HW8 Solutions, M22A, Spring Quarter 2014, Prof. Sornborger

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6.1: 3,4,8,9,12,14,15,27,28,36

3 Compute eigenvalues and eigenvector of A and A^{-1}. Check the trace!

$$A = \begin{bmatrix} 0, 2 \\ 1, 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -1/2, 1 \\ 1/2, 0 \end{bmatrix}$$

SOLUTION

We first solve for the eigenvalues of A by solving det $(A - \lambda I) = 0$

$$\det(A - \lambda I) = -\lambda * (1 - \lambda) - 2 = \lambda^2 - \lambda - 2 = 0$$

which has solutions $\lambda = -1, 2$

We next solve the equations $Ax = -1x$, and $Ax = 2x$ to find the eigenvector and eigenvalue pairs

$$\lambda_1 = -1, x_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \lambda_2 = 2, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

we do the same for A^{-1} to find

$$\lambda_1 = -1, x_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \lambda_2 = .5, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We notice that A^{-1} has the same eigenvectors as A. When A has eigenvectors λ_1, λ_2 , A^{-1} has eigenvalues $1/\lambda_1, 1/\lambda_2$. Why is this? Because if $Ax = \lambda x$ then

$$Ax = \lambda x \rightarrow x = \lambda A^{-1} x \rightarrow A^{-1} x = \frac{1}{\lambda} x$$

4 Compute the eigenvalues and Eigen vectors of A and A^2:

$$A = \begin{bmatrix} -1, 3 \\ 2, 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 7, -3 \\ -2, 6 \end{bmatrix}$$

SOLUTION

For A we have

$$\lambda_1 = -3, x_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad \lambda_2 = 2, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now since $Ax = \lambda x$ we have

$$A^2 x = A\lambda x = \lambda^2 x$$

so that A^2 has the same eigenvectors as A and its eigenvalues are just those of A's squared

The sums of the eigenvalues of a matrix is equal to the sum of its eigenvalues (p.289)so $\lambda_1^2 + \lambda_2^2 = 13$

8 (a) If you know that x as an eigenvector the way to find λ is to
SOLUTION

You simply find the λ that satisfies the equation $Ax = \lambda x$ (here A and x are givens)

(b) if you know that λ is an eigenvalue, the way to find x is to
SOLUTION

You simply find the x that satisfies the equation $Ax = \lambda x$ (here A and λ are givens), your solution will usually have a free variable, simply set it equal to any constant to select a single eigenvector.

9 What do you do to the equation $Ax = \lambda x$ in order to prove a,b,d

(a) λ^2 is an eigenvalue of A^2
SOLUTION

Multiply both sides by A to get $A^2 x = \lambda Ax = \lambda^2 x$

(b) λ^{-1} is an eigenvalue of A^{-1}

SOLUTION

Multiply both sides by A^{-1}, $x = \lambda A^{-1} x$, then divide by λ.

(c) $\lambda + 1$ is an eigenvalue of $A + I$

SOLUTION

Add x to both sides $Ax = \lambda x \to Ax + x = (\lambda + 1)x \to (A + I)x = (\lambda + 1)x$

12 Find three eigenvectors for this matrix P (projection matricie have $\lambda = 0, 1$)

$$P = \begin{bmatrix} .2, .4, 0 \\ .4, .8, 0 \\ 0, 0, 1 \end{bmatrix}$$

If two eigenvectors share the same λ so do all their linear combinations. Find an eigevector of P with no zero components.
SOLUTION

The set of eigenvalues and eigenvectors for P are

$$\lambda_1 = 1, x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \lambda_2 = 1, x_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \lambda_3 = 0, x_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

The problem is that all these components have a zero component, but since x_1, x_2 share the same eigenvalue we have that $x_1 + x_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 1.

14 Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos(\theta) \pm i \sin(\theta)$

$$Q = \begin{bmatrix} \cos\theta, -\sin\theta \\ \sin\theta, \cos\theta \end{bmatrix}$$

Q rotates the xy plane by the angle θ and has no real λ's. Find the eigenvectors of Q by solving $(Q - \lambda I)x = 0$ using $i^2 = -1$
SOLUTION

Solving the determinant formula gives us $\lambda = \cos(\theta) \pm i\sin(\theta)$ now we solve $Qx = \lambda x$ (equivalent to $(Q - \lambda I)x = 0$) to find the eigenvectors

$$Qx = \begin{bmatrix} \cos\theta x_1 - \sin\theta x_2 \\ \sin\theta x_1 + \cos\theta x_2 \end{bmatrix} = \begin{bmatrix} (\cos(\theta) \pm i\sin(\theta))x_2 \\ (\cos(\theta) \pm i\sin(\theta))x_2 \end{bmatrix} = \lambda x$$

solving this equation for the two different cases, $\pm$, and using $i^2 = -1$ gives the eigenvectors

$$\lambda_1 = \cos(\theta) - i\sin(\theta), x_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_2 = \cos(\theta) + i\sin(\theta), x_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

15 Every pemutaiton matrix leaves $x = (1, 1, \cdots, 1, 1)$ unchanged, so $\lambda = 1$.Find two more λ's for these P

$$P = \begin{bmatrix} 0,1,0 \\ 0,0,1 \\ 1,0,0 \end{bmatrix}, \quad P = \begin{bmatrix} 0,0,1 \\ 0,1,0 \\ 1,0,0 \end{bmatrix}$$

SOLUTION

Solving for $\det(A - \lambda I) = 0$ we find $\lambda^3 = 1$ who has solutions $\lambda = 1, \frac{-1 \pm i\sqrt{3}}{2}$

We do the same procedure for the second matrix and find $\lambda = 1, 1, -1$.

27 Find the rank and the four eigenvalues of A and C

$$A = \begin{bmatrix} 1,1,1,1 \\ 1,1,1,1 \\ 1,1,1,1 \\ 1,1,1,1 \end{bmatrix}, \quad C = \begin{bmatrix} 1,0,1,0 \\ 0,1,0,1 \\ 1,0,1,0 \\ 0,1,0,1 \end{bmatrix}$$

SOLUTION

Clearly the dimension of the column space of A is 1 so $\text{Rank}(A) = 1$, so we can expect only one eigenvalue with nonzero eigenvector, solving for them you find $\lambda = 4, 0, 0, 0$

Once again the rank of C can be seen to be 2 (since there are only 2 independent column vectors in C), this means we can only expect 2 nonzero eigenvalues, solving for them we find $\lambda = 2, 2, 0, 0$

28 Find the eigenvalues and determinants of

$$B = \begin{bmatrix} 0,1,1,1 \\ 1,0,1,1 \\ 1,1,0,1 \\ 1,1,1,0 \end{bmatrix} \quad C = -\begin{bmatrix} 0,1,1,1 \\ 1,0,1,1 \\ 1,1,0,1 \\ 1,1,1,0 \end{bmatrix}$$

SOLUTION

By computation we find for B, $\lambda = 3, -1, -1, -1$. and for C we simply have the opposite's since if $Ax = \lambda x$ then $-Ax = -\lambda x$.

36 Is there a real 2 by 2 matrix other then I with $A^3 = I$? Its eigenvalues must satisfy $\lambda^3 = 1$. What trace and determinant would this give? Construct a rotation matrix as A (which angle of rotation?)

SOLUTION

The solutions to $\lambda^3 = 1$ are $\lambda = 1, e^{2\pi i/3}, e^{-2\pi i/3}$ choosing the latter two
The determinant of this matrix would equal $\lambda_1\lambda_2 = 1$ and the trace would equal

$$\lambda_2 + \lambda_1 = \cos 2\pi/3 + i \sin 2\pi/3 + \cos + \cos -2\pi/3 + i \sin -2\pi/3 = 2\cos 2\pi/3 = -1$$

Defining a rotation matrix

$$A = \begin{bmatrix} \cos\theta, -\sin\theta \\ \sin\theta, \cos\theta \end{bmatrix}$$

we choose θ so that it satisfies the above constraints, in this case $\theta = 2\pi/3$

6.2: 1,2,3,18,26,35,36

1 (a) Factor these two matrices into $A = S\Lambda S^{-1}$

$$A = \begin{bmatrix} 1, 2 \\ 0, 3 \end{bmatrix}, \quad A = \begin{bmatrix} 1, 1 \\ 3, 3 \end{bmatrix}$$

SOLUTION

We first find the eigenvalues and eigenvectors of A

$$\lambda_1 = 3, x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 1, x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we then construct the S and Λ out of these.

$$\Lambda = \begin{bmatrix} 3, 0 \\ 0, 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1, 1 \\ 1, 0 \end{bmatrix}$$

notice how the 1st column of A and S contain a corresponding pair of eigenvalue and eigenvector, the same for the 2nd column.
Do the same procedure for the second matrix.

(b) If $A = S\Lambda S^{-1}$ then $A^3 =$ and $A^{-1} =$
SOLUTION

$$A^3 = S\Lambda S^{-1}S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^3 S^{-1}$$

$$A^{-1} = \left(S\Lambda S^{-1}\right)^{-1} = S\Lambda^{-1}S^{-1}$$

recall that since Λ is a diagonal matrix, finding its inverse is very easy, you simply invert each of it's elements on the diagonal (i.e. 4 becomes 1/4)

2 If A has $\lambda_1 = 2$ with eigenvector $x_1 = (1, 0)$ and $\lambda_2 = 5$ with $x_2 = (1, 1)$ use $S\lambda S^{-1}$ to find A. No other matrix has the same λ's and x's.
SOLUTION

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1, 1 \\ 0, 1 \end{bmatrix} \begin{bmatrix} 2, 0 \\ 0, 5 \end{bmatrix} \begin{bmatrix} 1, -1 \\ 0, 1 \end{bmatrix}$$

3 Suppose $A = S\Lambda S^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = ()()()^{-1}$
SOLUTION

$$A + 2I = S\Lambda S^{-1} + 2I = S\Lambda S^{-1} + 2SIS^{-1} = S(\Lambda + 2I)S^{-1}$$

18 Diagonalize A and compute $S\Lambda^k S^{-1}$ to prove this formula for A^k

$$A = \begin{bmatrix} 2, -1 \\ -1, 2 \end{bmatrix} \quad A^k = .5 \begin{bmatrix} 1 + 3^k, 1 - 3^k \\ 1 - 3^k, 1 + 3^k \end{bmatrix}$$

SOLUTION
computing the eigenvalues and eigenvectors we find

$$\lambda_1 = 3, \quad x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 1, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that

$$A = \begin{bmatrix} -1, 1 \\ 1, 1 \end{bmatrix} \begin{bmatrix} 3, 0 \\ 0, 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1, 1 \\ 1, -1 \end{bmatrix}$$

so that

$$A^k = S\Lambda^k S^{-1} = \begin{bmatrix} -1, 1 \\ 1, 1 \end{bmatrix} \begin{bmatrix} 3^k, 0 \\ 0, 1^k \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1, 1 \\ 1, -1 \end{bmatrix} = 1/2 \begin{bmatrix} 1 + 3^k, 1 - 3^k \\ 1 - 3^k, 1 + 3^k \end{bmatrix}$$

26 Suppose $Ax = \lambda x$. If $\lambda = 0$ then x is in the nullspace. If $\lambda \neq 0$ then x is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

SOLUTION

Because if a n x n square matrix had n linearly independent eigenvectors that would mean it had a n dimensional column space, we know that this is not true all square matrices.

35 The powers A^k approach zero if all $|\lambda_i| < 1$ and they blow up if any $\lambda_i| > 1$ Peter Lax gives these striking examples in his book Linear Algebra

$$A = \begin{bmatrix} 3, 2 \\ 1, 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3, 2 \\ -5, -3 \end{bmatrix}, \quad C = \begin{bmatrix} 5, 7 \\ -3, -4 \end{bmatrix}, \quad D = \begin{bmatrix} 5, 6.9 \\ -3, -4 \end{bmatrix}$$

$$\|A^{1024}\| > 10^{700}, \quad B^{1024} = I, \quad C^{1024} = -C, \quad \|D^{1024}\| < 10^{-78}$$

Find the eigenvalues $\lambda = e^{i\theta}$ of B and C to show $B^4 = I$ and $C^3 = -I$.

SOLUTION

Calculating the eigenvalues we find $\lambda = i, -i$. These two eigenvalues have two associated eigenvectors x, y, since any vector z can be constructed out of x, y we have $z = ax + by$ so that

$$B^4 z = aB^4 x + bB^4 y = a(i)^4 x + b(-i)^4 y = ax + by = z$$

so that clearly $B^4 = I$.

Do the same procedure for C^4.

36 The nth power of rotation through θ is rotation through $n\theta$

$$A^n = \begin{bmatrix} \cos\theta, -\sin\theta \\ \sin\theta, \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta, -\sin n\theta \\ \sin n\theta, \cos n\theta \end{bmatrix}$$

Prove that neat formula by diagonalizing A, the eigenvectors are $(1, i)$ and $(i, 1)$. Use Euler's formula.

SOLUTION

We have that

$$\lambda_1 = \cos\theta - i\sin\theta, \; x_1 = (1, i), \quad \lambda_2 = \cos\theta + i\sin\theta, \; x_2 = (i, 1)$$

Computing

$$A^k = S\Lambda^k S^{-1} = \begin{bmatrix} 1, i \\ i, 1 \end{bmatrix} \begin{bmatrix} (\cos\theta - i\sin\theta)^n, 0 \\ 0, (\cos\theta + i\sin\theta)^n \end{bmatrix} 1/2 \begin{bmatrix} 1, -i \\ -i, 1 \end{bmatrix}$$

will give the desired result.

6.3: 1,4

1 Find two λ's and x's so that $u = e^{\lambda t}x$ solves

$$du/dt = \begin{bmatrix} 4, 3 \\ 0, 1 \end{bmatrix} u$$

What combination $u = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$ starts from $u(0) = (5, -2)$.
SOLUTION

we first find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4, 3 \\ 0, 1 \end{bmatrix}$

$\lambda_1 = 1, x_1 = (1, -1)$ and $\lambda_2 = 4, x_2 = (1, 0)$
We construct our solution

$$u(t) = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{4t}$$

and choose C_1, C_2 to enforce the condition that $u(0) = (5, -2)$

$$u(0) = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

so that $C_1 = 2, C_2 = 3$

4 If $v(0) = 30, w(0) = 10$ and they are modeled by the differential equations ($v' = dv/dt$)

$$v' = w - v, \quad w' = v - w$$

Show that $v + w$ is constant ($= 40$). Find the matrix in A that models you would use to rewrite this problem as

$$u' = Au$$

and find its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

SOLUTION

$v + w$ can be shown to be constant by taking its derivative,

$$(v + w)' = v' + w' = w - v + v - w = 0$$

since its derivative is equal to 0, it is a constant, so that

$$v(t) + w(t) = v(0) + w(0) = 10 + 30 = 40$$

With

$$A = \begin{bmatrix} -1, 1 \\ 1, -1 \end{bmatrix}$$

the problem can be rewritten as $u' = Au$ where $u = (v, w)$. Solving for A's eigenvalues and eigenvectors we find

$$\lambda_1 = 0, \; x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = -2, \; x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

this gives us the solution

$$u(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

solving for C_1, C_2 using $u(0) = (30, 10)$ we find

$$u(0) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$$

so that

$$u(t) = 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 10 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

so that $u(1) = 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 10 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2}$ and $u(\infty) = 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

6.4: 2,4,5,7

2 If C is symmetric prove that A^TCA is also symmetric (transpose it). When A is 6 by 3 what are the shapes of C and A^TCA?

SOLUTION

Being symmetric means $C^T = C$ so since

$$(A^TCA)^T = A^TC^T(A^T)^T = A^TCA$$

we find that A^TCA is symmetric.

if A is 6x3, this means that C must be 6x6 (otherwise matrix multiplication wouldn't make sense), and so $A^TCA = (3x6) * (6x6) * (6x3) = (3x6) * (6x3) = 3x3$ matrix

4 Find an orthogonal matrix Q that diagonalizes $A = \begin{bmatrix} -2, 6 \\ 6, 7 \end{bmatrix}$. What is Λ

SOLUTION

we first find the eigenvalues and eigenvectors

$$\lambda_1 = 10,\ x_1 = (1,2) \quad \lambda_2 = -5,\ x_2 = (-2,1)$$

normalizing these eigenvectors we define the matrix

$$Q = 1/\sqrt{5} \begin{bmatrix} 1, -2 \\ 2, 1 \end{bmatrix}, \quad Q^{-1} = 1/\sqrt{5} \begin{bmatrix} 1, 2 \\ -2, 1 \end{bmatrix}$$

we compute $Q^{-1}AQ$ to find

$$\Lambda = \begin{bmatrix} 10, 0 \\ 0, -5 \end{bmatrix}$$

5 Find an orthogonal matrix Q that diagonalizes this symmetric matrix:

$$A = \begin{bmatrix} 1, 0, 2 \\ 0, -1, -2 \\ 2, -2, 0 \end{bmatrix}$$

SOLUTION

we first find the eigenvalues and eigenvectors

$$\lambda_1 = -3,\ x_1 = (-1,2,2)) \quad \lambda_2 = 3,\ x_2 = (2,-1,2), \quad \lambda_3 = 0,\ x_3 = (-2,-2,1)$$

normalizing the eigenvectors and placing them in Q gives us

$$Q = 1/3 \begin{bmatrix} -1, 2, -2 \\ 2, -1, -2 \\ 2, 2, 1 \end{bmatrix}, \quad Q^{-1} = 1/3 \begin{bmatrix} -1, 2, 2 \\ 2, -1, 2 \\ -2, -2, 1 \end{bmatrix}$$

computing gives us

$$\Lambda = Q^{-1}AQ = \begin{bmatrix} -3, 0, 0 \\ 0, 3, 0 \\ 0, 0, 0 \end{bmatrix}$$

7 (a) Find an symmetric marix $\begin{bmatrix} 1, b \\ b, 1 \end{bmatrix}$ that has a negative eigenvalue.

SOLUTION

The eigenvalues of this matrix solve the equation

$$\det(A - \lambda I) = (1 - \lambda)^2 - b^2 = 0$$

so that $\lambda = 1 \pm b$ which always has a negative eigenvalue whenever $|b| > 1$.

(b) How do you know it must have a negative pivot?

SOLUTION

On page 333 it states that # of positive eigenvalues of $A = A^T$ is equal to the number of its positive pivots.

(c) How do you know it can't have two negative eigenvalues?

SOLUTION

Because the determinant of A must equal the product of its eigenvalues and

$$1 - b^2 < 0$$

for $|b| > 1$ (which we are assuming from part a), this means that the two eigenvalues must have opposite signs.