THE TACOMA NARROWS BRIDGE

A mathematical model that attempts to capture the Tacoma Narrows Bridge incident was proposed recently by McKenna and Tuama [18]. The goal is to explain how torsional, or twisting, oscillations can be magnified by forcing that is strictly vertical.

Consider a roadway of width $2l$ hanging between two suspended cables, as in Figure 6.18(a). We will consider a two-dimensional slice of the bridge, ignoring the dimension of the bridge's length for this model, since we are only interested in the side-to-side motion. At rest, the roadway hangs at a certain equilibrium height due to gravity; let y denote the current distance the center of the roadway hangs below this equilibrium.

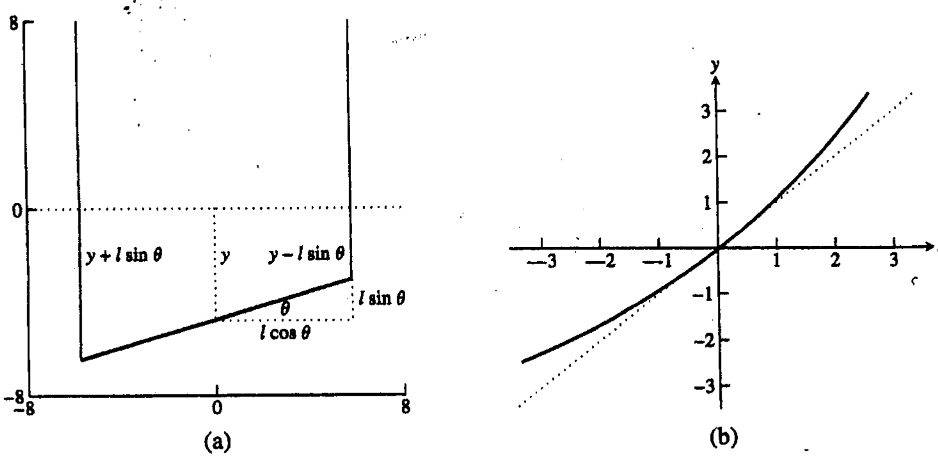

Figure 6.18 Schematics for the McKenna-Tuama model of the Tacoma Narrows Bridge. (a) Denote the distance from the roadway center of mass to its equilibrium position by y, and the angle of the roadway with the horizontal by θ. (b) Exponential Hooke's Law curve $f(y) = (K/a)(e^{ay} - 1)$.

Hooke's Law postulates a linear response, meaning that the restoring force the cables apply will be proportional to the deviation. Let θ be the angle the roadway makes with the horizontal. There are two suspension cables, stretched $y - l\sin\theta$ and $y + l\sin\theta$ from equilibrium, respectively. Assume that a viscous damping term is given that is proportional to the velocity. Using Newton's law $F = ma$ and denoting Hooke's constant by K, the equations of motion for y and θ are as follows:

$$
\begin{aligned}
y'' &= -dy' - \left[\frac{K}{m}(y - l\sin\theta) + \frac{K}{m}(y + l\sin\theta)\right] \\
\theta'' &= -d\theta' + \frac{3\cos\theta}{l}\left[\frac{K}{m}(y - l\sin\theta) - \frac{K}{m}(y + l\sin\theta)\right].
\end{aligned}
$$

However, Hooke's law is designed for springs, where the restoring force is more or less equal whether the springs are compressed or stretched. McKenna and Tuama hypothesize that cables pull back with more force when stretched than they push back when compressed. (Think of a string as an extreme example.) They replace the linear Hooke's Law restoring force $f(y) = Ky$ with a nonlinear force $f(y) = (K/a)(e^{ay} - 1)$, as shown in Figure 6.18(b). Both functions have the same slope K at $y = 0$; but for the nonlinear force, a positive y (stretched cable) causes a stronger restoring force than the corresponding negative y (slackened cable). Making this replacement in the preceding equations yields

$$
\begin{aligned}
y'' &= -dy' - \frac{K}{ma}\left[e^{a(y-l\sin\theta)} - 1 + e^{a(y+l\sin\theta)} - 1\right] \\
\theta'' &= -d\theta' + \frac{3\cos\theta}{l}\frac{K}{ma}\left[e^{a(y-l\sin\theta)} - e^{a(y+l\sin\theta)}\right].
\end{aligned}
\tag{6.54}
$$

As the equations stand, the state $y = y' = \theta = \theta' = 0$ is an equilibrium. Now turn on the wind. Add the forcing term $0.2W \sin\omega t$ to the right-hand side of the y equation, where W is the wind speed in km/hr. This adds a strictly vertical oscillation to the bridge.

Useful estimates for the physical constants can be made. The mass of a one-foot length of roadway was about 2500 kg, and the spring constant K has been estimated at 1000 Newtons. The roadway was about 12 meters wide. For this simulation, the damping coefficient was set at $d = 0.01$, and the Hooke's nonlinearity coefficient $a = 0.2$. An observer counted 38 vertical oscillations of the bridge in one minute shortly before the collapse—set $\omega = 2\pi(38/60)$. These coefficients are only guesses, but they suffice to show ranges of motion that tend to match photographic evidence of the bridge's final oscillations. MATLAB code that runs the model (6.54) is as follows:

```
%Program 6.6 Animation program for bridge using IVP solver
%Inputs: int=[a b] time interval,
%ic=[y(1,1) y(1,2) y(1,3) y(1,4)],
%h=stepsize, p=steps per point plotted
%Calls a one-step method such as trapstep.m
%Example usage: tacoma([0 1000],[1 0 0.001 0],.04,3)
function tacoma(inter,ic,h,p)
clf                          % clear figure window
a=inter(1);b=inter(2);n=ceil((b-a)/(h*p)); % plot n points
y(1,:)=ic;                   % enter initial conds in y
t(1)=a;len=6;
set(gca,'XLim',[-8 8],'YLim',[-8 8], ...
   'XTick',[-8 0 8],'YTick',[-8 0 8], ...
   'Drawmode','fast','Visible','on','NextPlot','add');


  cla;                         % clear screen
  axis square                  % make aspect ratio 1-1
  road=line('color','b','LineStyle','-','LineWidth',5,...
       'erase','xor','xdata',[],'ydata',[]);
  lcable=line('color','r','LineStyle','-','LineWidth',1,...
       'erase','xor','xdata',[],'ydata',[]);
  rcable=line('color','r','LineStyle','-','LineWidth',1,...
       'erase','xor','xdata',[],'ydata',[]);
  for k=1:n
    for i=1:p
      t(i+1)=t(i)+h;
      y(i+1,:)=trapstep(t(i),y(i,:),h);
    end
    y(1,:)=y(p+1,:);t(1)=t(p+1);
    z1(k)=y(1,1);z3(k)=y(1,3);
    c=len*cos(y(1,3));s=len*sin(y(1,3));
    set(road,'xdata',[-c c],'ydata',[-s-y(1,1) s-y(1,1)])
    set(lcable,'xdata',[-c -c],'ydata',[-s-y(1,1) 8])
    set(rcable,'xdata',[c c],'ydata',[s-y(1,1) 8])
    drawnow; pause(h)
  end

  function y=trapstep(t,x,h)
  %one step of the Trapezoid Method
  z1=ydot(t,x);
  g=x+h*z1;
  z2=ydot(t+h,g);
  y=x+h*(z1+z2)/2;

  function ydot=ydot(t,y)
  len=6;a=0.2; W=80; omega=2*pi*38/60;
  a1=exp(a*(y(1)-len*sin(y(3))));
  a2=exp(a*(y(1)+len*sin(y(3))));
  ydot(1)=y(2);
  ydot(2)=-0.01*y(2)-0.4*(a1+a2-2)/a+0.2*W*sin(omega*t);
  ydot(3)=y(4);
  ydot(4)=-0.01*y(4)+1.2*cos(y(3))*(a1-a2)/(len*a);
```

This problem and the following activities ~~is~~ are borrowed from Sauer.

1. Run `tacoma.m` with wind speed $W = 80$ km/hr and initial conditions $y = y' = \theta' = 0$, $\theta = 0.001$. The bridge is stable in the torsional dimension if small disturbances in θ die out; unstable if they grow far beyond original size. Which occurs for this value of W?
2. Replace the trapezoid method by fourth-order Runge Kutta to improve accuracy. Also, add new figure windows to plot $y(t)$ and $\theta(t)$.
3. The system is torsionally stable for $W = 50$ km/hr. Find the magnification factor for a small initial angle. That is, set $\theta(0) = 10^{-3}$ and find the ratio of the maximum angle $\theta(t)$, $0 \le t < \infty$, to $\theta(0)$. Is the magnification factor approximately consistent for initial angles $\theta(0) = 10^{-4}, 10^{-5}, \ldots$?
4. Find the minimum wind speed W for which a small disturbance $\theta(0) = 10^{-3}$ has a magnification factor of 100 or more. Can a consistent magnification factor be defined for this W?
5. Design and implement a method for computing the minimum wind speed in Step 4, to within 0.5×10^{-3} km/hr. You may want to use an equation solver from Chapter 1.
6. Try some larger values of W. Do all extremely small initial angles eventually grow to catastrophic size?

This project is an example of experimental mathematics. The equations are too difficult to derive closed-form solutions, and even too difficult to prove qualitative results about. Equipped with reliable ODE solvers, we can generate numerical trajectories for various parameter settings to illustrate the types of phenomena available to this model. Used in this way, differential equation models can predict behavior and shed light on mechanisms in scientific and engineering problems.

<u>**DUE DATE :**</u> *Monday,* *May* 28*th*, 2014 (*Classtime*)

<u>**Absolute Deadline :**</u> *Wednesday* *May* 30*th*, 2014 (*Classtime*)
There will be no credit for programs handed in after this date.