Topic 9b: Second Order Differential Equations

9.5 SOLVING SECOND ORDER DIFFERENTIAL EQUATIONS

A second-order differential equation is called ***linear*** if it can be written as

$$y'' + p(x)y' + q(x)y = r(x) \qquad (1)$$

where ***p, q, r*** are any given function of ***x***. Any *second* order differential equation that cannot be written in the above form is called ***nonlinear***.

If **$r(x) = 0$**, equation (1) becomes

$$y'' + p(x)y' + q(x)y = 0 \qquad (2)$$

and is called ***homogeneous***.

If **$r(x)$** is not identically zero, the equation is called ***non-homogeneous***.

Example 1

$y'' + 4y = e^{-x} \sin x$ --- *non-homogeneous linear d e*
$(1 - x^2)y'' - 2xy' + 6y = 0$ --- *homogeneous linear d.e*
$x(y''y + y'^2) + 2y'y = 0$ --- *homogeneous nonlinear d.e*

Theorem (*Fundamental theorem for the homogeneous equation*)
For a homogeneous linear differential equation (2), any linear combination of two solutions on an open interval I is again a solution of (2) on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Example 2

1. Verify that $y = e^x$ and $y = e^{-x}$ are solutions of the homogeneous linear differential equation

$$y'' - y = 0$$

2. Are $y = ce^x$, $y = de^{-x}$ and $y = ce^x + de^{-x}$ also solutions?

Solution:

1.

When $y = e^x$, $y' = e^x$ and $y'' = e^x$ Hence $y'' - y = e^x - e^x = 0$ Therefore, $y = e^x$ is a solution for the d.e.	When $y = e^{-x}$, $y' = -e^{-x}$ and $y'' = e^{-x}$ Hence $y'' - y = e^{-x} - e^{-x} = 0$ Therefore, $y = e^{-x}$ is also a solution for the d.e.

2.

When $y = ce^x$, $y' = ce^x$ and $y'' = ce^x$ Hence $y'' - y = ce^x - ce^x = 0$ Therefore, $y = ce^x$ is a solution for the d.e.	When $y = de^{-x}$, $y' = -de^{-x}$ and $y'' = de^{-x}$ Hence $y'' - y = de^{-x} - de^{-x} = 0$ Therefore, $y = de^{-x}$ is also a solution for the d.e.

Similarly,

$$y = ce^x + de^{-x}$$
$$y' = ce^x - de^{-x}$$
$$y'' = ce^x + de^{-x}$$
$$\therefore y'' - y = (ce^x + de^{-x}) - (ce^x + de^{-x}) = 0$$

Therefore, $y = ce^x + de^{-x}$ is another solution for the d.e.

Note: *This theorem does not hold for the non-homogeneous equation or for a nonlinear equation.*

General Solution

For second-order homogeneous linear equations (2), a ***general solution*** will be of the form

$$y = c_1 y_1 + c_2 y_2 \qquad (3)$$

a linear combination of two (suitable) solutions involving two arbitrary constants c_1, c_2. These two solutions (y_1 and y_2) form a ***basis*** (or ***fundamental set***) of solutions to the d.e. (2) on I.

Particular Solution

A ***particular solution*** of (2) on I is obtained if we assign specific values to c_1 and c_2 in (3).

Initial Value Problem

For second-order homogeneous linear equations, an ***initial value problem*** would consist of a homogeneous linear differential equation $y'' + p(x)y' + q(x)y = 0$

and two initial conditions $\quad y(x_0) = K_0, \quad y'(x_0) = K_1,$

Linear independence and dependence

Two functions $y_1(x)$, $y_2(x)$ are said to be linearly dependent on an interval I if there exist constants c_1, c_2 not all zero, such that

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

for every x in the interval.

It is said to be ***linearly independent*** on an interval I if it is not linearly dependent on the interval.

Example 3

The function $f_1(x) = \sin 2x$ and $f_2(x) = \sin x \cos x$ are linearly dependent on the interval $(-\infty, \infty)$ since

$$c_1 \sin 2x + c_2 \sin x \cos x = 0$$

is satisfied for every real x if we choose $c_1 = \frac{1}{2}$ and $c_2 = -1$.

Definition of a basis

A basis of solutions of (2) on an interval I is a pair y_1, y_2 of ***linearly independent*** solutions of (2) on I.

9.5.1 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

In this section, we show how to solve homogeneous second order linear equations

$$ay'' + by' + cy = 0 \qquad (4)$$

where the coefficients $a (\neq 0)$, b and c are constants.

We try a solution of the form $y = e^{\lambda x}$. Then $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$.
Equation (4) becomes

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$
$$(a\lambda^2 + b\lambda + c)e^{\lambda x} = 0.$$

Because $e^{\lambda x}$ is never zero for real values of x,

$$a\lambda^2 + b\lambda + c = 0.$$

This latter equation is called the ***auxiliary equation***, or ***characteristic equation***.

The roots of the auxiliary equation are

$$\lambda_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad \lambda_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

With that, we obtain

Case I: two real roots if $b^2 - 4ac > 0$
Case II: a real double root if $b^2 - 4ac = 0$
Case III: complex conjugate roots if $b^2 - 4ac < 0$

Consider these three cases, namely, the solutions of the auxiliary equation corresponding to distinct real roots, real but equal roots, and a conjugate pair of complex roots.

CASE 1: DISTINCT REAL ROOTS ($\lambda_1 \neq \lambda_2$)

The general solution of (4) on ***R*** is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

where c_1 and c_2 are arbitrary constants.

Example 4

Find the general solution of $y'' + 5y' + 6y = 0.$

Solution:

The characteristic equation is

$$\lambda^2 + 5\lambda + 6 = 0$$
$$(\lambda + 2)(\lambda + 3) = 0$$

$\lambda = -2$ or $\lambda = -3.$ The roots are -2 and -3.

Thus, the general solution is $y = c_1 e^{-2x} + c_2 e^{-3x}$.

CASE II: REPEATED REAL ROOTS ($\lambda_1 = \lambda_2$)

The general solution of (4) on ***R*** is

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

where c_1 and c_2 are arbitrary constants.

Example 5

Solve the differential equation $y'' + 4y' + 4y = 0.$

Solution:

The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0$$

$(\lambda + 2)^2 = 0$ So $\lambda = -2$ (repeated)

Thus, the general solution is $y = c_1 e^{-2x} + c_2 x e^{-2x}$.

***CASE III*: CONJUGATE COMPLEX ROOTS (λ_1, λ_2 are complex)**

If λ_1 and λ_2 are complex, then we can write

$$\lambda_1 = \alpha + i\beta \text{ and } \lambda_2 = \alpha - i\beta$$

where α and $\beta > 0$ are real.

Therefore, the general solution of (4) on $\boldsymbol{R}$ is

$$y = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x}$$

which can be expressed in the following form by using Euler's formula

$e^{i\theta} = \cos\theta + i\sin\theta$

$$\begin{aligned} y &= c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x). \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

Example 6

Find the general solution of $y'' + 9y = 0$.

Solution:

The characteristic equation is

$$\lambda^2 + 9 = 0$$

$$\lambda = \pm 3i$$

The general solution is $y = c_1\cos 3x + c_2\sin 3x$.

Summary of Case I, II, and III

$ay'' + by' + cy = 0$ …….. (4)

Case	Roots of ***characteristic equation*** $a\lambda^2 + b\lambda + c = 0$	Basis of solutions of (4)	General Solution of (4)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
II	Repeated real root $\lambda = \lambda_1 = \lambda_2$	$e^{\lambda x}, xe^{\lambda x}$	$y = (c_1 + c_2x)e^{\lambda x}$
III	Complex conjugates $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$	$e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$	$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$ or $y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x$

9.5.2 NON-HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

In this section, we show how to solve non-homogeneous linear differential equations

$$a\,y'' + by' + cy = r(x) \qquad (5)$$

where $a, b,$ *and* c are constants and $r(x) \neq 0$.

The corresponding homogeneous equation of (5) is

$$ay'' + by' + cy = 0 \qquad (6)$$

It can be shown that the ***general solution*** of the non-homogeneous equation (5) is given by

$$y = y_h(x) + y_p(x) \qquad (7)$$

where $y_h = c_1y_1(x) + c_2y_2(x)$ (also known as ***complementary function***) is the general solution of the homogeneous equation (6) and y_p is a ***particular solution*** of (5).

Method of solving nonhomogeneous DE with constant coefficients	
Step 1:	Solve for homogeneous equation (6).
Step 2:	Find any particular solution y_p of (5).
Step 3:	Form general solution $y = y_h + y_p$

Example 7
Find a particular solution of $y'' + 9y = 27$.

Solution: Since $r(x) = 27$ we assume that a particular solution is given by $y_p = A$ where A is a constant. Substituting $y_p = A$ into the above DE and noting that $y_p'' = 0$, we have

$$y_p'' + 9\,y_p = 0 + 9\,A = 27.$$

Therefore $A = 3$ *and a particular solution is given by* $y_p = 3$.

9.4.2.1 Method of Undetermined coefficients

The method of undetermined coefficient is a technique for determining a particular solution y_p.

Rules for the Method of Undetermined Coefficients

(a) Basic Rule.
If $r(x)$ is one of the functions in the first column in the table below, choose the corresponding function y_p in the second column and determine its undetermined coefficients by substituting y_p and its derivatives into (5).

Term in *r(x)*	Choice for y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \, (n = 0,1,\cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$	$K\cos\omega x + M\sin\omega x$
$k\sin\omega x$	
$ke^{\alpha x}\cos\omega x$	$e^{\alpha x}\left(K\cos\omega x + M\sin\omega x\right)$
$ke^{\alpha x}\sin\omega x$	
$x^n\cos\omega x$	$(K_n x^n + K_{n-1}x^{n-1} + \cdots + K_0)\cos\omega x + (L_n x^n + L_{n-1}x^{n-1} + \cdots + L_0)\sin\omega x$
$x^n\sin\omega x$	

Example 8

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6.$

Solution:

Step 1. ***We first solve the associated homogeneous equation***

$$y'' + 4y' - 2y = 0.$$

The characteristic equation is

$$\lambda^2 + 4\lambda - 2 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16+8}}{2} = -2 \pm \sqrt{6}$$

$$\therefore y_h = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}$$

Step 2. ***Solve for particular solution.***

Since $r(x) = 2x^2 - 3x + 6$ is a quadratic polynomial, we assume

$$y_p = Ax^2 + Bx + C.$$

Then $y_p' = 2Ax + B$ and $y_p'' = 2A.$

Substituting into the equation, we have

$$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

Equating coefficients: $-2A = 2,\ 8A - 2B = -3,\ 2A + 4B - 2C = 6$

Solving: $A = -1,\ B = -\frac{5}{2},\ C = -9$

$$\therefore y_P = -x^2 - \frac{5}{2}x - 9$$

Step 3. ***The general solution of the given equation is***

$$y(x) = y_h + y_p = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x} - x^2 - \tfrac{5}{2}x - 9$$

(b) Sum Rule.

If $r(x)$ consists of sum of m terms of the kind given in above table, the assumption for a particular solution of y_p consists of the sum of the trial forms $y_{p_1}, y_{p_2}, \cdots, y_{p_m}$ corresponding to these terms

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m}.$$

Example 9

Find the general solution of the equation

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}-6y=e^{-2x}+2-x.$$

Solution:

Step 1. ***We first solve the associated homogeneous equation***

The characteristic equation is

$$\lambda^2 + 5\lambda - 6 = 0$$
$$(\lambda - 1)(\lambda + 6) = 0$$
$$\lambda = 1 \text{ or } \lambda = -6$$
$$\therefore y_h = c_1e^x + c_2 e^{-6x}$$

Step 2. ***Solve for particular solution.***

Since $r(x) = e^{-2x} + 2 - x$ is the sum of two types of functions from the table in (a) (viz. exponential + polynomial) , we assume

$$y_{p_1} = Ae^{-2x},\ y_{p_2} = Bx + C$$

Let $y_p = Ae^{-2x} + Bx + C$

$$\therefore \quad y_p' = -2Ae^{-2x} + B$$
$$y_p'' = 4Ae^{-2x}$$

Substituting into the equation, we have

[*You are required to fill in the intermediate steps.*]

$$-12A = 1, \qquad -6B = -1, \qquad 5B - 6C = 2$$
$$A=-\frac{1}{12}, B=\frac{1}{6}, C=-\frac{7}{36}$$
$$\therefore y_P = \ldots$$

Step 3. ***The general solution of the given equation is***

$$y = y_h + y_p = c_1e^x+c_2e^{-6x}-\frac{e^{-2x}}{12}+\frac{x}{6}-\frac{7}{36}$$

(c) Modification Rule.

If a term in your choice for y_{p_i} contains terms that duplicate terms in y_h, then that y_{p_i} must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication.

Example 10

Find the general solution of the equation

$$\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=e^t$$

Solution:

Step 1. ***We first solve the associated homogeneous equation***

The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0$$

$(\lambda - 1)^2 = 1$ [*You are required to fill in the intermediate steps.*]

$\therefore y_h = c_1 e^t + c_2 te^t$

Step 2. ***Solve for particular solution.***

Since $r(t) = e^t$ is a term in y_c, we assume

$$y_p = At^2 e^t$$

$\therefore$ $y_p' = 2Ate^t + At^2 e^t$

$y_p'' = 2Ae^t + 4Ate^t + At^2 e^t$

[You are required to fill in the intermediate steps.]

Substituting into the equation, we have $A = \frac{1}{2}$

Step 3. ***The general solution of the given equation is***

$$y = y_h + y_p = c_1 e^t + c_2 te^t + \frac{1}{2} t^2 e^t .$$

Example 11

Given that the function $y_1(x) = e^{-5x}$ and $y_2(x) = e^{2x}$ are both the solutions of the homogeneous equation, find the general solution of the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = x(e^x + 1)$$

Solution:

Step 1. ***We first determine the solution of the associated homogeneous equation***

Since $y_1(x) = e^{-5x}$ and $y_2(x) = e^{2x}$ are both the solutions of the homogeneous equation

$\therefore y_h = c_1 e^{-5x} + c_2 e^{2x}$

Step 2. ***Solve for particular solution.***

Since $r(x) = x(e^x + 1)$ is a combination of two functions, we assume

$y_p = (Ax + B)e^x + Cx + D$ [*Do you understand how the rules are applied?*]

$\therefore$ $y_p' = (Ax + B)e^x + Ae^x + C$

$y_p'' = (Ax + B)e^x + 2Ae^x$

[You are required to fill in the intermediate steps.]

Substituting into the equation, we have

$$A = -\frac{1}{6} \qquad B = -\frac{5}{36} \qquad C = -\frac{1}{10} \qquad D = -\frac{3}{100}$$

Step 3. ***The general solution of the given equation is***

$$y = y_h + y_p = c_1 e^{-5x} + c_2 e^{2x} + \left(-\frac{1}{6}x - \frac{5}{36}\right)e^x - \frac{1}{10}x - \frac{3}{100}$$

---------------------------THE END----------------------------

(nby, July 2016)