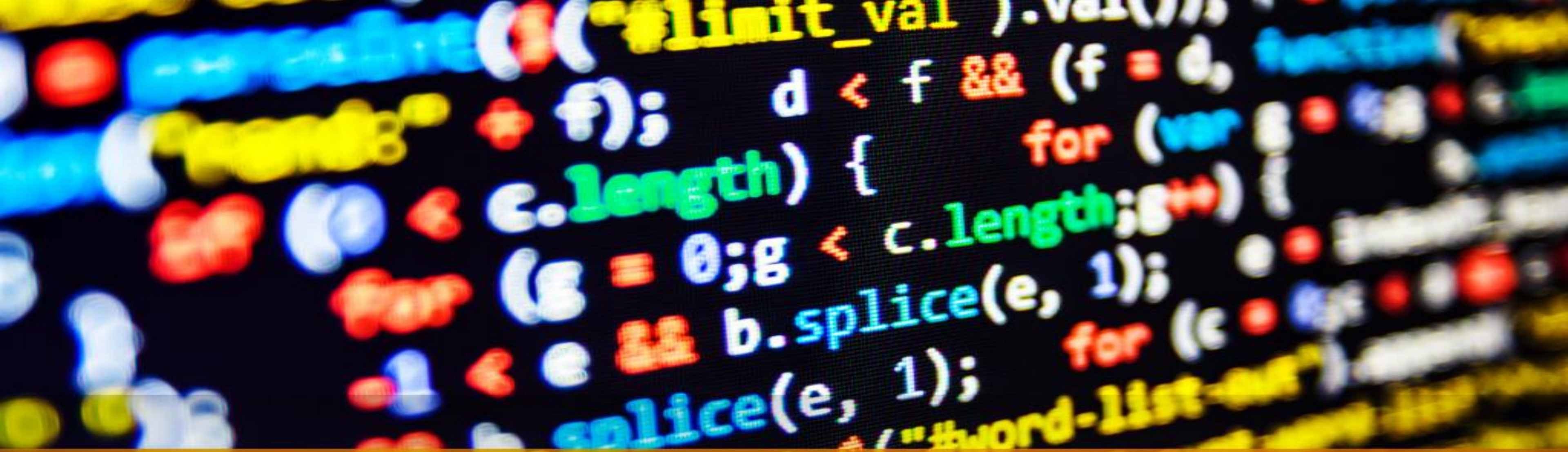


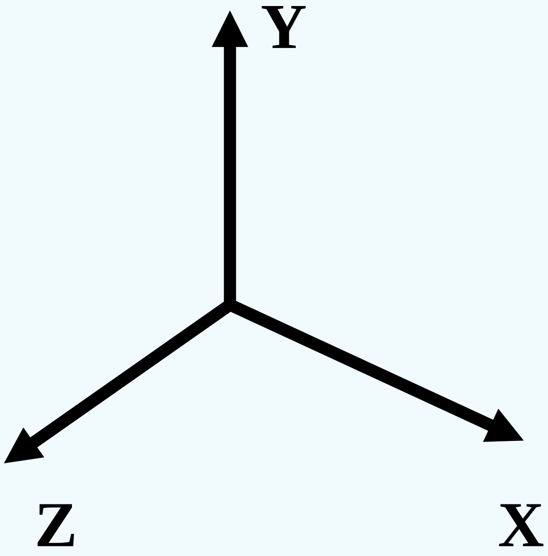
Lecture 03

Mathematical Fundamentals

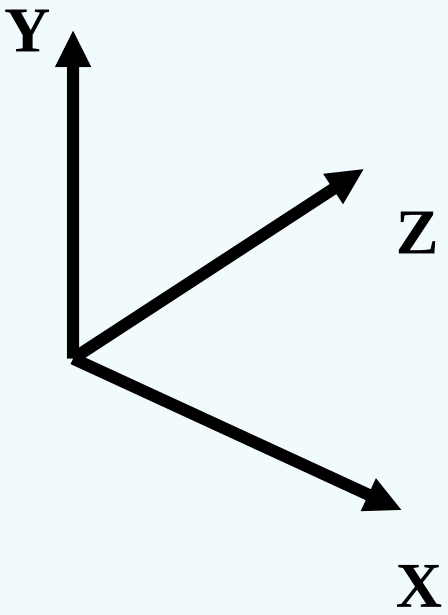
Prepared by Ban Kar Weng (William)

3D Coordinate Systems

3D Coordinate Systems



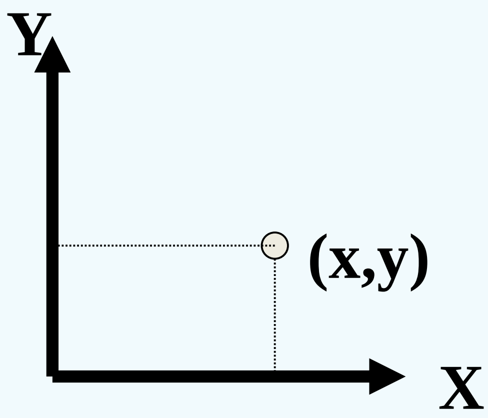
Right-handed System

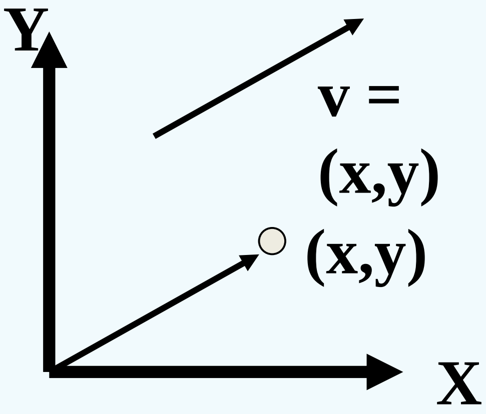


Left-handed System

Vector

Point and Vector





A **point** denotes a position in space.

A **vector** is a directed line segment, having both magnitude and direction.

Vector | Magnitude or Norm

Let vector $v = \langle x_v, y_v, z_v \rangle$

The magnitude or the norm of v:

$$\|v\| = \sqrt{x_v^2 + y_v^2 + z_v^2}$$

Vector | Unit Vector

Let vector $v = \langle x_v, y_v, z_v \rangle$

The unit vector of v:

$$\hat{v} = \frac{v}{\|v\|}$$

We **normalize** a vector when we convert the vector into a unit vector.

Vector | Dot Product

Let vector $v_1 = \langle x_{v_1}, y_{v_1}, z_{v_1} \rangle$ and $v_2 = \langle x_{v_2}, y_{v_2}, z_{v_2} \rangle$

The dot product of v_1 and v_2:

$$v_1 \cdot v_2 = x_{v_1} x_{v_2} + y_{v_1} y_{v_2} + z_{v_1} z_{v_2}$$

Vector | Cross Product

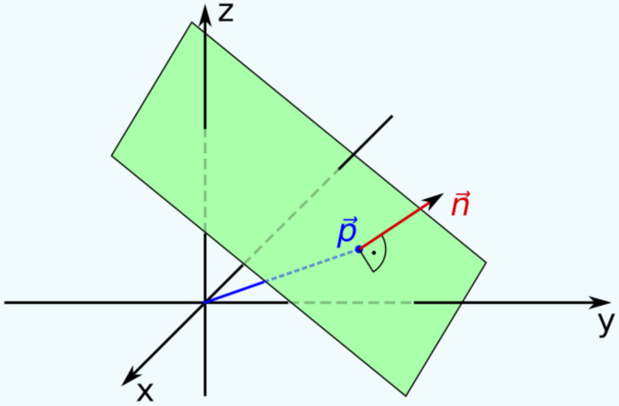
Let vector $v_1 = \langle x_{v_1}, y_{v_1}, z_{v_1} \rangle$ and $v_2 = \langle x_{v_2}, y_{v_2}, z_{v_2} \rangle$

The cross product of v_1 and v_2:

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ x_{v_1} & y_{v_1} & z_{v_1} \\ x_{v_2} & y_{v_2} & z_{v_2} \end{vmatrix} = \begin{bmatrix} y_{v_1} z_{v_2} - y_{v_2} z_{v_1} \\ z_{v_1} x_{v_2} - z_{v_2} x_{v_1} \\ x_{v_1} y_{v_2} - x_{v_2} y_{v_1} \end{bmatrix}$$

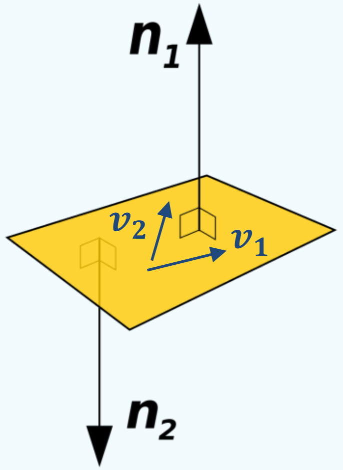
Vector | Plane

- A plane is a flat, 2D surface that extends infinitely far.
- Can be uniquely identified by any of the following:
  - Three non-collinear points (i.e. points not on a single line).
  - A line and a point not on that line.
  - Two distinct but intersecting lines.
  - Two distinct but parallel lines.
  - **Normal**



Vector | Normal

The normal of a plane spanned by two distinct vectors v_1 and v_2 is a vector **perpendicular** to the plane.

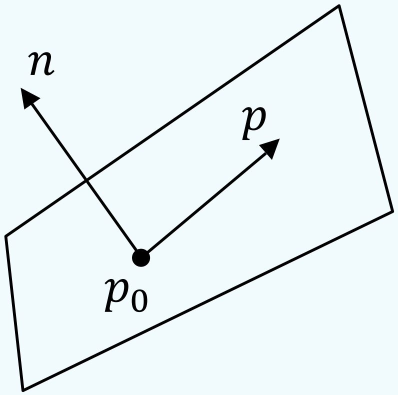


$$n_1 = v_1 \times v_2 \quad \text{or} \quad n_2 = v_2 \times v_1 = -n_1$$

NOTE ON NOTATIONS: Usually, normal is normalized because its magnitude are less useful in practice, so we use $\widehat{n_1}$ or $\widehat{n_2}$ to represent **unit normal**. In the context of *surface normal* of a 3D object, we use $\hat{n}$ to refer to unit normal that points towards the object's exterior.

Vector | Equation of a Plane

Given a normal $n = \langle x_n, y_n, z_n \rangle$ of a plane and a point $p_0 = \left(x_{p_0}, y_{p_0}, z_{p_0}\right)$ on the plane, we can derive the plane equation using **dot product**.



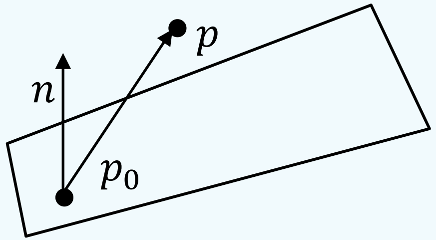
$$n \cdot (p - p_0) = 0$$

$$\boldsymbol{x_n\left(x_p - x_{p_0}\right) + y_n\left(y_p - y_{p_0}\right) + z_n\left(z_p - z_{p_0}\right) = 0}$$

Any point $p = (x_p, y_p, z_p)$ on the plane satisfies the above equation.

Vector | Distance between a Point and a Plane

The distance between a point p and a plane with normal n and a point p_0 on the plane, is given by:



$$D = \frac{|n \cdot (p - p_0)|}{\|n\|}$$

$$= \frac{\left|x_n(x_p - x_{p_0}) + y_n(y_p - y_{p_0}) + z_n(z_p - z_{p_0})\right|}{\sqrt{x_n^2 + y_n^2 + z_n^2}}$$

Matrices

Matrices | Vector specified in Matrix Notation

A vector $v = \langle x_v, y_v, z_v \rangle$ can also be specified in matrix notation as ***row vector*** or ***column vector***:

$$v = \begin{bmatrix} x_v & y_v & z_v \end{bmatrix}$$

Row vector

$$v = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}$$

Column vector

NOTE: This subject will use the **column vector** as the default vector notation, which is inline with the convention used in Linear Algebra.

Matrices | Identity Matrix

The identity of size n is the $n \times n$ square matrix with ones in the main diagonal and zeros elsewhere.

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \cdots,\ I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Definition of Identity Matrix and its images are taken from here.

Matrices | Transpose

Let a 3×3 matrix $M = \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix}$

The transpose of M:

$$M^T = \begin{bmatrix} m_{00} & m_{10} & m_{20} \\ m_{01} & m_{11} & m_{21} \\ m_{02} & m_{12} & m_{22} \end{bmatrix}$$

Matrices | Matrix Multiplication

Let $v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$v_1^T v_2 = ???$

Matrices | Matrix Multiplication

Let $v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$v_1 v_2^T = ???$

Matrices | Matrix Multiplication as Linear Combination

Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$Mv =$???

Matrices | Matrix Multiplication as Linear Combination

Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Linear combinations of **columns**

$$Mv = 1 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 9 \end{bmatrix}$$

Matrices | Matrix Multiplication as Linear Combination

Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$v^T M =$???

Matrices | Matrix Multiplication as Linear Combination

Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Linear combinations of **rows**

$$v^T M = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
$$= 1\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 2 & 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 9 \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{4} & \mathbf{1} & \mathbf{9} \end{bmatrix}$$

Matrices | Block Matrix Multiplication

Let $M_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$M_1 M_2 = ???$

Matrices | Block Matrix Multiplication

Partition M_1 into several submatrices or blocks. Example:

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & t_1 \\ \vec{0}^T & 1 \end{bmatrix}$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, t_1 = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}, \vec{0}^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Matrices | Block Matrix Multiplication

Partition M_2 into several submatrices or blocks. Example:

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & t_2 \\ \vec{0}^T & 1 \end{bmatrix}$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, t_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \vec{0}^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Matrices | Block Matrix Multiplication

$$
\begin{aligned}
M_1 M_2 &= \begin{bmatrix} I & t_1 \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} I & t_2 \\ \vec{0}^T & 1 \end{bmatrix} \\
&= \begin{bmatrix} II + t_1 \vec{0}^T & I t_2 + t_1 1 \\ \vec{0}^T I + 1 \vec{0}^T & \vec{0}^T t_2 + 1 \times 1 \end{bmatrix} \\
&= \begin{bmatrix} I & t_2 + t_1 \\ \vec{0}^T & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & -1+3 \\ 0 & 1 & 0 & -2+2 \\ 0 & 0 & 1 & -3+1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Matrices | Inverse

Let a 3×3 matrix $M = \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix}$

If M is invertible, there exists a 3×3 matrix M^{-1} such that:

$$M^{-1}M = MM^{-1} = I$$

Matrices | Orthogonal Matrices

A square matrix Q is an orthogonal matrix if its columns and rows are orthogonal unit vectors.

$$Q^T Q = Q Q^T = I$$

INTERESTING FACT: In linear algebra, Q is a common notation for representing an orthogonal matrix

Matrices | Orthogonal Matrices | Inverse

Interestingly, the inverse of an orthogonal matrix is equal to its transpose!

$$Q^{-1}Q = QQ^{-1} = I$$

$$Q^{T}Q = QQ^{T} = I$$

$$\therefore Q^{-1} = Q^{T}$$

Q & A

Acknowledgement

- This presentation has been designed using resources from PoweredTemplate.com