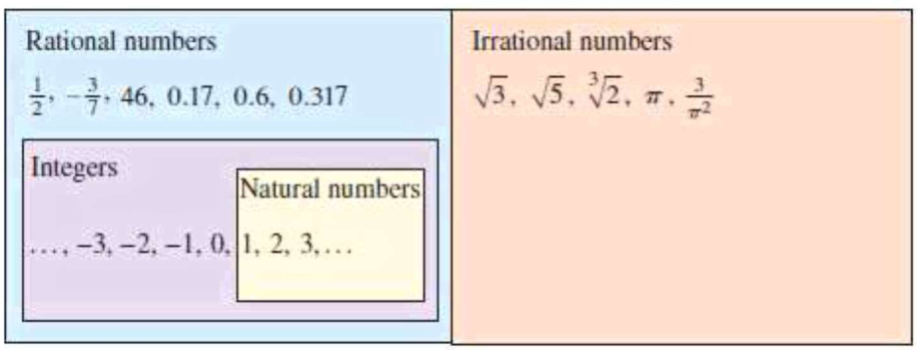
Fundamentals of Algebra

1.1 REAL NUMBERS (Adapted from "Precalculus" by Stewart et als.)

Properties of Real Numbers └ Addition and Subtraction └ Multiplication and Division └ The Real Line └ Sets and Intervals └ Absolute Value and Distance

Real numbers



Every real number has a decimal representation. If the number is rational, then its corresponding decimal is repeating.

$$\tfrac{1}{2} = 0.5000\ldots = 0.5\overline{0} \qquad \tfrac{2}{3} = 0.66666\ldots = 0.\overline{6}$$

$$\tfrac{157}{495} = 0.3171717\ldots = 0.3\overline{17} \qquad \tfrac{9}{7} = 1.285714285714\ldots = 1.\overline{285714}$$

(The bar indicates that the sequence of digits repeats forever.)
If the number is irrational, the decimal representation is nonrepeating:

$$\sqrt{2} = 1.414213562373095\ldots \qquad \pi = 3.141592653589793\ldots$$

Arithmetic operations

OPERATION	NOTATION	RESULT
Addition	$a+b$	Sum
Subtraction	$a-b$	Difference
Multiplication	$a \cdot b$ **or** ab **or** $(a)(b)$	Product
Division	$\frac{a}{b}$ $(b \neq 0)$	Quotient

- The form a/b for quotient is to be discouraged, especially in handwriting.
- Avoid using a diagonal slash to write a fraction.

Correct ***order of arithmetic operations***
when evaluating expressions involving real numbers

1. Start with the innermost grouping symbol and work outward.
(grouping symbols: parentheses ( ), brackets [ ], and braces { })
2. Perform all multiplications and divisions, working from left to right.
3. Perform all additions and subtractions, working from left to right.

Exponents are an important part of order of operations and will be discussed later.

Mnemonic: PEMDAS (*other versions?*)

P	Parentheses
E	Exponents
M, D	Multiplication, Division
A, S	Addition, Subtraction

PROPERTIES OF REAL NUMBERS

Property	Example	Description
Commutative Properties		
$a + b = b + a$	$7 + 3 = 3 + 7$	When we add two numbers, order doesn't matter.
$ab = ba$	$3 \cdot 5 = 5 \cdot 3$	When we multiply two numbers, order doesn't matter.
Associative Properties		
$(a + b) + c = a + (b + c)$	$(2 + 4) + 7 = 2 + (4 + 7)$	When we add three numbers, it doesn't matter which two we add first.
$(ab)c = a(bc)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$	When we multiply three numbers, it doesn't matter which two we multiply first.
Distributive Property		
$a(b + c) = ab + ac$ $(b + c)a = ab + ac$	$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5$ $(3 + 5) \cdot 2 = 2 \cdot 3 + 2 \cdot 5$	When we multiply a number by a sum of two numbers, we get the same result as we would get if we multiply the number by each of the terms and then add the results.

E X A M P L E 1 Using the Distributive Property

(a) $2(x + 3) = 2 \cdot x + 2 \cdot 3$ Distributive Property
$= 2x + 6$ Simplify

(b) $(a + b)(x + y) = (a + b)x + (a + b)y$ Distributive Property
$= (ax + bx) + (ay + by)$ Distributive Property
$= ax + bx + ay + by$ Associative Property of Addition

NOW TRY: State the property of real numbers being used. $(5x+1)3 = 15x+3$

▼ Addition and Subtraction

The number 0 is called the **additive identity** because $a+0=0+a=a$ for any real number a.

Every real number a has a **negative** $-a$, that satisfies $a+(-a)=0$

Subtraction is the operation that undoes addition; subtracting b is the same as adding the negative of b**.**

$$a-b=a+(-b)$$

To combine real numbers involving negatives, we use the following properties.

PROPERTIES OF NEGATIVES

Property	Example
1. $(-1)a=-a$	$(-1)5=-5$
2. $-(-a)=a$	$-(-5)=5$
3. $(-a)b=a(-b)=-(ab)$	$(-5)7=5(-7)=-(5\cdot 7)$
4. $(-a)(-b)=ab$	$(-4)(-3)=4\cdot 3$
5. $-(a+b)=-a-b$	$-(3+5)=-3-5$
6. $-(a-b)=b-a$	$-(5-8)=8-5$

E X A M P L E 2 Using Properties of Negatives

Let x, y, and z be real numbers.

(a) $-(x+5)=-x-5$

(b) $-(x+y-z)=-x-y-(-z)=-x-y+z$

NOW TRY: Use properties of real numbers to write the expression without parenthesis.

$-\frac{5}{2}(2x-4y)$

▼ Multiplication and Division

The number 1 is called the **multiplicative identity** because $a\cdot 1=1\cdot a=a$ for any real number a.

Every **nonzero** real number a has a **reciprocal** (or **multiplicative inverse**), $\frac{1}{a}$, that satisfies $a\cdot\frac{1}{a}=1$.

Division is the operation that undoes multiplication; dividing by a non-zero number b is the same as multiplying by the reciprocal of b.

$$a\div b=a\cdot\frac{1}{b}=\frac{a}{b}$$

In the **fraction** $\frac{a}{b}$ (a over b); a is the **numerator** and b is the **denominator** (or **divisor**).

PROPERTIES OF FRACTIONS

Property	Example	Description
1. $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$	$\frac{2}{3}\cdot\frac{5}{7}=\frac{2\cdot5}{3\cdot7}=\frac{10}{21}$	When **multiplying fractions**, multiply numerators and denominators.
2. $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$	$\frac{2}{3}\div\frac{5}{7}=\frac{2}{3}\cdot\frac{7}{5}=\frac{14}{15}$	When **dividing fractions**, invert the divisor and multiply.
3. $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$	$\frac{2}{5}+\frac{7}{5}=\frac{2+7}{5}=\frac{9}{5}$	When **adding fractions** with the **same denominator**, add the numerators.
4. $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$	$\frac{2}{5}+\frac{3}{7}=\frac{2\cdot7+3\cdot5}{35}=\frac{29}{35}$	When **adding fractions** with **different denominators**, find a common denominator. Then add the numerators.
5. $\frac{ac}{bc}=\frac{a}{b}$	$\frac{2\cdot5}{3\cdot5}=\frac{2}{3}$	**Cancel** numbers that are **common factors** in numerator and denominator.
6. If $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$	$\frac{2}{3}=\frac{6}{9}$, so $2\cdot9=3\cdot6$	**Cross-multiply**.

When adding fractions with different denominators, we don't usually use Property 4. Instead we rewrite the fractions so that they have the smallest possible common denominator (often smaller than the product of the denominators), and then we use Property 3.
This denominator is the **L**east **C**ommon **D**enominator (LCD) described in the next example.

E X A M P L E 3 Using the LCD to Add Fractions

Evaluate: $\frac{5}{36}+\frac{7}{120}$

S O LU T I O N Factoring each denominator into prime factors gives

$36=2^2\cdot3^2,\quad 120=2^3\cdot3\cdot5$

We find the least common denominator (LCD) by forming the product of all the factors that occur in these factorizations, using the highest power of each factor.
Thus the LCD is $2^3\cdot3^2\cdot5=360$.
So

$$\frac{7}{36}+\frac{11}{120}=\frac{7\cdot10}{36\cdot10}+\frac{11\cdot3}{120\cdot3}=\frac{70}{360}+\frac{33}{360}=\frac{103}{360}$$

Students are expected to write proper steps like this:

$36=2^2\cdot3^2,\quad 120=2^3\cdot3\cdot5$

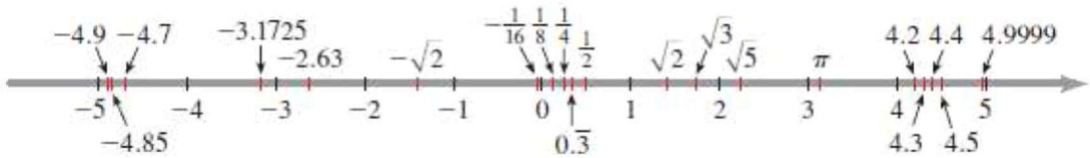
The LCD of 36 and 120 is $2^3\cdot3^2\cdot5=360$.
So

$$\frac{7}{36}+\frac{11}{120}=\frac{7\cdot10}{36\cdot10}+\frac{11\cdot3}{120\cdot3}=\frac{70}{360}+\frac{33}{360}=\frac{103}{360}$$

NOW TRY: $\frac{4}{15}+\frac{3}{10}$

▼ The Real Line

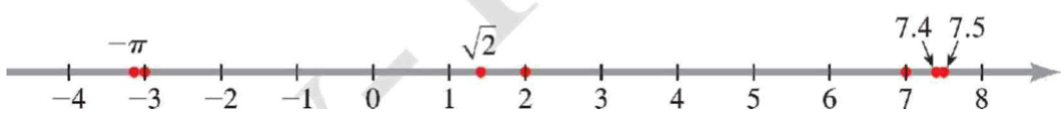
The real numbers can be represented by points on a line, as shown here.



The positive direction (toward the right) is indicated by an arrow.
We choose an arbitrary reference point O, called the **origin**, which corresponds to the real number 0.
Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and each negative number y is represented by the point $(-y)$ units to the left of the origin.
The number associated with the point P is called the coordinate of P, and the line is then called a **coordinate line**, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

The real numbers are <u>*ordered*</u>.
We say that ***a* is less than *b*** and write $a < b$ if $b - a$ is a positive number.
Geometrically, this means that a lies to the left of b on the number line.
Equivalently, we can say that ***b* is greater than *a*** and write $b > a$.
The symbol $a \le b$ (or $b \ge a$) means that either $a < b$ or $a = b$ and is read "a is less than or equal to b."
Examples:

$7 < 7.4 < 7.5$ $\qquad -\pi < -3$ $\qquad \sqrt{2} < 2$ $\qquad 2 \le 2$



Sets and Intervals

A **set** is a collection of objects, and these objects are called the **elements** of the set.
If S is a set, the notation $a \in S$ means that a is an element of S, and $b \notin S$ means that b is not an element of S.
For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$.

Some sets can be described by listing their elements within braces. For instance, the set A that consists of all positive integers less than 7 can be written as
$A = \{1, 2, 3, 4, 5, 6\}$

We could also write A in **set-builder notation** as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of all x such that x is an integer and $0 < x < 7$."

If S and T are sets, then their **union** $S \cup T$ is the set that consists of all elements that are in S *or* T (or in both).
The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S *and* T. In other words, $S \cap T$ is the common part of S and T.
The **empty set**, denoted by Ø, is the set that contains no element.

E X A M P L E 4 Union and Intersection of Sets
If $S = \{1, 2, 3, 4, 5\}$, $T = \{4, 5, 6, 7\}$, and $V = \{6, 7, 8\}$,
find the sets $S \cup T$, $S \cap T$, and $S \cap V$.
S O LU T I O N: ???

NOW TRY: If $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{2, 4, 6, 8\}$, find the sets $A \cup B$ and, $A \cap B$, and $S \cap V$.

Intervals (***Students are required to master all three versions shown in the table***.)
Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments.

Notation	Set description	Picture
(a, b)	$\{x \mid a < x < b\}$	a b
$[a, b]$	$\{x \mid a \leqslant x \leqslant b\}$	a b
$[a, b)$	$\{x \mid a \leqslant x < b\}$	a b
$(a, b]$	$\{x \mid a < x \leqslant b\}$	a b
(a, ∞)	$\{x \mid x > a\}$	a
$[a, \infty)$	$\{x \mid x \geqslant a\}$	a
$(-\infty, b)$	$\{x \mid x < b\}$	b
$(-\infty, b]$	$\{x \mid x \leqslant b\}$	b
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

If $a < b$, then the **open interval** from a to b consists of all numbers between a and b and is denoted (a, b).

Open interval (a, b)

[Note that parentheses in the interval notation and open circles on the graph indicate that endpoints are *excluded* from the interval.]

The **closed interval** from a to b includes the endpoints and is denoted $[a, b]$.

Closed interval $[a, b]$

[Note that square brackets and solid circles indicate that the endpoints are *included*.]

Intervals may also include one endpoint but not the other, or they may extend infinitely far in one direction or both.

Although the symbol ∞ ("infinity") is used in some of the notations, this does not mean that ∞ is a number (i.e., ∞ **is NOT a number**).
For example, the notation (a, ∞) stands for the set of all numbers that are greater than a, so the symbol ∞ simply indicates that the interval extends indefinitely far in the positive direction.

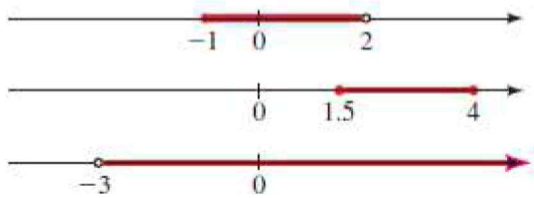
EXAMPLE 5 | Graphing Intervals

Express each interval in terms of inequalities, and then graph the interval.

(a) $[-1, 2) = \{x \mid -1 \le x < 2\}$

(b) $[1.5, 4] = \{x \mid 1.5 \le x \le 4\}$

(c) $(-3, \infty) = \{x \mid -3 < x\}$



NOW TRY: Express the interval in terms of inequalities and then graph the interval.
(–1, 3)

E X A M P L E 6 Finding Unions and Intersections of Intervals

Graph each set.

(a) $(1, 3) \cap [2, 7]$ **(b)** $(1, 3) \cup [2, 7]$

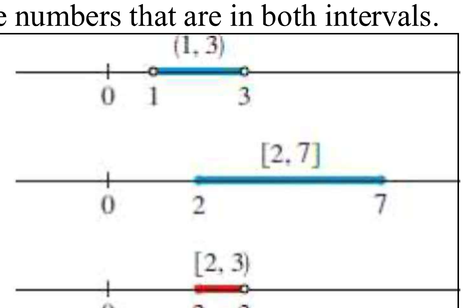
Solution:

(a) The intersection of two intervals consists of the numbers that are in both intervals.

Therefore

$$(1, 3) \cap [2, 7] = \{x \mid 1 < x < 3 \text{ and } 2 \le x \le 7\}$$

$$= \{x \mid 2 \le x < 3\} = [2, 3)$$



(b) ???

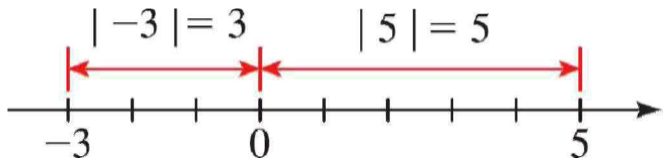
NOW TRY: Graph the set $(-1, 0) \cup (-1, 1)$.

Note: ***The union or intersection of intervals may not necessarily be an interval.***

Can you provide some examples?

▼ Absolute Value and Distance

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line.



Distance is always positive or zero, so we have $|a| \geq 0$ for every number a. Remembering that $-a$ is positive when a is negative, we have the following definition.

DEFINITION OF ABSOLUTE VALUE

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

EXAMPLE 7 Evaluating Absolute Values of Numbers

(a) $|3| = 3$

(b) $|-3| = -(-3) = 3$

(c) $|0| = 0$

(d) $|3 - \pi| = -(3 - \pi) = \pi - 3$ (since $3 < \pi \Rightarrow 3 - \pi < 0$)

NOW TRY: Evaluate $|100|$, $|\sqrt{5} - 5|$.

PROPERTIES OF ABSOLUTE VALUE

Property	Example	Description
1. $\lvert a \rvert \geq 0$	$\lvert -3 \rvert = 3 \geq 0$	The absolute value of a number is always positive or zero.
2. $\lvert a \rvert = \lvert -a \rvert$	$\lvert 5 \rvert = \lvert -5 \rvert$	A number and its negative have the same absolute value.
3. $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	$\lvert -2 \cdot 5 \rvert = \lvert -2 \rvert \lvert 5 \rvert$	The absolute value of a product is the product of the absolute values.
4. $\left\lvert \dfrac{a}{b} \right\rvert = \dfrac{\lvert a \rvert}{\lvert b \rvert}$	$\left\lvert \dfrac{12}{-3} \right\rvert = \dfrac{\lvert 12 \rvert}{\lvert -3 \rvert}$	The absolute value of a quotient is the quotient of the absolute values.

DISTANCE BETWEEN POINTS ON THE REAL LINE

If a and b are real numbers, then the **distance** between the points a and b on the real line is

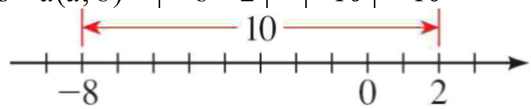
$$d(a, b) = |b - a|$$

Since $|b - a| = |a - b|$, the distance from a to b is the same as the distance from b to a.

E X A M P L E 8 Distance Between Points on the Real Line

The distance between the numbers –8 and 2 is $d(a, b) = |-8 - 2| = |-10| = 10$

We can check this calculation geometrically:



NOW TRY: Find the distance between the given numbers.

(a) 2 and 17 (b) -3 and 21 (c) $\frac{11}{8}$ and $-\frac{3}{10}$.

Exercise

1.1 Exercises

1–2 ▪ List the elements of the given set that are

(a) natural numbers
(b) integers
(c) rational numbers
(d) irrational numbers

1. $\{0, -10, 50, \frac{22}{7}, 0.538, \sqrt{7}, 1.2\overline{3}, -\frac{1}{3}, \sqrt[3]{2}\}$

2. $\{1.001, 0.333\ldots, -\pi, -11, 11, \frac{13}{15}, \sqrt{16}, 3.14, \frac{15}{3}\}$

3–10 ▪ State the property of real numbers being used.

3. $7 + 10 = 10 + 7$

4. $2(3 + 5) = (3 + 5)2$

5. $(x + 2y) + 3z = x + (2y + 3z)$

6. $2(A + B) = 2A + 2B$

7. $(5x + 1)3 = 15x + 3$

8. $(x + a)(x + b) = (x + a)x + (x + a)b$

9. $2x(3 + y) = (3 + y)2x$

10. $7(a + b + c) = 7(a + b) + 7c$

11–14 ▪ Rewrite the expression using the given property of real numbers.

11. Commutative Property of addition, $x + 3 = \boxed{}$

12. Associative Property of multiplication, $7(3x) = \boxed{}$

13. Distributive Property, $4(A + B) = \boxed{}$

14. Distributive Property, $5x + 5y = \boxed{}$

15–20 ▪ Use properties of real numbers to write the expression without parentheses.

15. $3(x + y)$

16. $(a - b)8$

17. $4(2m)$

18. $\frac{4}{3}(-6y)$

19. $-\frac{5}{2}(2x - 4y)$

20. $(3a)(b + c - 2d)$

21–26 ▪ Perform the indicated operations.

21. (a) $\frac{3}{10} + \frac{4}{15}$ **(b)** $\frac{1}{4} + \frac{1}{5}$

22. (a) $\frac{2}{3} - \frac{3}{5}$ **(b)** $1 + \frac{5}{8} - \frac{1}{6}$

23. (a) $\frac{2}{3}(6 - \frac{3}{2})$ **(b)** $0.25(\frac{8}{9} + \frac{1}{2})$

24. (a) $(3 + \frac{1}{4})(1 - \frac{4}{5})$ **(b)** $(\frac{1}{2} - \frac{1}{3})(\frac{1}{2} + \frac{1}{3})$

25. (a) $\dfrac{2}{\frac{2}{3}} - \dfrac{\frac{2}{3}}{2}$ **(b)** $\dfrac{\frac{1}{12}}{\frac{1}{8} - \frac{1}{9}}$

26. (a) $\dfrac{2 - \frac{3}{4}}{\frac{1}{2} - \frac{1}{3}}$ **(b)** $\dfrac{\frac{2}{5} + \frac{1}{2}}{\frac{1}{10} + \frac{3}{15}}$

27–28 ▪ Place the correct symbol ($<$, $>$, or $=$) in the space.

27. (a) $3 \;\boxed{}\; \frac{7}{2}$ **(b)** $-3 \;\boxed{}\; -\frac{7}{2}$ **(c)** $3.5 \;\boxed{}\; \frac{7}{2}$

28. (a) $\frac{2}{3} \;\boxed{}\; 0.67$ **(b)** $\frac{2}{3} \;\boxed{}\; -0.67$ **(c)** $|0.67| \;\boxed{}\; |-0.67|$

29–32 ▪ State whether each inequality is true or false.

29. (a) $-6 < -10$ **(b)** $\sqrt{2} > 1.41$

30. (a) $\dfrac{10}{11} < \dfrac{12}{13}$ **(b)** $-\dfrac{1}{2} < -1$

31. (a) $-\pi > -3$ **(b)** $8 \le 9$

32. (a) $1.1 > 1.\overline{1}$ **(b)** $8 \le 8$

33–34 ▪ Write each statement in terms of inequalities.

33. (a) x is positive
(b) t is less than 4
(c) a is greater than or equal to π
(d) x is less than $\frac{1}{3}$ and is greater than -5
(e) The distance from p to 3 is at most 5

34. (a) y is negative
(b) z is greater than 1
(c) b is at most 8

(d) w is positive and is less than or equal to 17

(e) y is at least 2 units from π

35–38 ■ Find the indicated set if

$$A = \{1, 2, 3, 4, 5, 6, 7\} \qquad B = \{2, 4, 6, 8\}$$
$$C = \{7, 8, 9, 10\}$$

35. **(a)** $A \cup B$ **(b)** $A \cap B$

36. **(a)** $B \cup C$ **(b)** $B \cap C$

37. **(a)** $A \cup C$ **(b)** $A \cap C$

38. **(a)** $A \cup B \cup C$ **(b)** $A \cap B \cap C$

39–40 ■ Find the indicated set if

$$A = \{x \mid x \geq -2\} \qquad B = \{x \mid x < 4\}$$
$$C = \{x \mid -1 < x \leq 5\}$$

39. **(a)** $B \cup C$ **(b)** $B \cap C$

40. **(a)** $A \cap C$ **(b)** $A \cap B$

41–46 ■ Express the interval in terms of inequalities, and then graph the interval.

41. $(-3, 0)$ **42.** $(2, 8]$

43. $[2, 8)$ **44.** $[-6, -\frac{1}{2}]$

45. $[2, \infty)$ **46.** $(-\infty, 1)$

47–52 ■ Express the inequality in interval notation, and then graph the corresponding interval.

47. $x \leq 1$ **48.** $1 \leq x \leq 2$

49. $-2 < x \leq 1$ **50.** $x \geq -5$

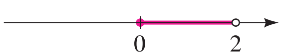
51. $x > -1$ **52.** $-5 < x < 2$

53–54 ■ Express each set in interval notation.

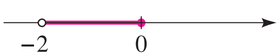
53. **(a)**

(b)

54. **(a)**



(b)



55–60 ■ Graph the set.

55. $(-2, 0) \cup (-1, 1)$ **56.** $(-2, 0) \cap (-1, 1)$

57. $[-4, 6] \cap [0, 8)$ **58.** $[-4, 6) \cup [0, 8)$

59. $(-\infty, -4) \cup (4, \infty)$ **60.** $(-\infty, 6] \cap (2, 10)$

61–66 ■ Evaluate each expression.

61. **(a)** $|100|$ **(b)** $|-73|$

62. **(a)** $|\sqrt{5} - 5|$ **(b)** $|10 - \pi|$

63. **(a)** $\big||-6| - |-4|\big|$ **(b)** $\dfrac{-1}{|-1|}$

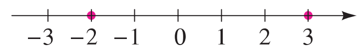
64. **(a)** $\big|2 - |-12|\big|$ **(b)** $-1 - \big|1 - |-1|\big|$

65. **(a)** $|(-2) \cdot 6|$ **(b)** $|(-\frac{1}{3})(-15)|$

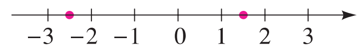
66. **(a)** $\left|\dfrac{-6}{24}\right|$ **(b)** $\left|\dfrac{7 - 12}{12 - 7}\right|$

67–70 ■ Find the distance between the given numbers.

67.



68.



69. **(a)** 2 and 17

(b) -3 and 21

(c) $\frac{11}{8}$ and $-\frac{3}{10}$

70. **(a)** $\frac{7}{15}$ and $-\frac{1}{21}$

(b) -38 and -57

(c) -2.6 and -1.8

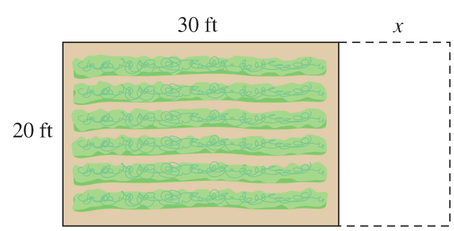
71–72 ■ Express each repeating decimal as a fraction. (See the margin note on page 2.)

71. **(a)** $0.\overline{7}$ **(b)** $0.2\overline{8}$ **(c)** $0.\overline{57}$

72. **(a)** $5.\overline{23}$ **(b)** $1.3\overline{7}$ **(c)** $2.1\overline{35}$

Applications

73. Area of a Garden Mary's backyard vegetable garden measures 20 ft by 30 ft, so its area is $20 \times 30 = 600$ ft^2. She decides to make it longer, as shown in the figure, so that the area increases to $A = 20(30 + x)$. Which property of real numbers tells us that the new area can also be written $A = 600 + 20x$?



74. Temperature Variation The bar graph shows the daily high temperatures for Omak, Washington, and Geneseo, New York, during a certain week in June. Let T_O represent the temperature in Omak and T_G the temperature in Geneseo. Calculate $T_O - T_G$ and $|T_O - T_G|$ for each day shown.

Answer

1. **(a)** 50 **(b)** 0, -10, 50 **(c)** 0, -10, 50, $\frac{22}{7}$, 0.538, $1.2\overline{3}$, $-\frac{1}{3}$ **(d)** $\sqrt{7}$, $\sqrt[3]{2}$ **3.** Commutative Property for addition
5. Associative Property for addition **7.** Distributive Property
9. Commutative Property for multiplication
11. $3 + x$ **13.** $4A + 4B$ **15.** $3x + 3y$ **17.** $8m$
19. $-5x + 10y$ **21.** **(a)** $\frac{17}{30}$ **(b)** $\frac{9}{20}$ **23.** **(a)** 3 **(b)** $\frac{25}{72}$
25. **(a)** $\frac{8}{3}$ **(b)** 6 **27.** **(a)** $<$ **(b)** $>$ **(c)** $=$ **29.** **(a)** False **(b)** True **31.** **(a)** False **(b)** True **33.** **(a)** $x > 0$ **(b)** $t < 4$ **(c)** $a \geq \pi$ **(d)** $-5 < x < \frac{1}{3}$ **(e)** $|p - 3| \leq 5$
35. **(a)** $\{1, 2, 3, 4, 5, 6, 7, 8\}$ **(b)** $\{2, 4, 6\}$
37. **(a)** $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ **(b)** $\{7\}$
39. **(a)** $\{x \mid x \leq 5\}$ **(b)** $\{x \mid -1 < x < 4\}$

41. $-3 < x < 0$

43. $2 \leq x < 8$

45. $x \geq 2$ 

47. $(-\infty, 1]$ 

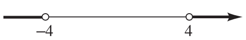
49. $(-2, 1]$ 

51. $(-1, \infty)$ 

53. **(a)** $[-3, 5]$ **(b)** $(-3, 5]$

55. 

57. 

59. 

61. **(a)** 100 **(b)** 73 **63.** **(a)** 2 **(b)** -1 **65.** **(a)** 12 **(b)** 5 **67.** 5 **69.** **(a)** 15 **(b)** 24 **(c)** $\frac{67}{40}$ **71.** **(a)** $\frac{7}{9}$ **(b)** $\frac{13}{45}$ **(c)** $\frac{19}{33}$ **73.** Distributive Property
75. **(a)** Yes, no **(b)** 6 ft

1.2 EXPONENTS AND RADICALS
(Adapted from "Precalculus" by Stewart et als.)

Integer Exponents └ Rules for Working with Exponents └ Radicals
└ Rational Exponents └ Rationalizing the Denominator

▼ Integer Exponents
A product of identical numbers is usually written in exponential notation.
For example, $5 \cdot 5 \cdot 5$ is written as 5^3.
In general, we have the following definition.

> **EXPONENTIAL NOTATION**
>
> If a is any real number and n is a positive integer, then the **nth power** of a is
>
> $$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$
>
> The number a is called the **base**, and n is called the **exponent**.

This gives only the definition for a^n when n is a positive integer.
Examples: 3^5 and 112^{15} make sense but not for 3^{-5} and 3^0.

EXAMPLE 1 | Exponential Notation

(a) $\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{32}$

(b) $(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$

(c) $-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$

NOW TRY : Evaluate each expression.

(a) -3^2 **(b)** $(-3)^2$ **(c)** $\left(\frac{1}{3}\right)^4(-3)^2$

▼ Rules for Working with Exponents
We can state several useful rules for working with exponential notation.

LAWS OF EXPONENTS

Law	Example	Description
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$	To multiply two powers of the same number, add the exponents.
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^5}{3^2} = 3^{5-2} = 3^3$	To divide two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$	To raise a power to a new power, multiply the exponents.
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2$	To raise a product to a power, raise each factor to the power.
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$	To raise a quotient to a power, raise both numerator and denominator to the power.

[Can you prove these laws for positive integers m and n?]

We would like these rules to be true even when m and n are 0 or negative integers. But we need to give meaning to a^n when n is 0 or a negative integer. These rules are true even when m and n are rational numbers, after we define a^n for rational number n.

ZERO AND NEGATIVE EXPONENTS

If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$

EXAMPLE 2 | Zero and Negative Exponents

(a) $\left(\frac{4}{7}\right)^0 = 1$

(b) $x^{-1} = \dfrac{1}{x^1} = \dfrac{1}{x}$

(c) $(-2)^{-3} = \dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8}$

NOW TRY : Evaluate each expression.

(a) $\left(\frac{5}{3}\right)^0 2^{-1}$ (b) $\dfrac{2^{-3}}{3^0}$ (c) $\left(\frac{1}{4}\right)^{-2}$

E X A M P L E 3 Using Laws of Exponents

(a) x^4x^7 **(b)** y^4y^{-7} **(c)** $\dfrac{c^9}{c^5}$ **(d)** $(b^4)^5$ **(e)** $(3x)^3$ **(f)** $\left(\dfrac{x}{2}\right)^5$

??????

NOW TRY : Simplify each expression.

(i) (a) x^8x^2 (b) $(3y^2)(4y^5)$ (c) x^2x^{-6}

(ii) (a) $\dfrac{y^{10}y^0}{y^7}$ (b) $\dfrac{x^6}{x^{10}}$ (c) $\dfrac{a^9a^{-2}}{a}$

(iii) (a) $(a^2a^4)^3$ (b) $\left(\dfrac{a^2}{4}\right)^3$ (c) $(3z)^2(6z^2)^{-3}$

E X A M P L E 4 Simplifying Expressions with Exponents

Simplify:

(a) $(2a^3b^2)(3ab^4)^3$ (b) $\left(\dfrac{x}{y}\right)^3\left(\dfrac{y^2x}{z}\right)^4$ **S O LU T I O N ???**

NOW TRY: Simplify the expression and eliminate any negative exponent(s).

(i) (a) $(5x^2y^3)(3x^2y^5)^4$ (b) $(2a^3b^2)^2(5a^2b^5)^3$

(ii) (a) $\left(\dfrac{a^2}{b}\right)^5\left(\dfrac{a^3b^2}{c^3}\right)^3$ (b) $\dfrac{(u^{-1}v^2)^2}{(u^3v^{-2})^3}$

Meaning of "simplify"?

When simplifying an expression, you will find that many different methods will lead to the same result; you should feel free to use any of the rules of exponents to arrive at your own method. We now give two additional laws that are useful in simplifying expressions with negative exponents.

Two additional laws that may be useful.

LAWS OF EXPONENTS

Law	Example	Description
6. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^{2}$	To raise a fraction to a negative power, invert the fraction and change the sign of the exponent.
7. $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$	$\frac{3^{-2}}{4^{-5}} = \frac{4^5}{3^2}$	To move a number raised to a power from numerator to denominator or from denominator to numerator, change the sign of the exponent.

(Do you know how to derive these two laws from the earlier laws?)

E X A M P L E 5 Simplifying Expressions with Negative Exponents
Eliminate negative exponents and simplify each expression.

(a) $\frac{6st^{-4}}{2s^{-2}t^2}$ (b) $\left(\frac{y}{3z^3}\right)^{-2}$??????? *More than one way? Which is better?*

NOW TRY: Simplify the expression and eliminate any negative exponent(s).

(a) $\frac{8a^3b^{-4}}{2a^{-5}b^5}$ (b) $\left(\frac{y}{5x^{-2}}\right)^{-3}$

▼ Radicals

We know what 2^n means whenever n is an integer. To give meaning to a power, such as $2^{\frac{4}{5}}$, whose exponent is a rational number, we need to discuss radicals.
The symbol $\sqrt{}$ means "the positive square root of." Thus

$$\sqrt{a} = b \quad \text{means} \quad b^2 = a \quad \text{and} \quad b \geq 0$$

Since $a = b^2 \geq 0$, the symbol $\sqrt{a}$ makes sense only when $a \geq 0$. Examples.???
Square roots are special cases of nth roots. The nth root of x is the number that, when raised to the nth power, gives x.

DEFINITION OF nth ROOT

If n is any positive integer, then the **principal nth root** of a is defined as follows:

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a$$

If n is even, we must have $a \geq 0$ and $b \geq 0$.

A radical sign $\sqrt[n]{}$ combined with a radicand is called a **radical**.

PROPERTIES OF nth ROOTS

Property	Example
1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8}\sqrt[3]{27} = (-2)(3) = -6$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\dfrac{16}{81}} = \dfrac{\sqrt[4]{16}}{\sqrt[4]{81}} = \dfrac{2}{3}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$
4. $\sqrt[n]{a^n} = a$ if n is odd	$\sqrt[3]{(-5)^3} = -5$, $\sqrt[5]{2^5} = 2$
5. $\sqrt[n]{a^n} = \lvert a \rvert$ if n is even	$\sqrt[4]{(-3)^4} = \lvert -3 \rvert = 3$

In each property we assume that all the given roots exist.

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$\sqrt{-8}$, $\sqrt[4]{-8}$, and $\sqrt[6]{-8}$ are not defined. (Not in the 'world ' of real numbers.)

$\sqrt{4^2} = \sqrt{16} = 4$ but

$\sqrt{(-4)^2} = \sqrt{16} = 4 = \lvert -4 \rvert$

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It is true that the number 9 has two square roots, 3 and -3, but the notation $\sqrt{9}$ is reserved for the *positive* square root of 9 (sometimes called the *principal square root* of 9). If we want the negative root, we must write $-\sqrt{9}$, which is -3.

******

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E X A M P L E 8 Simplifying Expressions Involving nth Roots

(a) $\sqrt[3]{x^4}$????? Meaning of "simplify"?

(b)

$$\sqrt[4]{81x^8y^4} = \sqrt[4]{81}\sqrt[4]{x^8}\sqrt[4]{y^4}$$
$$= 3\sqrt[4]{(x^2)^4}\lvert y \rvert$$
$$= 3x^2\lvert y \rvert$$

NOW TRY: (a) $\sqrt[4]{16x^8}$ (b) $\sqrt[6]{64a^6b^7}$

E X A M P L E 9 Combining Radicals

(a) $\sqrt{32} + \sqrt{200}$ **(b)** If $b > 0$, then $\sqrt{25b} - \sqrt{b^3}$ **?????**

NOW TRY:	Avoid making the following error:
(a) $\sqrt{32}+\sqrt{18}$ (b) $\sqrt[5]{96}+\sqrt[5]{3}$ (c) $\sqrt{16x}+\sqrt{x^5}$	$\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$ For instance, if we let $a=9$ and $b=16$, then we see the error: $\sqrt{9+16} \stackrel{?}{=} \sqrt{9}+\sqrt{16}$ $\sqrt{25} \stackrel{?}{=} 3+4$ $5 \stackrel{?}{=} 7$ Wrong!

Rational Exponents

To give meaning to the symbol $a^{\frac{1}{n}}$ in a way that is consistent with the Laws of Exponents, we would have to have $\left(a^{\frac{1}{n}}\right)^n = a^{\left(\frac{1}{n}\right)n} = a^1 = a$

So by the definition of nth root, $a^{\frac{1}{n}} = \sqrt[n]{a}$
In general, we define rational exponents as follows.

DEFINITION OF RATIONAL EXPONENTS

For any rational exponent $\frac{m}{n}$ in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \quad \text{or equivalently} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

If n is even, then we require that $a \geq 0$.

With this definition it can be proved that *the Laws of Exponents also hold for rational exponents*.

E X A M P L E 1 0 Using the Definition of Rational Exponents

(a) $4^{\frac{1}{2}}$ **(b)** $8^{\frac{2}{3}}$ (*Show 2 ways*) **(c)** $125^{-\frac{1}{3}}$ **(d)** $\dfrac{1}{\sqrt[3]{x^4}}$ (*Rewrite using rational exponent*)??

NOW TRY : Evaluate each expression.

(a) $\sqrt{\frac{4}{9}}$ (b) $\sqrt[4]{256}$ (c) $\sqrt[6]{\frac{1}{64}}$ (d) $\left(\frac{4}{9}\right)^{-1/2}$ (e) $(-32)^{2/5}$ (f) $-32^{2/5}$

E X A M P L E 1 1 Using the Laws of Exponents with Rational Exponents

(a) $a^{\frac{1}{3}}a^{\frac{7}{3}}$ **(b)** $\dfrac{a^{\frac{2}{5}}a^{\frac{7}{5}}}{a^{\frac{3}{5}}}$ **(c)** $(2a^3b^4)^{\frac{3}{2}}$ **(d)** $\left(\dfrac{2x^{\frac{3}{4}}}{y^{\frac{1}{3}}}\right)^3\left(\dfrac{y^4}{x^{-\frac{1}{2}}}\right)$

NOW TRY: Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

(a) $x^{3/4}x^{5/4}$ (b) $y^{2/3}y^{4/3}$ (c) $\dfrac{w^{4/3}w^{2/3}}{w^{1/3}}$ (d) $\dfrac{s^{5/2}(2s^{5/4})^2}{s^{1/2}}$

(e) $\dfrac{(8s^3t^3)^{2/3}}{(s^4t^{-8})^{1/4}}$ (f) $\dfrac{(32y^{-5}z^{10})^{1/5}}{(64y^6z^{-12})^{-1/6}}$ (g) $\left(\dfrac{x^{-2/3}}{y^{1/2}}\right)\left(\dfrac{x^{-2}}{y^{-3}}\right)^{1/6}$ (h) $\left(\dfrac{4y^3z^{2/3}}{x^{1/2}}\right)^2\left(\dfrac{x^{-3}y^6}{8z^4}\right)^{1/3}$

E X A M P L E 1 2 Simplifying by Writing Radicals as Rational Exponents

(a) $(2\sqrt{x})(3\sqrt[3]{x})$ **(b)** $\sqrt{x\sqrt{x}}$

NOW TRY: Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

(a) $\sqrt[6]{y^5}\sqrt[3]{y^2}$ (b) $(5\sqrt[3]{x})(2\sqrt[4]{x})$ (c) $\sqrt[3]{y\sqrt{y}}$ (d) $\sqrt{\dfrac{16u^3v}{uv^5}}$

▼ Rationalizing the Denominator

It is often useful to eliminate the radical in a denominator by multiplying both numerator and denominator by an appropriate expression. This procedure is called **rationalizing the denominator**. If the denominator is of the form $\sqrt{a}$, we multiply numerator and denominator by $\sqrt{a}$. In doing this we multiply the given quantity by 1, so we do not change its value. For instance,

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}}\cdot 1 = \frac{1}{\sqrt{a}}\cdot\frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

Note that the denominator in the last fraction contains no radical.

In general, if the denominator is of the form $\sqrt[n]{a^m}$ with $m < n$, then multiplying the numerator and denominator by $\sqrt[n]{a^{n-m}}$ will rationalize the denominator, because (for $a > 0$)

$$\sqrt[n]{a^m}\sqrt[n]{a^{n-m}} = \sqrt[n]{a^{m+n-m}} = \sqrt[n]{a^n} = a$$

E X A M P L E 1 3 Rationalizing Denominators

(a) $\dfrac{2}{\sqrt{3}}$ **(b)** $\dfrac{1}{\sqrt[3]{x^2}}$ **(c)** $\sqrt[7]{\dfrac{1}{a^2}}$

NOW TRY: Rationalize the denominator.

(a) $\dfrac{1}{\sqrt{10}}$ (b) $\sqrt{\dfrac{2}{x}}$ (c) $\sqrt{\dfrac{x}{3}}$ (d) $\dfrac{2}{\sqrt[3]{x}}$ (e) $\dfrac{1}{\sqrt[4]{y^3}}$ (f) $\dfrac{x}{y^{2/5}}$

(nby, Mar 2016)

Exercise

1.2 Exercises

1–8 ■ Write each radical expression using exponents, and each exponential expression using radicals.

	Radical expression	Exponential expression
1.	$\dfrac{1}{\sqrt{5}}$	
2.	$\sqrt[3]{7^2}$	
3.		$4^{2/3}$
4.		$11^{-3/2}$
5.	$\sqrt[5]{5^3}$	
6.		$2^{-1.5}$
7.		$a^{2/5}$
8.	$\dfrac{1}{\sqrt{x^5}}$	

9–18 ■ Evaluate each expression.

9. **(a)** -3^2 **(b)** $(-3)^2$ **(c)** $(-3)^0$

10. **(a)** $5^2 \cdot \left(\frac{1}{5}\right)^3$ **(b)** $\dfrac{10^7}{10^4}$ **(c)** $\dfrac{3}{3^{-2}}$

11. **(a)** $\dfrac{4^{-3}}{2^{-8}}$ **(b)** $\dfrac{3^{-2}}{9}$ **(c)** $\left(\frac{1}{4}\right)^{-2}$

12. **(a)** $\left(\frac{2}{3}\right)^{-3}$ **(b)** $\left(\frac{3}{2}\right)^{-2} \cdot \frac{9}{16}$ **(c)** $\left(\frac{1}{2}\right)^4 \cdot \left(\frac{5}{2}\right)^{-2}$

13. **(a)** $\sqrt{16}$ **(b)** $\sqrt[4]{16}$ **(c)** $\sqrt[4]{1/16}$

14. **(a)** $\sqrt{64}$ **(b)** $\sqrt[3]{-64}$ **(c)** $\sqrt[5]{-32}$

15. **(a)** $\sqrt[3]{\dfrac{8}{27}}$ **(b)** $\sqrt[3]{\dfrac{-1}{64}}$ **(c)** $\dfrac{\sqrt[5]{-3}}{\sqrt[5]{96}}$

16. **(a)** $\sqrt{7}\sqrt{28}$ **(b)** $\dfrac{\sqrt{48}}{\sqrt{3}}$ **(c)** $\sqrt[4]{24}\sqrt[4]{54}$

17. **(a)** $\left(\frac{4}{9}\right)^{-1/2}$ **(b)** $(-32)^{2/5}$ **(c)** $-32^{2/5}$

18. **(a)** $1024^{-0.1}$ **(b)** $\left(-\frac{27}{8}\right)^{2/3}$ **(c)** $\left(\frac{25}{64}\right)^{-3/2}$

19–22 ■ Evaluate the expression using $x = 3$, $y = 4$, and $z = -1$.

19. $\sqrt{x^2 + y^2}$

20. $\sqrt[4]{x^3 + 14y + 2z}$

21. $(9x)^{2/3} + (2y)^{2/3} + z^{2/3}$

22. $(xy)^{2z}$

23–26 ■ Simplify the expression.

23. $\sqrt{32} + \sqrt{18}$

24. $\sqrt{75} + \sqrt{48}$

25. $\sqrt[5]{96} + \sqrt[5]{3}$

26. $\sqrt[4]{48} - \sqrt[4]{3}$

27–44 ■ Simplify the expression and eliminate any negative exponent(s).

27. a^9a^{-5}

28. $(3y^2)(4y^5)$

29. $(12x^2y^4)\left(\frac{1}{2}x^5y\right)$

30. $(6y)^3$

31. $\dfrac{x^9(2x)^4}{x^3}$

32. $\dfrac{a^{-3}b^4}{a^{-5}b^5}$

33. $b^4\left(\frac{1}{3}b^2\right)(12b^{-8})$

34. $(2s^3t^{-1})\left(\frac{1}{4}s^6\right)(16t^4)$

35. $(rs)^3(2s)^{-2}(4r)^4$

36. $(2u^2v^3)^3(3u^3v)^{-2}$

37. $\dfrac{(6y^3)^4}{2y^5}$

38. $\dfrac{(2x^3)^2(3x^4)}{(x^3)^4}$

39. $\dfrac{(x^2y^3)^4(xy^4)^{-3}}{x^2y}$

40. $\left(\dfrac{c^4d^3}{cd^2}\right)\left(\dfrac{d^2}{c^3}\right)^3$

41. $\dfrac{(xy^2z^3)^4}{(x^3y^2z)^3}$

42. $\left(\dfrac{xy^{-2}z^{-3}}{x^2y^3z^{-4}}\right)^{-3}$

43. $\left(\dfrac{q^{-1}rs^{-2}}{r^{-5}sq^{-8}}\right)^{-1}$

44. $(3ab^2c)\left(\dfrac{2a^2b}{c^3}\right)^{-2}$

45–52 ■ Simplify the expression. Assume the letters denote any real numbers.

45. $\sqrt[4]{x^4}$

46. $\sqrt[5]{x^{10}}$

47. $\sqrt[4]{16x^8}$

48. $\sqrt[3]{x^3y^6}$

49. $\sqrt{a^2b^6}$

50. $\sqrt[3]{a^2b}\sqrt[3]{a^4b}$

51. $\sqrt[3]{\sqrt{64x^6}}$

52. $\sqrt[4]{x^4y^2z^2}$

53–70 ■ Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

53. $x^{2/3}x^{1/5}$

54. $(2x^{3/2})(4x)^{-1/2}$

55. $(-3a^{1/4})(9a)^{-3/2}$

56. $(-2a^{3/4})(5a^{3/2})$

57. $(4b)^{1/2}(8b^{2/5})$

58. $(8x^6)^{-2/3}$

59. $(c^2d^3)^{-1/3}$

60. $(4x^6y^8)^{3/2}$

61. $(y^{3/4})^{2/3}$

62. $(a^{2/5})^{-3/4}$

63. $(2x^4y^{-4/5})^3(8y^2)^{2/3}$

64. $(x^{-5}y^3z^{10})^{-3/5}$

65. $\left(\dfrac{x^6y}{y^4}\right)^{5/2}$

66. $\left(\dfrac{-2x^{1/3}}{y^{1/2}z^{1/6}}\right)^4$

67. $\left(\dfrac{3a^{-2}}{4b^{-1/3}}\right)^{-1}$

68. $\dfrac{(y^{10}z^{-5})^{1/5}}{(y^{-2}z^3)^{1/3}}$

69. $\dfrac{(9st)^{3/2}}{(27s^3t^{-4})^{2/3}}$

70. $\left(\dfrac{a^2b^{-3}}{x^{-1}y^2}\right)^3\left(\dfrac{x^{-2}b^{-1}}{a^{3/2}y^{1/3}}\right)$

71–72 ■ Write each number in scientific notation.

71. **(a)** 69,300,000 **(b)** 7,200,000,000,000
(c) 0.000028536 **(d)** 0.0001213

72. **(a)** 129,540,000 **(b)** 7,259,000,000
(c) 0.0000000014 **(d)** 0.0007029

73–74 ■ Write each number in decimal notation.

73. **(a)** 3.19×10^5 **(b)** 2.721×10^8
(c) 2.670×10^{-8} **(d)** 9.999×10^{-9}

74. **(a)** 7.1×10^{14} **(b)** 6×10^{12}
(c) 8.55×10^{-3} **(d)** 6.257×10^{-10}

75–76 ■ Write the number indicated in each statement in scientific notation.

75. **(a)** A light-year, the distance that light travels in one year, is about 5,900,000,000,000 mi.

(b) The diameter of an electron is about 0.0000000000004 cm.

(c) A drop of water contains more than 33 billion billion molecules.

76. **(a)** The distance from the earth to the sun is about 93 million miles.

(b) The mass of an oxygen molecule is about 0.000000000000000000000053 g.

(c) The mass of the earth is about 5,970,000,000,000,000,000,000,000 kg.

77–82 ■ Use scientific notation, the Laws of Exponents, and a calculator to perform the indicated operations. State your answer correct to the number of significant digits indicated by the given data.

77. $(7.2 \times 10^{-9})(1.806 \times 10^{-12})$

78. $(1.062 \times 10^{24})(8.61 \times 10^{19})$

79. $\dfrac{1.295643 \times 10^9}{(3.610 \times 10^{-17})(2.511 \times 10^6)}$

80. $\dfrac{(73.1)(1.6341 \times 10^{28})}{0.0000000019}$

81. $\dfrac{(0.0000162)(0.01582)}{(594{,}621{,}000)(0.0058)}$

82. $\dfrac{(3.542 \times 10^{-6})^9}{(5.05 \times 10^4)^{12}}$

83–86 ■ Rationalize the denominator.

83. **(a)** $\dfrac{1}{\sqrt{10}}$ **(b)** $\sqrt{\dfrac{2}{x}}$ **(c)** $\sqrt{\dfrac{x}{3}}$

84. **(a)** $\sqrt{\dfrac{5}{12}}$ **(b)** $\sqrt{\dfrac{x}{6}}$ **(c)** $\sqrt{\dfrac{y}{2z}}$

85. **(a)** $\dfrac{2}{\sqrt[3]{x}}$ **(b)** $\dfrac{1}{\sqrt[4]{y^3}}$ **(c)** $\dfrac{x}{y^{2/5}}$

86. **(a)** $\dfrac{1}{\sqrt[4]{a}}$ **(b)** $\dfrac{a}{\sqrt[3]{b^2}}$ **(c)** $\dfrac{1}{c^{3/7}}$

87. Let a, b, and c be real numbers with $a > 0$, $b < 0$, and $c < 0$. Determine the sign of each expression.

(a) b^5 **(b)** b^{10} **(c)** ab^2c^3

(d) $(b - a)^3$ **(e)** $(b - a)^4$ **(f)** $\dfrac{a^3c^3}{b^6c^6}$

88. Prove the given Laws of Exponents for the case in which m and n are positive integers and $m > n$.

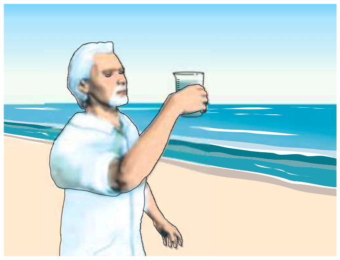
(a) Law 2 **(b)** Law 5 **(c)** Law 6

Applications

89. Distance to the Nearest Star Proxima Centauri, the star nearest to our solar system, is 4.3 light-years away. Use the information in Exercise 75(a) to express this distance in miles.

90. Speed of Light The speed of light is about 186,000 mi/s. Use the information in Exercise 76(a) to find how long it takes for a light ray from the sun to reach the earth.

91. Volume of the Oceans The average ocean depth is 3.7×10^3 m, and the area of the oceans is 3.6×10^{14} m^2. What is the total volume of the ocean in liters? (One cubic meter contains 1000 liters.)



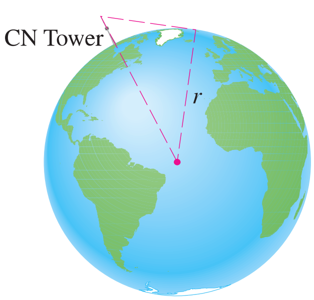
92. National Debt As of November 2004, the population of the United States was 2.949×10^8, and the national debt was 7.529×10^{12} dollars. How much was each person's share of the debt?

93. Number of Molecules A sealed room in a hospital, measuring 5 m wide, 10 m long, and 3 m high, is filled with pure oxygen. One cubic meter contains 1000 L, and 22.4 L of any gas contains 6.02×10^{23} molecules (Avogadro's number). How many molecules of oxygen are there in the room?

94. How Far Can You See? Due to the curvature of the earth, the maximum distance D that you can see from the top of a tall building of height h is estimated by the formula

$$D = \sqrt{2rh + h^2}$$

where $r = 3960$ mi is the radius of the earth and D and h are also measured in miles. How far can you see from the observation deck of the Toronto CN Tower, 1135 ft above the ground?

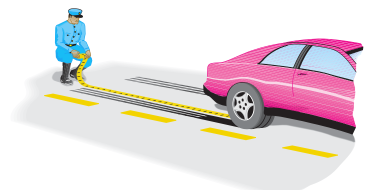


95. Speed of a Skidding Car Police use the formula $s = \sqrt{30fd}$ to estimate the speed s (in mi/h) at which a car is traveling if it skids d feet after the brakes are applied suddenly. The number f is the coefficient of friction of the road, which is a measure of the "slipperiness" of the road. The table gives some typical estimates for f.

	Tar	Concrete	Gravel
Dry	1.0	0.8	0.2
Wet	0.5	0.4	0.1

(a) If a car skids 65 ft on wet concrete, how fast was it moving when the brakes were applied?

(b) If a car is traveling at 50 mi/h, how far will it skid on wet tar?



96. Distance from the Earth to the Sun It follows from **Kepler's Third Law** of planetary motion that the average distance from a planet to the sun (in meters) is

$$d = \left(\frac{GM}{4\pi^2}\right)^{1/3} T^{2/3}$$

where $M = 1.99 \times 10^{30}$ kg is the mass of the sun, $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ is the gravitational constant, and T is the period of the planet's orbit (in seconds). Use the fact that the period of the earth's orbit is about 365.25 days to find the distance from the earth to the sun.

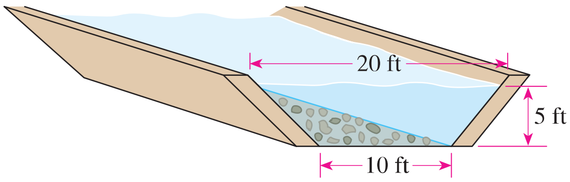
97. Flow Speed in a Channel The speed of water flowing in a channel, such as a canal or river bed, is governed by the **Manning Equation**

$$V = 1.486\,\frac{A^{2/3}S^{1/2}}{p^{2/3}n}$$

Here V is the velocity of the flow in ft/s; A is the cross-sectional area of the channel in square feet; S is the downward slope of the channel; p is the wetted perimeter in feet (the distance from the top of one bank, down the side of the channel, across the bottom, and up to the top of the other bank); and n is the roughness coefficient (a measure of the roughness of the channel bottom). This equation is used to predict the capacity of flood channels to handle runoff from heavy rainfalls. For the canal shown in the figure, $A = 75\ \text{ft}^2$, $S = 0.050$, $p = 24.1$ ft, and $n = 0.040$.

(a) Find the speed with which water flows through this canal.

(b) How many cubic feet of water can the canal discharge per second? [*Hint:* Multiply V by A to get the volume of the flow per second.]



Discovery • Discussion

98. How Big Is a Billion? If you have a million (10^6) dollars in a suitcase, and you spend a thousand (10^3) dollars each day, how many years would it take you to use all the money? Spending at the same rate, how many years would it take you to empty a suitcase filled with a *billion* (10^9) dollars?

99. Easy Powers That Look Hard Calculate these expressions in your head. Use the Laws of Exponents to help you.

(a) $\dfrac{18^5}{9^5}$ **(b)** $20^6 \cdot (0.5)^6$

100. Limiting Behavior of Powers Complete the following tables. What happens to the nth root of 2 as n gets large? What about the nth root of $\frac{1}{2}$?

n	$2^{1/n}$
1	
2	
5	
10	
100	

n	$\left(\frac{1}{2}\right)^{1/n}$
1	
2	
5	
10	
100	

Construct a similar table for $n^{1/n}$. What happens to the nth root of n as n gets large?

101. Comparing Roots Without using a calculator, determine which number is larger in each pair.

(a) $2^{1/2}$ or $2^{1/3}$ **(b)** $\left(\frac{1}{2}\right)^{1/2}$ or $\left(\frac{1}{2}\right)^{1/3}$

(c) $7^{1/4}$ or $4^{1/3}$ **(d)** $\sqrt[3]{5}$ or $\sqrt{3}$

Answer
Chapter 1

Section 1.1 ■ page 10

1. **(a)** 50 **(b)** 0, -10, 50 **(c)** 0, -10, 50, $\frac{22}{7}$, 0.538, $1.2\overline{3}$, $-\frac{1}{3}$ **(d)** $\sqrt{7}$, $\sqrt[3]{2}$ **3.** Commutative Property for addition
5. Associative Property for addition **7.** Distributive Property
9. Commutative Property for multiplication
11. $3 + x$ **13.** $4A + 4B$ **15.** $3x + 3y$ **17.** $8m$
19. $-5x + 10y$ **21.** **(a)** $\frac{17}{30}$ **(b)** $\frac{9}{20}$ **23.** **(a)** 3 **(b)** $\frac{25}{72}$
25. **(a)** $\frac{8}{3}$ **(b)** 6 **27.** **(a)** $<$ **(b)** $>$ **(c)** $=$ **29.** **(a)** False **(b)** True **31.** **(a)** False **(b)** True **33.** **(a)** $x > 0$ **(b)** $t < 4$ **(c)** $a \geq \pi$ **(d)** $-5 < x < \frac{1}{3}$ **(e)** $|p - 3| \leq 5$
35. **(a)** $\{1, 2, 3, 4, 5, 6, 7, 8\}$ **(b)** $\{2, 4, 6\}$
37. **(a)** $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ **(b)** $\{7\}$
39. **(a)** $\{x \mid x \leq 5\}$ **(b)** $\{x \mid -1 < x < 4\}$

41. $-3 < x < 0$


43. $2 \leq x < 8$


45. $x \geq 2$


47. $(-\infty, 1]$

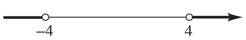

49. $(-2, 1]$


51. $(-1, \infty)$


53. **(a)** $[-3, 5]$ **(b)** $(-3, 5]$

55.


57.


59.


61. **(a)** 100 **(b)** 73 **63.** **(a)** 2 **(b)** -1 **65.** **(a)** 12 **(b)** 5 **67.** 5 **69.** **(a)** 15 **(b)** 24 **(c)** $\frac{67}{40}$ **71.** **(a)** $\frac{7}{9}$ **(b)** $\frac{13}{45}$ **(c)** $\frac{19}{33}$ **73.** Distributive Property
75. **(a)** Yes, no **(b)** 6 ft

Section 1.2 ■ page 21

1. $5^{-1/2}$ **3.** $\sqrt[3]{4^2}$ **5.** $5^{3/5}$ **7.** $\sqrt[5]{a^2}$ **9.** **(a)** -9 **(b)** 9 **(c)** 1 **11.** **(a)** 4 **(b)** $\frac{1}{81}$ **(c)** 16 **13.** **(a)** 4 **(b)** 2 **(c)** $\frac{1}{2}$
15. **(a)** $\frac{2}{3}$ **(b)** $-\frac{1}{4}$ **(c)** $-\frac{1}{2}$ **17.** **(a)** $\frac{3}{2}$ **(b)** 4 **(c)** -4
19. 5 **21.** 14 **23.** $7\sqrt{2}$ **25.** $3\sqrt[5]{3}$ **27.** a^4 **29.** $6x^7y^5$
31. $16x^{10}$ **33.** $4/b^2$ **35.** $64r^7s$ **37.** $648y^7$ **39.** $\dfrac{x^3}{y}$
41. $\dfrac{y^2z^9}{x^5}$ **43.** $\dfrac{s^3}{q^7r^6}$ **45.** $|x|$ **47.** $2x^2$ **49.** $|ab^3|$
51. $2|x|$ **53.** $x^{13/15}$ **55.** $\dfrac{-1}{9a^{5/4}}$ **57.** $16b^{9/10}$ **59.** $\dfrac{1}{c^{2/3}d}$
61. $y^{1/2}$ **63.** $\dfrac{32x^{12}}{y^{16/15}}$ **65.** $\dfrac{x^{15}}{y^{15/2}}$ **67.** $\dfrac{4a^2}{3b^{1/3}}$ **69.** $\dfrac{3t^{25/6}}{s^{1/2}}$
71. **(a)** 6.93×10^7 **(b)** 7.2×10^{12} **(c)** 2.8536×10^{-5} **(d)** 1.213×10^{-4} **73.** **(a)** 319,000 **(b)** 272,100,000 **(c)** 0.00000002670 **(d)** 0.000000009999
75. **(a)** 5.9×10^{12} mi **(b)** 4×10^{-13} cm **(c)** 3.3×10^{19} molecules **77.** 1.3×10^{-20} **79.** 1.429×10^{19}
81. 7.4×10^{-14} **83.** **(a)** $\dfrac{\sqrt{10}}{10}$ **(b)** $\dfrac{\sqrt{2x}}{x}$ **(c)** $\dfrac{\sqrt{3x}}{3}$
85. **(a)** $\dfrac{2\sqrt[3]{x^2}}{x}$ **(b)** $\dfrac{\sqrt[4]{y}}{y}$ **(c)** $\dfrac{xy^{3/5}}{y}$
87. **(a)** Negative **(b)** Positive **(c)** Negative **(d)** Negative **(e)** Positive **(f)** Negative **89.** 2.5×10^{13} mi
91. 1.3×10^{21} L **93.** 4.03×10^{27} molecules
95. **(a)** 28 mi/h **(b)** 167 ft **97.** **(a)** 17.707 ft/s **(b)** 1328.0 ft^3/s

Section 1.3 ■ page 31

1. Trinomial; x^2, $-3x$, 7; 2 **3.** Monomial; -8; 0
5. Four terms; $-x^4$, x^3, $-x^2$, x; 4 **7.** $7x + 5$
9. $5x^2 - 2x - 4$ **11.** $x^3 + 3x^2 - 6x + 11$ **13.** $9x + 103$
15. $-t^4 + t^3 - t^2 - 10t + 5$ **17.** $x^{3/2} - x$
19. $21t^2 - 29t + 10$ **21.** $3x^2 + 5xy - 2y^2$
23. $1 - 4y + 4y^2$ **25.** $4x^4 + 12x^2y^2 + 9y^4$
27. $2x^3 - 7x^2 + 7x - 5$ **29.** $x^4 - a^4$ **31.** $a - 1/b^2$
33. $1 + 3a^3 + 3a^6 + a^9$ **35.** $2x^4 + x^3 - x^2 + 3x - 2$
37. $1 - x^{2/3} + x^{4/3} - x^2$ **39.** $3x^4y^4 + 7x^3y^5 - 6x^2y^3 - 14xy^4$

1.3 ALGEBRAIC EXPRESSIONS, POLYNOMIALS & FACTORING
(Adapted from "Precalculus" by Stewart et als.)

A **variable** is a letter that can represent any number from a given set of numbers. If we start with variables, such as x, y, and z and some real numbers, and combine them using addition, subtraction, multiplication, division, powers, and roots, we obtain an **algebraic expression**. Here are some examples:

$$2x^2 - 3x + 4 \qquad \sqrt{x} + 10 \qquad \frac{y - 2z}{y^2 + 4}$$

A **monomial** is an expression of the form ax^k, where a is a real number and k is a nonnegative integer.
A **binomial** is a sum of two monomials and a **trinomial** is a sum of three monomials.

In general, a sum of monomials is called a ***polynomial***.
For example, the first expression listed above is a polynomial, but the other two are not.

POLYNOMIALS

A **polynomial** in the variable x is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a nonnegative integer. If $a_n \neq 0$, then the polynomial has **degree** n. The monomials $a_k x^k$ that make up the polynomial are called the **terms** of the polynomial.

Note that the degree of a polynomial is the highest power of the variable that appears in the polynomial

Polynomial	Type	Terms	Degree
$2x^2 - 3x + 4$	trinomial	$2x^2, -3x, 4$	2
$x^8 + 5x$	binomial	$x^8, 5x$	8
$3 - x + x^2 - \frac{1}{2}x^3$	four terms	$-\frac{1}{2}x^3, x^2, -x, 3$	3
$5x + 1$	binomial	$5x, 1$	1
$9x^5$	monomial	$9x^5$	5
6	monomial	6	0

▼ Adding and Subtracting Polynomials
We **add** and **subtract** polynomials using the properties of real numbers.

The idea is to combine **like terms** (that is, terms with the same variables raised to the same powers) using the Distributive Property.

For instance, $5x^7 + 3x^7 = (5 + 3)x^7 = 8x^7$

In subtracting polynomials, we have to remember that if a minus sign precedes an expression in parentheses, then the sign of every term within the parentheses is changed when we remove the parentheses: $-(b + c) = -b - c$

E X A M P L E 1 Adding and Subtracting Polynomials
(a) Find the sum $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$.
(b) Find the difference $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$.
S O LU T I O N:
(a) Group like terms Combine like terms

This line could be skipped.

(b) $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$

$$
\begin{aligned}
&= x^3 - 6x^2 + 2x + 4 - x^3 - 5x^2 + 7x && \text{Distributive Property}\\
&= (x^3 - x^3) + (-6x^2 - 5x^2) + (2x + 7x) + 4 && \text{Group like terms}\\
&= -11x^2 + 9x + 4 && \text{Combine like terms}
\end{aligned}
$$

NOW TRY: (a) $(3x^2 + x + 1) + (2x^2 - 3x - 5)$ (b) $(x^3 + 6x^2 - 4x + 7) - (3x^2 + 2x - 4)$

▼ Multiplying Algebraic Expressions

To find the **product** of polynomials or other algebraic expressions, we need to use the Distributive Property repeatedly. In particular, using it three times on the product of two binomials, we get
$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$

This says that we multiply the two factors by multiplying each term in one factor by each term in the other factor and adding these products. Schematically, we have

$$(a + b)(c + d) = \underset{\substack{\uparrow \\ \text{F}}}{ac} + \underset{\substack{\uparrow \\ \text{O}}}{ad} + \underset{\substack{\uparrow \\ \text{I}}}{bc} + \underset{\substack{\uparrow \\ \text{L}}}{bd}$$

The acronym **FOIL** helps us remember that the product of two binomials is the sum of the products of the **F**irst terms, the **O**uter terms, the **I**nner terms, and the **L**ast terms.

In general, we can multiply two algebraic expressions by using the Distributive Property and the Laws of Exponents.

EXAMPLE 2 | Multiplying Binomials Using FOIL

$$(2x+1)(3x-5) = \underset{\text{F}}{6x^2} - \underset{\text{O}}{10x} + \underset{\text{I}}{3x} - \underset{\text{L}}{5} \qquad \text{Distributive Property}$$

$$= 6x^2 - 7x - 5 \qquad \text{Combine like terms}$$

NOW TRY: $(3t-2)(7t-4)$

When we multiply trinomials or other polynomials with more terms, we use the Distributive Property. It is also helpful to arrange our work in table form.

The next example illustrates both methods.

E X A M P L E 3 | Multiplying Polynomials

Find the product: $(2x+3)(x^2-5x+4)$

S O LU T I O N 1: Using the Distributive Property

$(2x+3)(x^2-5x+4) = 2x(x^2-5x+4) + 3(x^2-5x+4) = \ldots$ (*and proceed*)

S O LU T I O N 2: Using Table Form

$$\begin{array}{rrrrl}
 & x^2 & -\ 5x & +\ 4 & \\
 & & 2x & +\ 3 & \\
\hline
 & 3x^2 & -\ 15x & +\ 12 & \quad \text{Multiply } x^2-5x+4 \text{ by } 3 \\
2x^3 & -\ 10x^2 & +\ 8x & & \quad \text{Multiply } x^2-5x+4 \text{ by } 2x \\
\hline
2x^3 & -\ 7x^2 & -\ 7x & +\ 12 & \quad \text{Add like terms}
\end{array}$$

NOW TRY: $(x+2)(x^2+2x+3)$

▼ Special Product Formulas

Certain types of products occur so frequently that you should memorize them. You can verify the following formulas by performing the multiplications.

SPECIAL PRODUCT FORMULAS

If A and B are any real numbers or algebraic expressions, then

1. $(A+B)(A-B) = A^2 - B^2$	Product of sun and difference
2. $(A+B)^2 = A^2 + 2AB + B^2$	Square of a sum
3. $(A-B)^2 = A^2 - 2AB + B^2$	Square of a difference
4. $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$	Cube of a sum
5. $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$	Cube of a difference

The key idea in using these formulas (or any other formula in algebra) is the **Principle of Substitution**: We may substitute any algebraic expression for any letter in a formula. For example, to find $(x^2+y^3)^2$ we use Product Formula 2, substituting x^2 for A

and y^3 for B, to get

$$\underbrace{(x^2 + y^3)^2}_{(A+B)^2} = \underbrace{(x^2)^2}_{A^2} + \underbrace{2(x^2)(y^3)}_{2AB} + \underbrace{(y^3)^2}_{B^2}$$

E X A M P L E 4 Using the Special Product Formulas
Use the Special Product Formulas to find each product.
(a) $(3x + 5)^2$ **(b)** $(x^2 - 2)^3$

NOW TRY: **(a)** $(3x + 4)^2$ **(b)** $(y + 2)^3$

EXAMPLE 5 | Using the Special Product Formulas

Find each product.
(a) $(2x - \sqrt{y})(2x + \sqrt{y})$ (b) $(x + y - 1)(x + y + 1)$

NOW TRY: $(\sqrt{a} - b)(\sqrt{a} + b)$

▼ Factoring Common Factors

We use the Distributive Property to expand algebraic expressions. We sometimes need to reverse this process (again using the Distributive Property) by **factoring** an expression as a product of simpler ones. For example, we can write

FACTORING →

$$x^2 - 4 = (x - 2)(x + 2)$$

← EXPANDING

We say that $x - 2$ and $x + 2$ are **factors** of $x^2 - 4$.

The easiest type of factoring occurs <u>when the terms have a common factor</u>.

E X A M P L E 6 Factoring Out Common Factors
Factor each expression.
(a) $3x^2 - 6x$ (b) $8x^4y^2 + 6x^3y^3 - 2xy^4$ (c) $(2x + 4)(x - 3) - 5(x - 3)$

S O LU T I O N

(a) The greatest common factor of the terms $3x^2$ and $-6x$ is $3x$, so we have

$$\boxed{3x^2 - 6x = 3x(x - 2)}$$

(b) We note that
8, 6, and –2 have the greatest common factor 2
x^4, x^3, and x have the greatest common factor x
y^2, y^3, and y^4 have the greatest common factor y^2

So the greatest common factor of the three terms in the polynomial is $2xy^2$, and we have

$$\boxed{\begin{aligned} 8x^4y^2 + 6x^3y^3 - 2xy^4 &= (2xy^2)(4x^3) + (2xy^2)(3x^2y) + (2xy^2)(-y^2) \\ &= 2xy^2(4x^3 + 3x^2y - y^2) \end{aligned}}$$

(c) The two terms have the common factor $x - 3$.

$$\boxed{(2x+4)(x-3) - 5(x-3) = [(2x+4) - 5](x-3) = (2x-1)(x-3)}$$

or

$$\boxed{(2x+4)(x-3) - 5(x-3) = (x-3)\,[(2x+4) - 5] = (x-3)\,(2x-1)}$$

NOW TRY: Factor out the common factor.

$-2x^3 + 16x$

$y(y-6) + 9(y-6)$

$2x^2y - 6xy^2 + 3xy$

▼ Factoring Trinomials

To factor a trinomial of the form $x^2 + bx + c$, we note that $x + r)(x + s) = x^2 + (r + s)x + rs$ so we need to choose numbers r and s so that $r + s = b$ *and* $rs = c$.

E X A M P L E 7 Factoring $x^2 + bx + c$ by Trial and Error

Factor: $x^2 + 7x + 12$

S O LU T I O N We need to find two integers whose product is 12 and whose sum is 7. By trial and error we find that the two integers are 3 and 4. Thus, the factorization is

$$x^2 + 7x + 12 = (x+3)(x+4)$$

factors of 12

CHECK YOUR ANSWER

Multiplying gives

$(x+3)(x+4) = x^2 + 7x + 12$ ✔

NOW TRY: Factor $x^2 + 2x - 3$.

To factor a trinomial of the form $ax^2 + bx + c$ with $a \neq 1$, we look for factors of the form $px + r$ and $qx + s$:

$$ax^2 + bx + c = (px + r)(qx + s) = pqx^2 + (ps + qr)x + rs$$

E X A M P L E 8 Factoring $ax^2 + bx + c$ by Trial and Error

Factor: $6x^2 + 7x - 5$

S O LU T I O N We can factor 6 as $6 \cdot 1$ or $3 \cdot 2$, and -5 as $-5 \cdot 1$ or $5 \cdot (-1)$. By trying these possibilities, we arrive at the factorization

factors of 6

$$6x^2 + 7x - 5 = (3x+5)(2x-1)$$

factors of -5

CHECK YOUR ANSWER

Multiplying gives

$(3x+5)(2x-1) = 6x^2 + 7x - 5$ ✔

NOW TRY: Factor: $8x^2 - 14x - 15$

E X A M P L E 9 Recognizing the Form of an Expression
Factor each expression.

(a) $x^2 - 2x - 3$ (b) $(5a + 1)^2 - 2(5a + 1) - 3$

S O LU T I O N

(a) $\mathbf{x^2 - 2x - 3 = (x - 3)(x + 1)}$ Trial and error

(b) This expression is of the form $\square^2 - 2\square - 3$ where $\square$ represents $5a + 1$.
This is the same form as the expression in part (a). **???**

NOW TRY: Factor: $3x^2 - 16x + 5$

▼ Special Factoring Formulas

Some special algebraic expressions can be factored using the following formulas. The first three are simply Special Product Formulas written backward.

SPECIAL FACTORING FORMULAS

Formula	Name
1. $A^2 - B^2 = (A - B)(A + B)$	Difference of squares
2. $A^2 + 2AB + B^2 = (A + B)^2$	Perfect square
3. $A^2 - 2AB + B^2 = (A - B)^2$	Perfect square
4. $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$	Difference of cubes
5. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$	Sum of cubes

E X A M P L E 1 0 Factoring Differences of Squares
Factor each expression.

(a) $4x^2 - 25$ (b) $(x + y)^2 - z^2$ (Hint: Use $(2x)^2 - 5^2$)

NOW TRY: (a) $9a^2 - 16$ (b) $(a + b)^2 - (a - b)^2$

E X A M P L E 1 1 Factoring Differences and Sums of Cubes
Factor each polynomial.

(a) $27x^3 - 1$ (b) $x^6 + 8$ (Hint: Use $(3x)^3 - 1^3$, $(x^2)^3 + 2^3$)

NOW TRY: (a) $27x^3 + y^3$ (b) $8x^3 - 125t^3$

A trinomial is a perfect square if it is of the form

$$A^2 + 2AB + B^2 \qquad \text{or} \qquad A^2 - 2AB + B^2$$

So we **recognize a perfect square** if the middle term ($2AB$ or $-2AB$) is plus or minus twice the product of the square roots of the outer two terms.

E X A M P L E 1 2 Recognizing Perfect Squares

Factor each trinomial.

(a) $x^2 + 6x + 9$ **(b)** $4x^2 - 4xy + y^2$

NOW TRY: Factor each expression completely.

(a) $t^2 - 6t + 9$ **(b)** $4x^2 + 4xy + y^2$

When we factor an expression, the result can sometimes be factored further. In general, *we first factor out common factors*, then inspect the result to see whether it can be factored by any of the other methods of this section We repeat this process until we have factored the expression completely.

E X A M P L E 1 3 Factoring an Expression Completely

Factor each expression completely.

(a) $2x^4 - 8x^2$ **(b)** $x^5y^2 - xy^6$

NOW TRY: **(a)** $x^3 + 2x^2 + x$ **(b)** $x^4y^3 - x^2y^5$

In the next example we factor out variables with fractional exponents. This type of factoring occurs in calculus.

E X A M P L E 1 4 Factoring Expressions with Fractional Exponents

Factor each expression.

(a) $3x^{\frac{3}{2}} - 9x^{\frac{1}{2}} + 6x^{-\frac{1}{2}}$ **(b)** $(2+x)^{-\frac{2}{3}}x + (2+x)^{\frac{1}{3}}$

S O LU T I O N

(a) Factor out the power of x with the *smallest exponent*, that is, $x^{-\frac{1}{2}}$.

Then ???...

(b) Factor out the power of $(2 + x)$ with the *smallest exponent*, that is, $(2+x)^{-\frac{2}{3}}$.

Then ???..

CHECK YOUR ANSWERS

To see that you have factored correctly, multiply using the Laws of Exponents.

(a) $3x^{-1/2}(x^2 - 3x + 2)$
$= 3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ ✔

(b) $(2 + x)^{-2/3}[x + (2 + x)]$
$= (2 + x)^{-2/3}x + (2 + x)^{1/3}$

...

NOW TRY: Factor each expression completely.

(a) $x^{-\frac{3}{2}} + 2x^{-\frac{1}{2}} + x^{\frac{1}{2}}$ **(b)** $(x^2+1)^{\frac{1}{2}} + 2(x^2+1)^{-\frac{1}{2}}$

▼ Factoring by Grouping Terms

Polynomials with at least four terms can sometimes be factored by grouping terms. The following example illustrates the idea.

E X A M P L E 1 5 Factoring by Grouping

Factor each polynomial.

(a) $x^3 + x^2 + 4x + 4$ (b) $x^3 - 2x^2 - 3x + 6$

NOW TRY: $x^3 + 4x^2 + x + 4$

(nby, Mar 2016)

1.3 Exercises

1–6 ■ Complete the following table by stating whether the polynomial is a monomial, binomial, or trinomial; then list its terms and state its degree.

Polynomial	Type	Terms	Degree
1. $x^2 - 3x + 7$			
2. $2x^5 + 4x^2$			
3. -8			
4. $\frac{1}{2}x^7$			
5. $x - x^2 + x^3 - x^4$			
6. $\sqrt{2}x - \sqrt{3}$			

7–42 ■ Perform the indicated operations and simplify.

7. $(12x - 7) - (5x - 12)$

8. $(5 - 3x) + (2x - 8)$

9. $(3x^2 + x + 1) + (2x^2 - 3x - 5)$

10. $(3x^2 + x + 1) - (2x^2 - 3x - 5)$

11. $(x^3 + 6x^2 - 4x + 7) - (3x^2 + 2x - 4)$

12. $3(x - 1) + 4(x + 2)$

13. $8(2x + 5) - 7(x - 9)$

14. $4(x^2 - 3x + 5) - 3(x^2 - 2x + 1)$

15. $2(2 - 5t) + t^2(t - 1) - (t^4 - 1)$

16. $5(3t - 4) - (t^2 + 2) - 2t(t - 3)$

17. $\sqrt{x}(x - \sqrt{x})$

18. $x^{3/2}(\sqrt{x} - 1/\sqrt{x})$

19. $(3t - 2)(7t - 5)$

20. $(4x - 1)(3x + 7)$

21. $(x + 2y)(3x - y)$

22. $(4x - 3y)(2x + 5y)$

23. $(1 - 2y)^2$

24. $(3x + 4)^2$

25. $(2x^2 + 3y^2)^2$

26. $\left(c + \dfrac{1}{c}\right)^2$

27. $(2x - 5)(x^2 - x + 1)$

28. $(1 + 2x)(x^2 - 3x + 1)$

29. $(x^2 - a^2)(x^2 + a^2)$

30. $(x^{1/2} + y^{1/2})(x^{1/2} - y^{1/2})$

31. $\left(\sqrt{a} - \dfrac{1}{b}\right)\left(\sqrt{a} + \dfrac{1}{b}\right)$

32. $(\sqrt{h^2 + 1} + 1)(\sqrt{h^2 + 1} - 1)$

33. $(1 + a^3)^3$

34. $(1 - 2y)^3$

35. $(x^2 + x - 1)(2x^2 - x + 2)$

36. $(3x^3 + x^2 - 2)(x^2 + 2x - 1)$

37. $(1 + x^{4/3})(1 - x^{2/3})$

38. $(1 - b)^2(1 + b)^2$

39. $(3x^2y + 7xy^2)(x^2y^3 - 2y^2)$

40. $(x^4y - y^5)(x^2 + xy + y^2)$

41. $(x + y + z)(x - y - z)$

42. $(x^2 - y + z)(x^2 + y - z)$

43–48 ■ Factor out the common factor.

43. $-2x^3 + 16x$

44. $2x^4 + 4x^3 - 14x^2$

45. $y(y - 6) + 9(y - 6)$

46. $(z + 2)^2 - 5(z + 2)$

47. $2x^2y - 6xy^2 + 3xy$

48. $-7x^4y^2 + 14xy^3 + 21xy^4$

49–54 ■ Factor the trinomial.

49. $x^2 + 2x - 3$

50. $x^2 - 6x + 5$

51. $8x^2 - 14x - 15$

52. $6y^2 + 11y - 21$

53. $(3x + 2)^2 + 8(3x + 2) + 12$

54. $2(a + b)^2 + 5(a + b) - 3$

55–60 ■ Use a Special Factoring Formula to factor the expression.

55. $9a^2 - 16$

56. $(x + 3)^2 - 4$

57. $27x^3 + y^3$

58. $8s^3 - 125t^6$

59. $x^2 + 12x + 36$

60. $16z^2 - 24z + 9$

61–66 ■ Factor the expression by grouping terms.

61. $x^3 + 4x^2 + x + 4$

62. $3x^3 - x^2 + 6x - 2$

63. $2x^3 + x^2 - 6x - 3$

64. $-9x^3 - 3x^2 + 3x + 1$

65. $x^3 + x^2 + x + 1$

66. $x^5 + x^4 + x + 1$

67–70 ■ Factor the expression completely. Begin by factoring out the lowest power of each common factor.

67. $x^{5/2} - x^{1/2}$

68. $x^{-3/2} + 2x^{-1/2} + x^{1/2}$

69. $(x^2 + 1)^{1/2} + 2(x^2 + 1)^{-1/2}$

70. $2x^{1/3}(x - 2)^{2/3} - 5x^{4/3}(x - 2)^{-1/3}$

71–100 ■ Factor the expression completely.

71. $12x^3 + 18x$

72. $5ab - 8abc$

73. $x^2 - 2x - 8$

74. $y^2 - 8y + 15$

75. $2x^2 + 5x + 3$

76. $9x^2 - 36x - 45$

77. $6x^2 - 5x - 6$

78. $r^2 - 6rs + 9s^2$

79. $25s^2 - 10st + t^2$

80. $x^2 - 36$

81. $4x^2 - 25$

82. $49 - 4y^2$

83. $(a + b)^2 - (a - b)^2$

84. $\left(1 + \dfrac{1}{x}\right)^2 - \left(1 - \dfrac{1}{x}\right)^2$

85. $x^2(x^2 - 1) - 9(x^2 - 1)$

86. $(a^2 - 1)b^2 - 4(a^2 - 1)$

87. $8x^3 + 125$

88. $x^6 + 64$

89. $x^6 - 8y^3$

90. $27a^3 - b^6$

91. $x^3 + 2x^2 + x$

92. $3x^3 - 27x$

93. $y^3 - 3y^2 - 4y + 12$

94. $x^3 + 3x^2 - x - 3$

95. $2x^3 + 4x^2 + x + 2$

96. $3x^3 + 5x^2 - 6x - 10$

97. $(x - 1)(x + 2)^2 - (x - 1)^2(x + 2)$

98. $y^4(y + 2)^3 + y^5(y + 2)^4$

99. $(a^2 + 1)^2 - 7(a^2 + 1) + 10$

100. $(a^2 + 2a)^2 - 2(a^2 + 2a) - 3$

101–104 ■ Factor the expression completely. (This type of expression arises in calculus when using the "product rule.")

101. $5(x^2 + 4)^4(2x)(x - 2)^4 + (x^2 + 4)^5(4)(x - 2)^3$

102. $3(2x - 1)^2(2)(x + 3)^{1/2} + (2x - 1)^3(\frac{1}{2})(x + 3)^{-1/2}$

103. $(x^2 + 3)^{-1/3} - \frac{2}{3}x^2(x^2 + 3)^{-4/3}$

104. $\frac{1}{2}x^{-1/2}(3x + 4)^{1/2} - \frac{3}{2}x^{1/2}(3x + 4)^{-1/2}$

105. **(a)** Show that $ab = \frac{1}{2}[(a + b)^2 - (a^2 + b^2)]$.

(b) Show that $(a^2 + b^2)^2 - (a^2 - b^2)^2 = 4a^2b^2$.

(c) Show that

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

(d) Factor completely: $4a^2c^2 - (a^2 - b^2 + c^2)^2$.

106. Verify Special Factoring Formulas 4 and 5 by expanding their right-hand sides.

Applications

107. Volume of Concrete A culvert is constructed out of large cylindrical shells cast in concrete, as shown in the figure. Using the formula for the volume of a cylinder given on the inside back cover of this book, explain why the volume of the cylindrical shell is

$$V = \pi R^2 h - \pi r^2 h$$

Factor to show that

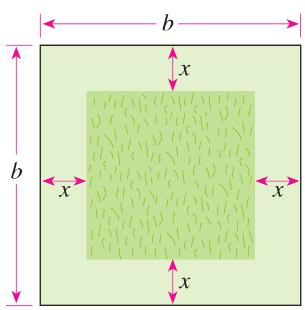
$$V = 2\pi \cdot \text{average radius} \cdot \text{height} \cdot \text{thickness}$$

Use the "unrolled" diagram to explain why this makes sense geometrically.

108. Mowing a Field A square field in a certain state park is mowed around the edges every week. The rest of the field is kept unmowed to serve as a habitat for birds and small animals (see the figure). The field measures b feet by b feet, and the mowed strip is x feet wide.

(a) Explain why the area of the mowed portion is $b^2 - (b - 2x)^2$.

(b) Factor the expression in (a) to show that the area of the mowed portion is also $4x(b - x)$.



Discovery • Discussion

109. Degrees of Sums and Products of Polynomials Make up several pairs of polynomials, then calculate the sum and product of each pair. Based on your experiments and observations, answer the following questions.

(a) How is the degree of the product related to the degrees of the original polynomials?

(b) How is the degree of the sum related to the degrees of the original polynomials?

110. The Power of Algebraic Formulas Use the Difference of Squares Formula to factor $17^2 - 16^2$. Notice that it is easy to calculate the factored form in your head, but not so easy to calculate the original form in this way. Evaluate each expression in your head:

(a) $528^2 - 527^2$ **(b)** $122^2 - 120^2$ **(c)** $1020^2 - 1010^2$

Now use the Special Product Formula

$$(A + B)(A - B) = A^2 - B^2$$

to evaluate these products in your head:

(d) $79 \cdot 51$ **(e)** $998 \cdot 1002$

111. Differences of Even Powers

(a) Factor the expressions completely: $A^4 - B^4$ and $A^6 - B^6$.

(b) Verify that $18{,}335 = 12^4 - 7^4$ and that $2{,}868{,}335 = 12^6 - 7^6$.

(c) Use the results of parts (a) and (b) to factor the integers 18,335 and 2,868,335. Then show that in both of these factorizations, all the factors are prime numbers.

112. Factoring $A^n - 1$ Verify these formulas by expanding and simplifying the right-hand side.

$$A^2 - 1 = (A - 1)(A + 1)$$
$$A^3 - 1 = (A - 1)(A^2 + A + 1)$$
$$A^4 - 1 = (A - 1)(A^3 + A^2 + A + 1)$$

Based on the pattern displayed in this list, how do you think $A^5 - 1$ would factor? Verify your conjecture. Now generalize the pattern you have observed to obtain a factoring formula for $A^n - 1$, where n is a positive integer.

113. Factoring $x^4 + ax^2 + b$ A trinomial of the form $x^4 + ax^2 + b$ can sometimes be factored easily. For example, $x^4 + 3x^2 - 4 = (x^2 + 4)(x^2 - 1)$. But $x^4 + 3x^2 + 4$ cannot be factored in this way. Instead, we can use the following method.

$$x^4 + 3x^2 + 4 = (x^4 + 4x^2 + 4) - x^2 \qquad \text{Add and subtract } x^2$$
$$= (x^2 + 2)^2 - x^2 \qquad \text{Factor perfect square}$$
$$= [(x^2 + 2) - x][(x^2 + 2) + x] \qquad \text{Difference of squares}$$
$$= (x^2 - x + 2)(x^2 + x + 2)$$

Factor the following using whichever method is appropriate.

(a) $x^4 + x^2 - 2$

(b) $x^4 + 2x^2 + 9$

(c) $x^4 + 4x^2 + 16$

(d) $x^4 + 2x^2 + 1$

Answer

Section 1.3 ■ page 31

1. Trinomial; $x^2, -3x, 7$; 2 **3.** Monomial; -8; 0
5. Four terms; $-x^4, x^3, -x^2, x$; 4 **7.** $7x + 5$
9. $5x^2 - 2x - 4$ **11.** $x^3 + 3x^2 - 6x + 11$ **13.** $9x + 103$
15. $-t^4 + t^3 - t^2 - 10t + 5$ **17.** $x^{3/2} - x$
19. $21t^2 - 29t + 10$ **21.** $3x^2 + 5xy - 2y^2$
23. $1 - 4y + 4y^2$ **25.** $4x^4 + 12x^2y^2 + 9y^4$
27. $2x^3 - 7x^2 + 7x - 5$ **29.** $x^4 - a^4$ **31.** $a - 1/b^2$
33. $1 + 3a^3 + 3a^6 + a^9$ **35.** $2x^4 + x^3 - x^2 + 3x - 2$
37. $1 - x^{2/3} + x^{4/3} - x^2$ **39.** $3x^4y^4 + 7x^3y^5 - 6x^2y^3 - 14xy^4$

41. $x^2 - y^2 - 2yz - z^2$ **43.** $2x(-x^2 + 8)$
45. $(y - 6)(y + 9)$ **47.** $xy(2x - 6y + 3)$
49. $(x - 1)(x + 3)$ **51.** $(2x - 5)(4x + 3)$
53. $(3x + 4)(3x + 8)$ **55.** $(3a - 4)(3a + 4)$
57. $(3x + y)(9x^2 - 3xy + y^2)$ **59.** $(x + 6)^2$
61. $(x + 4)(x^2 + 1)$ **63.** $(2x + 1)(x^2 - 3)$
65. $(x + 1)(x^2 + 1)$ **67.** $x^{1/2}(x + 1)(x - 1)$
69. $(x^2 + 3)(x^2 + 1)^{-1/2}$ **71.** $6x(2x^2 + 3)$
73. $(x - 4)(x + 2)$ **75.** $(2x + 3)(x + 1)$
77. $(3x + 2)(2x - 3)$ **79.** $(5s - t)^2$
81. $(2x - 5)(2x + 5)$ **83.** $4ab$
85. $(x + 3)(x - 3)(x + 1)(x - 1)$
87. $(2x + 5)(4x^2 - 10x + 25)$
89. $(x^2 - 2y)(x^4 + 2x^2y + 4y^2)$
91. $x(x + 1)^2$ **93.** $(y + 2)(y - 2)(y - 3)$
95. $(2x^2 + 1)(x + 2)$ **97.** $3(x - 1)(x + 2)$
99. $(a + 2)(a - 2)(a + 1)(a - 1)$
101. $2(x^2 + 4)^4(x - 2)^3(7x^2 - 10x + 8)$
103. $(x^2 + 3)^{-4/3}\left(\frac{1}{3}x^2 + 3\right)$
105. **(d)** $(a + b + c)(a + b - c)(a - b + c)(b - a + c)$

1.4 RATIONAL EXPRESSIONS (Adapted from "Precalculus" by Stewart et als.)

The Domain of an Algebraic Expression _ Simplifying Rational Expressions
_ Multiplying and Dividing Rational Expressions
_ Adding and Subtracting Rational Expressions _ Compound Fractions _
Rationalizing the Denominator or the Numerator _ Avoiding Common Errors

Fractional expression: a quotient of two algebraic expressions.

Examples: $\frac{2x-1}{x+3}$ $\quad$ $\frac{\sqrt{y}+4}{y^2+1}$ $\quad$ $\frac{z^3+1}{z^2+1}$

Rational expression: a fractional expression where both the numerator and denominator are polynomials.

Examples: $\frac{2x-1}{x+3}$ $\quad$ $\frac{y+y^{20}+4}{y^2+1}$ $\quad$ $\frac{z^3+1}{z^2+1}$

▼ The Domain of an Algebraic Expression

In general, an algebraic expression may not be defined for all values of the variable. The **domain** of an algebraic expression is the set of real numbers that the variable is permitted to have.

Examples:

Expression	Domain
$\frac{1}{x}$	$\{x \mid x \neq 0\}$
$\sqrt{x}$	$\{x \mid x \geq 0\}$
$\frac{1}{\sqrt{x}}$	$\{x \mid x > 0\}$

EXAMPLE 1 Finding the Domain of an Expression
Find the domains of the following expressions.

(a) $2x^2 + 3x - 1$ $\quad$ **(b)** $\frac{x}{x^2-5x+6}$ $\quad$ **(c)** $\frac{\sqrt{x}}{x-5}$

SOLUTION
(a) This polynomial is defined for every x. Thus, the domain is the set $\mathbb{R}$ of real numbers.
(b) We first factor the denominator.

$$\frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)}$$

Denominator would be 0 if $x = 2$ or $x = 3$

Since the denominator is zero when $x = 2$ or 3, the expression is not defined for these numbers. The domain is $\{x \mid x \neq 2 \text{ and } x \neq 3\}$.

$\dfrac{x}{x^2 - 5x + 6} = \dfrac{x}{(x-2)(x-3)}$ The domain is $\{x \mid x \neq 2 \text{ and } x \neq 3\}$.	The domain in interval notation is $(-\infty,2)\cup(2,3)\cup(3,\infty)$

**You must <u>master both ways</u> of stating the domain.

.**(c)** For the numerator to be defined, we must have $x \geq 0$.. Also, we cannot divide by zero, so $x \neq 5$.

Must have $x \geq 0$ to take square root $\qquad \dfrac{\sqrt{x}}{x-5} \qquad$ Denominator would be 0 if $x = 5$

Thus, the domain is $\{x \mid x \geq 0 \text{ and } x \neq 5\}$.

Given $\dfrac{\sqrt{x}}{x-5}$, we need $x \geq 0$ and $x \neq 5$. Thus, the domain is $\{x \mid x \geq 0 \text{ and } x \neq 5\}$.	The domain in interval notation?

.NOW TRY: Find the domain of the expression. $\dfrac{x^2+1}{x^2-x-2}$

▼ Simplifying Rational Expressions

To **simplify rational expressions**, we factor both numerator and denominator and use the following property of fractions:

$$\frac{AC}{BC} = \frac{A}{B}$$

[**Cancel** common factors from the numerator and denominator.]

EXAMPLE 2 Simplifying Rational Expressions by Cancellation

Simplify: $\dfrac{x^2 - 1}{x^2 + x - 2}$

SOLUTION

$$\frac{x^2-1}{x^2+x-2} = \frac{(x-1)(x+1)}{(x-1)(x+2)} \qquad \text{Factor}$$

$$= \frac{x+1}{x+2} \qquad \text{Cancel common factors}$$

We can't cancel the x^2's in $\dfrac{x^2-1}{x^2+x-2}$ because x^2 is not a factor.

NOW TRY: Simplify $\dfrac{x^2+6x+8}{x^2+5x+4}$

▼ Multiplying and Dividing Rational Expressions

To **multiply rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$$

[Multiply their numerators and multiply their denominators.]

EXAMPLE 3 Multiplying Rational Expressions

Perform the indicated multiplication and simplify:

$$\frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1}$$

SOLUTION We first factor.

$$\begin{aligned}
\frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1} &= \frac{(x-1)(x+3)}{(x+4)^2} \cdot \frac{3(x+4)}{x-1} && \text{Factor} \\
&= \frac{3(x-1)(x+3)(x+4)}{(x-1)(x+4)^2} && \text{Property of fractions} \\
&= \frac{3(x+3)}{x+4} && \text{Cancel common factors}
\end{aligned}$$

NOW TRY: Perform the indicated multiplication and simplify. $\frac{x^2 - 2x - 15}{x^2 - 9} \cdot \frac{x+3}{x-5}$

To **divide rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

[Invert the divisor and multiply.]

EXAMPLE 4 Dividing Rational Expressions

Perform the indicated division and simplify:

$$\frac{x - 4}{x^2 - 4} \div \frac{x^2 - 3x - 4}{x^2 + 5x + 6}$$

SOLUTION

$$\begin{aligned}
\frac{x - 4}{x^2 - 4} \div \frac{x^2 - 3x - 4}{x^2 + 5x + 6} &= \frac{x-4}{x^2-4} \cdot \frac{x^2 + 5x + 6}{x^2 - 3x - 4} && \text{Invert and multiply} \\
&= \frac{(x-4)(x+2)(x+3)}{(x-2)(x+2)(x-4)(x+1)} && \text{Factor} \\
&= \frac{x+3}{(x-2)(x+1)} && \text{Cancel common factors}
\end{aligned}$$

NOW TRY: Perform the indicated division and simplify. $\frac{x+3}{4x^2 - 9} \div \frac{x^2 + 7x + 12}{2x^2 + 7x - 15}$

▼ Adding and Subtracting Rational Expressions

To **add or subtract rational expressions**, we first find a common denominator and then use the following property of fractions:

$$\frac{A}{C} + \frac{B}{C} = \frac{A + B}{C}$$

Although any common denominator will work,
it is best to use the **least common denominator** (LCD).
The LCD is found by factoring each denominator and taking the product of the distinct factors, using the highest power that appears in any of the factors.

EXAMPLE 5 Adding and Subtracting Rational Expressions
Perform the indicated operations and simplify:

(a) $\dfrac{3}{x-1} + \dfrac{x}{x+2}$ **(b)** $\dfrac{1}{x^2-1} - \dfrac{2}{(x+1)^2}$

SOLUTION
(a) Here the LCD is simply the product $(x - 1)(x + 2)$.

$$\frac{3}{x-1} + \frac{x}{x+2} = \frac{???}{(x-1)(x+2)}$$

(b) $x^2 - 1 = (x - 1)(x + 1)$

The LCD of $x^2 - 1 = (x - 1)(x + 1)$ and $(x + 1)^2$ is $(x - 1)(x + 1)^2$.

$$\frac{1}{x^2-1} - \frac{2}{(x+1)^2} = \frac{1}{(x-1)(x+1)} - \frac{2}{(x+1)^2} = \frac{???}{(x-1)(x+1)^2}$$

NOW TRY: Perform the indicated operations and simplify.

(a) $\dfrac{1}{x+1} - \dfrac{1}{x+2}$ (b) $\dfrac{x}{(x+1)^2} + \dfrac{2}{x+1}$

▼ Compound Fractions

Compound fraction : a fraction in which the numerator, the denominator, or both, are themselves fractional expressions.

EXAMPLE 6 Simplifying a Compound Fraction

Simplify: $\dfrac{\dfrac{x}{y} + 1}{1 - \dfrac{y}{x}}$:

SOLUTION 1 We combine the terms in the numerator into a single fraction. We do the same in the denominator. Then we invert and multiply.

$$\frac{\frac{x}{y}+1}{1-\frac{y}{x}} = \frac{\frac{x+y}{y}}{\frac{x-y}{x}} = \frac{x+y}{y} \cdot \frac{x}{x-y}$$

$$= \frac{x(x+y)}{y(x-y)}$$

SOLUTION 2 We find the LCD of all the fractions in the expression, then multiply numerator and denominator by it. In this example the LCD of all the fractions is xy. Thus

$\frac{\frac{x}{y}+1}{1-\frac{y}{x}} = \frac{\frac{x}{y}+1}{1-\frac{y}{x}} \cdot \frac{xy}{xy}$ Multiply numerator and denominator by xy $= \frac{x^2+xy}{xy-y^2}$ Simplify $= \frac{x(x+y)}{y(x-y)}$ Factor	*There could be other ways of writing the steps.*

NOW TRY : Simplify the compound fractional expressions

(a) $\dfrac{x+\frac{1}{x+2}}{x-\frac{1}{x+2}}$ (b) $\dfrac{\frac{x+2}{x-1}-\frac{x-3}{x-2}}{x+2}$

Two examples from Calculus

The next two examples show situations in calculus that require the ability to work with fractional expressions.

EXAMPLES 7 & 8 Simplifying a Compound Fraction

Simplify: $\dfrac{\frac{1}{a+h}-\frac{1}{a}}{h}$ Simplify: $\dfrac{(1+x^2)^{1/2}-x^2(1+x^2)^{-1/2}}{1+x^2}$

NOW TRY: (a) $\dfrac{\frac{1}{1+x+h}-\frac{1}{1+x}}{h}$ (b) $\dfrac{2(1+x)^{\frac{1}{2}}-x(1+x)^{-\frac{1}{2}}}{x+1}$

▼ Rationalizing the Denominator or the Numerator

If a fraction has a denominator of the form $A+B\sqrt{C}$, we may rationalize the denominator by multiplying numerator and denominator by the **conjugate radical** $A-B\sqrt{C}$.

This works because the product of the denominator and its conjugate radical does not contain a radical: $(A+B\sqrt{C})(A-B\sqrt{C})=A^2-B^2C$

EXAMPLE 9 Rationalizing the Denominator

$$\frac{1}{1+\sqrt{2}}$$

SOLUTION We multiply both the numerator and the denominator by the conjugate radical of $1+\sqrt{2}$, which is $1-\sqrt{2}$.

$$\frac{1}{1+\sqrt{2}}=\frac{1}{1+\sqrt{2}}\cdot\frac{1-\sqrt{2}}{1-\sqrt{2}}$$

$$= \ldots$$

NOW TRY: Rationalize the denominator $\dfrac{1}{2-\sqrt{3}}$

EXAMPLE 1 0 Rationalizing the Numerator (*This example is useful in calculus.*)

$$\frac{\sqrt{4+h}-2}{h}$$

SOLUTION We multiply numerator and denominator by the conjugate radical $\sqrt{4+h}+2$

?????

NOW TRY: Rationalize the numerator $\dfrac{1-\sqrt{3}}{3}$

▼ Avoiding Common Errors

Don't make the mistake of applying properties of multiplication to the operation of addition. Many of the common errors in algebra involve doing just that.

The table on the right states several properties of multiplication and illustrates the error in applying them to addition

Correct multiplication property	Common error with addition
$(a\cdot b)^2=a^2\cdot b^2$	$(a+b)^2 \cancel{=} a^2+b^2$
$\sqrt{a\cdot b}=\sqrt{a}\sqrt{b}\quad(a,b\geq 0)$	$\sqrt{a+b}\cancel{=}\sqrt{a}+\sqrt{b}$
$\sqrt{a^2\cdot b^2}=a\cdot b\quad(a,b\geq 0)$	$\sqrt{a^2+b^2}\cancel{=}a+b$
$\frac{1}{a}\cdot\frac{1}{b}=\frac{1}{a\cdot b}$	$\frac{1}{a}+\frac{1}{b}\cancel{=}\frac{1}{a+b}$
$\frac{ab}{a}=b$	$\frac{a+b}{a}\cancel{=}b$
$a^{-1}\cdot b^{-1}=(a\cdot b)^{-1}$	$a^{-1}+b^{-1}\cancel{=}(a+b)^{-1}$

(nby, Mar 2016)

1.4 Exercises

1–6 ■ Find the domain of the expression.

1. $4x^2 - 10x + 3$

2. $-x^4 + x^3 + 9x$

3. $\dfrac{2x + 1}{x - 4}$

4. $\dfrac{2t^2 - 5}{3t + 6}$

5. $\sqrt{x + 3}$

6. $\dfrac{1}{\sqrt{x - 1}}$

7–16 ■ Simplify the rational expression.

7. $\dfrac{3(x + 2)(x - 1)}{6(x - 1)^2}$

8. $\dfrac{4(x^2 - 1)}{12(x + 2)(x - 1)}$

9. $\dfrac{x - 2}{x^2 - 4}$

10. $\dfrac{x^2 - x - 2}{x^2 - 1}$

11. $\dfrac{x^2 + 6x + 8}{x^2 + 5x + 4}$

12. $\dfrac{x^2 - x - 12}{x^2 + 5x + 6}$

13. $\dfrac{y^2 + y}{y^2 - 1}$

14. $\dfrac{y^2 - 3y - 18}{2y^2 + 5y + 3}$

15. $\dfrac{2x^3 - x^2 - 6x}{2x^2 - 7x + 6}$

16. $\dfrac{1 - x^2}{x^3 - 1}$

17–30 ■ Perform the multiplication or division and simplify.

17. $\dfrac{4x}{x^2 - 4} \cdot \dfrac{x + 2}{16x}$

18. $\dfrac{x^2 - 25}{x^2 - 16} \cdot \dfrac{x + 4}{x + 5}$

19. $\dfrac{x^2 - x - 12}{x^2 - 9} \cdot \dfrac{3 + x}{4 - x}$

20. $\dfrac{x^2 + 2x - 3}{x^2 - 2x - 3} \cdot \dfrac{3 - x}{3 + x}$

21. $\dfrac{t - 3}{t^2 + 9} \cdot \dfrac{t + 3}{t^2 - 9}$

22. $\dfrac{x^2 - x - 6}{x^2 + 2x} \cdot \dfrac{x^3 + x^2}{x^2 - 2x - 3}$

23. $\dfrac{x^2 + 7x + 12}{x^2 + 3x + 2} \cdot \dfrac{x^2 + 5x + 6}{x^2 + 6x + 9}$

24. $\dfrac{x^2 + 2xy + y^2}{x^2 - y^2} \cdot \dfrac{2x^2 - xy - y^2}{x^2 - xy - 2y^2}$

25. $\dfrac{2x^2 + 3x + 1}{x^2 + 2x - 15} \div \dfrac{x^2 + 6x + 5}{2x^2 - 7x + 3}$

26. $\dfrac{4y^2 - 9}{2y^2 + 9y - 18} \div \dfrac{2y^2 + y - 3}{y^2 + 5y - 6}$

27. $\dfrac{\dfrac{x^3}{x + 1}}{\dfrac{x}{x^2 + 2x + 1}}$

28. $\dfrac{\dfrac{2x^2 - 3x - 2}{x^2 - 1}}{\dfrac{2x^2 + 5x + 2}{x^2 + x - 2}}$

29. $\dfrac{x/y}{z}$

30. $\dfrac{x}{y/z}$

31–50 ■ Perform the addition or subtraction and simplify.

31. $2 + \dfrac{x}{x + 3}$

32. $\dfrac{2x - 1}{x + 4} - 1$

33. $\dfrac{1}{x + 5} + \dfrac{2}{x - 3}$

34. $\dfrac{1}{x + 1} + \dfrac{1}{x - 1}$

35. $\dfrac{1}{x + 1} - \dfrac{1}{x + 2}$

36. $\dfrac{x}{x - 4} - \dfrac{3}{x + 6}$

37. $\dfrac{x}{(x + 1)^2} + \dfrac{2}{x + 1}$

38. $\dfrac{5}{2x - 3} - \dfrac{3}{(2x - 3)^2}$

39. $u + 1 + \dfrac{u}{u + 1}$

40. $\dfrac{2}{a^2} - \dfrac{3}{ab} + \dfrac{4}{b^2}$

41. $\dfrac{1}{x^2} + \dfrac{1}{x^2 + x}$

42. $\dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3}$

43. $\dfrac{2}{x + 3} - \dfrac{1}{x^2 + 7x + 12}$

44. $\dfrac{x}{x^2 - 4} + \dfrac{1}{x - 2}$

45. $\dfrac{1}{x + 3} + \dfrac{1}{x^2 - 9}$

46. $\dfrac{x}{x^2 + x - 2} - \dfrac{2}{x^2 - 5x + 4}$

47. $\dfrac{2}{x} + \dfrac{3}{x - 1} - \dfrac{4}{x^2 - x}$

48. $\dfrac{x}{x^2 - x - 6} - \dfrac{1}{x + 2} - \dfrac{2}{x - 3}$

49. $\dfrac{1}{x^2 + 3x + 2} - \dfrac{1}{x^2 - 2x - 3}$

50. $\dfrac{1}{x + 1} - \dfrac{2}{(x + 1)^2} + \dfrac{3}{x^2 - 1}$

51–60 ■ Simplify the compound fractional expression.

51. $\dfrac{\dfrac{x}{y} - \dfrac{y}{x}}{\dfrac{1}{x^2} - \dfrac{1}{y^2}}$

52. $x - \dfrac{y}{\dfrac{x}{y} + \dfrac{y}{x}}$

53. $\dfrac{1 + \dfrac{1}{c - 1}}{1 - \dfrac{1}{c - 1}}$

54. $1 + \dfrac{1}{1 + \dfrac{1}{1 + x}}$

55. $\dfrac{\dfrac{5}{x - 1} - \dfrac{2}{x + 1}}{\dfrac{x}{x - 1} + \dfrac{1}{x + 1}}$

56. $\dfrac{\dfrac{a - b}{a} - \dfrac{a + b}{b}}{\dfrac{a - b}{b} + \dfrac{a + b}{a}}$

57. $\dfrac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}}$

58. $\dfrac{x^{-1} + y^{-1}}{(x + y)^{-1}}$

59. $\dfrac{1}{1 + a^n} + \dfrac{1}{1 + a^{-n}}$

60. $\dfrac{\left(a + \dfrac{1}{b}\right)^m \left(a - \dfrac{1}{b}\right)^n}{\left(b + \dfrac{1}{a}\right)^m \left(b - \dfrac{1}{a}\right)^n}$

61–66 ■ Simplify the fractional expression. (Expressions like these arise in calculus.)

61. $\dfrac{\dfrac{1}{a + h} - \dfrac{1}{a}}{h}$

62. $\dfrac{(x + h)^{-3} - x^{-3}}{h}$

63. $\dfrac{\dfrac{1 - (x + h)}{2 + (x + h)} - \dfrac{1 - x}{2 + x}}{h}$

64. $\dfrac{(x + h)^3 - 7(x + h) - (x^3 - 7x)}{h}$

65. $\sqrt{1 + \left(\dfrac{x}{\sqrt{1 - x^2}}\right)^2}$

66. $\sqrt{1 + \left(x^3 - \dfrac{1}{4x^3}\right)^2}$

67–72 ■ Simplify the expression. (This type of expression arises in calculus when using the "quotient rule.")

67. $\dfrac{3(x + 2)^2(x - 3)^2 - (x + 2)^3(2)(x - 3)}{(x - 3)^4}$

68. $\dfrac{2x(x + 6)^4 - x^2(4)(x + 6)^3}{(x + 6)^8}$

69. $\dfrac{2(1 + x)^{1/2} - x(1 + x)^{-1/2}}{x + 1}$

70. $\dfrac{(1 - x^2)^{1/2} + x^2(1 - x^2)^{-1/2}}{1 - x^2}$

71. $\dfrac{3(1 + x)^{1/3} - x(1 + x)^{-2/3}}{(1 + x)^{2/3}}$

72. $\dfrac{(7 - 3x)^{1/2} + \frac{3}{2}x(7 - 3x)^{-1/2}}{7 - 3x}$

73–78 ■ Rationalize the denominator.

73. $\dfrac{1}{2-\sqrt{3}}$ **74.** $\dfrac{2}{3-\sqrt{5}}$

75. $\dfrac{2}{\sqrt{2}+\sqrt{7}}$ **76.** $\dfrac{1}{\sqrt{x}+1}$

77. $\dfrac{y}{\sqrt{3}+\sqrt{y}}$ **78.** $\dfrac{2(x-y)}{\sqrt{x}-\sqrt{y}}$

79–84 ■ Rationalize the numerator.

79. $\dfrac{1-\sqrt{5}}{3}$ **80.** $\dfrac{\sqrt{3}+\sqrt{5}}{2}$

81. $\dfrac{\sqrt{r}+\sqrt{2}}{5}$ **82.** $\dfrac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$

83. $\sqrt{x^2+1}-x$ **84.** $\sqrt{x+1}-\sqrt{x}$

85–92 ■ State whether the given equation is true for all values of the variables. (Disregard any value that makes a denominator zero.)

85. $\dfrac{16+a}{16}=1+\dfrac{a}{16}$ **86.** $\dfrac{b}{b-c}=1-\dfrac{b}{c}$

87. $\dfrac{2}{4+x}=\dfrac{1}{2}+\dfrac{2}{x}$ **88.** $\dfrac{x+1}{y+1}=\dfrac{x}{y}$

89. $\dfrac{x}{x+y}=\dfrac{1}{1+y}$ **90.** $2\left(\dfrac{a}{b}\right)=\dfrac{2a}{2b}$

91. $\dfrac{-a}{b}=-\dfrac{a}{b}$ **92.** $\dfrac{1+x+x^2}{x}=\dfrac{1}{x}+1+x$

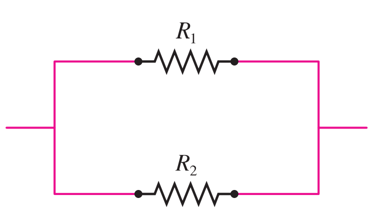
Applications

93. Electrical Resistance If two electrical resistors with resistances R_1 and R_2 are connected in parallel (see the figure), then the total resistance R is given by

$$R=\dfrac{1}{\dfrac{1}{R_1}+\dfrac{1}{R_2}}$$

(a) Simplify the expression for R.

(b) If $R_1 = 10$ ohms and $R_2 = 20$ ohms, what is the total resistance R?



94. Average Cost A clothing manufacturer finds that the cost of producing x shirts is $500 + 6x + 0.01x^2$ dollars.

(a) Explain why the average cost per shirt is given by the rational expression

$$A=\dfrac{500+6x+0.01x^2}{x}$$

(b) Complete the table by calculating the average cost per shirt for the given values of x.

x	Average cost
10	
20	
50	
100	
200	
500	
1000	

Discovery • Discussion

95. Limiting Behavior of a Rational Expression The rational expression

$$\dfrac{x^2-9}{x-3}$$

is not defined for $x = 3$. Complete the tables and determine what value the expression approaches as x gets closer and closer to 3. Why is this reasonable? Factor the numerator of the expression and simplify to see why.

x	$\dfrac{x^2-9}{x-3}$
2.80	
2.90	
2.95	
2.99	
2.999	

x	$\dfrac{x^2-9}{x-3}$
3.20	
3.10	
3.05	
3.01	
3.001	

96. Is This Rationalization? In the expression $2/\sqrt{x}$ we would eliminate the radical if we were to square both numerator and denominator. Is this the same thing as rationalizing the denominator?

97. Algebraic Errors The left-hand column in the table lists some common algebraic errors. In each case, give an example using numbers that show that the formula is not valid. An example of this type, which shows that a

Answer

1. $\mathbb{R}$ **3.** $x \neq 4$ **5.** $x \geq -3$ **7.** $\dfrac{x+2}{2(x-1)}$

9. $\dfrac{1}{x+2}$ **11.** $\dfrac{x+2}{x+1}$ **13.** $\dfrac{y}{y-1}$ **15.** $\dfrac{x(2x+3)}{2x-3}$

17. $\dfrac{1}{4(x-2)}$ **19.** $\dfrac{x+3}{3-x}$ **21.** $\dfrac{1}{t^2+9}$ **23.** $\dfrac{x+4}{x+1}$

25. $\dfrac{(2x+1)(2x-1)}{(x+5)^2}$ **27.** $x^2(x+1)$ **29.** $\dfrac{x}{yz}$

31. $\dfrac{3(x+2)}{x+3}$ **33.** $\dfrac{3x+7}{(x-3)(x+5)}$ **35.** $\dfrac{1}{(x+1)(x+2)}$

37. $\dfrac{3x+2}{(x+1)^2}$ **39.** $\dfrac{u^2+3u+1}{u+1}$ **41.** $\dfrac{2x+1}{x^2(x+1)}$

43. $\dfrac{2x+7}{(x+3)(x+4)}$ **45.** $\dfrac{x-2}{(x+3)(x-3)}$ **47.** $\dfrac{5x-6}{x(x-1)}$

49. $\dfrac{-5}{(x+1)(x+2)(x-3)}$ **51.** $-xy$ **53.** $\dfrac{c}{c-2}$

55. $\dfrac{3x+7}{x^2+2x-1}$ **57.** $\dfrac{y-x}{xy}$ **59.** 1 **61.** $\dfrac{-1}{a(a+h)}$

63. $\dfrac{-3}{(2+x)(2+x+h)}$ **65.** $\dfrac{1}{\sqrt{1-x^2}}$

67. $\dfrac{(x+2)^2(x-13)}{(x-3)^3}$ **69.** $\dfrac{x+2}{(x+1)^{3/2}}$ **71.** $\dfrac{2x+3}{(x+1)^{4/3}}$

73. $2+\sqrt{3}$ **75.** $\dfrac{2(\sqrt{7}-\sqrt{2})}{5}$ **77.** $\dfrac{y\sqrt{3}-y\sqrt{y}}{3-y}$

79. $\dfrac{-4}{3(1+\sqrt{5})}$ **81.** $\dfrac{r-2}{5(\sqrt{r}-\sqrt{2})}$ **83.** $\dfrac{1}{\sqrt{x^2+1}+x}$

1.5 COMPLEX NUMBERS

(Adapted from "Precalculus" by Stewart et als.)

Arithmetic Operations on Complex Numbers _ Square Roots of Negative Numbers _
Complex Solutions of Quadratic Equations (in a later section)

To solve the equation $x^2 - 4 = 0$, we get $x^2 = 4$ so $x = \pm\sqrt{4} = \pm 2$.
However, the equation $x^2 + 4 = 0$ has no real solution. If we try to solve this equation, we get $x^2 = -4$ so $x = \pm\sqrt{-4}$. This is impossible in the real number system since the square of any real number is positive. [For example, $(-2)^2 = 4$, a positive number.] Thus, negative numbers don't have real square roots.
To make it possible to solve *all* quadratic equations, mathematicians invented an expanded number system, called the *complex number system*. First they defined the new number

$$i = \sqrt{-1}$$

This means that $i^2 = -1$. A complex number is then a number of the form $a + bi$, where a and b are real numbers.

> **DEFINITION OF COMPLEX NUMBERS**
>
> A **complex number** is an expression of the form
>
> $$a + bi$$
>
> where a and b are real numbers and $i^2 = -1$. The **real part** of this complex number is a and the **imaginary part** is b. Two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

Note that both the real and imaginary parts of a complex number are real numbers.

E X A M P L E 1 | Complex Numbers
The following are examples of complex numbers.

$3 + 4i$	Real part 3, imaginary part 4
$\frac{1}{2} - \frac{2}{3}i$	Real part $\frac{1}{2}$, imaginary part $-\frac{2}{3}$
$6i$	Real part 0, imaginary part 6
-7	Real part -7, imaginary part 0

NOW TRY:
Find the real and imaginary parts of the complex number.
(a) $5 - 7i$ (b) 3

A number such as $6i$, which has real part 0, is called a **pure imaginary number**.A real number such as -7 can be thought of as a complex number with imaginary part 0.

In the complex number system every quadratic equation has solutions. The numbers $2i$ and $-2i$ are solutions of $x^2 = -4$ because

$$(2i)^2 = 2^2i^2 = 4(-1) = -4 \qquad \text{and} \qquad (-2i)^2 = (-2)^2i^2 = 4(-1) = -4$$

Although we use the term *imaginary* in this context, imaginary numbers should not be thought of as any less "real" (in the ordinary rather than the mathematical sense of thatword) than negative numbers or irrational numbers. All numbers (except possibly the positive integers) are creations of the human mind—the numbers -1 and $\sqrt{2}$ as well as the number i.

We study complex numbers because they complete, in a useful and elegant fashion, our study of the solutions of equations. In fact, imaginary numbers are useful not only in algebra and mathematics, but in the other sciences as well. To give just one example, in electrical theory the *reactance* of a circuit is a quantity whose measure is an imaginary number.

▼ Arithmetic Operations on Complex Numbers

Complex numbers are added, subtracted, multiplied, and divided just as we would any number of the form $a + b\sqrt{c}$. The only difference that we need to keep in mind is that $i^2 = -1$. Thus, the following calculations are valid.

$$\begin{aligned}(a + bi)(c + di) &= ac + (ad + bc)i + bdi^2 && \text{Multiply and collect like terms}\\ &= ac + (ad + bc)i + bd(-1) && i^2 = -1\\ &= (ac - bd) + (ad + bc)i && \text{Combine real and imaginary parts}\end{aligned}$$

We therefore define the sum, difference, and product of complex numbers as follows.

ADDING, SUBTRACTING, AND MULTIPLYING COMPLEX NUMBERS

Definition	Description
Addition	
$(a + bi) + (c + di) = (a + c) + (b + d)i$	To add complex numbers, add the real parts and the imaginary parts.
Subtraction	
$(a + bi) - (c + di) = (a - c) + (b - d)i$	To subtract complex numbers, subtract the real parts and the imaginary parts.
Multiplication	
$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$	Multiply complex numbers like binomials, using $i^2 = -1$.

E X A M P L E 2 Adding, Subtracting, and Multiplying Complex Numbers

Express the following in the form $a + bi$

(a) $(3+5i)+(4-2i)$ (b) $(3+5i)-(4-2i)$ (c) $(3+5i)(4-2i)$ (d) i^{23}

S O LU T I O N **[*Discuss different ways of writing steps.*]**

(a) According to the definition, we add the real parts and we add the imaginary parts.

$$(3+5i)+(4-2i) = (3+4)+(5-2)i = \ldots$$

(b) $(3+5i)-(4-2i)=(3-4)+[5-(-2)]i=\ldots$

(c) $(3+5i)(4-2i)=[3\cdot 4-5(-2)]+[3(-2)+5\cdot 4]i=\ldots$

(d) $i^{23}=i^{22+1}=(i^2)^{11}i=(-1)^{11}i=\ldots$

NOW TRY:

Evaluate the expression and write the result in the form $a+bi$

(a) $(2-5i)+(3+4i)$ (b) $(7-\frac{1}{2}i)-(5+\frac{3}{2}i)$ (c) $(7-i)(4+2i)$ (d) i^{100}

Division of complex numbers is much like rationalizing the denominator of a radical expression, which we considered in Section 1.4. For the complex number $z=a+bi$ we define its **complex conjugate** to be $\overline{z}=a-bi$.

Examples: $\overline{2-3i}=2+3i$, $\overline{2+3i}=2-3i$

Note that

$$z\cdot\overline{z}=(a+bi)(a-bi)=a^2+b^2$$

So the product of a complex number and its conjugate is always a nonnegative real number.

We use this property to divide complex numbers.

DIVIDING COMPLEX NUMBERS

To simplify the quotient $\frac{a+bi}{c+di}$, multiply the numerator and the denominator by the complex conjugate of the denominator:

$$\frac{a+bi}{c+di}=\left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right)=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

There is no real need to memorize this entire formula. Rather than memorizing this , it is easier to just remember the first step and then multiply out the numerator and the denominator as usual.

E X A M P L E 3 | Dividing Complex Numbers

Express the following in the form $a+bi$. **(a)** $\frac{3+5i}{1-2i}$ **(b)** $\frac{7+3i}{4i}$

S O LU T I O N We multiply both the numerator and denominator by the complex conjugate of the denominator to make the new denominator a real number.

(a) The complex conjugate of $1-2i$ is $1+2i$. $\frac{3+5i}{1-2i}=$	(b) The complex conjugate of $4i$ is $-4i$. $\frac{7+3i}{4i}=$

NOW TRY: Evaluate the expression and write the result in the form $a+bi$

(a) $\dfrac{2-3i}{1-2i}$ (b) $\dfrac{4+6i}{3i}$

▼ Square Roots of Negative Numbers

Just as every positive real number r has two square roots ($\sqrt{r}$ and $-\sqrt{r}$), every negative number has two square roots as well. If $-r$ is a negative number, then its square roots are $\pm i\sqrt{r}$ because $(i\sqrt{r})^{)} = i^2 r = -r$ and $(-i\sqrt{r})^{)} = (-1)^2 i^2 r = -r$.

SQUARE ROOTS OF NEGATIVE NUMBERS

If $-r$ is negative, then the **principal square root** of $-r$ is

$$\sqrt{-r} = i\sqrt{r}$$

The two square roots of $-r$ are $i\sqrt{r}$ and $-i\sqrt{r}$.

We usually write $i\sqrt{b}$ instead of $\sqrt{b}i$ to avoid confusion with . $\sqrt{bi}$

E X A M P L E 4 | Square Roots of Negative Numbers

(a) $\sqrt{-1} = i\sqrt{1} = i$ **(b)** $\sqrt{-16} = i\sqrt{16} = 4i$ **(c)** $\sqrt{-3} = i\sqrt{3}$

NOW TRY:

Evaluate the radical expression and express the result in the form $a + bi$.

(a) $\sqrt{-25}$ (b) $\sqrt{-3}\sqrt{-12}$

Special care must be taken in performing calculations that involve square roots of negative numbers. Although $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ when a and b are positive, this is *not* true when both are negative. For example,

$\sqrt{-2}\cdot\sqrt{-3} = i\sqrt{2}\cdot i\sqrt{3} = i^2\sqrt{6} = -\sqrt{6}$ but $\sqrt{(-2)(-3)} = \sqrt{6}$

So $\sqrt{-2}\cdot\sqrt{-3} \neq \sqrt{(-2)(-3)}$

When multiplying radicals of negative numbers, express them first in the form $i\sqrt{r}$ (where $r > 0$) to avoid possible errors of this type.

E X A M P L E 5 | Using Square Roots of Negative Numbers

Evaluate $(\sqrt{12}-\sqrt{-3})(3+\sqrt{-4})$ and express in the form $a + bi$.

S O LU T I O N ???

NOW TRY: Evaluate the radical expression and express the result in the form $a + bi$.

$(3-\sqrt{-5})(1+\sqrt{-1})$

(nby, Mar 2016)

Exercise

1–10 ■ Find the real and imaginary parts of the complex number.

1. $5 - 7i$

2. $-6 + 4i$

3. $\dfrac{-2 - 5i}{3}$

4. $\dfrac{4 + 7i}{2}$

5. 3

6. $-\frac{1}{2}$

7. $-\frac{2}{3}i$

8. $i\sqrt{3}$

9. $\sqrt{3} + \sqrt{-4}$

10. $2 - \sqrt{-5}$

11–22 ■ Perform the addition or subtraction and write the result in the form $a + bi$.

11. $(2 - 5i) + (3 + 4i)$

12. $(2 + 5i) + (4 - 6i)$

13. $(-6 + 6i) + (9 - i)$

14. $(3 - 2i) + \left(-5 - \frac{1}{3}i\right)$

15. $3i + (6 - 4i)$

16. $\left(\frac{1}{2} - \frac{1}{3}i\right) + \left(\frac{1}{2} + \frac{1}{3}i\right)$

17. $\left(7 - \frac{1}{2}i\right) - \left(5 + \frac{3}{2}i\right)$

18. $(-4 + i) - (2 - 5i)$

19. $(-12 + 8i) - (7 + 4i)$

20. $6i - (4 - i)$

21. $\frac{1}{3}i - \left(\frac{1}{4} - \frac{1}{6}i\right)$

22. $(0.1 - 1.1i) - (1.2 - 3.6i)$

23–56 ■ Evaluate the expression and write the result in the form $a + bi$.

23. $4(-1 + 2i)$

24. $2i\left(\frac{1}{2} - i\right)$

25. $(7 - i)(4 + 2i)$

26. $(5 - 3i)(1 + i)$

27. $(3 - 4i)(5 - 12i)$

28. $\left(\frac{2}{3} + 12i\right)\left(\frac{1}{6} + 24i\right)$

29. $(6 + 5i)(2 - 3i)$

30. $(-2 + i)(3 - 7i)$

31. $\dfrac{1}{i}$

32. $\dfrac{1}{1 + i}$

33. $\dfrac{2 - 3i}{1 - 2i}$

34. $\dfrac{5 - i}{3 + 4i}$

35. $\dfrac{26 + 39i}{2 - 3i}$

36. $\dfrac{25}{4 - 3i}$

37. $\dfrac{10i}{1 - 2i}$

38. $(2 - 3i)^{-1}$

39. $\dfrac{4 + 6i}{3i}$

40. $\dfrac{-3 + 5i}{15i}$

41. $\dfrac{1}{1 + i} - \dfrac{1}{1 - i}$

42. $\dfrac{(1 + 2i)(3 - i)}{2 + i}$

43. i^3

44. $(2i)^4$

45. i^{100}

46. i^{1002}

47. $\sqrt{-25}$

48. $\sqrt{\dfrac{-9}{4}}$

49. $\sqrt{-3}\sqrt{-12}$

50. $\sqrt{\frac{1}{3}}\sqrt{-27}$

51. $(3 - \sqrt{-5})(1 + \sqrt{-1})$

52. $\dfrac{1 - \sqrt{-1}}{1 + \sqrt{-1}}$

53. $\dfrac{2 + \sqrt{-8}}{1 + \sqrt{-2}}$

54. $(\sqrt{3} - \sqrt{-4})(\sqrt{6} - \sqrt{-8})$

55. $\dfrac{\sqrt{-36}}{\sqrt{-2}\sqrt{-9}}$

56. $\dfrac{\sqrt{-7}\sqrt{-49}}{\sqrt{28}}$

57–70 ■ Find all solutions of the equation and express them in the form $a + bi$.

57. $x^2 + 9 = 0$

58. $9x^2 + 4 = 0$

59. $x^2 - 4x + 5 = 0$

60. $x^2 + 2x + 2 = 0$

61. $x^2 + x + 1 = 0$

62. $x^2 - 3x + 3 = 0$

63. $2x^2 - 2x + 1 = 0$

64. $2x^2 + 3 = 2x$

65. $t + 3 + \dfrac{3}{t} = 0$

66. $z + 4 + \dfrac{12}{z} = 0$

67. $6x^2 + 12x + 7 = 0$

68. $4x^2 - 16x + 19 = 0$

69. $\frac{1}{2}x^2 - x + 5 = 0$

70. $x^2 + \frac{1}{2}x + 1 = 0$

71–78 ■ Recall that the symbol $\overline{z}$ represents the complex conjugate of z. If $z = a + bi$ and $w = c + di$, prove each statement.

71. $\overline{z} + \overline{w} = \overline{z + w}$

72. $\overline{zw} = \overline{z} \cdot \overline{w}$

73. $(\overline{z})^2 = \overline{z^2}$

74. $\overline{\overline{z}} = z$

75. $z + \overline{z}$ is a real number

76. $z - \overline{z}$ is a pure imaginary number

77. $z \cdot \overline{z}$ is a real number

78. $z = \overline{z}$ if and only if z is real

Discovery • Discussion

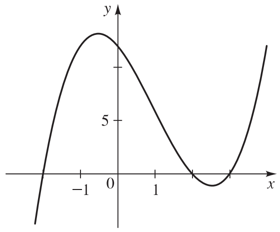
79. **Complex Conjugate Roots** Suppose that the equation $ax^2 + bx + c = 0$ has real coefficients and complex roots. Why must the roots be complex conjugates of each other? (Think about how you would find the roots using the quadratic formula.)

80. **Powers of i** Calculate the first 12 powers of i, that is, $i, i^2, i^3, \ldots, i^{12}$. Do you notice a pattern? Explain how you would calculate any whole number power of i, using the pattern you have discovered. Use this procedure to calculate i^{4446}.

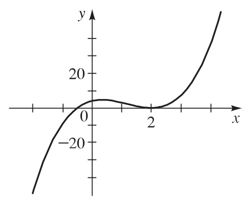
81. **Complex Radicals** The number 8 has one real cube root, $\sqrt[3]{8} = 2$. Calculate $(-1 + i\sqrt{3})^3$ and $(-1 - i\sqrt{3})^3$ to verify that 8 has at least two other complex cube roots. Can you find four fourth roots of 16?

49. $-1, -\frac{1}{2}, -3 \pm \sqrt{10}$

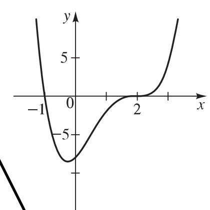
51. **(a)** $-2, 2, 3$ **(b)**



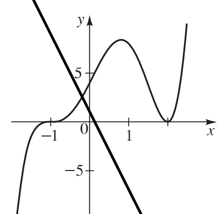
53. **(a)** $-\frac{1}{2}, 2$ **(b)**



55. **(a)** $-1, 2$ **(b)**



57. **(a)** $-1, 2$ **(b)**



59. 1 positive, 2 or 0 negative; 3 or 1 real
61. 1 positive, 1 negative; 2 real **63.** 2 or 0 positive, 0 negative; 3 or 1 real (since 0 is a zero but is neither positive nor negative) **69.** $3, -2$ **71.** $3, -1$ **73.** $-2, \frac{1}{2}, \pm 1$ **75.** $\pm\frac{1}{2}, \pm\sqrt{5}$
77. $-2, 1, 3, 4$ **83.** $-2, 2, 3$ **85.** $-\frac{3}{2}, -1, 1, 4$
87. $-1.28, 1.53$ **89.** -1.50 **93.** 11.3 ft **95.** **(a)** It began to snow again. **(b)** No **(c)** Just before midnight on Saturday night **97.** 2.76 m **99.** 88 in. (or 3.21 in.)

Section 3.4 ■ page 289

1. Real part 5, imaginary part -7 **3.** Real part $-\frac{2}{3}$, imaginary part $-\frac{5}{3}$ **5.** Real part 3, imaginary part 0 **7.** Real part 0, imaginary part $-\frac{2}{3}$ **9.** Real part $\sqrt{3}$, imaginary part 2
11. $5 - i$ **13.** $3 + 5i$ **15.** $6 - i$ **17.** $2 - 2i$
19. $-19 + 4i$ **21.** $-\frac{1}{4} + \frac{1}{2}i$ **23.** $-4 + 8i$ **25.** $30 + 10i$
27. $-33 - 56i$ **29.** $27 - 8i$ **31.** $-i$ **33.** $\frac{8}{5} + \frac{1}{5}i$
35. $-5 + 12i$ **37.** $-4 + 2i$ **39.** $2 - \frac{4}{3}i$ **41.** $-i$
43. $-i$ **45.** 1 **47.** $5i$ **49.** -6

51. $(3 + \sqrt{5}) + (3 - \sqrt{5})i$ **53.** 2 **55.** $-i\sqrt{2}$ **57.** $\pm 3i$

59. $2 \pm i$ **61.** $-\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$ **63.** $\frac{1}{2} \pm \frac{1}{2}i$ **65.** $-\dfrac{3}{2} \pm \dfrac{\sqrt{3}}{2}i$

67. $\dfrac{-6 \pm \sqrt{6}i}{6}$ **69.** $1 \pm 3i$

Section 3.5 ■ page 298

1. **(a)** $0, \pm 2i$ **(b)** $x^2(x - 2i)(x + 2i)$
3. **(a)** $0, 1 \pm i$ **(b)** $x(x - 1 - i)(x - 1 + i)$
5. **(a)** $\pm i$ **(b)** $(x - i)^2(x + i)^2$
7. **(a)** $\pm 2, \pm 2i$ **(b)** $(x - 2)(x + 2)(x - 2i)(x + 2i)$
9. **(a)** $-2, 1 \pm i\sqrt{3}$
(b) $(x + 2)(x - 1 - i\sqrt{3})(x - 1 + i\sqrt{3})$
11. **(a)** $\pm 1, \frac{1}{2} \pm \frac{1}{2}i\sqrt{3}, -\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$
(b) $(x - 1)(x + 1)(x - \frac{1}{2} - \frac{1}{2}i\sqrt{3})(x - \frac{1}{2} + \frac{1}{2}i\sqrt{3}) \times (x + \frac{1}{2} - \frac{1}{2}i\sqrt{3})(x + \frac{1}{2} + \frac{1}{2}i\sqrt{3})$

In answers 13–30, the factored form is given first, then the zeros are listed with the multiplicity of each in parentheses.
13. $(x - 5i)(x + 5i)$; $\pm 5i$ (1)
15. $[x - (-1 + i)][x - (-1 - i)]$; $-1 + i$ (1), $-1 - i$ (1)
17. $x(x - 2i)(x + 2i)$; 0 (1), $2i$ (1), $-2i$ (1)
19. $(x - 1)(x + 1)(x - i)(x + i)$; 1 (1), -1 (1), i (1), $-i$ (1)
21. $16(x - \frac{3}{2})(x + \frac{3}{2})(x - \frac{3}{2}i)(x + \frac{3}{2}i)$;
$\frac{3}{2}$ (1), $-\frac{3}{2}$ (1), $\frac{3}{2}i$ (1), $-\frac{3}{2}i$ (1)
23. $(x + 1)(x - 3i)(x + 3i)$; -1 (1), $3i$ (1), $-3i$ (1)
25. $(x - i)^2(x + i)^2$; i (2), $-i$ (2)
27. $(x - 1)(x + 1)(x - 2i)(x + 2i)$; 1 (1), -1 (1), $2i$ (1), $-2i$ (1)
29. $x(x - i\sqrt{3})^2(x + i\sqrt{3})^2$; 0 (1), $i\sqrt{3}$ (2), $-i\sqrt{3}$ (2)
31. $P(x) = x^2 - 2x + 2$ **33.** $Q(x) = x^3 - 3x^2 + 4x - 12$
35. $P(x) = x^3 - 2x^2 + x - 2$
37. $R(x) = x^4 - 4x^3 + 10x^2 - 12x + 5$
39. $T(x) = 6x^4 - 12x^3 + 18x^2 - 12x + 12$

41. $-2, \pm 2i$ **43.** $1, \dfrac{1 \pm i\sqrt{3}}{2}$ **45.** $2, \dfrac{1 \pm i\sqrt{3}}{2}$

47. $-\frac{3}{2}, -1 \pm i\sqrt{2}$ **49.** $-2, 1, \pm 3i$ **51.** $1, \pm 2i, \pm i\sqrt{3}$
53. 3 (multiplicity 2), $\pm 2i$ **55.** $-\frac{1}{2}$ (multiplicity 2), $\pm i$
57. 1 (multiplicity 3), $\pm 3i$ **59.** **(a)** $(x - 5)(x^2 + 4)$
(b) $(x - 5)(x - 2i)(x + 2i)$
61. **(a)** $(x - 1)(x + 1)(x^2 + 9)$
(b) $(x - 1)(x + 1)(x - 3i)(x + 3i)$
63. **(a)** $(x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4)$