Obs & Stats HW 5

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Gaussian Point Spread Functions (Jointly Gaussian Random Variables)

1. First, simplify the matrix terms:

$$|P| = \sqrt{ac - b^2} \tag{1}$$

$$\begin{aligned} z^T P^{-1} z &= \begin{pmatrix} x & y \end{pmatrix} \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{1}{ac - b^2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} cx - by \\ -bx + ay \end{pmatrix} \\ &= \frac{1}{ac - b^2} (cx^2 - 2bxy + ay^2) \end{aligned} \tag{2}$$

Then substitute into the PDF:

$$f_{XY}(x, y) = \left(\frac{1}{2\pi \sqrt{ac - b^2}} \right) \exp \left[-\frac{cx^2 - 2bxy + ay^2}{2(ac - b^2)} \right] \tag{3}$$

Obtain $f_X(x)$ by integrating over all y:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} dy \ f_{XY}(x, y) \\ &= \frac{1}{2\pi \sqrt{ac - b^2}} \int_{-\infty}^{\infty} dy \ \exp \left[-\frac{acx^2 - 2abxy + a^2 y^2}{2a(ac - b^2)} \right] \\ &= \frac{1}{2\pi \sqrt{ac - b^2}} \int_{-\infty}^{\infty} dy \ \exp \left[-\frac{acx^2 - 2abxy + a^2 y^2 - b^2 x^2 + b^2 x^2}{2a(ac - b^2)} \right] \\ &= \frac{1}{2\pi \sqrt{ac - b^2}} \int_{-\infty}^{\infty} dy \ \exp \left[-\frac{(ay - bx)^2 + (ac - b^2) x^2)}{2a(ac - b^2)} \right] \\ &= \frac{1}{2\pi \sqrt{ac - b^2}} \int_{-\infty}^{\infty} dy \ \exp \left[-\frac{a(y - \frac{b}{a} x)^2}{2(ac - b^2)} - \frac{x^2}{2a} \right] \\ &= \frac{1}{2\pi \sqrt{ac - b^2}} \exp \left[-\frac{x^2}{2a} \right] \int_{-\infty}^{\infty} dy \ \exp \left[-\frac{a(y - \frac{b}{a} x)^2}{2(ac - b^2)} \right] \\ &= \frac{1}{2\pi \sqrt{ac - b^2}} \exp \left[-\frac{x^2}{2a} \right] \sqrt{\frac{2\pi (ac - b^2)}{a}} \\ &= \frac{1}{\sqrt{2\pi a}} \exp \left[-\frac{x^2}{2a} \right] \end{aligned} \tag{4}$$

This is a Gaussian with mean $\mu_X = 0$ and variance $\sigma_X^2 = a$.

2. The mean E$[x]$ and variance E$[x^2]$ can be obtained as follows:

$$\begin{aligned} \mathrm{E}[x] &= \int_{-\infty}^{\infty} dx\ x f_X(x) & \mathrm{E}[x^2] &= \int_{-\infty}^{\infty} dx\ x^2 f_X(x) \\ &= \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} dx\ x \exp\left[-\frac{x^2}{2a}\right] & &= \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} dx\ x^2 \exp\left[-\frac{x^2}{2a}\right] \\ &= \frac{1}{\sqrt{2\pi a}}\ 0 = 0 & &= \frac{1}{\sqrt{2\pi a}} \frac{2a}{2} \sqrt{2\pi a} = a \end{aligned} \tag{5}$$

As expected, the mean is 0 and the variance is a.

3. The procedure for obtaining $f_Y(y)$ is the same as for $f_X(x)$: complete the square, separate terms, simplify, and integrate. The only difference is an exchange of a with c, and x with y. Thus, we can write down $f_Y(y)$, E$[y]$, and E$[y^2]$:

$$f_Y(y) = \frac{1}{\sqrt{2\pi c}} \exp\left[-\frac{y^2}{2c}\right] \qquad \mathrm{E}[y] = 0 \qquad \mathrm{E}[y^2] = c \tag{6}$$

Simple Error Propagation for People Sick of Gaussians By Now

1. The variance σ_z^2 is obtained from the matrix equation $C_z = TCT^T$, where $T = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix}$:

$$C_z = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} y & x \end{pmatrix} \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} \tag{8}$$

$$= y^2\sigma_x^2 + x^2\sigma_y^2 = \sigma_z^2 \tag{9}$$

2. Starting with the definition of the variance,

$$\begin{aligned} V[z] &= E[(z-\mu_z)^2] \\ &= E[z^2] - E[z]^2 \\ &= E[x^2y^2] - E[xy]^2 \\ &= E[x^2]E[y^2] - E[x]^2E[y]^2 \\ &= E[x^2]E[y^2] - E[x]^2E[y^2] + E[x]^2E[y^2] - E[x]^2E[y]^2 \\ &= V[x]E[y^2] + E[x^2]V[y] \\ &= V[x]E[y^2] + E[x^2]V[y] - V[x]E[y]^2 + V[x]E[y]^2 \\ &= V[x]V[y] + E[x^2]V[y] + V[x]E[y]^2 \\ &= V[x]V[y] + E[x^2]V[y] + V[x]E[y]^2 - E[x]^2V[y] + E[x]^2V[y] \\ &= V[x]V[y] + V[x]E[y]^2 + E[x]^2V[y] \\ &= \sigma_x\sigma_y + \sigma_x\mu_y^2 + \mu_x^2\sigma_y \end{aligned} \tag{10}$$

This is not generally equivalent to the expression obtained in part 1. However, if we assume that the variances are small compared to the squares of the means, and let $\mu_x = x$ and $\mu_y = y$, we recover the approximate form obtained in part 1.