Liftings modulo p^2 and decomposition of the de Rham complex*

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0 Introduction

Let X be a smooth proper scheme over a field k. The de Rham cohomology of X/k, $\mathrm{H}^*_{\mathrm{dR}}(X/k) := \mathrm{H}^*(X, \Omega^\bullet_{X/k})$, is the result of the Hodge-de Rham spectral sequence

$$E_1^{ij} = \mathrm{H}^j(X, \Omega^i_{X/k}) \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{dR}}(X/k). \tag{0.1}$$

We know that if k has characteristic 0, (0.1) degenerates at E_1: for X projective, this is a result of Hodge theory, and the proper case reduces to the projective case by Chow's lemma and the resolution of singularities (cf. [5, 5.5]).

The first proof of this fact not using Hodge theory was given by Faltings [8], as an application of his theory of the existence of a Hodge-Tate decomposition for the p-adic étale cohomology on smooth proper varieties over local fields of different characteristic.

If k has characteristic $p > 0$, it is possible that (0.1) does not degenerate at E_1 (cf. Mumford [22] and 2.5(i)). However, Kato has recently shown [14] that, given k perfect of characteristic $p > 0$ and X smooth projective over k, if we assume that X has dimension $< p$ and lifts to the ring $W(k)$ of Witt vectors of k, then this "pathological" phenomena does not occur. Fontaine and Messing [10] have extended this result to the proper case, and deduced, by a standard argument, the degeneration of (0.1) in characteristic 0. This second proof uses crystalline techniques.

We give here a basic proof of a more precise result than that of Kato or Fontaine-Messing, and which is the following. Suppose k perfect, of characteristic $p > 0$, and let X be a smooth k-scheme (of arbitrary dimension, and not necessarily proper). Let X' be induced from X by extention of scalars $k \xrightarrow{\sim} k$, $\lambda \mapsto \lambda^p$, and $F : X \to X'$ the relative Frobenius (1.1). To prove that each smooth lift $\widetilde{X}$ of X to the ring $W_2(k)$ of Witt vectors of length 2 determines an isomorphism

$$\varphi_{\widetilde{X}} : \bigoplus_{0 \le i < p} \Omega^i_{X'/k}[-i] \xrightarrow{\sim} \tau_{<p} F_* \Omega^\bullet_{X/k} \tag{0.2}$$

in the derived category $D(X', \mathscr{O})$. It follows, by counting dimensions, that if X admits a smooth lift $\widetilde{X}$ to $W_2(k)$, and is moreover assumed proper and of dimension $< p$, the spectral sequence (0.1) degerates at E_1.

Here is the principle of the construction of $\varphi = \varphi_{\widetilde{X}}$. Let $\sigma : W_2(k) \xrightarrow{\sim} W_2(k)$ be the lift $(\lambda_0, \lambda_1) \mapsto (\lambda_0^p, \lambda_1^p)$ of the automorphism $\lambda \mapsto \lambda^p$ of k and $\widetilde{X}'$ be induced from $\widetilde{X}$ by extension of scalars by σ. If F lifts to $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$, the homomorphism $\widetilde{F}^* : \Omega^1_{\widetilde{X}/W_2(k)} \to \widetilde{F}_* \Omega^1_{\widetilde{X}/W_2(k)}$ provides, after division by p, a morphism of complexes

$$f : \Omega^1_{X'/k}[-i] \longrightarrow F_* \Omega^\bullet_{X/k}$$

inducing on $\mathscr{H}^1$ the Cartier isomorphism C^{-1} (1.2) (this idea, which goes back to Mazur [20], has already been widely exploited). If $\widetilde{F}_1$ and $\widetilde{F}_2$ are two lifts, their "difference" is a homomorphism of $\Omega^1_{X'/k}$ into $F_* \mathscr{O}_X$. It provides a homotopy between the maps f_1 and f_2 associated to $\widetilde{F}_1$ and $\widetilde{F}_2$. These homotopies verify a transitivity condition. This makes it possible to globalize the construction by means of a covering of X by open spaces where F lifts. We thus obtain the component φ^1 of φ. We define the component φ^0 as the map induced by F^*, and we construct the φ^i from φ^0 and φ^1 thanks to the multiplicative structure of the de Rham complex (here is where the restriction $i < p$ comes in).

This construction is explained in n° 2, after a brief reminder, in n° 1, of the definition of the relative Frobenius and the Cartier isomorphism. We also give, in n° 2, the "standard" argument allowing us to deduce from (0.2) the degeneration of (0.1) in characteristic zero. For X of dimension $< p$, liftable to $W_2(k)$, (0.2) gives a decomposition in $D(X', \mathscr{O})$

$$\bigoplus_i \Omega^i_{X'/k}[-i] \xrightarrow{\sim} F_* \Omega^\bullet_{X/k}. \tag{0.3}$$

We show that we still have such decomposition for X of dimension p. Finally, we deduce from (0.2) a Kodaira vanishing theorem in characteristic p: if X is a smooth projective 6-dimensional k-scheme over $W_2(k)$, and if L is an ample inverible sheaf on X, then

$$\mathrm{H}^j(X, \Omega^i_{X/k} \otimes L^{-1}) = 0 \quad \text{for} \quad i + j < \inf(p, \dim X).$$

This result and its proof are due to Michael Raynaud. The classical theorem of Kodaira-Akizuki-Nakano en characteristic zero follows from the usual argument.

*If you notice a typo or translation error, please visit https://github.com/ryankeleti/Deligne-Illusie/issues.

The obstruction to the existence of a decomposition as in (0.2) and the dependence of $\varphi_{\widetilde{X}}$ relative to $\widetilde{X}$ are studied in n° 3, where the construction of φ is taken over a base S of characteristic $p>0$. We show in particular the following result: if S admits a flat lift $\widetilde{S}$ to $\mathbf{Z}/p^2$, if X is a smooth S-scheme and X' is its inverse image under the absolute Frobenius of S, then X' admits a smooth lift to $\widetilde{S}$ if and only if there exists a map $\Omega^1_{X'/S}[-1] \to F_*\Omega^\bullet_{X/S}$ in $D(X', \mathcal{O})$ inducing the Cartier isomorphism C^{-1} on $\mathcal{H}^1$. An application to the degeneration of the Hodge-de Rham spectral sequence is given in n° 4, where we also briefly treat the variant of the previous results for the de Rham complex at logarithmic poles along a divisor at normal crossings.

1 Notation and reminders

Let p be a prime number.

1.1. If S is a scheme of characteristic p, we denote by F_S the *Frobenius endomorphism* of S (given by the identity on the underlying topological space and $a \mapsto a^p$ on $\mathcal{O}_S$). If $u : X \to S$ is a morphism of schemes, with S of characteristic p, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/S}"] \arrow[rr, bend left, "F_X"] \arrow[dr, "u"'] & X' \arrow[r] \arrow[d] & X \arrow[d, "u"] \\ & S \arrow[r, "F_S"] & S, \end{tikzcd}$$

where the square is Cartesian; the morphism $F_{X/S}$ is by definition the *relative Frobenius morphism* of X/S; it will be simply denoted F when there is no cause for confusion. For x a local section of $\mathcal{O}_X$, $x \otimes 1$ its image in $\mathcal{O}_{X'}$, we have $F^*_{X/S}(x \otimes 1) = F^*_X(x) = x^p$. Example: if X is defined by equations $f_\alpha = \sum_m a_{\alpha,m} T^m$ in the affine space $S[T_1, \dots, T_n] = \mathbf{A}^n_S$, X' is defined by the equations $f_\alpha^{(p)} = \sum_m a^p_{\alpha,m} T^m$ in $\mathbf{A}^n_S$, and $F_{X/S}$ is given by $T_i \mapsto T_i^p$.

Let $\Omega^\bullet_{X/S}$ be the de Rham complex of X/S. We will systematically use relative de Rham complexes, and later sometimes abbreviate $\Omega^\bullet_{X/S}$ to $\Omega^\bullet_X$, or even $\Omega^\bullet$. The complex $F_*\Omega^\bullet_{X/S}$ (where $F = F_{X/S}$) is a complex of $\mathcal{O}_{X'}$-modules, with linear differential. If X is smooth over S, the $\mathcal{O}_X$-modules $\Omega^i_{X/S}$ are locally free of finite type (as well as the $\mathcal{O}_{X'}$-modules $\Omega^i_{X'/S}$), and the same is true of the $\mathcal{O}_{X'}$-modules $F_*\Omega^i_{X/S}$, since F is finite locally free (of rank p^r if X is of relative dimension r over S). In addition, we have the following basic result, thanks to Cartier [4]:

Theorem 1.2 (Cartier)**.** *Let $X \to S$ be a smooth morphism, with S of characteristic p. There exists a unique morphism of graded $\mathcal{O}_{X'}$-algebras*

$$C^{-1} : \bigoplus_i \Omega^i_{X'/S} \longrightarrow \bigoplus_i \mathcal{H}^i F_* \Omega^\bullet_{X/S}$$

such that $C^{-1} d(x \otimes 1) =$ class of $x^{p-1} dx$ for each local section x of $\mathcal{O}_{X'}$, and C^{-1} is an isomorphism.

(The existence and uniqueness of C^{-1} are easy, and it is verified that C^{-1} is an isomorphism by reduction to the case of the affine line, and by a direct calculation: cf. Katz [15, 7.2].)

1.3. If k is a perfect field of characteristic p, we denote by $W(k)$ the ring of Witt vectors of k, and $W_n(k) = W(k)/p^n$. The ring $W_n(k)$ is flat over $\mathbf{Z}/p^n$, given by an isomorphism $W_n(k)/pW_n(k) \xrightarrow{\sim} k$, and is characterized up to unique isomorphism by these properties; we have $W(k) = \varprojlim W_n(k)$. The Frobenius automorphism of k induces, by functoriality, an automorphism σ of $W(k)$ (resp. $W_n(k)$), given by $\sigma(a_0, a_1, \dots) = (a_0^p, a_1^p, \dots)$.

1.4. Let A be an abelian category. For $n \in \mathbf{Z}$, the truncation $\tau_{\leq n} L$ of a complex L in A is the subcomplex of L of components L_i for $i < n$, $\ker(d)$ for $i = n$, and 0 for $i > n$. We have $\mathrm{H}^i \tau_{\leq n} L = \mathrm{H}^i L$ (resp. 0) if $i \leq n$ (resp. $i > n$). We put $\tau_{<n} L := \tau_{\leq n-1} L$. We define dually $\tau_{\geq n} L$, a quotient of L with $\mathrm{H}^i \tau_{\geq n} = \mathrm{H}^i L$ (resp. 0) if $i \geq n$ (resp. $i < n$).

The shift $L[n]$ is the complex of components $L[n]^i = L^{i+n}$ with differential $d_{L[n]} = (-1)^n d_L$. For M an object of A, we still denote by M the reduced complex of M concentrated in degree 0; $M[n]$ is then the reduced complex M concentrated in degree $-n$.

1.5. If X is a scheme, we write $D(X) := D(X, \mathcal{O}_X)$ for the derived category of the category of $\mathcal{O}_X$-modules.

1.6. Let $\widetilde{S}$ be a scheme and S a closed subscheme defined by a squarefree ideal. If X is a flat S-scheme, we say that X is *liftable* to S if X admits a lifting on $\widetilde{S}$, i.e. a flat $\widetilde{S}$-scheme $\widetilde{X}$ with an isomorphism $\widetilde{X} \times_{\widetilde{S}} S \xrightarrow{\sim} X$; if X is smooth over S, $\widetilde{X}$ is automatically smooth over $\widetilde{S}$.

2 Decomposition of the de Rham complex and applications (for a perfect field)

Theorem 2.1. *Let k be a perfect field of characteristic $p > 0$, $S = \mathrm{Spec}(k)$, $\widetilde{S} = \mathrm{Spec}(W_2(k))$ (1.3), and let X be a smooth S-scheme. For each smooth $\widetilde{S}$-scheme $\widetilde{X}$ lifting to X there is a canonically associated isomorphism*

$$\varphi_{\widetilde{X}} : \bigoplus_{i<p} \Omega^i_{X'/S}[-i] \xrightarrow{\sim} \tau_{<p} F_* \Omega^\bullet_{X/S}$$

in $D(X')$, such that $\mathscr{H}^i \varphi_{\widetilde{X}} = C^{-1}$ (1.2) for $i < p$.

The proof will be done in four steps.
(a) *Reduction to the definition of $\varphi^1_{\widetilde{X}}$.* The data of $\varphi_{\widetilde{X}}$ is equivalent to the data, for each $i < p$, of a map $\varphi^i_{\widetilde{X}} : \Omega^i_{X'/S}[-i] \to F_* \Omega^\bullet_{X/S}$ in $D(X')$ such that $\mathscr{H}^i \varphi^i_{\widetilde{X}} = C^{-1}$. The map $\varphi^0_{\widetilde{X}}$ is necessarily the composite

$$\mathscr{O}_{X'} \xrightarrow{C^{-1}} \mathscr{H}^0 F_* \Omega^\bullet_{X/S} \hookrightarrow F_* \Omega^\bullet_{X/S}.$$

Suppose we define $\varphi^1_{\widetilde{X}}$ such that $\mathscr{H}^1 \varphi^1_{\widetilde{X}} = C^{-1}$. For $i \geq 1$, consider the product map

$$(\Omega^1_{X'/S})^{\otimes i} \longrightarrow \Omega^i_{X'/S}, \quad \omega_1 \otimes \cdots \otimes \omega_i \longmapsto \omega_1 \wedge \cdots \wedge \omega_i;$$

for $i < p$, this admits an "antisymmetric" section a, given by

$$a(\omega_1 \wedge \cdots \wedge \omega_i) = (1/i!) \sum_{s \in S_i} \mathrm{sgn}(s) \omega_{s(1)} \otimes \cdots \otimes \omega_{s(i)}.$$

For $1 \leq i < p$, define $\varphi^i_{\widetilde{X}}$ as the composite map

$$\begin{tikzcd}
(\Omega^1_{X'/S})^{\otimes i}[-i] \arrow[r, "(\varphi^1_{\widetilde{X}})^{\otimes i}"] & (F_* \Omega^\bullet_{X/S})^{\otimes i} \arrow[d, "\text{product}"] \\
\Omega^i_{X'/S}[-i] \arrow[u, "{a[-i]}"] \arrow[r, "\varphi^i_{\widetilde{X}}"] & F_* \Omega^\bullet_{X/S}.
\end{tikzcd}$$

It follows from the multiplicative property of the Cartier isomorphism that $\mathscr{H}^i \varphi^i_{\widetilde{X}} = C^{-1}$, and $\varphi_{\widetilde{X}} = \sum_{i<p} \varphi^i_{\widetilde{X}}$, with $\varphi^0_{\widetilde{X}}$ as above, answer the question. If sufficies to define $\varphi^1_{\widetilde{X}} : \Omega^1_{X'/S}[-1] \to F_* \Omega^\bullet_{X/S}$ such that $\mathscr{H}^1 \varphi^1_{\widetilde{X}} = C^{-1}$, which we will do in the next three steps.
(b) *The case where $F : X \to X'$ lifts.* Let $\widetilde{X}'$ be a S-scheme induced from $\widetilde{X}$ by the change of base $\sigma : \widetilde{S} \to \widetilde{S}$ (1.3). Suppose we define an $\widetilde{S}$-morphism $\widetilde{F} : \widetilde{X} \to \widetilde{X}'$ lifting F. Since $F^* : \Omega^1_{X'/S} \to F_* \Omega^1_{X/S}$ is zero, the image of $\widetilde{F}^* : \Omega^1_{\widetilde{X}'/\widetilde{S}} \to \widetilde{F}_* \Omega^1_{\widetilde{X}/\widetilde{S}}$ is continuous in $p \widetilde{F}_* \Omega^1_{\widetilde{X}/\widetilde{S}}$. The multiplication by p induces an isomorphism $\boldsymbol{p} : F_* \Omega^1_{X/S} \xrightarrow{\sim} p \widetilde{F}_* \Omega^1_{\widetilde{X}/\widetilde{S}}$, so there exists a unique map

$$f = p^{-1} \widetilde{F}^* : \Omega^1_{X'/S} \to F_* \Omega^1_{X/S}$$

rendering commutative the square

$$\begin{tikzcd}
\Omega^1_{\widetilde{X}'/\widetilde{S}} \arrow[r, "\widetilde{F}^*"] \arrow[d, two heads] & p \widetilde{F}_* \Omega^1_{\widetilde{X}/S} \\
\Omega^1_{X'/S} \arrow[r, "f"] & F_* \Omega^1_{X/S}. \arrow[u, "\boldsymbol{p}", "\simeq"']
\end{tikzcd}$$

If x is a local section of $\mathscr{O}_{\widetilde{X}}$, by reduction x_0 mod p, we have

$$\widetilde{F}^*(x \otimes 1) = x^p + \boldsymbol{p} u(x) \tag{1}$$

with $u(x)$ a section of $\mathscr{O}_{\widetilde{X}}$ (and $\boldsymbol{p} : \mathscr{O}_X \xrightarrow{\sim} p \mathscr{O}_{\widetilde{X}}$ multiplication by p), and

$$f(dx_0 \otimes 1) = x_0^{p-1} dx_0 + du(x). \tag{2}$$

In particular, we have

$$df = 0, \tag{3}$$

so that f defines a morphism of complexes

$$f : \Omega^1_{X'/S}[-1] \longrightarrow F_* \Omega^\bullet_{X/S},$$

such that $\mathscr{H}^1 f = C^{-1}$ according to (2).
(c) *Homotopies*. Let, for $i = 1, 2$, $\widetilde{F}_i : \widetilde{X} \to \widetilde{X}'$ be an $\widetilde{S}$-morphism lifting F. Then $\widetilde{F}_2^* - \widetilde{F}_1^* : \mathscr{O}_{\widetilde{X}'} \to p\widetilde{F}_*\mathscr{O}_{\widetilde{X}} = \boldsymbol{p}F_*\mathscr{O}_X$ is a derivation, which determines a $\mathscr{O}_{X'}$-linear map

$$h_{12} : \Omega^1_{X'/S} \longrightarrow F_*\mathscr{O}_X$$

rendering commutative the diagram

$$\begin{tikzcd}
\mathscr{O}_{\widetilde{X}'} \arrow[r, "\widetilde{F}_2^* - \widetilde{F}_1^*"] \arrow[d, two heads] & p\widetilde{F}_*\mathscr{O}_{\widetilde{X}} \\
\mathscr{O}_{X'} \arrow[d, "d"'] & \\
\Omega^1_{X'/S} \arrow[r, "h_{12}"] & F_*\mathscr{O}_X . \arrow[uu, "\simeq"', "\boldsymbol{p}"]
\end{tikzcd}$$

If, for x as in (b), $\widetilde{F}_i^*(x \otimes 1) = x^p + \boldsymbol{p}u_i(x)$, then

$$h_{12}(dx_0 \otimes 1) = u_2(x) - u_1(x),$$

from which, given (2),

$$f_2 - f_1 = dh_{12}, \tag{4}$$

where $f_i = p^{-1}\widetilde{F}_i^* : \Omega^1_{X'/S} \to F_*\Omega^1_{X/S}$.
If, for $i = 1, 2, 3$, $\widetilde{F}_i : \widetilde{X} \to \widetilde{X}'$ an S-morphism lifting F, and h_{ij} corresponds to $\widetilde{F}_j - \widetilde{F}_i$, then we have

$$h_{12} + h_{23} = h_{12}. \tag{5}$$

(d) *The general case*. As X'/S is smooth, F admits, locally for the Zariski topology on X, a lifting $\widetilde{F}$ [(SGA 1 III) or (EGA IV §17)]. So we can find an open cover $\mathscr{U} = (U_i)_{i\in I}$ of X, and, for each i, an S-morphism $\widetilde{F}_i : \widetilde{U}_i \to \widetilde{U}'_i$ lifting F ($X, X', \widetilde{X}, \widetilde{X}'$ have the same underlying spaces and we denote by $\mathscr{U}'$, $\widetilde{\mathscr{U}}$, $\widetilde{\mathscr{U}}'$ the open covers of X', $\widetilde{X}$, $\widetilde{X}'$). Let, as in (b) and (c),

$$f_i = p^{-1}\widetilde{F}_i^* : \Omega^1_{X'/S}|U'_i \longrightarrow F_*\Omega^1_{U_i/S},$$

and, for $U'_{ij} = U'_i \cap U'_j$, $h_{ij} : \Omega^1_{X'/S}|U'_{ij} \to F_*\Omega^1_{U'_{ij}/S}$ correspond to $\widetilde{F}_j^* - \widetilde{F}_i^*$. We have

$$df_i = 0, \quad f_j - f_i = dh_{ij} \text{ (on } U'_{ij}), \quad h_{ij} + h_{jk} = h_{ik} \text{ (on } U'_{ijk} = U'_i \cap U'_j \cap U'_k). \tag{6}$$

Let $\check{\mathscr{C}}(\mathscr{U}, \Omega^\bullet_{X/S})$ be the ordinary complex associated to the double Čech complex of $\Omega^\bullet_{X/S}$, defined as the following. Put $\Delta_n = \{0, \dots, n\}$, and, for $s : \Delta_n \to I$, denote by U_s the intersection of the $U_{s(n)}$ and by j_s the inclusion of U_s into X. The component of degree n of $\check{\mathscr{C}}(\mathscr{U}, \Omega^\bullet_{X/S})$ is

$$\check{\mathscr{C}}(\mathscr{U}, \Omega^\bullet_{X/S})^n = \bigoplus_{a+b=n} \check{\mathscr{C}}^b(\mathscr{U}, \Omega^a_{X/S}),$$

where $\check{\mathscr{C}}^b(\mathscr{U}, \Omega^a_{X/S})$ is the product, extended by $s : \Delta_b \to I$, of $j_{s*}j_s^*\Omega^a_{X/S}$. The differential of $\check{\mathscr{C}}(\mathscr{U}, \Omega^\bullet_{X/S})$ is $d = d_1 + d_2$, with $d_1 : \check{\mathscr{C}}^b(\mathscr{U}, \Omega^a_{X/S}) \to \check{\mathscr{C}}^b(\mathscr{U}, \Omega^{a+1}_{X/S})$ induced by the differential of the de Rham complex, and $d_2 : \check{\mathscr{C}}^b(\mathscr{U}, \Omega^a_{X/S}) \to \check{\mathscr{C}}^{b+1}(\mathscr{U}, \Omega^a_{X/S})$ equal to $(-1)^a \sum_i (-1)^i \partial_i$. The evident morphisms $\Omega^a_{X/S} \to \check{\mathscr{C}}^0(\mathscr{U}, \Omega^a_{X/S})$ define a quasi-isomorphism $\Omega^\bullet_{X/S} \to \check{\mathscr{C}}(\mathscr{U}, \Omega^\bullet_{X/S})$, and consequently a quasi-isomorphism

$$F_*\Omega^\bullet_{X/S} \longrightarrow F_*\check{\mathscr{C}}(\mathscr{U}, \Omega^\bullet_{X/S}). \tag{7}$$

Define

$$\varphi^1_{(\mathscr{U},(\widetilde{F}_i))} = (\varphi_1, \varphi_2) : \Omega^1_{X'/S} \longrightarrow F_*\check{\mathscr{C}}(\mathscr{U}, \Omega^\bullet_{X/S})^1 = F_*\check{\mathscr{C}}^1(\mathscr{U}, \mathscr{O}_X) \oplus F_*\check{\mathscr{C}}^0(\mathscr{U}, \Omega^1_{X/S})$$

by

$$(\varphi_1\omega)(i,j) = h_{ij}(\omega|U'_{ij}), \quad (\varphi_2\omega)(i) = f_i(\omega|U'_i).$$

The relations in (6) tell us that $d\varphi^1_{(\mathscr{U},(\widetilde{F}_i))} = 0$, i.e. $\varphi^1_{(\mathscr{U},(\widetilde{F}_i))}$ is a morphism of complexes

$$\varphi^1_{(\mathscr{U},(\widetilde{F}_i))} : \Omega^1_{X'/S}[-1] \longrightarrow F_*\check{\mathscr{C}}(\mathscr{U}, \Omega^\bullet_{X/S}). \tag{8}$$

Finally, define

$$\varphi^1_{\widetilde{X}} : \Omega^1_{X'/S}[-1] \longrightarrow F_*\Omega^\bullet_{X/S} \tag{9}$$

as the composite map in $D(X')$ of (8) and the inverse of (7). We verify that (9) does not depend on the choice of $(\mathscr{U}, (\widetilde{F}_i))$. It is clear that (9) does not change if we replace $\mathscr{U}$ by a finner covering and the $\widetilde{F}_i$ by the induced lifts. If $(\mathscr{U} = (U_i)_{i\in I}, (\widetilde{F}_i)_{i\in I})$

and $(\mathscr{V}=(V_i)_{i\in J},(\widetilde{F}_i)_{i\in J})$ are two choices, the coverings $\mathscr{U}$ and $\mathscr{V}$ are finner than $\mathscr{U}\amalg\mathscr{V}$, indexed by $I\amalg J$, and $(\mathscr{U},(\widetilde{F}_i)_{i\in I})$ and $(\mathscr{V},(\widetilde{F}_i)_{i\in J})$ define the same map (9) that $(\mathscr{U}\amalg\mathscr{V},(\widetilde{F}_i)_{i\in I\amalg J})$ does.

The only thing left is to show that $\mathscr{H}^1\varphi^1_{\widetilde{X}}=C^{-1}$. This is a local question, so we can suppose that F admits a lift $\widetilde{F}:\widetilde{X}\to\widetilde{X}'$. The map $\varphi^1_{\widetilde{X}}$ is then defined by the morphism of complexes f of (b), and we have seen that $\mathscr{H}^1 f=C^{-1}$. This completes the proof of (2.1).

Remark 2.2. (i) We have not actually used the fact that S is the spectrum of a perfect field: the proof provides in fact an isomorphism $\varphi_{\widetilde{X}}$ for S equal to the mod p reduction of a scheme $\widetilde{S}$ to $\mathbf{Z}/p^2$ with an endomorphism $F_{\widetilde{S}}$ lifting F_S ($\widetilde{X}'$ is then defined as induced from $\widetilde{X}$ by the change of base $F_{\widetilde{S}}$). We will study further (3.7) the dependence of $\varphi^1_{\widetilde{X}}$ on X.

(ii) In the case where $F:X\to X'$ admits a lift $\widetilde{F}:\widetilde{X}\to\widetilde{X}'$, the map $f=p^{-1}\widetilde{F}^*$ of (b) extend to a quasi-isomorphism of complexes of $\mathscr{O}_{X'}$-modules

$$\varphi_{(\widetilde{X},\widetilde{F})}:\bigoplus_{i\geq 0}\Omega^i_{X'/S}[-i]\longrightarrow F_*\Omega^\bullet_{X/S}$$

inducing C^{-1} on $\mathscr{H}^i$ (and such that $\tau_{<p}\varphi_{(\widetilde{X},\widetilde{F})}$ has the image $\varphi_{\widetilde{X}}$ in $D(X')$): it is indeed enough to define the component of $\varphi_{(\widetilde{X},\widetilde{F})}$ of degree i

$$\varphi_{(\widetilde{X},\widetilde{F})}:\Omega^i_{X'/S}\longrightarrow ZF_*\Omega^i_{X/S}$$

as C^{-1} for $i=0$, and, for $i\geq 1$, as the composition of $\Lambda^i f:\Omega^i_{X'/S}\to\Lambda^i ZF_*\Omega^1_{X/S}$ and the product map $\Lambda^i ZF_*\Omega^1_{X/S}\to ZF_*\Omega^i_{X/S}$ (Z denotes the kernel of d).

(iii) The local lifts of F form a torsor on X' under the sheaf $\mathscr{H}om(\Omega^1_{X'/S},F_*\mathscr{O}_X)=\Theta_{X'/S}\otimes F_*\mathscr{O}_X$ of derivations of X' with values in $F_*\mathscr{O}_X$. The class c of this torsor in $\mathrm{H}^1(X',\Theta_{X'/S}\otimes F_*\mathscr{O}_X)$ is the obstruction to the existence of a global lift $\widetilde{F}:\widetilde{X}\to\widetilde{X}'$ of F. With the notations as in (d) above, and with a sign depending on the chosen conventions, c is the class of the cocycle (h_{ij}). By the construction of $\varphi^1_{\widetilde{X}}$, considered as the map of $\Omega^1_{X'/S}[-1]$ into $F_*\mathscr{O}_X$ in $D(X')$ is none other than the composition of $\varphi^1_{\widetilde{X}}$ and the natural projection $F_*\Omega^\bullet_{X/S}\to F_*\mathscr{O}_X$, i.e. the obstruction to representing $\varphi^1_{\widetilde{X}}$ by a morphism of complexes.

(iv) Suppose that X admits a formal smooth lift $X^\wedge$ to $W(k)$, and let m be an integer $<p-1$. The isomorphism ψ_ε of Ogus' theorem [3, 8.20] gives the truncation $\tau_{\leq m}$ of the isomorphism $\varphi_{\widetilde{X}}$, where $\widetilde{X}$ is the reduction of $X^\wedge$ modulo p^2: if ε_m denotes the "gauge" $i\mapsto\langle m-i\rangle$ [3], 8.18.3, the subcomplex $(\Omega^\bullet_{X^{\wedge\prime}/W})_{\varepsilon_r}$ of $\Omega^\bullet_{X^{\wedge\prime}/W}$, for $r=m,m+1$, is written

$$p^r\mathscr{O}_{X^{\wedge\prime}}\longrightarrow p^{r-1}\Omega^1_{X^{\wedge\prime}/W}\longrightarrow\cdots;$$

this identifies with $Ru_{X'/W*}\mathscr{S}^{[r]}_{X'/W}$; applying ψ_ε with $\varepsilon=\varepsilon_m,\varepsilon_{m+1}$ and passing to the quotient, we obtain the desired decomposition. It is this observation, implicit in Kato [14], proved in 2.6.1, that is the origin of this article.

If X is of dimension $N<p$, the isomorphism ψ_{ε_N}, moreover, provides for all integers $n\geq 1$, an analog of the decomposition of 2.1 for the de Rham complex $\Omega^\bullet_{X_n}$ of $X_n/W_n(k)$, where X_n is the reduction of $X^\wedge$ modulo p^n. More precisely, denote by W_nX the scheme with the same underlying space as X and with structure sheaf $W_n\mathscr{O}_X$ the sheaf the Witt vectors of length n over $\mathscr{O}_X$. The ring homomorphism

$$W_n\mathscr{O}_X\longrightarrow\mathscr{O}_{X_n},\quad(a_0,\dots,a_{n-1})\longmapsto\widetilde{a}_0^{p^{n-1}}+p\widetilde{a}_1^{p^{n-2}}+\cdots+p^{n-1}\widetilde{a}_{n-1},$$

where $\widetilde{a}_i$ lift a_i, allows us to consider the components of $\Omega^\bullet_{X_n}$ as modules over W_nX. The Frobenius endomorphism of $W_n\mathscr{O}_X$ defines a relative Frobenius morphism $W_nX\to W_nX'$, which is a linear differential. Let X'_n be induced from X_n by the extention of scalars $\sigma:W_n(k)\xrightarrow{\sim}W_n(k)$ (1.3), and denote by $(\Omega^*_{X'_n},pd)$ the complex induced from the de Rham complex of $X'_n/W_n(k)$ by multipication by p of the differential. It is also a complex of modules, with linear differential, over W_nX'. The construction of the isomorphism ψ_{ε_N} gives, by reduction modulo p^n, an isomorphism in $D(W_nX')$:

$$\varphi_{X^\wedge}:(\Omega^*_{X'_n},pd)\xrightarrow{\sim}F_*\Omega^\bullet_{X_n}.\tag{2.2.1}$$

For $n=1$, $X_1=X$, $(\Omega^*_{X'_n},pd)=\bigoplus_i\Omega^i_{X'/k}[-i]$ and (2.2.1) coincides with φ_{X_2}. We refer to the articles of Fontaine-Messing [10] and Kato [14] for variants and applications of (2.2.1), in particular the degeneration of the Hodge-de Rham spectral sequence of $X_n/W_n(k)$ for X proper over k.

Corollary 2.3. *With the notation of 2.1, let X be a smooth k-scheme of dimension $\leq p$, liftable to $W_2(k)$. Then the complex $F_*\Omega^\bullet_{X/S}$ is isomorphic, in $D(X')$, to a complex with zero differential.*

The result furthermore means that $F_*\Omega^\bullet_{X/S}$ is isomorphic, in $D(X')$, to the sum of its $\mathscr{H}^i[-i]$, or that there exists, in $D(X')$, an isomorphism

$$\bigoplus_i\Omega^i_{X'/S}[-i]\xrightarrow{\sim}F_*\Omega^\bullet_{X/S}$$

inducing C^{-1} on $\mathscr{H}^i$ (cf. 3.1 below).

We prove 2.3. We reduce to the case of X connected. If $\dim X<p$, it suffices to apply 2.1. Suppose $\dim X=p$. On X, the locally free sheaves $\Omega^i_{X/S}$ and $\Omega^{p-i}_{X/S}$ satisfy Serre duality: the product $\alpha\wedge\beta$ is a perfect duality, with values in $\Omega^p_{X/S}$, between $\Omega^i_{X/S}$

and $\Omega^{p-i}_{X/S}$. The morphism $F = F_{X/S}$ is finite and flat, and as a result, by Grothendieck duality, $F_*\Omega^i_{X/S}$ and $F_*\Omega^{p-i}_{X/S}$ are still in duality (with values in $\Omega^p_{X'/S}$), given by the pairing $(\alpha,\beta)\mapsto C(\alpha\wedge\beta)$, where here we denote by $C : F_*\Omega^p_{X/S}\to\Omega^p_{X'/S}$ the composition of $F_*\Omega^p_{X/S}\to\mathscr{H}^pF_*\Omega^\bullet_{X/S}$ and the inverse of C^{-1} in degree p (1.2): the point is that C is none other than trace morphism. It is easy to verify directly that this pairing is perfect. The transpose, for this duality, of the differential d of $F_*\Omega^\bullet_{X/S}$ is still d (with a sign depending on the conventions). This expresses that

$$C(d\alpha\wedge\beta)\pm C(\alpha\wedge d\beta)=C(d(\alpha\wedge\beta))=0.$$

According to 2.1, $\tau_{<p}F_*\Omega^\bullet_{X/S}$ is isomorphic, in $D(X')$, to the sum of its $\mathscr{H}^i[-i]$. By duality, $\tau_{\geq 1}F_*\Omega^\bullet_{X/S}$ has the same property. We have a distinguished triangle

$$\tau_{<p}F_*\Omega^\bullet_{X/S}\longrightarrow F_*\Omega^\bullet_{X/S}\longrightarrow\mathscr{H}^p[-p]\xrightarrow{e}. \qquad (*)$$

Since $\tau_{<p}F_*\Omega^\bullet_{X/S}$ is the sum of its $\mathscr{H}^i[-i]$, the conclusion of 2.3 is equivalent to the nullity of

$$e:\mathscr{H}^p[-p]\longrightarrow\left(\bigoplus_{i<p}\mathscr{H}^i[-i]\right)[1].$$

Let e_i $(0\leq i\leq p-1)$ be the components of e:

$$e_i\in\mathrm{Hom}(\mathscr{H}^p[-p],\mathscr{H}^i[-i+1])=\mathrm{H}^{p-i+1}(X',\mathscr{H}om(\mathscr{H}^p,\mathscr{H}^i)).$$

The triangle $(*)$ maps to the triangle

$$\tau_{[1,p-1]}F_*\Omega^\bullet_{X/S}\longrightarrow\tau_{\geq 1}F_*\Omega^\bullet_{X/S}\longrightarrow\mathscr{H}^p[-p]\longrightarrow,$$

where $\tau_{[1,p-1]}=\tau_{\geq 1}\tau_{<p}=\tau_{<p}\tau_{\geq 1}$. As $\tau_{\geq 1}F_*\Omega^\bullet_{X/S}$ is the sum of its $\mathscr{H}^i[-i]$, we have $e_i=0$ for $i\neq 0$. For $i=0$, the class e_i lives in $\mathrm{H}^{p+1}(X',\mathscr{H}om(\mathscr{H}^p,\mathscr{H}^0))$, and this group is zero because $\dim X=p$.

Corollary 2.4. *With the notation of 2.1, let X be a proper smooth k-scheme of dimension $\leq p$, liftable to $W_2(k)$. Then the Hodge-de Rham spectral sequence*

$$E_1^{ij}=\mathrm{H}^j(X,\Omega^i)\Longrightarrow\mathrm{H}^*_{\mathrm{dR}}(X/k) \qquad (2.4.1)$$

degenerates at E_1.

As the $\mathrm{H}^j(X,\Omega^i)$ are finite-dimensional k-vector spaces, the result means that we have, for each n,

$$\sum_{i+j=n}\dim\mathrm{H}^j(X,\Omega^i)=\dim\mathrm{H}^n_{\mathrm{dR}}(X/k).$$

According to 2.3, we have an isomorphism in $D(X')$

$$\bigoplus_i\Omega^i_{X'/S}[-i]\xrightarrow{\sim}F_*\Omega^\bullet_{X/S}.$$

We therefore get, for each n, an isomorphism

$$\bigoplus_i\mathrm{H}^{n-i}(X',\Omega^i_{X'/S})\xrightarrow{\sim}\mathrm{H}^n(X',F_*\Omega^\bullet_{X/S}).$$

Or, as F is finite, we have $\mathrm{H}^n(X',F_*\Omega^\bullet_{X/S})=\mathrm{H}^n(X,\Omega^\bullet_{X/S})$. On the other hand, X' being induced from X by an extenstion of field scalars, we have $\dim\mathrm{H}^{n-i}(X',\Omega^i_{X'/S})=\dim\mathrm{H}^{n-i}(X,\Omega^i_{X/S})$, hence the conclusion.

Corollary 2.5. *With the notation of 2.1, let X be a proper smooth k-scheme liftable to $W_2(k)$. Then the Hodge-de Rham spectral sequence satisfies $E_1^{ij}=E_\infty^{ij}$ for $i+j<p$.*

The result means that, for $n<p$, we have

$$\sum_{i+j=n}\dim\mathrm{H}^j(X,\Omega^i)=\dim\mathrm{H}^n_{\mathrm{dR}}(X/k).$$

The exact sequence of complexes

$$0\longrightarrow\tau_{<p}F_*\Omega^\bullet_X\longrightarrow F_*\Omega^\bullet_X\longrightarrow F_*\Omega^\bullet_X/\tau_{<p}F_*\Omega^\bullet_X\longrightarrow 0$$

provides, for $n<p$, an isomorphism

$$\mathrm{H}^n(X',\tau_{<p}F_*\Omega^\bullet_X)\xrightarrow{\sim}\mathrm{H}^n(X',F_*\Omega^\bullet_X)=\mathrm{H}^n_{\mathrm{dR}}(X/k).$$

Applying 2.2, we obtain, for $n<p$,

$$\bigoplus_i\mathrm{H}^{n-i}(X',\Omega^\bullet_{X'})\xrightarrow{\sim}\mathrm{H}^n(X',F_*\Omega^\bullet_X)$$

and we conclude as in 2.4.
See 4.1.4 for the generalizations of 2.4 and 2.5.

Remark 2.6. (i) Mumford [22] has given examples of smooth projective surfaces X/k having non-closed global 1-forms. Other examples have later been given by Lang [19], Raynaud and Szpiro [11]. These surfaces does not therefore do not rely on $W_2(k)$. It is the same, for $p=2$, for the Enriques surfaces of type α_2, for which the differential $d_1 : \mathrm{H}^1(\mathscr{O}) \to \mathrm{H}^1(\Omega^1)$ is nonzero (cf. for example [13, II 7.3.8]). See also Suwa [26] for an interpretation of the degeneration condition of (2.4.1) at E_1, in terms of the structure of $\mathrm{Pic}^\tau/\mathrm{Pic}^0$ for surfaces such that

$$\chi(\mathscr{O}) - 1 + (b_1/2) = 0.$$

(ii) Under the hypotheses of 2.4, it is not true in general that the Hodge symmetry $h^{ij} = h^{ji}$ is valid (where $h^{ij} = \dim \mathrm{H}^j(X, \Omega^i)$), cf. Serre [25], §20.

(iii) If X is a smooth k-scheme of dimension $\geq p+1$, liftable to $W_2(k)$ (or even a smooth formal scheme over $W(k)$) it is probably not true in general that $F_*\Omega^\bullet_{X/k}$ is still a sum, in $D(X')$, of its $\mathscr{H}^i[-i]$. However, we do not have a counterexample. We do not even know if, for such an X, supposing proper, that the Hodge-de Rham spectral sequence degenerates at E_1.

(iv) If X is a smooth k-scheme, we say that the complex $F_*\Omega^\bullet_{X/k}$ is *decomposible* if it is the sum, in $D(X')$, of its $\mathscr{H}^i[-i]$ (cf. 3.1). If $F_*\Omega^\bullet_{X/k}$ is decomposible, the same is true of $F_*\Omega^\bullet_{Y/k}$ for Y étale over X. If $F_*\Omega^\bullet_{X/k}$ and $F_*\Omega^\bullet_{Y/k}$ are decomposible, so is $F_*\Omega^\bullet_{(X\times Y)/k}$. If for each n the natural morphism $\bigotimes^n \Omega^1_{X'/k} \to \Omega^n_{X'/k}$ admits a section, and X lifts to $W_2(k)$, an analagous argument to 2.1(a) shows that $F_*\Omega^\bullet_{X/k}$ is decomposible; this applies if X is an abelian variety over k (the decomposition of $F_*\Omega^\bullet_{X/k}$ in this case was reported by M. Raynaud, with another proof).

Corollary 2.7. *Let K be a field of characteristic 0 and X a proper smooth K-scheme. Then the Hodge-de Rham spectral sequence*

$$E_1^{ij} = \mathrm{H}^j(X, \Omega^i) \Longrightarrow \mathrm{H}^*_{\mathrm{dR}}(X/K) \tag{2.7.1}$$

degenerates at E_1.

The result still means that we have, for each n,

$$\sum_{i+j=n} h^{ij} = h^n, \tag{$*$}$$

where $h^{ij} = \dim \mathrm{H}^j(X, \Omega^i)$ and $h^n = \dim \mathrm{H}^n_{\mathrm{dR}}(X/K)$. We can assume that X is connected, with d its dimension. The standard argument show that there is an integral ring A of finite type over $\mathbf{Z}$, a proper smooth morphism $f : \mathscr{X} \to \mathrm{Spec}(A)$, of relative dimension d, and a homomorphism $A \to K$ such that $X = \mathscr{X} \otimes_A K$. The sheaves $R^j f_* \Omega^i_{\mathscr{X}/A}$, $R^n f_* \Omega^\bullet_{\mathscr{X}/A}$ are coherent, therefore, replacing A by $A[s^{-1}]$ for suitable $s \in A$, we can assume that they are locally free (therefore compatible under any change of base) and of constant ranks, h^{ij} and h^n respectively. Let T be the schematic closure, in $\mathrm{Spec}(A)$, of a closed point of $\mathrm{Spec}(A \otimes \mathbf{Q})$; this is a quasi-finite scheme flat over $\mathrm{Spec}(\mathbf{Z})$. Choose a closed point s of T at which T is étale over $\mathbf{Z}$, and such that the characteristic p of the finite field $k = k(s)$ is $\geq d$. If $\mathscr{O}_s$ denotes the local ring of s in T, its maximal ideal $\mathfrak{m}_s$ is generated by p, and $\mathscr{O}_s/\mathfrak{m}_s^2 = W_2(k)$. The scheme $\mathscr{X} \otimes_A (\mathscr{O}_s/\mathfrak{m}_s^2)$ is a smooth lift of $\mathscr{X}_s = \mathscr{X} \otimes_A k(s)$. Given the hypothesis stated above, we have $\dim_{k(s)} \mathrm{H}^j(\mathscr{X}_s, \Omega^i) = h^{ij}$, and $\dim_{k(s)} \mathrm{H}^n_{\mathrm{dR}}(\mathscr{X}_s/k(s)) = h^n$. But, as $d \leq p$, the relation $(*)$ is satisfied by 2.4.

Corollary 2.8 (Raynaud)**.** *With the notation of 2.1, let X be a smooth projective k-scheme, of pure dimension d, liftable to $W_2(k)$, and let L be an invertible sheaf on X. We make one of the following hypotheses:*

(i) *L is ample;*

(ii) *$d = 2$ and L is numerically positive (i.e. we have $L \cdot L > 0$ and $L \cdot \mathscr{O}(D) \geq 0$ for each effective divisor D of X).*

Then we have:

$$\mathrm{H}^j(X, \Omega^i \otimes L) = 0 \quad \textit{for } i+j > \sup(d, 2d-p), \tag{2.8.1}$$

$$\mathrm{H}^j(X, \Omega^i \otimes L^{-1}) = 0 \quad \textit{for } i+j < \inf(d, p). \tag{2.8.2}$$

Note that (2.8.1) and (2.8.2) are equivalent by Serre duality.
The heart of the proof is the following lemma:

Lemma 2.9. *Let X be a smooth k-scheme, M an invertible sheaf on X an b an integer. Suppose that $\tau_{<b}F_*\Omega^\bullet_X$ is isomorphic, in $D(X')$, to a complex with zero differential, and that*

$$\mathrm{H}^j(X, \Omega^i_X \otimes M^{\otimes p}) = 0 \quad \textit{for } i+j < b. \tag{$*$}$$

Then we have

$$\mathrm{H}^j(X, \Omega^i_X \otimes M) = 0 \quad \textit{for } i+j < b.$$

Let M' on X' be induced from M by a change of base. We have $F^*M' = M^{\otimes p}$, from where, by the projection formula,

$$\mathrm{H}^j(X, M^{\otimes p} \otimes_{\mathscr{O}_X} \Omega^i_X) = \mathrm{H}^j(X', M' \otimes_{\mathscr{O}_{X'}} F_*\Omega^i_X)$$

for each (i,j). As $F_*\Omega^\bullet_X$ is a complex of $\mathcal{O}_{X'}$-modules (with $\mathcal{O}_{X'}$-linear differential), we can consider the tensor product $M'\otimes_{\mathcal{O}_{X'}} F_*\Omega^\bullet_X$, and we have a spectral sequence

$$E_1^{ij} = \mathrm{H}^j(X', M'\otimes_{\mathcal{O}_{X'}} F_*\Omega^i_X) \Longrightarrow \mathrm{H}^{i+j}(X', M'\otimes_{\mathcal{O}_{X'}} F_*\Omega^\bullet_X).$$

The hypothesis $(*)$ implies therefore that $\mathrm{H}^n(X', M'\otimes_{\mathcal{O}_{X'}} F_*\Omega^\bullet_X)=0$ for $n<b$. But, for $n<b$, we have

$$\mathrm{H}^n(X', M'\otimes_{\mathcal{O}_{X'}} F_*\Omega^\bullet_X) = \mathrm{H}^n(X', M'\otimes_{\mathcal{O}_{X'}} \tau_{<b}F_*\Omega^\bullet_X).$$

Now we have, by hypothesis, an isomorphism in $D(X')$

$$\bigoplus_{i<b}\Omega^i_{X'}[-i] \xrightarrow{\sim} \tau_{<b}F_*\Omega^\bullet_X.$$

As a result, for $n<b$, we have

$$0=\mathrm{H}^n(X', M'\otimes_{\mathcal{O}_{X'}} F_*\Omega^\bullet_X) = \bigoplus_i \mathrm{H}^{n-i}(X', M'\otimes_{\mathcal{O}_{X'}} \Omega^i_{X'}).$$

Since (X',M') are induced from (X,M) by a change of base $F_S : S \xrightarrow{\sim} S$, we have as well that

$$\mathrm{H}^{n-i}(X, M\otimes_{\mathcal{O}_X}\Omega^i_X)=0$$

for $n<b$ and each i.

We prove (2.8.2). According to 2.1, for $b\le p$, $\tau_{<b}F_*\Omega^\bullet_X$ is isomorphic, in $D(X')$, to a complex with zero differential. Under the hypothesis (i), we have $\mathrm{H}^j(X,\Omega^i_X\otimes L^{\otimes(-N)})=0$ for N large enough and $j<d$, in particular for $N=p^n$ with n large enough and $i+j<d$. Let M be the dual of L. Applying 2.9 to $M^{\otimes p^n}$ and to $b=\inf(p,d)$, we obtain (2.8.2) at the end of a decending induction on n. Under the hypothesis (ii), we still have $\mathrm{H}^j(X,\Omega^i_X\otimes L^{\otimes(-N)})=0$ for N large enough and $i+j<2$. Indeed, it is trivial for $i=j=0$; for $j=1$, $i=0$, the assertion is due to Szpiro [21, Prop. 2]; finally, for $j=0$, $i=1$, it stems from the fact that, by Riemann-Roch, the dimension of $\mathrm{H}^0(X,L^{\otimes N})$ tends to infinity when N tends to infinity. We then conclude as in the case of (i).

Remark 2.10. (i) The idea of obtaining cancellation results by descending induction on the Frobenius morphisms is due to Szpiro (cf. [21, 27, 28]). Otherwise, Esnault and Viehweg [7] have recently shown that, over $\mathbf{C}$, there exists a narrow link between the degeneration at E_1 of certain spectral sequences of type "Hodge-de Rham" and the Kodaira vanishing theorems.

(ii) Raynaud [24] and Szpiro [9] have constructed examples of couples (X,L), where X/k is a smooth projective surface and L is an ample invertible sheaf on X such that $\mathrm{H}^1(X,L^{-1})\neq 0$. These surfaces do not lift to $W_2(k)$.

(iii) Let X be a smooth k-scheme. We will see in 3.6 that if X does not lift to $W_2(k)$, then $\tau_{\ge 1}F_*\Omega^\bullet_{X/k}$—and a fortiori $F_*\Omega^\bullet_{X/k}$—is not isomorphic, in $D(X')$, to a complex with zero differential. This "pathology" may be invisible at the level of the Hodge spectral sequence or Kodaira's vanishing statements: for $p\ge 7$, Raynaud can construct, by the method of Godeaux-Serre [25], a smooth projective surface X over $\mathbf{F}_p$, that does not lift to $\mathbf{Z}/p^2$, and such: (a) the Hodge-de Rham spectral sequence of X degenerates at E_1 (b) each ample invertible sheaf on X satisfies the Kodaira-Akizuki-Nakano vanishing theorem.

Corollary 2.11 (Kodaira-Akizuki-Nakano [1, 18], Ramanujam [23]). *Let K be a field of characteristic 0, X a smooth projective K-scheme, of pure dimension d, and L an invertible sheaf on X. Suppose that L is ample, or $d=2$ and L is numerically positive. Then we have*

$$\mathrm{H}^j(X,\Omega^i\otimes L)=0 \quad \textit{for } i+j>d$$

(or, equivalently, by Serre duality, $\mathrm{H}^j(X,\Omega^i\otimes L^{-1})=0$ for $i+j<d$).

We deduce 2.11 from 2.8 as we deduced 2.6 from 2.1; we ommit the details (for the numerically positive case, cf. [21, p. 42]).

Corollary 2.12 ("Weak Lefschetz", cf. Berthelot [2]). *With the notation of 2.1, let X be a smooth projective k-scheme of pure dimension d, and $D\subset X$ a smooth divisor. Suppose that X and D are liftable to $W_2(k)$ and that D is ample. Then the restriction map $\mathrm{H}^n_{\mathrm{dR}}(X/k)\to\mathrm{H}^n_{\mathrm{dR}}(D/k)$ is an isomorphism for $n<\inf(p,d)-1$ and an injection for $n=\inf(p,d)-1$.*

The kernel $\Omega^\bullet_X(\log D)(-D)$ of $\Omega^\bullet_X\to\Omega^\bullet_D$ [cf. 4.2.2(c)] admits a dévissage ("weight filtration")

$$0\longrightarrow\Omega^\bullet_X(-D)\longrightarrow\Omega^\bullet_X(\log D)(-D)\longrightarrow\Omega^{\bullet-1}_D(-D)\longrightarrow 0.$$

The result of 2.12 means that

$$\mathrm{H}^n(X,\Omega^\bullet_X(\log D)(-D))=0 \quad \text{for } n<\inf(p,d).$$

We apply 2.8 to $(X,\mathcal{O}_X(D))$ and $(D,\mathcal{O}_D(D))$.

3 Gerbes of liftings and splittings, and the decomposition of the de Rham complex (general case)

3.1. Let A be an abelian category. We say that an object K of $D^b(A)$ is *decomposable* if K is isomorphic (in $D^b(A)$) to a complex with zero differential. For K to be decomposable, it is necessary and sufficient that there exist, in $D(A)$, morphisms $f_i : \mathrm{H}^i K[-i] \to K$ such that $\mathrm{H}^i(f_i)$ are the identity maps of $\mathrm{H}^i K$ (this condition is necessary, because it is verified by $\bigoplus_i \mathrm{H}^i K[-i]$, and sufficient, because the f_i induce an isomorphism $\bigoplus_i \mathrm{H}^i K[-i] \xrightarrow{\sim} K$). If K is decomposible, we call the *decomposition* of K a morphism $f = \sum_i f_i : \bigoplus_i \mathrm{H}^i K[-i] \to K$ such that $\mathrm{H}^i f$ are the identity maps of $\mathrm{H}^i K$.

Let K be a decomposible complex such that $K^i = 0$ for $i \neq 0, 1$. A decomposition of K is entirely determined by the data of $f_1 : \mathrm{H}^1 K[-1] \to K$ such that $\mathrm{H}^1 f_1$ is the identity (f_0 is given by the injection $\mathrm{H}^0 K \hookrightarrow K^0$). The distinguished triangle

$$\mathrm{H}^0 K \longrightarrow K \longrightarrow \mathrm{H}^1 K[-1] \longrightarrow$$

provides the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathrm{H}^1 K[-1], \mathrm{H}^0 K) \longrightarrow \mathrm{Hom}(\mathrm{H}^1 K[-1], K) \xrightarrow{a} \mathrm{Hom}(\mathrm{H}^1 K, \mathrm{H}^1 K),$$

which shows that the set of decompositions of K is an affine space in $\mathrm{Hom}(\mathrm{H}^1 K[-1], \mathrm{H}^0 K) = \mathrm{Ext}^1(\mathrm{H}^1 K, \mathrm{H}^0 K)$ (the inverse image by a of Id).

3.2. Let $(T, \mathscr{O})$ be a ringed site and K a complex of $\mathscr{O}$-modules over T with $K^i = 0$ for $i \neq 0, 1$. Suppose that $\mathscr{H}^1 K$ is locally free of finite rank; the projection of K^1 onto $\mathscr{H}^1 K$ therefore admits a local section. Furthermore, suppose that the projection of K^0 onto $\mathrm{im}(d)$ admits a local section.

Let $\mathrm{sc}'(K)$[1] be the following fibered category over T: an object over U is a splitting (over U) $s : \mathscr{H}^1 K \to K^1$ of the projection of K^1 onto $\mathscr{H}^1 K$; a map from s' to s'' is a homomorphism $h : \mathscr{H}^1 K \to K^0$ (over U) such that $s'' = s' + dh$. If s' and s'' are two objects over U, the morphisms of s' to s'' form a sheaf on U: $\mathrm{sc}'(K)$ is a prestack (in groupoids). Furthermore, the hypotheses made on K imply that: (a) each U admits a covering R such that the fibered category $\mathrm{sc}'(K)(R)$ is nonempty (b) any two objects of $\mathrm{sc}'(K)$ are locally isomorphic. The stack associated to the prestack $\mathrm{sc}'(K)$ is a gerbe (Giraud [12, III § 2]), the *gerbe of splittings* of K, denoted $\mathrm{sc}(K)$.

For each object s of $\mathrm{sc}(K)$, $\mathscr{H}om(s, s)$ is the abelian sheaf $\mathscr{H}om(\mathscr{H}^1 K, \mathscr{H}^0 K)$. The gerbe $\mathrm{sc}(K)$ admits a global object[2] if its class [12, IV 3.1, 3.5]

$$\mathrm{cl}\,\mathrm{sc}(K) \in \mathrm{H}^2(T, \mathscr{H}om(\mathscr{H}^1 K, \mathscr{H}^0 K)) = \mathrm{Ext}^2(\mathscr{H}^1 K, \mathscr{H}^0 K)$$

is zero. If this is the case, the set of isomorphism classes of global objects of $\mathrm{sc}(K)$ is an affine space in $\mathrm{H}^1(T, \mathscr{H}om(\mathscr{H}^1 K, \mathscr{H}^0 K))$ $= \mathrm{Ext}^1(\mathscr{H}^1 K, \mathscr{H}^0 K)$ (if s and t are two global objects, their "difference" $t - s$ is the torsor class of local isomorphisms of s to t, cf. [12, III 2.2.6]).

We also denote

$$\mathrm{e}(K) \in \mathrm{Ext}^2(\mathscr{H}^1 K, \mathscr{H}^0 K)$$

the class defined by the degree 1 morphism of the distinguished triangle

$$\mathscr{H}^0 K \longrightarrow K \longrightarrow \mathscr{H}^1 K[-1] \xrightarrow{+1} .$$

We have $\mathrm{e}(K) = 0$ if and only if K is decomposible (3.1). On the other hand, we have seen that, if K is decomposible, the set of decompositions of K is an affine space in $\mathrm{Ext}^1(\mathscr{H}^1 K, \mathscr{H}^0 K)$.

Proposition 3.3. *With the preceding notation:* (a) *we have*

$$\mathrm{cl}\,\mathrm{sc}(K) = -\mathrm{e}(K).$$

(b) *There exists a canonical affine bijection α, defined below, from the set of isomorphism classes of global objects of* $\mathrm{sc}(K)$ *to the set of decompositions of K, inducing the indentity of the group of translations* $\mathrm{Ext}^1(\mathscr{H}^1 K, \mathscr{H}^0 K)$.

We prove (a). Choose a hypercover $U_\bullet \to T$, over U_0 a section $f : \mathscr{H}^1 K \to K^1$ of the projection of K^1 onto $\mathscr{H}^1 K$, and over U_1, $g : \mathscr{H}^1 K \to K^0$ such that $d_1^* f - d_0^* f = g$. Then $h = d_0^* g - d_1^* g + d_2^* g$ is a 2-cocycle of $U_\bullet$ with values in $\mathscr{H}om(\mathscr{H}^1 K, \mathscr{H}^0 K)$, whose image in $\mathrm{H}^2(T, \mathscr{H}om(\mathscr{H}^1 K, \mathscr{H}^0 K))$ is $\mathrm{cl}\,\mathrm{sc}(K)$ (cf. [12, IV 3.5]). On the other hand, with the sign convention of [J.-L. Verdier, Catégories dérivées, Etat 0, p. 269, in SGA 4 1/2, Springer Lecture Notes 569], $\mathrm{e}(K)$ is the class of the "composite" morphism

$$\mathscr{H}^1 K[-1] \xleftarrow{q} E \xrightarrow{-\mathrm{pr}} \mathscr{H}^0 K[1],$$

where

$$E = \left(\mathscr{H}^0 K \longrightarrow K^0 \xrightarrow{d} K^1 \right)$$

[1] [Trans]. sc is an abbreviation for the French *scindages*, for "splittings".
[2] See 3.3(b) for a description of global objects.

is the cone (concentrated in degrees $-1,0,1$) of $\mathscr{H}^0K \to K$, q the quasi-isomorphism given by the projection of K^1 onto $\mathscr{H}^1K$, and pr the evident projection. We put $M = \mathscr{H}^0K$, $N = \mathscr{H}^1K$, and denote by $\check{N}$ the dual of N. The class $\mathrm{e}(K)$ is still the image of $\mathrm{Id}_N \in \mathrm{H}^0(T, \check{N} \otimes N)$ in $\mathrm{H}^2(T, \check{N} \otimes M) = \mathrm{H}^2(T, \mathscr{H}\!om(\mathscr{H}^1K, \mathscr{H}^0K))$ under the composite morphism

$$\mathrm{H}^0(T, \check{N} \otimes N) \underset{\simeq}{\xleftarrow{q}} \mathrm{H}^1(T, \check{N} \otimes K) \xrightarrow{-\mathrm{Id}_N \otimes \mathrm{pr}} \mathrm{H}^2(T, \check{N} \otimes M).$$

Or

$$c = (h, -g, f) \in \check{C}^1(U_\bullet, \check{N} \otimes M) = \check{C}^2(U_\bullet, \check{N} \otimes M) \oplus \check{C}^1(U_\bullet, \check{N} \otimes K^0) \oplus \check{C}^0(U_\bullet, \check{N} \otimes K^1)$$

is a 1-cocycle, of the image of Id_N under q, and $-h$ under $\mathrm{Id}_N \otimes \mathrm{pr}$. Thus $\mathrm{e}(K) = -\mathrm{cl}\,\mathrm{sc}(K)$.

We now construct α. Let s be a global object of $\mathrm{sc}(K)$, described by a hyper $U_\bullet \to T$, over U_0 a section $f : \mathscr{H}^1K \to K^1$ of the projection of K^1 onto $\mathscr{H}^1K$, and over U_1, $h : \mathscr{H}^1K \to K^0$ such that $d_0^*f - d_1^*f = dh$ and $d_0^*h - d_1^*h + d_2^*h = 0$. Then

$$(h, f) : \mathscr{H}^1K \longrightarrow \check{\mathscr{C}}^1(U_\bullet, K) = \check{\mathscr{C}}^1(U_\bullet, K^0) \oplus \check{\mathscr{C}}^0(U_\bullet, K^1)$$

is a morphism from $\mathscr{H}^1K[-1]$ to $\check{\mathscr{C}}(U_\bullet, K)$, whose image $\alpha(s)$ in $\mathrm{Hom}_{D(T)}(\mathscr{H}^1K[-1], K)$ is a decomposition of K. By arguments similar to those of the proof of 2.1(d), it is verified that $\alpha(s)$ is independent of the choices of $(U_\bullet, f, h)$, and depends only on the isomorphism class of s. It remains to prove that, if t is a second global object of $\mathrm{sc}(K)$, we have $\alpha(t) - \alpha(s) = t - s$ in $\mathrm{Ext}^1(\mathscr{H}^1K, \mathscr{H}^0K)$. We can assume that t is described by $(U_\bullet, g, k)$ and that we have, over U_0, $u : \mathscr{H}^1K \to K^0$ such that $g - f = du$. The square

$$\begin{array}{ccc} d_1^*f & \xrightarrow{h} & d_0^*f \\ {\scriptstyle d_1^*u}\downarrow & & \downarrow{\scriptstyle d_0^*u} \\ d_1^*g & \xrightarrow{k} & d_0^*g \end{array}$$

gives

$$v = d_1^*u + k - h - d_0^*u : \mathscr{H}^1K \longrightarrow \mathscr{H}^0K \quad \text{over } U_1,$$

a 1-cocycle of $U_\bullet$ with values in $\mathscr{H}\!om(\mathscr{H}^1K, \mathscr{H}^0K)$, whose image in $\mathrm{H}^1(T, \mathscr{H}\!om(\mathscr{H}^1K, \mathscr{H}^0K))$ is (with the adequate conventions) the class of $t - s$. With the above notation, we then have

$$(k, g) - (h, f) = v + du : \mathscr{H}^1K \longrightarrow \check{\mathscr{C}}^1(U_\bullet, K),$$

where u is considered as a map from $\mathscr{H}^1K$ to $\check{\mathscr{C}}^0(U_\bullet, K)$ and d denotes the total differential of the complex $\check{\mathscr{C}}(U_\bullet, K)$. So we have $\alpha(t) - \alpha(s) = t - s$, which completes the proof of (b).

3.4. Let S be a scheme of characteristic $p > 0$, X a smooth scheme over S, and $F : X \to X'$ the relative Frobenius (1.1). Suppose we are given a scheme $\widetilde{S}$ flat over $\mathbf{Z}/p^2$ whose reduction modulo p is S. We propose to describe the gerbe of splittings of $K = \tau_{\leq 1}F_*\Omega^\bullet_{X/S}$ in terms of lifts of X' to $\widetilde{S}$.

For each smooth scheme Y over S, the *gerbe of lifts* of Y to $\widetilde{S}$, denoted $\mathrm{Rel}(Y, \widetilde{S})$[3], is the gerbe over Y having for objects over an open U the schemes $\widetilde{U}$ flat over $\widetilde{S}$ whose reduction modulo p is U (i.e. equipped with an isomorphism of their reduction modulo p to U). A morphism $\widetilde{U}' \to \widetilde{U}''$ is a morphism of $\widetilde{S}$-schemes, with reduction modulo p the identity. The sheaf of automorphisms of any lift of U is an abelian sheaf $\Theta_{U/S} = \mathscr{H}\!om(\Omega^1_{U/S}, \mathscr{O}_U)$ of stacks of relative vectors on U.

Theorem 3.5. *There exists a canonical equivalence of gerbes, defined below,*

$$\Phi : \mathrm{Rel}(X', \widetilde{S}) \longrightarrow \mathrm{sc}(\tau_{\leq 1}F_*\Omega^\bullet_{X/S}),$$

inducing the identity on the sheaves of automorphisms of objects.

(Note that, for the two gerbes considered, the sheaf of automorphims of an object is the sheaf $\Theta_{X'/S}$ of stacks of relative vectors.) Taking into account 3.3, we deduce:

Corollary 3.6. (a) *For X' to admit a lift to $\widetilde{S}$, it is necessary and sufficient that $\tau_{\leq 1}F_*\Omega^\bullet_{X/S}$ is decomposable.*

(b) *If this is the case, $\alpha \circ \Phi$ is an affine bijection from the set of isomorphism classes of lifts of X' to $\widetilde{S}$ to the set of decompositions of $\tau_{\leq 1}F_*\Omega^\bullet_{X/S}$, inducing the identity on the group of translations* $\mathrm{Ext}^1(\mathscr{H}^1F_*\Omega^\bullet_{X/S}, \mathscr{H}^0F_*\Omega^\bullet_{X/S})$ *(isomorphic, by Cartier, to* $\mathrm{Ext}^1(\Omega^1_{X'/S}, \mathscr{O}_{X'}) = \mathrm{H}^1(X', \Theta)$*).*

The arguments of 2.1(a) and the proof of 2.3 give as well:

Corollary 3.7. (a) *For each lift of X' to $\widetilde{S}$*

Thanks. We are grateful to J.-M. Fontaine and W. Messing for sharing with us their results of degenerence, which, together with that of K. Kato, served as a catalyst for this paper. It is a pleasure to thank G. Laumon for the work he has done since the beginning; his critical reading of a first version of this article and the suggestions he made to us were priceless. Finally, we would like to thank M. Raynaud for many discussions.

[3] [Trans]. Rel is an abbreviation for the French *relèvements*, for "lifts".

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Acronyms

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