BRUHAT DECOMPOSITION

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Theorem. *The general linear group $GL_n(\mathbb{R})$ is the union of double cosets BPB, where B is the group of upper triangular matricies and P is a permutation matrix.*

1. Describing $GL_n(\mathbb{R})$

Proposition 1.1. *Let $n \in \mathbb{N}$ and let $m \leq n$. For all j such that $1 \leq j \leq m$ let $A_j := (a_{1,j}, \ldots, a_{n,j})$ such that $A_1, \ldots, A_m$ are m linearly independent vectors of n elements. For all j such that $1 \leq j \leq m-1$ let $\mathfrak{a}_j := (a_{i(1),j}, \ldots, a_{i(m-1),j})$ for some $i(1), \ldots, i(m-1)$ such that $\mathfrak{a}_1, \ldots, \mathfrak{a}_{m-1}$ are $m-1$ linearly independent vectors of $m-1$ elements.*

Let k be such that $1 \leq k \leq m$ and $k \neq i(j)$ for all $1 \leq j \leq m-1$. For all j such that $1 \leq j \leq m$, if $i(l) \leq k \leq i(l+1)$ for some $0 \leq l \leq m-1$, then define

$$a_j(k) := (a_{i(1),j}, \ldots, a_{i(l),j}, a_{k,j}, a_{i(l+1),j}, \ldots, a_{i(m-1),j}).$$

Then there exists k such that $a_1(k), \ldots, a_m(k)$ are m linearly independent vectors of m elements.

Proof. Let $k \neq i(j)$ for all $1 \leq j \leq m-1$. Suppose $a_1(l), \ldots, a_m(l)$ are linearly dependent for all $l \neq k$. Then there exist scalars $x_1(l), \ldots, x_m(l)$ not all zero such that

$$x_m(l)a_m(l) = \sum_{1 \leq j \leq m-1} x_j(l)a_j(l).$$

Claim. $x_m(l) \neq 0$ for all $l \neq k$.

Proof. We distinguish two cases :

(1) $\sum\limits_{1 \leq j \leq m-1} x_j(l)a_j(l) \neq 0 \implies x_m(l)a_m(l) \neq 0 \implies x_m(l) \neq 0$

(2) Since $\mathfrak{a}_1, \ldots, \mathfrak{a}_{m-1}$ are linearly independent

$$\begin{aligned} \sum_{1 \leq j \leq m-1} x_j(l)a_j(l) = 0 \quad &\implies \quad \textstyle\sum_{1 \leq j \leq m-1} x_j(l)\mathfrak{a}_j = 0 \\ &\implies \quad x_j(l) = 0 \text{ for all } 1 \leq j \leq m-1 \\ &\implies \quad x_m(l) \neq 0 \end{aligned}$$

$\square$

Define $\mathfrak{a}_m := (a_{i(1),m}, \ldots, a_{i(m-1),m})$. Then $a_m(l) = \sum_{1 \leq j \leq m-1} \frac{x_j(l)}{x_m(l)} a_j(l)$ implies $\mathfrak{a}_m = \sum_{1 \leq j \leq m-1} \frac{x_j(l)}{x_m(l)} \mathfrak{a}_j$.

Claim. For all $1 \leq j \leq m-1$ if $l_1, l_2 \neq k$, then $\frac{x_j(l_1)}{x_m(l_1)} = \frac{x_j(l_2)}{x_m(l_2)}$.

Proof. Since $\mathfrak{a}_1, \ldots, \mathfrak{a}_{m-1}$ are linearly independent

$$\sum_{1 \le j \le m-1} \left(\frac{x_j(l_1)}{x_m(l_1)} - \frac{x_j(l_2)}{x_j(l_2)} \right) \mathfrak{a}_j = \mathfrak{a}_m - \mathfrak{a}_m = 0 \implies \frac{x_j(l_1)}{x_m(l_1)} = \frac{x_j(l_2)}{x_m(l_2)}.$$

□

For each $1 \le j \le m-1$ let $\frac{x_j}{x_m} = \frac{x_j(l)}{x_m(l)}$ for some l. Then $\frac{x_j}{x_m} = \frac{x_j(l)}{x_m(l)}$ for all l. Therefore, for each $a_{i,m}$ such that $i \neq k$

$$a_m(i) = \sum_{1 \le j \le m-1} \frac{x_j}{x_m} a_j(i) \implies a_{i,m} = \sum_{1 \le j \le m-1} \frac{x_j}{x_m} a_{i,j}.$$

But since $A_1, \ldots, A_m$ are linearly independent

$$A_m - \sum_{1 \le j \le m-1} \frac{x_j}{x_m} A_j \neq 0 \implies b_{k,m} := a_{k,m} - \sum_{1 \le j \le m-1} \frac{x_j}{x_m} a_{k,j} \neq 0.$$

Let $b_m(k) := a_m(k) - \sum_{1 \le j \le m-1} \frac{x_j}{x_m} a_j(k)$ so that $b_m(k) = (0, \ldots, 0, b_{k,m}, 0, \ldots, 0) \neq 0$. If $w_1, \ldots, w_m$ are such that

$$w_m b_m(k) + \sum_{1 \le j \le m-1} w_j a_j(k) = 0$$

then, since $b_{k,m} \notin \mathfrak{a}_j$

$$\begin{array}{rcl} \sum_{1 \le j \le m-1} w_j \mathfrak{a}_j & \implies & w_j = 0 \text{ for all } 1 \le j \le m-1 \\ & \implies & w_m b_m(k) = 0 \\ & \implies & w_m = 0 \\ & \implies & a_1(k), \ldots, a_{m-1}(k), b_m(k) \text{ are linearly independent.} \end{array}$$

Therefore, the vectors $a_1(k), \ldots, a_m(k)$ are linearly independent. □

Corollary 1.2. *Let $A \in GL_n(\mathbb{R})$. Let $m \le n$ and let $\mathfrak{a} \in GL_{m-1}(\mathbb{R})$ be a submatrix of A composed of the columns $1, \ldots, m-1$ and rows $i(1), \ldots, i(m-1)$ of A for some $1 \le i(1), \ldots, i(m-1) \le n$. Then*

- **(C):** *there exists $i(m)$ such that there exists a submatrix $a \in GL_m(\mathbb{R})$ of A composed of the columns $1, \ldots, m$ and rows $i(1), \ldots, i(m)$ of A.*

Proof. Let A_j be the j^{th} column of A for all $1 \le j \le m$. Since A is invertible, $A_1, \ldots, A_m$ are linearly independent. Let $\mathfrak{a}_j$ be the j^{th} column of $\mathfrak{a}$ for all $1 \le j \le m-1$. Since $\mathfrak{a}$ is invertible, $\mathfrak{a}_1, \ldots, \mathfrak{a}_{m-1}$ are linearly independent. By the proposition, there exists k such that $a_1(k), \ldots, a_m(k)$ are linearly independent. Let a be a matrix with columns $a_1(k), \ldots, a_m(k)$. Then a is invertible. □

The matrix $a|_0 := [\,]$ is trivially invertible. For all $1 \le j \le n$ let $a|_j \in GL_j(\mathbb{R})$ contain $a|_{j-1}$ and have the additional row $i(j)$ such that $i(j)$ is *maximal* with respect to property **C**.

Definition 1.3. The elements $a_{i(j),j}$ of A are called *characterizing*. If $a_{i,j}$ is such that $i > i(j)$ and for all $a_{i,k}$ such that $k < j$, $a_{i,k}$ is not characterizing, then $a_{i,j}$ is called *secondary*. If $a_{i,j}$ is not secondary, it is *primary*.

Note. The position of a characterizing element is unique with respect to its row or column.

The preceeding observations describe a method of finding the characterizing elements of a given matrix (the significance of which will become apparent in the following sections). That the characterizing elements are those elements whose row index $i(j)$ is maximal with respect to property **C** says that $i(j)$ is the greatest row that we may choose that will give an invertible submatrix, which we call $a|_j$. So, we take determinants of submatricies by adding the column j and then sequentially considering an additional row, beginning with the greatest row not in $a|_{j-1}$, until we find a submatrix with a non-zero determinant. By the corollary such a submatrix assuredly exists.

Example 1.4. Let A be the 5×5 matrix

$$\begin{bmatrix} 1 & 1 & 4 & 3 & 28 \\ 1 & 6 & 7 & 11 & 97 \\ 1 & 8 & 9 & 9 & 125 \\ 1 & 1 & 2 & 2 & 26 \\ 0 & 10 & 10 & 10 & 140 \end{bmatrix}.$$

A is invertible. First we attempt to find $a|_1$. To do so we consider the submatricies in column 1. The submatrix $[0]$ at row 5 has determinant 0. The submatrix $[1]$ at row 4 has determinant 1. Thus, $a|_1 = [1]$ and $a_{4,1}$ is characterizing.

Now we attempt to find $a|_2$; we consider the square submatricies containing columns 1, 2 and row 4. The submatrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 10 \end{bmatrix}$$

at row 5 has determinant 10, which is thus $a|_2$. Then $a_{5,2}$ is characterizing.

The submatricies containing columns 1, 2, 3 and rows 4, 5 have a zero determinant if they also contain row 3 or 2. The submatrix

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 2 \\ 0 & 10 & 10 \end{bmatrix}$$

with the additional row 1 has determinant 20. This submatrix is $a|_3$ and $a_{1,3}$ is characterizing.

Continuing in this manner we find that $a|_4$ is the submatrix containing the additional row 2, and that $a|_5$ necessarily contains the additional row 3. Hence, $a_{4,1}$, $a_{5,2}$, $a_{1,3}$, $a_{2,4}$, $a_{3,5}$ are the characterizing elements of the matrix A. By definition, the secondary elements are those whose row was considered but gave a non-invertible submatrix.

We may picture the position of these elements within the matrix thusly:

$$\begin{bmatrix} 1 & 1 & [4] & 3 & 28 \\ 1 & 6 & (7) & [11] & 97 \\ 1 & 8 & (9) & (9) & [125] \\ [1] & 1 & 2 & 2 & 26 \\ (0) & [10] & 10 & 10 & 140 \end{bmatrix}.$$

Elements in round brackets are secondary and elements in square brackets are characterizing.

2. The Matricies $[\alpha], P, [\beta]$

With A we associate a permutation $p \in S_n$ such that $p(j) = i(j)$ for each characterizing element $a_{i(j),j}$, $1 \leq j \leq n$.

Proposition 2.1. *For all* $1 \leq i, j \leq n$

(1) $a_{i,j}$ secondary $\Leftrightarrow p^{-1}(i) > j$ and $i > p(j)$
(2) $a_{i,j}$ primary $\Leftrightarrow p^{-1}(i) \leq j$ or $i \leq p(j)$.

Proof. For all $1 \leq i, j \leq n$

$$(1)\quad \begin{aligned} a_{i,j} \text{ secondary} &\Leftrightarrow i < i(j) = p(j) \text{ and } i \neq p(k) \text{ for all } 1 \leq k < j \\ &\Leftrightarrow i > p(j) \text{ and } p^{-1}(i) \neq k \text{ for all } 1 \leq k < j \\ &\Leftrightarrow i > p(j) \text{ and } p^{-1}(i) > j. \end{aligned}$$

(2) is the negation of (1) by definition.

□

Let P be the permutation matrix which permutes rows upon left multiplication as the permutation p. Then P permutes row indicies as p^{-1} and column indicies as p. Let $[\alpha_{i,j}]$ and $[\beta_{i,j}]$ be $n \times n$ matricies whose nonconstant elements are formal variables such that $\alpha_{i,j} = 0$ if $i > j$, and such that $\beta_{i,j} = 0$ if $i > j$ or $p(i) < p(j)$, and $\beta_{i,j} = 1$ if $i = j$. Then $\alpha_{i,j}$ is nonconstant if $i \leq j$, and $\beta_{i,j}$ is nonconstant if $i < j$ and $p(i) > p(j)$; equivalently, $\alpha_{i,p(j)}$ is nonconstant if $i \leq p(j)$, and $\beta_{p^{-1}(i),j}$ is nonconstant if $p^{-1}(i) < j$ and $i > p(j)$. Moreover,

$$[\alpha_{i,j}]P[\beta_{i,j}] = \left[\sum_{1\leq k\leq n} \alpha_{i,k}\beta_{p^{-1}(k),j}\right] = \left[\sum_{1\leq k\leq n} \alpha_{i,p(k)}\beta_{k,j}\right].$$

Finally, we may restrict the range of summation as

$$\sum_{1\leq k\leq n} \alpha_{i,k}\beta_{p^{-1}(k),j} = \sum_{i\leq k\leq n} \alpha_{i,k}\beta_{p^{-1}(k),j}$$

or as

$$\sum_{1\leq k\leq n} \alpha_{i,p(k)}\beta_{k,j} = \sum_{1\leq k\leq j} \alpha_{i,p(k)}\beta_{k,j}.$$

Example 2.2. Continuing from Example 1.4 we find that the permutation p associated with A is $(1\,4\,2\,5\,3)$. The matrix $[\alpha]$ is simply an upper-triangular matrix and does not depend on p. The matricies P and $[\beta]$ are given by

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \beta_{1,3} & \beta_{1,4} & \beta_{1,5} \\ 0 & 1 & \beta_{2,3} & \beta_{2,4} & \beta_{2,5} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

respectively. Note how the positions of the non-zero elements in P correspond with the characterizing elements of A. The product $[\alpha]P[\beta]$ is the matrix

$$\begin{bmatrix} \alpha_{1,4} & \alpha_{1,5} & \alpha_{1,1} + \alpha_{1,4}\beta_{1,3} + \alpha_{1,5}\beta_{2,3} \\ \alpha_{2,4} & \alpha_{2,5} & \alpha_{2,4}\beta_{1,3} + \alpha_{2,5}\beta_{2,3} \\ \alpha_{3,4} & \alpha_{3,5} & \alpha_{3,4}\beta_{1,3} + \alpha_{3,5}\beta_{2,3} \\ \alpha_{4,4} & \alpha_{4,5} & \alpha_{4,4}\beta_{1,3} + \alpha_{4,5}\beta_{2,3} \\ 0 & \alpha_{5,5} & \alpha_{5,5}\beta_{2,3} \end{bmatrix}$$

$$\begin{matrix} \alpha_{1,2}+\alpha_{1,4}\beta_{1,4}+\alpha_{1,5}\beta_{2,4} & \alpha_{1,3}+\alpha_{1,4}\beta_{1,5}+\alpha_{1,5}\beta_{2,5} \\ \alpha_{2,2}+\alpha_{2,4}\beta_{1,4}+\alpha_{2,5}\beta_{2,4} & \alpha_{2,3}+\alpha_{2,4}\beta_{1,5}+\alpha_{2,5}\beta_{2,5} \\ \alpha_{3,4}\beta_{1,4}+\alpha_{3,5}\beta_{2,4} & \alpha_{3,3}+\alpha_{3,4}\beta_{1,5}+\alpha_{3,5}\beta_{2,5} \\ \alpha_{4,4}\beta_{1,4}+\alpha_{4,5}\beta_{2,4} & \alpha_{4,4}\beta_{1,5}+\alpha_{4,5}\beta_{2,5} \\ \alpha_{5,5}\beta_{2,4} & \alpha_{5,5}\beta_{2,5} \end{matrix}\Bigg] .$$

3. The Equations

For all $1 \le j \le n$ define τ_j to be the mapping from the set $\{i|a_{i,j} \text{ primary}\}$ to the set of nonconstant $\alpha_{i,p(j)}$ and $\beta_{p^{-1}(i),j}$ for any i given by

$$\tau_j(i) := \begin{cases} \beta_{p^{-1}(i),j} & \text{if } p^{-1}(i) > j \text{ and } i > p(j) \\ \alpha_{i,p(j)} & \text{otherwise} \end{cases} .$$

Proposition 3.1. *τ_j is a bijection for all $1 \le j \le n$.*

Proof. Let $1 \le j \le n$. Suppose $a_{i,j}$ and $a_{r,s}$ are primary and $i \ne r$. Then $\alpha_{i,p(j)} \ne \alpha_{r,p(j)}$ and $p^{-1}(i) \ne p^{-1}(r)$ implies $\beta_{p^{-1}(i),j} \ne \beta_{p^{-1}(r),j}$. Furthermore, $\alpha_{i,j} \ne \beta_{r,j}$ for all i, r. Hence, τ_j is injective.

Let $\beta_{p^{-1}(i),j}$ be nonconstant. Then $p^{-1}(i) < j$ implies $a_{i,j}$ is primary, so that $\tau_j(i) = \beta_{p^{-1}(i),j}$. Let $\alpha_{i,p(j)}$ be nonconstant. Then $i \le p(j)$ implies $a_{i,j}$ is primary so that $\tau_j(i) \ne \beta_{p^{-1}(i),j}$. Then $\tau_j(i) = \alpha_{i,p(j)}$. Hence, τ_j is surjective.

Thus τ_j is bijective. □

Define $\mathcal{T}_j := \{\tau_j(i)|a_{i,j} \text{ primary}\}$.

Corollary 3.2. *For all $1 \le j \le n$, $\#\{i|a_{i,j}$ primary$\} = \#\mathcal{T}_j$.*

Define m_j to be this common number.

Let $A_j := [a_{i,j}|1 \le i \le n]^T$. Let E_j be the system of equations

$$A_j = \left[\sum_{1\le k\le n} \alpha_{i,k}\beta_{p^{-1}(k),j} \right]$$

in the variables $\mathcal{T}_j$. Similarly, let $A'_j := [a_{i,j} \text{ primary}|1 \le i \le n]^T$ and let E'_j be the system of equations

$$A'_j = \left[\sum_{1\le k\le n} \alpha_{i,k}\beta_{p^{-1}(k),j} \,\middle|\, a_{i,j} \text{ primary} \right]$$

in the variables $\mathcal{T}_j$. (We will sometimes consider the elements $a_{i,j}$ to be functions of the variables $\mathcal{T}_j$ or speak of the equation *associated* with a particular element.) Let E' be the aggregate system of equations of all E'_j.

Proposition 3.3. *The coefficient of $\tau_s(r)$ in the equation associated with $a_{i,j}$ inE' is $\partial a_{i,j}/\partial \tau_s(r)$.*

Proof. As $\sum_{1\le k\le n} \alpha_{i,k}\beta_{p^{-1}(k),j}$ is a polynomial with first order terms, the partial derivative with respect to a term gives its coefficient. □

Corollary 3.4. *The matrix*

$$\left[\frac{\partial a_{i,j}}{\partial \tau_j(r)}\right]$$

is the coefficient matrix for E'_j.

Therefore, by Cramer's rule there exists a unique solution to E_j' if the determinant of $[\partial a_{i,j}/\partial\tau_j(r)]$ is nonzero. (By 3.2 the determinant exists.) We analyze this matrix and its determinant presently.

For each $1 \le j \le n$ define

$$\sigma_j(i) := \begin{cases} \alpha_{i,i} & \text{if } \tau_j(i) = \beta_{p^{-1}(i),j} \\ 1 & \text{otherwise} \end{cases} .$$

Proposition 3.5. *For all* $1 \le i, j, r, s \le n$

(1) $\frac{\partial a_{i,j}}{\partial \tau_j(i)} = \sigma_j(i)$
(2) $\frac{\partial a_{i,j}}{\partial \tau_s(r)} = \sigma_j(i) \implies (r,s) = (i,j)$
(3) $\frac{\partial a_{i,j}}{\partial \tau_s(r)} = \sigma_s(r) \implies (i,j) = (r,s)$
(4) If $r < i$ or $s > j$, then $\frac{\partial a_{i,j}}{\partial \tau_s(r)} = 0$.

Proof. (1) The term $\tau_j(i)$ is coincident with $\sigma_j(i)$ in $a_{i,j}$ implying that the partial derivative with respect to $\tau_j(i)$ is $\sigma_j(i)$. (2) As $\tau_s(r)$ has as a coefficient either $\alpha_{i,i}$ or $1 = \beta_{j,j}$ and as $\tau_s(r)$ must occur in $a_{i,j}$ (since the partial derivative is nonzero), both $r = i$ and $s = j$. (3) As both $\sigma_s(r)$ and $\tau_s(r)$ must occur in $a_{i,j}$, both $i = r$ and $j = s$. (4) Either the index of $\tau_s(r)$ does not occur in $a_{i,j}$ or its coefficient is zero. □

Notation 3.6. If M is a matrix, then let $M[[i,\ldots,j;k,\ldots,l]]$ denote the matrix obtained from M by eliminating rows $i,\ldots,j$ and columns $k,\ldots,l$.

Proposition 3.7. *Let j be such that $1 \le j \le n$. Then*

$$det\left[\frac{\partial a_{i,j}}{\partial \tau_j(r)}\right] = \pm \prod_{1 \le k \le j-1} \alpha_{p(k),p(k)}^{s(p(k))}$$

where $s(k) = 1$ if $\beta_{p^{-1}(k),j}$ nonconstant and 0 otherwise.

Proof. Define the sequence $i := (n-k+1 | 1 \le k \le n$ and $a_{n-k+1,j}$ primary$)$. Let $i(x)$ denote the x^{th} term of i. Then $x > y$ implies $i(x) < i(y)$.

Let $C_j := [\partial a_{i,j}/\partial \tau_j(r)]$. We apply induction on the terms of i:

(B) $detC_j[[i(1),\ldots,i(m_j);i(1),\ldots,i(m_j)]] = det[\,] = 1$

(I) Let $0 \le k < m_j$ and assume the induction hypothesis for numbers greater than k. Then by row expansion

$$\begin{aligned} & detC_j[[i(1),\ldots,i(k);i(1),\ldots,i(k)]] \\ = \; & \pm\tfrac{\partial a_{i(k+1),j}}{\partial \tau_j(i(k+1))} detC_j[[i(1),\ldots,i(k+1);i(1),\ldots,i(k+1)]] \\ & \pm \sum_{k+2 \le l \le m_j} (-1)^l \tfrac{\partial a_{i(k+1),j}}{\partial \tau_j(i(l))} detC_j[[i(1),\ldots,i(k+1);i(1),\ldots,i(k),i(l)]] \\ = \; & \pm\tfrac{\partial a_{i(k+1),j}}{\partial \tau_j(i(k+1))} detC_j[[i(1),\ldots,i(k+1);i(1),\ldots,i(k+1)]] \\ = \; & \pm\sigma_j(i(k+1)) \cdots \sigma_j(i(m_j)). \end{aligned}$$

Step 2 follows because $l > k+1$ implies $i(l) < i(k+1)$ so that $\partial a_{i(k+1),j}/\partial \tau_j(i(l)) = 0$ by 3.5 (4); step 3 follows by the induction hypothesis and 3.5 (1). Hence

$$\begin{aligned} detC_j &= \pm \prod_{1\le k\le m_j} \sigma_j(i(k)) \\ &= \pm \prod_{1\le k\le n} \alpha_{k,k}^{s(k)} \\ &= \pm \prod_{1\le k\le n} \alpha_{p(k),p(k)}^{s(p(k))} \\ &= \pm \prod_{1\le k\le j-1} \alpha_{p(k),p(k)}^{s(p(k))} \end{aligned}$$

since $\sigma_j(p(k)) = \alpha_{p(k),p(k)}$ implies $\tau_j(p(k)) = \beta_{k,j}$ so that $k < j$. □

Notation 3.8. Let M be an $n \times n$ matrix. Let $M\langle i,j\rangle$ denote the element at position (i,j). Let $M\langle j\rangle$ denote the j^{th} column.

Define:

(1) $\rho(M\langle i,j\rangle) := M\langle i,j\rangle - \sum_{1\le k\le j-1} \rho(M\langle i,k\rangle)\beta_{k,j}$
(2) $\rho(M\langle j\rangle) := M\langle j\rangle - \sum_{1\le k\le j-1} \rho(M\langle k\rangle)\beta_{k,j}$
(3) $\rho(M)$ to be the matrix composed of columns $\rho(M\langle 1\rangle), \ldots, \rho(M\langle n\rangle)$.

Proposition 3.9. *For all* $1 \le i,j \le n$

(1) $detM = det\rho(M)$
(2) $\rho(M\langle j\rangle) = [\rho(M\langle 1,j\rangle), \ldots, \rho(M\langle n,j\rangle)]^T$
(3) *If there exists a unique solution to* $M\langle i,j\rangle = \sum_{1\le k\le n} \alpha_{i,k}\beta_{p^{-1}(k),j}$, *then* $\rho(M\langle i,j\rangle) = \alpha_{i,p(j)}$.

Proof. (1) $\rho(M)$ and M differ only by linear combinations of columns. (2) and (3) follow from induction. □

Proposition 3.10. *Suppose there exists a solution to the equations associated with all elements of* $a|_j$. *Then*

$$|det\, a|_j| = \alpha_{p(1),p(1)} \cdots \alpha_{p(j),p(j)}.$$

Proof. For all $1 \le r \le m$ let $i(r) = p(s)$ for some $s \in \{1,\ldots,j\}$ such that $i(1) < \cdots < i(m)$. We proceed by induction:

(B) $det\,\rho(a|_j)[[i(1),\ldots,i(j); p^{-1}(i(1)),\ldots,p^{-1}(i(j))]] = det[\,] = 1$

(I) Let $1 \le r \le m$ and assume the induction hypothesis for numbers greater than $r-1$. By column expansion

$$\begin{aligned} & det\,\rho(a|_j)[[i(1),\ldots,i(r-1); p^{-1}(i(1)),\ldots,p^{-1}(i(r-1))]] \\ = \; & \pm\rho(a|_j\langle i(r), p^{-1}(i(r))\rangle) det\,\rho(a|_j)[[i(1),\ldots,i(r); p^{-1}(i(1)),\ldots,p^{-1}(i(r))]] \\ & \pm \sum_{r+1\le k\le m} (-1)^k \rho(a|_j\langle i(k), p^{-1}(i(r))\rangle) \\ & \times det\,\rho(a|_j)[[i(1),\ldots,i(r-1),i(k); p^{-1}(i(1)),\ldots,p^{-1}(i(r))]] \\ = \; & \pm\rho(a|_j\langle i(r), p^{-1}(i(r))\rangle) det\,\rho(a|_j)[[i(1),\ldots,i(r); p^{-1}(i(1)),\ldots,p^{-1}(i(r))]] \\ = \; & \pm\rho(a|_j\langle i(r), p^{-1}(i(r))\rangle) \cdots \rho(a|_j\langle i(m), p^{-1}(i(m))\rangle) \end{aligned}$$

Step 2 follows because $k > r$ implies $i(k) > i(r)$ so that $\rho(a|_j\langle i(k), p^{-1}(i(r))\rangle) = \alpha_{i(k),i(r)} = 0$; step 3 follows from the induction hypothesis. Therefore,

$$\begin{aligned} |det\, a|_j| &= |det\rho(a|_j)| \\ &= \rho(a|_j\langle i(r), p^{-1}(i(r))\rangle) \cdots \rho(a|_j\langle i(m), p^{-1}(i(m))\rangle) \\ &= \alpha_{p(1),p(1)} \cdots \alpha_{p(m),p(m)}. \end{aligned}$$

□

Proposition 3.11. *Let B be a submatrix of A formed from $a|_{j-1}$ for some j by adding the additional row i and column j. Assume B has a unique solution to the equations associated with each of its elements except the equation associated with $a_{i,j}$. If $detB = 0$, then*

$$a_{i,j} = \sum_{1 \leq k \leq j-1} \alpha_{i,p(k)} \beta_{k,j}.$$

Proof. Following the proof of 3.10 we find that

$$\alpha_{p(1),p(1)} \cdots \alpha_{p(j-1),p(j-1)} \left(a_{i,j} - \sum_{1 \leq k \leq j-1} \alpha_{i,p(k)} \beta_{k,j} \right) = detB = 0;$$

whence $a_{i,j} = \sum_{1 \leq k \leq j-1} \alpha_{i,p(k)} \beta_{k,j}$. □

Theorem 3.12. *$A = [\alpha]P[\beta]$ has a unique solution.*

Proof. We proceed by induction:

(B) E_0 has the trivial solution.

(I) Assume E_k has a unique solution for all $1 \leq k \leq j-1$. Then the system of equations associated with $a|_{j-1}$ has a unique solution. Therefore by 3.10

$$\alpha_{p(1),p(1)} \cdots \alpha_{p(j-1),p(j-1)} = |det\, a|_{j-1}| \neq 0.$$

Hence, E'_j is solvable by Cramer's rule and 3.7.

For each secondary $a_{i,j}$ consider the square matrix $B_{i,j}$ formed from $a|_{j-1}$ by adding the additional row i and column j. As each E_k has a solution and as all elements in column j other than $a_{i,j}$ are primary, $B_{i,j}$ has solutions to the equations associated with all elements other than $a_{i,j}$. Since $a_{i,j}$ is secondary, $i > p(j)$. But since the rows of the characterizing elements were chosen to be maximal, this implies $detB_{i,j} = 0$. By 3.11, the equation $a_{i,j} = \sum_{1 \leq k \leq j-1} \alpha_{i,k} \beta_{k,j}$ has a unique solution. Therefore there are solutions for the equations associated with every primary and secondary $a_{i,j}$. Thus, E_j has a solution.

Whence, we conclude that the system of equations $A = [\alpha]P[\beta]$ has a unique solution. □

Let B be the group of upper triangular matricies. It is clear that $BPB \subset GL_n(\mathbb{R})$ for all permutation matricies P. As $A \in GL_n(\mathbb{R})$, $detA \neq 0$ so that $det[\alpha]$ and $det[\beta]$ are nonzero. Futhermore, $[\alpha]$ and $[\beta]$ are upper triangular matricies. Hence, $[\alpha]$ and $[\beta]$ are in the group B. Thus, 3.12 implies every element of $GL_n(\mathbb{R})$ is an element of some double coset BPB.

Example 3.13. We conclude the previous examples by finding explicitly the decomposition of A. The systems of equations E_1 and E_1' are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_{1,4} \\ \alpha_{2,4} \\ \alpha_{3,4} \\ \alpha_{4,4} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_{1,4} \\ \alpha_{2,4} \\ \alpha_{3,4} \\ \alpha_{4,4} \end{bmatrix}$$

respectively. The systems of equations E_2 and E_2' are both

$$\begin{bmatrix} 1 \\ 6 \\ 8 \\ 1 \\ 10 \end{bmatrix} = \begin{bmatrix} \alpha_{1,5} \\ \alpha_{2,5} \\ \alpha_{3,5} \\ \alpha_{4,5} \\ \alpha_{5,5} \end{bmatrix}.$$

The solutions to E_1 and E_2 are obvious. The system of equations E_3 is

$$\begin{bmatrix} 4 \\ 7 \\ 9 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} \alpha_{1,1} + \beta_{1,3} + \beta_{2,3} \\ \beta_{1,3} + 6\beta_{2,3} \\ \beta_{1,3} + 8\beta_{2,3} \\ \beta_{1,3} + \beta_{2,3} \\ 10\beta_{2,3} \end{bmatrix}$$

and E_3' is

$$\begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} \alpha_{1,1} + \beta_{1,3} + \beta_{2,3} \\ \beta_{1,3} + \beta_{2,3} \\ 10\beta_{2,3} \end{bmatrix}.$$

E_3' is a system of equations in the variables $\{\alpha_{1,1}, \beta_{1,3}, \beta_{2,3}\}$. The coefficient matrix C_3 is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 10 \end{bmatrix}$$

with determinant 10. So, E_3' is solvable. Cramer's rule gives the solution $\alpha_{1,1} = 2$, $\beta_{1,3} = 1$, $\beta_{2,3} = 1$. We may check as well that this is a solution to the equations associated with the secondary elements; that is, those equations not appearing in E_3'. Hence, the solution to E_3' is the solution to E_3.

The system of equations E_4 is

$$\begin{bmatrix} 3 \\ 11 \\ 9 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} \alpha_{1,2} + \beta_{1,4} + \beta_{2,4} \\ \alpha_{2,2} + \beta_{1,4} + 6\beta_{2,4} \\ \beta_{1,4} + 8\beta_{2,4} \\ \beta_{1,4} + \beta_{2,4} \\ 10\beta_{2,4} \end{bmatrix}$$

and the system of equations E_4' is

$$\begin{bmatrix} 3 \\ 11 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} \alpha_{1,2} + \beta_{1,4} + \beta_{2,4} \\ \alpha_{2,2} + \beta_{1,4} + 6\beta_{2,4} \\ \beta_{1,4} + \beta_{2,4} \\ 10\beta_{2,4} \end{bmatrix}$$

which is in the variables $\{\alpha_{1,2}, \alpha_{2,2}, \beta_{1,4}, \beta_{2,4}\}$. The coefficient matrix C_4 is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

which has determinant 10. The solution to this system of equations is $\alpha_{1,2} = 1$, $\alpha_{2,2} = 4$, $\beta_{1,4} = 1$, $\beta_{2,4} = 1$. Again, we may check that this is a solution to E_4.

Both E_5 and E_5' are the system of equations

$$\begin{bmatrix} 28 \\ 97 \\ 125 \\ 26 \\ 140 \end{bmatrix} = \begin{bmatrix} \alpha_{1,3} + \beta_{1,5} + \beta_{2,5} \\ \alpha_{2,3} + \beta_{1,5} + 6\beta_{2,5} \\ \alpha_{3,3} + \beta_{1,5} + 8\beta_{2,5} \\ \beta_{1,5} + \beta_{2,5} \\ 10\beta_{2,5} \end{bmatrix}$$

in the variables $\{\alpha_{1,3}, \alpha_{2,3}, \alpha_{3,3}, \beta_{1,5}, \beta_{2,5}\}$ with the coefficient matrix C_5 equal to

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

which has determinant 10. The solution to both systems of equations is $\alpha_{1,3} = 2$, $\alpha_{2,3} = 1$, $\alpha_{3,3} = 1$, $\beta_{1,5} = 12$, $\beta_{2,5} = 14$.

We conclude that

$$\begin{bmatrix} 1 & 1 & 4 & 3 & 28 \\ 1 & 6 & 7 & 11 & 97 \\ 1 & 8 & 9 & 9 & 125 \\ 1 & 1 & 2 & 2 & 26 \\ 0 & 10 & 10 & 10 & 140 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 0 & 4 & 1 & 1 & 6 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 & 12 \\ 0 & 1 & 1 & 1 & 14 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

4. Further Investigations

We now observe some properties of the system of equations E'. Its coefficient matrix is

$$\left[\frac{\partial a_{i,j}}{\partial \tau_s(r)}\right].$$

Proposition 4.1. *The determintant of the coefficient matrix of the aggregate system of equations E' is*

$$det\left[\frac{\partial a_{i,j}}{\partial \tau_s(r)}\right] = \prod_{1 \le k \le n} \alpha_{k,k}^{r(k)},$$

where $r(k)$ is the number of nonconstant $\beta_{p^{-1}(k),j}$. Moreover, $0 \le r(k) \le k-1$.

Proof. Define the double sequence $ij := ((n+1-k, j)|1 \leq j \leq n, 1 \leq k \leq n)$ such that $n+1-k \leq p(j)$ or $p^{-1}(n+1-k) \leq j$. Note that k goes through $1, \ldots, n$ for every j; *that is,* k increases first. Then $ij(x) = (i,j)$ and $ij(y) = (r,s)$ implies $r < i$ or $s > j$ for all $x > y$.

The remainder of the proof follows that of 3.7 but we note in addition that since $[\beta]$ is upper triangular with 1's along the diagonal, for a row k, there can be at most $k-1$ nonconstant $\beta_{k,j}$ and at least 0. $\square$

We would like to know how the determinant of the coefficient matrix is related to its associated permutation. We will use transpositions of consecutive numbers to generate permutations with any desired set of nonconstant β; that is, a permutation can be found which will give any determinant.

Let $R(i;p) := \{j|\beta_{p^{-1}(i),j}$ nonconstant under $p\}$ and let $r(i;p) := \#R(i;p)$.

Proposition 4.2. *Let $p \in S_n$. If there exist x and y such that $x < y$ and $p(x)+1 = p(y)$, and if $t := (p(x), p(y))$ is a transposition, then*

(1) $r(p(i); tp) = r(p(i); p)$ *for* $i \neq x$ *or* y
(2) $r(p(x); tp) = r(p(y); p)$
(3) $r(p(y); tp) = r(p(x); p) + 1$.

Proof. (1) Note that $tp(i) = p(i)$.

Suppose $p^{-1}(p(i)) < j$ and $p(i) > p(j)$ for some j. Then $(tp)^{-1}(p(i)) = (tp)^{-1}(tp(i)) < j$. And, $p(i) \geq tp(j)$ with equality only if $p(i) = p(x)$ or $p(y)$. So $p(i) > tp(j)$. Therefore, $R(p(i);p) \subset R(p(i);tp)$.

Suppose $(tp)^{-1}(p(i)) < j$ and $p(i) > tp(j)$ for some j. Then $p^{-1}(p(i)) < j$. And, $p(i) \geq p(j)$ with equality only if $p(i) = p(x)$ or $p(y)$. So $p(i) > p(j)$. Therefore $R(p(i);tp) \subset R(p(i);p)$.

Hence, $r(p(i);tp) = r(p(i);p)$.

(2) Suppose $p^{-1}(p(y)) < j$ and $p(y) > p(j)$ for some j. Then $(tp)^{-1}(p(x)) = (tp)^{-1}(tp(y)) < j$. And, $p(x) = tp(y) \geq p(j)$ with equality only if $j = x$. But since $x < y < j$, $tp(j) = p(j)$ and $p(x) > tp(j)$. Therefore $R(p(y);p) \subset R(p(x);tp)$.

Suppose $(tp)^{-1}(p(x)) < j$ and $p(x) > tp(j)$ for some j. Then $p^{-1}(p(y)) > j$. And $p(y) > p(x) \geq p(j)$. Therefore $R(p(x);tp) \subset R(p(y);p)$.

Hence, $r(p(x);tp) = r(p(y);p)$.

(3) Suppose $p^{-1}(p(x)) < j$ and $p(x) > p(j)$ for some j. Then $(tp)^{-1}(p(y)) < j$. And, $p(y) > p(x) \geq tp(j)$ and $j \neq y$. Therefore, $R(p(x);p) \subset R(p(y);tp) \setminus \{y\}$.

Suppose $(tp)^{-1}(p(y)) > j$ and $p(y) > tp(j)$ for some $j \neq y$. Then $p^{-1}(p(x)) < j$. And, $p(y) > p(j)$ with equality only if $j = y$. So $p(y) > p(j)$. But then $p(x)+1 = p(y)$ implies $p(x) \geq p(j)$ with equality only if $j = x$. So $p(x) > p(j)$. Therefore, $R(p(y);tp) \setminus \{y\} \subset R(p(x);p)$.

Now, $(tp)^{-1}(p(y)) = x < y$ and $p(y) > p(x) = tp(y)$ implies $y \in R(p(y);tp)$.

Hence, $r(p(y);tp) = r(p(x);p) + 1$. $\square$

Lemma 4.3. *Let p be a permutation such that $r(i;p) = 0$ for some $i \neq 1$. Then there exist x, y such that $x < y$ and $p(x) = i-1$ and $p(y) = i$.*

Proof. $r(i;p) = 0$ implies $p^{-1}(i) \geq j$ or $i \leq p(j)$ for all j. Let $p(x) = i-1$ and $p(y) = i$. Then $x \neq y$. Thus $p(y) > p(x)$ implies $x \leq p^{-1}(p(y)) = y$ and so $x < y$. $\square$

Proposition 4.4. *Let $r_1, \ldots, r_n$ be such that $0 \leq r_i \leq i-1$ for all $1 \leq i \leq n$. Then there exists a permutation $p \in S_n$ such that $r(i;p) = r_i$ for all $1 \leq i \leq n$.*

Proof. Define $p_i := (i, i-1)(i-1, i-2)\cdots(i-r_i+1, i-r_i)$ and define $t_k := (i-k, i-k+1)$, and $s_k := t_k t_{k+1} \cdots t_{r_{i-1}}$.

Claim. Let q be a permutation such that $r(l;q) = 0$ for $1 \le l \le i$. For all k, $r(i-k; s_k q) = r_i - k$ and $r(l; s_k q) = r(l;q)$ for all $l \neq i-k$.

Proof. By induction on k:

(B) $s_{r_i} = (1)$ implies $r(i-r_i; s_{r_i} q) = r(i-r(i); q) = 0 = r_i - r_i$, and, $r(l; s_{r_i} q) = r(l;q)$ for all $l \neq i - r_i$.

(I) Let $1 \le k \le i$ and assume the induction hypothesis for numbers less than k. Then by Lemma 4.3

$$r(i-k; s_{k-1} q) = r(i-k; q) = 0$$

implies there exists x, y such that $x < y$ and $s_{k-1}q(x) = i-k-1$ and $s_{k-1}q(y) = i-k$. By 4.2,

$$\begin{aligned} r(l; s_k q) &= r(l; t_k s_{k-1} q) \\ &= r(l; s_{k-1} q) \\ &= r(l; q) \end{aligned}$$

for $l \neq i-k-1, i-k$; and,

$$\begin{aligned} r(i-k-1; s_k q) &= r(i-k-1; t_k s_{k-1} q) \\ &= r(t_k(i-k); t_k s_{k-1} q) \\ &= r(i-k; s_{k-1} q) \\ &= r(i-k; q) \\ &= 0 \\ &= r(i-k-1; q). \end{aligned}$$

Finally,

$$\begin{aligned} r(i-k; s_k q) &= r(i-k; t_k s_{k-1} q) \\ &= r(t_k(i-k-1); t_k s_{k-1} q) \\ &= r(i-k-1; s_{k-1} q) + 1 \\ &= r_i - k - 1 + 1 \\ &= r_i - k. \end{aligned}$$

Therefore, $r(i; p_i q) = r_i$ and $r(l; p_i q) = r(l; q)$ for $l \neq i$. □

Define $q_i := p_i p_{i+1} \cdots p_n$.

Claim. For all i, $r(i; q_i) = r_i$ and $r(l; q_i) = 0$ for all $l < i$ and $r(l; q_i) = r(l; q_{i+1})$ for $l > i$.

Proof. By induction on i:

(B) $q_{n+1} = (1)$ implies $r(l; (1)) = 0$ for $1 \le l \le n$.

(I) Let $1 \le k \le n$ and assume the induction hypothesis for numbers greater than k. Then by the prior claim $r(l; q_{k+1}) = 0$ for $1 \le l \le k$ implies

$$r(k; q_k) = r(k; p_k q_{k+1}) = r_k;$$

and for $l \neq i$, $r(l; q_k) = r(l; p_k q_{k+1}) = r(l; q_{k+1})$ implies

$$r(l; q_k) = 0 \text{ for } 1 \le l \le k-1$$

and

$$r(l;q_k) = r(l;q_{k+1}) \text{ for } k+1 \le l \le n.$$

□

Therefore, $r(p;i) = r(q_1;i) = r_i$ for all i. □

Corollary 4.5. *There is a bijective correspondence between the set S_n of permutations and the set*

$$\left\{ \prod_{1 \le k \le n} \alpha_{kk}^{r_k} \,\middle|\, 0 \le r_k \le k-1 \right\}$$

given by the association of a permutation with the determinant of the coefficient matrix of the associated system of equations.

Given that $p(x) + 1 = p(y)$, we saw in 4.2 the effect of a transposition when $x < y$. We will now observe what happens when $x > y$. Notice the symmetry in the proofs.

Proposition 4.6. *Let $p \in S_n$. If there exists x and y such that $x > y$ and $p(x) + 1 = p(y)$, and if $t := (p(x), p(y))$ is a transposition, then*

(1) $r(p(i);tp) = r(p(i);p)$ *for* $i \ne x$ *or* y
(2) $r(p(x);tp) = r(p(y);p) - 1$
(3) $r(p(y);tp) = r(p(x);p)$.

Proof. (1) is the same as in 4.2.

(2) Suppose $p^{-1}(p(y)) < j$ and $p(y) > p(j)$ for some $j \ne x$. Then $(tp)^{-1}(p(x)) < j$. And, $p(x) \ge p(j)$ with equality only if $j = x$ implies $p(x) > p(j)$. But then $p(x) \ge tp(j)$ with equality only if $j = y$. So $p(x) > tp(j)$. Therefore $R(p(y);p) \smallsetminus \{x\} \subset R(p(x);tp)$.

Suppose $(tp)^{-1}(p(x)) < j$ and $p(x) > tp(j)$ for some j. Then $p^{-1}(p(y)) < j$. And, $p(y) > p(x) \ge p(j)$ and $j \ne x$. Therefore, $R(p(x);tp) \subset R(p(y);p) \smallsetminus \{x\}$.

Now, $p^{-1}(p(y)) < x$ and $p(y) > p(x)$. Then $x \in R(p(y);p)$.

Hence, $r(p(x);tp) = r(p(y);p) - 1$.

(3) Suppose $p^{-1}(p(x)) < j$ and $p(x) > p(j)$ for some j. Then $(tp)^{-1}(p(y)) < j$. And, $p(y) > p(x) \ge tp(j)$. Therefore, $R(p(x);p) \subset R(p(y);tp)$.

Suppose $(tp)^{-1}(p(y)) < j$ and $p(y) > tp(j)$ for some j. Then $p^{-1}(p(x)) < j$. And, $p(x) \ge tp(j)$ with equality only if $j = y$. But then $y < x < j$ implies $p(x) > tp(j)$. So, $p(x) \ge p(j)$ with equality only if $j = x$. Then $p(x) > p(j)$. Therefore $R(p(y);tp) \subset R(p(x);p)$.

Hence, $r(p(y);tp) = r(p(x);p)$. □

For all $p \in S_n$ define $r(p) := r(1;p) + \cdots + r(n;p)$.

Proposition 4.7. *$r(p)$ is the minimum length of a product of transpositions that can be equal to p.*

Proof. By 4.2 and 4.6 we see that $r(tp)$ of a composition of a transposition t with a permutation p will either be one more one less than $r(p)$. A product of no transpositions is length 0, which is $r((1))$. If a product p_k of k transpositions has $r(p_k) \le k$, then a product p_{k+1} of $k+1$ transpositions has $r(p_{k+1}) \le k+1$. Hence, by induction, if q is a product of n transpositions and $n < r(p)$ for some permutation p, then $q \ne p$. □

Example 4.8. We may extract from the proof of 4.4 a method of generating permutations from a given set of r_i. Suppose we wish to find the permutation $p \in S_8$ associated with the determinant $\alpha_{2,2}\alpha_{4,4}^3\alpha_{7,7}^4\alpha_{8,8}^2$. By the second claim

$$\begin{aligned} p &= (2\,1)(4\,3)(3\,2)(2\,1)(7\,6)(6\,5)(5\,4)(4\,3)(8\,7)(7\,6) \\ &= (2\,1)(4\,3\,2\,1)(7\,6\,5\,4\,3)(8\,7\,6) \\ &= (1\,4)(3\,7\,5)(6\,8). \end{aligned}$$

We note too that p is a product of $r(p) = 10$ transpositions of consecutive numbers.

If we compute the determinant associated with the permutation $(1\,4\,2\,5\,3)$ we used in our previous examples we find that it is $\alpha_{4,4}^3\alpha_{5,5}^3$; and truely

$$(4\,3)(3\,2)(2\,1)(5\,4)(4\,3)(3\,2) \quad = \quad (1\,4\,2\,5\,3).$$

5. References and Acknowledgements

I made use of no references other than *Algebra* by Michael Artin (from which the problem was assigned). Two things must be mentioned, however. The idea to use 1's along the diagonal of the matrix $[\beta]$ is from page 37, exercise 6, that states in part "Most invertible matricies can be written as a product $A = LU$ of a lower triangular matrix L and an upper triangular matrix U, where in addition all diagonal entries of U are 1." Consideration of transpositions began with exercise 15 on page 233, that suggests we "Prove that the transpositions $(1\,2), (2\,3), \ldots, (n-1, n)$ generate the symmetric group S_n."

I wish to thank especially and generally Dr. Andy Magid and in particular for suggesting the use and application of Cramer's Rule, when previously I had been attempting to use the Inverse Function Theorem.