HOMOLOGY OF COMMUTATIVE RINGS

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Chapter I.

§1. Differentials and derivations. All rings are commutative, associative, with identity and all modules are unitary unless stated otherwise. $\mathscr{R}$ denotes the category of rings. Also all diagrams are understood to be commutative.

Let A be a ring and let $\mathscr{C} = A\backslash\mathscr{R}$ be the category of A-algebras. An object of $\mathscr{C}$ is a ring B together with a map $i_B : A \to B$ of rings, and a map $f : B \to B'$ in $\mathscr{C}$ is a map of rings such that $f i_B = i_{B'}$. $\mathscr{C}$ is a category of universal algebras so the general theory of [HA, §5, p. 5.14] applies to define cohomology groups for an object B of $\mathscr{C}$ with values in a "B-module," where "B-module" in the general theory means an abelian group object in $\mathscr{C}/B$. We show in this section that this notion of B-module coincides with the usual one. Also we calculate the abelianization functor on $\mathscr{C}/B$ in terms of differentials.

Let B denote a fixed A-algebra and let $\mathscr{C}/B$ be the category of A-algebras over B. Here "over" is used as in category theory, so that an object of $\mathscr{C}/B$ is an A-algebra together with a map $u_X : X \to B$ of A-algebras. If M is a B-module, let $B \oplus M$ be the A-algebra over B with

$$(b \oplus m)(b' \oplus m') = bb' \oplus (bm' + b'm),$$
$$i_{B\oplus M}(a) = i_B(a),$$
$$u_{B\oplus M}(b \oplus m) = b.$$

If $\theta : Y \to B \oplus M$ is a map in $\mathscr{C}/B$, then $\theta y = u_Y y \oplus Dy$ where $D : Y \to M$ satisfies

$$Di_Y(a) = 0,$$
$$D(yy') = u_Y(y)Dy' + u_Y(y')Dy.$$

In other words D is an A-derivation of Y with values in M considered as a B-module via u_Y. Conversely given such a D be obtain a θ and therefore

$$\mathrm{Hom}_{\mathscr{C}/B}(Y, B \oplus M) \simeq \mathrm{Der}(Y/A, M). \tag{1.1}$$

This is an isomorphism of functors of Y. As $\mathrm{Der}(Y/A, M)$ is an abelian group under addition we see that $B \oplus M$ is an abelian group object of $\mathscr{C}/B$.

Recall that an H-object (H for H. Hopf) in a category is an object endowed with an operation having a two-sided identity. An H-object of $\mathscr{C}/B$ is therefore an object X together with maps

$$\varepsilon : B \longrightarrow X,$$
$$\mu : X \times_B X \longrightarrow X,$$

such that

$$\mu \circ (\varepsilon u_X, \mathrm{id}_X) = \mu \circ (\mathrm{id}_X, \varepsilon u_X) = \mathrm{id}_X\,. \tag{1.2}$$

For the abelian group object $B \oplus M$ one calculates that

$$\varepsilon(b) = b \oplus 0,$$
$$\mu(b \oplus m, b \oplus m') = b \oplus (m + m'). \tag{1.3}$$

Proposition 1.4. *Any H-object of $\mathscr{C}/B$ is isomorphic to $B \oplus M$ with multiplication (1.3) for some B-module M. In particular any H-object is an abelian group object.*

Proof. Given an H-object X, let $M = \ker(u_X : X \to B)$ considered as a B-module via $\varepsilon_X : B \to X$. If $x, y \in M$, then by (1.2)

$$\mu(x, 0) = x, \quad \mu(0, y) = y$$

so as μ is a homomorphism

$$xy = \mu(x, 0)\mu(0, y) = \mu((x, 0)(0, y)) = \mu(0, 0) = 0.$$

Thus M has the zero multiplication. It follows easily that the map $B \oplus M \to X$ given by $b \oplus m \mapsto \varepsilon_X b + m$ is an isomorphism in $\mathscr{C}/B$. Abbreviating ε_X to ε we have

$$\mu(\varepsilon b + m, \varepsilon b + m') = \mu(\varepsilon b, \varepsilon b) + \mu(m, 0) + \mu(0, m')$$
$$= \varepsilon b + m + m',$$

which shows that the multiplication μ on X is the same as (1.3) on $B \oplus M$. □

Let $(\mathscr{C}/B)_{\mathrm{ab}}$ be the category of abelian group objects in $\mathscr{C}/B$ and let $\mathscr{M}_B$ be the abelian category of B-modules. From 1.4 we obtain

Proposition 1.5. *There is an equivalence of categories*

$$\begin{gathered} B\oplus M \longleftarrow M \\ (\mathscr{C}/B)_{\mathrm{ab}} \simeq \mathscr{M}_B \\ X \longmapsto \ker u_X. \end{gathered}$$

In particular $(\mathscr{C}/B)_{\mathrm{ab}}$ is an abelian category.

If Y is an A-algebra, let $\Omega_{Y/A}$ be the Y-module of A-differentials of Y. There is a canonical A-derivation $d : Y \to \Omega_{Y/A}$ such that

$$\begin{gathered} \mathrm{Hom}_{\mathscr{M}_Y}(\Omega_{Y/A}, N) \simeq \mathrm{Der}(Y/A, N) \\ \theta \longmapsto \theta \circ d \end{gathered}$$

for any Y-module N. From 1.1 we have

$$\mathrm{Hom}_{\mathscr{C}/B}(Y, B\oplus M) \simeq \mathrm{Der}(Y/A, M) \simeq \mathrm{Hom}_{\mathscr{M}_Y}(\Omega_{Y/A}, M) \simeq \mathrm{Hom}_{\mathscr{M}_B}(\Omega_{Y/A}\otimes_Y B, M). \tag{1.6}$$

Thus

Proposition 1.7. *With respect to the equivalences of categories of 1.5 we have*

$$\begin{gathered} \Omega_{Y/B}\otimes_Y B \longleftarrow Y \\ \mathscr{M}_B \simeq (\mathscr{C}/B)_{\mathrm{ab}} \underset{i}{\overset{\mathrm{ab}}{\rightleftarrows}} \mathscr{C}/B \\ M \longmapsto B\oplus M, \end{gathered}$$

where i is the natural faithful functor and ab *is the abelianization functor, the left adjoint of i.*

From now on we will identify $(\mathscr{C}/B)_{\mathrm{ab}}$ and $\mathscr{M}_B$ by the equivalence of Prop. 1.5. 1.7 shows that $Y \mapsto \Omega_{Y/A}\otimes_Y B$ is identified with the abelianization functor on $\mathscr{C}/B$.

§2. Homology and cohomology. The q-th cohomology group of the A-algebra B with values in the abelian group object $B\oplus M$ of $\mathscr{C}/B$ [HA, II, p. 5.14] will be denoted $\mathrm{D}^q(B/A, M)$ and called simply the q-th cohomology group of the A-algebra B with values in the B-module M. According to Theorem 5 (loc. cit.) this may be defined is two different but equivalent ways—as shead cohomology from a Grothendieck topology and by (semi-)simplicial resolutions. Both definitions for $\mathrm{D}^q(B/A, M)$ will be given in this section. The former will be used in globalizing the definition to preschemes and the latter leads to a notion of homology for the A-algebra B.

(2.1). Let $\mathscr{T}$ be the Grothendieck topology [GT] whose underlying category is $\mathscr{C}/B$ and where a covering of Y is a family consisting of a single map $Z \to Y$ which is set-theoretically surjective. Representable functors are sheaves for $\mathscr{T}$ hence by 1.1 $Y \mapsto \mathrm{Der}(Y/A, M)$ is a sheaf of B-modules on $\mathscr{T}$. The first definition for the cohomology of B with values in M is

$$\mathrm{D}^q(B/A, M) \simeq \mathrm{H}^q_{\mathscr{T}}(B, \mathrm{Der}(-/A, M)) \tag{2.2}$$

where the right side denotes sheaf cohomology for the topology $\mathscr{T}$.

We will give other definitions for D^q in 2.14. It is first necessary to state some facts about simplicial objects.

(2.3). If X is a simplicial object in an abelian category, its homology in dimension q, denoted $\mathrm{H}_q(X)$, is defined to be the homology in dimension q of the chain complex contructed from X with differential $d = \sum_i (-1)^i d_i$. By the normalization theorem this is the same as $\mathrm{H}_q(NX)$ where NX is the normalized subcomplex of X. Hence when X is a simplicial abelian group, $\mathrm{H}_q(X)$ is the q-th homology group of X in the sense of Moore.

(2.4). If $f : X \to Y$ and $i : U \to V$ are maps in a category, we say that f has the right lifting property (RLP) with respect to i and that i has the left lifting property (LLP) with respect to f if given any commutative square of solid arrows

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ V \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

a dotted arrow exists such that the whole diagram is commutative. Let $\Delta(n)$ be the "standard n-simplex' simplicial set and let $i_n : \dot\Delta(n) \to \Delta(n)$ be the inclusion of its $(n-1)$-skeleton. The following proposition characterizes those maps of simplicial abelian groups which as trivial fibrations in the sense of [HA].

Proposition 2.5. *The following assertions are equivalent for a map $f : X \to Y$ of simplicial abelian groups.*

(i) *f is surjective (in each dimension) and $\mathrm{H}_*(f) : \mathrm{H}_*(X) \simeq \mathrm{H}_*(Y)$.*
(ii) *As a map of simplicial sets f has the RLP with respect to $i_n : \dot\Delta(n) \to \Delta(n)$ for all $n \geq 0$.*
(iii) *As a map of simplicial sets f has the RLP with respect to any injective (in each dimension) map of simplicial sets.*

This results from [HA, II, §3, Prop. 2].

(2.6). A map of simplicial rings is said to be a trivial fibration if as a map of simplicial abelian groups it satisfies the equivalent conditions of 2.5. A map of simplicial rings is called a cofibration if it has the LLP with respect to all trivial fibrations of simplicial rings. Cofibrations may be described in the following alternative way. Call a map $i : R \to S$ of simplicial rings free if there are subsets $C_q \subset S_q$, $q \geq 0$, such that (i) $\eta^* C_q \subset C_p$ whenever $\eta : [p] \to [q]$ is a surjective monotone map and (ii) S_q is a free R_q-algebra (polynomial ring) with generators C_q. Then Theorem 6.1 of [?] implies:

Proposition 2.7. *Any free map of simplicial rings is a cofibration.* The proof of Propostion 3 [HA, II, §4] yields

Proposition 2.8. *Any map f of simplicial rings may be factored $f = pi$ where i is free and p is a trivial fibration.*

Proposition 2.9. *A map $i : R \to S$ of simplicial rings is a cofibration if and only if there is a free map $j : R \to T$ and maps $u : T \to S$, $v : S \to T$ of simplicial rings under R such that $uv = \mathrm{id}_S$.*

Proof. The sufficiency is clear. If i is a cofibration let $i = uj$ be a factorization as in 2.8. Then v is obtained as the dotted arrow in

$$\begin{tikzcd} R \arrow[r, "j"] \arrow[d, "i"'] & T \arrow[d, "u"] \\ S \arrow[r, "\mathrm{id}_S"'] \arrow[ur, dotted] & S. \end{tikzcd}$$

□

(2.10). If K is a simplicial set and Y is a simplicial ring, then the function complex Y^K has a natural structure as a simplcial ring. Let $I = \Delta(1)$ and let $j_e : Y^I \to Y$, $e = 0, 1$ (resp. $\sigma : Y \to Y^I$) be the map induced by the inclusion of the e-th vertex $\Delta(0) \to \Delta(1)$ (resp. the unique map $\Delta(1) \to \Delta(0)$). If $f, g : X \rightrightarrows Y$ are two maps of simplicial rings then a (simplicial) homotopy from f to g may be identified with a map of simplicial rings $h : X \to Y^I$ such that $j_0 h = f$ and $j_1 h = g$. $\sigma : Y \to Y^I$ is of course the "constant" homotopy from id_Y to id_Y.

Proposition 2.11. *Let $i : R \to S$ be a cofibration and $f : X \to Y$ a trivial fibration of simplicial rings. Let $h : R \to X^I$ and $k : S \to Y^I$ be homotopies such that $ki = f^I h$, and let $\theta_0, \theta_1 : S \rightrightarrows X$ be maps with $\theta_e i = j_e h$ and $f\theta_e = j_e k$, $e = 0, 1$. Then there is a homotopy $H : S \to X^I$ with $Hi = h$, $f^I H = k$, and $j_e H = \theta_e$, $e = 0, 1$.*

Proof. Consider the square

$$\begin{tikzcd} R \arrow[r, "h"] \arrow[d, "i"'] & X^I \arrow[d, "(f^I, j_0, j_1)"] \\ S \arrow[r, "(k, \theta_0, \theta_1)"'] \arrow[ur, dotted, "H"] & Y^I \times_{Y^{\dot{I}}} X^{\dot{I}} \end{tikzcd}$$

where $X^{\dot{I}} = X \times X$. The map (f^I, j_0, j_1) is seen to be a trivial fibration using 2.5(iii), hence the dotted arrow exists and gives te desired homotopy. □

(2.12). Let $u : R \to S$ be a map of simplicial rings. By a cofibrant factorization of u we mean a factorization $R \xrightarrow{i} T \xrightarrow{p} S$ of u where i is a cofibration and p is a trivial fibration. If $R \to T' \to S$ is another cofibrant factorization of u, then by the definition of cofibration there are maps $\varphi : T \to T'$ and $\psi : T' \to T$ in the category $R \backslash s\mathscr{R}/S$. By 2.11 $\varphi\psi$ and $\psi\varphi$ are homotopic to $\mathrm{id}_{T'}$ and id_T respectively in this category. Therefore a cofibrant factorization of a map is unique up to simplicial homotopy under the source and over the target of the map.

(2.13). If X is an object of a category we let cX denote the "constant" simplicial object with $(cX)_q = X$, $\varphi^* = \mathrm{id}_X$ for all q, φ.

(2.14). Let A, B, M, and $\mathscr{C}$ be as in §1. A simplicial A-algebra over B (i.e. simplicial object of $\mathscr{C}/B$) P is the same as a factorization $cA \to P \to cB$ of the map $cA \to cB$. We shall call P a projective A-algebra resolution of B if this factorization is a cofibrant factorization. By 2.8 projective A-algebra resolutions of B exist. Choose one P, and let $\mathbf{L}\Omega_{B/A}$ be the simplicial B-module $\Omega_{P/A} \otimes_P B$ obtained by applying the "abelianization" functor $Y \mapsto \Omega_{Y/A} \otimes_Y B$ dimension-wise to P. The simplicial homotopy type of $\mathbf{L}\Omega_{B/A}$ is independent of the choice of P by 2.12. The underlying chain complex of $\mathbf{L}\Omega_{B/A}$ is called the cotangent complex of the A-algebra B and is unique up to chain homotopy equivalence of the choice of P. The following groups are then well-defined:

$$\mathrm{D}_q(B/A, M) = \mathrm{H}_q(\mathbf{L}\Omega_{B/A} \otimes_B M), \tag{2.15}$$

$$\mathrm{D}^q(B/A, M) = \mathrm{H}^q(\mathrm{Hom}_B(\mathbf{L}\Omega_{B/A}, M)). \tag{2.16}$$

We call these the q-th homology and cohomology of the A-algebra B with values in M. When $M = B$ we write simply $\mathrm{D}_q(B/A)$ and $\mathrm{D}^q(B/A)$.

Proposition 2.14. *The definitions 2.2 and 2.16 are consistent.*

Proof. We have

$$\mathrm{D}^q(B/A, M) = \mathrm{H}^q(\mathrm{Hom}_B(\Omega_{P/A} \otimes_P B, M)) = \mathrm{H}^q(\mathrm{Der}(P/A, M)),$$

so the result follows from [HA, II, §5, th. 5]. Alternatively it follows from a general theorem of Verdier (SGA, 1963–1964, Expose 5, Appendix) once one notes that a $\mathscr{T}$-hypercovering of B is the same as a simplicial A-algebra resolution and that a projective resolution is by (2.14) a cofinal object in the category of $\mathscr{T}$-hypercoverings of B. □

§3. Elementary properties. In this section we establish properties of the homology and cohomology groups which are elementary in the sense that they do not use special properties of commutative rings and hence are true without essential modification for all kinds of universal algebras.

Proposition 3.1. *If B is a free A-algebra, then $\mathbf{L}\Omega_{B/A} \simeq c\Omega_{B/A}$ and hence*

$$\mathrm{D}^q(B/A,M) = \mathrm{D}_q(B/A,M) = 0, \quad q > 0,$$
$$\mathrm{D}^0(B/A,M) = \mathrm{Der}(B/A,M),$$
$$\mathrm{D}_0(B/A,M) = \Omega_{B/A} \otimes_B M.$$

Proof. The identity map $cB \to cB$ makes cB a free simplicial A-algebra resolution of B, hence $\mathbf{L}\Omega_{B/A} \simeq c\Omega_{B/A}$ and the proposition follows. □

(3.2). In analogy with 2.6 we say that a simplicial module X over a simplicial ring R is free if there are subsets $C_q \subset X_q$ such that (i) $\eta^* C_q \subset C_p$ if $\eta : [p] \to [q]$ is a surjective monotone map and (ii) X_q is a free R_q-module with base C_q. X will be called projective if it is a direct summand of a free simplicial R-module.

If P is a free A-algebra resolution of B with generators $C_q \subset P_q$ as in 2.6, then $\{dx \otimes 1 : x \in C_q\} \subset \Omega_{P_q/A} \otimes_{P_q} B$ is a set of generators as in 3.2. Hence $\Omega_{P/A} \otimes_P B$ is a free simplicial B-module. If Q is a projective A-algebra resolution, then by 2.9 Q is a retract of a free one P and so $\Omega_{Q/A} \otimes_Q B$ is a projective simplicial B-module. In particular as a chain complex $\Omega_{Q/A} \otimes_Q B$ is projective in each dimension. Thus

Proposition 3.3. *$\mathbf{L}\Omega_{B/A}$ is a projective simplicial B-module.*

Corollary. *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of B-modules then there are long exact sequences*

$$0 \longrightarrow \mathrm{D}^0(B/A,M') \longrightarrow \mathrm{D}^0(B/A,M) \longrightarrow \mathrm{D}^0(B/A,M'') \longrightarrow \mathrm{D}^1(B/A,M') \longrightarrow \cdots, \tag{3.4}$$
$$\cdots \longrightarrow \mathrm{D}_1(B/A,M'') \longrightarrow \mathrm{D}_0(B/A,M') \longrightarrow \mathrm{D}_0(B/A,M) \longrightarrow \mathrm{D}_0(B/A,M'') \longrightarrow 0. \tag{3.5}$$

Corollary. *There are universal coefficient spectral sequences*

$$E^2_{pq} = \mathrm{Tor}^B_p(\mathrm{D}_q(B/A),M) \Longrightarrow \mathrm{D}_{p+q}(B/A,M), \tag{3.6}$$
$$E_2^{pq} = \mathrm{Ext}^p_B(\mathrm{D}_q(B/A),M) \Longrightarrow \mathrm{D}^{p+q}(B/A,M). \tag{3.7}$$

Proposition 3.8. $\mathrm{D}^0(B/A,M) = \mathrm{Der}(B/A,M)$, $\mathrm{D}_0(B/A,M) = \Omega_{B/A} \otimes_B M$.

Proof. As $Y \mapsto \Omega_{Y/A} \otimes_Y B$ is a left adjoint functor by 1.6, it is right exact. Hence $\mathrm{D}_0(B/A) = \Omega_{B/A}$ and the proposition follows from 3.6 and 3.7. □

Remark 3.9. The spectral sequence 3.7 shows that $M \mapsto \mathrm{D}^q(B/A,M)$ is the q-th derived functor of $M \mapsto \mathrm{Der}(B/A,M)$ if and only if $\mathrm{D}_q(B/A) = 0$ for $q > 0$. This can fail to be so even if A is a field, and in this respect commutative ring cohomology is quite distinct from group, Lie algebra, and associative algebra cohomology.

(3.10). By an extension of the A-algebra B by M we mean an exact sequence

$$0 \longrightarrow M \xrightarrow{i} X \xrightarrow{u} B \longrightarrow 0 \tag{3.11}$$

where u is a map of A-algebras such that $(\ker u)^2 = 0$ and where i induces an isomorphism of B-modules $M \xrightarrow{\sim} \ker u$, $\ker u$ being endowed with the B-module structure $u(x)y = x \cdot y$, $x \in X$, $y \in \ker u$. Let $\mathrm{Exalcomm}(B/A,M)$ be the set of isomorphism classes of extentions of the A-algebra B by M.

Proposition 3.12. *There is a canonical bijection*

$$\mathrm{D}^1(B/A,M) \simeq \mathrm{Exalcomm}(B/A,M).$$

Proof. Given a free A-algebra resolution P of B and an extention 3.11, choose a map $\theta : P_0 \to X$ of A-algebras over B. Then $i^{-1}\{\theta d_0 - \theta d_1\} : P_1 \to M$ is a normalized 1-cocycle of $\mathrm{Der}(P/A,M)$ whose cohomology class is independent of the choice of θ. This gives a map $\Phi : \mathrm{Exalcomm}(B/A,M) \to \mathrm{D}^1(B/A,M)$ which one may show is independent of the choice of P. Conversely given an A-derivation $D : P_1 \to M$ let

$$X = \mathrm{coker}\Big(P_1 \overset{(d_0,D)}{\underset{(d_1,0)}{\rightrightarrows}} P_0 \oplus M\Big),$$

the cokernel being in the category of A-modules, and let $p : P_0 \oplus M \to X$ be the canonical projection. Then X is a quotient A-algebra of $P_0 \oplus M$ and if D is a cocycle of $\mathrm{Der}(P/A,M)$ we obtain an extention 3.11 with $u(p(y \oplus m)) = \varepsilon y$, $i(m) = p(0 \oplus m)$. It is staightforward to verify that this procedure gives an inverse to Φ. □

Remark 3.13. It is also possible to prove 3.12 using the interpretation of the sheaf cohomology in terms of "torsors." By [?] one also gets a general interpretation of D^2 this way, however for rings the methods of the second chapter seem better for handling the higher cohomology.

Corollary 3.14. *Suppose that $A/I \simeq B$ where I is an ideal in A. Then $\mathrm{D}_0(B/A) = 0$ and $\mathrm{D}_1(B/A) \simeq I/I^2$.*

Proof. $\mathrm{D}_0(B/A) = 0$ by 3.8. From 3.7 and 3.12 we have $\mathrm{Hom}_B(\mathrm{D}_1(B/A), M) \simeq \mathrm{D}^1(B/A, M) \simeq \mathrm{Exalcomm}(B/A, M)$.

Let $\chi \in \mathrm{Exalcomm}(B/A, I/I^2)$ be the isomorphism class of the extention $0 \to I/I^2 \to A/I^2 \to B \to 0$. By functorality χ defines a natural transformation $\chi_* : \mathrm{Hom}_N(I/I^2, M) \to \mathrm{Exalcomm}(B/A, M)$. Conversely given an extention 3.11, choose a map $\theta : A \to X$ with $u\theta = i_B$; the restriction of θ to I gives rise to a homomorphism of B-modules $I/I^2 \to M$. This procedure is easily seen to define an inverse to χ_*, hence $\mathrm{Hom}_B(\mathrm{D}_1(B/A), M) \simeq \mathrm{Hom}_B(I/I^2, M)$ and so $\mathrm{D}_1(B/A) \simeq I/I^2$. ☐

§4. Further properties. In this section we establish properties of the cohomology theory which are peculiar to commutative rings, since they depend on the fact that the direct sum in the category of A-algebras is given by the tensor product. We begin by recalling facts about simplicial modules which are proved in [HA, II, §6].

(4.1). Let R be a simplicial ring and let $\mathscr{M}_R$ be the abelian category of simplicial R-modules. The homotopy category $\mathrm{Ho}(\mathscr{M}_R)$ is obtained from $\mathscr{M}_R$ by formally adjoining the inverses of the weak equivalences (maps which induce isomorphisms on homology). $\mathrm{Ho}(\mathscr{M}_R)$ is equivalent to the category whose objects are the projective simplicial R-modules (3.2) with homotopy classes of maps for morphisms. When $R = cA$, $\mathrm{Ho}(\mathscr{M}_R)$ is equivalent to the full subcategory of the derived category of A-modules consisting of the chain complexes.

(4.2). If X, Y are two simplicial R-modules, let $\mathrm{Tor}_i^R(X, Y)$ be the simplicial R-module obtained by first applying the tri-functor $\mathrm{Tor}_i^-(-,-)$ dimension-wise. Let $X \otimes_R^{\mathbf{L}} Y$ denote the derived tensor product of X and Y. $X \otimes_R^{\mathbf{L}} Y$ is isomorphic in $\mathrm{Ho}(\mathscr{M}_R)$ to $P \otimes_R Q$, where P and Q are projective resolutions of X and Y respectively. There is a spectral sequence ([HA, II, §6, th. 6])

$$E^2_{pq} = \mathrm{H}_p(\mathrm{Tor}_q^R(X, Y)) \Longrightarrow \mathrm{H}_{p+q}(X \otimes_R^{\mathbf{L}} Y) \tag{4.3}$$

one of whose edge homomorphisms is the map on homology induced by the canonical map $X \otimes_R^{\mathbf{L}} Y \simeq P \otimes_R Q \to X \otimes_R Y$. Consequently we have

Proposition 4.3. *If $\mathrm{Tor}_q^R(X, Y) = 0$ for $q > 0$, then $X \otimes_R^{\mathbf{L}} Y \simeq X \otimes_R Y$.*

(4.4). Let $u : R \to S$ be a map of simplicial rings. Define $\mathbf{L}\Omega_{S/R}$ to be the projective simplicial S-module $\Omega_{P/R} \otimes_P S$ where $R \to P \to S$ is a cofibrant factorization of u (2.12). As an object of $\mathrm{Ho}(\mathscr{M}_S)$ it is independent up to isomorphism of the choice of factorization. If X is a simplicial S-module we define

$$\mathrm{D}_q(S/R, X) = \mathrm{H}_q(\mathbf{L}\Omega_{S/R} \otimes_S^{\mathbf{L}} X) \simeq \mathrm{H}_q(\mathbf{L}\Omega_{S/R} \otimes_S X), \tag{4.5}$$

where the last isomorphism is from 4.3. It is clear that this definition specializes to 2.15 in the case where u is the map $cA \to cB$ and $X = cM$. Moreover the obvious generalization of 3.5 holds and 3.6 generalizes by ([HA, II, th. 6(c)]) to a spectral sequence

$$E^2_{pq} = \mathrm{H}_p(\mathrm{D}_q(S/R) \otimes_S^{\mathbf{L}} X) \Longrightarrow \mathrm{D}_{p+q}(S/R, X), \tag{4.6}$$

where $\mathrm{D}_q(S/R) = \mathrm{D}_q(S/R, S)$.

Proposition. *Suppose $u : R \to R'$ and $v : R \to S$ are maps of simplicial rings such that $\mathrm{Tor}_q^R(R', S) = 0$ for $q > 0$. If $S' = S \otimes_R R'$, then there are canonical isomorphisms in $\mathrm{Ho}(\mathscr{M}_{S'})$*

$$\mathbf{L}\Omega_{S/R} \otimes_R R' \simeq \mathbf{L}\Omega_{S'/R'}, \tag{4.7}$$

$$\mathbf{L}\Omega_{S'/R} \simeq \mathbf{L}\Omega_{S/R} \otimes_R R' \oplus \mathbf{L}\Omega_{R'/R} \otimes_R S. \tag{4.8}$$

Proof. First observe that if $R \to P$ is a cofibration (resp. free map) of simplicial rings, then P is a projective (resp. free) R-module. Using 2.9, one reduces to the case where P is free, whence if C_* is an R-algebra basis for P as in 2.6, then the monomials in the elements of C_* form an R-module basis for P as in 3.2.

Now let $R \to P \to S$ be a cofibrant factorization of $R \to S$. As $P \to S$ is a weak equivalence of R-modules, so is $P \otimes_R^{\mathbf{L}} R' \to S \otimes_R^{\mathbf{L}} R'$, since $- \otimes_R^{\mathbf{L}} -$ is a functor on $\mathrm{Ho}(\mathscr{M}_R)$. By hypothesis, the fact that P is a projective R-module by the above remarks, and 4.3, this map is isomorphic to the map $P \otimes_R R' \to S'$. Hence this last map is a weak equivalence; as it is a trivial fibration. As cobase extension preserves cofibrations, it follows that $R' \to P \otimes_R R' \to S'$ is a cofibrant factorization of $R' \to S'$ and hence setting $P' = P \otimes_R R'$

$$\mathbf{L}\Omega_{S'/R'} = \Omega_{P'/R'} \otimes_{R'} S' \simeq (\Omega_{P/R} \otimes_R S) \otimes_R R' \simeq \mathbf{L}\Omega_{S/R} \otimes_R R'$$

which proves 4.7. To prove 4.8, let $R \to Q \to R'$ be a cofibrant factorization of $R \to R'$. By the same argument as above we find that $R \to P \otimes_R Q \to S'$ is a cofibrant factorization of $R \to S'$ and so

$$\begin{aligned}
\mathbf{L}\Omega_{S'/R'} &= \Omega_{P \otimes_R Q/R} \otimes_{P \otimes_R Q} S' \simeq (\Omega_{P/R} \otimes_R Q \oplus \Omega_{Q/R} \otimes_R P) \otimes_{P \otimes_R Q} S' \\
&\simeq (\Omega_{P/R} \otimes_P S) \otimes_R R' \oplus (\Omega_{Q/R} \otimes_Q R') \otimes_R S \\
&\simeq \mathbf{L}\Omega_{S/R} \otimes_R R' \oplus \mathbf{L}\Omega_{R'/R} \otimes_R S.
\end{aligned}$$

The resulting isomorphisms in $\mathrm{Ho}(\mathscr{M}_{S'})$ are canonical since they are independent of the choices of P, Q by 2.11. ☐

Corollary 4.9. *Let B and C be A-algebras and let N be a $B \otimes_A C$-module. If $\mathrm{Tor}_q^A(B,C) = 0$ for $q > 0$, then there are isomorphisms*

$$\mathrm{D}^q(B \otimes_A C/C, N) = \mathrm{D}^q(B/A, N),$$

$$\mathrm{D}^q(B \otimes_A C/A, N) = \mathrm{D}^q(B/A, N) \oplus \mathrm{D}^q(C/A, M),$$

and similarly for homology.

(4.10). If $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathscr{M}_R$, then in $\mathrm{Ho}(\mathscr{M}_R)$ there is a cofibration sequence ([HA, II, §6])

$$X' \longrightarrow X \longrightarrow X'' \xrightarrow{\partial} \Sigma X', \tag{4.11}$$

where Σ denotes the suspension functor on $\mathrm{Ho}(\mathscr{M}_R)$. When $R = cB$ and we identify $\mathrm{Ho}(\mathscr{M}_R)$ with the subcategory of the derived category of B-modules consisting of chain complexes, then the suspension functor shifts a complex to the left and the sequence 4.11 is the distinguished triangle associated to the exact sequence.

Theorem. *Let $R \xrightarrow{u} S \xrightarrow{v} T$ be maps of simplicial rings. Then there is a canonical cofibration sequence* $\mathrm{Ho}(\mathscr{M}_T)$

$$\mathbf{L}\Omega_{S/R} \otimes_S T \longrightarrow \mathbf{L}\Omega_{T/R} \longrightarrow \mathbf{L}\Omega_{T/S} \longrightarrow \Sigma(\mathbf{L}\Omega_{S/R} \otimes_S T). \tag{4.12}$$

Proof. Form a diagram

$$\begin{tikzcd}
 & & Q \arrow[d, "i_2"] \arrow[ddrr, "q"] & & \\
 & P \arrow[ur, "j"] \arrow[d, "p"] & S \otimes_P Q \arrow[drr, "r"] & & \\
R \arrow[ur, "i"] \arrow[r, "u"] & S \arrow[ur, "i_1"] \arrow[rrr, "v"] & & & T
\end{tikzcd} \tag{4.13}$$

by choosing cofibrant factorizations $u = pi$ and $vp = qj$, and then filling in the rest of the diagram in the obvious way.

As j is a cofibration Q is a projective P-module, hence by 4.3 the map i_2 is isomorphic to $p \otimes_P^{\mathbf{L}} \mathrm{id}_Q : P \otimes_P^{\mathbf{L}} Q \to S \otimes_P^{\mathbf{L}} Q$, which is a weak equivalence since p is. As i_2 and q are weak equivalences so is r; r is also surjective since q is and therefore r is a trivial fibration of simplicial S-algebras. As j is a cofibration so is i_1; therefore $v = ri_1$ is a cofibrant factorization.

Suppose for the moment that 4.13 is a diagram of rings where Q is a free P-algebra. If N is a T-module, then there is an exact sequence

$$0 \longrightarrow \mathrm{Der}(Q/P, N) \longrightarrow \mathrm{Der}(Q/R, N) \xrightarrow{j^*} \mathrm{Der}(P/R, N) \longrightarrow 0$$

where j is onto because Q is a free P-algebra and hence there is an R-algebra map from Q to P which is left inverse to j. As this sequence is functorial in N it comes from an exact sequence of T-modules

$$0 \longrightarrow (\Omega_{P/R} \otimes_P S) \otimes_S T \longrightarrow \Omega_{Q/R} \otimes_Q T \longrightarrow \Omega_{S \otimes_P Q/S} \otimes_{S \otimes_P Q} T \longrightarrow 0. \tag{4.14}$$

This last sequence is seen to be functorial in the diagram of rings 4.13, hence applying this functorial exact sequence dimension-wise to 4.13 now considered as a diagram of simplicial rings we obtain an exact sequence 4.14 of simplicial T-modules.

Now using that $S \otimes_P Q$ is a projective S-algebra resolution of T, that Q is a projective R-algebra resolution of T, and that P is a projective R-algebra resolution of S, we see that the cofibration sequence in $\mathrm{Ho}(\mathscr{M}_T)$ associated to 4.14 is the desired cofibration sequence 4.12.

It remains to show this cofibration sequence is independent of the choice of diagram 4.13. Suppose given a diagram of simplicial rings

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & T \arrow[d] \\
R' \arrow[r] & S' \arrow[r] & T'
\end{tikzcd} \tag{4.15}$$

and suppose given a diagram $(4.13)'$ similar to 4.13 but with primes. By lifting successively in the diagrams

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & P' \arrow[d] \\
P \arrow[r] \arrow[ur, dotted] & S'
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r] \arrow[d] & Q' \arrow[d] \\
Q \arrow[r] \arrow[ur, dotted] & T'
\end{tikzcd} \tag{4.16}$$

we obtain a map from 4.13 to $(4.13)'$, hence a map from the exact sequence 4.14 to the corresponding exact sequence $(4.14)'$, and finally a map of cofibration sequences. The resulting map is independent of the choices of liftings in 4.16 because by 2.11 two liftings in the first square are joined by a homotopy which may then be extended to a homotopy between the liftings in the second square. It follows that the map of exact sequences 4.14 to $(4.14)'$ is unique up to homotopy and hence the map of cofibration sequences is well-defined. □

Specializing to constant simplicial rings, we have

Corollary 4.17. *If $A \to B \to C$ are maps of rings, then there is a canonical exact triangle in the derived category of C-modules*

$$\begin{tikzcd} \mathbf{L}\Omega_{B/A} \otimes_B C \arrow[rr] & & \mathbf{L}\Omega_{C/A} \arrow[dl] \\ & \mathbf{L}\Omega_{C/B}. \arrow[ul, dotted] & \end{tikzcd}$$

Hence if M is a C-module, there are canonical exact sequences

$$0 \longrightarrow \mathrm{D}^0(C/B, M) \longrightarrow \mathrm{D}^0(C/A, M) \longrightarrow \mathrm{D}^0(B/A, M) \longrightarrow \mathrm{D}^1(C/B, M) \longrightarrow \cdots,$$

$$\cdots \longrightarrow \mathrm{D}_1(C/B, M) \longrightarrow \mathrm{D}_0(B/A, M) \longrightarrow \mathrm{D}_0(C/A, M) \longrightarrow \mathrm{D}_0(C/B, M) \longrightarrow 0.$$

§5. Some applications. In this section we extend to all q certain vanishing results for $\mathrm{D}^q(B/A, M)$ which we proved in [?] for $q = 1$ and in [?] for $q = 1, 2$. We shall state these results only for the cotangent complex $\mathbf{L}\Omega_{B/A}$ leaving the translation for the functors D^q and D_q to the reader.

Proposition 5.1. *If S is a multiplicative system in A, then $\mathbf{L}\Omega_{S^{-1}A/A} \simeq 0$.*

Proof. (after André [?, 20.1]) Let $C = S^{-1}A$. As $C \otimes_A C \simeq C$ we have an isomorphism $\mathbf{L}\Omega_{C/A} \otimes_A C \simeq \mathbf{L}\Omega_{C/A}$ of projective simplicial C-modules. As C is flat over A the former complex by 4.7 is isomorphic to $\mathbf{L}D_{C\otimes_A C/C} \simeq \mathbf{L}\Omega_{C/C} \simeq 0$. □

Corollary 5.2. *Suppose T is a multiplicative system in B and that S is a multiplicative system in A which is carried into T by the homomorphism $A \to B$. Then*

$$\mathbf{L}\Omega_{T^{-1}B/S^{-1}A} \simeq \mathbf{L}\Omega_{B/A} \otimes_B T^{-1}B.$$

Proof. Applying 4.17 to $A \to B \to T^{-1}B$ and $A \to S^{-1}A \to T^{-1}B$ and using 5.1 we have

$$\mathbf{L}\Omega_{B/A} \otimes_B T^{-1}B \simeq \mathbf{L}\Omega_{T^{-1}B/A} \simeq \mathbf{L}\Omega_{T^{-1}B/S^{-1}A}.$$

□

Proposition 5.3. *Suppose that A is Noetherian and B is of finite type as an A-algebra. Then*

B is étale over A $\Longleftrightarrow$ $\mathbf{L}\Omega_{B/A} \simeq 0$;

B is smooth over A $\Longleftrightarrow$ $\mathbf{L}\Omega_{B/A} \simeq c\Omega_{B/A}$ and $\Omega_{B/A}$ is a projective B-module.

Proof. ($\Longrightarrow$) Suppose B étale over A, i.e. B is flat over A and $\Delta : \operatorname{Spec} B \to \operatorname{Spec} B \otimes_A B$ is an open immersion. Let $\mathfrak{p}$ be a prime ideal of B and let $\mathfrak{q} = \Delta(\mathfrak{p})$ so that $(B \otimes_A B)_\mathfrak{q} \simeq B_\mathfrak{p}$. Then there is an isomorphism of projective $B_\mathfrak{p}$-modules

$$\begin{aligned} (\mathbf{L}\Omega_{B/A})_\mathfrak{p} &\simeq (\mathbf{L}\Omega_{B/A} \otimes_A B)_\mathfrak{q} \\ &\simeq (\mathbf{L}\Omega_{B\otimes_A B/B})_\mathfrak{q} \quad \text{by 4.9 since } B \text{ is flat over } A. \\ &\simeq \mathbf{L}\Omega_{(B\otimes_A B)_\mathfrak{q}/B_\mathfrak{p}} \quad \text{by 5.2} \\ &\simeq 0. \end{aligned}$$

Thus $\mathrm{H}_*((\mathbf{L}\Omega_{B/A})_\mathfrak{p}) = \mathrm{H}_*(\mathbf{L}\Omega_{B/A})_\mathfrak{p} = 0$ as as $\mathfrak{p}$ is an arbitrary prime ideal of B, we have $\mathbf{L}\Omega_{B/A} \simeq 0$.

Now suppose B smooth over A. As $\mathrm{H}_0(\mathbf{L}\Omega_{B/A}) \simeq \Omega_{B/A}$ (3.8) there is a canonical map of projective simplicial B-modules $\mathbf{L}\Omega_{B/A} \to c\Omega_{B/A}$. To prove this is an isomorphism we reduce by localizing on B to the case where $A \to B$ may be factoried $A \to P \to B$ where P is a polynomial ring over A and $P \to B$ is étale. Then by 4.17, the étale case of 5.3, and 3.1 we have

$$\begin{tikzcd} \mathbf{L}\Omega_{P/A} \otimes_P B \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathbf{L}\Omega_{B/A} \arrow[d] \\ \Omega_{P/A} \otimes_P B \arrow[r, "\sim"] & c\Omega_{B/A} \end{tikzcd}$$

which proves the assertion.

($\Longleftarrow$) The spectral sequence 3.7 degenerates yielding $\mathrm{D}^1(B/A, M) = 0$ for all B-modules M. By 3.12 all A-algebra extentions of B by an ideal of square zero split hence B is smooth over A by SGA, 1960–1961, III, 2–3. If also $\Omega_{B/A} = 0$, then B is étale over A. □

We know wish to give a reasonably "geometric" example where $\mathrm{D}_1(B/A) \neq 0$. The following results from 4.17 and 3.14.

Proposition 5.4. *Suppose that $A \to P \to B$ is a factorization of $A \to B$ where P is a polynomial ring over A and $P \to B$ is surjective with kernel I. Then*

$$\mathrm{D}_q(B/A, M) \simeq \mathrm{D}_q(B/P, M), \quad q \geq 2,$$

and there is an exact sequence

$$0 \longrightarrow \mathrm{D}_1(B/A, M) \longrightarrow I/I^2 \otimes_B M \longrightarrow \Omega_{P/A} \otimes_P M \longrightarrow \Omega_{B/A} \otimes_B M \longrightarrow 0$$

and similar assertions hold for cohomology.

Example 5.5. Supose that k is an algebraically closed field and that R is the coordinate ring of the curve $x = t^3$, $y = t^4$, $z = t^5$. Then R is an integral domain finitely generated over k. We show that $\mathrm{D}_1(R/k) \neq 0$. $R = P/I$ where $P = k[X,Y,Z]$ and $I = (Y^2 - XZ, YZ - X^3, Z^2 - XY)$. The element $u = XY(Y^2 - XZ) - X(YZ - X^3) + Z(Z^2 - X^2Y)$ is in I but not in I^2. In effect if $\mathfrak{m} = (X,Y,Z)$, $I^2 \subset \mathfrak{m}^4$ and $u \notin \mathfrak{m}^4$. But by 5.4 we have the exact sequence

$$0 \longrightarrow \mathrm{D}_1(R/k) \longrightarrow I/I^2 \xrightarrow{\delta} \Omega_{P/k} \otimes_P R \longrightarrow \Omega_{R/k} \longrightarrow 0$$

and a short calculation shows that $\delta(u + I^2) = 0$. Hence $\mathrm{D}_1(R/k) \neq 0$ as asserted. It may be worth remarking that I is a prime ideal in P such that I^2 is not primary; in fact $\operatorname{im}\delta = I/I^{(2)}$ where $I^{(2)}$ is the I-primary component of I^2.

Chapter II. The fundamental spectral sequence

In order to calculate $\mathrm{D}_*(B/A, M)$ one is reduced by 4.17 to the case where $B = A/I$, I an ideal in A. In this case there is a spectral sequence which relates these groups to $\mathrm{Tor}^A_*(B, M)$, which is more easily computable. In this chapter we derive this spectral sequence and give some of its applications.

§6. Construction of the spectral sequence. We retain the notations of the preceding chapter except that certain rings will not be commutative, but skew-commutative with respect to a canonical grading.

Let P be a free simplicial A-algebra resolution of B. Then $Q = P \otimes_A B$ is a simplicial augmented B-algebra. If $J = \ker(P \otimes_A B \to B)$ is the augmentation ideal, then

$$Q \supset J \supset J^2 \supset \cdots \tag{6.1}$$

is a filtration of Q by simplicial ideals. By means of the shuffle operation $\underline{\otimes}$ ([HA, II, p. 6.6, (6)]), Q with differential $d = \sum_i (-1)^i d_i$ becomes a skew-commutative differential graded ring and 6.1 is a filtration of Q by differential graded ideals. Consequently we obtain a spectral sequence of alegbras where

$$E^2_{pq} = \mathrm{H}_{p+q}(J^q/J^{q+1}), \quad d^r : E^r_{pq} \longrightarrow E^r_{p-r, q+r-1}, \tag{6.2}$$

and ignoring for the moment questions of convergence, whose abutment is $\mathrm{H}_*(Q) \simeq \mathrm{Tor}^A_*(B,B)$. As P is free over A, Q is free over B hence there is an isomorphism of graded simplicial algebras

$$\bigoplus_q S^B_q(J/J^2) \simeq \bigoplus_q J^q/J^{q+1}, \tag{6.3}$$

where the left side the symmetric algebra functor over B applied dimension-wise to the simplicial B-module J/J^2. Applying 5.4 dimension-wise to the maps $cA \to P \to cB$ we obtain isomorphisms of simplicial B-modules

$$J/J^2 \simeq \Omega_{P/A} \otimes_P B \simeq \mathbf{L}\Omega_{B/A}. \tag{6.4}$$

We now turn to the convergence of this spectral sequence.

Lemma 6.5. *Suppose that Q is a projective augmented simplicial B-algebra with augmentation idela J. If $\mathrm{H}_0(J) = 0$, then $\mathrm{H}_k(J^n) = 0$ for $k < n$.*

A more general proof of this lemma will be proven in 8.8. An alternative proof of 6.5 in outline is as follows. The arguments in [?, §4] are very general and show that it is sufficient to prove 6.5 when $Q = S^B X$ and $X = K(B,1)^r$ where $K(B,1) = B\Delta(1)/B\dot{\Delta}(1)$ is the simplicial B-module whose normalization is the complex with B in dimension 1 and 0 elsewhere. In this case one may apply known results on the connectivity of the symmetric algebra functor [?], in particular the following which will be proved in 7.32.

Lemma. *Suppose that X is a flat simplicial B-module with $\mathrm{H}_0(X) = 0$. Then*

$$\mathrm{H}_q(S^B_n X) = 0, \quad q < n,$$

and there is a graded algebra isomorphism

$$\bigoplus_n \wedge^B_n \mathrm{H}_1(X) \simeq \bigoplus_n \mathrm{H}_n(S^B_n X),$$

where $\wedge^B$ is the exterior algebra functor on B-modules.

In virtue of the augmentation, $\mathrm{H}_0(J) = 0$ is equivalent to $\mathrm{H}_0(Q) \simeq B$, which when $Q = P \otimes_A B$ means $B \otimes_A B \simeq B$. In this case the spectral sequence 6.2 consructed from the J-adic filtration on Q converges by 6.5, that is, $E^r_{pq} = E^\infty_{pq}$ for $r > p + q$ and moreover $E^2_{pq} = 0$ if p of $q < 0$ by 6.5. Combining 6.2–6.4, we therefore have

Theorem 6.8. *If $B \otimes_A B \simeq B$, then there is a first quadrant spectral sequence*

$$E^2_{pq} = \mathrm{H}_{p+q}(S^B_q \mathbf{L}\Omega_{B/A}) \Longrightarrow \mathrm{Tor}^A_{p+q}(B,B)$$

of bigraded algebras, skew-commutative for the total degree.
Picture of spectral sequence:

$$\begin{array}{c|c|c|c|c|}
\cline{2-5}
q & & & & \\
\cline{2-5}
 & \wedge^3 \mathrm{D}_1 & & & \\
\cline{2-5}
 & \wedge^2 \mathrm{D}_1 & & & \\
\cline{2-5}
 & \mathrm{D}_1 & \mathrm{D}_2 & \mathrm{D}_3 & \\
\cline{2-5}
 & B & 0 & 0 & \\
\cline{2-5}
\multicolumn{4}{c}{} & \multicolumn{1}{c}{p}
\end{array}\ .$$

Edge homomorphisms

$$\mathrm{Tor}_n^A(B,B) \longrightarrow \mathrm{D}_n(B/A), \quad n>0, \tag{6.9}$$
$$\wedge_n^B \mathrm{D}_1(B/A) \longrightarrow \mathrm{Tor}_n^A(B,B). \tag{6.10}$$

Low dimensional isomorphisms

$$\mathrm{D}_0(B/A)=0,$$
$$\mathrm{D}_1(B/A) \simeq \mathrm{Tor}_1^A(B,B). \tag{6.11}$$

Five-term exact sequence

$$\mathrm{Tor}_3^A(B,B) \longrightarrow \mathrm{D}_3(B/A) \xrightarrow{d_2} \wedge_2^B \mathrm{D}_1(B/A) \longrightarrow \mathrm{Tor}_2^A(B,B) \longrightarrow \mathrm{D}_2(B/A) \longrightarrow 0. \tag{6.12}$$

6.10 is the unique graded B-algebra morphism extending the isomorphism 6.11.

When $B=A/I$ where I is an ideal in A the hypothesis of 6.8 holds and in this case we may avail ourselves of the isomorphism

$$\mathrm{D}_1(B/A) \simeq \mathrm{Tor}_1^A(B,B) \simeq I/I^2, \tag{6.13}$$

and rewrite the edge homomorphism 6.10 in the form

$$\wedge_n^B(I/I^2) \longrightarrow \mathrm{Tor}_n^A(B,B). \tag{6.14}$$

Proposition 6.15. *The edge homomorphism (6.9) annihilates the decomposable elements of* $\mathrm{Tor}_*^A(B,B)$*.*

Proof. For $n>0$, $\mathrm{Tor}_n^A(B,B)=\mathrm{H}_n(Q)\simeq \mathrm{H}_n(J)$. If $\alpha \in \mathrm{Tor}_p$, $p>0$, is represented by $x \in J_p$ and $\beta \in \mathrm{Tor}_q$, $q>0$, is represented by $y \in J_q$, then $\alpha\cdot\beta \in \mathrm{Tor}_{p+q}$ is represented by $\mu(x\underline{\otimes}y) \in J^2_{p+q}$, where $\mu : Q\otimes Q \to Q$ is the multiplication. But the edge homomorphism 6.9 is induced by $J \to J/J^2$, hence the image of $\alpha\cdot\beta$ in $\mathrm{D}_{p+q}(B/A)$ is zero. □

If M is a B-module the as J^q and J/J^2 are projective simplicial B-modules

$$Q\otimes_B M \supset J\otimes_B M \supset \cdots$$

is a filtered simplicial module over the filtered simplicial ring Q with

$$\mathrm{gr}\, Q\otimes_B M \simeq S^B(J/J^2)\otimes_B M.$$

Hence

Theorem 6.16. *If $B\otimes_A B \simeq B$, then there is a spectral sequence*

$$E^2_{pq} = \mathrm{H}_{p+q}(S_q^B \mathbf{L}\Omega_{B/A}\otimes_B M) \Longrightarrow \mathrm{Tor}^A_{p+q}(B,M)$$

which is a spectral sequence of modules over the spectral sequence 6.12.

It is easy to verify that this spectral sequence has the following properties:

Edge homomorphisms:

$$\mathrm{Tor}_n^A(B,M) \longrightarrow \mathrm{D}_n(B/A,M), \quad n>0,$$
$$\wedge_n^B \mathrm{D}_1(B/A)\otimes_B M \longrightarrow \mathrm{Tor}_n^A(B,M).$$

Low-dimensional isomorphisms:

$$\mathrm{D}_0(B/A,M)=0,$$
$$\mathrm{D}_1(B/A,M)=\mathrm{Tor}_1^A(B.M)\simeq \mathrm{D}_1(B/A)\otimes_B M.$$

Five-term exact sequence:

$$\mathrm{Tor}_3^A(B,M) \longrightarrow \mathrm{D}_3(B/A,M) \longrightarrow \wedge^2 \mathrm{D}_1(B/A)\otimes_B M \longrightarrow \mathrm{Tor}_2^A(B,M) \longrightarrow \mathrm{D}_2(B/A,M) \longrightarrow 0$$

and that $\mathrm{Tor}_+^A(B,B)\,\mathrm{Tor}_+^A(B,M) \subset \mathrm{Tor}_+^A(B,M)$ is annihilated by the edge homomorphism.

Remark 6.17. The condition $B\otimes_A B \simeq B$ is necessary as is shown by the Example 5.5 where B is flat over A and $\mathrm{D}_1(B/A) \neq 0$.

Remark 6.18. The spectral sequences 6.8 and 6.16 are functorial in the triple A, B, M since the only choice made in their construction was the free A-algebra resolution P of B which is seen to be unique and functorial in A, B using 2.11.

§7. Homology of the symmetric algebra. In order to use the spectral sequence 6.8 it is necessary to have results relating to the homology of the symmetric algebra of a simplicial module with the homology of the module. In this rather long section we collect the results that we need. They include a connectivity assertion (7.3), calculation of the first non-vanishing homology groups (7.27), and a calculation in the case where the ground ring is of characteristic zero (7.43). The symmetric algebra functor is closely connected with Eilenberg-MacLane spaces in topology and at tthe end of this section we outline this connection.

(7.1). Let F be a functor defined on the category of ring-modules (B,M) consisting of a ring B and a B-module M and having values in an abelian category $\mathscr{A}$. The functors we have in mind are the symmetric algebra S, the exterior algebra $\wedge$, the divided power algebra Γ, as well as any of the tensor products built up from homogeneous components of these functors. If R is a simplicial ring and X is a simplicial R-module, then applying F dimension-wise to X we obtain a simplicial object $F(R,X)$. For the most part R will be fixed and we will write simply $F(X)$ when there is no possibility of confusion.

The left-derived functor $\mathbf{L}F$ of F is defined by $\mathbf{L}F(X)=F(P)$, where $P\to X$ is a projective resolution of X. The homotopy type of the simplicial object $\mathbf{L}F(X)$ is independent of the choice of P and $\mathbf{L}F$ is a functor from $\mathrm{Ho}(\mathscr{M}_R)$ to the category $\pi_0\, s\mathscr{A}$ whose objects are the same as $s\mathscr{A}$ but with homotopy classes of maps for morphisms. The map $P\to X$ gives rise to a natural transformation $\mathbf{L}F(X)\to F(X)$.

(7.2). A map $f:X\to Y$ of simplicial objects in an abelian category will be called a k-equivalence if $f_*:\mathrm{H}_q(X)\to\mathrm{H}_q(Y)$ is an isomorphism for $q\le k$. X is said to be k-connected if $\mathrm{H}_q(X)=0$ for $q<k$.

Proposition 7.3. *If $f:X\to Y$ is a k-equivalence so is $\mathbf{L}F(f)$.*

Proof. We may assume X and Y are free simplicial R-modules and drop the $\mathbf{L}$. By "attaching cells" to X we will now construct a free map $X\to X'$ which is an isomorphism in dimensions $\le k+1$ such that $\mathrm{H}_q(X')=0$ for $q>k$. If X is a simplicial set, let RK be the free simplicial R-module generated by K ($K\mapsto RK$ is left adjoint to the forgetful functor $\mathscr{M}_R\to\mathrm{sSet}$). Let $\alpha_i\in\mathrm{H}_{k+1}(X)$ be elements which generate $\mathrm{H}_{k+1}(X)$ as an $\mathrm{H}_0(X)$-module and choose a representative $x_i\in N_{k+1}(X)$ for α_i. Here $N(X)$ is the normalized chain complex of X. Let $u_i:R\Delta(k+2)\to X$ be the unique simplicial R-module map sending $d_j\,\mathrm{id}_{[k+2]}$ to x_i for $j=0$ and 0 for $j=1,\dots,k+2$. Define $X\to X^{(1)}$ by a co-cartesian diagram

$$\begin{tikzcd} \bigoplus_i R\Delta(k+2) \arrow[r] \arrow[d, "\sum_i u_i"'] & \bigoplus_i R\Delta(k+2) \arrow[d] \\ X \arrow[r] & X^{(1)}. \end{tikzcd}$$

The map $X\to X^{(1)}$ is an isomorphism in dimensions $\le k+1$. The cokernel of both horizontal maps of this square is

$$\textstyle\bigoplus_i R\Delta(k+2)/R\Delta(k+2)$$

whose homology is a free H_*R-module on generators of dimension $k+2$ corresponding to the elements of I (see [HA, II, p. 5.11, assertion A]). The long exact sequence in homology for the exact sequence containing $X\to X^{(1)}$ is thus

$$\cdots\longrightarrow\textstyle\bigoplus_I \mathrm{H}_0(R)\xrightarrow{\delta}\mathrm{H}_{k+1}(X)\longrightarrow\mathrm{H}_{k+1}(X^{(1)})\longrightarrow 0\longrightarrow\cdots.$$

By contruction δ is surjective hence $\mathrm{H}_{k+1}(X^{(1)})=0$. Repeating this contruction we obtain free maps $X^{(n)}\to X^{(n+1)}$ which are isomorphisms in dimension $\le k+n+1$ such that $\mathrm{H}_q(X^{(n+1)})=0$ for $k<q\le k+n+1$. Then $g:X\to X'=\lim_n\{X\to X^{(n)}\}$ is a free map with $\mathrm{H}_q(X')=0$ for $q>k$ which is an isomorphism in dimensions $\le k$.

Form a co-cartesian diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \tag{7.4}$$

and note that g' is an isomorphism in dimension $\le k+1$, since g is.

Constructing a map $Y'\to Y''$ which is an isomorphism in dimension $\le k+1$ and has $\mathrm{H}_q(Y'')=0$ for $q>k$, and replacing Y' by Y'' we obtain a diagram (7.4) where the vertical maps are free and isomorphisms in dimensions $\le k+1$ and $\mathrm{H}_q(X')=\mathrm{H}_q(Y')=0$ for $q>k$.

If f is a k-equivalence, then as g, g' are so is f'. Thus f' is a weak equivalence, hence a homotopy equivalence since X' and Y' are free. Thus $F(g)$, $F(f')$, $F(g')$ are k-equivalences so $F(f)$ is. □

Corollary 7.5. *If X is k-connected to is $\mathbf{L}F(X)$.*

(7.6). For simplicial objects in an abelian category $\mathscr{A}$ the 0-th homology functor $\mathrm{H}_0: s\mathscr{A}\to\mathscr{A}$ is left adjoint to the functor $c:\mathscr{A}\to s\mathscr{A}$. This also holds for more general categories, such as categories of universal algebras havig an underlying abelian group law, and in particular for the category of ring-modules. Hence given a simplicial ring and module (R,X), the canonical adjunction map $(R,X)\to(c\mathrm{H}_0R,c\mathrm{H}_0X)$ gives rise to a map $F(R,X)\to F(\mathrm{H}_0R,\mathrm{H}_0X)$ and hence to a canonical map

$$\mathrm{H}_0(F(R,X))\longrightarrow F(\mathrm{H}_0R,\mathrm{H}_0X). \tag{7.7}$$

We shall say that F is right exact if this map is always an isomorphism. $F = S$, $\wedge$, and Γ are all right exact because they are left adjoint functors. For example if $F = S$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Alg}_{\mathrm{H}_0R}}(S^{\mathrm{H}_0R}(\mathrm{H}_0X),A) &= \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{H}_0R}}(\mathrm{H}_0X,A)\\ &= \mathrm{Hom}_{\mathscr{M}_R}(X,cA)\\ &= \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{simp},R}}(S^RX,cA)\\ &= \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{H}_0R}}(\mathrm{H}_0(S^RX),A).\end{aligned}$$

(7.8). If (B,M) is a ring-module let $L_qF(B,M) = \mathrm{H}_q(\mathbf{L}F(B,M))$. If F is right exact clearly

$$L_0F(B,M) \simeq F(B,M).$$

Proposition 7.9. *There is a spectral sequence*

$$E^2_{pq} = \mathrm{H}_p((L_qF)(R,X)) \Longrightarrow \mathrm{H}_{p+q}(\mathbf{L}F(R,X))$$

which when F is right exact has the edge homomorphism

$$\mathrm{H}_n(\mathbf{L}F(R,X)) \longrightarrow E^2_{n0} = \mathrm{H}_n(F(R,X)) \tag{7.10}$$

which is the map on homology induced by the canonical map $\mathbf{L}F(X) \to F(X)$.

Proof. This spectral sequence is similar to the Künneth spectral sequence th.6(b) of [HA, II] and is constructed in pretty much the same way. We construct an exact sequence in $\mathscr{M}_R$

$$\cdots \longrightarrow P_{(2)} \longrightarrow P_{(1)} \longrightarrow P_{(0)} \longrightarrow X \longrightarrow 0 \tag{7.11}$$

by recursion, letting $X_{(0)} = X$, $P_{(q)} \to X_{(q)}$ be a free resolution of $X_{(q)}$, and $X_{(q+1)} = \ker(P_{(q)} \to X_{(q)})$. Let $Q_{(*)} = N^{-1}_{(*)}(P_{(*)})$ be the simplicial object in $\mathscr{M}_R$ obtained by applying the inverse of the normalization functor to the complex $P_{*()}$ ([?, §3]). Then

$$Q_{(k)} = \bigoplus\nolimits_\eta P_{(t\eta)}$$

where η runs over all surjective morphisms with source $[k]$ and target $[t\eta]$. From this we see that (i) $Q_{(k)}$ is a free ? module (ii) the inclusion $P_{(0)} \to Q_{(k)}$, coming from $\eta =$ unique map: $[k] \to [0]$, is a homotopy equivalence. Indeed by contruction $\mathrm{H}_*(P_{(k)}) = 0$ for $k > 0$ hence $P_{(k)}$ is contractible.

Now consider the bisimplicial abelian group $K_{pq} = F(R_q, Q_{(p)q})$ and the two associated spectral sequences having the homology of the diagonal simplicial abelian group K_{nn} (for common abutment see [?] Satz. 2.15 or [?]). Using the property (ii) we have

$$\mathrm{H}^v_q(K_{p*}) = \mathrm{H}_qF(R, P_{(0)})$$

for all p hence

$$\mathrm{H}^h_p\mathrm{H}^v_q(K_{**}) = \begin{cases}0, & p > 0,\\ \mathrm{H}_qF(R,P_{(0)}), & p = 0.\end{cases}$$

Thus the spectral sequence with this as E^2 degenerates showing that the map $F(R,P_{(0)}) \to \Delta K$ is a weak equivalence. For fixed ?, the exactness of (7.11) together with property (i) imply that $Q_{(*)n}$ is a free simplicial R_m-module resolution of X_m, hence

$$\mathrm{H}^h_q(K_{*m}) = (L_qF)(R_m, X_m)$$

and

$$E^2_{pq} = \mathrm{H}^v_p\mathrm{H}^h_q(K_{**}) = \mathrm{H}_p((L_qF)(R,X)) \Longrightarrow \mathrm{H}_{p+q}(\Delta K).$$

□

References

[1] [HA] D. Quillen, *Homotopical algebra*.
[2] [GT] M. Artin, *Grothendieck topologies*.