Topic 4.2 Recursion

TMA1201 Discrete Structures & Probability
Faculty of Computing & Informatics
Multimedia University

What you will learn in this lecture:

- Sequence
- Examples of sequence
- Explicit Formula for a sequence
- Recursive definition of a sequence

Sequence

• Definition

A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

Typically a sequence is represented as a set of elements written in a row.

An example of a sequence is,

This is generally referred as a_k

$$\mathbf{2^0, 2^1, 2^2, 2^3, 2^4}, \ldots, 2^k$$

each individual element, a_k (read "*a* sub *k*") is called a **term**.

Defining a Sequence – Method 1 (Listing)

A sequence can be defined in a variety of different ways.

One informal way is to write the first few terms with the expectation that the general pattern will be obvious.

We might say, for instance, "consider the sequence 3, 5, 7, ..." Unfortunately, misunderstandings can occur when this approach is used.

The next term of the sequence could be 9 if we mean a sequence of odd integers, or it could be 11 if we mean the sequence of odd prime numbers.

Defining a Sequence – Method 2 (Explicit Formula)

The second way to define a sequence is to give an explicit formula for its *n*th term.

For example, a sequence $a_0, a_1, a_2 \ldots$ can be specified by writing

$$a_n = \frac{(-1)^n}{n+1} \quad \text{for all integers } n \geq 0.$$

The advantage of defining a sequence by such an explicit formula is that each term of the sequence is uniquely determined and can be computed in a fixed, finite number of steps, by substitution.

Defining a Sequence – Method 3 (Recursion)

The third way to define a sequence is to use recursion.

All recursive definition consist of two parts:

1) Initial value
2) Recurrence relation

The **recurrence relation** defines each later term in the sequence by reference to earlier terms and one or more initial values for the sequence.

Example

Given the sequence $2^0, 2^1, 2^2, 2^3, 2^4, \ldots$

The explicit formula for the above is:

$$a(n) = 2^n \quad \text{for } n \in N$$

The recursive definition is:

This is the initial value

$$f(0) = 1$$

$$f(n+1) = 2 \times f(n) \quad \text{for } n \in N$$

This is the recurrence relation

Example: Arithmetic Sequence

The recursive definition of the arithmetic sequence is,
initial value: a_1
recurrence relation: $a_{n+1} = a_n + d$

Say $a_1 = 2$ and $d = 4$, then we will have
$a_1 = 2$
$a_2 = a_1 + d = 2 + 4 = 6$
$a_3 = a_2 + d = 6 + 4 = 10$
$a_4 = a_3 + d = 10 + 4 = 14$
$a_5 = a_4 + d = 14 + 4 = 18$
$a_6 = a_5 + d = 18 + 4 = 22$

The explicit formula for the arithmetic sequence,

$$a_n = a + (n - 1)d$$

Say a = 2, d = 4, then we will have
$a_1 = 2 + (1 - 1)4 = 2$
$a_2 = 2 + (2 - 1)4 = 6$
$a_3 = 2 + (3 - 1)4 = 10$
$a_4 = 2 + (4 - 1)4 = 14$
$a_5 = 2 + (5 - 1)4 = 18$
$a_6 = 2 + (6 - 1)4 = 22$

Why recursion?

Recursion is one of the central ideas of computer science.

The idea is to divide and conquer:

> To solve a problem recursively means to find a way to break it down into smaller sub-problems, each having the same form as the original problem, and to do this in such a way that when the process is repeated many times, the last of the sub-problems are small and easy to solve and the solutions of the sub-problems can be woven together to form a solution to the original problem.

Factorial Sequence

The Factorial is defined recursively as

initial value: $f_1 = 1$

recurrence relation: $f_n = n \times f_{n-1}$, for $n \geq 2$

n	f_n
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
⋮	⋮

Fibonacci Sequence

The Fibonacci sequence is defined recursively as

initial values: $f_0 = 1, f_1 = 1$

recurrence relation: $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$

n	f_n
0	1
1	1
2	2
3	3
4	5
5	8
6	13
⋮	⋮

Proving of Explicit Formula for Recursive Sequence

Suppose you have a sequence that satisfies a certain initial value and recurrence relation

It is often helpful to know an explicit formula for the sequence, especially if you need to compute terms with very large subscripts or if you need to examine general properties of the sequence.

However, explicit formulas can be difficult to determine, and once found, needs proving for its correctness.

Proving of explicit formula can be done by using the Principles of Mathematical Induction discussed in lecture 07.

Example

Show that $a_n = 1 + 2n$ for all $n \in$ N is the explicit formula for the recursion with initial value $a_0 = 1$ and recurrence relation $a_n = a_{n-1} + 2$ for all $n>0$

Solution:

Let $b_n = 1 + 2n$

Inductive base: $n = 0$

$b_0 = 1 + 2(0) = 1 = a_0$

$a_n = b_n$ is true for $n = 0$

Inductive hypothesis: $n = k$

Assume that $a_k = b_k$ is true for n = k

Inductive step: $n = k + 1$

$a_{k+1} = a_k + 2$

$a_{k+1} = b_k + 2 = 1 + 2(k) + 2$ (from Hypothesis)

$a_{k+1} = 1 + 2(k + 1) = b_{k+1}$

Hence, by mathematical induction $a_n = b_n$ is true for all $n \in$ N

Example

Given the recursion: $S_0 = 2,$
$S_1 = 4,$
$S_n = 4S_n - 3S_{n-1}$, for $n \geq 2$.

Use **strong induction** method to prove that $S_n = 1 + 3^n \;\; \forall n \geq 0$ is the explicit formula for the above recursion.

Solution:

Let the explicit formula $T_n = 1 + 3^n$.

Inductive base: $T_0 = 1 + 3^0 = 2 = S_0,$
$T_1 = 1 + 3^1 = 4 = S_1 \quad \therefore T_n$ is true for n=0, 1.

Inductive hypothesis:

Assume that the explicit formula $T_n = S_n$ is true for *n* = 0,1,2, …, *k*-1, *k*

Inductive step:

(Note: We want to show that $S_{k+1} = T_{k+1}$).

$S_{k+1} = 4S_k - 3S_{k-1}$ (from recursion)
$= 4T_k - 3\, T_{k-1}$ (from hypothesis)
$= 4(1 + 3^k) - 3(1 + 3^{k-1})$
$= 4 + 4(3^k) - 3 - 3(3^{k-1}) \quad = 4 + 4(3^k) - 3 - 3^k \quad = 1 + 3(3^k)$
$= 1 + 3^{k+1} = T_{k+1}$

T_{k+1} is true whenever T_k and T_{k-1} are true.

$\therefore$ The explicit formula $T_n = 1 + 3^n$ is true for $\forall n \geq 0$.

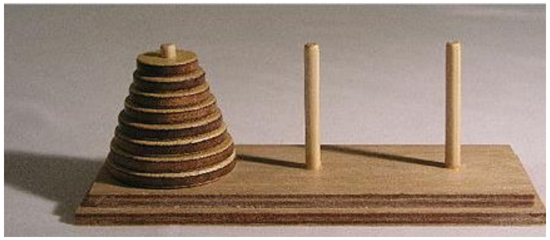
Try this

Show that $\frac{3^{n+1}-1}{2}$ is the explicit formula for the recursion defined below:

Initial value: $g(0) = 1$
Recurrence relation: $g(n) = g(n - 1) + 3^n$

Solution:

The Tower of Hanoi



- The Tower of Hanoi (also called the Tower of Brahma or Lucas' Tower) is a mathematical game or puzzle. It consists of three rods and a number of disks of different sizes, which can slide onto any rod. The puzzle starts with the disks in a neat stack in ascending order of size on one rod, the smallest at the top, thus making a conical shape.
- The objective of the puzzle is to move the entire stack to another rod, obeying the following rules:
  1) Only one disk can be moved at a time.
  2) Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod.
  3) No larger disk may be placed on top of a smaller disk.

The Tower of Hanoi

- The minimal number of moves required to move a stack of n disks from one rod to another is $2^n - 1$.
- Can you come out with a recursive formula that compute the minimum number of moves? Verify your answer with the explicit formula for number of disks up to 5.

Summary

We have learnt the following concepts related to recursion:

- Apart from listing the elements, a sequence can be defined by an **explicit formula** or by **recursion**(recurrence definition).
- The recurrence definition involves two parts, the initial value and the recurrence relation.
- A proof by induction can be applied to prove the correctness of an explicit formula for a recurrence definition.

Exercise 1

For the sequence 10, 50, 250, 1250, …

1. Determine the first term ***a*** and the common ratio **r**.
2. Infer an explicit formula for this sequence.
3. Write its recursive definition.
4. Proof that the explicit formula in part 2 is the correct formula for the recursive definition you have written in part 3.

Solution:

Exercise 2

Given the recursion $r_0 = 1$, $r_1 = 7$ and $r_n = 2r_{n-1} + 3r_{n-2}, n \geq 2$.

a) Find the value r_3 and r_4.

b) By mathematical induction show that $r_n = (-1)^{n+1} + 2 \cdot 3^n$ for $n \geq 0$.