HW1 Solutions, M22A, Spring Quarter 2014, Prof. Sornborger
1.1: 5, 7, 13, 28, 30
1.2: 1, 2, 4, 7, 9, 16, 20
1.3: 2, 7, 8, 12

1.1: 5, 7, 13, 28, 30

5 Compute $\mathbf{u}+\mathbf{v}+\mathbf{w}$ and $2\mathbf{u}+2\mathbf{v}+\mathbf{w}$.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

How do you know $\mathbf{u}, \mathbf{v}, \mathbf{w}$ lie in a plane?
SOLUTION

$$\mathbf{u}+\mathbf{v}+\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2\mathbf{u}+2\mathbf{v}+\mathbf{w} &= 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \end{aligned}$$

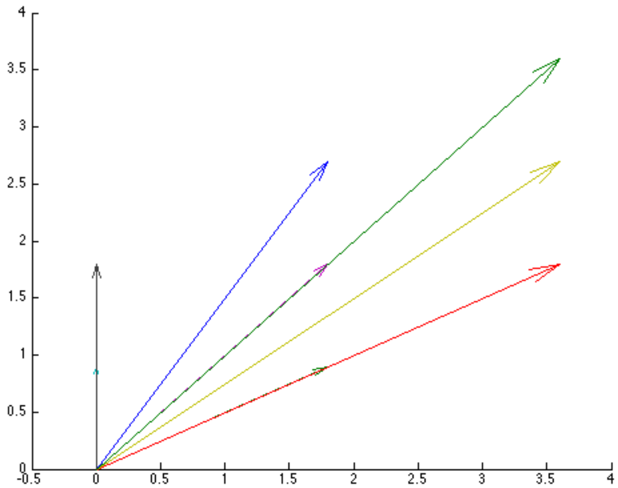
Why do they lie in a plane? Think about what forms a plane, you need two unique vectors (or lines). In this case we have two unique vectors $\mathbf{u}$ and $\mathbf{v}$, while $\mathbf{w}$ is just a linear combination of the $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{w} = -\mathbf{v} - \mathbf{u}$$

this means that $\mathbf{w}$ is simply a line that lies on the plane formed by $\mathbf{u}$ and $\mathbf{v}$.

7 In the xy plane mark all nine of these linear combinations

$$c\begin{bmatrix} 2 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{where } c = 0, 1, 2 \quad d = 0, 1, 2.$$



13 (a) What is the sum $\mathbf{V}$ of the twelve vectors that go from the center of a clock to the hours $1:00, 2:00, ..., 12:00$?

SOLUTION Each vector has an opposite vector, for example the $3:00$ vector is the opposite of the $9:00$ vector, so they add up to 0. So the sum of all the vectors is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(b) If the $2:00$ vector is removed why do the 11 remaining vectors at to $8:00$?

SOLUTION When you remove the $2:00$ vector, it is no longer able to cancel with the $8:00$ vector, so the $8:00$ vector remains.

(c) What are the components of that $2:00$ vector $\mathbf{v} = (\cos(\theta), \sin(\theta))$?

SOLUTION The angle between $2:00$ and $3:00$ is $\pi/6$ (30 degrees). This is because the circle is split up into 12 equal segements, so that the angle between each segment $= 2\pi/12 = \pi/6$.

By the unit circle we know $\mathbf{v} = (\cos(\pi/6), \sin(\pi/6))$

28 Find vectors $\mathbf{v}$ and $\mathbf{w}$ so that $\mathbf{v} + \mathbf{w} = (4, 5, 6)$ and $\mathbf{v} - \mathbf{w} = (2, 5, 8)$.

This is a question with BLANK unknown numbers, and an equal number of equations to find those numbers.

SOLUTION

The quesiton has 6 unknown numbers, because v and w both have 3 unknown values each. There are also 6 corresponding equations.

We need

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

and

$$\mathbf{v} - \mathbf{w} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

adding the two equations together gives

$$2\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\mathbf{v} = \frac{1}{2}\begin{bmatrix} 6 \\ 10 \\ 14 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

similarily we can subtract the two equations we are given to get

$$-2\mathbf{w} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\mathbf{w} = -.5\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

30 The linear combinations of $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$ fill the plane unless BLANK. Find four vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$ with four components each so that their combinations

$$c\mathbf{u} + d\mathbf{v} + e\mathbf{w} + f\mathbf{z}$$

produce all vectors (b_1, b_2, b_3, b_4) in four dimensional space.

SOLUTION

The two vectors $\mathbf{v}$ and $\mathbf{w}$ fill the plane unless they are linearly dependent i.e.

$$\mathbf{v} = c\mathbf{w}$$

where c is any constant.

The simplest choice of vectors that fill four dimensional space is the standard basis vectors.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

you can create any vector in four dimensions out of linear combinations of these four vectors, for example:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

These are the "standard" basis (i.e. buildling blocks) for n dimensional spaces.

1.2: 1, 2, 4, 7, 9, 16, 20

1 Calculate the dot products $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{w}$, $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$, and $\mathbf{w} \cdot \mathbf{v}$.

$$\mathbf{u} = \begin{bmatrix} -.6 \\ .8 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

SOLUTION

$$\mathbf{u} \cdot \mathbf{v} = -.6 * 3 + .8 * 4 = -1.4$$

$$\mathbf{u} \cdot \mathbf{w} = -.6 * 8 + .8 * 6 = 0$$

$$\mathbf{w} \cdot \mathbf{v} = 8 * 3 + 6 * 4 = 1.4$$

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \begin{bmatrix} 11 \\ 10 \end{bmatrix} = -.6 * 11 + .8 * 10 = -58$$

2 Compute the lengths of the above 3 vectors. Check the Schwartz inequalities $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$ and $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$

SOLUTION

$$\|\mathbf{u}\| = \sqrt{(-6)^2 + .8^2} = 6.053098$$

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5$$

$$\|\mathbf{w}\| = \sqrt{8^2 + 6^2} = 10$$

Now we verify that the Schwartz inequalities hold.

$$14.8 = |\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| = 30.25$$

$$48 = |\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\| = 50$$

4 For any unit vectors $\mathbf{v}$ and $\mathbf{w}$, find the dot products

$$a)\ \mathbf{v} \cdot (-\mathbf{v}), \quad b)\ (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}), \quad c)\ (\mathbf{v} - 2\mathbf{w}) \cdot (\mathbf{v} + 2\mathbf{w})$$

SOLUTION

$$a)\ -\mathbf{v} \cdot \mathbf{v} = -\|\mathbf{v}\|^2 = -1$$

$$b)\ (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot v - \mathbf{w} \cdot \mathbf{w} = \|v\|^2 - \|w\|^2 = 1 - 1 = 0$$

$$c)\ (\mathbf{v} - 2\mathbf{w}) \cdot (\mathbf{v} + 2\mathbf{w}) = v \cdot v + 2v \cdot w - 2w \cdot v - 4w \cdot w = \|v\|^2 - 4\|w\|^2 = 1 - 4 = -3$$

7 Find the angel θ (from its cosine) between these pairs of vectors

(a)

$$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

SOLUTION

We use the equation

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\mathbf{w}\|} = \cos\theta$$

where θ is the angle between the two vectors.

$$\theta = \cos^{-1}(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}) = \cos^{-1}(1/2) = \pi/3$$

(b)

$$\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

SOLUTION

$$\theta = \cos^{-1}(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}) = \cos^{-1}((4-2-2)/(3*3)) = \cos^{-1}(0) = \pi/2$$

(c)

$$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \quad \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$$

SOLUTION

$$\theta = \cos^{-1}(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}) = \cos^{-1}(-1+3)/(2*2) = \cos^{-1}(1)/(2) = \pi/3$$

(d)

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

SOLUTION

$$\theta = \cos^{-1}(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}) = \cos^{-1}(-3-2)/(\sqrt{10}\sqrt{5}) = \cos^{-1}(-\sqrt{5}\sqrt{5})/(\sqrt{5}\sqrt{2}\sqrt{5}) = \cos^{-1}(-1/\sqrt{2}) = 3\pi/4$$

9 The slopes of the arrows from $(0, 0$ to (v_1, v_2) and $(w_1, w_2$ are v_2/v_1 and w_2/w_1. Suppose the product v_2w_2/v_1w_1 of these slopes is -1. Show that $\mathbf{v} \cdot \mathbf{w} = 0$ and the vectors are perpendicular.

SOLUTION

The problem tells us that $v_2w_2/v_1w_1 = -1$, this means

$$v_2w_2 = -v_1w_1$$

$$\mathbf{v}\ cdot\mathbf{w} = (v_1, v_2) \cdot (w_1, w_2) = v_1w_1 + v_2w_2 = v_1w_1 - v_1w_1 = 0$$

and we know that two vectors are perpendicular if their dot product is 0 so v and w must be perpendicular.

16 How long is the vector

$$\mathbf{v} = (1, 1, 1, \cdots, 1)$$

in 9 dimensions? Find a unit vectors $\mathbf{u}$ in the same direction as $\mathbf{v}$ and a unit vector $\mathbf{w}$ that is perpedicular to $\mathbf{v}$

SOLUTION

$$\|\mathbf{v}\| = \sqrt{1^2+1^2+1^2+\cdots+1^2} = \sqrt{9} = 3$$

A unit vector that goes in the same direction as **v** is found by simplyly dividing **v** by its length, 3.

$$\mathbf{u} = (1,1,\cdots,1)/3 = (1/3,1/3,\cdots,1/3)$$

We next want to find a vector **w** who is perpendicular to **v**, this means that

$$\mathbf{w}\cdot\mathbf{v} = (a_1,a_2,a_3,\cdots,a_9)\cdot(1,1,1,\ldots,1) = 0$$

One thing to ask is how many answers do we expect? Think about 3 dimensions, how many vectors are orthogonal to any given vector? An infinite amount! the same is true in 9 dimensions, the point is that instead of solving an equation to find the perpendicular vector, you just pick the simplest one that satisfies the dot product being 0.

One very simple choice is the vector

$$\mathbf{w} = (1,0,0\cdots,0,-1)$$

Clearly

$$\mathbf{w}\cdot\mathbf{v} = (1,0...,0,-1)\cdot(1,1,\cdots,1) = 1-1 = 0$$

20 The law of cosines comes from

$$(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - 2\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w}$$

COSINE LAW

$$\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta + \|\mathbf{w}\|^2$$

If $\theta < 90$ show that $\|v\|^2 + \|w\|^2$ is larger than $\|v-w\|^2$

SOLUTION

If $\theta < 90$ what is true about $cos(\theta)$? It lies between 1 and 0.

This means that

$$\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta \le \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$

1.3: 2,7,8,12

2 Solve these equations

$$\begin{bmatrix} 1,0,0 \\ 1,1,0 \\ 1,1,1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1,0,0 \\ 1,1,0 \\ 1,1,1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

The sum of the first n odd numbers is?

SOLUTION

Multiplying the first equation gives us

$$\begin{bmatrix} y_1 \\ y_1 + y_2 \\ y_1 + y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which yields

$$y_1 = 1, \quad y_2 = 1 - y_1 = 1 - 1 = 0, \quad y_3 = 1 - y_1 - y_2 = 1 - 1 - 0 = 0$$

for the second equation we have

$$\begin{bmatrix} y_1 \\ y_1 + y_2 \\ y_1 + y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

which yields

$$y_1 = 1, \quad y_2 = 4 - y_1 = 3, \quad y_3 = 9 - y_1 - y_2 = 9 - 1 - 3 = 5$$

What is the sum of the first n odd numbers? Let us do a few examples to get a feel for the problem.

$$n = 1, \quad 1$$
$$n = 2, \quad 1 + 3 = 4$$
$$n = 3, \quad 1 + 3 + 5 = 9$$
$$n = 4, \quad 1 + 3 + 5 + 7 = 16$$
$$n = 5, \quad 1 + 3 + 5 + 7 + 9 = 25$$

The pattern should be obvious now, the sum of the first n odd numbers is n^2. Wonder why this is true? Take a look at the first proof at http://www.9math.com/book/sum-first-n-odd-natural-numbers for an interesting geometric interpretation.

7 If the columns combine into $A\mathbf{x} = \mathbf{0}$ then each row has $\mathbf{r} \cdot \mathbf{x} = 0$

$$\begin{bmatrix} \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by rows

$$\begin{bmatrix} \mathbf{r_1} \cdot \mathbf{x} \\ \mathbf{r_2} \cdot \mathbf{x} \\ \mathbf{r_3} \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The three rows also lie in a plane. Why is that plane perpendicular to x?

SOLUTION

Because the three planes form a plane, every vector that lies on the plane is equal to a linear combination of the row vectors. But we already know that x is perpendicular to the rwo vectors because $r \cdot x = 0$ for each row vector r.

This means that every vector that lies on the plane is perpendicular to x and thus, the entire plane.

8 Moving to a 4 by 4 difference equation, $A\mathbf{x} = \mathbf{b}$, find the four components x_1, x_2, x_3, x_4. Then write this solution as $\mathbf{x} = S\mathbf{b}$ to find the inverse matrix $S = A^{-1}$;

$$A\mathbf{x} = \begin{bmatrix} 1,0,0,0 \\ -1,1,0,0 \\ 0,-1,1,0 \\ 0,0,-1,1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{b}$$

SOLUTION

Multiplying the matrix out gives

$$\begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \\ -x_3 + x_4 \end{bmatrix} = \mathbf{b}$$

this tells us

$$x_1 = b_1, \quad x_2 = b_2 + x_1 = b_2 + b_1, \quad x_3 = b_3 + x_2 = b_3 + b_2 + b_1, \quad x_4 = b_4 + x_3 = b_4 + b_3 + b_2 + b_1$$

we can write this out as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1,0,0,0 \\ 1,1,0,0 \\ 1,1,1,0 \\ 1,1,1,1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

where the matrix here is A^{-1}

12 The last lines of the Worked Example say that the 4 by 4 centered difference matrix in (16) is invertible. Solve $C\mathbf{x} = (b_1, b_2, b_3, b_4)$ to find its inverse in $\mathbf{x} = C^{-1}\mathbf{b}$.

SOLUTION

We want to solve

$$\begin{bmatrix} 0,1,0,0 \\ -1,0,1,0 \\ 0,-1,0,1 \\ 0,0,-1,0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

If we multiply the matrix we get

$$\begin{bmatrix} x_2 = b_1 \\ -x_1 + x_3 = b_2 \\ -x_2 + x_4 = b_3 \\ -x_3 = b_4 \end{bmatrix}$$

so that $x_2 = b_1, \quad x_3 = -b_4, \quad x_1 = x_3 - b_2 = -b_4 - b_2 \quad , x_4 = b_3 + x_2 = b_3 + b_1$

We can write this out as a matrix to get

$$x = \begin{bmatrix} 0,-1,0,-1 \\ 1,0,0,0 \\ 0,0,0,-1 \\ 1,0,1,0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

where the matrix above is C^{-1}