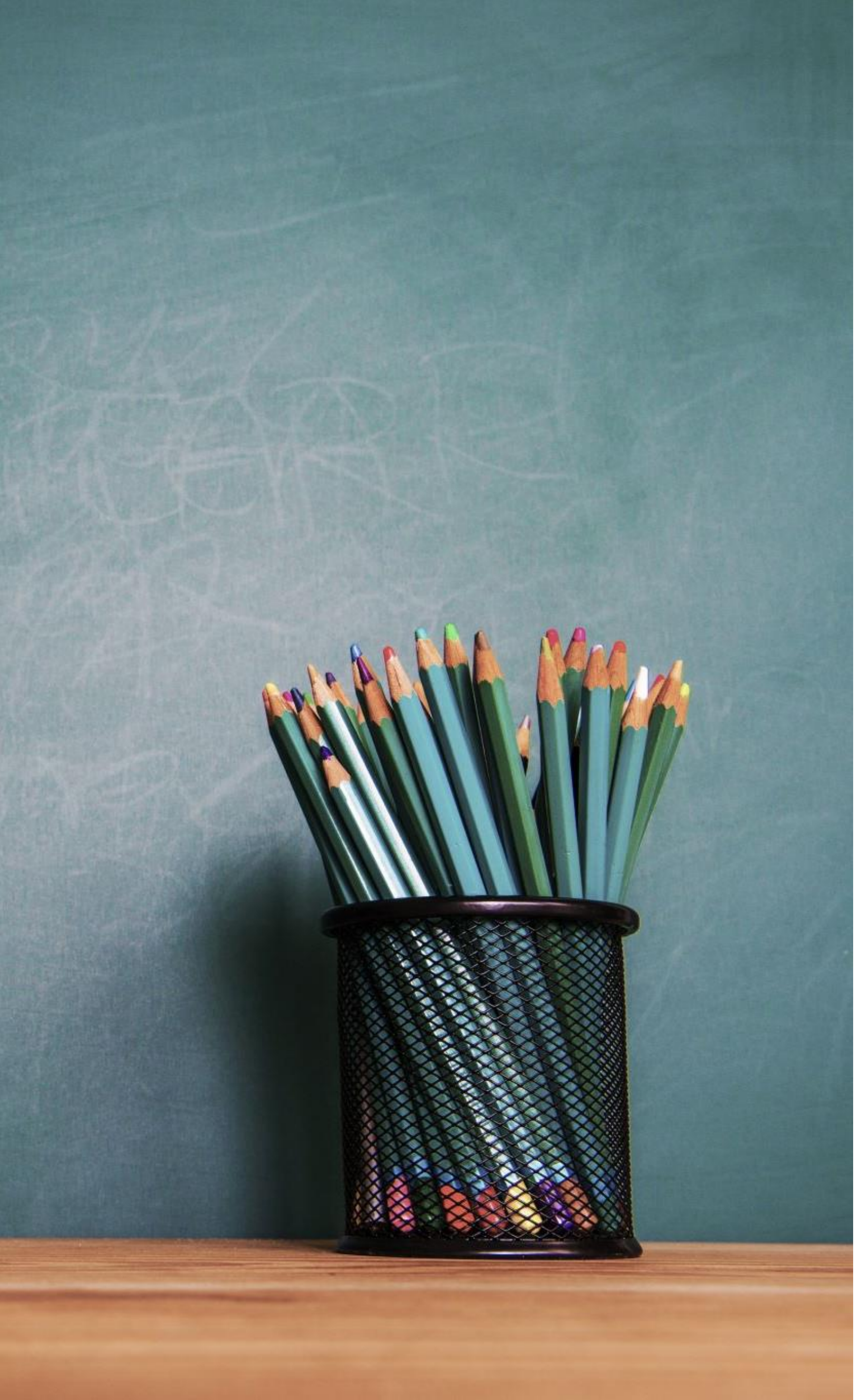


# CHAPTER 3
# Limits & Continuity

Lecture 11 – 14.12.2022

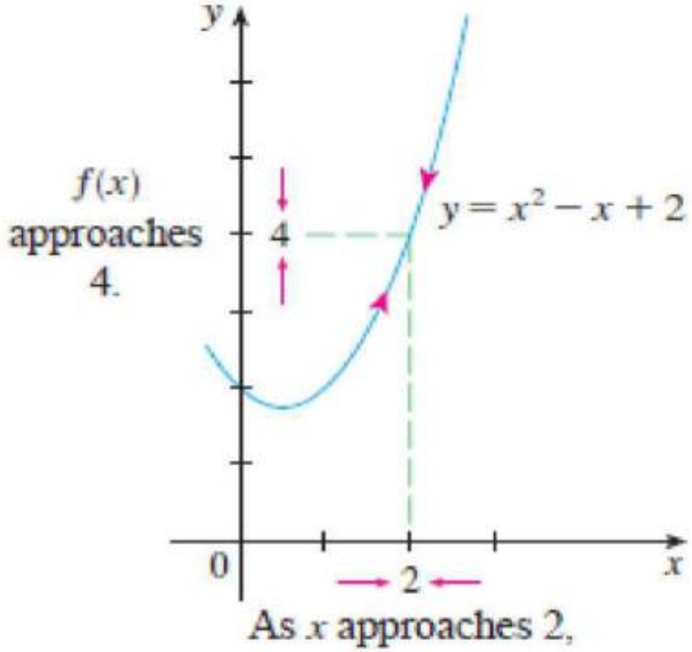
- Limit of Functions
- Limit Laws
- One-sided limits, Infinity, and Asymptotes
- Limits of Trigonometric Functions
- Continuity

# A. Limit of a Function

This topic is to investigate the limits of functions and their properties. We can evaluate limit of a function (i) numerically by constructing a table (ii) graphically by drawing a graph and (iii) analytically using properties of limits.

First, we will examine the limit of a function numerically and graphically. Supposed function $f$ given by $f(x) = x^2 - x + 2$.

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 1.0 | 2.000000 | 3.0 | 8.000000 |
| 1.5 | 2.750000 | 2.5 | 5.750000 |
| 1.8 | 3.440000 | 2.2 | 4.640000 |
| 1.9 | 3.710000 | 2.1 | 4.310000 |
| 1.95 | 3.852500 | 2.05 | 4.152500 |
| 1.99 | 3.970100 | 2.01 | 4.030100 |
| 1.995 | 3.985025 | 2.005 | 4.015025 |
| 1.999 | 3.997001 | 2.001 | 4.003001 |



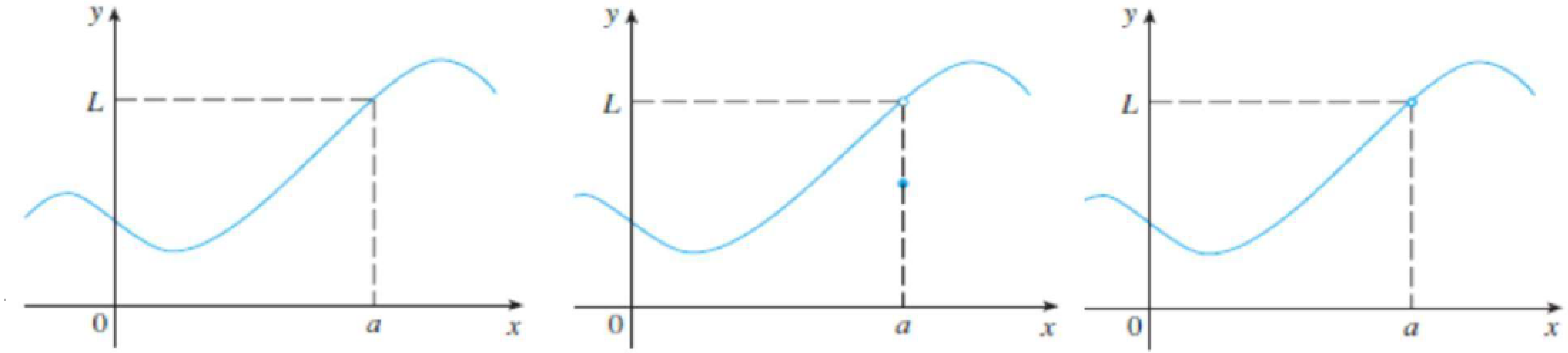
From the table and the graph shown above we see that when $x$ is close to 2 (on either side of 2), is $f(x)$ is close to 4.

We express this by saying "the limit of the function $f(x)$ as $x$ approaches 2 is equal to 4." The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

Note:

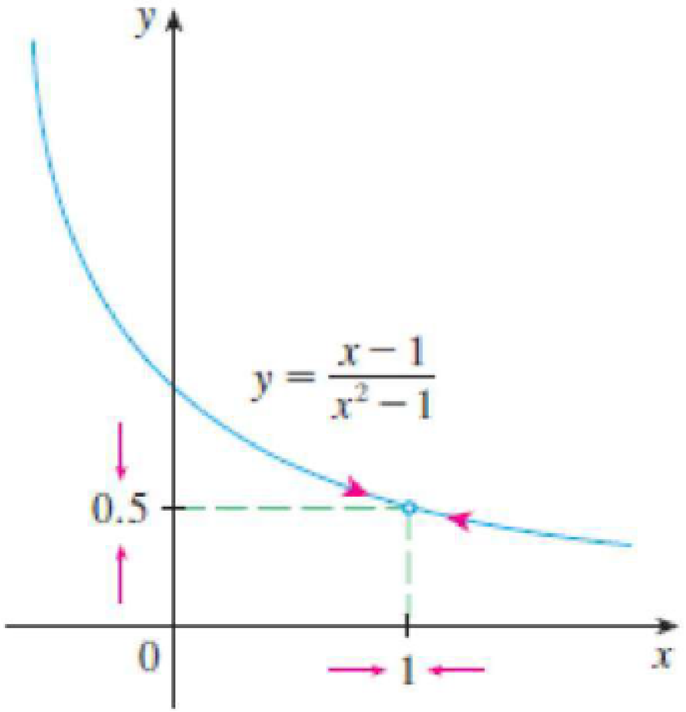
- $\lim_{x \to a} f(x) = L$ is also read as "the limit of $f$ at $a$ is $L$"
- If no number $L$ exists, we say that $f$ has no limit at $a$. i.e. $\lim_{x \to a} f(x)$ **does not exist.**
- In finding the limit of $f(x)$ as $x$ approaches $a$, we never consider $x=a$. In fact, $f(x)$ need not even be defined when $x=a$. The only thing that matters is how $f$ is defined near $a$. In the other word, the limit may exist even if the value of $f$ at $x=a$ is unknown or undefined.




For each function $y=f(x)$ in the figure above, $\lim_{x \to a} f(x) = L$ thus the limit exist.

**Example:** Given $f(x)=\dfrac{x-1}{x^2-1}$ and the graph of *f(x)* is shown. Estimate the value of

$$\lim_{x\to 1} f(x).$$




SOLUTION

$$\lim_{x\to 1} f(x)=0.5\,.$$

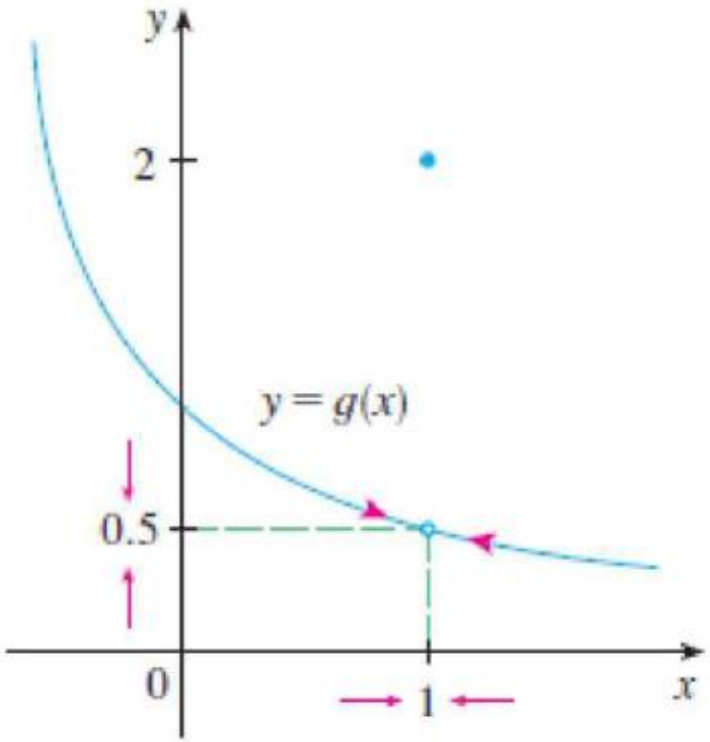
**Example:**

Consider $g(x) = \begin{cases} \dfrac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

Given the graph of $g(x)$, estimate the value of $\lim_{x\to 1} g(x)$.

SOLUTION

$\lim_{x\to 1} g(x) = 0.5$



## B. Limit Laws

The limit can be evaluated by **direct substitution**. That is,

$$\lim_{x \to c} f(x) = f(c).$$ Substitute $c$ for $x$.

### Calculate Limits Using the Limit Laws

If $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$ both exist, and $k \in \Re$, then

1) Sum Rule: $\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$

2) Difference Rule: $\lim_{x \to c}[f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M$

3) Product Rule: $\lim_{x \to c}[f(x).g(x)] = \lim_{x \to c} f(x).\lim_{x \to c} g(x) = LM$

4) Constant Multiple Rule: $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x) = k \cdot L$

5) Quotient Rule: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \dfrac{L}{M}, \quad M \neq 0$

6) Power Rule: $\lim_{x \to c}[f(x)]^n = L^n$, $n$ is a positive integer

7) Root Rule: $\lim_{x \to c} \sqrt[n]{(f(x))} = \sqrt[n]{L} = L^{\frac{1}{n}}$, $n$ is a positive integer

(If $n$ is even, we assume that $\lim_{x \to c} f(x) = L > 0$)

**Some useful limits**

1) $\lim_{x \to c} a = a$

→e.g $\lim_{x \to 7} 2 = 2$

2) $\lim_{x \to c} x = c$

→ e.g. $\lim_{x \to 4} x = 4$

3) $\lim_{x \to c} x^n = c^n$ , where $n$ is a positive integer

→ e.g. $\lim_{x \to 3} x^2 = 3^2$

4) $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ , where $n$ is a positive integer (and if $n$ is even, we assume $c > 0$)

→e.g. $\lim_{x \to 7} \sqrt[3]{x} = \sqrt[3]{7}$

## Limits of Polynomials

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ is a polynomial, then

$$\lim_{x \to c} p(x) = p(c) = a_n c^n + a_{n-1} c^{n-1} + \ldots + a_0$$

→e.g. $p(x) = x^3 + 2x^2 - 1$

$\lim_{x \to c} p(x) = \lim_{x \to c} x^3 + 2x^2 - 1 = (1)^3 + 2(1)^2 - 1 = 2$ , which is $p(1)=2$.

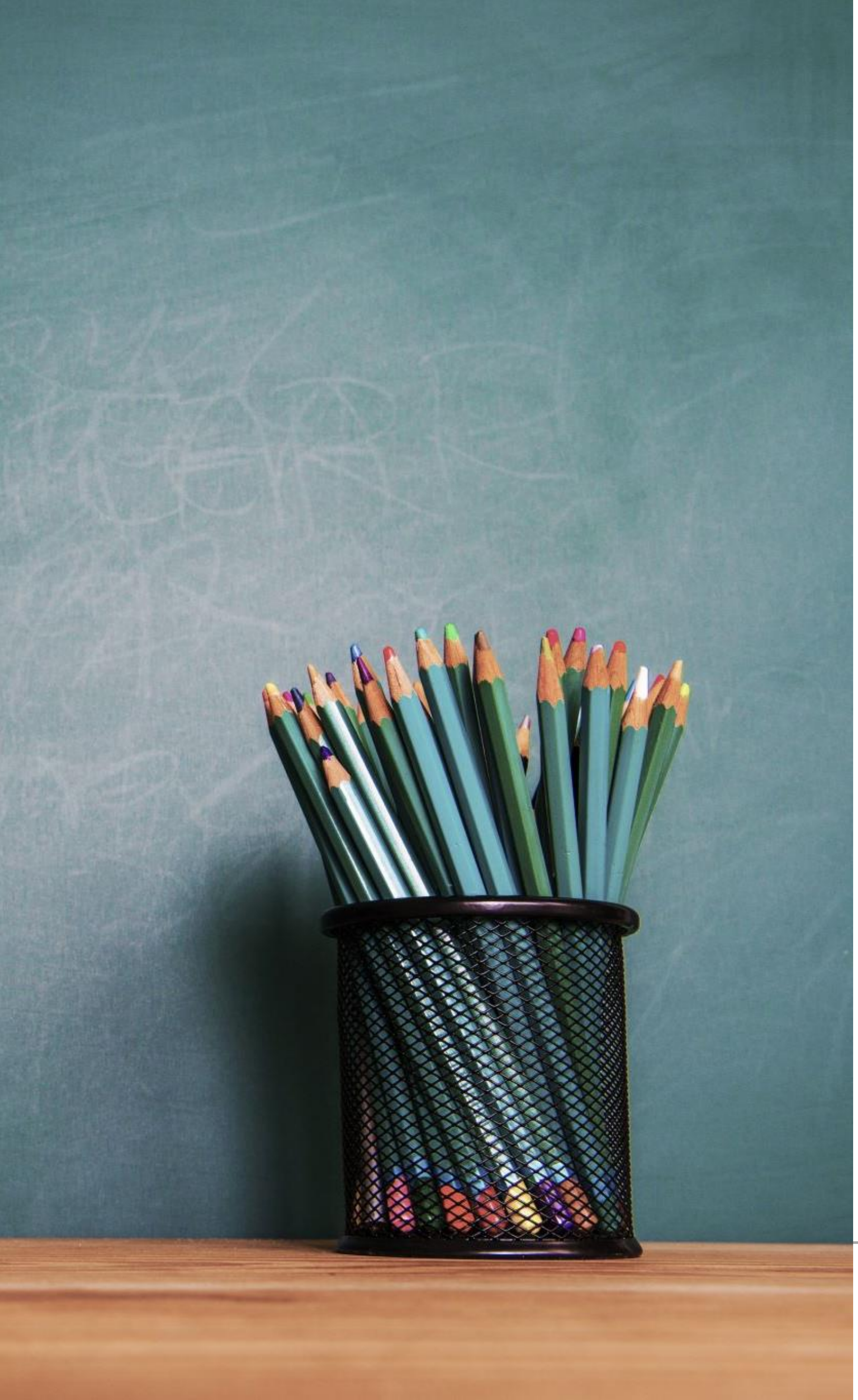
**Example**:

Find the following limits, if they exist. Apply *Limit Laws* to evaluate the limits.

| | | |
|---|---|---|
| 1) | $\lim_{x\to 3} x =$ 3 | |
| 2) | $\lim_{x\to 0} 4(e^x) = 4\lim_{x\to 0} e^x$ <br> $= 4(e^0) = 4(1) = 4$ | *(constant multiple rule)* <br> $e^0 = 1$ |
| 3) | $\lim_{x\to 0} x^3(2x^2+15)$ <br> $= \left(\lim_{x\to 0} x^3\right)\left(\lim_{x\to 0} 2x^2+15\right) = 0^3 \cdot [2(0^2)+15] = 0$ # | *(product rule)* <br> $\lim_{x\to 0} x^3(2x^2+15) = \lim_{x\to 0} 2x^5+15x^3$ <br> $= 2(0)^5+15(0)^3$ <br> $= 0$ # |
| 4) | $\lim_{x\to 5}(x^3-3x+5)$ <br> $= \lim_{x\to 5} x^3 - \lim_{x\to 5} 3x + \lim_{x\to 5} 5 = 5^3 - 3(5) + 5 = 115$ # | *(sum/difference rule)* |

| | | |
|---|---|---|
| 5) | $\lim_{x\to c}\left(x^3+4x^2-3\right)$ <br> $= c^3+4c^2-3$ | *(sum/difference rule)* |
| 6) | $\lim_{x\to 2}\frac{5x^3+4}{x-3}=\frac{\lim_{x\to 2}(5x^3+4)}{\lim_{x\to 2}(x-3)}$ <br> $= \frac{5(2)^3+4}{2-3} = \frac{44}{-1} = -44$ | *(quotient rule. **Make sure the denominator is NOT ZERO)* |
| 7) | $\lim_{x\to -2}\frac{3x+4}{x+5}=\frac{\lim_{x\to -2}3x+4}{\lim_{x\to -2}x+5}$ <br> $= \frac{3(-2)+4}{-2+5} = \frac{-2}{3}$ | *(quotient rule, **Make sure the denominator is NOT ZERO)* |

| | | |
|---|---|---|
| 8) | $\lim_{x\to -1} \sqrt[3]{x^3 + 8x^2 - 3}$ <br> $= \lim_{x\to -1} (x^3 + 8x^2 - 3)^{1/3}$ <br> $= \left(\lim_{x\to -1} x^3 + 8x^2 - 3\right)^{1/3} = (4)^{1/3} = \sqrt[3]{4}$ | *(root rule)* |
| 9) | $\lim_{x\to 0} (2x+2)^3$ <br> $= \left[\lim_{x\to 0} 2x+2\right]^3 = [2(0)+2]^3 = 2^3 = 8_{\#}$ | *(power rule)* |



# CHAPTER 3
# Limits & Continuity

Lecture 12 – 16.12.2022

- Limit of Functions
- Limit Laws
- One-sided limits, Infinity, and Asymptotes
- Limits of Trigonometric Functions
- Continuity

## Special case for limit of fractional functions

If the denominator of a fractional expression is 0, then quotient rule is not applicable. The limits of the expression can be solved using:

(i) Dividing out technique by **factorizing** expression

(ii) Rationalizing the numerator/denominator technique by **multiplication of conjugates** (useful when the limit of a fractional expression contains surd.)

**Example:** (Dividing Out Technique)
Determine if the following limits exist. If the limit exists, state the limit.

1. $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

SOLUTION

Since $\lim\limits_{x \to 2} x - 2 = 0$ (limit of the denominator is ZERO), quotient rule is not applicable.

$\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = \lim\limits_{x \to 2} \dfrac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim\limits_{x \to 2}(x+2) = 4$, hence the limit exist.

*****Note:** The expression is undefined at $x = 2$, but the limit exist.*

2. $\lim_{x\to 1}\frac{x^2-1}{x-1}$

$$= \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

$$= \lim_{x\to 1} x+1$$

$$= 2_{\#}$$

3. $\lim_{x\to 1}\frac{x^2+4x-5}{x-1}$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+5)}{\cancel{x-1}}$$

$$= \lim_{x\to 1} x+5$$

$$= 6_{\#}$$

4. $\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h}$

$= \lim_{h\to 0} \frac{h(6+h)}{h}$

$= \lim_{h\to 0} 6+h$

$= 6_{\#}$

**Note: $\lim_{x \to 0} \frac{2}{x^2}$ does not exist. <u>**WHY?**</u>

Since the denominator's limit $\lim_{x \to 0} x^2 = 0$ the quotient rule is not applicable. Moreover, the expression cannot be simplified any further.

Hence, $\lim_{x \to 0} \frac{2}{x^2}$ does not exist. In this case we can also deduce $\lim_{x \to 0} \frac{2}{x^2} = \infty$

(To be discussed in section of Infinite Limit)

**Example:** (Rationalizing the numerator/denominator)

1. Find $\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$

SOLUTION

We can't apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}=\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$$ *(MULTIPLICATION OF CONJUGATES)*

$$=\lim_{t\to 0}\frac{t^2+9-9}{t^2(\sqrt{t^2+9}+3)}$$

$$=\lim_{t\to 0}\frac{1}{(\sqrt{t^2+9}+3)}=\frac{1}{6}$$

Hence the limit exist.

2. Find $\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2}$ $= \lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2} \cdot \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}$

$$= \lim_{x\to 2} \frac{(x-2)(\sqrt{x+2}+2)}{(x+2)-4}$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(\sqrt{x+2}+2)}{\cancel{x-2}}$$

$$= \sqrt{2+2}+2$$

$$= 4_{\#}$$

3. Find $\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}$ $= \lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}$

$$= \lim_{x\to 1} \frac{\cancel{1-x}}{\cancel{(1-x)}(1+\sqrt{x})}$$

$$= \frac{1}{1+\sqrt{1}}$$

$$= \frac{1}{2} \#$$

## C. One-sided limits

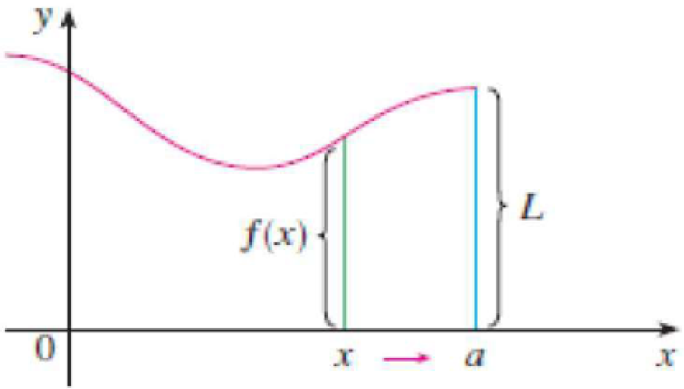
1. If the value of $f(x)$ approaches the number $L$ as $x$ **approaches** $\boldsymbol{a}$ **from the right (right-hand limit)**, we write

$$\boxed{\lim_{x \to a^+} f(x) = L}$$

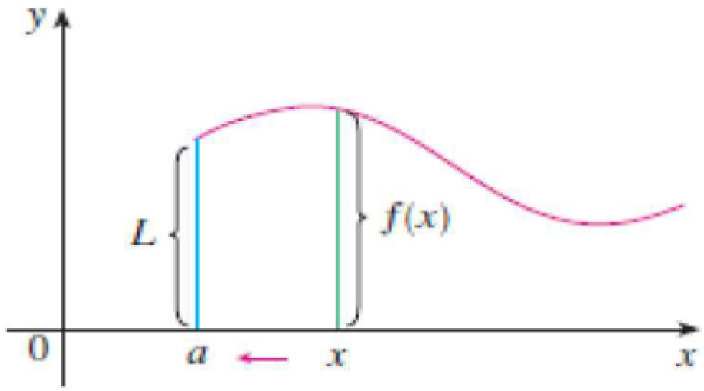
2. If the value of $f(x)$ approaches the number $M$ **as** $\boldsymbol{x}$ **approaches** $\boldsymbol{a}$ **from the left (left-hand limit)**, we write

$$\boxed{\lim_{x \to a^-} f(x) = M}$$

Thus the symbol "$x \to a^+$" means that we consider only $x > a$.




(a) $\lim_{x \to a^-} f(x) = L$




(b) $\lim_{x \to a^+} f(x) = L$

**Theorem:**

A function $f(x)$ has a limit as $x$ approaches $a$ *if and only if* it has left-hand and right–hand limits are equal:

$$\lim_{x \to a} f(x) = L \quad \textbf{\textit{if and only if}} \quad \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

****Note**:

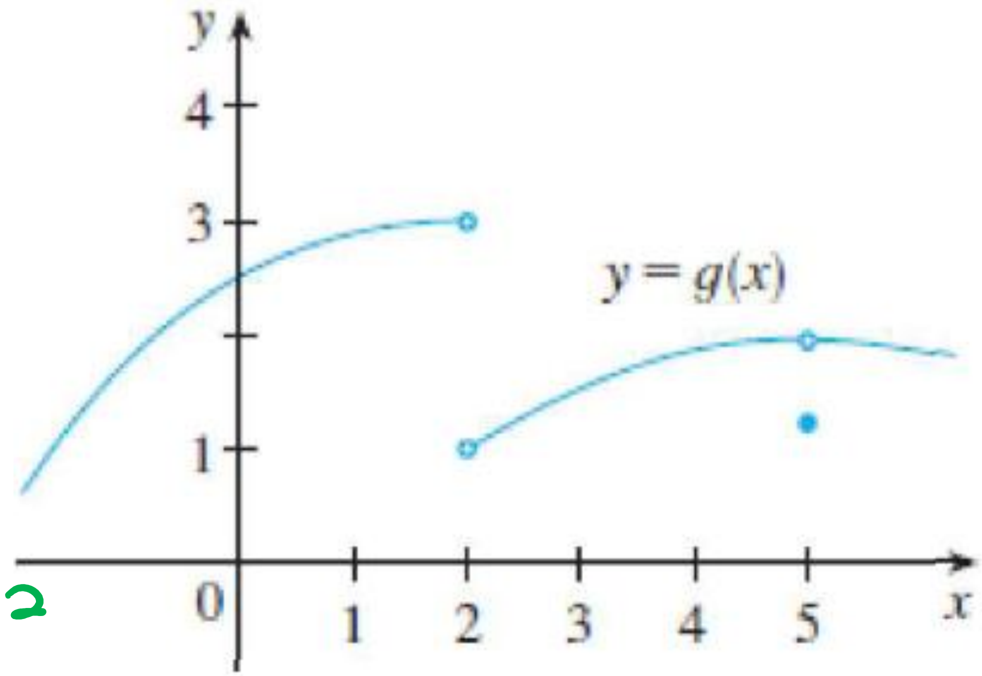
1. This would be very useful when dealing with piecewise-defined functions.
2. If both one-sided limits **do not have the same value**, then $\lim_{x \to a} f(x)$ does not exist.
3. The limit of a function $f(x)$ as $x$ approaches $a$ does not depend on the value of the function at $a$.

**Example:**

The graph of a function $g(x)$ is shown the figure.
Use it to state the values (if they exist) of the following:

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$

(d) $\lim_{x\to 5^-} g(x)$ (e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$




a) $\lim_{x\to 2^-} g(x) = 3$

b) $\lim_{x\to 2^+} g(x) = 1$

c) Since $\lim_{x\to 2^-} g(x) \neq \lim_{x\to 2^+} g(x)$

$\lim_{x\to 2} g(x)$ does not exist #

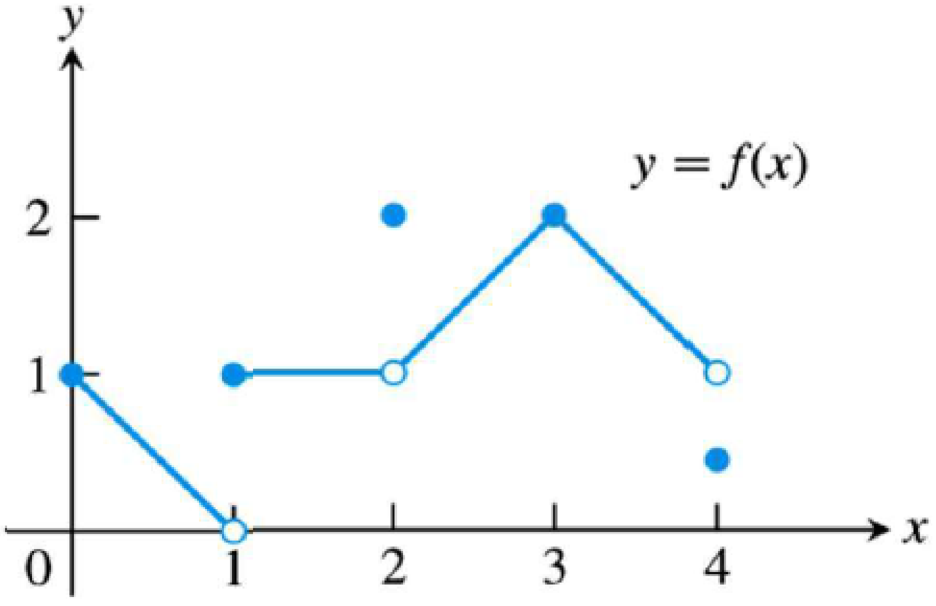
d) $\lim_{x\to 5^-} g(x) = 2$

e) $\lim_{x\to 5^+} g(x) = 2$

f) $\lim_{x\to 5^-} g(x) = \lim_{x\to 5^+} g(x)$ $\therefore \lim_{x\to 5} g(x) = 2$ #

**Example:**

Refer to the figure below. Find (a) $\lim_{x\to 1} f(x)$, (b) $\lim_{x\to 2} f(x)$, (c) $\lim_{x\to 3} f(x)$ and (d) $\lim_{x\to 4} f(x)$.




a) $\lim_{x\to 1^-} f(x) = 0$

$\lim_{x\to 1^+} f(x) = 1$

Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

$\therefore \lim_{x\to 1} f(x)$ does not exist. #

b) $\lim_{x\to 2^-} f(x) = 1$

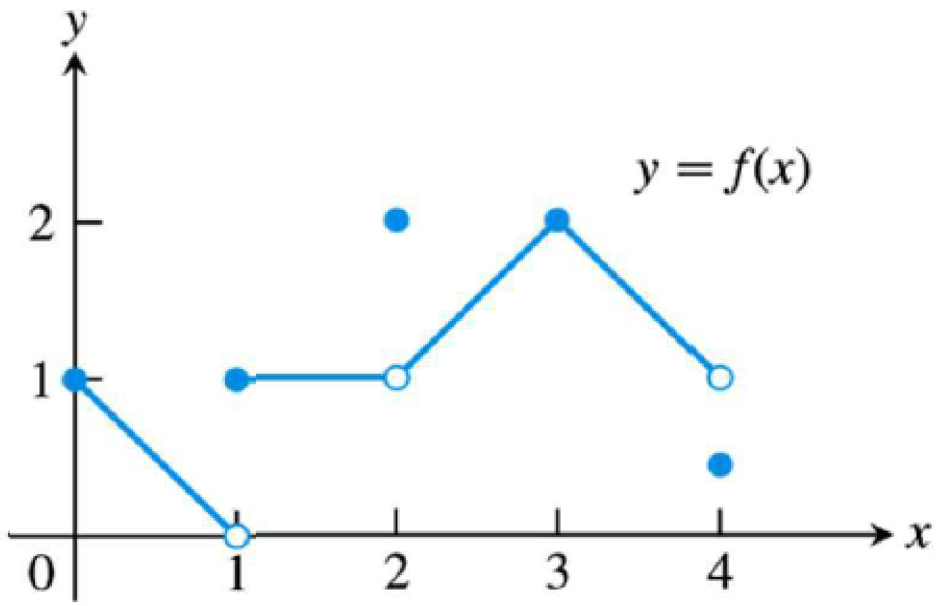
$\lim_{x\to 2^+} f(x) = 1$

Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\therefore \lim_{x\to 2} f(x) = 1$ #

**Example:**

Refer to the figure below. Find (a) $\lim_{x\to 1} f(x)$, (b) $\lim_{x\to 2} f(x)$, (c) $\lim_{x\to 3} f(x)$ and (d) $\lim_{x\to 4} f(x)$.




c) $\lim_{x\to 3^-} f(x) = 2$

$\lim_{x\to 3^+} f(x) = 2$

Since $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$\therefore \lim_{x\to 3} f(x) = 2$ #

d) $\lim_{x\to 4^-} f(x) = 1$

$\therefore \lim_{x\to 4} f(x) = 1$ #

**Example:**

Find $\lim_{x\to 3} f(x)$ for $f(x)=\begin{cases} x^2-5, & x\le 3 \\ \sqrt{x+13}, & x>3 \end{cases}$

SOLUTION

Since the *f(x)* is a **piecewise-defined function**, we must check the left-hand and right–hand limits:

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(x^2-5)=4$$

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}\sqrt{x+13}=4$$

Both one-sided limits are the same, hence the **limit exists** and $\lim_{x\to 3} f(x)=4$

**Example:**

Determine if the limits exist.

1) Consider $f(x)=\begin{cases} x+2 & , \quad x\le 0 \\ x-1 & , \quad x>0 \end{cases}$

a) $\lim_{x\to 0^-} f(x)$

SOLUTION

$\lim_{x\to 0^-} f(x) = 0+2$

$= 2$ #

b) $\lim_{x\to 0^+} f(x)$

$\lim_{x\to 0^+} f(x) = 0-1$

$= -1$ #

c) $\lim_{x\to 0} f(x)$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\therefore \lim_{x\to 0} f(x)$ does not exist. #

2) Consider $g(x)=\begin{cases} 5x-1 & , \quad x \neq 4 \\ 4x+3 & , \quad x = 4 \end{cases}$

a) $\lim\limits_{x \to 4^-} g(x)$

SOLUTION

$$\lim_{x \to 4^-} g(x) = 5(4)-1$$
$$= 19_{\#}$$

b) $\lim\limits_{x \to 4^+} g(x)$

$$\lim_{x \to 4^+} g(x) = 5(4)-1$$
$$= 19_{\#}$$

c) $\lim\limits_{x \to 4} g(x)$

Since $\lim\limits_{x \to 4^-} g(x) = \lim\limits_{x \to 4^+} g(x)$

$\therefore \lim\limits_{x \to 4} g(x) = 19_{\#}$

3) Consider $h(x)=\begin{cases} x-1 & , \quad x<2 \\ 0 & , \quad x=2 \\ x^2-3 & , \quad x>2 \end{cases}$

a) $\lim_{x\to 2^-} h(x)$

SOLUTION

$$\lim_{x\to 2^-} h(x) = 2-1$$
$$= 1_{\#}$$

b) $\lim_{x\to 2^+} h(x)$

$$\lim_{x\to 2^+} h(x) = 2^2-3$$
$$= 1_{\#}$$

c) $\lim_{x\to 2} h(x)$

Since $\lim_{x\to 2^-} h(x) = \lim_{x\to 2^+} h(x)$

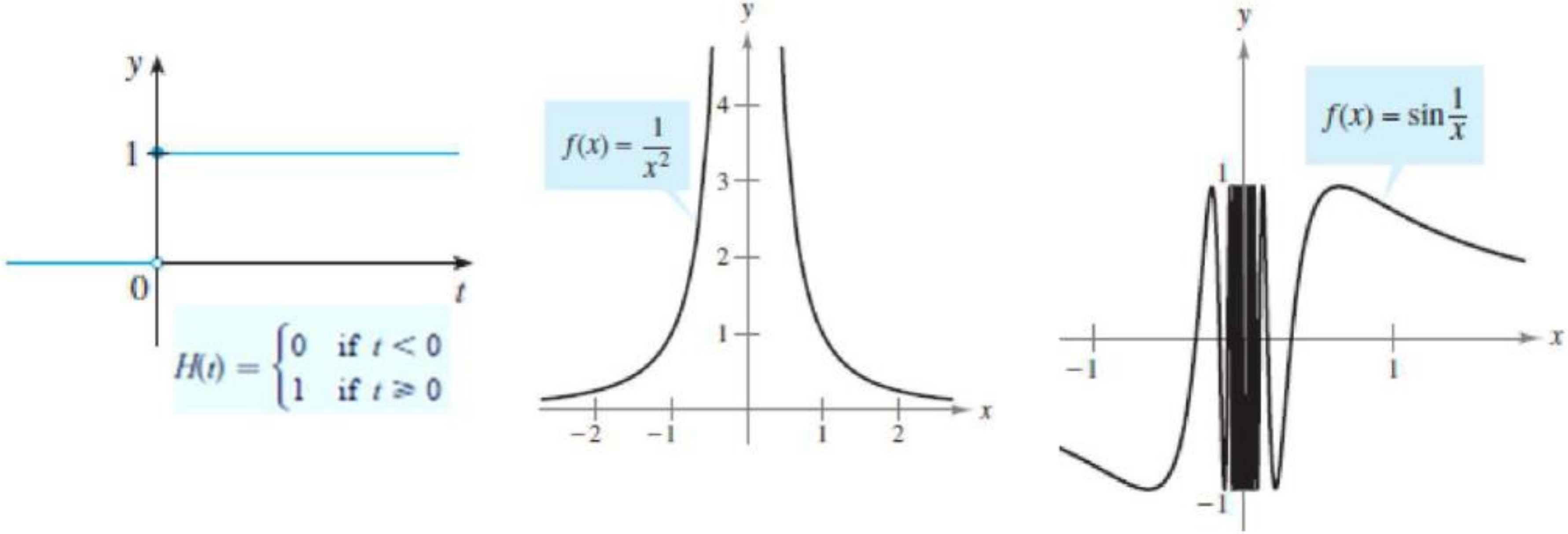
$\therefore \lim_{x\to 2} h(x) = 1_{\#}$

## Limits That Fail to Exist

**Common Types of Behavior Associated with Nonexistence of a Limit**

1. $f(x)$ approaches a different number from the right side of $c$ than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as $x$ approaches $c$.
3. $f(x)$ oscillates between two fixed values as $x$ approaches $c$.

The figures below show graphs of functions of the 3 cases where the **limits fail to exist**.



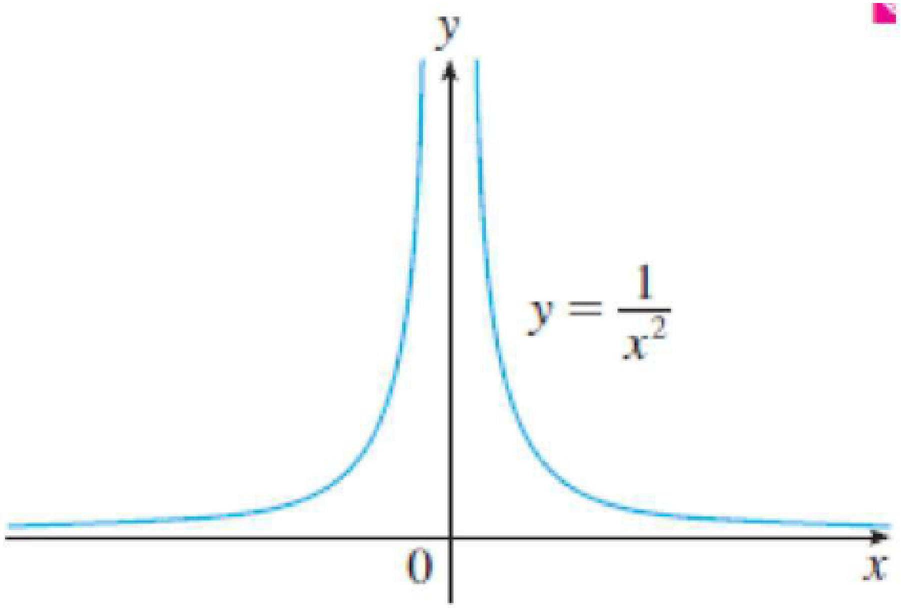
# D. Infinite Limit and Vertical Asymptote

Let's first investigate the infinite limit numerically and graphically. Consider $f(x) = \dfrac{1}{x^2}$.

As shown in table and graph below, the values of $f(x) = \dfrac{1}{x^2}$ do not approach a number as $x$ becomes close to $0$, so $\lim_{x\to 0} f(x)$ does not exist. We use the notation $\lim_{x\to 0} f(x) = \infty$.

**<u>**Note:</u>** This does not mean that we are regarding $\infty$ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist.

| $x$ | $\dfrac{1}{x^2}$ |
|---|---|
| $\pm 1$ | 1 |
| $\pm 0.5$ | 4 |
| $\pm 0.2$ | 25 |
| $\pm 0.1$ | 100 |
| $\pm 0.05$ | 400 |
| $\pm 0.01$ | 10,000 |
| $\pm 0.001$ | 1,000,000 |

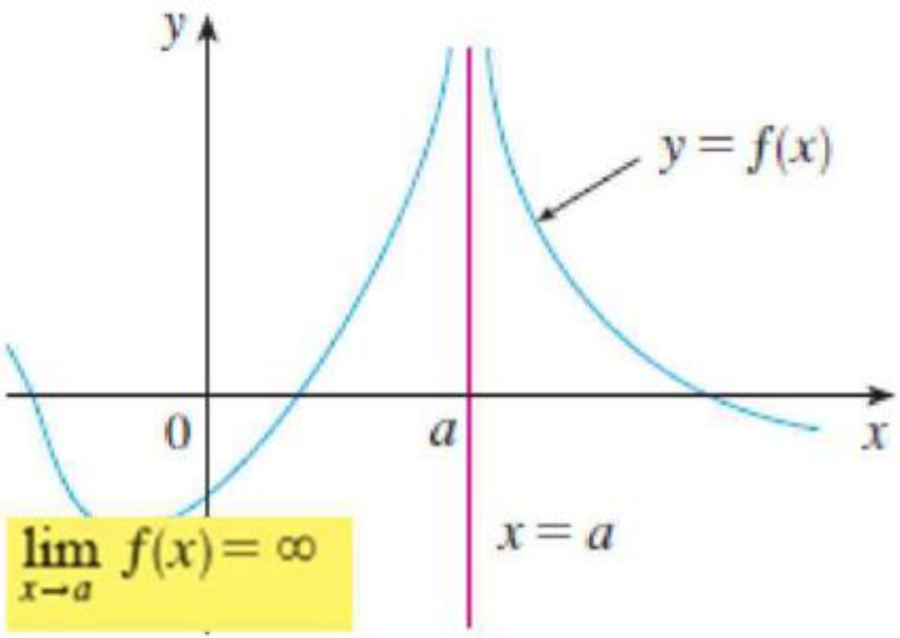


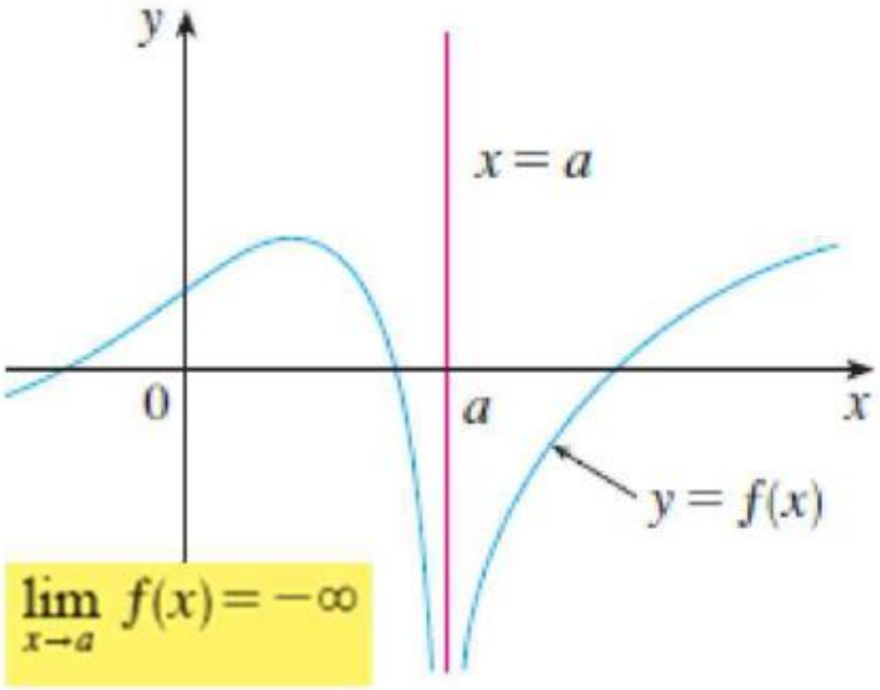
$$\Rightarrow \lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking $x$ sufficiently close to $a$, but not equal to $a$.

$$\Rightarrow \lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking $x$ sufficiently close to $a$, but not equal to $a$.

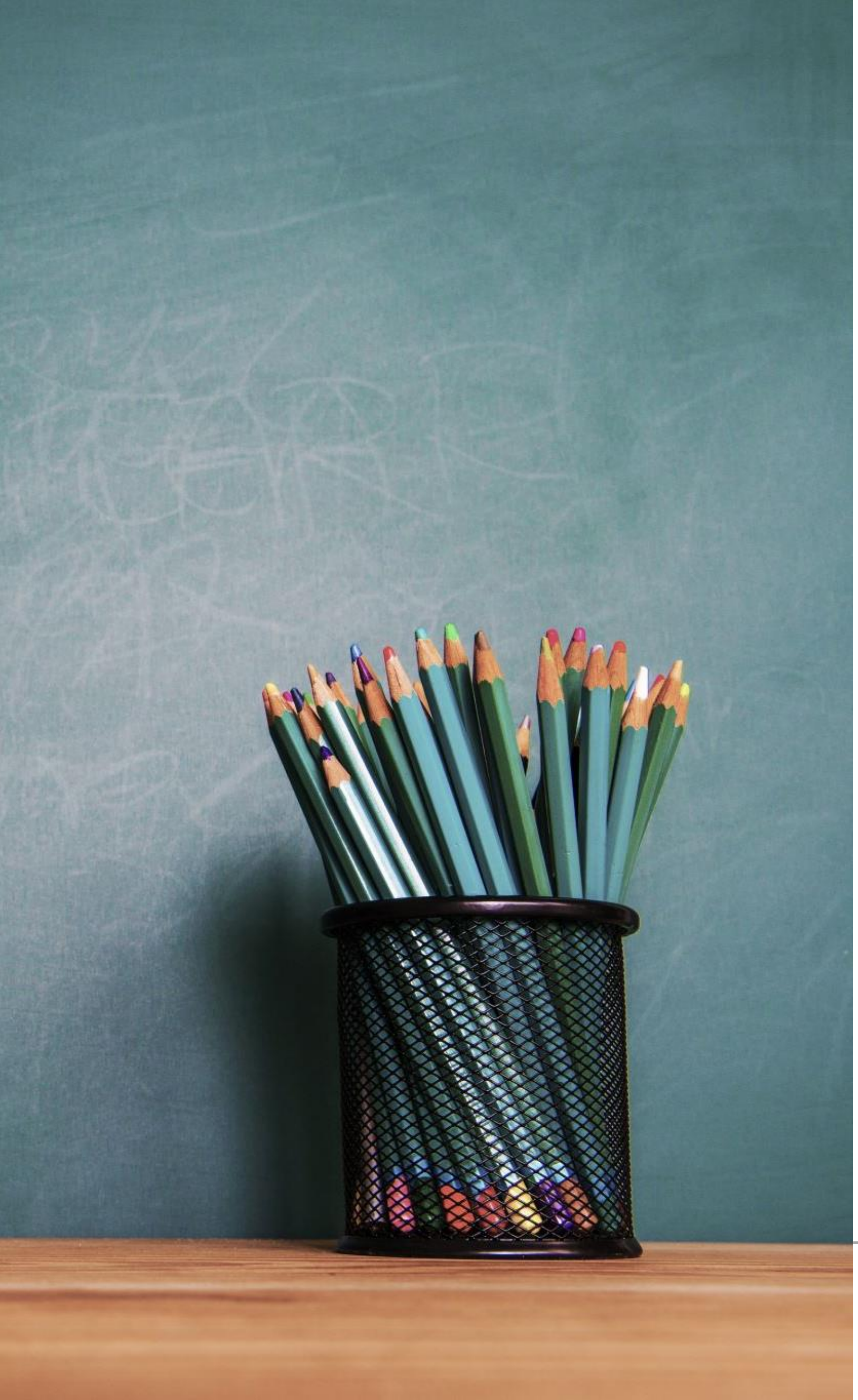



Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$



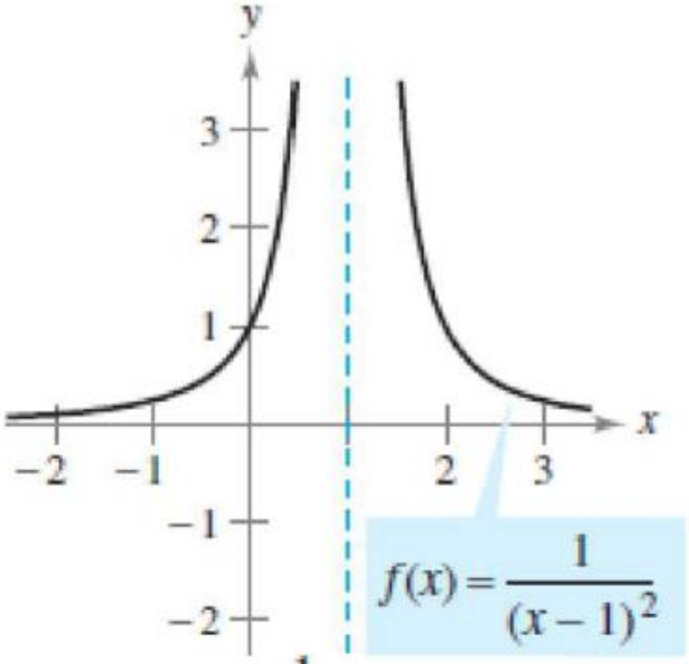
# CHAPTER 3
# Limits & Continuity

Lecture 13 – 21.12.2022

- Limit of Functions
- Limit Laws
- One-sided limits, Infinity, and Asymptotes
- Limits of Trigonometric Functions
- Continuity

## Definition of Vertical Asymptote

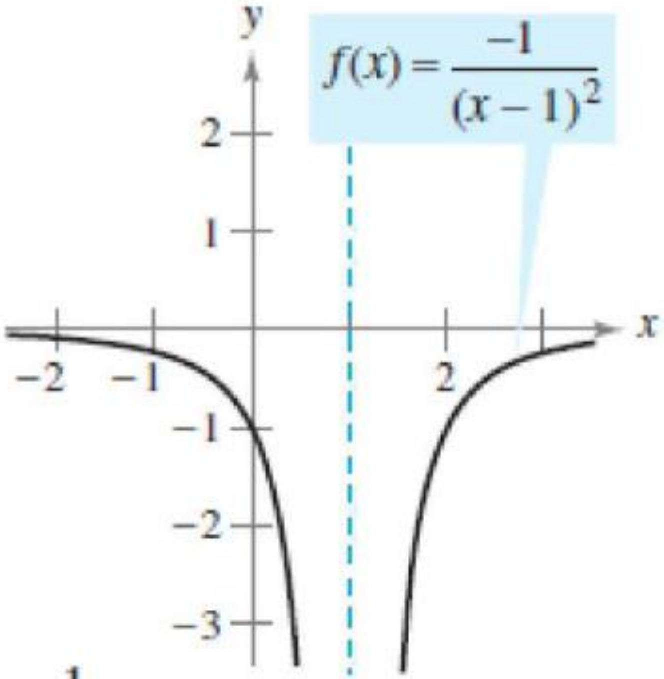
If $f(x)$ approaches infinity (or negative infinity) as $x$ approaches $c$ from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of $f$.




$$\lim_{x \to 1} \frac{-1}{(x-1)^2} = -\infty$$

Limit from each side is $-\infty$.

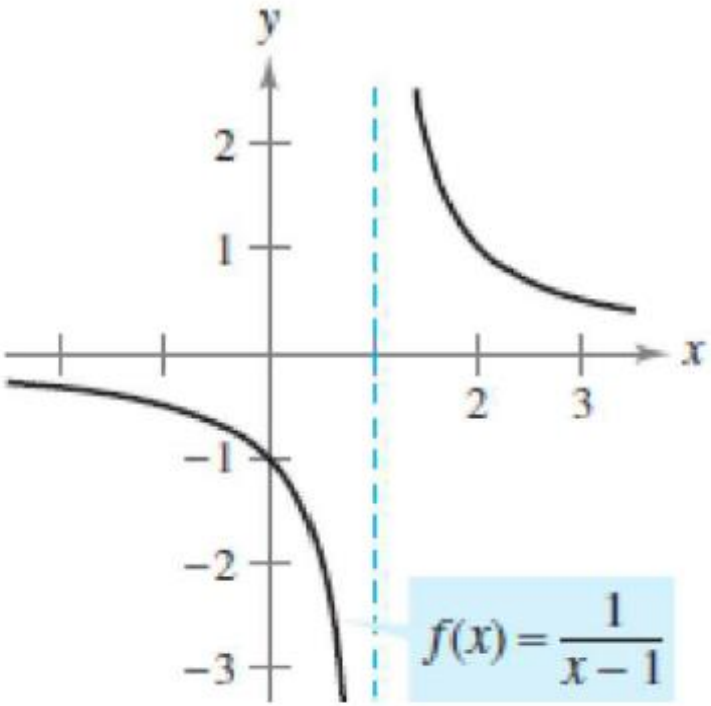
Line $x=1$ is a vertical asymptote




$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$$

Limit from each side is $\infty$.

Line $x=1$ is a vertical asymptote




$$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty \qquad \lim_{x \to 1^+} \frac{1}{x-1} = \infty$$

Line $x=1$ is a vertical asymptote

****<u>NOTE:</u>** In this case it is WRONG to say

$$\cancel{\lim_{x \to 1} \frac{1}{x-1} = \infty} \quad \text{or} \quad \cancel{\lim_{x \to 1} \frac{1}{x-1} = -\infty}$$

We say: $\displaystyle\lim_{x \to 1} \frac{1}{x-1}$ does not exist.

**Example:**

Consider $y = \dfrac{2x}{x-3}$.

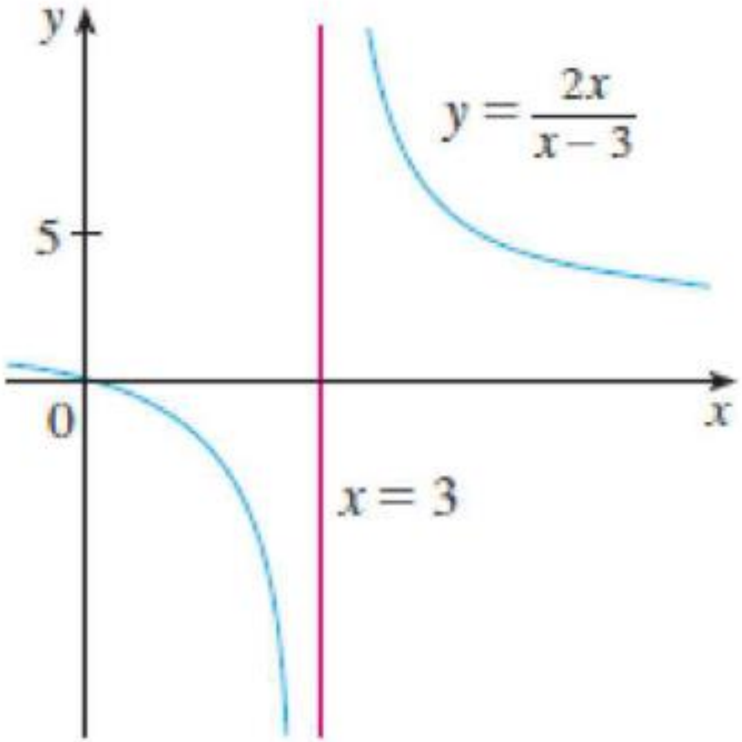
(i) Find $\lim_{x\to 3^-} \dfrac{2x}{x-3}$ and $\lim_{x\to 3^+} \dfrac{2x}{x-3}$.

(ii) Then find $\lim_{x\to 3} \dfrac{2x}{x-3}$.

(iii) State the vertical asymptote(s), if any.

i) $\lim_{x\to 3^-} \dfrac{2x}{x-3} = -\infty$

$\lim_{x\to 3^+} \dfrac{2x}{x-3} = +\infty$

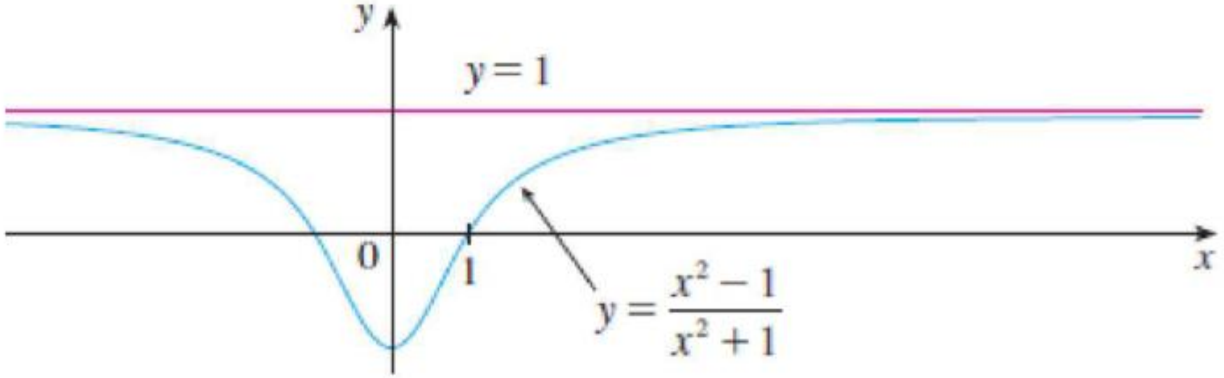



ii) $\lim_{x\to 3} \dfrac{2x}{x-3}$ does not exist.

iii) Vertical asymptote, $x = 3$

# E. Limits at Infinity and Horizontal Asymptotes

Let's investigate the behavior of the function $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$




| $x$ | $f(x)$ |
|---|---|
| 0 | −1 |
| ±1 | 0 |
| ±2 | 0.600000 |
| ±3 | 0.800000 |
| ±4 | 0.882353 |
| ±5 | 0.923077 |
| ±10 | 0.980198 |
| ±50 | 0.999200 |
| ±100 | 0.999800 |
| ±1000 | 0.999998 |

From the table and the graph, we find that $\lim_{x \to \infty} \dfrac{x^2 - 1}{x^2 + 1} = 1$

Hence, line *y=1* is a horizontal asymptote.

=> $\lim_{x \to \infty} f(x) = L$

means that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large.

=> $\lim_{x \to -\infty} f(x) = L$

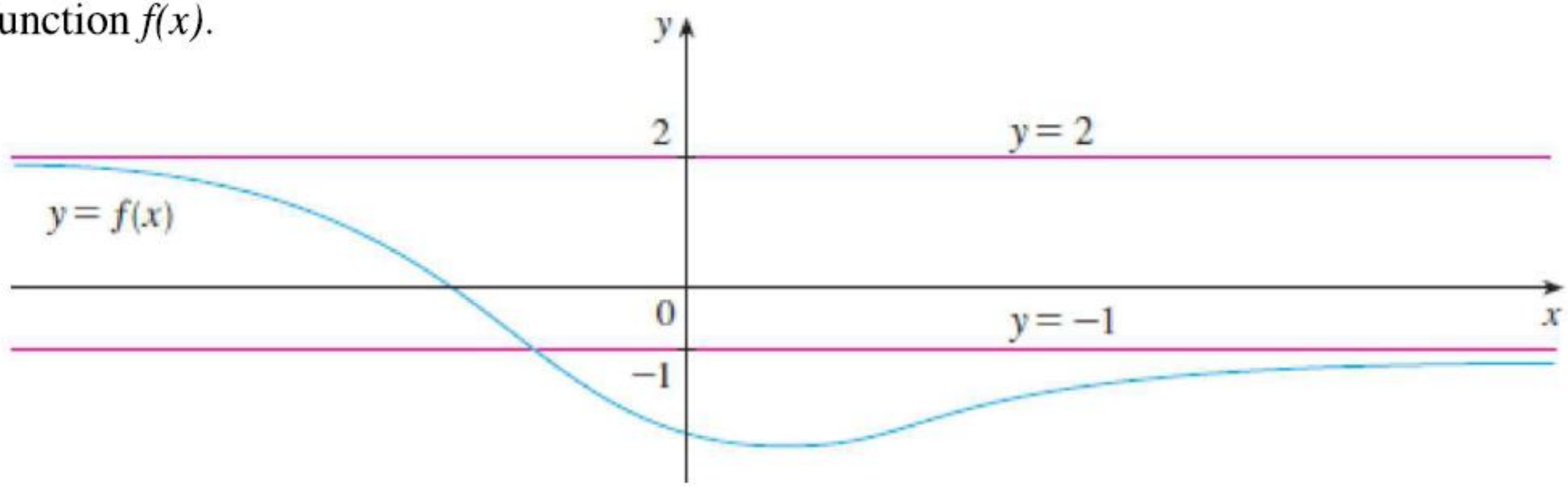
means that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large negative.

## Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of $f$ if

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L.$$

Refer to the figure below. Find limits at infinity and horizontal asymptotes for the function *f(x)*.




$$\lim_{x \to \infty} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

Hence, *y=-1* and *y=2* are the horizontal asymptotes.

**Example:** Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$.

Then, state the horizontal asymptote(s), if any.

SOLUTION

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

$\therefore$ Horizontal asymptote.

$y = 0$

$$\lim_{x \to \pm\infty} \frac{a}{x^n} = 0$$

$a$ = constant

$n$ = positive integer

## Limit at infinity of rational function

If $r$ is a positive rational number and $c$ is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if $x^r$ is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of $x$ that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of $x$.)

**Example:**

Find $\lim\limits_{x \to \infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$ *(Note that Degree of numerator = degree of denominator)*

Then state the horizontal asymptote, if any.

$$\lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{3 - \cancelto{0}{\frac{1}{x}} - \cancelto{0}{\frac{2}{x^2}}}{5 + \cancelto{0}{\frac{4}{x}} + \cancelto{0}{\frac{1}{x^2}}} = \frac{3}{5} \#$$

$x \to \pm\infty$

$\frac{a}{x^n} \to 0$

$\therefore$ Horizontal asymptote, $y = \frac{3}{5}$ #.

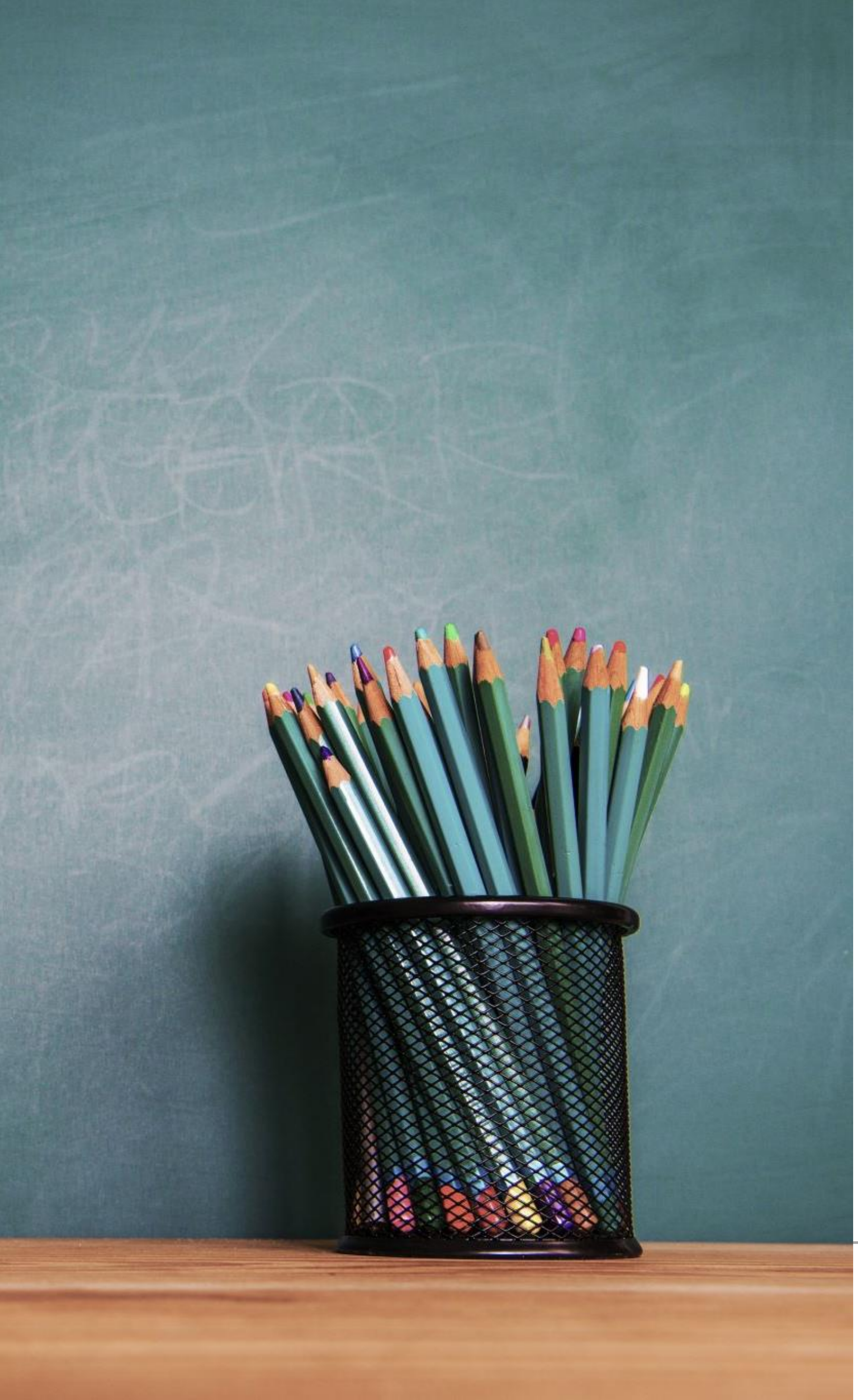
**Example**:

(a) Find $\lim\limits_{x\to-\infty} \dfrac{3x+2}{5x^3-4}$ *(Note that Degree of numerator < degree of denominator)*

(b) State the horizontal asymptote of $f(x)=\dfrac{3x+2}{5x^3-4}$ , if any.

a) $$\lim_{x\to\infty} \frac{\frac{3x}{x^3}+\frac{2}{x^3}}{\frac{5x^3}{x^3}-\frac{4}{x^3}} = \lim_{x\to\infty} \frac{\cancel{\frac{3}{x^2}}^{0}+\cancel{\frac{2}{x^3}}^{0}}{5-\cancel{\frac{4}{x^3}}^{0}} = \frac{0}{5} = 0_{\#}$$

b) Horizontal asymptote , $y=0_{\#}$



# CHAPTER 3
# Limits & Continuity

Lecture 14 – 23.12.2022

- Limit of Functions
- Limit Laws
- One-sided limits, Infinity, and Asymptotes
- Limits of Trigonometric Functions
- Continuity

**Example**:

(a) Find $\lim_{x \to \infty} \frac{2x^2+5}{3x+1}$ *(Note that Degree of numerator > degree of denominator)*

(b) State the horizontal asymptote of $f(x) = \frac{2x^2+5}{3x+1}$, if any.

a) $\lim_{x \to \infty} \frac{2x^2+5}{3x+1} = \lim_{x \to \infty} \frac{\frac{2x^2}{x} + \frac{5}{x}}{\frac{3x}{x} + \frac{1}{x}} = \lim_{x \to \infty} \frac{2x + \frac{5}{x} \to 0}{3 + \frac{1}{x} \to 0} = \lim_{x \to \infty} \frac{2x}{3} = \infty$ #

b) There is NO horizontal asymptote.

**Example:**

(a) Find $\lim_{x \to -\infty} \frac{x^2 - 3}{2x - 4}$ *(Note that Degree of numerator > degree of denominator)*

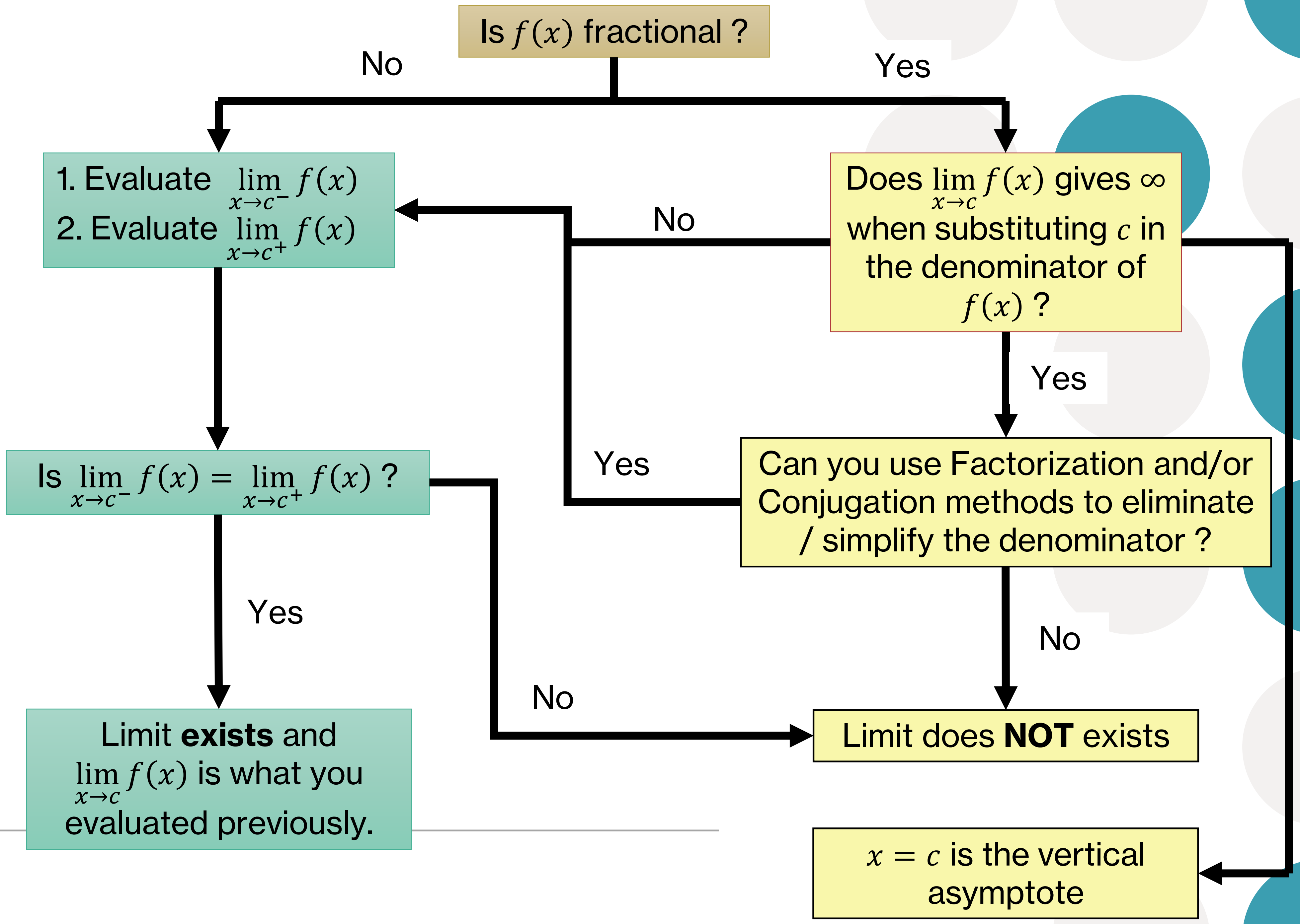
(b) State the horizontal asymptote of $f(x) = \frac{x^2 - 3}{2x - 4}$, if any.

SOLUTION

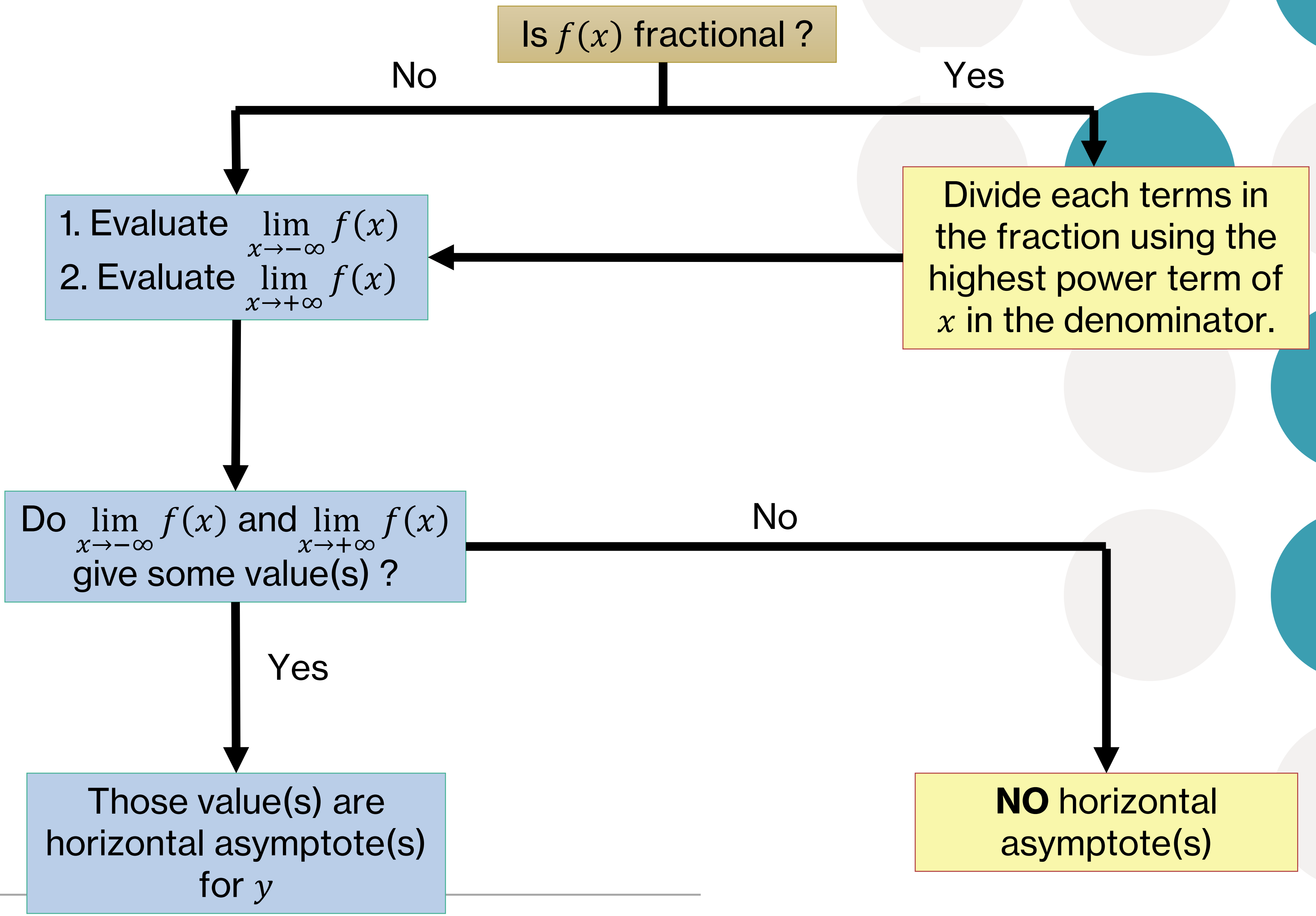
a) $\lim_{x \to -\infty} \frac{x^2-3}{2x-4} = \lim_{x \to -\infty} \frac{\frac{x^2}{x} - \frac{3}{x}}{\frac{2x}{x} - \frac{4}{x}} = \lim_{x \to -\infty} \frac{x - \frac{3}{x}}{2 - \frac{4}{x}} = \lim_{x \to -\infty} \frac{x}{2} = -\infty$ #

b) There is No horizontal asymptote.

# Evaluating Limits given a Function $f(x)$ as $x$ approaches $c$ and the Vertical Asymptote



# Evaluating Limits given a Function $f(x)$ as $x$ approaches $\infty$ and the Horizontal Asymptote



# F. Limit involving trigonometric functions

**Limits of Trigonometric Functions**

Let $c$ be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \cot x = \cot c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \csc x = \csc c$

**Example:**

a. $\lim_{x \to 0} \tan x = \tan(0) = 0$

b. $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi \cos(\pi) = -\pi$

c. $\lim_{x \to 0} \sin^2 x = \lim_{x \to 0} (\sin x)^2 = 0^2 = 0$

**Special Trigonometric Limits**

$$\text{1. } \lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{2. } \lim_{x\to 0} \frac{1 - \cos x}{x} = 0$$

**Example:** Find the limit: $\lim_{x\to 0} \frac{\tan x}{x}$.

SOLUTION

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

Now, because

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{1}{\cos x} = 1$$

you can obtain

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x}{x} &= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right)\\ &= (1)(1)\\ &= 1.\end{aligned}$$

# Special Trigonometric Limits

**1.** $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

Area of $\Delta OAB = \frac{1}{2}(1)(1)\sin\theta = \frac{1}{2}\sin\theta$

Area of sector $OAB = \frac{1}{2}(1)^2\theta = \frac{1}{2}\theta$

Area of $\Delta OBC = \frac{1}{2}(1)\tan\theta = \frac{1}{2}\tan\theta$

$$\cancel{\tfrac{1}{2}}\sin\theta < \cancel{\tfrac{1}{2}}\theta < \cancel{\tfrac{1}{2}}\tan\theta$$

$$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta}$$

$$1 \quad \frac{\cancel{\sin\theta}}{\cancel{\sin\theta}} < \frac{\theta}{\sin\theta} < \frac{\frac{\cancel{\sin\theta}}{\cos\theta}}{\cancel{\sin\theta}}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

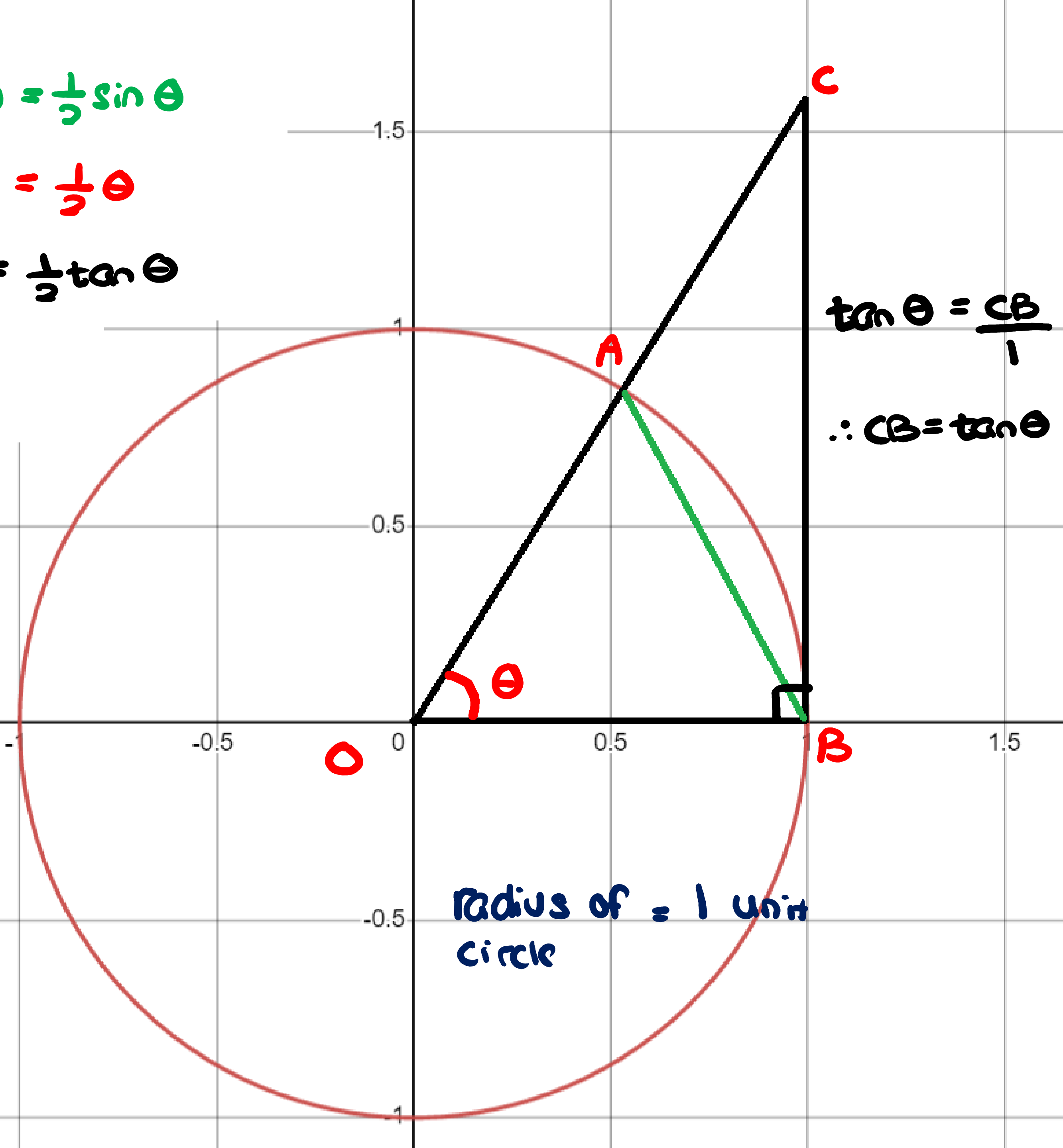
$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

$$\lim_{x\to 0} 1 > \boxed{\lim_{x\to 0}\frac{\sin\theta}{\theta}} > \lim_{x\to 0}\cancelto{1}{\cos\theta}$$

Because $\cos\theta = 1$

Since $\lim\limits_{x\to 0}\dfrac{\sin\theta}{\theta}$ is 'squeezed' by 1

$\therefore \lim\limits_{x\to 0}\dfrac{\sin\theta}{\theta} = 1$



**Example:** Find the limit: $\lim_{x\to 0} \frac{\sin 4x}{x}$.

SOLUTION

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right).$$

Multiply and divide by 4.

Now, by letting $y = 4x$ and observing that $x \to 0$ if and only if $y \to 0$, you can write

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 4x}{x} &= 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right) \\
&= 4\left(\lim_{y\to 0} \frac{\sin y}{y}\right) \\
&= 4(1) \\
&= 4.
\end{aligned}$$

**Example:**

Find $\lim_{x \to 0} \dfrac{\cos x - \cos^2 x}{x}$

SOLUTION

$$= \lim_{x \to 0} \frac{\cos x \, (1 - \cos x)}{x}$$

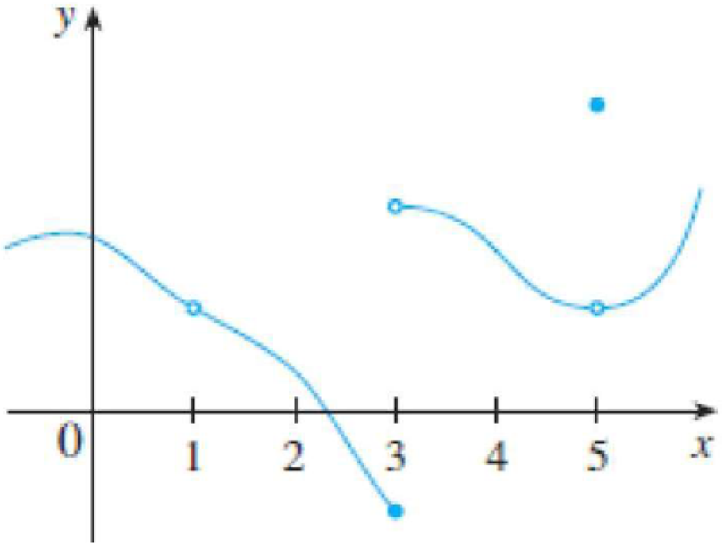
$$= \lim_{x \to 0} \cos x \cdot \lim_{x \to 0} \frac{1 - \cos x}{x}$$

$$= 1 \, (0)$$

$$= 0 \,_{\#}$$

# G. Continuity

A function $f$ is continuous at $x=c$ means that there is no interruption in the graph of $f$ at $c$. That is, its graph is unbroken at $c$ and there are no holes, jumps, or gaps. Figure shows the graph of a function $f$.




There are discontinuities when $x=1$, $x=3$ and $x=5$ because the graph has breaks there.

- $f(1)$ is not defined.
- $f(3)$ is defined, but $\lim_{x\to 3} f(x)$ does not exist (because left and right limits are different.)
- $f(5)$ is defined and $\lim_{x\to 5} f(x)$ exits (because left and right limits are the same), but $f(5)$ *is not equal to* $\lim_{x\to 5} f(x)$.

## Definition of Continuity

*Continuity at a Point:* A function $f$ is **continuous at $c$** if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

*Continuity on an Open Interval:* A function is **continuous on an open interval $(a, b)$** if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is **everywhere continuous.**

**Note: If at least one of the three conditions is violated, then the function $f$ is discontinuous at $x=c$.

**Example:**

Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

## SOLUTION

(a) $f(x)$ discontinuous at $x=2$ because $f(2)$ is undefined.

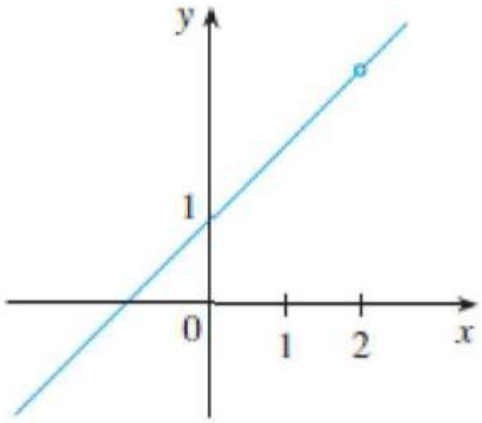
(b) $f(x)$ discontinuous at $x=0$ because $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2}$ does not exist.

(c) $\Rightarrow f(2)=1$

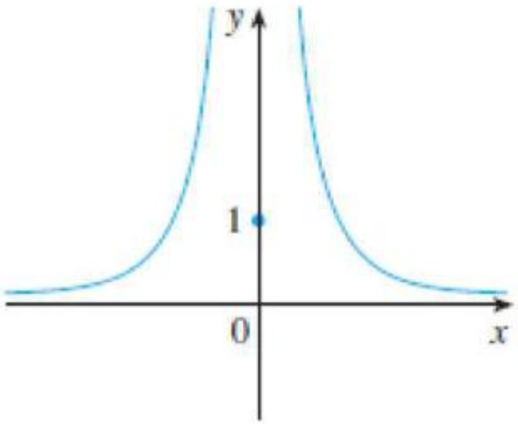
$$\Rightarrow \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2}$$

$$= \lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} = \lim_{x\to 2}(x+1) = 3$$

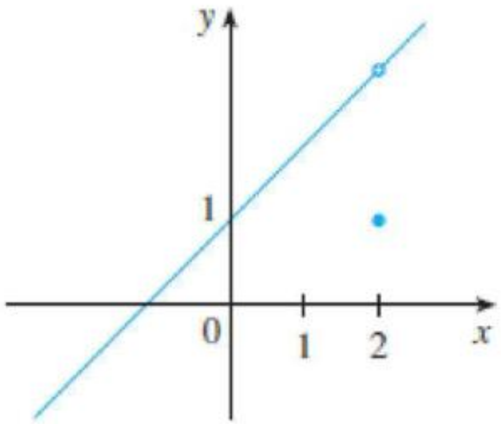
$f(x)$ is discontinuous at $x=2$ because $\lim_{x\to 2} f(x) \neq f(2)$.




(a) $f(x)=\dfrac{x^2-x-2}{x-2}$




(b) $f(x)=\begin{cases} \dfrac{1}{x^2} & \text{if } x\neq 0 \\ 1 & \text{if } x=0 \end{cases}$




(c) $f(x)=\begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x\neq 2 \\ 1 & \text{if } x=2 \end{cases}$

**Example:**

Given $h(x)=\begin{cases} x^2+2 & x<2 \\ 8 & x=2 \\ 3x & x>2 \end{cases}$

(a) Find $h(2)$, $\lim_{x\to 2^-} h(x)$, $\lim_{x\to 2^+} h(x)$.

(b) Does $\lim_{x\to 2} h(x)$ exist?

(c) Determine if the function $h(x)$ continuous at $x=2$? Explain your answer.

a) $h(2) = 8$ #

$\lim_{x\to 2^-} h(x) = (2)^2+2 = 6$ #

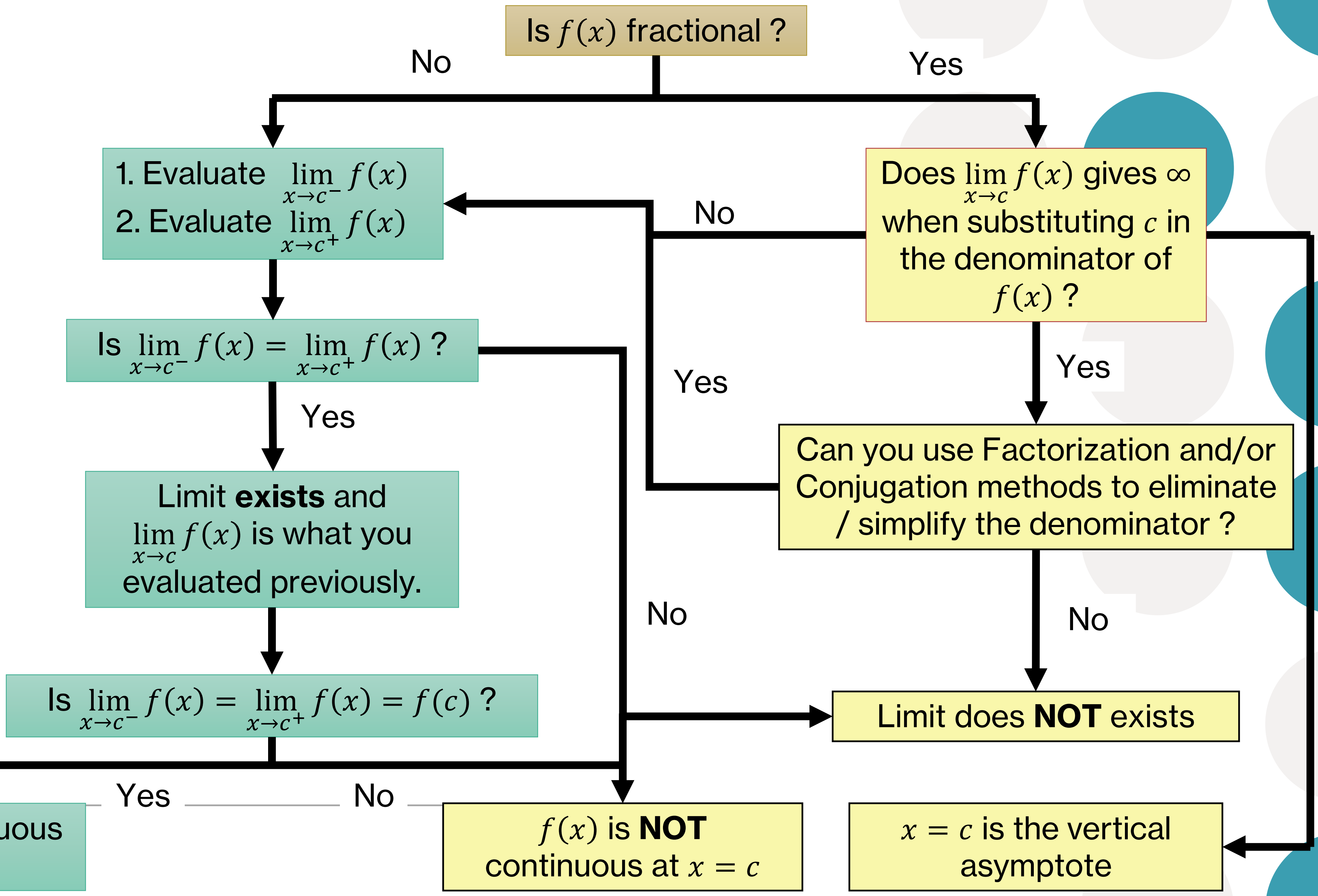
$\lim_{x\to 2^+} h(x) = 3(2) = 6$ #

b) Since $\lim_{x\to 2^-} h(x) = \lim_{x\to 2^+} h(x) = 6$

$\therefore$ limit exist and $\lim_{x\to 2} h(x) = 6$ #

c) Since $\lim_{x\to 2} h(x) = 6 \neq h(2)$, $\therefore h(x)$ is NOT continuous at $x = 2$ #

# Limits and Continuity of a Function $f(x)$ when $x \to c$ and when $x = c$

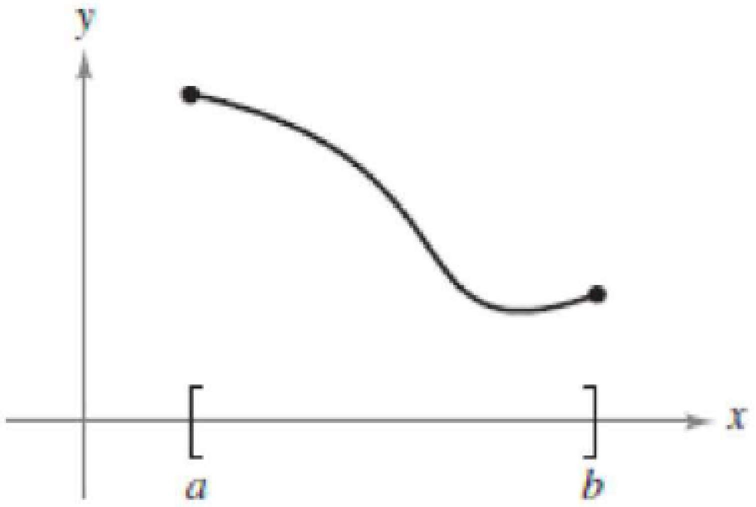


## Definition of Continuity on a Closed Interval

A function $f$ is **continuous on the closed interval** $[a, b]$ if it is continuous on the open interval $(a, b)$ and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

The function $f$ is **continuous from the right** at $a$ and **continuous from the left** at $b$




Continuous function on a closed interval

## Properties of Continuity

If $b$ is a real number and $f$ and $g$ are continuous at $x = c$, then the following functions are also continuous at $c$.

1. Scalar multiple: $bf$
2. Sum and difference: $f \pm g$
3. Product: $fg$
4. Quotient: $\dfrac{f}{g}$, if $g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

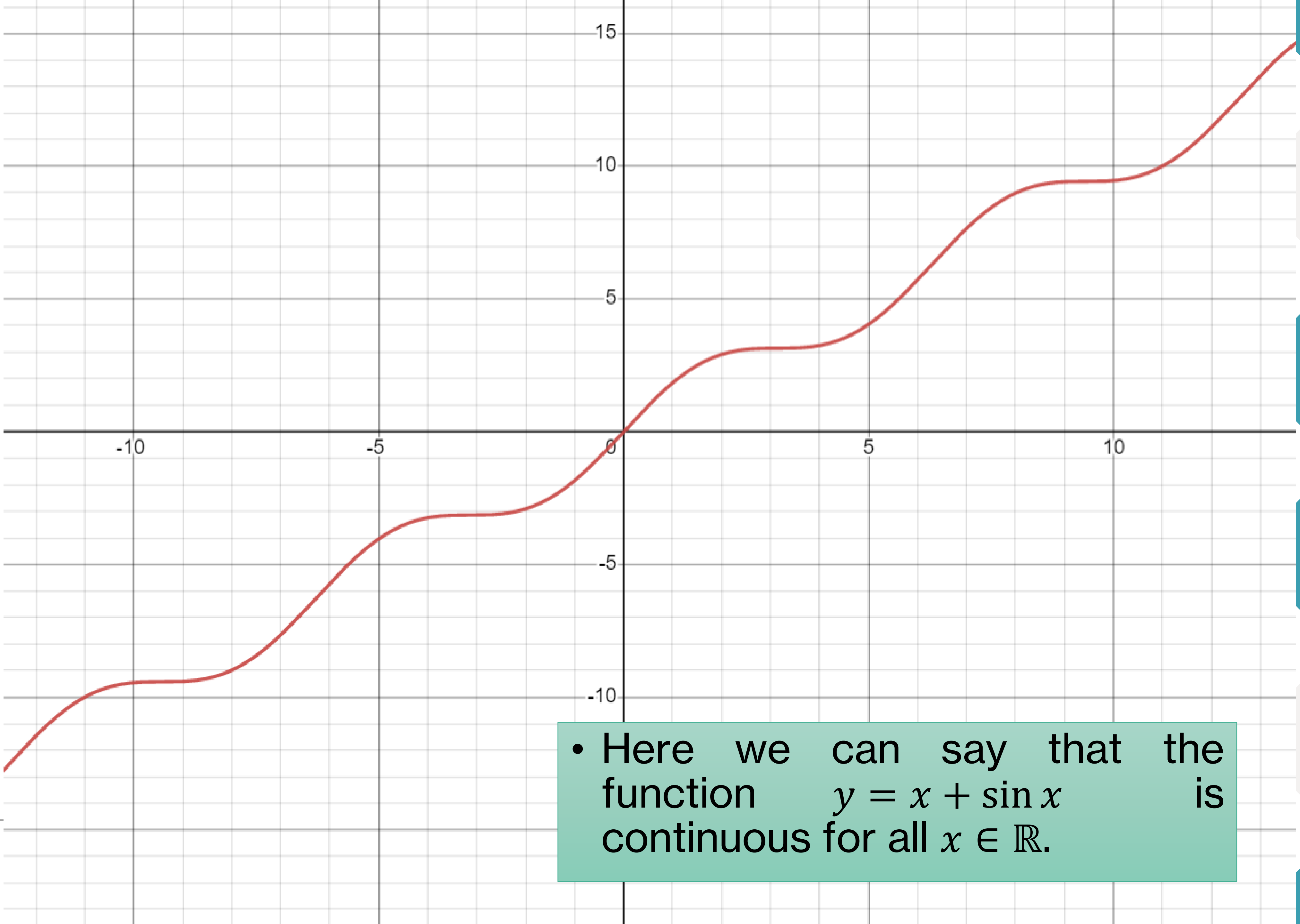
1. Polynomial functions: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
2. Rational functions: $r(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$
3. Radical functions: $f(x) = \sqrt[n]{x}$
4. Trigonometric functions: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

**Example:**

It follows that each of the following functions is continuous at every point in its domain.

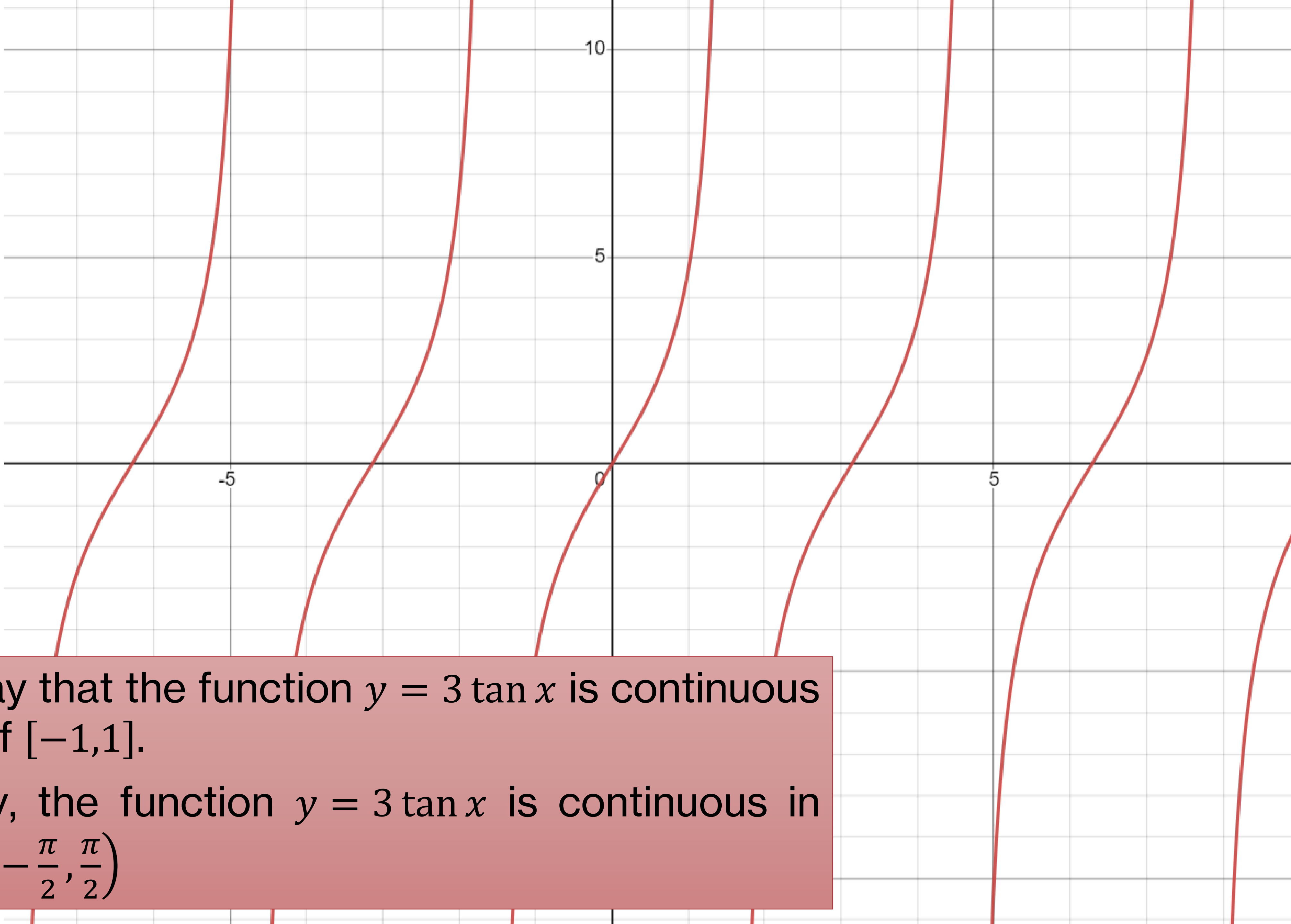
$$f(x) = x + \sin x, \qquad f(x) = 3 \tan x, \qquad f(x) = \frac{x^2 + 1}{\cos x}$$

$y = x + \sin\ x$

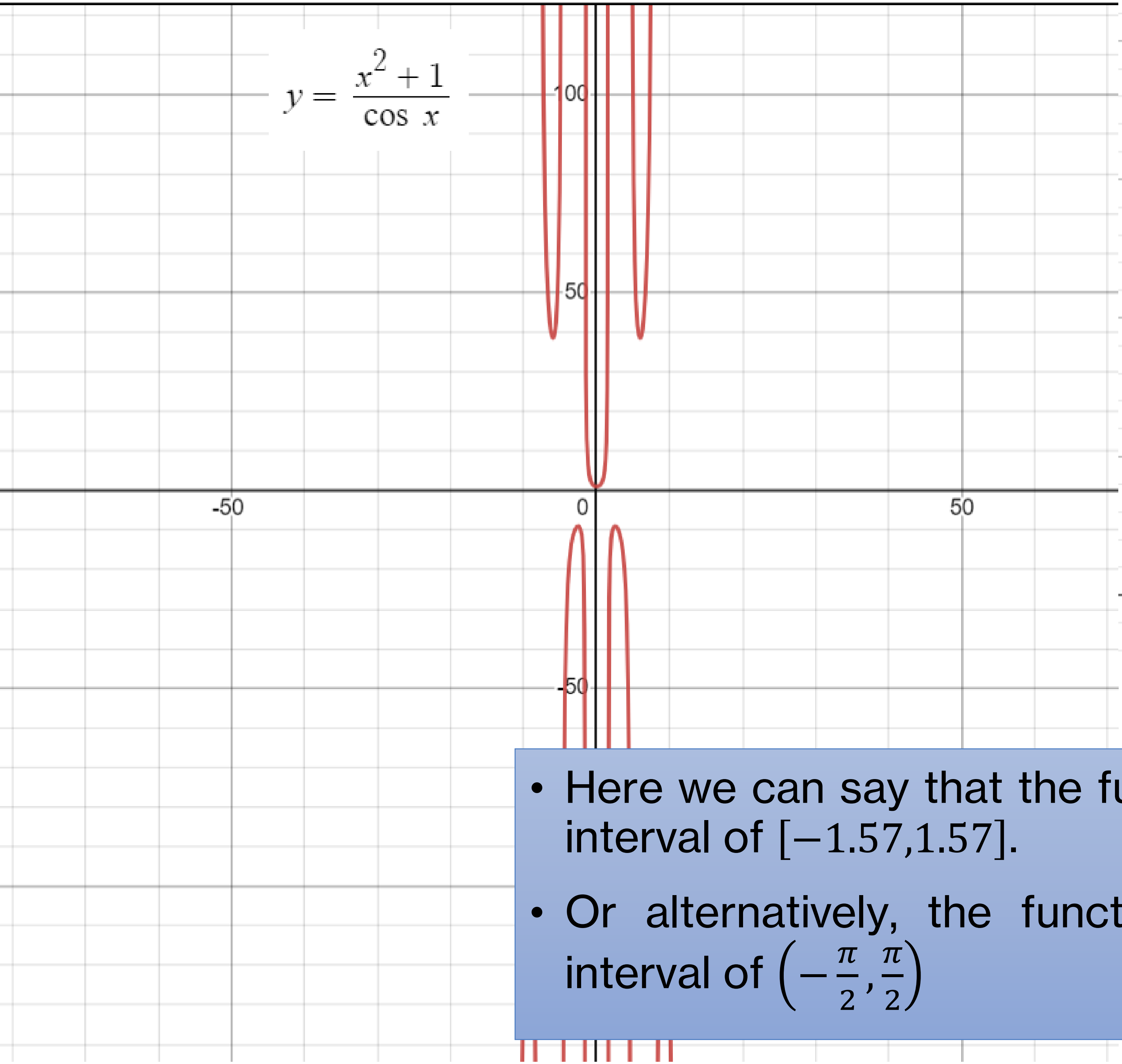


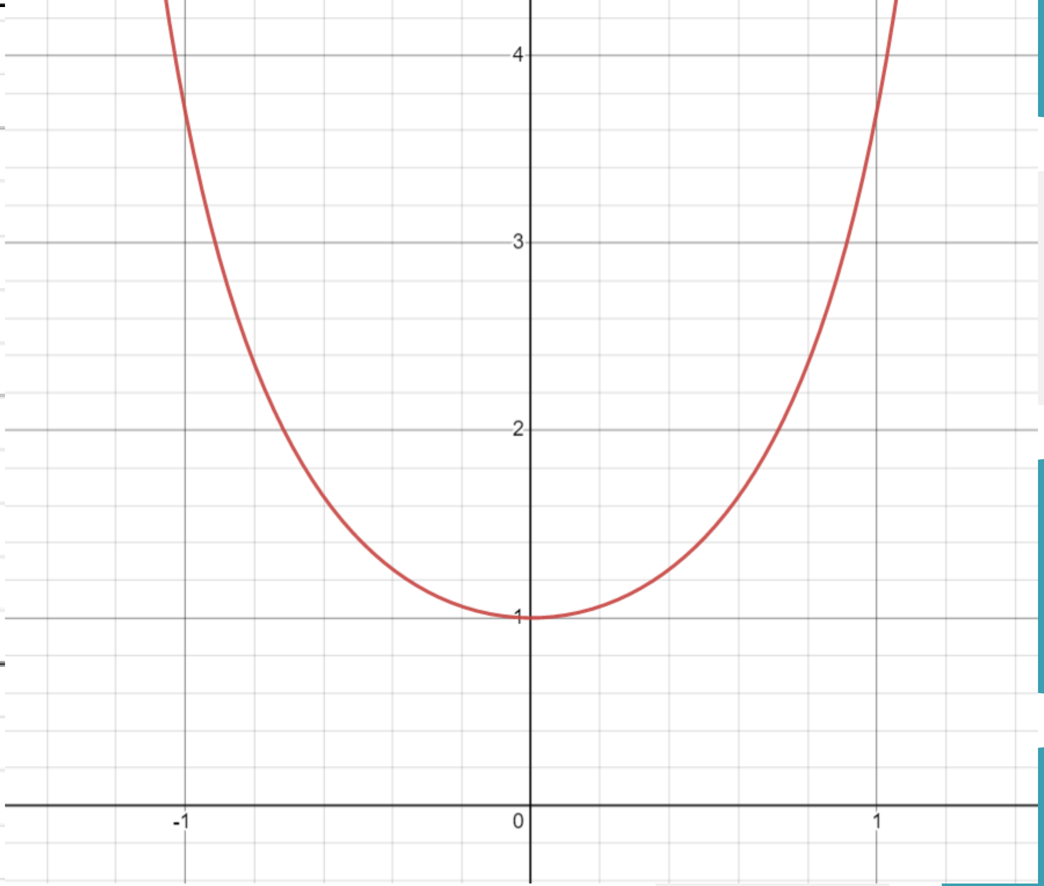

- Here we can say that the function $y = x + \sin x$ is continuous for all $x \in \mathbb{R}$.

$y = 3 \ \tan \ x$




- Here we can say that the function $y = 3\tan x$ is continuous in the interval of $[-1,1]$.
- Or alternatively, the function $y = 3\tan x$ is continuous in the interval of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$




- Here we can say that the function $y = \frac{x^2+1}{\cos x}$ is continuous in the interval of $[-1.57, 1.57]$.
- Or alternatively, the function $y = \frac{x^2+1}{\cos x}$ is continuous in the interval of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

☺ ~ THE END ~ ☺