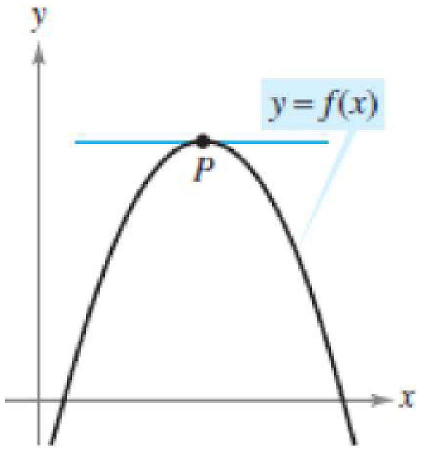
- DEFINITIONS OF DERIVATIVES
- DIFFERENTIATION RULES
- THE DERIVATIVES AS A RATE OF CHANGE
- MINIMUM & MAXIMUM VALUES

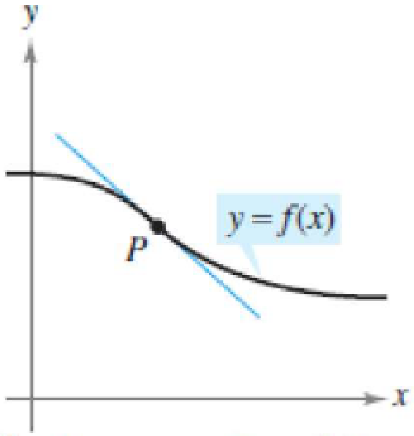
CHAPTER 4
DERIVATIVES

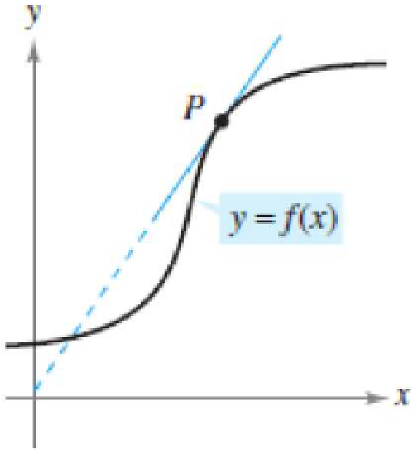
LECTURE 15 – 28.12.2022

A. Definition of derivatives

Definition of tangent line to the curve at a point





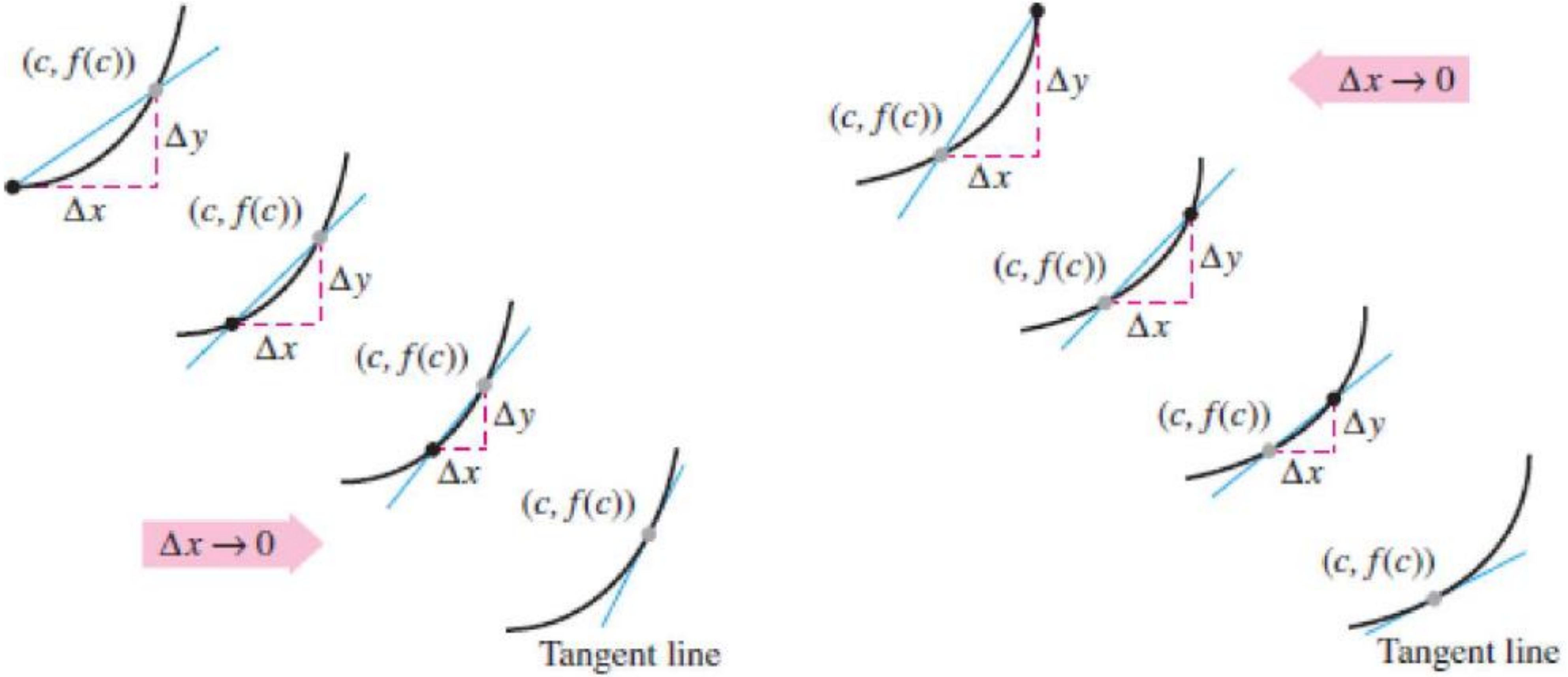


Tangent line to a curve at a point

Problem of finding the tangent line at a point P is closely related to the problem of finding the **slope** of the tangent line at point P. We can approximate this slope using a **secant line** through the point of tangency and a second point on the curve. The slope of the secant line, m_{sec}, through the two points is given by

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}. \qquad \frac{\text{Change in } y}{\text{Change in } x}$$

The denominator Δx is the change in x, and numerator $\Delta y = f(c + \Delta x) - f(c)$ is the change in y. The figure below shows a sequence of secant lines approaching a tangent line at $x=c$.



Tangent line approximations

Definition of Tangent Line with Slope m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

Definition of Derivative

We now investigate the derivative as a function derived from f by considering the limit at each point of the domain of f. The process of finding the derivative of a function is called **differentiation**. Let change in x, $\Delta x = h$.

Definition of the Derivative of a Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

The derivative of a function *f* is itself a function, which can be used to find the slope of the tangent line at the point $(x, f(x))$ on the graph of f.

Other Notations of Derivatives

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Notation for derivatives

The notation dy/dx is read as "the derivative of y *with respect to* x" or simply "$dy - dx$".

Example: Use the **definition of derivative** to find the derivative $f'(x)$ if $f(x) = 2x - 1$

SOLUTION

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{[2(x+h)-1]-[2x-1]}{h}$$

$$= \lim_{h\to 0}\frac{(2x+2h-1)-2x+1}{h}$$

$$= \lim_{h\to 0}\frac{2h}{h} = \lim_{h\to 0} 2 = 2$$

What is the ***slope of line*** *f(x)=2x-1?*

Example: Use the **definition of derivative** to find the derivative of the function $f(x)=x^2-8x+9$.

SOLUTION

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left[(x+h)^2-8(x+h)+9\right]-[x^2-8x+9]}{h}$$

$$=\lim_{h\to 0}\frac{(\cancel{x^2}+2xh+h^2-\cancel{8x}-8h\cancel{+9})\cancel{-x^2}\cancel{+8x}\cancel{-9}}{h}$$

$$=\lim_{h\to 0}\frac{2xh+h^2-8h}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+h-8)}{\cancel{h}}$$

$$=2x+0-8$$

$$=2x-8\ \#$$

Example:

(a) Using the **definition of derivative** to find the derivative of $f(x)=\sqrt{x}$ for $x>0$.

(b) Hence, what is the slope of the tangent to the graph of $f(x)=\sqrt{x}$ at the point ~~(1,3)~~? (1, 1)

SOLUTION

a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

$= \lim_{h\to 0} \dfrac{(\cancel{x}+h) - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim_{h\to 0} \dfrac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$

$= \dfrac{1}{\sqrt{x+0} + \sqrt{x}} = \dfrac{1}{\sqrt{x} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$ #

b) $f'(x) = \dfrac{1}{2\sqrt{x}}$

at (1,1), the slope:

$f'(1) = \dfrac{1}{2\sqrt{1}}$

$= \dfrac{1}{2}$ #

Example: Use the **definition of derivative** to find the derivative $f'(2)$ if $f(x)=3x^2$.

SOLUTION

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{\left[3(2+h)^2\right]-\left[3(2)^2\right]}{h}=\lim_{h\to 0}\frac{\left[3(4+4h+h^2)\right]-[12]}{h}$$

$$=\lim_{h\to 0}\frac{\left[\cancel{12}+12h+3h^2\right]-[\cancel{12}]}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(12+3h)}{\cancel{h}}$$

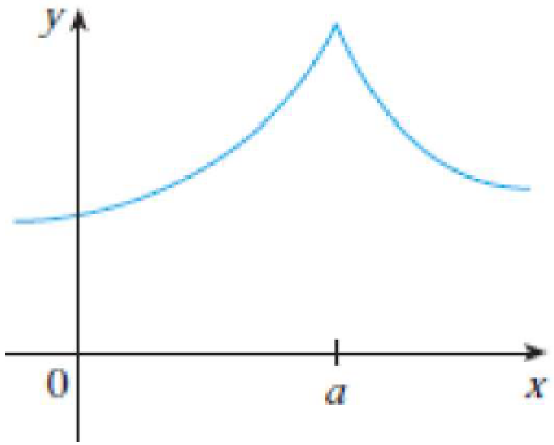
$$=12+3(0)$$

$$=12$$

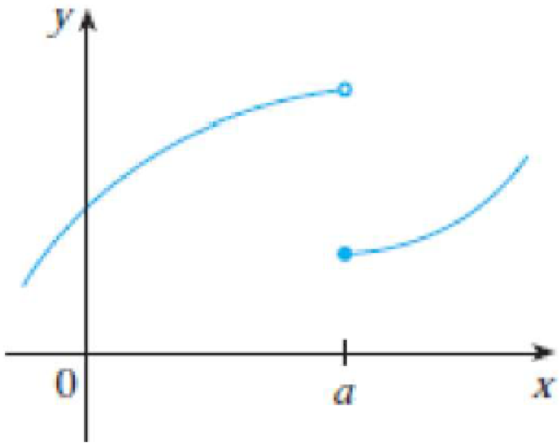
How Can a Function Fail to Be Differentiable?

Three ways for f not to be differentiable at $x=a$:

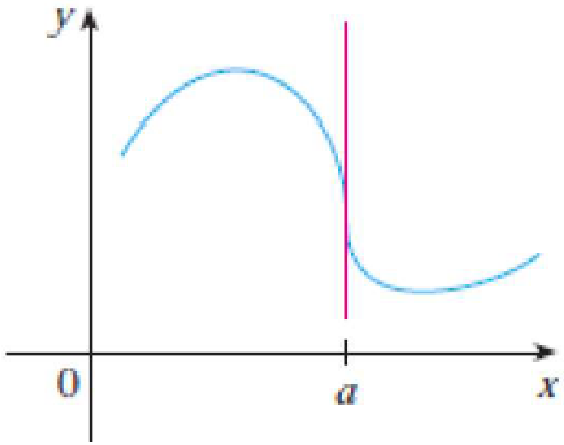
A Graph with a Sharp Turn



A Graph with a Discontinuity



A Graph with a Vertical Tangent Line



- DEFINITIONS OF DERIVATIVES
- DIFFERENTIATION RULES
- THE DERIVATIVES AS A RATE OF CHANGE
- MINIMUM & MAXIMUM VALUES

CHAPTER 4 DERIVATIVES

LECTURE 16 – 30.12.2022

B. Differentiation Rules

The Constant Rule

If $f(x) = c$, then $\Rightarrow$ $$\frac{d}{dx}[c] = 0$$

The Power Rule

If n is any real number, then $\Rightarrow$ $$\frac{d}{dx}[x^n] = nx^{n-1}, \quad n \in \Re$$

The Constant Multiple Rule

Let c be a constant, then $\Rightarrow$ $$\frac{d}{dx}[cf(x)] = c.\frac{d}{dx}[f(x)]$$

The Sum and Difference Rule

If f and g are both differentiable at x, then

$$\Rightarrow \quad \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

The Product Rule

If f and g are both differentiable at x, then

$$\Rightarrow \frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The Quotient Rule

If f and g are both differentiable at x, and $g(x) \neq 0$, then

$$\Rightarrow \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Note:

$$\frac{d}{dx}[f(x).g(x)] \neq \frac{df(x)}{dx}.\frac{dg(x)}{dx}$$

WRONG!

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{\frac{df(x)}{dx}}{\frac{dg(x)}{dx}}$$

Example: Using the Constant Rule

Function	*Derivative*
$y = 7$	$\dfrac{dy}{dx} = 0$
$y = k\pi^2$, k is constant	$y' = 0$

Example: Using the Power Rule

Function	*Derivative*
$f(x) = x^3$	$f'(x) = 3x^2$
$g(x) = \sqrt[3]{x}$	$g'(x) = \dfrac{d}{dx}[x^{1/3}] = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$
$y = \dfrac{1}{x^2}$	$\dfrac{dy}{dx} = \dfrac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\dfrac{2}{x^3}$

Example: Using the Constant Multiple Rule

Function	*Derivative*
$y = \frac{2}{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{-1}] = 2\frac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\frac{2}{x^2}$
$f(t) = \frac{4t^2}{5}$	$f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8}{5}t$
$y = 2\sqrt{x}$	$\frac{dy}{dx} = \frac{d}{dx}[2x^{1/2}] = 2\left(\frac{1}{2}x^{-1/2}\right) = x^{-1/2} = \frac{1}{\sqrt{x}}$
$y = \frac{1}{2\sqrt[3]{x^2}}$	$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}x^{-2/3}\right] = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-5/3} = -\frac{1}{3x^{5/3}}$
$y = -\frac{3x}{2}$	$y' = \frac{d}{dx}\left[-\frac{3}{2}x\right] = -\frac{3}{2}(1) = -\frac{3}{2}$

Example: Using the Sum and Difference Rules

Function	*Derivative*
$f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
$g(x) = -\frac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$

Example: Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

SOLUTION

$$h'(x) = \overbrace{(3x - 2x^2)}^{\text{First}}\overbrace{\frac{d}{dx}[5 + 4x]}^{\text{Derivative of second}} + \overbrace{(5 + 4x)}^{\text{Second}}\overbrace{\frac{d}{dx}[3x - 2x^2]}^{\text{Derivative of first}}$$

$$= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$$

$$= (12x - 8x^2) + (15 - 8x - 16x^2)$$

$$= -24x^2 + 4x + 15 \quad \text{Simplify.}$$

Example: Using the Product Rule

Find the derivative of $f(x)=(6x^3)(7x^4)$. Show the detailed steps.

SOLUTION

Let $u = 6x^3$ $\quad v = 7x^4$

$\frac{du}{dx} = 18x^2$ $\quad \frac{dv}{dx} = 28x^3$

$f'(x) = u.\frac{dv}{dx} + v.\frac{du}{dx}$

$= (6x^3)(28x^3) + (7x^4)(18x^2)$

$= 168x^6 + 126x^6$

$= 294x^6$ #

Alternatively, $f(x) = (6x^3)(7x^4)$

$= 42x^7$

$\therefore f'(x) = 7(42x^6)$

$= 294x^6$ #

Index Laws:

1.) $x^a . x^b = x^{a+b}$

2.) $\frac{x^a}{x^b} = x^{a-b}$

3.) $x^{-n} = \frac{1}{x^n}$

4.) $\sqrt[n]{x} = x^{\frac{1}{n}}$

Example: Using the Quotient Rule

Find the derivative of $y = \dfrac{5x - 2}{x^2 + 1}$.

SOLUTION

$$\frac{d}{dx}\left[\frac{5x - 2}{x^2 + 1}\right] = \frac{(x^2 + 1)\frac{d}{dx}[5x - 2] - (5x - 2)\frac{d}{dx}[x^2 + 1]}{(x^2 + 1)^2}$$

$$= \frac{(x^2 + 1)(5) - (5x - 2)(2x)}{(x^2 + 1)^2}$$

$$= \frac{(5x^2 + 5) - (10x^2 - 4x)}{(x^2 + 1)^2} = \frac{-5x^2 + 4x + 5}{(x^2 + 1)^2}$$ Simplify.

Example: Using the Quotient Rule

Find the derivative of $y = \dfrac{t^2 - 1}{t^2 + 1}$.

SOLUTION

Let $u = t^2 - 1$ $\quad v = t^2 + 1$

$\dfrac{du}{dt} = 2t \quad \dfrac{dv}{dt} = 2t$

$$\frac{dy}{dt} = \frac{v \cdot \frac{du}{dt} - u \cdot \frac{dv}{dt}}{v^2}$$

$$= \frac{(t^2+1)(2t) - (t^2-1)(2t)}{(t^2+1)^2}$$

$$= \frac{\cancel{2t^3} + 2t - \cancel{2t^3} + 2t}{(t^2+1)^2}$$

$$= \frac{4t}{(t^2+1)^2}$$

#

Derivatives of Trigonometric Functions

$\frac{d}{dx}[\sin x] = \cos x$	$\frac{d}{dx}[\csc x] = -\csc x \cot x$
$\frac{d}{dx}[\cos x] = -\sin x$	$\frac{d}{dx}[\sec x] = \sec x \tan x$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\frac{d}{dx}[\cot x] = -\csc^2 x$

Derivatives of Logarithmic and Exponential Functions

$\frac{d}{dx}[\ln x] = \frac{1}{x}; \quad x > 0$	$\frac{d}{dx}[e^x] = e^x$
$\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}, \quad x \neq 0$	

Note: The derivatives of sine and cosine functions can be proven using formal definition of the derivative. Knowing the derivatives of the sine and cosine functions, we can use the Quotient Rule to find the derivatives of the four remaining trigonometric functions.

- For example, show that $\dfrac{d}{dx}[\sec x] = \sec x \tan x$

Since $\sec x = \dfrac{1}{\cos x}$

Let $u = 1$ $\quad v = \cos x$

$\dfrac{du}{dx} = 0 \quad \dfrac{dv}{dx} = -\sin x$

$$\therefore \frac{d}{dx}[\sec x] = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

$$= \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)$$

$$= \sec x \tan x \ \#$$

Example: Differentiating Trigonometric Functions

Function	*Derivative*	
$y = 2 \sin x$	$y' = 2 \cos x$	
$y = \dfrac{\sin x}{2} = \dfrac{1}{2} \sin x$	$y' = \dfrac{1}{2} \cos x = \dfrac{\cos x}{2}$	
$y = x + \cos x$	$y' = 1 - \sin x$	
$y = x - \tan x$	$\dfrac{dy}{dx} = 1 - \sec^2 x$	
$y = x \sec x$	$y' = x(\sec x \tan x) + (\sec x)(1)$	Product Rule
	$= (\sec x)(1 + x \tan x)$	Simplify.

Example: Differentiating Logarithmic and Exponential Functions

Function	*Derivative*
$y=3e^x$	$y'=3\frac{d}{dx}\left(e^x\right)=3e^x$
$y=3e^x+x^{-2}$	$y'=$ $3e^x+(-2)x^{-3}=3e^x-\frac{2}{x^3}$ #
$y=3\ln x$	$y'=$ $3\left(\frac{1}{x}\right)=\frac{3}{x}$ #
$y=x-3\ln x$	$y'=$ $1-3\left(\frac{1}{x}\right)=1-\frac{3}{x}$ #

What if $y=\log_{10}x$?

1. change the base to $\ln=\log_e$
2. differentiate $\ln$ w.r.t. x

$\therefore\ y=\log_{10}x=\dfrac{\log_e x}{\log_e 10}$

$=\dfrac{\ln x}{\ln 10}$

this is a constant

$\frac{1}{\ln 10}=0.4342\ldots$

$\therefore\ \dfrac{dy}{dx}=\left(\dfrac{1}{\ln 10}\right)\left(\dfrac{1}{x}\right)$

$=\dfrac{1}{x\ln 10}$ #

Example: Find the derivative of $y = 2x\cos x - 2\sin x$.

SOLUTION

$$\frac{dy}{dx} = \overbrace{(2x)\left(\frac{d}{dx}[\cos x]\right) + (\cos x)\left(\frac{d}{dx}[2x]\right)}^{\text{Product Rule}} - \overbrace{2\frac{d}{dx}[\sin x]}^{\text{Constant Multiple Rule}}$$

$$= 2x(-\sin x) + (\cos x)(2) - 2(\cos x)$$

$$= -2x\sin x + \cancel{2\cos}\, x - \cancel{2\cos}\, x$$

$$= -2x\sin x$$ Simplify.

Example:

Consider $\tan x = \dfrac{\sin x}{\cos x}$, use the Quotient Rule to prove the derivative $\dfrac{d}{dx}(\tan x) = \cancel{\dfrac{1}{\sec^2 x}}$ $\sec^2 x$

SOLUTION

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{(\cos x)\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x}$$

Quotient Rule.

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

→ identity $\sin^2 x + \cos^2 x = 1$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

#

Example: Find the derivative of $f(x) = (x^2 - 4x)\cos x$

SOLUTION

$$f'(x) = (x^2 - 4x)\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2 - 4x) \quad \text{Product Rule}$$

$$= (x^2 - 4x)(-\sin x) + (\cos x)(2x - 4)$$

$$= -(x^2 - 4x)\sin x + (2x - 4)\cos x \quad \text{Simplify.}$$

Example: Find the derivative of $f(x)=\dfrac{e^x-1}{\cos x}$.

SOLUTION

Quotient Rule.

$$f'(x)=\frac{\cos x\dfrac{d}{dx}\left(e^x-1\right)-\left(e^x-1\right)\dfrac{d}{dx}(\cos x)}{\cos^2 x}=\frac{(\cos x)(e^x)-(e^x-1)(-\sin x)}{\cos^2 x}$$

$$=\frac{e^x\cos x+e^x\sin x-\sin x}{\cos^2 x}$$

$$=\frac{e^x(\cos x+\sin x)-\sin x}{\cos^2 x}$$

Simplify.

Higher Derivatives

First derivative of f:	$(f)' = f'$	$\dfrac{dy}{dx} = y'$
Second derivative of f:	$(f')' = f''$	$\dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] = \dfrac{d^2 y}{dx^2} = y'' = D^2 f(x)$
Third derivative of f:	$(f'')' = f'''$	$\dfrac{d}{dx}\left[\dfrac{d^2 y}{dx^2}\right] = \dfrac{d^3 y}{dx^3} = y''' = D^3 f(x)$
Fourth derivative of f:	$(f''')' = f^{(4)}$	
$\vdots$		
nth derivative of f:	$f^{(n)}$	$\dfrac{d^n y}{dx^n} = \dfrac{d^n}{dx^n}[f(x)] = y^{(n)} = D^n f(x)$

Example:

Find the **first and second derivatives** of $y = x^3 + 5x^2 - 4x$.

SOLUTION

$$\frac{dy}{dx} = 3x^2 + 10x \quad , \quad \frac{d^2y}{dx^2} = 6x + 10$$

Example:

If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

SOLUTION

$f^1(x) =$ $3x^2 - 1$

$f^2(x) =$ $6x$

$f^3(x) =$ 6

$f^4(x) =$ 0

Example:

Find the **fourth derivative** of $y = \sin x$.

SOLUTION

$f^1(x) = \cos x$

$f^2(x) = -\sin x$

$f^3(x) = -\cos x$

$f^4(x) = \sin x = y$

- DEFINITIONS OF DERIVATIVES
- DIFFERENTIATION RULES
- THE DERIVATIVES AS A RATE OF CHANGE
- MINIMUM & MAXIMUM VALUES

CHAPTER 4 DERIVATIVES

LECTURE 17 – 4.1.2023

The Chain Rule

This rule deals with derivatives of composite functions. For example, compare the functions shown below. Those on the left can be differentiated without the Chain Rule, and those on the right are best done with the Chain Rule.

Without the Chain Rule	*With the Chain Rule*
$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
$y = \sin x$	$y = \sin 6x$
$y = 3x + 2$	$y = (3x + 2)^5$

THEOREM The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

or, equivalently, $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$.

Example: Use the Chain Rule to find the derivative of the functions. **Show all steps.**

<table>
<tr>
<td>

(a) $y = \sin 6x$

SOLUTION

$u = 6x, \qquad \dfrac{du}{dx} = 6$

$y = \sin u, \qquad \dfrac{dy}{du} = \cos u$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = 6\cos u = 6\cos 6x$

</td>
<td>

(b) $y = e^{6x}$

SOLUTION

$u = 6x, \qquad \dfrac{du}{dx} = 6$

$y = e^{u}, \qquad \dfrac{dy}{du} = e^{u}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = 6e^{u} = 6e^{6x}$

</td>
</tr>
</table>

Example: Use the Chain rule to find the derivative of the functions, **without showing the detailed steps**.

Function	*Derivative*	*Function*	*Derivative*
$y=\cos 2x$	$y'=-2\sin 2x$	$y=\sin 3x$	$y'=3\cos 3x$
$y=\tan 2x$	$y'=2\sec^2 2x$	$y=\tan 3x$	$y'=3\sec^2 3x$
$y=e^{2x}$	$y'=2e^{2x}$	$y=e^{3x}$	$y'=3e^{3x}$
$y=e^{-2x}$	$y'=-2e^{-2x}$	$y=e^{-3x}$	$y'=-3e^{-3x}$
$y=\ln(2x)$	$y'=2\dfrac{1}{(2x)}=\dfrac{1}{x}$	$y=\ln(3x)$	$y'=\left(\dfrac{1}{3x}\right)\cdot 3 = \dfrac{1}{x}$

Example:

$y=\ln(3x+1)$; $y'=\dfrac{1}{3x+1}\cdot 3$

$= \dfrac{3}{3x+1}$

Example: Find dy/dx for $y = (x^2 + 1)^3$.

SOLUTION

$$u = x^2 + 1, \quad \frac{du}{dx} = 2x$$

$$y = u^3, \quad \frac{dy}{du} = 3u^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (3u^2)(2x) \quad = 3(x^2+1)^2(2x) = 6x(x^2+1)^2$$

***These intermediate steps are required

Simplify.

Example: Find the derivative for $f(x) = (5x^2 - 3x + 3)^{-5}$.

SOLUTION

$$u = 5x^2 - 3x + 3, \quad \frac{du}{dx} = 10x - 3$$

$$y = u^{-5}, \quad \frac{dy}{du} = -5u^{-6}$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -5u^{-6} \cdot (10x-3) = \frac{-5(10x-3)}{(5x^2-3x+3)^6} = \frac{-50x+15}{(5x^2-3x+3)^6}$$

$$= \frac{-5(5x^2-3x+3)^{-6}}{10x-3}$$

Simplify.

Example: Find the derivative for $y = \sin^3 x$.

SOLUTION

$$y = \sin^3 x = (\sin x)^3$$

$$u = \sin x, \quad \frac{du}{dx} = \cos x$$

$$y = u^3, \quad \frac{dy}{du} = 3u^2$$

$$y'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot \cos x = 3 \sin^2 x \cos x$$

$$= 3(\sin^2 x)\cos x$$ Simplify.

Example: Find the derivative for $y=(2x+3)e^{-2x}$.

SOLUTION

Let $y=(2x+3)e^{-2x}=f(x)g(x)$, hence we use Product Rule $y'=fg'+gf'$

$f(x)=(2x+3)$, $g(x)=e^{-2x}$

$f'(x)=2$ $g'(x)=-2e^{-2x}$ Chain Rule

$$y'=fg'+gf'=(2x+3)\left(-2e^{-2x}\right)+\left(e^{-2x}\right)(2) \quad \text{Product Rule}$$

$$= -4xe^{-2x}-6e^{-2x}+2e^{-2x}$$

$$= -4xe^{-2x}-4e^{-2x}$$

$$=-4e^{-2x}(x+1) \quad \text{Simplify.}$$

****Note:** The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. See the following examples.

Example:

Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

SOLUTION

Instead of using Quotient Rule,

First rewrite f: $\quad f(x) = (x^2 + x + 1)^{-1/3}$

Then, use the Chain Rule

$$f'(x) = -\frac{1}{3}(x^2+x+1)^{-\frac{1}{3}-1} \cdot (2x+1)$$

$$= -\frac{1}{3}(2x+1)(x^2+x+1)^{-\frac{4}{3}}$$

$$= \frac{-(2x+1)}{3(x^2+x+1)^{\frac{4}{3}}} = \frac{-(2x+1)}{3\sqrt[3]{(x^2+x+1)^4}}$$

Example: Find the derivative of the following functions.

(a) $y = \dfrac{(x-1)(x^2-2x)}{x^4}$

SOLUTION

Instead of using *Quotient Rule*, we expand the numerator and divide by x^4:

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3-3x^2+2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}$$

$$\therefore \frac{dy}{dx} = -x^{-2} + 6x^{-3} - 6x^{-4} = -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}$$

(b) $y = \dfrac{1 - \cos x}{\sin x}$

SOLUTION

Solution I: Using Quotient Rule

$$y = \frac{1 - \cos x}{\sin x}$$

$$y' = \frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x}$$

$$= \frac{1 - \cos x}{\sin^2 x} \quad \text{Simplify.}$$

Solution II:

Instead of using *Quotient Rule*, first rewrite

$$y = \frac{1 - \cos x}{\sin x} = \csc x - \cot x.$$

$$y' = -\csc x \cot x + \csc^2 x$$

SAME...

$$= \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = \csc^2 x - \csc x \cot x$$

- DEFINITIONS OF DERIVATIVES
- DIFFERENTIATION RULES
- THE DERIVATIVES AS A RATE OF CHANGE
- MINIMUM & MAXIMUM VALUES

CHAPTER 4 DERIVATIVES

LECTURE 18 – 6.1.2023

C. The Derivative as a Rate of Change

Motion in a straight line

The derivative can also be used to determine the rate of change of one variable with respect to another. A common use for rate of change is to describe the **motion of an object moving in a straight line**. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\Delta s/\Delta t$ represents the average velocity over a time period Δt, and $v = ds/dt$ represents the instantaneous **velocity** (the rate of change of displacement with respect to time). The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$.

$s(t)$	Position function
$v(t) = s'(t)$	Velocity function
$a(t) = v'(t) = s''(t)$	Acceleration function

The speed of an object is the absolute value of its velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

Example: Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 2.21). The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32 \qquad \text{Position function}$$

where s is measured in feet and t is measured in seconds.

a. When does the diver hit the water?

b. What is the diver's velocity at impact?

a) when the diver hits the water, $s(t)=0$

$\therefore -16t^2 + 16t + 32 = 0$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$t-2=0$ or $t+1=0$

$t=2$ # $\qquad t=-1$ (rejected)

b) velocity, $v(t) = s'(t)$

$= -32t + 16$

at $t=2$

$\therefore v(2) = -32(2)+16$

$= -64+16$

$= -48$ #

Example:

The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (that is, in the positive direction)?

(e) Find the acceleration at time t and after 4 s.

a) Velocity, $v(t) = s'$

$= 3t^2 - 12t + 9$

b) at $t=2$

$v(2) = 3(2)^2 - 12(2) + 9$

$= 12 - 24 + 9$

$= -3 \text{ ms}^{-1}$

at $t=4$

$v(4) = 3(4)^2 - 12(4) + 9$

$= 48 - 48 + 9$

$= 9 \text{ ms}^{-1}$

Example:

The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.
(b) What is the velocity after 2 s? After 4 s?
(c) When is the particle at rest?
(d) When is the particle moving forward (that is, in the positive direction)?
(e) Find the acceleration at time t and after 4 s.

e) acceleration, $a(t) = v'(t)$

$= 6t - 12$

at $t = 4$

$a(4) = 6(4) - 12 = 12\ ms^{-2}$ #

c) when the particle at rest,

$v(t) = 0$

$\therefore 3t^2 - 12t + 9 = 0$

$t^2 - 4t + 3 = 0$

$(t-3)(t-1) = 0$

$t - 3 = 0$ or $t - 1 = 0$

$t = 3$ # $\quad t = 1$ #

d) when the particle is moving in the positive direction,

$v(t) > 0$

$\therefore 3t^2 - 12t + 9 > 0$

$(t-1)(t-3) > 0$

$t > 1$ or $t > 3$

$(t-1)$	$-$		$+$		$+$
$(t-3)$	$-$		$-$		$+$
$(t-1)(t-3)$	$+$	1	$-$	3	$+$

$\therefore$ the solution : $t < 1$ or $t > 3$ #

D. Minimum and maximum values

Absolute maximum or minimum

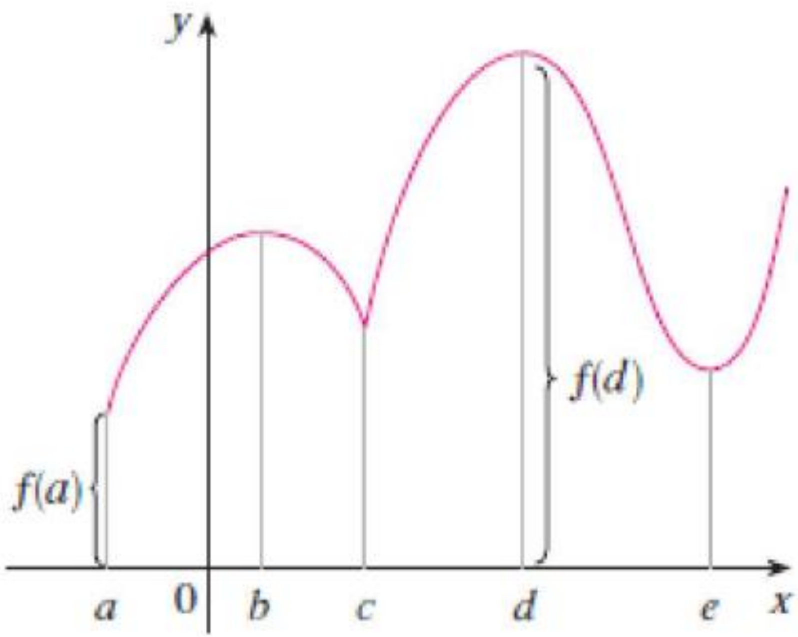
An absolute maximum or minimum is sometimes called a global maximum or minimum. The maximum and minimum values of f are called **extreme values** of f.

Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

Definition The number $f(c)$ is a

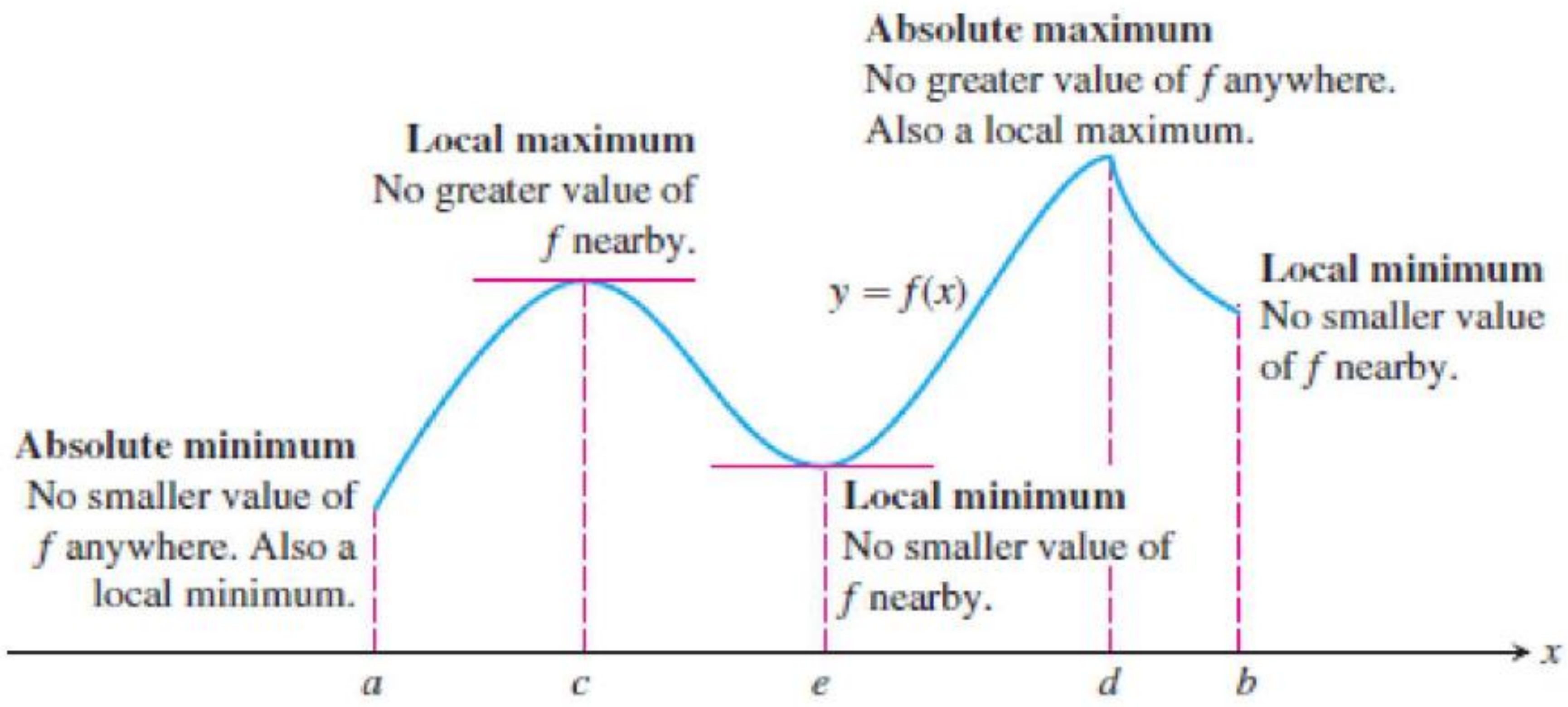
- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.



Abs min $f(a)$, abs max $f(d)$,
loc min $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$

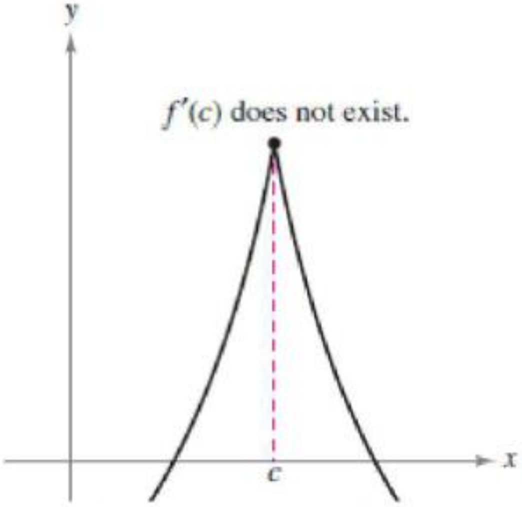
The graph of a function f has **absolute maximum** at d and **absolute minimum** at a. Note that $(d,f(d))$ is the highest point on the graph and $(a,f(a))$ is the lowest point.

If we consider only values of x near b [for instance, if we restrict our attention to the interval (a,c)] then $f(b)$is the largest of those values of $f(x)$ and is called a **local maximum value** of f. Likewise, $f(c)$ is called a **local minimum** value of f because $f(c) \leq f(x)$ for x near c[in the interval (b,d)], for instance]. The function f also has a local minimum at e.

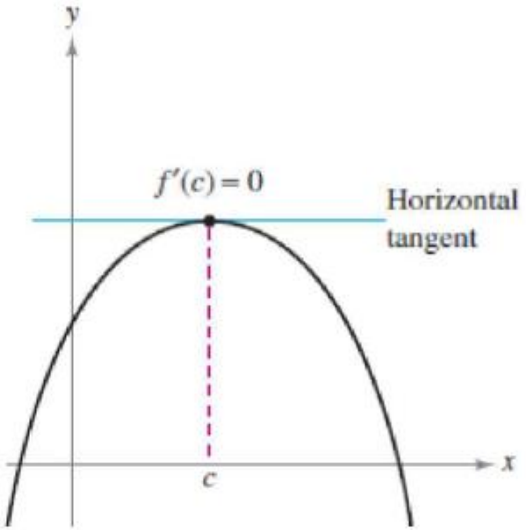

Absolute maximum
No greater value of f anywhere.
Also a local maximum.
Local maximum
No greater value of
f nearby.
y = f(x)
Local minimum
No smaller value
of f nearby.
Absolute minimum
No smaller value of
f anywhere. Also a
local minimum.
Local minimum
No smaller value of
f nearby.
x
a
c
e
d
b

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a critical number of f.



c is a critical number of f.



- ***f* is not differentiable at *c*** means that after finding $f'(x)$, we search for the value of x such that $f'(x)$ is undefined.

Example: Find the critical number(s) of the following functions.

(a) $f(x) = 8x - x^4$. SOLUTION $f'(x) = 8 - 4x^3 = 0$ Critical number $x = \sqrt[3]{2}$	(b) $f(x) = 2x^3 - 3x^2 - 12x$. SOLUTION $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$ $= 6(x-2)(x+1) = 0$ Critical numbers $x = -1, 2$
(c) $f(x) = x^3 - 3x^2 + 1$. SOLUTION $f'(x) = 3x^2 - 6x = 3x(x-2)$ at $f'(x) = 0$; $3x(x-2) = 0 \Rightarrow 3x = 0$ or $x-2=0$ $x = 0$ $x = 2$ Critical numbers $x = 0,2$	(d) $f(x) = 3x^4 - 4x^3$. SOLUTION $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$ at $f'(x) = 0$: $12x^2(x-1) = 0 \Rightarrow 12x^2 = 0$ or $x-1=0$ $x = 0$ $x = 1$ Critical numbers $x = 0, 1$

Example:

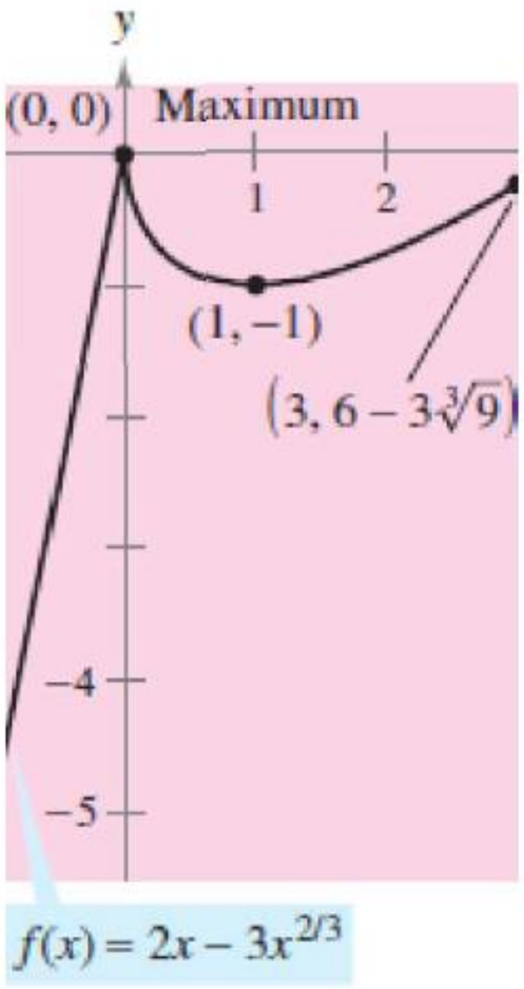
Find the critical number(s) of the function $f(x) = 2x - 3x^{2/3}$

$$f(x) = 2x - 3x^{2/3}$$ Write original function.

$$f'(x) = 2 - \frac{2}{x^{1/3}} = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right)$$ Differentiate.

Critical numbers $x = 0, 1$

The number *1* is a critical number because *f '(1)=0*, and the number *0* is a critical number because *f '(0)*does not exist.

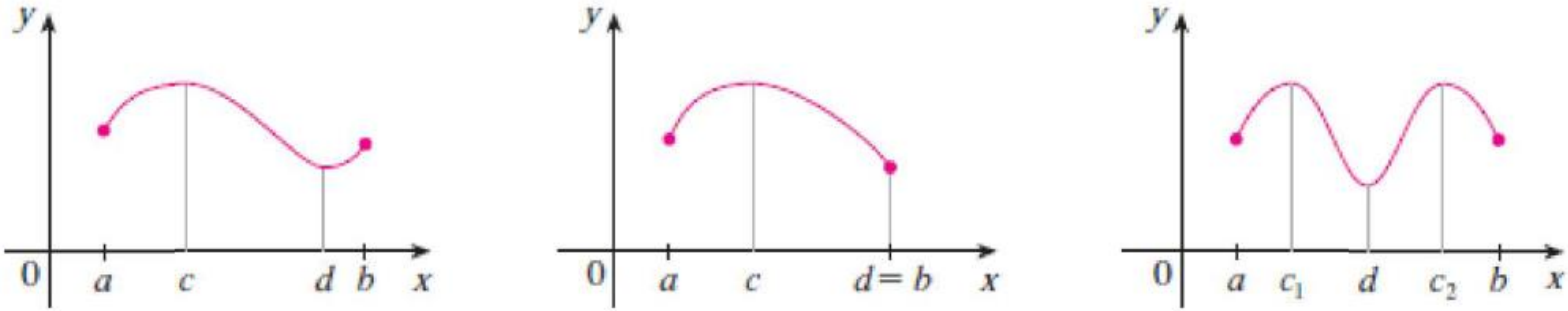


<u>Finding Extrema (*absolute max or min*) on a Closed Interval</u>

Some functions have extreme values, whereas others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

THEOREM The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.



If f has a local maximum or minimum at c, then c is a critical number of f. Hence, to find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local [in which case it occurs at a critical number] or it occurs at an endpoint of the interval. Thus the following three-step procedure always works.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example: Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \leqslant x \leqslant 4 \Leftrightarrow \left[-\tfrac{1}{2}, 4\right]$$

$f'(x) = 3x^2 - 6x$

For critical numbers, $f'(x) = 0$

$\therefore\ 3x^2 - 6x = 0$

$3x(x-2) = 0$

$3x = 0$ or $x - 2 = 0$

$x = 0$ $\qquad x = 2$

x	f(x)
$x=0$	$f(0) = 0^3 - 3(0)^2 + 1 = 1$
$x=2$	$f(2) = 2^3 - 3(2)^2 + 1 = -3$ (min)
$x=-\frac{1}{2}$	$f(-\frac{1}{2}) = (-\frac{1}{2})^3 - 3(-\frac{1}{2})^2 + 1 = \frac{1}{8}$
$x=4$	$f(4) = 4^3 - 3(4)^2 + 1 = 17$ (max)

$\therefore$ absolute minimum $= -3$

absolute maximum $= 17$

Example: Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

$f'(x) = 12x^3 - 12x^2$

For critical numbers, $f'(x) = 0$

$12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$12x^2 = 0$ or $x - 1 = 0$

$x = 0$ $\quad x = 1$

x	$f(x)$
$x = 0$	$f(0) = 3(0)^4 - 4(0)^3 = 0$
$x = 1$	$f(1) = 3(1)^4 - 4(1)^3 = -1$ (min)
$x = -1$	$f(-1) = 3(-1)^4 - 4(-1)^3 = 7$
$x = 2$	$f(2) = 3(2)^4 - 4(2)^3 = 16$ (max)

$\therefore$ absolute minimum $= -1$

absolute maximum $= 16$

Example: Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

SOLUTION

$f'(x) = 2x$

For critical numbers, $f'(x) = 0$

$\therefore 2x = 0$

$x = 0$

x	$f(x)$
$x = 0$	$f(0) = 0^2 = 0$ (min)
$x = -2$	$f(-2) = (-2)^2 = 4$ (max)
$x = 1$	$f(1) = 1^2 = 1$

$\therefore$ absolute minimum $= 0$ #

absolute maximum $= 4$ #

Example:

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

$= \sqrt[3]{x^2}$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

Since $f'(x)$ is undefined when $x = 0$

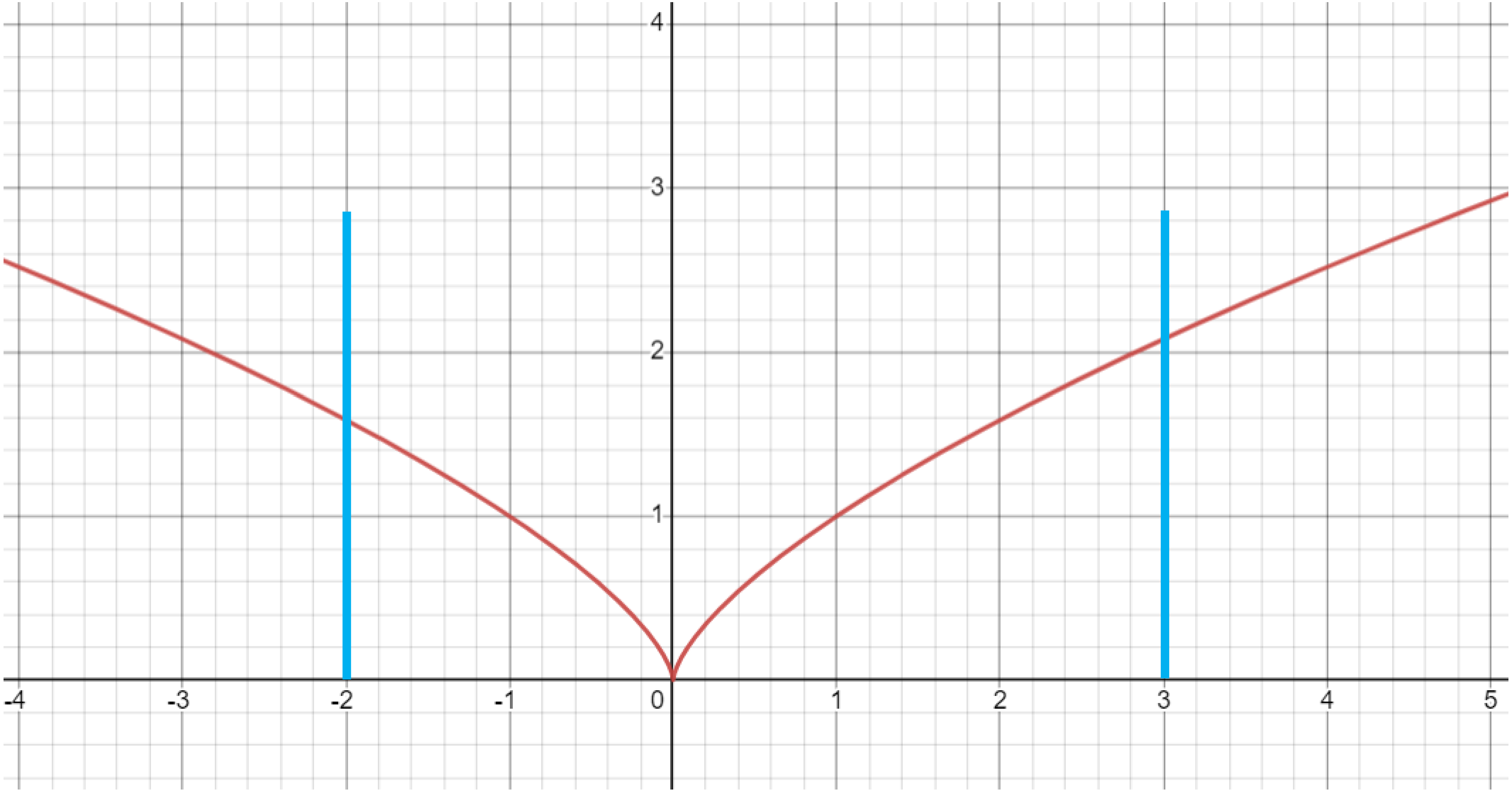
$\therefore$ Critical number, $x = 0$

x	f(x)
$x = 0$	$f(0) = \sqrt[3]{0^2} = 0$ (min)
$x = -2$	$f(-2) = \sqrt[3]{(-2)^2} = \sqrt[3]{4}$
$x = 3$	$f(3) = \sqrt[3]{3^2} = \sqrt[3]{9}$ (max)

$\therefore$ absolute minimum $= 0$ \#

absolute maximum $= \sqrt[3]{9}$ \#

$f(x) = x^{\frac{2}{3}}$



☺ ~ THE END ~ ☺