Topic 4.1
Induction

TMA1201 Discrete Structures & Probability
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What you will learn in this lecture:

- Mathematical Induction
- Strong Induction

Motivation

Consider the following property of positive integer:

The sum of the first n positive integers is $\frac{(n)(n+1)}{2}$

Often such property (and many others) can be represented as a predicate, *P(n)*, and the property said that *P(n)* is true for all positive integers *n*.

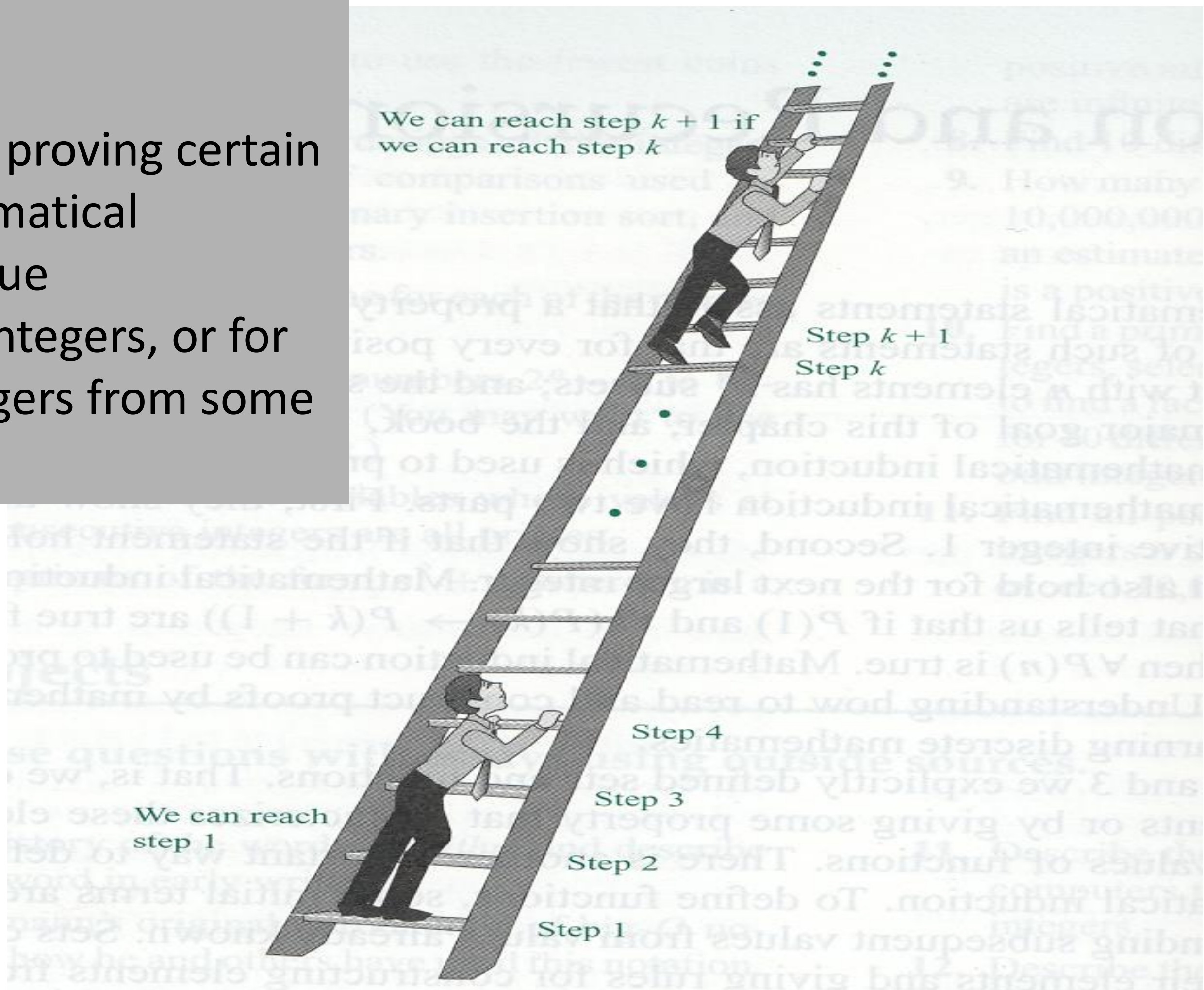
How to prove them?

→ Use mathematical induction

The Principles of Mathematical Induction

<u>The Principles of Mathematical Induction:</u>

A technique for proving certain types of mathematical statements is true
for all positive integers, or for all positive integers from some point on.



The Principles of Mathematical Induction

Let $P(n)$ be a proposition depending on n, where n is a **positive** integer.

To prove $P(n)$ is true for all positive integers, it suffices to prove:

- $P(1)$ is true
- For all $k \geq 1$, $P(k + 1)$ is true whenever $P(k)$ is true: $P(k) \rightarrow P(k + 1)$

$$\boxed{[P(1) \wedge \forall k \, \{P(k) \rightarrow P(k+1)\}] \Rightarrow \forall n \, P(n)}$$

The 3 Steps of Mathematical Induction (for nonnegative or positive integers)

1. Inductive base:
Show that P(base value) is true. Note that base value not always =1.

2. Inductive hypothesis:
Assume $P(k)$ is true.

3. Inductive step:
Show that $P(k + 1)$ is true on the basis of the inductive hypothesis.

Example 1: Use Mathematical Induction to prove that the sum of the first *n* positive integers is $\frac{n(n+1)}{2}$.

Solution:

Let *P(n)* be the proposition that $1+2+3+\cdots+n=\frac{n(n+1)}{2}$, for *n* = 1, 2, 3, ….

Proof by induction.

1) Inductive base: *n = 1*

Left side of the equation *= 1*

Right side of the equation = $\frac{1(1+1)}{2}=1$

Hence *P(1)* is true.

2) Inductive hypothesis: *n* = *k*

Assume that *P(k)* is true, i.e. $1+2+3+\ldots+k=\frac{k(k+1)}{2}$

3) Inductive step: *n* = *k* + 1

$$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

We have shown that *P(k + 1)* is true whenever *P(k)* is true.

By mathematical induction, *P(n)* is true for all *n* ≥ 1 .

Example 2: Use mathematical induction to prove the inequality $n < 2^n$ for all positive integers *n*.

Solution:

Let *P(n)* be the proposition $n < 2^n$, for $n = 1, 2, 3, \ldots$.

Proof by Induction

1) Inductive base: $n = 1$

LHS of the inequality = 1, RHS of the inequality = $2^1 = 2$

Since $1 < 2$, *P(1)* is true.

2) Inductive hypothesis: $n = k$

Assume *P(k)* is true, which means $k < 2^k$.

3) Inductive step: $n = k + 1$

We want to show that *P(k+1)* is true, i.e. we want to show that $k+1 < 2^{k+1}$.

Now, $k + 1 \le k + k = 2k < 2(2^k) = 2^{k+1}$.

By assuming *P(k)* is true, we have shown that *P(k+1)* is true.

Thus by mathematical induction, P(*n*) is true for all $n \geq 1$.

Strong Induction

- Similar steps as the ordinary mathematical induction.
- However, the basis step may contain the proof for more than one initial values.
- Also the assumption is made for not just one value of n, but for all values throughout k.

Example 3:

Prove that every integer in the sequence a_0, a_1, a_2, … that is defines as

$$a_0 = 2,\ a_1 = 6,\ a_n = a_{n-1} + a_{n-2} \text{ for all } n >= 2 \text{ is even.}$$

Solution:

1) Inductive base: $n = 0$ and $n = 1$

Given that $a_0 = 2$, $a_1 = 6$ thus a_n is even for $n = 0$ and $n = 1$

2) Inductive hypothesis: $n = 0, 1, 2, \ldots k$

Assume that a_n is even for $n = 0, 1, 2, \ldots k$

3) Inductive step: $n = k + 1$

From inductive hypothesis, a_k is even so $a_k = 2m$ for integer m and $a_{k-1} = 2n$ for integer n

$$a_{k+1} = a_k + a_{k-1} = 2m + 2n = 2(m + n)$$

Thus a_{k+1} is even whenever a_n is even for $n = 0, 1, 2, \ldots k$

Try this:

Use mathematical induction to prove that 3 divides $n^3 - n$, where n is positive integer.

Summary

We have discussed the concepts related to the principles of mathematical induction:

- There are 3 steps to prove by induction (ordinary or strong):
    1. Inductive base
    2. Inductive hypothesis
    3. Inductive step

- You should always take note on the differences between ordinary induction and strong induction.

Exercise 1

Infer a formula for the sum of the first *n* positive odd integers. Then using mathematical induction, prove your inference.

Solution:

Exercise 2

Use mathematical induction to show that for all positive integers $n \geq 4$,

$$n! > 2^n$$

Solution: