Topic 9.1 Introduction to Probability

TMA1201 Discrete Structures & Probability
Faculty of Computing & Informatics
Multimedia University

Outline

- Introduction to Probability Theory
- Conditional probability
- Representing conditional probabilities using tree Diagram
- Bayes' Theorem

Introduction to Probability Theory

Probability:

- The study of randomness and uncertainty
  - In other words, probability is a numerical measure of chance for the occurrence of an event.

Experiment:

- a procedure that yields one of a given set of possible outcomes.

Sample space:

- The set of all possible outcomes of an experiment, denoted by *S*.

Event:

- An event is any collection (subset) of outcomes contained in the sample space *S*.

Some Relation from Set Theory

Complement:

- The complement of an event A, denoted by A' (or $\bar{A}$), is the set of all outcomes in *S* that are not contained in A.

Intersection:

- The *intersection* of two events *A* and *B*, denoted by $A \cap B$, is the event containing all outcomes that are in both *A* and *B*.

Union:

- The *union* of the events *A* and *B*, denoted by $A \cup B$, is the event consisting of all outcomes that is either in *A* or in *B* or in both.

Mutually exclusive:

- When two events *A* and *B* have no outcomes in common, they are said to be mutually exclusive, or disjoint events. In other words, $\boldsymbol{A \cap B = \phi}$

Example 1

Consider an experiment of rolling a 6-sided die. Let A be an event of an odd number is rolled and B be an event of a number less than four is rolled.

Outcome: 1, 2, 3, 4, 5, or 6.

Sample space, S = {1, 2, 3, 4, 5, 6}

$A = \{1,3,5\}$ and $B = \{1,2,3\}$

(i) Find A', $A \cap B$ and $A \cup B$.

$$A' = S - A = \{2,4,6\}$$

$$A \cap B = \{1,3,5\} \cap \{1,2,3\} =$$

$$A \cup B = \{1,3,5\} \cup \{1,2,3\} =$$

(ii) Are the events A and B mutually exclusive? Explain your answer.

Example 2

Let A be an event of obtaining 'at least three heads' when four coins are tossed. Find A, A' , $|A|$ and $|A'|$.

- ***S*** **= {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, THTH, THHT, HTTH, HTHT, HHTT, TTTH, TTHT, THTT, HTTT, TTTT};**

  $|S|$= 16
- ***A*** **= {HHHH, HHHT, HHTH, HTHH, THHH};**

  $|A|$ = 5
- ***A'*** **= {HHTT, THTH, THHT, HTTH, HTHT, HHTT, TTTH, TTHT, THTT, HTTT, TTTT};**

  $|A'|$ = 11

Axioms of Probability

- Given an experiment and a sample space S, the objective of probability is to assign to each event A a number P(A), called the probability of the event A, which will give a precise measure of the chance that A will occur. To ensure that the probability assignments will be consistent with our intuitive notions of probability, all assignments should satisfy the following axioms (basic properties) of probability.

(i) For any event A, $P(A) \geq 0$

(ii) $P(S) = 1$

(iii) If $A_1, A_2, A_3 \cdots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(Ai)$$

Try This

- An experiment has four possible outcomes, A, B, C, D, which are mutually exclusive. Explain why the following assignments of probabilities are not permissible:

(a) $P(A) = 0.12$ $P(B) = 0.23$ $P(C) = 0.15$ $P(D) = -0.2$

(b) $P(A) = 1/12$ $P(B) = 5/12$ $P(C) = 4/12$ $P(D) = 7/12$

Example 3

In an experiment of rolling a fair die, let G denotes the event that a number greater than 3 has occurred on a single roll of die. Find *P(G)*.

<u>Solution</u>

S = {1, 2, 3, 4, 5, 6}; G = {4, 5, 6}

As it is a fair die, each of the outcome is equal likely

$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$

Consider the event, G

$P(G) = \sum_{s \in G} P(s) = P(4) + P(5) + P(6) = 1/6 + 1/6 + 1/6 = 1/2$

Example 4

A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find *P(G)*, where *G* is the event that a number greater than 3 occurs on a single roll of die.

<u>Solution</u>

$S = \{1, 2, 3, 4, 5, 6\}$; $G = \{4, 5, 6\}$

Probability of even number: $P(2) = P(4) = P(6) = p_e$

Probability of odd number: $P(1) = P(3) = P(5) = p_o = 2p_e$

$$P(S) = \sum_{s \in S} P(s) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$$

$$= 2p_e + p_e + 2p_e + p_e + 2p_e + p_e = 1 \Rightarrow p_e = 1/9$$

Consider the set of events, *G*

$$P(G) = P(4) + P(5) + P(6) = 1/9 + 2/9 + 1/9 = 4/9$$

Probabilities of Complements and Unions of Events

- Suppose A' denotes complement of *A*

$$P(A') + P(A) = 1 \text{ or } P(A') = 1 - P(A)$$

For any two events *A* and *B*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If the events *A* and *B* are disjoint then

$$P(A \cap B) = 0$$

which implies

$$P(A \cup B) = P(A) + P(B)$$

Example 5

Assume that the engine component of a spacecraft consists of two engines in parallel. If the main engine is 95% reliable, the backup is 80% reliable, and the engine component as a whole is 99% reliable, what is the probability that

(i) Both engines will be operable.

(ii) The main engine will fail but the backup will be operable.

(iii) The engine component will fail.

<u>Solution</u>

Let M be main engine is operatable and B be backup engine is operatable

Engine reliability: $P(M) = 0.95$, $P(B) = 0.80$, $P(M \cup B) = 0.99$

(i) Both engines operable:

$$P(M \cap B) = P(M) + P(B) - P(M \cup B) = 0.76$$

(ii) Main engine fails but backup operable:

$$P(M' \cap B) = P(B) - P(M \cap B) = 0.04$$

(iii) Engine component fails: $P(\overline{M \cup B}) = 1 - P(M \cup B) = 0.01$

Conditional Probability

- Let A and B be events with $P(B) > 0$. The conditional probability of A given B, denoted by $P(A|B)$ is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example 6

In experiment of rolling a fair die, suppose E denotes the event of an odd number is rolled and F denotes a number grater than 4 is rolled. Find

i) $E \cup F$ and $E \cap F$

E={1,3,5} and F={5,6}
$E \cup F$={1,3,5,6} and $E \cap F$={5}

ii) $P(E)$, $P(F)$, $P(E \cap F)$ $P(E|F)$ and $P(F|E)$

$P(E)=3/6=1/2$ and $P(F)=1/3$

$$P(E \cup F) = \frac{4}{6} = \frac{2}{3}\,, \qquad P(E \cap F) = P(5) = \frac{1}{6}$$

$$P(E|F) = \frac{P(E \cap F)}{p(F)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}, \qquad P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

Independence

- The events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Example 7

Suppose E is the event that a randomly generated bit string of length four begins with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent, if the 16 bit strings of length four are equally likely?

Solution

E = {1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}.

F = {0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111}

$\mathrm{E} \cap F$={1001,1010,1100, 1111}

$P(E) = 8/16 = 1/2$

$P(F) = 8/16 = 1/2$

Because $P(E) \times P(F)=1/4 = P(\mathrm{E} \cap F)$

We conclude that E and F are independent.

Representing Conditional Probabilities with a Tree Diagram

We can understand conditional probability better by using a tree diagram.

Tree diagram:

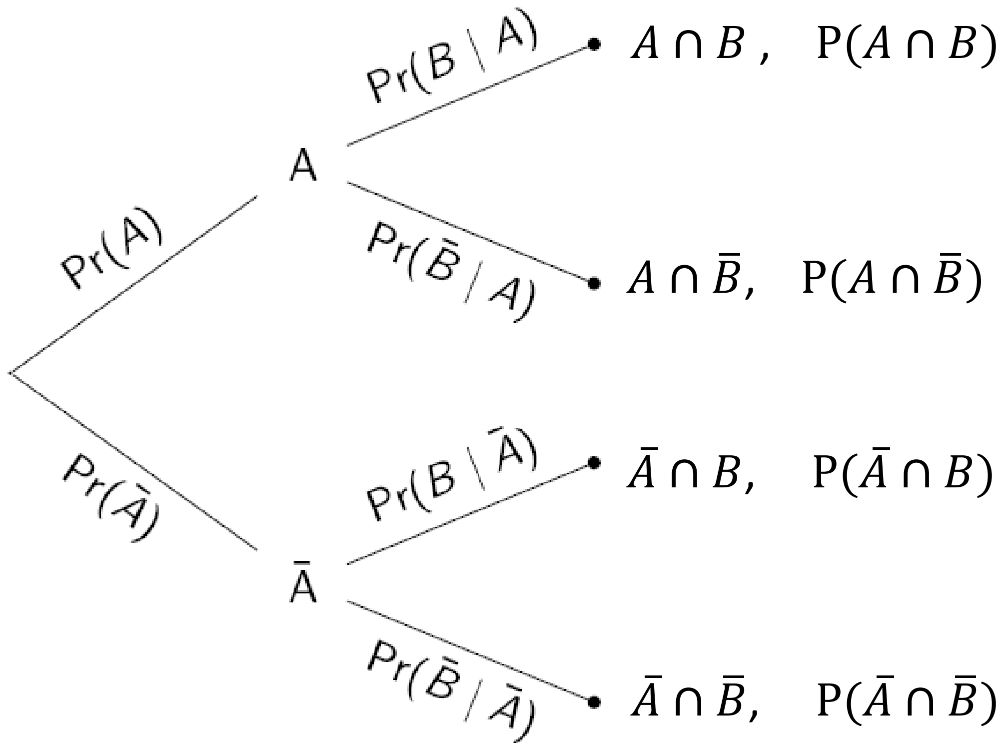
An illustrative way to view conditional probability.

Especially useful for determining probabilities involving events that are not independent.

Conditional probabilities are the probabilities on the second tier of branches.

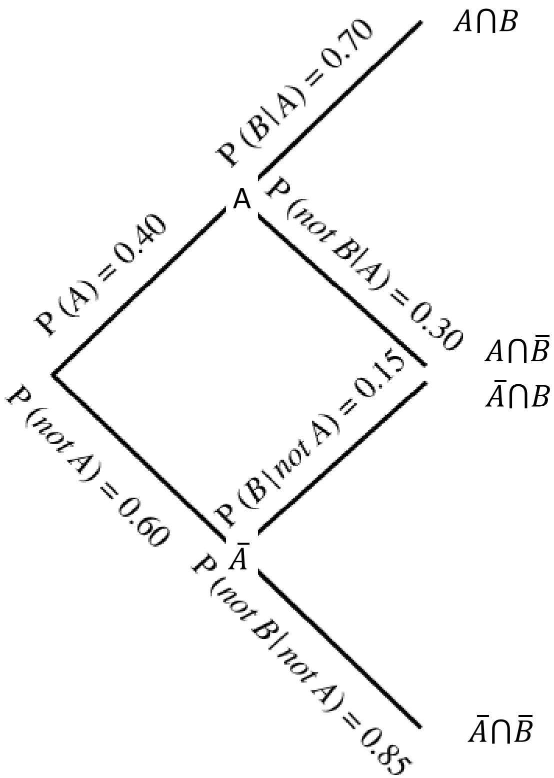
Tree Diagram

- Multiply across; add down



Tree Diagram

- Multiply across; add down



$P(B|A)\,P(A) = 0.70 \times 0.40 = 0.28$

$P(not\ B|A)\,P(A) = 0.30 \times 0.40 = 0.12$

$P(B|not\ A)\,P(not\ A) = 0.15 \times 0.60 = 0.09$

$P(not\ B|not\ A)\,P(not\ A) = 0.85 \times 0.60 = \underline{0.51}$

$\text{Total} = 1.00$

Bayes' Theorem

- Suppose that A and B are events from a sample space S such that $P(A) \neq 0$ and $P(B) \neq 0$. Then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

Proof

From the definition of conditional probability we have $P(A|B)=\dfrac{P(A\cap B)}{P(B)}$ and $P(A\cap B)=P(B\cap A)=P(B\,|\,A)P(A)$.

Observe that $B=B\cap S=B\cap(A\cup\overline{A})=(B\cap A)\cup(B\cap\overline{A})$

Since $(B\cap A)$ and $(B\cap\overline{A})$ are disjoint

$$\begin{aligned}P(B)&=P((B\cap A)\cup(B\cap\overline{A}))\\&=P(B\cap A)+P(B\cap\overline{A})\\&=P(B\,|\,A)P(A)+P(B|\overline{A})P(\overline{A})\end{aligned}$$

Thus $P(A|B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{P(B\,|\,A)P(A)}{P(B|A)P(A)+P(B|\overline{A})P(\overline{A})}$

Example 8

A certain disease occurs in mild (denoted as M) or severe form (denoted as S); three-quarter of patients have the mild form. A new drug is available. The probability that a mild case of the disease responds to the drug ($P(R|M)$) is 0.9, and the probability that a severe case responds ($P(R|S)$) is 0.5.

(i) What is the probability that a randomly chosen case will respond to the drug, $P(R)$?

(ii) You are told that a certain patient has responded to the drug. What is the probability that the patient has the mild form of disease, $P(M|R)$?

Example 8 (Cont.)

Solution

Let M and N denote disease is mild and severe, respectively, and R be patient response to the drug.

Mild cases: $P(M) = 0.75$; Severe cases: $P(S) = 0.25$

Response: $P(R/M) = 0.9$; $p(R|S) = 0.5$

(i) $P(R) = P(R \cap M) + P(R \cap S)$

$P(R \cap M) = P(R|M) \cdot P(M) = (0.9)(0.75) = 0.675$

$P(R \cap S) = P(R|S) \cdot P(S) = (0.5)(0.25) = 0.125$

$P(R) = 0.8$

(ii) $P(M|R) = \frac{P(M \cap R)}{P(R)} = \frac{0.675}{0.8} = 0.8438$

Example 8 (Using tree diagram)

Summary

Materials covered in this lecture

- Probability theory
- Sample space
- Event
- Mutually exclusive event
- Conditional probability
- Independent events
- Bayes' Theorem

Exercise 1

An urn contains four blue balls and five red balls. What is the probability that a ball chosen from the run is blue?

Exercise 2

A sequence of ten bits is randomly generated. What is the probability that at least one of these bits is 0?

Exercise 3

M&M sweets are of varying colors and the different colors occur in different proportions. The table below gives the probability that a randomly chosen M&M has each color, but the value for blue candies is missing.

Color	Brown	Red	Yellow	Green	Orange	Blue
Probability	0.3	0.2	0.2	0.1	0.1	?

(a) Find the missing probability.

(b) You draw and M&M at random from a packet. What is the probability of each of the following events?

(i) You get a brown one or a red one.

(ii) You don't get a yellow one.

(iii) You don't get either an orange one or a blue one.

(iv) You get one that is brown or red or yellow or green or orange or blue.

Exercise 4

A computer assembling company has two assembly plants, plant S and T. 30% of the company's products are assembled at plant S, and the remaining 70% at plant T. 5% of computers assembled at plant S and 6% of computers assembled at plant T are defective. A customer bought a computer from this company. Let S and T denote the events that a computer was assembled at plant S and T, respectively. Also, let D be the event that a computer is defective.

1) Find the probability that the computer bought was assembled at plant S and is defective.

2) Find the probability that the computer bought is defective.

3) Find the probability that the computer bought was assembled at plant S given that it is defective.