Topic 7 Sequences and Series

7.1 SEQUENCES

7.1.1. SEQUENCES OF REAL NUMBERS

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \ldots, \quad a_n, \quad \ldots$$

The number a_1 is known as the first term, a_2 as the second term, and in general a_n is the n^{th} term.

A sequence can be defined as a function whose domain is the set of **positive integers.**
It may also be denoted by defining the n-th term a_n, or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.
Elimination of a certain number of terms from a sequence forms a new sequence.
The new sequence is called a **subsequence** of the old one.

Examples of sequences denoted by different descriptions:

1. $\{n^2\}_{n=1}^{\infty}$ can also be denoted as $a_n = n^2$ or $\{1, 4, 9, 16, 25, \ldots\}$
2. $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ can also be denoted as $a_n = \dfrac{n}{n+1}$ or $\left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$
3. $\{\sqrt{n-3}\}_{n=3}^{\infty}$ can be denoted as $a_n = \sqrt{n-3}, n \ge 3$ or $\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\}$

<u>**New Sequences from the Old Ones**</u>

Suppose we are given two sequences $\{a_n\}, \{b_n\}$ and a number k. We shall denote by $\{a_n + b_n\}$ the sequence whose n^{th} term is $a_n + b_n$. Similarly, $\{a_n - b_n\}, \{a_n b_n\}$ and $\{ka_n\}$ are respectively the sequences whose n^{th} terms are $a_n - b_n$, $a_n b_n$ and ka_n.

If $b_n \neq 0$ for all n, then $\left\{\dfrac{a_n}{b_n}\right\}$ is the sequence whose n^{th} term is $\dfrac{a_n}{b_n}$.

7.1.2. PROPERTIES OF SEQUENCES

Bounded and Unbounded Sequences

A sequence $\{a_n\}$ is said to be **bounded** if there is a number M such that $|a_n| \le M$ for all $n \ge 1$. Otherwise, it is said to be **unbounded.**

Example of bounded sequences: $\left\{\dfrac{1}{n}\right\}$, $\{(-1)^n\}, \{e^{-n}\}$

Example of unbounded sequences: $\{n^2\}, \{(-1)^n n\}, \{\sqrt{n}\}$

If $\{a_n\}$ and $\{b_n\}$ are bounded, then so are the following sequences:

1. $\{k \cdot a_n\}$ for any constant k; 2. $\{-a_n\}$; 3. $\{a_n \pm b_n\}$; 4. $\{a_n b_n\}$; and
5. any subsequence $\{a_{n_k}\}$ of $\{a_n\}$

Monotonic

A sequence $\{a_n\}$ is called **nondecreasing** if $a_n \le a_{n+1}$ for all $n \ge 1$, that is, $a_1 \le a_2 \le a_3 \le \dots$. It is called **nonincreasing** if $a_n \ge a_{n+1}$ for all $n \ge 1$. It is called **monotonic** (or **monotone**) if it is either nondecreasing or nonincreasing.

Examples of monotonic sequences:

1. $\left\{\dfrac{3}{n+5}\right\}$ (nonincreasing) 2. $\left\{\dfrac{1}{n}\right\}$ (nonincreasing)
3. $\{-n\}$ (nonincreasing) 4. $\{n^2\}$ (nondecreasing)
5. $\{\sqrt{n}\}$ (nondecreasing) 6. $\{c\}$ (constant)

]

<u>Examples 1</u>: Determine whether the following sequence is bounded and monotonic.

(a) $\{n(n-1)\}$ (b) $\left\{\dfrac{2^n}{3^{n+1}}\right\}$

Solution (a)

$a_n = n(n-1) \ge (n-1)^2$ *for* $n \ge 1$ **.** The sequence is therefore <u>unbounded</u>.

$a_{n+1} = (n+1)n \ge n(n-1) = a_n$ *for* $n \ge 1$ [Alternatively, one can try to show that $a_n - a_{n+1} \le 0$]

Therefore the sequence is <u>nondecreasing, hence monotonic</u>.

Solution (b)

$$\left\{\frac{2^n}{3^{n+1}}\right\}$$

Note that $\dfrac{2^n}{3^{n+1}} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n \le \dfrac{1}{3}(1)^n = \dfrac{1}{3}$

Hence $\left|\dfrac{2^n}{3^{n+1}}\right| = \dfrac{2^n}{3^{n+1}} \le \dfrac{1}{3}$ and therefore $\left\{\dfrac{2^n}{3^{n+1}}\right\}$ is <u>bounded.</u>

Now $a_{n+1} = \dfrac{2^{n+1}}{3^{n+2}} = \left(\dfrac{2}{3}\right)\dfrac{2^n}{3^{n+1}} \le \dfrac{2^n}{3^{n+1}} = a_n$ *for* $n \ge 1$

Hence $\left\{\dfrac{2^n}{3^{n+1}}\right\}$ is <u>nonincreasing, hence monotonic</u>.

[Alternatively, one can try to show that $a_n - a_{n+1} \ge 0$]

7.1.3. CONVERGENCE AND DIVERGENCE OF SEQUENCES

A sequence $\{a_n\}$ **converges** (or is convergent) to the number L if to every positive number $\varepsilon > 0$, there corresponds an integer N such that for all n,

$$|a_n - L| < \varepsilon, \text{ whenever } n > N$$

If $\{a_n\}$ converges to L, we write $\lim\limits_{n\to\infty} a_n = L$ or simply $a_n \to L$, and call L the **limit** of the sequence.

The above definition says that a sequence $\{a_n\}$ has the limit L if we can make the terms a_n as close to L as we like by taking n sufficiently large. If a sequence is not convergent, then we say it **diverges** (or is divergent).

Theorems

1. A convergent sequence has only **one** limit.
2. A subsequence of a convergent sequence is also convergent and has the **same** limit. In other words, if a sequence has a divergent subsequence, then the sequence is divergent.
3. **A monotonic and bounded sequence is convergent.**
4. A convergent sequence is bounded, but is **not** necessarily monotonic.

Refer back to Example 1

(a) Since $\{n(n-1)\}$ is unbounded, it is divergent.

(b) Since $\left\{\dfrac{2^n}{3^{n+1}}\right\}$ is bounded and monotonic, it is convergent.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences such that $\lim\limits_{n\to\infty} a_n = A$, $\lim\limits_{n\to\infty} b_n = B$ and c is a constant, then:

1. $\lim\limits_{n\to\infty}(a_n + b_n) = A + B$
2. $\lim\limits_{n\to\infty}(a_n - b_n) = A - B$
3. $\lim\limits_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. $\lim\limits_{n\to\infty}(c \cdot a_n) = c \cdot A$
5. $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{A}{B}$, if $b_n \neq 0$ for all n and $B \neq 0$
6. $\lim\limits_{n\to\infty} c = c$

Some limits that arise frequently [Here, x is a FIXED number]

1. $\lim\limits_{n\to\infty}\dfrac{\ln n}{n} = 0$ [You can apply L'hopital's rule to get this result.]
2. $\lim\limits_{n\to\infty}\sqrt[n]{n} = 1$
3. $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1 \quad (x > 0)$
4. $\lim\limits_{n\to\infty} x^n = 0 \quad |x| < 1$
5. $\lim\limits_{n\to\infty}\left(1 + \dfrac{x}{n}\right)^n = e^x$ for any x
6. $\lim\limits_{n\to\infty}\dfrac{x^n}{n!} = 0$ for any x

Example 2:

Determine whether the following sequences converge.

(a) $\left\{\dfrac{n+(-1)^n}{n}\right\}$ (b) $\left\{(-1)^n\left(1-\dfrac{1}{n}\right)\right\}$

Solution (a)

$$\lim_{n\to\infty}\left(\frac{n+(-1)^n}{n}\right)=\lim_{n\to\infty}\left(1+\frac{(-1)^n}{n}\right)=1.$$

The limit exists.

The sequence $\left\{\dfrac{n+(-1)^n}{n}\right\}$ is convergent.

Solution (b)

$\displaystyle\lim_{n\to\infty}\left((-1)^n\left(1-\frac{1}{n}\right)\right)$ doesn't exist.

The sequence $\left\{(-1)^n\left(1-\dfrac{1}{n}\right)\right\}$ is divergent.

$$(-1)^n\left(1-\frac{1}{n}\right)=\begin{cases}1-\dfrac{1}{n} & \text{if } n \text{ is even}\\[2mm] -1+\dfrac{1}{n} & \text{if } n \text{ is odd}\end{cases}$$

cannot approach a particular number.

The Squeeze Theorem/Sandwich Theorem for Sequences

Let $\{a_n\},\{b_n\},\{c_n\}$ be sequences. If there is an integer N such that $a_n \le b_n \le c_n$ for $n \ge N$ and $\displaystyle\lim_{n\to\infty} a_n=\lim_{n\to\infty} c_n = L$, then $\displaystyle\lim_{n\to\infty} b_n = L$.

Example 3: Find the limit.

(a) $\left\{\dfrac{(-1)^{n-1}n}{n^2+1}\right\}$ (b) $\left\{\dfrac{n\cos n}{n^2+1}\right\}$

Solution (a)

Let $b_n=\dfrac{(-1)^{n-1}n}{n^2+1}$

Choose $a_n=\dfrac{-n}{n^2+1}$ and $c_n=\dfrac{n}{n^2+1}$.

We have $\dfrac{-n}{n^2+1}\le\dfrac{(-1)^{n-1}n}{n^2+1}\le\dfrac{n}{n^2+1}$

$$\lim_{n\to\infty}\left(\frac{-n}{n^2+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{-1}{n}}{1+\frac{1}{n^2}}\right)=\frac{\lim\limits_{n\to\infty}\frac{-1}{n}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n^2}\right)}=\frac{0}{1}=0$$

Also $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n^2+1}\right)=0$.

Hence, by the Sandwich Theorem,

$$\lim_{n\to\infty}\left(\frac{n\cos n}{n^2+1}\right)=0.$$

Solution (b)

Let $b_n=\dfrac{n\cos n}{n^2+1}$

Choose $a_n=\dfrac{-n}{n^2+1}$ and $c_n=\dfrac{n}{n^2+1}$.

We have $\dfrac{-n}{n^2+1}\le\dfrac{n\cos n}{n^2+1}\le\dfrac{n}{n^2+1}$

$$\lim_{n\to\infty}\left(\frac{-n}{n^2+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{-1}{n}}{1+\frac{1}{n^2}}\right)=\frac{\lim\limits_{n\to\infty}\frac{-1}{n}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n^2}\right)}=\frac{0}{1}=0$$

Also $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n^2+1}\right)=0$.

Hence, by the Sandwich Theorem,

$$\lim_{n\to\infty}\left(\frac{n\cos n}{n^2+1}\right)=0.$$

7.2 SERIES

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$, we get an expression of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

which is called an **infinite series** (or just a **series**) and is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$

7.2.1. PARTIAL SUMS and CONVERGENCE OF SERIES

Suppose that we are given a sequence $\{a_n\}$. We can construct a new sequence $\{S_n\}$ from $\{a_n\}$ such that

$$S_1 = a_1 \qquad S_2 = a_1 + a_2 \qquad S_3 = a_1 + a_2 + a_3$$

and in general,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

We call S_n the ***n*^*th*** **partial sum** of the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_k + \ldots$

If the sequence $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = S$ exists as a real number,

then the series $\sum a_n$ is said to be **convergent** and we write

$$a_1 + a_2 + a_3 + \ldots + a_k + \ldots = \sum_{n=1}^{\infty} a_n = S \quad \text{(sum of the series)}$$

Otherwise we say that the series is **divergent** or the series diverges.

<u>An example using the partial sums to determine convergence</u>

Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Solution

Let $S_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{k(k+1)}$

By using partial fraction decomposition,

$$\begin{aligned} S_k &= \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= 1 - \frac{1}{k+1} \end{aligned}$$

We now see that $S_k \to 1$ as $k \to \infty$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and the sum is 1, i.e., $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

7.2.2. GEOMETRIC SERIES

An important example of infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

where each term is obtained from the preceding term by multiplying it by the common ratio r.

This geometric series is **convergent if** $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the geometric series is divergent.

Example 4: Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$$

Solution

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} + \frac{-3}{4^2} + \frac{(-3)^2}{4^3} + \frac{(-3)^3}{4^4} + \ldots + \frac{(-3)^{k-1}}{4^k} + \ldots$$

$$= \frac{1}{4}\left\{1 + \left(\frac{-3}{4}\right) + \left(\frac{-3}{4}\right)^2 + \left(\frac{-3}{4}\right)^3 + \left(\frac{-3}{4}\right)^4 + \ldots + \left(\frac{-3}{4}\right)^{k-1} + \ldots\right\} = \frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{-3}{4}\right)^{n-1}$$

Obviously $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{3}{4}$

Since $|r| = \left|-\frac{3}{4}\right| = \frac{3}{4} < 1$, $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^{n-1}}$ is convergent.

The sum of $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^{n-1}}$ is $\frac{a}{1-r} = \frac{1}{1-\left(-\frac{3}{4}\right)} = \frac{4}{7}$.

Thus the sum of the given series $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^{n-1}}$ is $\frac{1}{4}\left(\frac{4}{7}\right) = \frac{1}{7}$

Limit Laws for Series

If $\sum a_n$ and $\sum b_n$ are convergent series with the sum A and B respectively, then the following series are also convergent:

1. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n = cA$

2. $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$

3. $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = A - B$

7.2.3. DIVERGENCE TEST

Theorem: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

In other words, if $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

But the converse of the above theorem is not true,

that is, $\lim_{n\to\infty} a_n = 0$ **does not imply the convergence of** $\sum_{n=1}^{\infty} a_n$ **.**

<u>Divergence test (the *n*-th term test for divergence)</u>

For a given series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

<u>Example 5</u>: Show that the following series diverges.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ (b) $\sum_{n=1}^{\infty} n^2$ (c) $\sum_{n=1}^{\infty} (-1)^{n+1}$

Solution (a) $\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5} \neq 0$

Hence the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges by the Divergence Test.

Solution (b) $\lim_{n\to\infty} n^2 = \infty$ The limit does not exist.

Hence the series $\sum_{n=1}^{\infty} n^2$ diverges by the Divergence Test.

Solution (c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges since $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

7.2.4 THE INTEGRAL TEST AND *p*-SERIES TEST

(**The Integral Test**) Suppose that f is a **continuous, positive, and decreasing** function for $x \geq 1$. Suppose also that $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)dx$ **both converge or both diverge**. In other words:

i) If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

ii) If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

***p*-Series** By using the integral test, we can show that the following result is true.

(**The *p*-Series Test**) The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the ***p*-series.**

The *p*-series is **convergent if *p* > 1 and divergent if** $p \leq 1$

Example 6: Determine whether the following series converges?

(a) $\sum_{n=1}^{\infty} \frac{5}{n+1}$ (b) $\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$

Solution (a) $\sum_{n=1}^{\infty} \frac{5}{n+1}$

The function $f(x) = \frac{5}{x+1}$ is positive, continuous and decreasing for $x \geq 1$

Hence we can apply the Integral Test.

$$\int_{n=1}^{\infty} f(x)dx = \int_{n=1}^{\infty} \frac{5}{x+1}dx = \lim_{b\to\infty} \int_{n=1}^{b} \frac{5}{x+1}dx = 5\lim_{b\to\infty}[\ln(x+1)]_1^b = \infty$$

Thus $\int_{n=1}^{\infty} f(x)dx$ is divergent and $\sum_{n=1}^{\infty} \frac{5}{n+1}$ is also divergent by the Integral Test.

Solution (b) $\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \left(\frac{5}{n^3} - \frac{2\sqrt{n}}{n^3}\right) = 5\sum_{n=1}^{\infty} \frac{1}{n^3} - 2\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}}$ are *p*-series with p = 3 and $p = \frac{5}{2}$ respectively, they are both convergent. Thus $\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$ is convergent.

7.2.5 THE COMPARISON TESTS

The Direct Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms, then:

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all *n*, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all *n*, then $\sum a_n$ is also divergent.

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms, then

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where *c* is a **positive** number,

then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and $\sum b_n$ converges, then $\sum a_n$ converges

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, and $\sum b_n$ diverges, then $\sum a_n$ diverges

In this course, you are not required to know the limit comparison test. It is discussed here for completeness.

Example 7: Determine whether the following series converges or diverges?

(a) $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$ (b) $\sum_{n=1}^{\infty}\frac{1}{2^n-1}$

Solution (a)

Let $a_n=\left(\frac{n}{3n+1}\right)^n$, $b_n=\left(\frac{n}{3n}\right)^n=\left(\frac{1}{3}\right)^n$.

Since $\left(\frac{n}{3n+1}\right)^n<\left(\frac{1}{3}\right)^n$, we have $a_n\le b_n$

The series $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a convergent series since it is a geometric series with initial term $a=\frac{1}{3}$ and common ratio $r=\frac{1}{3}$, which is less than 1.

Therefore $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$ is convergent by the Direct Comparison Test.

The Limit Comparison Test can also be used.

$$\lim_{n\to\infty}\frac{\left(\frac{n}{3n+1}\right)^n}{\left(\frac{1}{3}\right)^n}=\lim_{n\to\infty}\left(\frac{3n}{3n+1}\right)^n=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{3n}}\right)^n=1>0$$

Since $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is convergent, then $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$ is also convergent.

Solution (b)

Let $a_n=\frac{1}{2^n-1}$ and $b_n=\frac{1}{2^n}$.

Since $\frac{1}{2^n-1}>\frac{1}{2^n}$, we have $a_n\ge b_n$

The series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a convergent series since it is a geometric series with initial term $a=\frac{1}{2}$ and common ratio $r=\frac{1}{2}$, which is less than 1.

The Direct Comparison Test doesn't apply. (Can you compare with $\frac{1}{2^{n-1}}$?)

Using the Limit Comparison Test: $\lim_{n\to\infty}\frac{\frac{1}{2^n-1}}{\frac{1}{2^n}}=\lim_{n\to\infty}\frac{2^n}{2^n-1}=\lim_{n\to\infty}\frac{1}{1-\frac{1}{2^n}}=1>0$

Since $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is convergent, then $\sum_{n=1}^{\infty}\frac{1}{2^n-1}$ is also convergent.

Remark: The terms of the series being tested must be **smaller** than those of a convergent series or **larger** than those of the divergent series.
If the terms are larger than the terms of a convergent series or smaller than the term of a divergent series, then the direct comparison test doesn't apply.

7.2.6 THE ALTERNATING SERIES TEST (Optional, i.e., not required)

An alternating series is a series whose terms are alternately positive and negative.

Example: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots+\frac{(-1)^{n+1}}{n}+\ldots$

$1-2+3-4+5-6+\ldots+(-1)^{n+1}n+\ldots$

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1-a_2+a_3-a_4+\ldots.(-1)^{n+1}a_n+\ldots \qquad a_n \geq 0$$

satisfies (i) $a_{n+1} \leq a_n$ for all n (which means a_n is nonincreasing), **AND** (ii) $\lim_{n\to\infty} a_n = 0$ then the series is convergent.

Example 8: Determine whether the following series converges or diverges?

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin\left(n-\frac{1}{2}\right)\pi}{n!}$

Solution (a) The series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}} = 1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots+\frac{(-1)^{n-1}}{\sqrt{n}}$ is alternating and

satisfies (i) $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ $[a_{n+1} \leq a_n]$ (ii) $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ $[\lim_{n\to\infty} a_n = 0]$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ is convergent by the Alternating Series Test.

Note: $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ doesn't imply that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is convergent.

In fact, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series with $p=\frac{1}{2}$

Solution (b) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin\left(n-\frac{1}{2}\right)\pi}{n!}$

We know $\displaystyle\sum_{n=1}^{\infty}\sin\left(n-\frac{1}{2}\right)\pi = 1-1+1-1+1\ldots = (-1)^{n+1}$

Hence $\displaystyle\sum_{n=1}^{\infty}\frac{\sin\left(n-\frac{1}{2}\right)\pi}{n!} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}$ is alternating and satisfies

(i) $\frac{1}{(n+1)!} < \frac{1}{n!}$ $[a_{n+1} \leq a_n]$ (ii) $\lim_{n\to\infty}\frac{1}{n!} = 0$ $[\lim_{n\to\infty} a_n = 0]$

Therefore the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sin\left(n-\frac{1}{2}\right)\pi}{n!} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}$ is convergent by the Alternating Series Test.

7.2.7 THE RATIO AND ROOT TESTS

Absolute Convergence Versus Conditional Convergence

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
If $\sum |a_n|$ is divergent but $\sum a_n$ is convergent, then $\sum a_n$ is said to be **conditionally convergent.**
If a series $\sum a_n$ is **absolutely convergent**, then $\sum a_n$ is **convergent**; the converse is **not** true.

Note: For a series $\sum a_n$ with non-negative terms (i.e. $a_n \geq 0$),
absolute convergence and **convergence mean the same thing.**

Example 9: Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent or divergent.

Solution This series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \ldots$ has positive and negative terms but it is not alternating (first term is positive, next three are negative, then the following three are positive…).

We can apply comparison test to the series of absolute values $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$

Since $|\cos n| \leq 1$, $\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (p-series with p = 2) and therefore $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ is convergent by the Comparison Test.

Thus the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolute convergent and is therefore convergent.

The Ratio Test

For a given series $\sum a_n$, let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

i) If $L < 1$, then $\sum a_n$ is absolutely convergent.
ii) If $L > 1$ or $L = \infty$, then $\sum a_n$ is divergent
iii) If $L = 1$, the test is inconclusive

Example 10: Test the convergence of series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

Solution We let $a_n = (-1)^n \frac{n^3}{3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(n+1)^3}{3^{(n+1)}}}{(-1)^n\frac{n^3}{3^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{3^{(n+1)}}\cdot\frac{3^n}{n^3} = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \lim_{n\to\infty}\frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3}(1+0)^3 = \frac{1}{3} < 1$$

Thus, by the Ratio Test, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is absolutely convergent and therefore convergent.

<u>The Root Test</u>

For a given series $\sum a_n$, let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

i) If $L < 1$, then $\sum a_n$ is absolutely convergent.

ii) If $L > 1$ or $L = \infty$, then $\sum a_n$ is divergent

iii) If $L = 1$, the test is inconclusive

<u>Example 11</u>: Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

Solution: We use $a_n = \left(\frac{2n+3}{3n+2}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \lim_{n\to\infty}\left(\frac{2n+3}{3n+2}\right) = \lim_{n\to\infty}\left(\frac{2+\frac{3}{n}}{3+\frac{2}{n}}\right) = \frac{2}{3} < 1$. Thus, by the Root Test, the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is absolutely convergent and therefore convergent.

<u>Examples 12</u>: Determine whether the series is absolutely convergent, conditional convergent or divergent

(a) $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution (a):

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|} = \lim_{n\to\infty}\left(\frac{n^2+1}{2n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1+1/n^2}{2+1/n^2}\right) = \frac{1}{2} < 1$$

Thus, the series $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

Solution (b): The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is alternating and

satisfies (i) $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ (ii) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Therefore the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent by the Alternating Test.

But $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (p-series with p<1). So $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditional convergent.

In this course, we shall apply the ratio test and the root test only to infinite series with positive terms so that the question of conditional convergence does not arise.

<u>Restate the root test and ratio test for infinite series with positive terms.</u>

Root Test:

Ratio Test:

7.2.8 POWER SERIES

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where x is a variable and c_n's are constants is called a **power series**. The constants c_n's are called the **coefficients** of the power series. We adopt the convention $x^0 = 1$ even if $x = 0$.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots + c_n(x-a)^n + \ldots$$

is called a **power series centered at *a*** or a **power series about *a***. Here we adopt the convention that $(x-a)^0 = 1$ when $x = a$.

The Convergence Theorem for Power Series

For a given **power series about *a*,** there are only three possibilities:

(i) The series converges only when $x = a$

(ii) The series converges for all x

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R is called the **radius of convergence** of the power series. We adopt the convention that in case (i), we let $R = 0$ and in case (ii), we let $R = \infty$.

The **interval of convergence *I*** of a power series is the interval that contains all values of x for which the power series converges. In case (i), the "interval I" *is actually a singleton set* $\{a\}$ and in case (ii), $I = (-\infty, \infty)$. In case (iii) I can be one of the following intervals:
$(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$, or $[a-R, a+R]$

7.3 TAYLOR SERIES AND MACLAURIN SERIES

If a function $f(x)$ has a power series representation (expansion) at a, that is, if it can be expressed in the following form:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for } |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!} \qquad \text{for all } n$$

Substituting this formula for c_n back into the series, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{for } |x-a| < R$$

The power series on the right hand side is called the **Taylor series of the function *f* at *a***.

In other words,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

In the case when a = 0, we have

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \qquad \text{for } |x|<R$$

which is called the **Maclaurin series of function *f*.**

The series becomes

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$$

Example 15: Find the Taylor series for $f(x)=e^x$ centered at a = 2.

Solution $f(x)=e^x,\ f^{(1)}(x)=e^x,\ f^{(2)}(x)=e^x,\ f^{(3)}(x)=e^x,\ldots,\ f^{(n)}(x)=e^x$

$f(2)=e^2,\ f^{(1)}(2)=e^2,\ f^{(2)}(2)=e^2,\ f^{(3)}(2)=e^2,\ldots,\ f^{(n)}(2)=e^2$

$f^{(n)}(2)=e^2$ for all n

Thus the Taylor series for $f(x)=e^x$ centered at a = 2 is

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n=\sum_{n=0}^{\infty}\frac{e^2}{n!}(x-2)^n$$

or $e^x=e^2+\frac{e^2}{1!}(x-2)+\frac{e^2}{2!}(x-2)^2+\frac{e^2}{3!}(x-2)^3+\ldots$

Example 16: Find the Maclaurin series for $f(x)=e^x$ and its radius of convergence.

Solution As in Example 15, $f^{(n)}(x)=e^x$ for all n. For the Maclaurin series, a = 0 and so

$$f(0)=f^{(1)}(0)=f^{(2)}(0)=\ldots=f^{(n)}(0)=e^0=1$$

Therefore the Maclaurin Series is $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

To find the radius of convergence, use the Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}.\frac{n!}{x^n}\right|=\lim_{n\to\infty}\frac{|x|}{(n+1)}=|x|\lim_{n\to\infty}\frac{1}{(n+1)}=0<1$$

So the series converges for all x by the Ratio Test and the radius of convergence is $R=\infty$. The interval of convergence is $I=(-\infty,\infty)$.

Note: Some important Maclaurin series and their intervals of convergence

$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\ldots \quad (-1,1)$	$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots \quad (-\infty,\infty)$
$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots \quad (-\infty,\infty)$	$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots \quad (-\infty,\infty)$

Taylor polynomial of order/degree *n* : $P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$

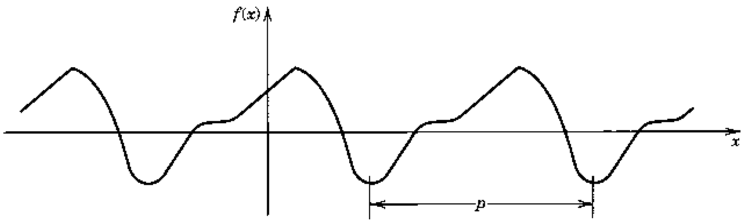
Maclaurin polynomial of order/degree *n* : $P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k$

7.4 Fourier Series

7.4.1 Periodic Functions

A function f is **periodic** if there is some positive number p such that

$$f(x+p) = f(x) \text{ for all real } x.$$



Examples of periodic functions

The trigonometric functions cos x and sin x are periodic functions of period 2π. We note that 4π, 6π, 8π,…, are also periods of the functions. In fact any multiple of 2π is a period of the functions. But the smallest period is 2π. The smallest period is known as the *fundamental period*.

The functions cos $2x$ and sin $2x$ are periodic functions of period π. In general cos kx and sin kx are periodic functions of period $\dfrac{2\pi}{k}$.

Frequency of a periodic function: $\text{frequency} = \dfrac{1}{\text{period}} = \dfrac{1}{p}$

7.4.2 Fourier Series

We assume that a periodic function can be represented by an infinite series in terms of sine and cosine functions. Such a representation is known as a Fourier series. It can be shown that:

A Fourier series of a function f of period $p = 2L$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega x + b_n \sin n\omega x), \qquad \text{where } \omega = \frac{2\pi}{2L} = \frac{\pi}{L},$$

where $a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx = \frac{1}{L}\int_{-L}^{L} f(x)\cos n\omega xdx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx = \frac{1}{L}\int_{-L}^{L} f(x)\sin n\omega xdx$$

What do these formulas look like when the period is 2π.

Periodic functions in problems for this course shall be confined to only those with period 2π.

The a_0, a_n, and b_n are known as **Fourier coefficients**. These coefficients are unknown and we have to find their values by integration.

Even though that the "=" sign is usually used to equate a periodic function and its Fourier series, we need to be a little careful. The function f and its Fourier series "representation" are only equal to each other if, and whenever, f is continuous.

Example:

Find the Fourier coefficients of the following periodic function

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 \le x < \pi \end{cases}$$

Solution

Recall the definition of Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi}{L} x + b_n \sin\frac{n\pi}{L} x)$$

For the given $f(x)$, $-\pi < x < \pi$, we have $L = \pi$. The Fourier coefficients are:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)dx + \int_{0}^{\pi}(k)dx\right] = 0$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nxdx + \int_{0}^{\pi} k\cos nxdx\right] = \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\bigg|_{-\pi}^{0} + k\frac{\sin nx}{n}\bigg|_{0}^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{0}(-k)\sin nxdx + \int_{0}^{\pi}(k)\sin nxdx$$

$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - k\frac{\cos nx}{n}\bigg|_{0}^{\pi}\right]$$

$$= \frac{k}{n\pi}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{nk}(1 - \cos n\pi)$$

We also note that $\cos\pi = -1,\ \cos 2\pi = 1,\ \cos 3\pi = -1,\ \cdots$

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$$

and thus

$$1 - \cos nx = \begin{cases} 2 & \text{for odd } n \\ 0 & \text{for even } n \end{cases}.$$

Hence the Fourier coefficient b_n is

$$b_n = \frac{2k}{n\pi}(1 - \cos n\pi) = \begin{cases} \dfrac{2k}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Hence

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

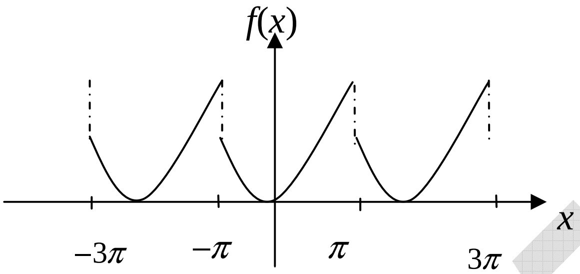
Example

Consider a 2π-periodic function

$$f(x) = x^2 + x, \quad -\pi < x < \pi, \quad \text{and} \quad f(x) = f(x+2\pi).$$

Sketch a graph of the function $f(x)$ for values of x from $x = -3\pi$ to $x = 3\pi$ and obtain a Fourier series expansion of the function.

Solution



The Fourier series of f with period 2π:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with Fourier coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x)dx = \frac{3}{2}\pi^2$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x)\cos nx\, dx$$

which on integration by parts, gives

$$a_n = \frac{1}{\pi}\left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx + \frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\frac{4\pi}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

and $\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x)\sin nx\, dx$

which on integration by parts, gives

$$b_n = \frac{1}{\pi}\left[-\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx - \frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi}$$

$$= -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n$$

Hence we get : $\quad f(x) = \frac{1}{3}\pi^3 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx - \sum_{n=1}^{\infty}\frac{2}{n}(-1)^n \sin nx$

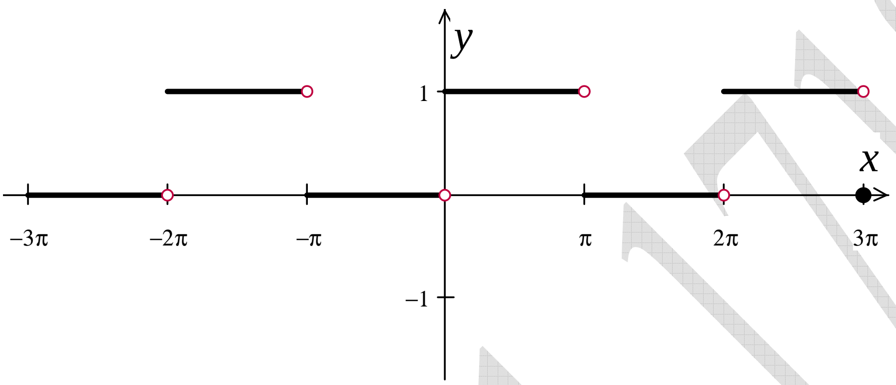
Example

A periodic function $f(x)$ of period 2π (that is, $f(x+2\pi) = f(x)$ is defined in the interval $-\pi < x < \pi$ by

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

Sketch a graph of $f(x)$ for $-3\pi \le x \le 3\pi$ and obtain a Fourier series expansion for the function.

Solution A graph of $f(x)$ for $-3\pi \le x \le 3\pi$ is shown below:



The Fourier series for f

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Thus the Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} 1 dx\right) = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} 1 \cos nx\, dx\right) = 0$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} \sin nx\, dx\right)$$

$$= \frac{1}{n\pi}(1 - \cos n\pi) = \frac{1}{n\pi}[1 - (-1)^n]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Thus we get: $$f(x) = \frac{1}{2} + \frac{2}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots)$$

$$= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$$

7.4.3 Odd and Even Function

Odd functions:

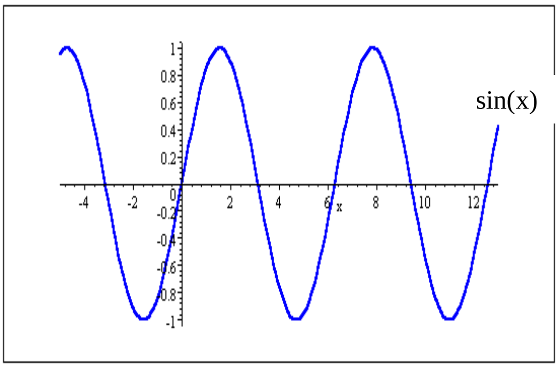
$f(x)$ is an odd function if and only if (iff)

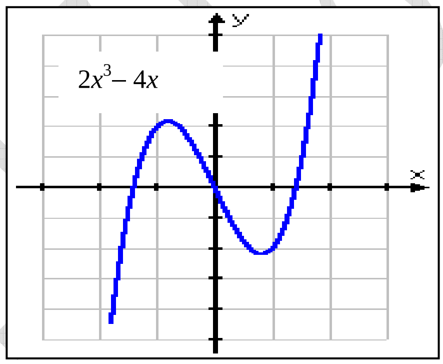
$$f(-x) = -f(x) \qquad \text{for} \quad -L \le x \le L$$

The graph of an odd function is symmetric about the origin. If we already have the graph of f for $x \ge 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

Examples of odd functions:

$f(x) = \sin(x)$ is odd because $\sin(-x) = -\sin(x)$ $\qquad$ $f(x) = 2x^3 - 4x$





Fourier Series of Odd function

Fourier series of an odd function, $f(x)$ defined in $-L \le x \le L$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

with $\quad a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\,dx = 0$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0, \quad \text{for } n = 1, 2, 3, \ldots$$

Thus $\quad f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ $\qquad$ (also known as *Fourier sine series*)

Example:

Determine the Fourier series for $f(x) = x^3$ in $-4 \le x \le 4$.

Recall that $\quad f(x) = \dfrac{a_o}{2} + \displaystyle\sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right)\right].$

Since $f(x) = x^3$ is an odd function, thus $a_0 = 0$ and $a_n = 0$, for $n = 1,2,3,\ldots$.

The Fourier coefficient is

$$b_n = \frac{1}{4}\int_{-4}^{4} x^3 \sin\left(\frac{n\pi x}{4}\right)dx = \frac{1}{2}\int_0^4 x^3 \sin\left(\frac{n\pi x}{4}\right)dx$$

$$= (-1)^{n+1}128\frac{n^2\pi^2 - 6}{n^3\pi^3}$$

The Fourier series for $f(x) = x^3$ in $-4 \le x \le 4$ is

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}128\frac{(n^2\pi^2 - 6)}{n^3\pi^3}\sin\left(\frac{n\pi x}{4}\right).$$

Even functions:

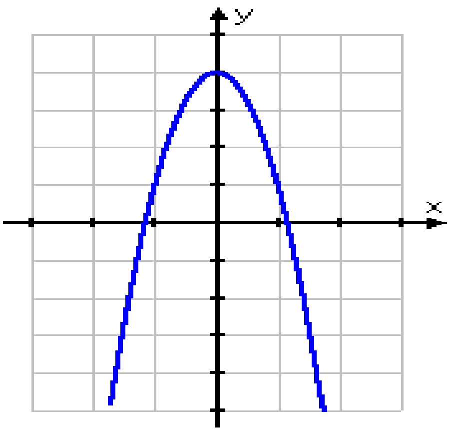
$f(x)$ is an even function if and only if (iff)

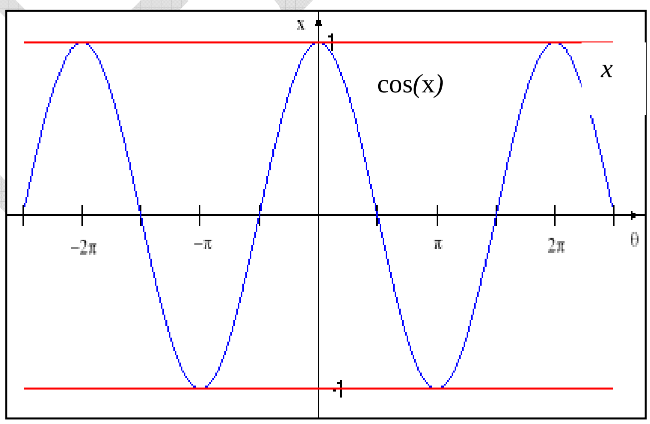
$$f(-x) = f(x) \qquad \text{for} \quad -L \le x \le L$$

The graph of an even function is symmetric with respect to the y-axis.

Examples of even functions

$f(x) = -3x^2+4$, $f(x) = \cos x$.





Fourier Series of Even Function

Fourier series for even function $f(x)$ defined in $-L \le x \le L$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x)$$

where $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0$, for $n = 1, 2, 3, \ldots$

Thus $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})$ (also known as *Fourier cosine series*)

Example:

Determine the Fourier series of $f(x) = x^2$ in $-2 \le x \le 2$

Write

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right)\right]$$

where

$$a_0 = \frac{1}{2}\int_{-2}^{2} x^2 dx = \frac{8}{3}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} x^2 \cos\left(\frac{n\pi x}{2}\right)dx = \frac{16}{n^2\pi^2}(-1)^n, \quad n = 1,2,3,\ldots$$

$$b_n = \frac{1}{2}\int_{-2}^{2} x^2 \sin\left(\frac{n\pi x}{2}\right)dx = 0$$

Thus the Fourier series for $f(x) = x^2$ in $-2 \le x \le 2$ is

$$f(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right)$$

Properties of Even and Odd Functions

From calculus we have

1. $\int_{-L}^{L} f(x)dx = 0$ if $f(x)$ is odd on $-L \le x \le L$

2. $\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$ if $f(x)$ is even on $-L \le x \le L$.

---------------------------THE END---------------------------

-(nby, Jun 2017)