Obs & Stats HW 4

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24 February 2017

Adding Poisson Distributions

1. Let $m = j + k$, then the distribution of the sum is

$$f_{AB}(j,k;\lambda_{AB}) = \sum_{j=0}^{m} f_A(j) f_B(m-j) \tag{1}$$

$$= \sum_{j=0}^{m} \frac{\lambda_A^j}{j!} e^{-\lambda_A} \frac{\lambda_B^{m-j}}{(m-j)!} e^{-\lambda_B} \tag{2}$$

$$= e^{-(\lambda_A+\lambda_B)} \sum_{j=0}^{m} \frac{\lambda_A^j \lambda_B^{m-j}}{j!(m-j)!} \tag{3}$$

Use the binomial formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \tag{4}$$

with $a = \lambda_A$, $b = \lambda_B$, $n = m$, and $k = j$, multiplying top and bottom of Eq. 3 by $m!$:

$$f_{AB}(j,k;\lambda_{AB}) = e^{-(\lambda_A+\lambda_B)} \sum_{j=0}^{m} \frac{\lambda_A^j \lambda_B^{m-j} m!}{j!(m-j)!m!} \tag{5}$$

$$= e^{-(\lambda_A+\lambda_B)} \frac{(\lambda_A+\lambda_B)^m}{m!} \tag{6}$$

$$= \frac{\lambda_{AB}^{j+k}}{(j+k)!} e^{-\lambda_{AB}} \tag{7}$$

2. The Poisson distribution describes the likelihood of rare events. One possible situation where a joint Poisson distribution might be encountered is in the probability of conjunctions between different object types. For example, the supernova Refsdal was a conjunction of a supernova and a massive galaxy cluster.

Uniform Circular Joint Probability Distribution

1. The 68% confidence interval is located at the radius $r^2 = 0.68$, or $r_{68} = 0.825$. Similarly, the 95% confidence interval is located at $r_{95} = 0.975$.

2. The 68% marginal confidence interval for $f(x)$ is found by first integrating $f(x, y)$ over all y to obtain $f(x)$:

$$f(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dx = \frac{2}{\pi}\sqrt{1-x^2} \tag{8}$$

   Then integrate $f(x)$ over the desired interval:

$$\begin{aligned} \int_{-x_{68}}^{x_{68}} f(x)dx &= 2\int_0^{x_{68}} \frac{2}{\pi}\sqrt{1-x^2}dx & (9) \\ &= \frac{4}{\pi}\int_0^{x_{68}} \sqrt{1-x^2}dx & (10) \\ &= \frac{2}{\pi}\left(x\sqrt{1-x^2} + \arcsin x\right) & (11) \end{aligned}$$

   Finally, solve numerically or graphically for x_{68}:

$$\begin{aligned} 0.68 &= \frac{2}{\pi}\left(x\sqrt{1-x^2} + \arcsin x\right) & (12) \\ x &= 0.566 & (13) \end{aligned}$$

3. The conditional confidence interval can be found by integrating the conditional probability:

$$\begin{aligned} 0.68 &= \int_{-x_{68}}^{x_{68}} f(x|y=y_0)dx = \int_{-x_{68}}^{x_{68}} \frac{f(x,y)}{f(y_0)}dx & (14) \\ &= \int_{-x_{68}}^{x_{68}} \frac{\frac{1}{\pi}}{\frac{2}{\pi}\sqrt{1-y_0^2}}dx & (15) \\ &= \int_0^{x_{68}} \frac{1}{\sqrt{1-y_0^2}}dx & (16) \\ &= \frac{x_{68}}{\sqrt{1-y_0^2}} & (17) \\ \implies x_{68} &= 0.68\sqrt{1-y_0^2} & (18) \end{aligned}$$

   At $y_0 = 0$, the conditional 68% confidence interval is $[-0.68, 0.68]$, as expected.

Covariance Associated with Rotation

1. Use the vector generalized form for the covariance matrix, along with the fact that X_1 and X_2 are uncorrelated:

$$\begin{aligned}
\Sigma_Y &= \mathrm{E}\left[YY^T\right] = \mathrm{E}\left[(RX)(RX)^T\right] \\
&= \mathrm{E}\left[RXX^TR^T\right] = R\ \mathrm{E}\left[XX^T\right]R^T \\
&= R\ \mathrm{E}\left[\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \end{pmatrix}\right]R^T \\
&= R\ \mathrm{E}\left[\begin{pmatrix} X_1^2 & X_1X_2 \\ X_1X_2 & X_2^2 \end{pmatrix}\right]R^T \\
&= R\begin{pmatrix} \sigma_{X_1}^2 & 0 \\ 0 & \sigma_{X_2}^2 \end{pmatrix}R^T \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \sigma_{X_1}^2 & 0 \\ 0 & \sigma_{X_2}^2 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \sigma_{X_1}^2\cos\theta & \sigma_{X_1}^2\sin\theta \\ -\sigma_{X_2}^2\sin\theta & \sigma_{X_2}^2\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \sigma_{X_1}^2\cos^2\theta + \sigma_{X_2}^2\sin^2\theta & (\sigma_{X_1}^2 - \sigma_{X_2}^2)\sin\theta\cos\theta \\ (\sigma_{X_1}^2 - \sigma_{X_2}^2)\sin\theta\cos\theta & \sigma_{X_1}^2\sin^2\theta + \sigma_{X_2}^2\cos^2\theta \end{pmatrix}
\end{aligned} \tag{19}$$

2. There are two special sets of conditions for which Y_1 and Y_2 are uncorrelated. First, if $\theta = \frac{n\pi}{2}$, n odd, then the rotation simply exchanges $\sigma_{X_1}^2$ and $\sigma_{X_2}^2$; if θ is an integer multiple of π, there is no effect. Second, if $\sigma_{X_1}^2 = \sigma_{X_2}^2$, then any rotation has no visible effect.

3. For ground-based astrometry, the independent variables are azimuth and elevation, and the dependent variables are RA and DEC. Since rotations from ALT/AZ to RA/DEC are almost never a half-integer multiple of π, the only way for the uncertainties to remain uncorrelated under rotation is if they are equal. Since positional uncertainties are not necessarily of equal magnitude, the uncertainties in RA/DEC will not, in general, be uncorrelated. Thus, care must be taken to account for this when estimating the uncertainties of object positions in the image.