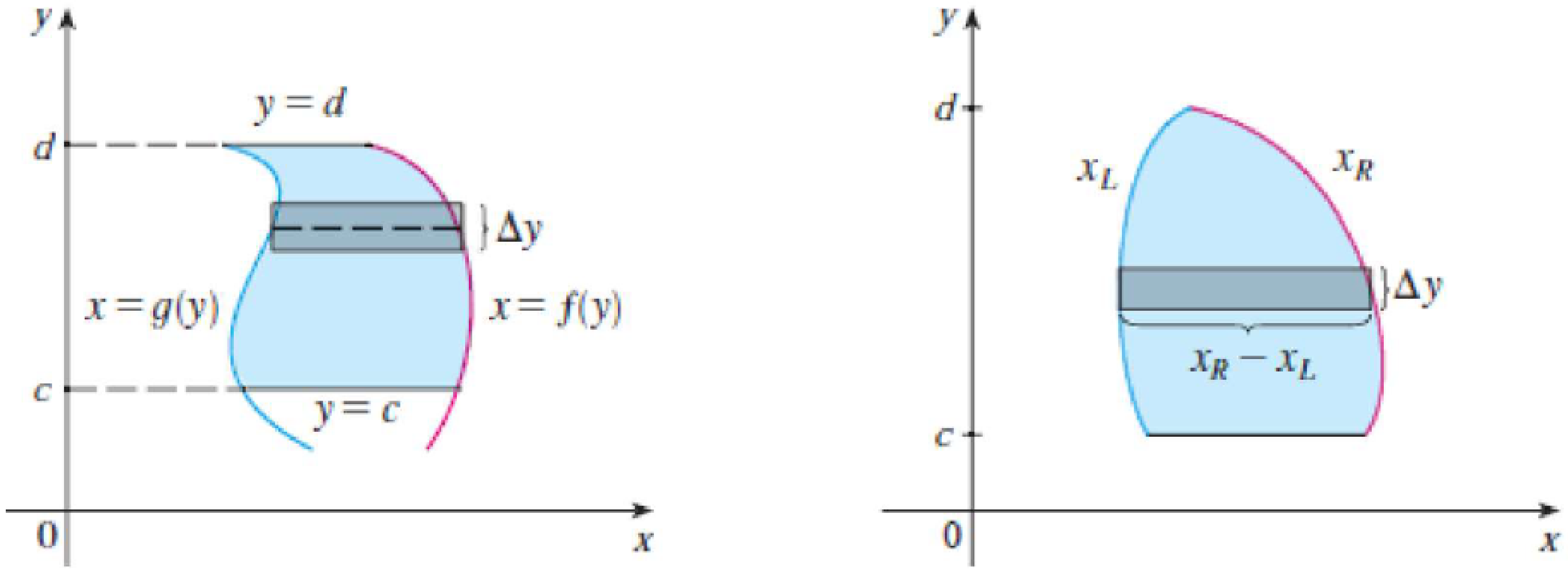
$$= \frac{7}{6} - \left(-\frac{10}{3}\right)$$

$$= \frac{9}{2} \text{ unit}^2$$

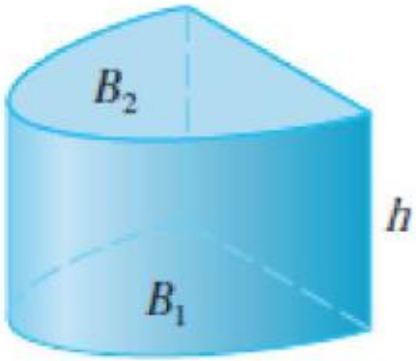


Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$ (see Figure 11), then its area is

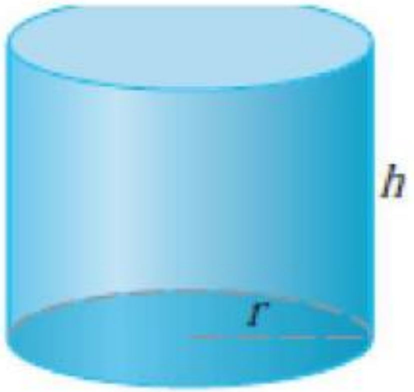
$$A = \int_c^d [f(y) - g(y)]\, dy$$



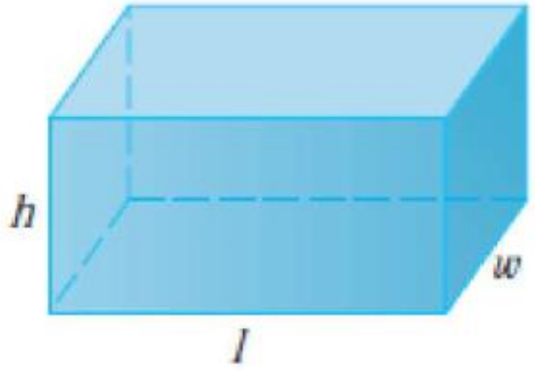
Volumes



(a) Cylinder $V = Ah$



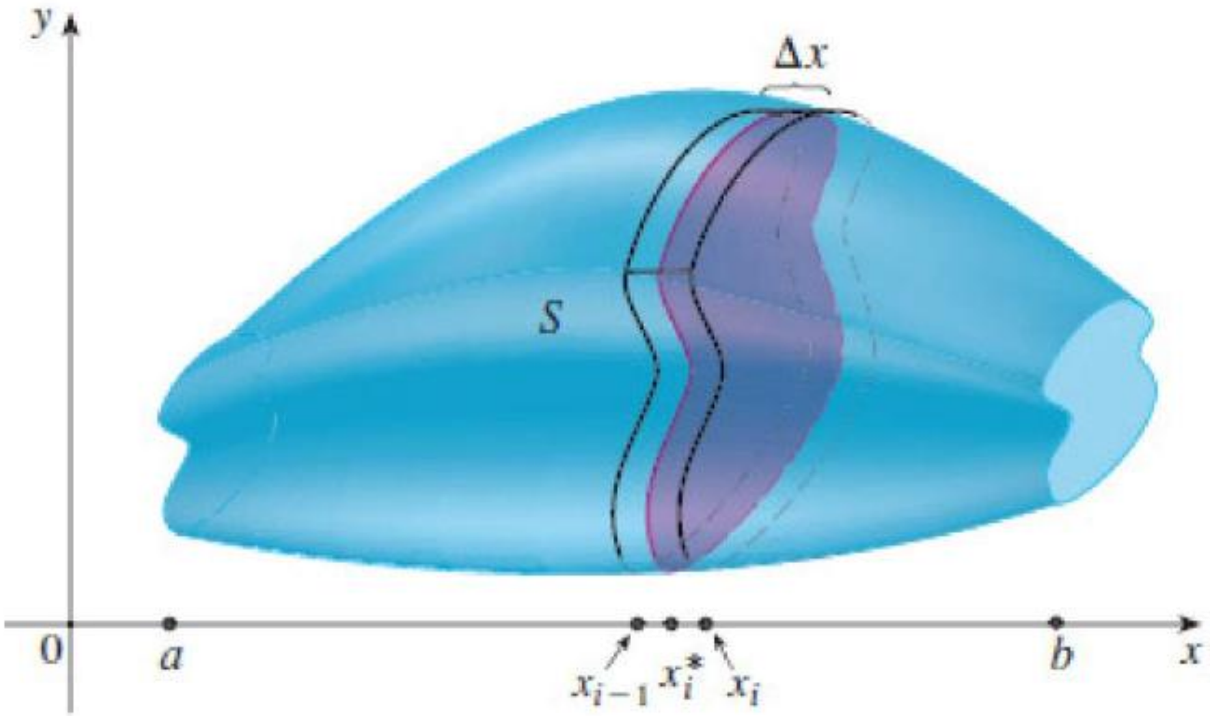
(b) Circular cylinder $V = \pi r^2 h$



(c) Rectangular box $V = lwh$

Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\, \Delta x = \int_a^b A(x)\, dx$$



The solids that are obtained by revolving a region about a line are called **solids of revolution**.

If the cross-section is a disk (*as in examples below*), we find the radius of the disk (in terms of x or y) and use $A = \pi(\text{radius})^2$

If the cross-section is a washer, we find the inner radius r_{in} and outer radius r_{out} from a sketch and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk: $A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$

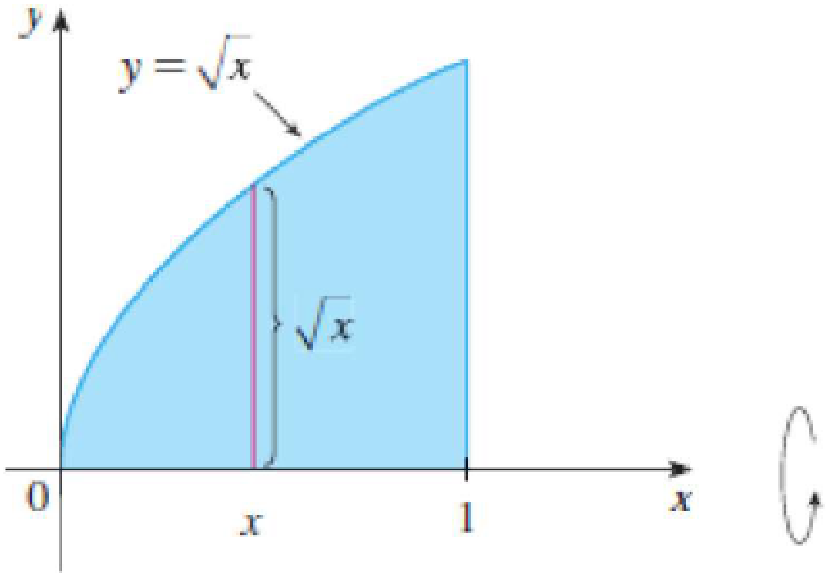
Example:

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

SOLUTION

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi x\,dx = \pi \frac{x^2}{2}\bigg]_0^1 = \frac{\pi}{2}$$



If a solid is formed by rotating a region about the y-axis, then it makes sense to slice the solid perpendicular to the y-axis and therefore to integrate with respect to y. See example below.

Example

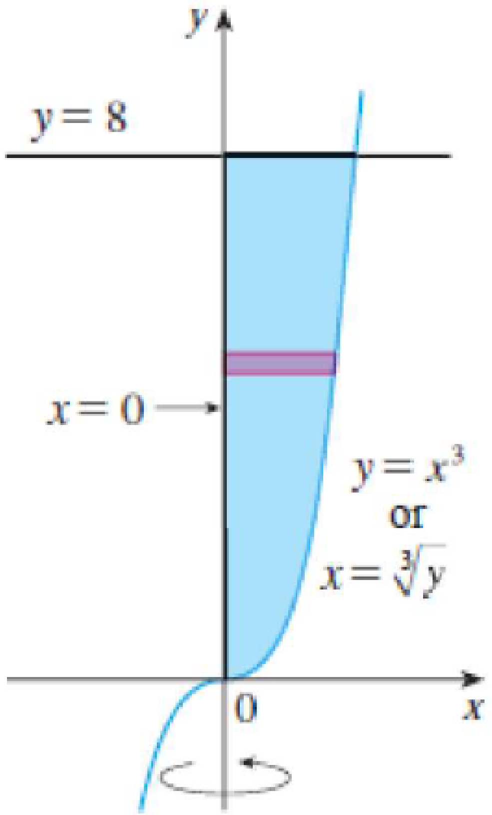
Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y=8$ and $x=0$ and about the y-axis.

SOLUTION

$$V = \int_0^8 A(y)\,dy$$

$$= \int_0^8 \pi y^{2/3}\,dy$$

$$= \pi\left[\tfrac{3}{5}y^{5/3}\right]_0^8 = \frac{96\pi}{5}$$



- **Finding Volume generated when a region bounded by 2 curves and is rotated about x or y-axis:**

1. **Sketch the 2 curves correctly on the same graph.**
2. **Find the points of intersection between these 2 curves, they are the upper and lower limits of the integral.**
3. **Identify the region bounded these 2 curves correctly.**
4. **Express the functions with respect to the integral, either the volume is rotated about x-axis ($\pi \int y^2\,dx$) or rotated about x-axis rotated about y-axis $\pi \int x^2\,dy$.**
5. **Perform the definite integration.**

- **Another example:**

➢ **Find the volume generated by the solid between the curves $y = 4x^2$ and $y^2 = 2x$ that is rotated completely about the x-axis.**

$$y^2 = 2x$$

$$y = \sqrt{2x}$$

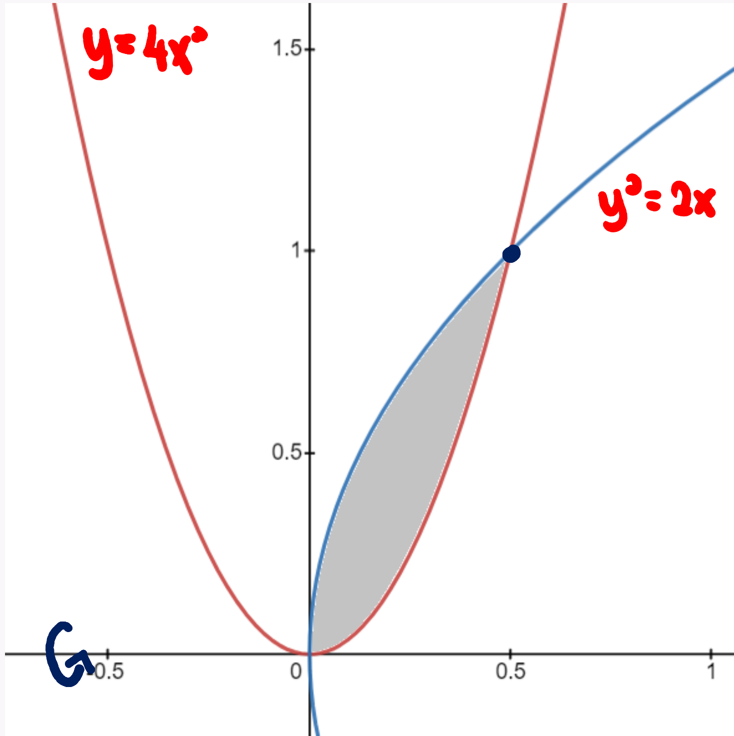
$$\text{Volume generated} = \pi \int_0^{\frac{1}{2}} (\sqrt{2x})^2 - (4x^2)^2 \, dx$$

$$= \pi \int_0^{\frac{1}{2}} 2x - 16x^4 \, dx$$

$$= \pi \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{16x^5}{5} \right]_0^{\frac{1}{2}}$$

$$= \pi \left[\left(\tfrac{1}{2}\right)^2 - \tfrac{16}{5}\left(\tfrac{1}{2}\right)^5 \right]$$

$$= \pi \left[\tfrac{1}{4} - \tfrac{16}{5}\left(\tfrac{1}{32}\right) \right] = \frac{3}{20}\pi \text{ unit}^3$$



- **Another example:**

➢ **Find the volume generated by the solid between the curves $y = 4x^2$ and $y^2 = 2x$ that is rotated completely about the y-axis.**

$4x^2 = y$

$x = \sqrt{\frac{y}{4}}$

$y^2 = 2x$

$x = \frac{y^2}{2}$

Volume generated $= \pi \int_0^1 \left(\sqrt{\frac{y}{4}}\right)^2 - \left(\frac{y^2}{2}\right)^2 dy$

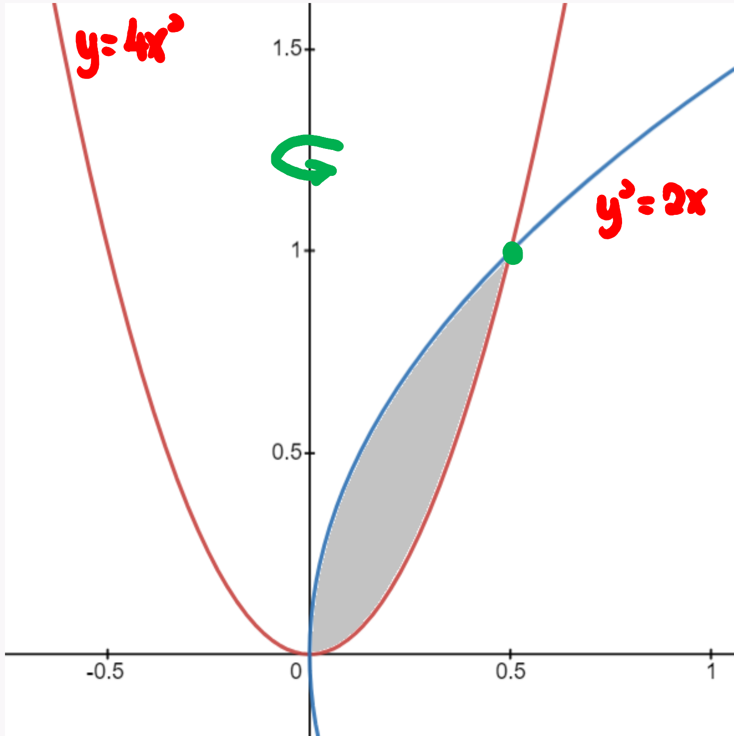
$= \pi \int_0^1 \frac{y}{4} - \frac{y^4}{4} \, dy$

$= \frac{1}{4}\pi \int_0^1 y - y^4 \, dy$

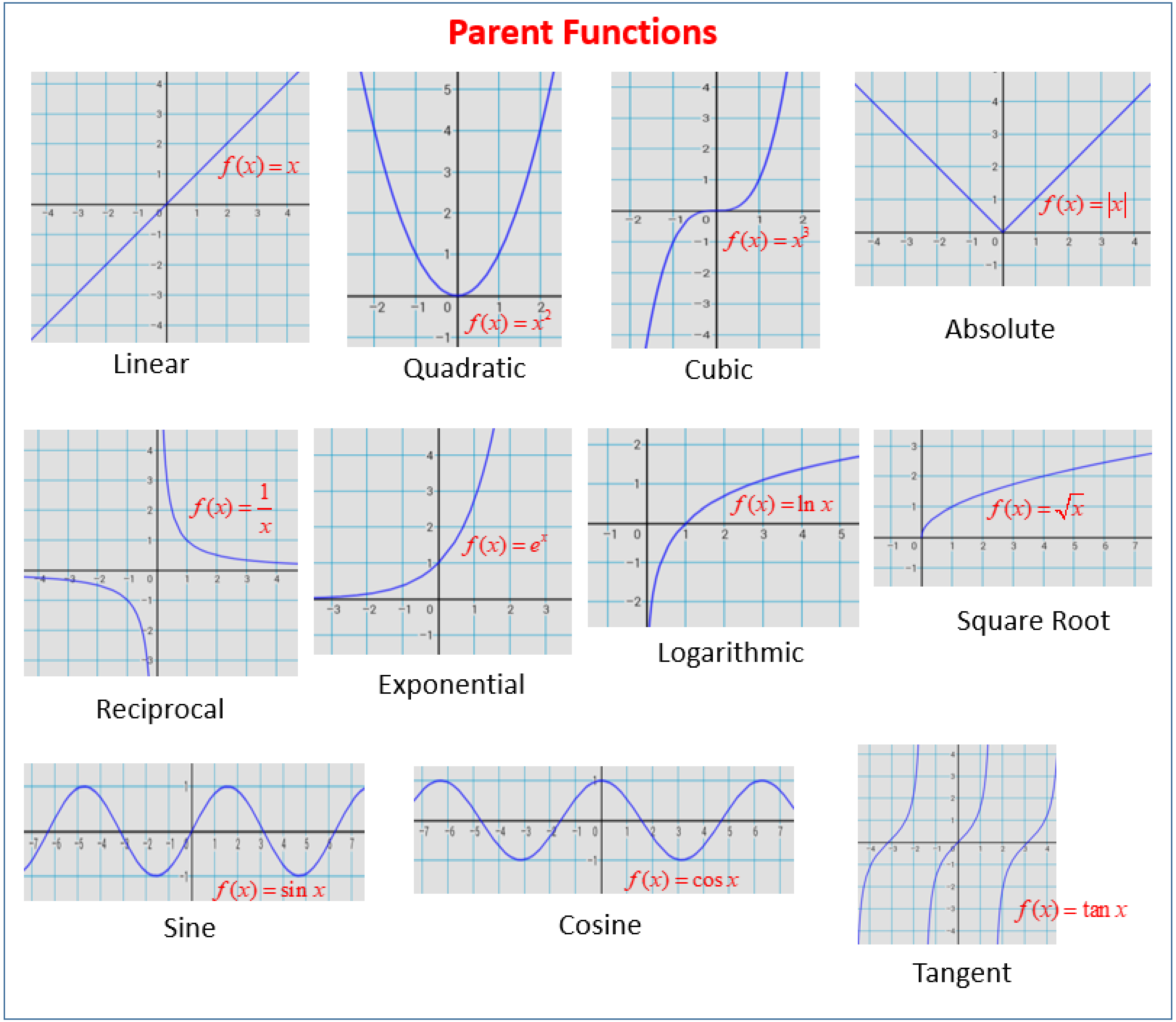
$= \frac{1}{4}\pi \left[\frac{y^2}{2} - \frac{y^5}{5}\right]_0^1$

$= \frac{1}{4}\pi \left[\frac{1^2}{2} - \frac{1^5}{5}\right]$

$= \frac{1}{4}\pi \left(\frac{3}{10}\right) = \frac{3}{40}\pi \text{ unit}^3$



- Some general parent functions, the variable and its degree determine the pattern and shape of the graph.



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