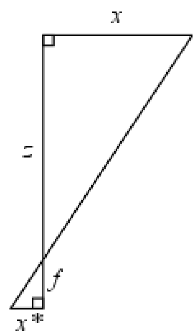

Figure 1: Converting camera coordinates to real-world coordinates.

1 Triangulation

1.1 Relating image coordinates to world coordinates

Solving the system requires knowledge of θ_C, the angle the incoming ray creates with the camera's optical axis. Figure 1 demonstrates the geometry involved in this operation.

By itself, the camera can directly measure neither x_C nor z_C. (If it could, this project would be rather pointless.) However, figure 1 demonstrates that

$$\frac{x_C}{z_C} = \frac{x_C^\star}{f_C} \tag{1}$$

f_C is the distance between the focal point and the camera sensor. The sensor is composed of many tiny pixel sensors. These sensors give the camera sensor a resolution R_C, measured in px/mm. These sensors convert the image into pixel coordinates with the following relationship:

$$u'_C = R_C x_C^\star \tag{2}$$

Combining (1) with (2) and defining a scale factor $S_C = R_C f_C$, we get

$$u'_C = S_C \frac{x_C}{z_C} \tag{3}$$

Knowing the scale factor S_C, we can easily calculate θ_C:

$$\tan\theta_C = \frac{x_C}{z_C} \tag{4}$$

$$\tan\theta_C = \frac{u'_C}{S_C} \tag{5}$$

Similarly,

$$\tan\phi_C = \frac{v'_C}{S_C} \tag{6}$$

$$\tan\theta_P = \frac{u'_P}{S_P} \quad (7)$$

$$\tan\phi_P = \frac{v'_P}{S_P} \quad (8)$$

1.2 Recovering 3D information

All 3D work in this section is performed using the camera's coordinate system. We start by assuming a distance vector measured from the camera to the projector.

$$\mathbf{D} = \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix}$$

$\hat{\mathbf{M}}$ is the unit vector pointing from the camera to the measured point $\mathbf{M}$.

$$1 = \hat{M}_x^2 + \hat{M}_y^2 + \hat{M}_z^2 \quad (9)$$

$$\hat{M}_z = \left[\tan^2\theta_C + \tan^2\phi_C + 1\right]^{-\frac{1}{2}} \quad (10)$$

$$\hat{\mathbf{M}} = \begin{pmatrix} \hat{M}_z \tan\theta_C \\ \hat{M}_z \tan\phi_C \\ \hat{M}_z \end{pmatrix} \quad (11)$$

Even though the fringe appears as a contour where it illuminates a surface, it may be conceptualized as a plane extending from the projector through every possible point of illumination. Any point on the fringe plane may be identified uniquely and parametrically as a linear combination of two vectors $\mathbf{F_1}$ and $\mathbf{F_2}$.

$$\mathbf{F}(t_1, t_2) = t_1\mathbf{F_1} + t_2\mathbf{F_2} \quad (12)$$

Vertical fringes will always pass through the y-axis, meaning

$$\mathbf{F_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

To define the other vector, we can use an angle ψ, defined as

$$\psi \equiv \pi + \theta_P - \beta$$

The physical significance of ψ is that it is the angle formed in the $y = 0$ plane by the fringe plane's trace and the camera's optical axis ($x = y = 0$).

We can now relate $\mathbf{F_2}$ to the projector's image coordinates:

$$\mathbf{F_2} = \begin{pmatrix} \cos\psi \\ 0 \\ \sin\psi \end{pmatrix}$$

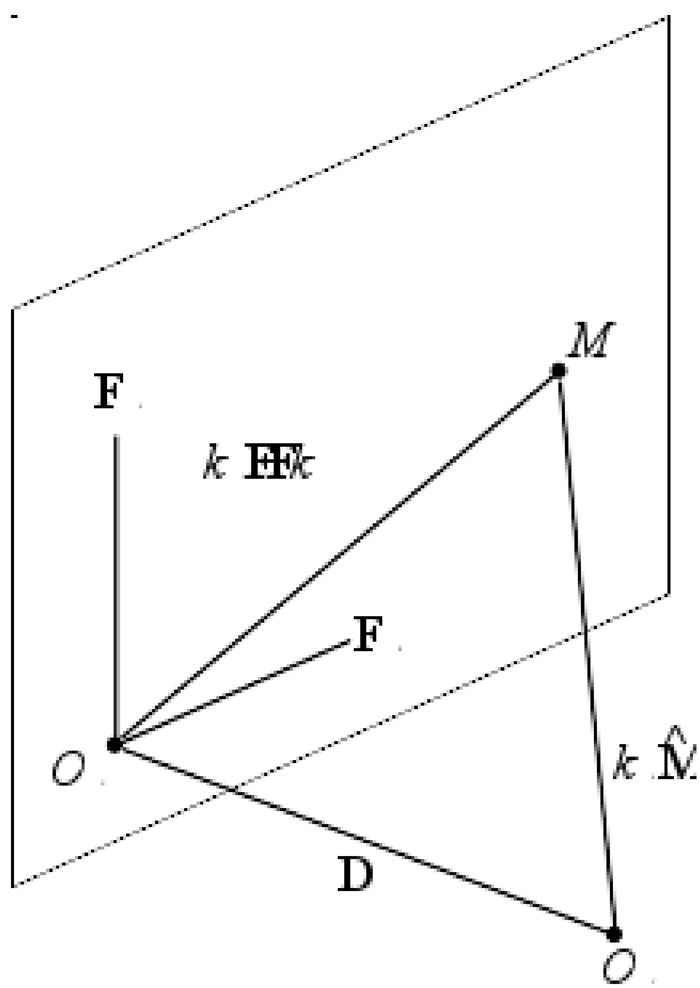

Figure 2: Triangulation geometry.

Now we have three vectors which sum to $\mathbf{D}$ in some linear combination. k_1 and k_2 are not necessarily positive.

$$k_0\hat{\mathbf{M}} + k_1\mathbf{F_1} + k_2\mathbf{F_2} = \mathbf{D}$$

This may be solved by using the equation

$$\begin{aligned} \begin{pmatrix} \hat{\mathbf{M}} & \mathbf{F_1} & \mathbf{F_2} \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \\ k_2 \end{pmatrix} &= \begin{pmatrix} \mathbf{D} \end{pmatrix} \\ \begin{pmatrix} \hat{\mathbf{M}} & \mathbf{F_1} & \mathbf{F_2} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{D} \end{pmatrix} &= \begin{pmatrix} k_0 \\ k_1 \\ k_2 \end{pmatrix} \end{aligned}$$

Now that we have our constants, the point $\mathbf{M}$ is simply

$$\mathbf{M} = k_0\hat{\mathbf{M}} \tag{13}$$

in the camera's coordinate space.