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HOMOLOGY OF COMMUTATIVE RINGS

Homology of Commutative Rings
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Chapter I.

§1. Differentials and derivations.

All rings are commutative, associative, with identity and all modules are unitary unless stated otherwise. $\underline{R}$ denotes the category of rings. Also all diagrams are understood to be commutative.

Let A be a ring and let $\underline{C} = A\backslash\underline{R}$ be the category of A-algebras. An object of $\underline{C}$ is a ring B together with a map $i_B: A \to B$ of rings, and a map $f: B \to B'$ in $\underline{C}$ is a map of rings such that $fi_B = i_{B'}$. $\underline{C}$ is a category of universal algebras so the general theory of [HA, §5, p.5.14] applies to define cohomology groups for an object B of $\underline{C}$ with values in a "B-module," where "B-module" in the general theory means an abelian group object in $\underline{C}/B$. We show in this section that this notion of B-module coincides with the usual one. Also we calculate the abelianization functor on $\underline{C}/B$ in terms of differentials.

Let B denote a fixed A-algebra and let $\underline{C}/B$ be the category of A-algebras over B. Here "over" is used as in category theory, so that an object of $\underline{C}/B$ is an A-algebra X together with a map $u_X: X \to B$ of A-algebras. If M is a B-module, let $B \oplus M$ be the A-algebra over B with

$$\boxed{\begin{aligned} (b\oplus m)(b'\oplus m') &= bb'\oplus(bm'+b'm) \\ i_{B\oplus M}(a) &= i_B(a) \\ u_{B\oplus M}(b\oplus m) &= b \end{aligned}}$$

If $\theta\colon Y \to B\oplus M$ is a map in $\underline{C}/B$, then $\theta y = u_Y y \oplus Dy$ where $D\colon Y \to M$ satisfies

$$Di_Y(a) = 0$$

$$D(yy') = u_Y(y)Dy' + u_Y(y')Dy .$$

In other words D is an A-derivation of Y with values in M considered as a B-module via u_Y. Conversely given such a D we obtain a θ and therefore

(1.1) $$\boxed{\mathrm{Hom}_{\underline{C}/B}(Y,B\oplus M) \simeq \mathrm{Der}(Y/A,M)}$$

This is an isomorphism of functors of Y. As $\mathrm{Der}(Y/A,M)$ is an abelian group under addition we see that $B\oplus M$ is an abelian group object of $\underline{C}/B$. ((H - Hopf) ;

Recall that an H-object in a category is an object endowed with an operation having a two-sided identity. An H-object of $\underline{C}/B$ is therefore an object X together with maps

$$\varepsilon\colon B \to X$$

$$\mu\colon X \times_B X \to X$$

such that

$$\mu\circ(\epsilon u_X, id_X) = \mu\circ(id_X, \epsilon u_X) = id_X \tag{1.2}$$

For the abelian group object $B\oplus M$ one calculates that

$$\begin{aligned}\epsilon(b) &= b\oplus 0\\ \mu(b\oplus m, b\oplus m') &= b\oplus(m+m')\end{aligned} \tag{1.3}$$

Proposition 1.4: Any H-object of $\underline{C}/B$ is isomorphic to $B\oplus M$ with multiplication (1.3) for some B-module M. In particular any H-object is an abelian group object.

Proof: Given an H-object X, let $M = \mathrm{Ker}(u_X\colon X \to B)$ considered as a B module via $\epsilon_X\colon B \to X$. If $x,y \in M$, then by (1.2)

$$\mu(x,0) = x \qquad \mu(0,y) = y$$

so as μ is a homomorphism

$$xy = \mu(x,0)\mu(0,y) = \mu((x,0)(0,y)) = \mu(0,0) = 0$$

Thus M has the zero multiplication. It follows easily that the map $B\oplus M \to X$ given by $b\oplus m \to \epsilon_X b+m$ is an isomorphism in $\underline{C}/B$. Abbreviating ϵ_X to ϵ we have

$$\begin{aligned}\mu(\epsilon b+m, \epsilon b+m') &= \mu(\epsilon b, \epsilon b) + \mu(m,0) + \mu(0,m')\\ &= \epsilon b + m + m'\end{aligned}$$

which shows that the multiplication μ on X is the same as (1.3) on $B\oplus M$. Q.E.D.

Let $(\underline{C}/B)_{ab}$ be the category of abelian group objects in $\underline{C}/B$ and let $\underline{M}_B$ be the abelian category of B modules. From 1.4 we obtain

Proposition 1.5: *There is an equivalence of categories*

$$\begin{array}{c} B\oplus M \longleftarrow\!\shortmid\; M \\ (\underline{C}/B)_{ab} \simeq \underline{M}_B \\ X \longmapsto \operatorname{Ker} u_X \end{array}$$

In particular $(\underline{C}/B)_{ab}$ *is an abelian category.*

If Y is an A-algebra, let $D_{Y/A}$ be the Y module of A-differentials of Y. There is a canonical A-derivation $d\colon Y \to D_{Y/A}$ such that

$$\operatorname{Hom}_{\underline{M}_Y}(D_{Y/A},N) \simeq \operatorname{Der}(Y/A,N)$$
$$\theta \to \theta\circ d$$

for any Y module N. From 1.1 we have

(1.6)

$$\operatorname{Hom}_{\underline{C}/B}(Y,B\oplus M) \simeq \operatorname{Der}(Y/A,M) \simeq \operatorname{Hom}_{\underline{M}_Y}(D_{Y/A},M) \simeq \operatorname{Hom}_{\underline{M}_B}(D_{Y/A}\otimes_Y B,M) .$$

Thus

Proposition 1.7: *With respect to the equivalences of categories of 1.5 we have*

$$\begin{array}{c} D_{Y/B}\otimes_Y B \longleftarrow\!\shortmid\; Y \\ \underline{M}_B \simeq (\underline{C}/B)_{ab} \;\underset{i}{\overset{ab}{\rightleftarrows}}\; \underline{C}/B \\ M \longmapsto B\oplus M \end{array}$$

where i is the natural faithful functor and ab is the abelianization functor, the left adjoint of i .

From now on we will identify $(\underline{C}/B)_{ab}$ and $\underline{M}_B$ by the equivalence of Prop. 1.5. 1.7 shows that $Y \mapsto D_{Y/A} \otimes_Y B$ is identified with the abelianization functor on $\underline{C}/B$.

§2. Homology and cohomology

The qth cohomology group of the A-algebra B with values in the abelian group object $B\oplus M$ of $\underline{C}/B$ [HA,II,p.5.14] will be denoted $D^q(B/A,M)$ and called simply the qth cohomology group of the A-algebra B with values in the B module M. According to Theorem 5 (loc.cit.) this may be defined in two different but equivalent ways--(1) as sheaf cohomology from a Grothendieck topology and (2) by (semi-) simplicial resolutions. Both definitions for $D^q(B/A,M)$ will be given in this section. The former will be used in globalizing the definition to pre-schemes and the latter leads to a notion of homology for the A-algebra B.

2.1. Let $\underline{T}$ be the Grothendieck topology [G T] whose underlying category is $\underline{C}/B$ and where a covering of Y is a family consisting of a single map $Z \to Y$ which is set-theoretically surjective. Representable functors are sheaves for $\underline{T}$ hence by 1.1 $Y \mapsto \mathrm{Der}(Y/A,M)$ is a sheaf of B modules on $\underline{T}$. The first definition for the cohomology of B with values in M is

(2.2) $$\boxed{D^q(B/A,M) \simeq H^q_{\underline{T}}(B,\mathrm{Der}(\cdot/A,M))}$$

where the right side denotes sheaf cohomology for the topology $\underline{T}$.

We will give the other definition for D^q in 2.14. It is first necessary to state some facts about simplicial objects.

2.3. If X is a simplicial object in an abelian category, its homology in dimension q, denoted $H_q(X)$, is defined to be the homology in dimension q of the chain complex constructed from X with differential $d = \Sigma(-1)^i d_i$. By the normalization theorem this is the same as $H_q(NX)$ where NX is the normalized subcomplex of X. Hence when X is a simplicial abelian group $H_q(X)$ is the qth homotopy group of X in the sense of Moore.

2.4. If $f\colon X \to Y$ and $i\colon U \to V$ are maps in a category, we say that f has the right lifting property (RLP) with respect to i and that i has the left lifting property (LLP) with respect to f if given any commutative square of solid arrows

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ V \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

a dotted arrow exists such that the whole diagram is commutative. Let $\Delta(n)$ be the "standard n simplex" simplicial set and let $i_n\colon \dot{\Delta}(n) \to \Delta(n)$ be the inclusion of its $n-1$ skeleton. The following proposition characterizes those maps of simplicial abelian groups which are trivial fibrations in the sense of [HA].

Proposition 2.5: The following assertions are equivalent for a map $f\colon X \to Y$ of simplicial abelian groups.

(i) f is surjective (in each dimension) and $H_*(f): H_*(X) \simeq H_*(Y)$.

(ii) As a map of simplicial sets f has the RLP with respect to $i_n: \dot{\Delta}(n) \to \Delta(n)$ for all $n \geq 0$.

(iii) As a map of simplicial sets f has the RLP with respect to any injective (in each dimension) map of simplicial sets.

This results from [HA], II, §3, Prop. 2.

(2.6). A map of simplicial rings is said to be a trivial fibration if as a map of simplicial abelian groups it satisfies the equivalent conditions of 2.5. A map of simplicial rings is called a cofibration if it has the LLP with respect to all trivial fibrations of simplicial rings. Cofibrations may be described in the following alternative way. Call a map $i: R \to S$ of simplicial rings free if there are subsets $C_q \subset S_q$ $q \geq 0$ such that (i) $\eta^* C_q \subset C_p$ whenever $\eta: [p] \to [q]$ is a surjective monotone map and (ii) S_q is a free R_q algebra (polynomial ring) with generators C_q. Then Theorem 6.1 of [ ] implies:

Proposition 2.7: Any free map of simplicial rings is a cofibration.

The proof of Proposition 3, §4, [HA], II, yields

Proposition 2.8: Any map f of simplicial rings may be factored $f = pi$ where i is free and p is a trivial fibration.

Corollary 2.9: A map $i: R \to S$ of simplicial rings is a cofibration if and only if there is a free map $j: R \to T$ and maps $u: T \to S$, $v: S \to T$ of simplicial rings under R such that $uv = id_S$.

Proof: The sufficiency is clear. If i is a cofibration let $i = uj$ be a factorization as in 2.8. Then v is obtained as the dotted arrow in

$$\begin{tikzcd}
R \arrow[r, "j"] \arrow[d, "i"'] & T \arrow[d, "u", dotted] \\
S \arrow[r, "="] & S
\end{tikzcd}$$

2.10. If K is a simplicial set and Y is a simplicial ring, then the function complex Y^K has a natural structure as a simplicial ring. Let $I = \Delta(1)$ and let $j_e: Y^I \to Y$ $e = 0,1$ (resp. $\sigma: Y \to Y^I$) be the map induced by the inclusion of the eth vertex $\Delta(0) \to \Delta(1)$ (resp. the unique map $\Delta(1) \to \Delta(0)$). If $f, g: X \rightrightarrows Y$ are two maps of simplicial rings then a (simplicial) homotopy from f to g may be identified with a map of simplicail rings $h: X \to Y^I$ such that $j_0 h = f$ and $j_1 h = g$. $\sigma: Y \to Y^I$ is of course the "constant" homotopy from id_Y to id_Y.

Proposition 2.11: Let $i: R \to S$ be a cofibration and $f: X \to Y$ a trivial fibration of simplicial rings. Let $h: R \to X^I$ and $k: S \to Y^I$ be homotopies such that $ki = f^I h$, and let $\theta_0, \theta_1: S \rightrightarrows X$ be maps with $\theta_e i = j_e h$ and $f\theta_e = j_e k$ $e = 0,1$. Then there is a homotopy $H: S \to X^I$ with $Hi = h$, $f^I H = k$, and

$j_e H = \theta_e \quad e = 0,1$.

Proof: Consider the square

$$\begin{tikzcd} R \arrow[r, "h"] \arrow[d, "i"'] & X^I \arrow[d, "(f^I, j_0, j_1)"] \\ S \arrow[r, "(k,\theta_0,\theta_1)"'] \arrow[ur, dashed, "H"] & Y^I \times_{Y^{\dot{I}}} X^{\dot{I}} \end{tikzcd}$$

where $X^{\dot{I}} = X \times X$. The map (f^I, j_0, j_1) is seen to be a trivial fibration using 2.5(iii), hence the dotted arrow exists and gives the desired homotopy. Q.E.D.

2.12. Let $u: R \to S$ be a map of simplicial rings. By a cofibrant factorization of u we mean a factorization $R \xrightarrow{i} T \xrightarrow{p} S$ of u where i is a cofibration and p is a trivial fibration. If $R \to T' \to S$ is another cofibrant factorization of u , then by the definition of cofibration there are maps $\varphi: T \to T'$ and $\psi: T' \to T$ in the category $R\backslash sR/S$. By 2.11 $\varphi\psi$ and $\psi\varphi$ are homotopic to $id_{T'}$ and id_T respectively in this category. Therefore a cofibrant factorization of a map is unique up to simplicial homotopy under the source and over the target of the map.

2.13. If X is an object of a category we let cX denote the "constant" simplicial object with $(cX)_q = X$, $\varphi^* = id_X$ for all q, φ .

2.14. Let A,B,M, and $\underline{C}$ be as in §1. A simplicial

A-algebra over B (i.e. simplicial object of $\underline{C}/B$) P is the same as a factorization $cA \to P \to cB$ of the map $cA \to cB$. We shall call P a projective A-algebra resolution of B if this factorization is a cofibrant factorization. By 2.8 projective A-algebra resolutions of B exist. Choose one P and let $\mathbb{L}D_{B/A}$ be the simplicial B module $\boxed{D_{P/A}\otimes_P B}$ obtained by applying the "abelianization" functor $Y \mapsto D_{Y/A}\otimes_Y B$ dimension-wise to P. The simplicial homotopy type of $\mathbb{L}D_{B/A}$ is independent of the choice of P by 2.12. The underlying chain complex of $\mathbb{L}D_{B/A}$ is called the cotangent complex of the A-algebra B and is unique up to chain homotopy equivalence of the choice of P. The following groups are then well-defined:

(2.15) $$D_q(B/A,M) = H_q(\mathbb{L}D_{B/A}\otimes_B M)$$

(2.16) $$D^q(B/A,M) = H^q(\mathrm{Hom}_B(\mathbb{L}D_{B/A},M))$$

We call these the qth homology and cohomology groups of the A-algebra B with values in M. When $M = B$ we write simply $D_q(B/A)$ and $D^q(B/A)$.

Proposition 2.17: The definitions 2.2 and 2.16 are consistent.

Proof: We have

$$D^q(B/A,M) = H^q\{\mathrm{Hom}_B(D_{P/A}\otimes_P B,M)\} = H^q\{\mathrm{Der}(P/A,M)\}$$

p. 5.6

so the result follows from [HA], II, §5, th.5. Alternatively it follows from a general theorem of Verdier (SGA, 1963-64, Expose 5, Appendix) once one notes that a $\underline{T}$-hypercovering of B is the same as a simplicial A-algebra resolution and that a projective resolution is by (2.14) a cofinal object in the category of T-hypercoverings of B .

§3. Elementary properties

In this section we establish properties of the homology and cohomology groups which are elementary in the sense that they do not use special properties of commutative rings and hence are true without essential modification for all kinds of universal algebras.

Proposition 3.1: If B is a free A-algebra, then $\underline{\underline{L}}D_{B/A} \simeq cD_{B/A}$ and hence

$$D^q(B/A,M) = D_q(B/A,M) = 0 \qquad q > 0$$

$$D^o(B/A,M) = \mathrm{Der}(B/A,M)$$

$$D_o(B/A,M) = D_{B/A} \otimes_B M$$

Proof: The identity map $cB \to cB$ makes cB a free simplicial A-algebra resolution of B, hence $\underline{\underline{L}}D_{B/A} \simeq cD_{B/A}$ and the proposition follows.

3.2. In analogy with 2.6 we say that a simplicial module X over a simplicial ring R is free if there are subsets $C_q \subset X_q$ such that (i) $\eta^* C_q \subset C_p$ if $\eta: [p] \to [q]$ is a surjective monotone map and (ii) X_q is a free R_q module with base C_q. X will be called projective if it is a direct summand of a free simplicial R module.

If P is a free A-algebra resolution of B with generators $C_q \subset P_q$ as in 2.6, then $\{dx \otimes 1 | x \in C_q\} \subset D_{P_q/A} \otimes_{P_q} B$ is a set of generators as in 3.2. Hence $D_{P/A} \otimes_P B$ is a free simplicial B

module. If Q is a projective A-algebra resolution, then by 2.9 Q is a retract of a free one P and so $D_{Q/A}\otimes_Q B$ is a projective simplicial B module. In particular as a chain complex $D_{Q/A}\otimes_Q B$ is projective in each dimension. Thus

Proposition 3.3: $\underline{\underline{L}}D_{B/A}$ is a projective simplicial B module.

Corollary: If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of B modules then there are long exact sequences.

(3.4)

$$0 \to D^0(B/A,M') \to D^0(B/A,M) \to D^0(B/A,M'') \to D^1(B/A,M') \to \cdots$$

(3.5)

$$\cdots \to D_1(B/A,M'') \to D_0(B/A,M') \to D_0(B/A,M) \to D_0(B/A,M'') \to 0$$

Corollary: There are universal coefficient spectral sequenc

(3.6) $$E^2_{pq} = \mathrm{Tor}^B_p(D_q(B/A),M) \Longrightarrow D_{p+q}(B/A,M)$$

(3.7) $$E_2^{pq} = \mathrm{Ext}^p_B(D_q(B/A),M) \Longrightarrow D^{p+q}(B/A,M)$$

Proposition 3.8: $D^0(B/A,M) = \mathrm{Der}(B/A,M)$

$D_0(B/A,M) = D(B/A)\otimes_B M$

Proof: As $Y \mapsto D_{Y/A}\otimes_Y B$ is a left adjoint functor by 1.6, it is right exact. Hence $D_0(B/A) = D_{B/A}$ and the proposition follows from 3.6 and 3.7.

Remark 3.9: The spectral sequence 3.7 shows that $M \mapsto D^q(B/A,M)$ is the qth derived functor of $M \mapsto \mathrm{Der}(B/A,M)$ <u>if</u>

and only if $D_q(B/A) = 0$ for $q > 0$. This can fail to be so even if A is a field, and in this respect commutative ring cohomology is quite distinct from group, Lie algebra, and associative algebra cohomology.

3.10. By an extension of the A-algebra B by M we mean an exact sequence

$$(3.11) \qquad 0 \to M \xrightarrow{i} X \xrightarrow{u} B \to 0$$

where u is a map of A algebras such that $(\mathrm{Ker}\, u)^2 = 0$ and where i induces an isomorphism of B modules $M \cong \mathrm{Ker}\, u$, $\mathrm{Ker}\, u$ being endowed with the B module structure $u(x)y = x \cdot y$ $x \in X$, $y \in \mathrm{Ker}\, u$. Let $\mathrm{Exalcomm}(B/A,M)$ be the set of isomorphism classes of extensions of the A-algebra B by M.

Proposition 3.12: There is a canonical bijection

$$\boxed{D^1(B/A,M) \simeq \mathrm{Exalcomm}(B/A,M)}.$$

Proof: Given a free A-algebra resolution P of B and an extension 3.11, choose a map $\theta: P_0 \to X$ of A algebras over B. Then $i^{-1}\{\theta d_0 - \theta d_1\}: P_1 \to M$ is a normalized 1-cocycle of $\mathrm{Der}(P/A,M)$ whose cohomology class is independent of the choice of θ. This gives a map $\Phi: \mathrm{Exalcomm}(B/A,M) \to D^1(B/A,M)$ which one may show is independent of the choice of P. Conversely given an A-derivation $D: P_1 \to M$ let

$$X = \operatorname{Coker}\{P_1 \underset{(d_1,0)}{\overset{(d_o,D)}{\rightrightarrows}} P_o \oplus M\}$$

the cokernel being in the category of A modules, and let $p: P_o \oplus M \to X$ be the canonical projection. Then X is a quotient A-algebra of $P_o \oplus M$ and if D is a cocycle of $\operatorname{Der}(P/A,M)$ we obtain an extension 3.11 with $u(p(y \oplus m)) = \epsilon y$, $i(m) = p(0 \oplus m)$. It is straightforward to verify that this procedure gives an inverse to Φ. Q.E.D.

Remark 3.13: It is also possible to prove 3.12 using the interpretation of the sheaf cohomology in terms of "torsors." By [ ] one also gets a general interpretation of D^2 this way, however for rings the methods of the second chapter seem better for handling the higher cohomology.

Corollary 3.14: Suppose that $A/I \simeq B$ where I is an ideal in A. Then $D_o(B/A) = 0$ and $D_1(B/A) \simeq I/I^2$.

Proof: $D_o(B/A) = 0$ by 3.8. From 3.7 and 3.12 we have $\operatorname{Hom}_B(D_1(B/A),M) \simeq D^1(B/A,M) \simeq \operatorname{Exalcomm}(B/A,M)$.
Let $\chi \in \operatorname{Exalcomm}(B/A,I/I^2)$ be the isomorphism class of the extension $0 \to I/I^2 \to A/I^2 \to B \to 0$. By functorality χ defines a natural transformation $\chi_*: \operatorname{Hom}_B(I/I^2,M) \to \operatorname{Exalcomm}(B/A,M)$. Conversely given an extension 3.11, choose a map $\theta: A \to X$ with $u\theta = i_B$; the restriction of θ to I gives rise a homomorphism of B modules $I/I^2 \to M$. This procedure is easily seen to define an inverse to χ_*, hence $\operatorname{Hom}_B(D_1(B/A),M) \simeq \operatorname{Hom}_B(I/I^2,M)$ and so $D_1(B/A) \simeq I/I^2$. Q.E.D.

§4. Further properties

In this section we establish properties of the cohomology theory which are peculiar to commutative rings, since they depend on the fact that direct sum in the category of A-algebras is given by tensor product. We begin by recalling facts about simplicial modules which are proved in [HA], II, §6.

4.1. Let R be a simplicial ring and let $\underline{M}_R$ be the abelian category of simplicial R modules. The homotopy category $Ho(\underline{M}_R)$ is obtained from $\underline{M}_R$ by formally adjoining the inverses of the weak equivalences (maps which induce isomorphisms on homology). $Ho(\underline{M}_R)$ is equivalent to the category whose objects are the projective simplicial R modules (3.2) with homotopy classes of maps for morphisms. When $R = cA$, $Ho(\underline{M}_R)$ is equivalent to the full subcategory of the derived category of A modules consisting of the chain complexes.

4.2. If X, Y are two simplicial R modules, let $\mathrm{Tor}_i^R(X,Y)$ be the simplicial R module obtained by applying the tri-functor $\mathrm{Tor}_i^{\cdot}(\cdot,\cdot)$ dimension-wise. Let $X \overset{L}{\otimes}_R Y$ denote the derived tensor product of X and Y. $X \overset{L}{\otimes}_R Y$ is isomorphic in $Ho(\underline{M}_R)$ to $P \otimes_R Q$, where P and Q are projective resolutions of X and Y respectively. There is a spectral sequence ([HA], II, §6, th. 6)

$$(4.3) \qquad E^2_{pq} = H_p(\mathrm{Tor}_q^R(X,Y)) \Longrightarrow H_{p+q}(X \overset{L}{\otimes}_R Y)$$

one of whose edge homomorphisms is the map on homology induced by the canonical map $X \overset{L}{\otimes}_R Y \simeq P \otimes_R Q \to X \otimes_R Y$. Consequently we have

<u>Proposition 4.3</u>: If $\mathrm{Tor}_q^R(X,Y) = 0$ for $q > 0$, then $X\overset{L}{\otimes}_R Y \simeq X\otimes_R Y$.

4.4. Let $u: R \to S$ be a map of simplicial rings. Define $\underline{\underline{L}}D_{S/R}$ to be the projective simplicial S module $D_{P/R}\otimes_P S$ where $R \to P \to S$ is a cofibrant factorization of u (2.). As an object of $\mathrm{Ho}(\underline{M}_S)$ it is independent up to isomorphism of the choice of the factorization. If X is a simplicial S module we define

$$(4.5)\qquad D_q(S/R,X) = H_q(\underline{\underline{L}}D_{S/R}\overset{L}{\otimes}_S X) \simeq H_q(\underline{\underline{L}}D_{S/R}\otimes_S X)$$

where the last isomorphism is from 4.3. It is clear that this definition specializes to 2.15 in the case where u is the map $cA \to cB$ and $X = cM$. Moreover the obvious generalization of 3.5 holds and 3.6 generalizes by ([HA], II, Th. 6(c)) to a spectral sequence

$$(4.6)\qquad E^2_{pq} = H_p(D_q(S/R)\overset{L}{\otimes}_S X) \Longrightarrow D_{p+q}(S/R,X)$$

where $D_q(S/R) = D_q(S/R,S)$.

<u>Proposition</u>: Suppose $u: R \to R'$ and $v: R \to S$ are maps of simplicial rings such that $\mathrm{Tor}_q^R(R',S) = 0$ for $q > 0$. If $S' = S\otimes_R R'$, then there are canonical isomorphisms in $\mathrm{Ho}(\underline{M}_{S'})$

$$(4.7)\qquad \underline{\underline{L}}D_{S/R}\otimes_R R' \simeq \underline{\underline{L}}D_{S'/R'}$$

$$(4.8)\qquad \underline{\underline{L}}D_{S'/R} \simeq \underline{\underline{L}}D_{S/R}\otimes_R R' \oplus \underline{\underline{L}}D_{R'/R}\otimes_R S$$

<u>Proof</u>: First observe that if $R \to P$ is a cofibration (resp. free map) of simplicial rings, then P is a projective (resp.

free) R module. Using 2.9, one reduces to the case where P is free, whence if C_* is an R-algebra basis for P as in 2.6, then the monomials in the elements of C_* form an R module basis for P as in 3.2.

Now let $R \to P \to S$ be a cofibrant factorization of $R \to S$. As $P \to S$ is a weak equivalence of R modules, so is $P\overset{L}{\otimes}_R R' \to S\overset{L}{\otimes}_R R'$, since $\overset{L}{\otimes}_R$ is a functor on $\mathrm{Ho}(\underline{M}_R)$. By hypothesis, the fact that P is a projective R module by the above remarks, and 4.3, this map is isomorphic to the map $P\otimes_R R' \to S'$. Hence this last map is a weak equivalence; as it is clearly surjective it is a trivial fibration. As cobase extension preserves cofibrations, it follows that $R' \to P\otimes_R R' \to S'$ is a cofibrant factorization of $R' \to S'$ and hence setting $P' = P\otimes_R R'$

$$\underline{\underline{L}}D_{S'/R'} = D_{P'/R'}\otimes_{R'} S' \simeq (D_{P/R}\otimes_R S)\otimes_R R' \simeq \underline{\underline{L}}D_{S/R}\otimes_R R'$$

which proves 4.7. To prove 4.8, let $R \to Q \to R'$ be a cofibrant factorization of $R \to R'$. By the same argument as above we find that $R \to P\otimes_R Q \to S'$ is a cofibrant factorization of $R \to S'$ and so

$$\begin{aligned}\underline{\underline{L}}D_{S'/R} &= D_{P\otimes_R Q/R}\otimes_{P\otimes_R Q} S' \simeq (D_{P/R}\otimes_R Q \oplus D_{Q/R}\otimes_R P)\otimes_{P\otimes_R Q} S' \\ &\simeq (D_{P/R}\otimes_P S)\otimes_R R' \oplus (D_{Q/R}\otimes_Q R')\otimes_R S \\ &\simeq \underline{\underline{L}}D_{S/R}\otimes_R R' \oplus \underline{\underline{L}}D_{R'/R}\otimes_R S .\end{aligned}$$

The resulting isomorphisms in $\mathrm{Ho}(\underline{M}_{S'})$ are canonical since they are independent of the choices of P, Q by 2.11. Q.E.D.

Corollary 4.9: Let B and C be A-algebras and let N be a $B\otimes_A C$ module. If $\mathrm{Tor}_q^A(B,C) = 0$ for $q > 0$, then there are isomorphisms

$$D^q(B\otimes_A C/C, N) = D^q(B/A, N)$$

$$D^q(B\otimes_A C/A, N) = D^q(B/A, N) \oplus D^q(C/A, M)$$

and similarly for homology.

4.10. If $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\underline{M}_R$, then in $\mathrm{Ho}(\underline{M}_R)$ there is a cofibration sequence ([HA], II, §6)

$$X' \to X \to X'' \xrightarrow{\partial} \Sigma X' \tag{4.11}$$

where Σ denotes the suspension functor on $\mathrm{Ho}(\underline{M}_R)$. When $R = cB$ and we identify $\mathrm{Ho}(\underline{M}_R)$ with the subcategory of the derived category of B modules consisting of chain complexes, then the suspension functor shifts a complex to the left and the sequence 4.11 is the distinguished triangle associated to the exact sequence.

Theorem: Let $R \xrightarrow{u} S \xrightarrow{v} T$ be maps of simplicial rings. Then there is a canonical cofibration sequence in $\mathrm{Ho}(\underline{M}_T)$

$$\underline{\mathbb{L}D}_{S/R}\otimes_S T \to \underline{\mathbb{L}D}_{T/R} \to \underline{\mathbb{L}D}_{T/S} \to \Sigma(\underline{\mathbb{L}D}_{S/R}\otimes_S T) \tag{4.12}$$

Proof: Form a diagram

(4.13)
$$\begin{tikzcd}
& & Q \arrow[d, "i_2"] \arrow[ddr, "q"] & \\
& P \arrow[ur, "j"] \arrow[d, "p"] & S\otimes_P Q \arrow[dr, "r"] & \\
R \arrow[ur, "i"] \arrow[r, "u"] & S \arrow[ur, "i_1"] \arrow[rr, "v"] & & T
\end{tikzcd}$$

by choosing cofibrant factorizations $u = pi$ and $vp = qj$, and then filling in the rest of the diagram in the obvious way.

As j is a cofibration Q is a projective P module, hence by 4.3 the map i_2 is isomorphic to $p\overset{L}{\otimes}_P id_Q : P\overset{L}{\otimes}_P Q \to S\overset{L}{\otimes}_P Q$, which is a weak equivalence since p is. As i_2 and q are weak equivalences so is r; r is also surjective since q is and therefore r is a trivial fibration of simplicial S-algebras. As j is a cofibration so is i_1; therefore $v = ri_1$ is a cofibrant factorization.

Suppose for the moment that 4.13 is a diagram of rings where Q is a free P-algebra. If N is a T module, then there is an exact sequence

$$0 \to \mathrm{Der}(Q/P,N) \to \mathrm{Der}(Q/R,N) \xrightarrow{j^*} \mathrm{Der}(P/R,N) \to 0$$

where j is onto because Q is a free P-algebra and hence there is an R-algebra map from Q to P which is left inverse to j. As this sequence is functorial in N it comes from an exact sequence of T modules

(4.14) $$0 \to (D_{P/R}\otimes_P S)\otimes_S T \to D_{Q/R}\otimes_Q T \to D_{S\otimes_P Q/S}\otimes_{S\otimes_P Q} T \to 0$$

This last sequence is seen to be functorial in the diagram of rings 4.13, hence applying this functorial exact sequence dimension-wise to 4.13 now considered as a diagram of simplicial rings we obtain an exact sequence 4.14 of simplicial T modules.

Now using that $S\otimes_P Q$ is a projective S-algebra resolution of T, that Q is a projective R-algebra resolution of T and that P is a projective R-algebra resolution of S, we see that the cofibration sequence in $\mathrm{Ho}(\underline{M}_T)$ associated to 4.14 is the desired cofibration sequence 4.12.

It remains to show this cofibration sequence is independent of the choice of the diagram 4.13. Suppose given a diagram of simplicial rings

(4.15)
$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & T \arrow[d] \\ R' \arrow[r] & S' \arrow[r] & T' \end{tikzcd}$$

and suppose given a diagram (4.13)' similar to 4.13 but with primes. By lifting successively in the diagrams

(4.16)
$$\begin{tikzcd} R \arrow[r] \arrow[d] & P' \arrow[d] \\ P \arrow[r] \arrow[ur, dashed] & S' \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] \arrow[d] & Q' \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & T' \end{tikzcd}$$

we obtain a map from 4.13 to (4.13)', hence a map from the exact sequence 4.14 to the corresponding exact sequence (4.14)', and finally a map of cofibration sequences. The resulting map is independent of the choices of the liftings in 4.16 because by 2.11 two liftings in the first square are joined by a homotopy which may then be extended to a homotopy between the liftings in the second square. It follows that the map of exact sequences 4.14 to (4.14)' is unique up to homotopy and hence the map of cofibration sequences is well-defined. Q.E.D.

Specializing to constant simplicial rings, we have

<u>Corollary 4.17</u>: If $A \to B \to C$ are maps of rings, then there is a canonical exact triangle in the derived category of C modules

$$\begin{tikzcd}
\underline{\underline{L}}D_{B/A} \otimes_B C \arrow[rr] & & \underline{\underline{L}}D_{C/A} \arrow[dl] \\
& \underline{\underline{L}}D_{C/B} \arrow[ul, dashed] &
\end{tikzcd}$$

Hence if N is a C module, there are canonical exact sequences

$$0 \to D^0(C/B,M) \to D^0(C/A,M) \to D^0(B/A,M) \to D^1(C/B,M) \to \cdots$$

$$\cdots \to D_1(C/B,M) \to D_0(B/A,M) \to D_0(C/A,M) \to D_0(C/B,M) \to 0$$

§5. Some applications

In this section we extend to all q certain vanishing results for $D^q(B/A,M)$ which were proved in [ ] for $q = 1$ and in [ ] for $q = 1,2$. We shall state these results only for the cotangent complex $\underline{\underline{L}}D_{B/A}$ leaving the translation for the functors D^q and D_q to the reader.

Proposition 5.1: If S is a multiplicative system in A, then $\underline{\underline{L}}D_{S^{-1}A/A} \simeq 0$.

Proof: (after André [ ], 20.1) Let $C = S^{-1}A$. As $C \otimes_A C \simeq C$ we have an isomorphism $\underline{\underline{L}}D_{C/A} \otimes_A C \simeq \underline{\underline{L}}D_{C/A}$ of projective simplicial C modules. As C is flat over A the former complex by 4.7 is isomorphic to $\underline{\underline{L}}D_{C \otimes_A C/C} \simeq \underline{\underline{L}}D_{C/C} \simeq 0$.

Corollary 5.2: Suppose T is a multiplicative system in B and that S is a multiplicative system in A which is carried into T by the homomorphism $A \to B$. Then

$$\underline{\underline{L}}D_{T^{-1}B/S^{-1}A} \simeq \underline{\underline{L}}D_{B/A} \otimes_B T^{-1}B .$$

Proof: Applying 4.17 to $A \to B \to T^{-1}B$ and $A \to S^{-1}A \to T^{-1}B$ and using 5.1 we have

$$\underline{\underline{L}}D_{B/A} \otimes_B T^{-1}B \simeq \underline{\underline{L}}D_{T^{-1}B/A} \simeq \underline{\underline{L}}D_{T^{-1}B/S^{-1}A}$$

Proposition 5.3: Suppose that A is noetherian and B is of finite type as an A algebra. Then : $\longrightarrow$

$\boxed{B \text{ is etale over } A \iff \mathbb{L}D_{B/A} \simeq 0}$

B <u>is smooth over</u> A $\iff \mathbb{L}D_{B/A} \simeq cD_{B/A}$ <u>and</u> $D_{B/A}$ <u>is a projective</u> B <u>module.</u>

<u>Proof</u>: $(\Longrightarrow)$ Suppose B etale over A, i.e. B is flat over A and Δ: Spec B $\to$ Spec $B \otimes_A B$ is an open immersion. Let $\underline{p}$ be a prime ideal of B and let $\underline{q} = \Delta(\underline{p})$ so that $(B \otimes_A B)_{\underline{q}} \simeq B_{\underline{p}}$. Then there is an isomorphism of projective $B_{\underline{p}}$ modules

$$
\begin{aligned}
(\mathbb{L}D_{B/A})_{\underline{p}} &\simeq (\mathbb{L}D_{B/A} \otimes_A B)_{\underline{q}} \\
&\simeq (\mathbb{L}D_{B\otimes_A B/B})_{q} && \text{by 4.9 since } B \text{ is flat over } A. \\
&\simeq \mathbb{L}D_{(B\otimes_A B)_{\underline{q}}/B_{\underline{p}}} && \text{by 5.2} \\
&\simeq 0
\end{aligned}
$$

Thus $H_*((\mathbb{L}D_{B/A})_{\underline{p}}) = H_*(\mathbb{L}D_{B/A})_{\underline{p}} = 0$ and as $\underline{p}$ is an arbitrary prime ideal of B, we have $\mathbb{L}D_{B/A} \simeq 0$.

Now suppose B smooth over A. As $H_0(\mathbb{L}D_{B/A}) \simeq D_{B/A}$ (3.8) there is a canonical map of projective simplicial B modules $\mathbb{L}D_{B/A} \to cD_{B/A}$. To prove this is an isomorphism we reduce by localizing on B to the case where $A \to B$ may be factored $A \to P \to B$ where P is a polynomial ring over A and $P \to B$ is etale. Then by 4.17, the etale case of 5.3, and 3.1 we have

$$\begin{array}{ccc} \underline{\underline{L}}D_{P/A}\otimes_P B & \xrightarrow{\sim} & \underline{\underline{L}}D_{B/A} \\ \big\downarrow\wr & & \big\downarrow \\ D_{P/A}\otimes_P B & \xrightarrow{\sim} & cD_{B/A} \end{array}$$

which proves the assertion.

(<==) The spectral sequence 3.7 degenerates yielding $D^1(B/A,M) = 0$ for all B modules M. By 3.12 all A-algebra extensions of B by an ideal of square zero split hence B is smooth over A by SGA, 1960-61, III, 2-3. If also $D_{B/A} = 0$, then B is etale over A. Q.E.D.

We now wish to give a reasonably "geometric" example where $D_1(B/A) \neq 0$. The following results from 4.17 and 3.14.

Proposition 5.4: Suppose that $A \to P \to B$ is a factorization of $A \to B$ where P is a polynomial ring over A and $P \to B$ is surjective with kernel I. Then

$$D_q(B/A,M) \simeq D_q(B/P,M) \qquad \boxed{q \geq 2}$$

and there is an exact sequence

$$0 \to D_1(B/A,M) \to I/I^2\otimes_B M \to D_{P/A}\otimes_P M \to D_{B/A}\otimes_B M \to 0$$

and similar assertions hold for cohomology.

Example 5.5: Suppose that k is an algebraically closed field and that R is the coordinate ring of the curve $x = t^3$, $y = t^4$, $z = t^5$. Then R is an integral domain finitely generated over k. We show that $D_1(R/k) \neq 0$. $R = P/I$ where $P = k[X,Y,Z]$ and $I = (Y^2-XZ, YZ-X^3, Z^2-XY)$. The element $u = XY(Y^2-XZ) - X(YZ-X^3) + Z(Z^2-X^2Y)$ is in I but not in I^2. In effect if $m = (X,Y,Z)$, $I^2 \subset m^4$ and $u \notin m^4$. But by 5.4 we have the exact sequence

$$0 \to D_1(R/k) \to I/I^2 \xrightarrow{\delta} D_{P/k} \otimes_P R \to D_{R/k} \to 0$$

and a short calculation shows that $\delta(u+I^2) = 0$. Hence $D_1(R/k) \neq 0$ as asserted. It may be worth remarking that I is a prime ideal in P such that I^2 is not primary; in fact $\operatorname{Im}\delta = I/I^{(2)}$ where $I^{(2)}$ is the I primary component of I^2.

Chapter II. The fundamental spectral sequence

In order to calculate $D_*(B/A,M)$ one is reduced by 4.17 to the case where $B = A/I$, I an ideal in A . In this case there is a spectral sequence which relates these groups to $\mathrm{Tor}_*^A(B,M)$, which is more easily computable. In this chapter we derive this spectral sequence and give some of its applications.

§6. Construction of the spectral sequence

We retain the notations of the preceding chapter except that certain rings will not be commutative, but skew-commutative with respect to a canonical grading.

Let P be a free simplicial A-algebra resolution of B. Then $Q = P \otimes_A B$ is a simplicial augmented B-algebra. If $J = \operatorname{Ker} P\otimes_A B \to B$ is the augmentation ideal, then

$$(6.1) \qquad Q \supset J \supset J^2 \supset \ldots$$

is a filtration of Q by simplicial ideals. By means of the shuffle operation $\underline{\otimes}$ ([HA], II, p.6.6, (6)), Q with differential $d = \Sigma(-1)^i d_i$ becomes a skew-commutative differential graded ring and 6.1 is a filtration of Q by differential graded ideals. Consequently we obtain a spectral sequence of algebras where

$$(6.2) \qquad E^2_{pq} = H_{p+q}(J^q/J^{q+1}) \ , \ d^r\colon E^r_{pq} \to E^r_{p-r,q+r-1} \ ,$$

and ignoring for the moment questions of convergence, whose abutment is $H_*(Q) \simeq \operatorname{Tor}^A_*(B,B)$. As P is free over A, Q is free over B hence there is an isomorphism of graded simplicial algebras

$$(6.3) \qquad \oplus_q S^B_q(J/J^2) \simeq \oplus_q J^q/J^{q+1}$$

where the left side is the symmetric algebra functor over B applied dimension-wise to the simplicial B module J/J^2. Applying 5.4 dimension-wise to the maps $cA \to P \to cB$ we obtain isomorphisms of simplicial B modules

$$J/J^2 \simeq D_{P/A} \otimes_P B \simeq \underline{\underline{L}}D_{B/A} \tag{*}$$

We now turn to the convergence of this spectral sequence.

Lemma 6.5: Suppose that Q is a projective augmented simplicial B algebra with augmentation ideal J. If $H_0(J) = 0$, then $H_k(J^n) = 0$ for $k < n$.

A more general form of this lemma will be proved in 8.8. An alternative proof of 6.5 in outline is as follows. The arguments of [ ], §4 are very general and show that it is sufficient to prove 6.5 when $Q = S^B X$ and $X = K(B,1)^r$ where $K(B,1) = BM(1)/B\dot{\Delta}(1)$ is the simplicial B module whose normalization is the complex with B in dimension 1 and 0 elsewhere. In this case one may apply known results on the connectivity of the symmetric algebra functor [ ], in particular the following which will be proved in 7.32.

Lemma: Suppose that X is a flat simplicial B module with $H_0(X) = 0$. Then

$$H_q(S_n^B X) = 0 \qquad q < n$$

and there is a graded algebra isomorphism

$$\oplus_n \wedge_n^B H_1(X) \simeq \oplus_n H_n(S_n^B X)$$

where $\wedge^B$ is the exterior algebra functor on B modules.

In virtue of the augmentation, $H_o(J) = 0$ is equivalent to $H_o(Q) \simeq B$, which when $Q = P\otimes_A B$ means $B\otimes_A B \simeq B$. In this case the spectral sequence 6.2 constructed from the J adic filtration on Q converges by 6.5, that is, $E^r_{pq} = E^\infty_{pq}$ for $r > p+q$ and moreover $E^2_{pq} = 0$ if p or $q < 0$ by 6.5. Combining 6.2-6.4 we therefore have

Theorem 6.8: *If* $B\otimes_A B \simeq B$, *then there is a first quadrant spectral sequence*

$$E^2_{pq} = H_{p+q}(S^B_q LD_{B/A}) \Longrightarrow \mathrm{Tor}^A_{p+q}(B,B)$$

of bigraded algebras, skew-commutative for the total degree.

Picture of spectral sequence:

q

$\Lambda^3 D_1$			
$\Lambda^2 D_1$			
D_1	D_2	D_3	
B	0	0	

p

Edge homomorphisms

(6.9) $$\boxed{\mathrm{Tor}_n^A(B,B) \to D_n(B/A)} \qquad n > 0$$

(6.10) $$\wedge_n^B D_1(B/A) \to \mathrm{Tor}_n^A(B,B)$$

Low dimensional isomorphisms

(6.11) $$D_o(B/A) = 0$$
$$D_1(B/A) \simeq \mathrm{Tor}_1^A(B,B)$$

5 term exact sequence

(6.12)
$$\boxed{\mathrm{Tor}_3^A(B,B) \to D_3(B/A) \xrightarrow{d_2} \wedge_2^B D_1(B/A) \to \mathrm{Tor}_2^A(B,B) \to D_2(B/A) \to 0}$$

6.10 is the unique graded B-algebra morphism extending the isomorphism 6.11.

When $B = A/I$ where I is an ideal in A the hypothesis of 6.8 holds and in this case we may avail ourselves of the isomorphism

(6.13) $$D_1(B/A) \simeq \mathrm{Tor}_1^A(B,B) \simeq I/I^2$$

rewrite the edge homomorphism 6.10 in the form

(6.14) $$\wedge_n^B(I/I^2) \to \mathrm{Tor}_n^A(B,B)$$

Proposition 6.15: The edge homomorphism (6.9) annihilates the decomposable elements of $\mathrm{Tor}_*^A(B,B)$.

Proof: For $n > 0$ $\mathrm{Tor}_n^A(B,B) = H_n(Q) \approx H_n(J)$. If $\alpha \in \mathrm{Tor}_p$ $p > 0$ is represented by $x \in J_p$ and $\beta \in \mathrm{Tor}_q$ $q > 0$ is represented by $y \in J_q$, then $\alpha \cdot \beta \in \mathrm{Tor}_{p+q}$ is represented by $\mu(x \otimes y) \in J^2_{p+q}$, where $\mu\colon Q\otimes Q \to Q$ is the multiplication. But the edge homomorphism 6.9 is induced by $J \to J/J^2$, hence the image of $\alpha \cdot \beta$ in $D_{p+q}(B/A)$ is zero. Q.E.D.

If M is a B module then as J^q and J/J^2 are projective simplicial B modules

$$Q\otimes_B M \supset J\otimes_B M \supset \ldots$$

is a filtered simplicial module over the filtered simplicial ring Q with

$$\mathrm{gr}\ Q\otimes_B M \simeq S^B(J/J^2)\otimes_B M .$$

Hence

Theorem 6.16: If $B\otimes_A B \simeq B$, then there is a spectral sequence

$$E^2_{pq} = H_{p+q}(S^B_q \underline{LD}_{B/A}\otimes_B M) \Longrightarrow \mathrm{Tor}^A_{p+q}(B,M)$$

which is a spectral sequence of modules over the spectral sequence 6.?.

It is easy to verify that this spectral sequence has the following properties:

edge homomorphisms:

$$\mathrm{Tor}_n^A(B,M) \to D_n(B/A,M) \qquad n > 0$$

$$\wedge_n^B D_1(B/A)\otimes_B M \to \mathrm{Tor}_n^A(B,M)$$

low-dimensional isomorphisms:

$$D_o(B/A,M) = 0 \qquad D_1(B/A,M) = \mathrm{Tor}_1^A(B,M) \simeq D_1(B/A)\otimes_B M$$

5-term exact sequence:

$$\mathrm{Tor}_3^A(B,M) \to D_3(B/A,M) \to \wedge^2 D_1(B/A)\otimes_B M \to \mathrm{Tor}_2^A(B,M) \to D_2(B/A,M) \to 0$$

and that $\mathrm{Tor}_+^A(B,B)\mathrm{Tor}_+^A(B,M) \subset \mathrm{Tor}_+^A(B,M)$ is annihilated by the edge homomorphism.

Remark 6.17: The condition $B\otimes_A B \simeq B$ is necessary as is shown by the example 5.5 where B is flat over A and $D_1(B/A) \neq 0$.

Remark 6.18: The spectral sequences 6.8 and 6.16 are functorial in the triple A,B,M since the only choice made in their construction was the free A-algebra resolution P of B which is seen to be unique and functorial in A,B using 2.11.

§7. Homology of the symmetric algebra

In order to use the spectral sequence 6.8 it is necessary to have results relating the homology of the symmetric algebra of a simplicial module with the homology of the module. In this rather long section we collect the results that we need. They include a connectivity assertion (7.3), calculation of the first non-vanishing homology groups (7.27), and a calculation in the case where the ground ring is of characteristic zero (7.43). The symmetric algebra functor is closely connected with Eilenberg-MacLane spaces in topology and at the end of this section we outline this connection.

7.1. Let F be a functor defined on the category of ring-modules (B,M) consisting of a ring B and a B module M and having values in an abelian category $\underline{A}$. The functors we have in mind are (1) the symmetric algebra S, (2) the exterior algebra Λ, (3) the divided power algebra Γ as well as any of tensor products built up from homogeneous components of these functors. If R is a simplicial ring and X is a simplicial R module, then applying F dimension-wise to X we obtain a simplicial object $F(R,X)$. For the most part R will be fixed and we will write simply $F(X)$ when there is no possibility of confusion.

The left-derived functor $\underline{\underline{L}}F$ of F is defined by $\underline{\underline{L}}F(X) = F(P)$ where $P \to X$ is a projective resolution of X. The homotopy type of the simplicial object $\underline{\underline{L}}F(X)$ is independent of the choice of P and $\underline{\underline{L}}F$ is a functor from $\mathrm{Ho}(\underline{M}_R)$ to the category $\pi_0 s\underline{A}$ whose objects are the same as $s\underline{A}$ but with homotopy classes of maps for morphisms. The map $P \to X$ gives rise to a natural transformation $\underline{\underline{L}}F(X) \to F(X)$.

7.2. A map $f: X \to Y$ of simplicial objects in an abelian category will be called a k-equivalence if $f_*: H_q(X) \to H_q(Y)$ is an isomorphism for $q \leq k$. X is said to be k-connected if $H_q(X) = 0$ for $q < k$.

Proposition 7.3: If $f: X \to Y$ is a k-equivalence so is $\underline{\underline{L}}F(f)$.

Proof: We may assume X and Y are free simplicial R modules and drop the $\underline{\underline{L}}$. By "attaching cells" to X we will now construct a free map $X \to X'$ which is an isomorphism in dimensions $\leq k+1$ such that $H_q(X') = 0$ for $q > k$. If K is a simplicial set, let RK be the free simplicial R module generated by K ($K \mapsto RK$ is left adjoint to the forgetful functor $\underline{M}_R \to s$ (sets)). Let $\alpha_i \in H_{k+1}(X)$ be elements which generate $H_{k+1}(X)$ as an $H_0(X)$ module and choose a representative $x_i \in N_{k+1}(X)$ for α_i. Here $N(X)$ is the normalized chain complex of X. Let $u_i: R\dot{\Delta}(k+2) \to X$ be the unique simplicial R module map sending $d_j id_{[k+2]}$ to x_i for $j = 0$ and 0 for $j = 1,\ldots,k+2$. Define $X \to X^{(1)}$ by a co-cartesian diagram

$$\begin{array}{ccc}
\oplus_i R\dot{\Delta}(k+2) & \longrightarrow & \oplus_i R\Delta(k+2) \\
\Big\downarrow{\scriptstyle \Sigma u_i} & & \Big\downarrow \\
X & \longrightarrow & X^{(1)}
\end{array}$$

The map $X \to X^{(1)}$ is an isomorphism in dimension $\leq k+1$. The cokernel of both horizontal maps of this square are $\oplus_I R\Delta(k+2)/R\dot{\Delta}(k+2)$ whose homology is a free H_*R module on generators of dimension $k+2$ corresponding to the elements of I (see [HA], II, p.6.11, assertion A). The long exact sequence in homology for the exact sequence containing $X \to X^{(1)}$ is thus

$$\cdots \to \oplus_I H_o(R) \xrightarrow{\delta} H_{k+1}(X) \to H_{k+1}(X^{(1)}) \to 0 \to \cdots$$

By construction δ is surjective hence $H_{k+1}(X^{(1)}) = 0$. Repeating this construction we obtain free maps $X^{(n)} \to X^{(n+1)}$ which are isomorphisms in dimension $\leq k+n+1$ such that $H_q(X^{(n+1)}) = 0$ for $k < q \leq k+n+1$. Then $g\colon X \to X' = \lim_n\{X \to X^{(n)}\}$ is a free map with $H_q(X') = 0$ for $q > k$ which is an isomorphism in dimensions $\leq k$.

Form a co-cartesian diagram

(7.4)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

and note that g' is an isomorphism in dimension $\leq k+1$, since g is.

Constructing a map $Y' \to Y''$ which is an isomorphism in dimension $\leq k+1$ and has $H_q(Y'') = 0$ for $q > k$, and replacing Y' by Y'' we obtain a diagram (7.4) where the vertical maps are free and isomorphisms in dimensions $\leq k+1$ and $H_q(X') = H_q(Y') = 0$ for $q > k$.

If f is a k-equivalence, then as g, g' are so is f'. Thus f' is a weak equivalence, hence a homotopy equivalence since X' and Y' are free. Thus $F(g), F(f'), F(g')$ are k equivalences so $F(f)$ is. Q.E.D.

Corollary 7.5: *If* X *is* k-connected *so is* $\underline{L}F(X)$.

7.6. For simplicial objects in an abelian category $\underline{A}$ the 0th homology functor $H_0 : s\underline{A} \to \underline{A}$ is left adjoint to the functor $c: \underline{A} \to s\underline{A}$. This also holds for more general categories, such as categories of universal algebras having an underlying abelian group law, and in particular for the category of ring-modules. Hence given a simplicial ring and module (R,X), the canonical adjunction map $(R,X) \to (cH_0R, cH_0X)$ gives rise to a map $F(R,X) \to F(H_0R, H_0X)$ and hence to a canonical map

$$H_0(F(R,X)) \to F(H_0R, H_0X) \tag{7.7}$$

We shall say that F is right exact if this map is always an isomorphism. $F = S, \Lambda$, and Γ are all right exact because they are left adjoint functors. For example if $F = S$ we have

$$\begin{aligned}
\mathrm{Hom}_{H_0R\text{-alg}}(S^{H_0R}(H_0X), A) &= \mathrm{Hom}_{H_0R\text{-mod}}(H_0X, A) \\
&= \mathrm{Hom}_{\underline{M}_R}(X, cA) = \mathrm{Hom}_{s\text{-}R\text{-alg}}(S^R X, cA) \\
&= \mathrm{Hom}_{H_0R\text{-alg}}(H_0(S^R X), A) .
\end{aligned}$$

7.8. If (B,M) is a ring-module let $L_qF(B,M) = H_q(\underline{\underline{L}}F(B,M))$. If F is right exact clearly

$$L_oF(B,M) \simeq F(B,M)$$

Proposition 7.9: There is a spectral sequence

$$E^2_{pq} = H_p\{(L_qF)(R,X)\} \Longrightarrow H_{p+q}(\underline{\underline{L}}F(R,X))$$

which when F is right exact has the edge homomorphism

(7.10) $$\boxed{H_n(\underline{\underline{L}}F(R,X)) \to E^2_{no} = H_n\{F(R,X)\}}$$

which is the map on homology induced by the canonical map $\underline{\underline{L}}F(X) \to F(X)$.

Proof: This spectral sequence is similar to the Kunneth spectral sequence th.6(b) of [HA], II and is constructed in pretty much the same way. We construct an exact sequence in $\underline{M}_R$

(7.11) $$\cdots \to P_{(2)} \to P_{(1)} \to P_{(0)} \to X \to 0$$

by recursion, letting $X_{(o)} = X$, $P_{(q)} \to X_{(q)}$ be a free resolution of $X_{(q)}$, and $X_{(q+1)} = \operatorname{Ker} P_{(q)} \to X_{(q)}$. Let $Q_{(\cdot)} = N^{-1}_{(\cdot)}\{P_{(\cdot)}\}$ be the simplicial object in $\underline{M}_R$ obtained by applying the inverse of the normalization functor to the complex $P_{(\cdot)}$ ([ ], §3). Then

$$Q_{(k)} = \oplus_\eta P_{(t\eta)}$$

where η runs over all surjective monotone maps with source $[k]$ and target $[t\eta]$. From this we see that (i) $Q_{(k)}$ is a free module and (ii) the inclusion $P_{(o)} \to Q_{(k)}$, coming from η = unique map: $[k] \to [0]$, is a homotopy equivalence. Indeed by construction $H(P_{(k)}) = 0$ for $k > 0$ hence $P_{(k)}$ is contractible.

Now consider the bisimplicial abelian group $K_{pq} = F(R_q, Q_{(p)q})$ and the two associated spectral sequences having the homology of the diagonal simplicial abelian group K_{nn} for common abutment (see [ ] Satz. 2.15 or [ ]). Using the property (ii) we have

$$H_q^v(K_{p\cdot}) = H_q F(R, P_{(o)})$$

for all p hence

$$H_p^h H_q^v(K_{\cdot\cdot}) = \begin{cases} 0 \,; & p > 0 \\ H_q F(R, P_{(o)}) \,; & p = 0 \end{cases}$$

Thus the spectral sequence with this as E^2 degenerates showing that the map $F(R, P_{(o)}) \to \Delta K$ is a weak equivalence. For fixed m, the exactness of 7.11 together with property (i) imply that $Q_{(\cdot)m}$ is a free simplicial R_m module resolution of X_m, hence

$$H_q^h(K_{\cdot m}) = (L_q F)(R_m, X_m) \qquad \text{and}$$

$$E_{pq}^2 = H_p^v H_q^h(K_{\cdot\cdot}) = H_p\{(L_q F)(R,X)\} \Longrightarrow H_{p+q}(\Delta K)$$

Combining this spectral sequence with the weak equivalence $(R,X) = F(R, P_{(o)}) \to \Delta K$ we obtain the desired spectral sequence.

It remains to note that the edge homomorphism $H_n(\Delta K) \to$ ... is induced by the map $\Delta K \to H_o^h(K_{\cdot\cdot}) = F(R,X)$ which when composed with $F(R, P_{(o)}) \to \Delta K$ gives the map $F(R, P_{(o)}) \to F(R,X)$, alternatively the natural map $\underline{\underline{L}}F(R,X) \to F(R,X)$. This proves the assertion about the edge homomorphism so completes the proof of 7.9. Q.E.D.

Corollary 7.12: If F is right exact and $L_qF(R,X) = 0$ for $q > 0$, then $\underline{\underline{L}}F(R,X) \to F(R,X)$ is a weak equivalence.

Proposition 7.13: Suppose that filtered inductive limits in the target abelian category $\underline{A}$ of F are exact and that F commutes with filtered inductive limits. Then if M_i , $i \in I$ is a filtered inductive system of B modules

$$\lim L_qF(B,M_i) \approx L_qF(B,\lim M_i)$$

Proof: Let $C(M)$ be the cotriple resolution of M with respect to the free B-module--underlying set pair of adjoint functors. As both functors commute with filtered inductive limits $\lim C(M_i) \simeq C(\lim M_i)$ hence $\lim L_qF(B,M_i) = H_q\{\lim F(B,C(M_i))\}$ (since filtered inductive limits are exact in $\underline{A}$) = $H_q\{F(B,\lim C(M_i)\} = H_q\{F(B,C(\lim M_i)\} = L_qF(B,\lim M_i)$. Q.E.D.

Corollary 7.14: If M is a flat B module satisfies the hypotheses of 7.13, then $L_qF(B,M) = 0$ for $q > 0$.

Proof: This is clearly true if M is a free B module, hence for any flat module since it is a filtered inductive limit of free modules by a theorem of Lazard.

7.15. A simplicial module X over R will be called flat if each X_q is a flat R_q module. Combining 7.12 and 7.14 we obtain

Corollary 7.16: Suppose F is right exact and satisfies the hypotheses of 7.13. Then if X is a flat R module the canonical homomorphism $\underline{\underline{L}}F(R,X) \to F(R,X)$ is a weak equivalence.

7.17. Recall that the symmetric and divided power algebras of a B module M are commutative graded algebras

$$S^B M = \bigoplus_{n\geq 0} S_n^B M \qquad \Gamma^B M = \bigoplus_{n\geq 0} \Gamma_n^B M$$

and the exterior algebra is a graded algebra

$$\Lambda^B M = \bigoplus_{n\geq 0} \Lambda_n^B M$$

which is skew-commutative with respect to the grading. Consequently if X is a simplicial module over a simplicial ring R, the bigraded algebras

$$H_q(S_n^R X) \ , \ H_q(\Gamma_n^R X) \ , \ H_q(\Lambda_n^R X) \qquad q \geq 0 \ , \ n \geq 0$$

are skew-commutative for the degrees q, q and $n+q$ respectively.

7.18. Let C and Σ be the cone and suspension functors on $\underline{M}_R$ given by

$$CX = X\otimes_{\mathbb{Z}} \mathbb{Z}\Delta(1)/X\otimes_{\mathbb{Z}}\mathbb{Z}\{0\}$$

$$\Sigma X = X\otimes_{\mathbb{Z}} \mathbb{Z}\Delta(1)/X\otimes_{\mathbb{Z}}\mathbb{Z}\dot{\Delta}(1)$$

so that there is a canonical exact sequence

(7.19) $$0 \to X \to CX \to \Sigma X \to 0$$

which splits in each dimension. Σ induces the suspension functor (again denoted by Σ) on $\mathrm{Ho}(\underline{M}_R)$ ([HA], II, p.6.5), and as CX is contractible 7.19 gives rise to the suspension isomorphism

$$H_q(X) \simeq H_{q+1}(\Sigma X) \ . \tag{7.20}$$

<u>Proposition 7.21</u>: <u>There are canonical bigraded algebra isomorphisms</u>

$$H_q(\underline{\underline{L}}\wedge_n^R X) \simeq H_{q+n}(\underline{\underline{L}}S_n^R \Sigma X) \qquad q,n \geq 0 \tag{7.22}$$

$$H_q(\underline{\underline{L}}\Gamma_n^R X) \simeq H_{q+n}(\underline{\underline{L}}\wedge_n^R \Sigma X) \qquad q,n \geq 0 \tag{7.23}$$

which reduce to the suspension isomorphism 7.20 when $n = 1$. Moreover

$$H_q(\underline{\underline{L}}S_n^R \Sigma X) = 0 \qquad 0 \leq q < n \tag{7.24}$$

$$H_q(\underline{\underline{L}}\wedge_n^R \Sigma X) = 0 \qquad 0 \leq q < n \ . \tag{7.25}$$

Before proving this we deduce some corollaries. Recall that $S, \wedge$, and Γ are left adjoint functors, hence right exact (7.7) and there are canonical isomorphisms of graded algebras

$$\begin{aligned} H_o(\underline{\underline{L}}S_n^R X) &\simeq S_n^{H_oR}(H_oX) \\ H_o(\underline{\underline{L}}\wedge_n^R X) &\simeq \wedge_n^{H_oR}(H_oX) \\ H_o(\underline{\underline{L}}\Gamma_n^R X) &\simeq \Gamma_n^{H_oR}(H_oX) \end{aligned} \tag{7.26}$$

<u>Corollary 7.27</u>: If $H_oX = 0$, then

$$H_q(\underline{\underline{L}}S_n^R X) = H_q(\underline{\underline{L}}\wedge_n^R X) = 0 \qquad 0 \leq q < n$$

and there are canonical graded algebra isomorphisms

(7.28) $$\wedge_n^{H_oR}(H_1X) \simeq H_n(\underline{\underline{L}}S_n^RX)$$

(7.29) $$\Gamma_n^{H_oR}(H_1X) \simeq H_n(\underline{\underline{L}}\wedge_n^RX)$$

Proof: As $H_oX = 0$, X is isomorphic in $Ho(\underline{M}_R)$ to ΣY for some Y by [HA], II, §6, prop.1. The corollary follows from the proposition using 7.26 and the suspension isomorphism $H_1X \simeq H_oY$.

In a similar way one may prove

Corollary 7.30 If $H_oX = H_1X = 0$, then

$$H_q(\underline{\underline{L}}S_n^RX) = 0 \qquad q < 2n$$

and there is a canonical graded algebra isomorphism

(7.31) $$\Gamma_n^{H_oR}(H_2X) \simeq H_{2n}(\underline{\underline{L}}S_n^RX)$$

7.32. 6.6 and 6.7 now follow from 7.27 and 7.28, using 7.16 to drop the $\underline{\underline{L}}$.

Remark 7.33: 7.28 is the unique algebra map which extends the canonical isomorphism for $n = 1$. Similarly by means of a suitable shuffle formula it is possible to define divided power operations on the right sides of 7.29 and 7.31 and then these maps are the unique homomorphisms of divided power algebras extending the canonical isomorphism for $n = 1$.

<u>7.34. Proof of the proposition</u>: If M is a B module let d be the Koszul differential on the bigraded algebra $\wedge_q M \otimes S_n M$ $q,n \geq 0$ where $\otimes, \wedge,$ and S are taken over B. d is the unique endomorphism of this bigraded algebra which is a skew-derivation with respect to the exterior degree q and is such that $d(m\otimes 1) = 1\otimes m$, $d(1\otimes m) = 0$. If M is a flat B module, then

$$(7.35) \qquad \cdots \xrightarrow{d} \wedge_2 M\otimes SM \xrightarrow{d} \wedge_1 M\otimes SM \xrightarrow{d} SM \to 0 \to \cdots$$

is a flat differential graded skew-commutative algebra which is a resolution of B considered as an SM algebra via the augmentation $SM \to B$. In effect one reduces by Lazard to the case where M is a finitely generated free B module, then to $M = B$ by the Kunneth formula, in which case the fact that 7.35 is a resolution is clear.

If

$$0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$$

is an exact sequence of <u>flat</u> B modules, we may define a differential d on the bigraded algebra $\wedge_q M'\otimes S_n M$ by requiring it to be a skew-derivation with respect to the exterior degree q such that $d(m'\otimes 1) = 1\otimes im'$, $d(1\otimes m) = 0$. Then

$$(7.36) \qquad \cdots \xrightarrow{d} \wedge_2 M'\otimes SM \xrightarrow{d} \wedge_1 M'\otimes SM \xrightarrow{d} SM \to 0 \cdots$$

is a flat differential graded skew-commutative algebra which is a resolution of SM'' considered as an SM algebra via $Sj: SM \to SM''$. In effect we may assume M'' free by Lazard in which case

$M \simeq M' \oplus M''$ so 7.36 is the tensor product $(\Lambda M' \otimes SM') \otimes SM''$, which by Kunneth has homology SM'' in dimension 0.

To prove the proposition we may assume that X is a free simplicial R module and drop the $\underline{\underline{L}}$. Applying the sequence 7.36 dimension-wise to the exact sequence 7.19 we obtain exact sequences

$$(7.37)\quad 0 \to \Lambda_n^R X \to \dots \to \Lambda_1^R X \otimes_R S_{n-1}^R CX \to S_n^R CX \to S_n^R \Sigma X \to 0$$

of R modules. Using the fact that CX is contractible we obtain canonical isomorphisms

$$(7.38)\qquad \begin{aligned} H_q(S_n^R \Sigma X) &\simeq H_{q-n}(\Lambda_n^R X) & q &\geq n \\ &\simeq 0 & q &< n \end{aligned}$$

To show these isomorphisms constitute an isomorphism of graded algebras, let $K_{pq}(n) = \Lambda_p^{R_q} X_q \otimes_{R_q} S_{n-p}^{R_q} X_q$ be considered a graded double complex, whose horizontal differential is the d in 7.37 and whose vertical differential comes from the simplicial structure. $K_{pq} = \oplus_n K_{pq}(n)$ is a bigraded ring for which the total differential is a derivation and hence the two spectral sequences of the double complex K_{pq} are algebra spectral sequences. But as 7.37 is exact and CX is contractible both spectral sequence collapse yielding algebra isomorphisms

$$H_m(S_n^R \Sigma X) \simeq H_m(K(n)) \simeq H_{m-n}(\Lambda_n^R X)$$

whose composition is nothing but 7.38. Thus 7.38 is an algebra isomorphism and we have proved half of the proposition.

The proof of the other half is similar where the sequence 7.36 is replaced by

$$(7.39) \qquad \to \Gamma_2 M' \otimes \wedge M \xrightarrow{d} \Gamma_1 M' \otimes \wedge M \xrightarrow{d} \wedge M \to 0 \ldots$$

where d is the unique endomorphism of $\Gamma M' \otimes \wedge M$ which is a skew-derivation for the exterior degree and is such that $d(\gamma_k(m') \otimes 1) = \gamma_{k-1}(m') \otimes im'$, $d(1 \otimes m) = 0$. Q.E.D.

Corollary 7.40: If X is k-connected and $n > 0$, then

$$\underline{\underline{L}}S_n^R X \text{ is } 2(n-1)+k \text{ connected} \quad k \geq 1$$

$$\underline{\underline{L}}\wedge_n^R X \text{ is } (n-1)+k \text{ connected} \quad k \geq 1$$

$$\underline{\underline{L}}\Gamma_n^R X \text{ is } k \text{ connected} \quad k \geq 0 .$$

Proof: The last assertion follows from 7.3, and the first two may be deduced from the last setting $X = \Sigma Y$ and using the proposition.

Remark 7.41: The connectivity assertions 7.40 are the best possible in general. In characteristic zero (i.e. when R is a simplicial Q algebra), then $\Gamma \simeq S$ hence iterating the proposition one finds that if X is k-1 connected, then $\underline{\underline{L}}\wedge_n X$ and $\underline{\underline{L}}S_n X$ are nk-1 connected.

7.42. In characteristic zero the homological properties of the symmetric algebra are simpler because S_n is a canonical direct summand of the n-fold tensor product. For example, if U is a graded skew-commutative ring let $N \mapsto \tilde{S}^U N$ be the (skew-commutative) symmetric algebra functor, that is, the left adjoint of the forgetful functor from skew-commutative graded U-algebras to graded U-modules. Then

<u>Proposition 7.43</u>: Suppose that R is a simplicial ring of characteristic zero (i.e. $cQ \subset R$) and that X is a flat simplicial R-module such that HX is a flat HR module. Then there is a canonical isomorphism of bi-graded HR algebras

$$\tilde{S}_n^{HR}(HX) \simeq H(S_n^R X) \qquad n \geq 0$$

<u>Proof</u>: Let $T = \oplus T_n$ be the tensor algebra functor on ring-modules and let $\tilde{T} = \oplus \tilde{T}_n$ be the tensor algebra functor on graded ring-modules. The symmetric group Σ_n on n letters acts on T_n in the obvious way and on $\tilde{T}_n$ with the skew-commutative sign rule, and the shuffle map $\underline{\otimes}$ induces a Σ_n equivariant map of chain complexes

$$\tilde{T}_n^R X \to T_n^R X \tag{7.44}$$

As X is R-flat and HX is HR flat the Kunneth spectral sequences show that 7.53 gives rise to Σ_n equivariant isomorphisms

$$\tilde{T}_n^{HR}(HX) \simeq H(\tilde{T}_n^R X) \simeq H(T_n^R X) .$$

In general the largest Σ_n-invariant quotient of $\tilde{T}_n$ (resp. T_n) is $\tilde{S}_n$ (resp. S_n) and in characteristic zero the symmetrization operator $\frac{1}{n!}\sum_{\sigma\in\Sigma_n}\sigma$ allows us to canonically identify this quotient as a direct summand of $\tilde{T}_n$ (resp. T_n). As homology commutes with direct sums we obtain isomorphisms

$$\tilde{S}_n^{HR}(HX) \simeq H(S_n^R X)$$

and it is easily seen that these isomorphisms are compatible with the algebra structures. Q.E.D.

Remark 7.45: This proposition enables one to compute $H(S^R X)$ in terms of HX when R is ck and k is a field of characteristic zero. There is a more complicated formula if k is of characteristic p which we now briefly describe. This formula is based on the Dold-Thom theorem which in its simplicial form asserts that if X is a reduced simplicial set then

(7.46) $$SP^\infty X \hookrightarrow \mathbb{Z}X$$

is a weak homotopy equivalence of simplicial sets, where $SP^\infty X$ (resp. $\mathbb{Z}X$) is the free abelian monoid (resp. group) generated by X with the basepoint put equal to the identity. Applying the B module functor to both sides of 7.46 we obtain an isomorphism

(7.47) $$H(S^B(\bar{B}X)) \simeq H_*(\mathbb{Z}X,B)$$

where $\bar{B}X = BX/B_*$ and where the right side denotes the homology of the generalize Eilenberg-MacLane space $\mathbb{Z}X$ with coefficients in B. In particular if $X = \Delta(n)/\dot{\Delta}(n)$, then

$$(7.48) \qquad H(S^B K(B,n)) \simeq H_*(K(\mathbb{Z},n),B)$$

where $K(B,n)$ is the simplicial B module whose normalization is the complex with B located in dimension n and zero elsewhere. Now Cartan [ ] has given a formula for the right side of 7.48 when $B = \mathbb{Z}/p\mathbb{Z}$ which may be used by the Kunneth theorem to calculate $H(S^B X)$ when B is any field.

7.49. Finally we want to point out that the convergence theorem 8.8 can be used to give a proof of the Dold-Thom theorem (7.46). As this is a map of connected H-spaces it suffices to prove that 7.46 induces a map on homology ([ ], 1959-60, p.16-08) that is, that $\mathbb{Z}(SP^\infty X) \to \mathbb{Z}(\bar{\mathbb{Z}}X)$ is a weak equivalence. Filtering both sides by powers of the augmentation ideal we obtain a map of spectral sequences with the same E^1 term. But the augmentation ideals of both rings are regular so 8.8 implies these spectral sequences converge and hence the map is a weak equivalence.

§8. Regular ideals and the convergence theorem

Let I be an ideal in a ring A and let $B = A/I$. Let P be a flat differential graded or simplicial A-algebra resolution of B. Filtering P by I^nP $n \geq 0$ we obtain a spectral sequence of algebras whose E^1 term

$$(8.1) \qquad H_q(gr_n^I P) = \mathrm{Tor}_q^A(B, I^n/I^{n+1})$$

is a bigraded B-algebra anti-commutative for the homology degree q and whose first differential

$$(8.2) \qquad d: \mathrm{Tor}_q^A(B, I^n/I^{n+1}) \to \mathrm{Tor}_{q-1}^A(B, I^{n+1}/I^{n+2})$$

is a derivation of this algebra. It is clear that this differential bigraded B-algebra structure on $\mathrm{Tor}_*^A(B, gr_*^I A)$ is independent of the choice of the resolution P.

For $q = 1, n = 0$ the differential d yields isomorphisms

$$\mathrm{Tor}_1^A(B,B) \approx \mathrm{Tor}_0^A(B, I/I^2) \simeq I/I^2$$

which extend naturally to a canonical homomorphism of differential bigraded B-algebras

$$(8.3) \qquad \theta_{q,n}: \Lambda_q(I/I^2) \otimes S_n(I/I^2) \to \mathrm{Tor}_q^A(B, I^n/I^{n+1})$$

where $\otimes, \Lambda, S$ are over B and where the left side of 8.3 is endowed with the Koszul differential (7.34).

Definition 8.1. I is said to be quasi-regular (resp. regular) if I/I^2 is a flat (resp. projective) B module and if the canonical map

$$\theta_{q,o}: \Lambda_q(I/I^2) \to \mathrm{Tor}_q^A(B,B) \qquad q \geq 0$$

is an isomorphism.

Example: Suppose M is a flat B module, $A = SM$, and $I = S_+M$. Then we may take P to be the Koszul complex and we find that I is quasi-regular and regular iff M is projective.

Proposition 8.5: If I is quasi-regular, then $\theta = \{\theta_{q,n}\}$ is an isomorphism. In particular

$$S_n(I/I^2) \simeq I^n/I^{n+1}$$

Proof: To simplify notation let $N = I/I^2$ and $T_q(\cdot) = \mathrm{Tor}_q^A(B,\cdot)$. We shall prove by induction on m, the following assertions

A_m: $S_kN \simeq I^k/I^{k+1}$ for $k \leq m$

B_m: $T_q(A/I^k) \to T_q(A/I^{k-1})$ is the 0 map for $k \leq m$, $q > 0$.

Note first that as N is flat, A_m implies that I^k/I^{k+1} for $k \leq m$ is flat B module, hence $\mathrm{Tor}_q^A(B,I^k/I^{k+1}) \simeq \mathrm{Tor}_q^A(B,B) \otimes (I^k/I^{k+1}) \simeq \Lambda_q N \otimes S_k N$. Thus A_m implies that $\theta_{q,k}$ is an isomorphism for $k \leq m$.

A_o, B_o are trivial

$A_m, B_m \Longrightarrow B_{m+1}$. Let ∂_m be the boundary operator for the T_* long exact sequence associated to

(8.6) $$0 \to I^{m-1}/I^m \xrightarrow{i_m} A/I^m \xrightarrow{j_m} A/I^{m-1} \to 0$$

and consider the diagram

(8.7)

$$\begin{tikzcd}
 & T_{q+2}(A/I^{m-1}) \arrow[dr, "\partial_m"] & & T_{q+1}(A/I^m) \arrow[dr, "(\partial_{m+1})_*"] & \\
T_{q+2}(I^{m-2}/I^{m-1}) \arrow[ur, "(i_{m-1})_*"] \arrow[rr, "d"] & & T_{q+1}(I^{m-1}/I^m) \arrow[ur, "(i_m)_*"] \arrow[rr, "d"] & & T_q(I^m/I^{m+1}) \\
\Lambda_{q+2}N\otimes S_{m-2}N \arrow[u, "\theta_{q+2,m-2}"'] \arrow[rr, "d"] & & \Lambda_{q+1}N\otimes S_{m-1}N \arrow[u, "\theta_{q+1,m-1}"'] \arrow[rr, "d"] & & \Lambda_q N\otimes S_m N \arrow[u, "\theta_{q,m}"']
\end{tikzcd}$$

The bottom row is exact since N is flat (see 7.35). A_m implies that the θ maps are isomorphisms, hence the middle row is exact. By B_m $(i_m)_*$ is surjective. Suppose $x \in \mathrm{Im}\{T_{q+1}(A/I^{m+1}) \to T_{q+1}(A/I^m)\}$, that is $(\partial_{m+1})_* x = 0$. If $x = (i_m)_* y$, then $dy = 0$ so $y = dz$. Then $x = (i_m)_* \partial_m (i_{m-1})_* y = 0$, and as x is arbitrary $T_{q+1}(A/I^{m+1}) \to T_q(A/I^m)$ is 0, and we have proved B_{m+1}.

$A_{m-1}, B_m \implies A_m$. Consider the diagram above with $q = 0$. By B_m, $(i_{m-1})_*$ and $(i_m)_*$ are surjective and ∂_m is injective. But $(\partial_{m+1})_*\colon T_1(A/I^m) \cong T_o(I^m/I^{m+1}) \simeq I^m/I^{m+1}$ which proves the middle row of the diagram is exact and the right hand d is surjective. As the same is true for the bottom row and as $\theta_{2,m-2}$ and $\theta_{1,m-1}$ are isomorphisms by A_{m-1}, we see $\theta_{o,m}$ is an isomorphism and hence have proved A_m.

Thus A_m is true for all m and 8.5 is proved. Moreover as the diagram 8.7 is functorial in the pair A,I we obtain from the proof of 8.5 the following.

Proposition 8.7: If I is quasi-regular there are exact sequences

$$\cdots \xrightarrow{d} \Lambda_{q+1}N \otimes S_{n-1}N \xrightarrow{d} \Lambda_q N \otimes S_n N \to \mathrm{Tor}_q^A(B, A/I^{n+1}) \to 0$$

where $N = I/I^2$, $q > 0$ which are functorial in the pair (A,I) .

Convergence theorem 8.8. Let R be a simplicial ring and J be a simplicial ideal in R such that $H_o(J) = 0$. If J_q is quasi-regular in R_q for each q , then $H_k(J^n) = 0$ for $k < n$.

Proof: We use induction on n , the case $n = 1$ being a hypothesis. Consider the exact and spectral sequences

(8.9) $$0 \to \mathrm{Tor}_1^R(R/J, J^n) \to J \otimes_R J^n \to J^{n+1} \to 0$$

(8.10) $$E_{pq}^2 = H_p\{\mathrm{Tor}_q^R(J, J^n)\} \Longrightarrow H_{p+q}(J \overset{L}{\otimes}_R J^n)$$

(8.11) $$E_{pq}^2 = \mathrm{Tor}_p^{H_*R}(H_*J, H_*J^n)_q \Longrightarrow H_{p+q}(J \overset{L}{\otimes}_R J^n)$$

(8.12)

$$\to \Lambda_{q+1}N \otimes S_{n-2}N \xrightarrow{d} \Lambda_q N \otimes S_{n-1}N \to \mathrm{Tor}_q^R(R/J, R/J^n) \to 0 \qquad q > 0$$

where 8.10 and 8.11 are Kunneth spectral sequences [HA], II, th.6 and where 8.12 follows from 8.7, the $\Lambda, S, \otimes$ being over R/J and $N = J/J^2$. As N is a quotient of J we have $H_o(N) = 0$, hence as N is flat over $\underline{L}S_k N \simeq S_k N$ by 7.16 and so $S_k N$ is (k-1)-connected by 7.5. Similarly $\Lambda_k N$ is (k-1)-connected. Again by flatness of N $\Lambda_q N \otimes S_k N = \Lambda_q N \overset{L}{\otimes} S_k N$, so $\Lambda_q N \otimes S_k N$ is (q+k-1)-connected by the Kunneth spectral sequence analogous to 8.11. From 8.12 one therefore finds that

$$\mathrm{Tor}_q^R(R/J, R/J^n) = \begin{cases} \mathrm{Tor}_{q-1}^R(R/J, J^n) & \text{if } q > 1 \\ \mathrm{Tor}_{q-2}^R(J, J^n) & \text{if } q > 2 \end{cases}$$

is (q+n-2)-connected. Hence $E^2_{pq} = 0$ for $q > 0$, $p < q+n$ in 8.10. By induction hypothesis J^n is (n-1)-connected so by 8.11 $J \overset{L}{\otimes}_R J^n$ is n-connected. Thus 8.10 shows that $J \otimes_R J^n$ is n-connected, whence by 8.9 and the fact that $\mathrm{Tor}_1^R(R/J, J^n)$ is (n+1)-connected, we see that J^{n+1} is n-connected which completes the induction. Q.E.D.

For noetherian rings there is a close relation between regular ideals and ideals generated by regular sequences.

Proposition 8.13: If A (commutative) is noetherian the following conditions are equivalent:

(i) I is regular

(ii) I is quasi-regular

(iii) I/I^2 is a projective B module and $\Lambda^2(I/I^2) \to \mathrm{Tor}_2^A(B,B)$ is surjective.

(iv) I/I^2 is a projective B module and $S(I/I^2) \simeq \mathrm{gr}^I A$.

(v) For each maximal ideal p in A containing I, the ideal IA_p in A_p is generated by a regular sequence.

<u>Proof</u>: (i) <==> (ii) I/I^2 is a finitely presented B module, hence it is flat iff projective iff locally free.

As Tor's, $\wedge$, S, gr are all compatible with localization we may suppose that A is a local noetherian ring and that I is contained in the maximal ideal of A. (i) ==> (iii) is obvious. (v) ==> (i). Let $\underline{f} = \{f_1, \ldots, f_n\}$ be a regular sequence generating I. Then the Koszul complex $K(\underline{f};A)$ is a free differential graded algebra resolution of B over A. Hence

$$\mathrm{Tor}_*^A(B,B) \simeq H_*(\underline{f};B) \simeq \wedge H_1(\underline{f};B)$$

and

$$I/I^2 \simeq \mathrm{Tor}_1^A(B,B) \simeq H_1(\underline{f};B) \simeq B^n .$$

Thus I is regular.

(iii) ==> (v). Let $\underline{f} = \{f_1, \ldots, f_n\}$ be a minimal system of generators for I. The Koszul complex $K(\underline{f};A)$ is a free differential graded algebra over A with an augmentation to $A/I = B$, hence there is a canonical homomorphism of graded B algebras

$$\theta_*: H_*(\underline{f};B) \to \mathrm{Tor}_*^A(B,B)$$

This homomorphism is an edge homomorphism in the spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^A(H_q(\underline{f};A),B) \Longrightarrow H_{p+q}(\underline{f};B)$$

whose five term exact sequence is

$$H_2(\underline{f};B) \xrightarrow{\theta_2} \mathrm{Tor}_2^A(B,B) \to H_1(\underline{f};A)\otimes_A B \to H_1(\underline{f};B) \xrightarrow{\theta_1} \mathrm{Tor}_1^A(B,B) \to 0$$

As I/I^2 is free over B, θ_1 is an isomorphism. The isomorphism $H_2(\underline{f};B) \approx \wedge^2 H_1(\underline{x},B)$ and the fact that θ is an algebra homomorphism show that θ_2 is isomorphic to the map $\wedge^2 I/I^2 \to \mathrm{Tor}_2^A(B,B)$ which is surjective by hypothesis. Hence $H_1(\underline{f};A)\otimes_A B = 0$ so by Nakyama $H_1(f;A) = 0$ and so $f_1,\ldots,f_n$ is a regular sequence ([ ], IV, prop.3).

(iv) <==> (v). See EGA, 0 IV, 15.1.11.

§9. Some applications of the spectral sequence.

In this section we give applications of the fundamental spectral sequence 6.8 of the vanishing or mod-$\underline{C}$ theory type.

9.1. If $B = S^{-1}A$, then $B \otimes_A B = B$ so the spectral sequence 6.8 may be applied. By 7.5, $E^2_{p1} = D_{p+1}(B/A) = 0$ for $p < n \Longrightarrow E^2_{pq} = 0$ for $p < n$, $q \geq 1$, and hence as $\mathrm{Tor}^A_+(B,B) = 0 \Longrightarrow D_{n+1}(B/A) = 0$. Thus by induction we find that $D_*(B/A) = 0$ which gives an alternative proof of 5.1.

Proposition 9.2: *If A is noetherian and B is a localization of a finite type A-algebra, then $D_q(B/A)$ is a finitely generated B module for each q. Consequently if M is a finitely generated B module, so are $D_q(B/A,M)$ and $D^q(B/A,M)$.*

Proof: The second statement follows from the first by means of the spectral sequences 3.6, 3.7 and the fact that B is noetherian. The first statement reduces by 5.2 to the case where B is a finite type A algebra. Choosing a polynomial ring P over A with finitely many generators mapping onto B we are then reduced by 4.17 to the case where $A = P$, in which case $B = A/I$ and we can apply the spectral sequence. The abutment is a finitely generated B module in each dimension, hence working modulo the class of finitely generated B modules in the spectral sequence, it suffices to show that $D_k(B/A)$ finitely generated for $k \leq n \Longrightarrow H_p(S_q \underline{\underline{L}}D_{B/A})$ finite generated for $p \leq n$. But if a complex X of projective B modules, such as $\underline{\underline{L}}D_{B/A}$, has finitely generated homology in dimensions $\leq n$, it is homotopy equivalent to a complex F of free B modules which is finite type in dimensions $\leq n$. In effect construct inductively a q-equivalence $F^{(q)} \to X$ by attaching q-dimensional generators to $F^{(q-1)}$ to obtain a (q-1)-equivalence $F^{(q-1)'} \to X$ which is surjective on H_q; then add (q+1)-dimensional generators to $F^{(q-1)'}$ to obtain a q-equivalence $F^{(q)} \to X$. If $F^{(q-1)}$ and $H_q X$ are finitely generated, we may assume $F^{(q)}$ is also as B is noetherian; hence setting $F = \lim F^{(n)}$ we obtain a weak equivalence $F \to X$, where in dimensions $\leq n$, $F = F^{(n)}$ is finitely generated. As X is projective $F \to X$ is a homotopy equivalence. This shows that up to simplicial homotopy we may replace $\underline{\underline{L}}D_{B/A}$ by a free simplicial B module $N^{-1}F$ which is finitely generated in dimensions $\leq n$. Hence $E^2_{pq} = H_p(S_q \underline{\underline{L}}D_{B/A}) \simeq H_p(S_q N^{-1}F)$ is finitely generated for $p \leq n$ and the proof of 9.2 is complete. Q.E.D.

Remark 9.3: For a different proof of 9.2 see [ ], prop.17.2. That proof yields the stronger result that when B is a finite type A-algebra and A is noetherian, there is a free simplicial A-algebra resolution P of B with only finitely many generators in each dimension.

Theorem 9.4: (Nilpotent Extension Theorem) Suppose that A is noetherian, B is a localization of a finite type A-algebra, and M is a (not necessarily finitely generated) B module. If $u \in D^q(B/A,M)$ $q > 0$, then there is a surjective map $p: B' \to B$ of A-algebras, where B' is a localization of a finite type A-algebra and the kernel of p is nilpotent, such that $p^* u = 0$

Proof: Choose an A-algebra P and a surjection $P \to B$ where P is a localization of a finitely generated polynomial ring over A. Let I = Kernel of $P \to B$ and set $P_n = B/I^{n+1}$. We are going to show that for n sufficiently large we may take $B' = P_n$. First note that

$$D^q(P_n/P,M) \simeq D^q(P_n/A,M)$$

for $q > 0$ by 4.17 and 5.3, hence we may assume that $P = A$. Secondly

$$\begin{aligned} D^q(A_n/A,M) &= H^q(\mathrm{Hom}_{A_n}(\underline{\mathbb{L}}D_{A_n/A},M)) \\ &= H^q(\mathrm{Hom}_B(\underline{\mathbb{L}}D_{A_n/A}\otimes_{A_n} B,M)) \end{aligned}$$

and hence there is an inverse system of spectral sequences

$$(9.5) \qquad E_2^{pq} = \mathrm{Ext}_B^p(H_q(\underline{\mathbb{L}}D_{A_n/A}\otimes_{A_n} B),M) \Longrightarrow D^{p+q}(A_n/A,M)$$

9.6. We shall say that an inverse system

$$\cdots \to M_{n+1} \to M_n \to \cdots$$

of objects in an abelian category with indexing set the integers ≥ 0 is a strict-essentially-zero inverse system if there is an integer N such that $M_n \to M_m$ is 0 for $n \geq m+N$. It is clear that the strict-essentially-zero inverse systems form a thick subcategory and are preserved by additive functors. Hence in virtue of the spectral sequence 9.5, theorem 9.4 will follow from

<u>Theorem 9.7</u>: <u>Suppose</u> A <u>is a noetherian ring,</u> I <u>is an ideal in</u> A , $A_n = A/I^{n+1}$, <u>and</u> $B = A_o$. <u>Then for each</u> q <u>the inverse system</u> $H_q(\underline{\underline{L}}D_{A_n/A}\otimes_{A_n} B)$ <u>is strict-essentially-zero.</u>

In virtue of the inverse system of spectral sequences

$$E^2_{pq} = H_{p+q}(S^B_q(\underline{\underline{L}}D_{A_n/A}\otimes_{A_n} B)) \Longrightarrow \mathrm{Tor}^A_{p+q}(A_n,B)$$

which results from 6.16, it suffices to prove the following two lemmas.

<u>Lemma 9.8</u>: If X_n is an inverse system in $Ho(\underline{M}_B)$ such that $H_k(X_n)$ is strict-essentially-zero for $k \leq r$, then $H_k(\underline{\underline{L}}S^B_q X_n)$ is strict-essentially-zero for $k \leq r$ and $q > 0$.

<u>Lemma 9.9</u>: If M is a finitely generated A module, then the inverse system $\mathrm{Tor}^A_q(A_n,M)$ is essentially zero for $q > 0$.

<u>Proof of 9.8</u>: By a step-by-step construction we may represent the inverse system X_n in $Ho(\underline{M}_B)$ by an inverse system in $\underline{M}_R$ of free simplicial R modules which we denote again by X_n . Let $X_n \to X_n(\ell,0)$ be the Postnikov quotient of X_n with $H_q(X_n) \simeq H_q(X_n(\ell,0))$ for $q \leq \ell$ and $H_q(X_n(\ell,0)) = 0$ for $q > \ell$. Suppose N is large enough so that $H_q(X_n) \to H_q(X_m)$ is zero for $n \geq m+N$, $q \leq r$. We show by induction on ℓ that the map $X_n \to X_m \to X_m(\ell,0)$ is homotopic to zero if $n \geq m + \ell N$, $\ell \leq r$. This is clear for $\ell = 0$ and if true for $\ell - 1$, then

consider the diagram

$$\begin{tikzcd} & X_n \arrow[dl, dashed, "\exists"'] \arrow[d] \arrow[dr, "0 \text{ if } n \geq m'+(\ell-1)N"] & \\ X_{m'}(\ell,\ell) \arrow[r] \arrow[d, "0 \text{ if } m' \geq m+N"] & X_{m'}(\ell,0) \arrow[r] \arrow[d] & X_{m'}(\ell-1,0) \arrow[d] \\ X_m(\ell,\ell) \arrow[r] & X_m(\ell,0) \arrow[r] & X_m(\ell-1,0) \end{tikzcd},$$

where the rows are fibration sequences in $Ho(\underline{M}_B)$ and where $X_m(\ell,\ell)$ has only the homology group $H_\ell(X)$ in dimension ℓ and hence has the property that maps of $\ell - 1$ connected complexes to it are determined by the map on homology. The diagram completes the induction. Thus the composite $\underline{\underline{LS}}_q X_n \to \underline{\underline{LS}}_q X_m \to \underline{\underline{LS}}_q X_m(r,0)$ for $q > 0$ and $n \geq m+rN$ is zero in $Ho(\underline{M}_B)$. As the latter map is an r-equivalence by 7.3, the former induces the zero map on homology. Q.E.D.

<u>Proof of 9.9</u>: By dimension-shifting we may assume that $q = 1$, in which case choosing an exact sequence

$$0 \to K \to F \to M \to 0$$

with F a free finitely generated A module we have

$$\mathrm{Tor}_1^A(A/I^n, M) = \frac{K \cap I^n F}{I^n K}$$

By Artin-Rees there is an integer N with $K \cap I^nF \subset I^mK$ for $n \geq m+N$, hence $\mathrm{Tor}_1^A(A_n,M) \to \mathrm{Tor}_1^A(A_m,M)$ is zero for $n \geq m+N$. Q.E.D.

n = nilpotent

Corollary 9.10: Suppose A noetherian, B is a finite type A-algebra and M is a B module. Let $\underline{T}_n$ be the Grothendieck topology on the category of finite type A-algebras over B in which the covering families are single maps $X \to Y$ which are surjective and have nilpotent kernel. Then

$$\boxed{D^q(B/A,M) \simeq H^q_{\underline{T}_n}(B,\mathrm{Der}(\cdot/A,M))}$$

Proof: With the notations of 2.1 there is an obvious map of topologies $f\colon \underline{T} \to \underline{T}_n$ which gives rise to a Leray spectral sequence

$$E_2^{pq} = H^p_{\underline{T}_n}(B,R^qf_*(\mathrm{Der}(\cdot/A,M))) \Longrightarrow D^{p+q}(B/A,M)$$

where $R^qf_*(\mathrm{Der}(\cdot/A,M))$ is the sheaf on $\underline{T}_n$ associated to the presheaf $X \to H^q_{\underline{T}}(X,\mathrm{Der}(\cdot/A,M)) = D^q(X/A,M)$. Theorem 9.7 thus shows this sheaf is zero for $q > 0$, hence the spectral sequence degenerates and the corollary follows.

Remark 9.11: In general for $q = 1$ any element $u \in D^1(B/A,M)$ may be killed by a nilpotent extension $X \to B$, namely take X to be the extension of B by M corresponding to u by 3.12. If the $D^q(B/A,\cdot)$ are derived functors of $D^0(B/A,\cdot)$, then dimension shifting would permit one to conclude the same for $q \geq 1$ (and this is the case for group and Lie algebra cohomology). But

9.7 for $q = 2$ is generally false. For example suppose $I = fA \subset A$ and M is an injective B module containing $D_2(B/A)$. Let $u \in D^2(B/A,M) \simeq \mathrm{Hom}_B(D_2(B/A),M)$ be the element corresponding to the inclusion. As the map $A_n = A/I^{n+1} \to B$ are cofinal in all nilpotent extensions of B as an A-algebra, to be able to kill u by a nilpotent extension means that

$$D_2(A_n/A,B) \to D_2(B/A)$$

is zero for n large. However as I is principal the 5 term exact sequence 6.12 shows that this map is isomorphic to

$$\mathrm{Tor}_2^A(A_n,B) \to \mathrm{Tor}_2^A(B,B) ,$$

which after some calculation is seen to be isomorphic to

$$\frac{Af \cap \mathrm{Ann}\, f^{n+1}}{f\ \mathrm{Ann}\, f^{n+1}} \xrightarrow{f^n} \frac{Af \cap \mathrm{Ann}\, f}{0} ,$$

which is zero if and only if $\mathrm{Ann}\, f^{n+1} = \mathrm{Ann}\, f^{n+2}$. But it is easy to produce examples of elements f in non-noetherian rings for which the increasing sequence $\mathrm{Ann}\, f^n$ does not stabilize.

§10. Local complete intersections

In this section we study when the cotangent complex $\underline{\mathbb{L}}_{B/A}$ is of projective dimension ≤ 1. In the noetherian case we show this is equivalent to B being a local complete intersection over A. When A is a local noetherian ring with residue field k we show that A is regular (resp. a complete intersection) if and only if $D_2(k/A)$ (resp. $D_3(k/A)$) is zero. These results are not entirely new (compare [ ] and [André]), but they are rather nice corollaries of the fundamental spectral sequence.

10.1. If M is a B module we let $K(M,q)$ $q \geq 0$ be the simplicial B module whose normalization is the chain complex $M[q]$ with M in dimension q and zero elsewhere. For any $(q-1)$-connected simplicial B module X we have

$$\operatorname{Hom}_{Ho(\underline{M}_B)}(X, K(M,q)) \simeq \operatorname{Hom}_{D(B)}(NX, M[q])$$
$$\simeq \operatorname{Hom}_B(H_q X, M)$$

hence if $B = A/I$ there is a canonical map

(10.2) $$\underline{\mathbb{L}}_{B/A} \to K(I/I^2, 1)$$

which is a 1-equivalence.

Theorem 10.3: The following assertions are equivalent when $B = A/I$:

(i) I is quasi-regular

(ii) $D_q(B/A,M) = 0$ for all B modules M and $q \geq 2$.

(iii) I/I^2 is a flat B module and $\underline{\mathbb{L}}_{B/A} \simeq K(I/I^2,1)$.

Proof: (ii) and (iii) are equivalent by the universal coefficient spectral sequence 3.

(iii) <==> (i). $K(I/I^2,1)$ is homotopy equivalent to $\Sigma(c(I/I^2))$, hence by 7.22

$$H_{p+q}(\underline{\underline{L}}S_q^B K(I/I^2,1)) \simeq H_p(\underline{\underline{L}} \wedge_q^B c(I/I^2)) \simeq \begin{cases} 0 & p > 0 \\ \wedge_q(I/I^2) & p = 0 \end{cases}$$

As I/I^2 is flat both $\underline{\underline{L}}$'s may be dropped by 7.16. Therefore assuming 10.2 is an n-equivalence $n \geq 1$, we obtain from 7.3 the formula

$$E_{pq}^2 = H_{p+q}(S_q^B \underline{\underline{L}} D_{B/A}) = H_{p+q}(S_q^B K(I/I^2,1)) = \begin{cases} 0 & p+q \leq n \text{ , } p > 0 \\ \wedge_q(I/I^2) & p = 0 \end{cases}$$

If (iii) holds, then we may take $n = \infty$, so $E_{pq}^2 = 0$ for $p > 0$ and the fundamental spectral sequence degenerates showing that the edge homomorphism $\wedge(I/I^2) \to \mathrm{Tor}^A(B,B)$ is an isomorphism and hence I is quasi-regular. If I is quasi-regular and 10.2 is an n-equivalence, then as the edge homomorphism is an isomorphism the only possible non-zero differential issuing from $E_{n,1}^2$ namely $d_n: E_{n,1}^2 \to E_{o,n}^2$ is zero, hence $E_{n,1}^2 = D_{n+1}(B/A) = 0$ and so 10.2 is an (n+1)-equivalence. Thus induction yields (iii). Q.E.D.

Corollary 10.4: The following are equivalent when $B = A/I$:

(i) I is regular

(ii) $D^q(B/A,M) = 0$ for all B modules M and $q \geq 2$.

(iii) I/I^2 is a projective B module and $\underline{\underline{L}}D_{B/A} \simeq K(I/I^2,1)$.

Corollary 10.5: If A is noetherian and $B = A/I$, the following are equivalent:

(i) I is regular

(ii) $D_2(B/A,M) = 0$ for all finitely generated B-modules M

(ii)' $D^2(B/A,M) = 0$ " " " " " - " "

(iii) I/I^2 is a projective B-module and $D_2(B/A) = 0$.

Proof: (ii) (ii)' and (iii) are equivalent by the universal coefficient spectral sequences. (i) ==> (ii) is clear from 10.3 while (iii) ==> (i) follows from 8.13 (iii) and the 5 term exact sequence 6.12.

10.6. Suppose A is noetherian and B is a finite type A-algebra. Choose a surjection of A-algebras $P \to B$ where P is a polynomial ring over A of finite type and I be the kernel of this map. B is said to be a local complete intersection over A if I is regular. By 5.4 we have $D_q(B/A,M) = D_q(B/P,M)$ for $q > 0$ hence by 10.3 this condition on B is independent of P and we obtain the following.

Theorem 10.7: Suppose A is noetherian and B is an A-algebra of finite type. The following assertions are equivalent:

(i) B is a local complete intersection over A

(ii) $D_q(B/A,M) = D^q(B/A,M) = 0$ for all B-modules M and $q \geq 2$.

(iii) $D_2(B/A,M) = 0$ for all finitely generated B modules

(iii)' $D^2(B/A,M) = 0$ " " " " " "

(iv) $\underline{\underline{L}}D_{B/A}$ has projective dimension ≤ 1

Here a chain complex X of B modules is said to be of projective dimension $\leq r$ if it is isomorphic in the derived category to a chain complex of projective B modules which is zero in dimension $> r$, or equivalently $H^q\{\mathrm{Hom}_B(X,M)\} = 0$ for all B modules M.

Corollary 10.8: If A is a local noetherian ring with residue class field k, the following are equivalent:

(i) A is regular

(ii) $D_q(k/A) = 0$ for $q \geq 2$

(iii) $D_2(k/A) = 0$

A local ring is regular if and only if its maximal ideal $\underline{m}$ is generated by regular sequence, i.e. iff $\underline{m}$ is regular, so this follows from 10.5.

10.9. If A is a local noetherian ring, with residue field k, then by Cohen $\hat{A} = P/I$ where P is a complete regular local ring. A is said to be a complete intersection if I is a regular ideal. By the change of rings exact sequence associated to $P \to \hat{A} \to k$ (4.17) and flat base extension (4.9)

$$\begin{array}{ccccccc} \to D_q(\hat{A}/P,k) & \to & D_q(k/P) & \to & D_q(k/\hat{A}) & \to & \cdots \\ & & \parallel & & \wr\!\parallel & & \\ & & 0 \text{ if } q\geq 2 & & D_q(k/A) & & \end{array}$$

we find that

$$(10.10) \qquad D_q(k/A) \simeq D_{q-1}(\hat{A}/P,k) \qquad q \geq 3$$

$$(10.11) \qquad 0 \to D_2(k/A) \to I/I^2 \otimes_{\hat{A}} k \xrightarrow{\alpha} \underline{m}_p/\underline{m}_p^2 \to D_1(k/A) \to 0$$

where α is induced by the inclusion $I \subset m$. Now as $\hat{A}$ is local noetherian, the projective dimension of $\underline{\underline{LD}}_{\hat{A}/P}$ is the largest q for which $H_q(\underline{\underline{LD}}_{\hat{A}/P} \otimes_{\hat{A}} k) = D_q(\hat{A}/P,k) \neq 0$. Thus from 10.10 and 10.5 we find

Corollary 10.12: If A is a local noetherian ring with residue field k , then the following conditions are equivalent:

i A is a complete intersection;

ii $D_q(k/A) = 0$ for all $q \geq 3$

iii $D_3(k/A) = 0$

Remark 10.13: One may always arrange that the dimension of P is minimal such that $P/I = \hat{A}$. This is equivalent to $I \subset m_p^2$, whence the map α in 10.11 is zero, and so we obtain an interpretation of $\dim_k D_1(k/A)$ as the minimal number of generators for A and $\dim_k D_2(k/A)$ as the minimal number of relations. We will generalize this in the following chapter.

Remark 10.14: Suppose A noetherian and B finite type over A . Then 10.7 and 5.3 characterize when $\underline{\underline{LD}}_{B/A}$ has projective dimension 0 or 1 . In the next chapter we will give evidence for the following conjecture.

Conjecture 10.15: B is a local complete intersection over A iff B is finite Tor dimension over A and $\underline{\underline{L}}D_{B/A}$ has finite projective dimension.

The implication ==> is clear. One reduces to the case where $B = A/I$, whence if I is generated by r elements, Tor dim of A/I over A is $\leq r$ by means of the Koszul complex. For the converse, the hypothesis that B be of finite Tor dimension over A is necessary, as may be seen from the example of a local noetherian ring A which is a complete intersection but which is not regular; then the projective dimension of $\underline{\underline{L}}D_{k/A}$ is two. (Incidentally in all examples where $\underline{\underline{L}}D_{B/A}$ has finite projective dimension known to the author the projective dimension is ≤ 2 and one might conjecture this is always true.) By the argument used to prove 10.12 the conjecture 10.15 implies the following.

Conjecture 10.16: If A is a local noetherian ring with residue field k and $D_q(k/A) = 0$ for q sufficient large, then A is a complete intersection.

§11. Local rings in characteristic zero

A denotes a local noetherian ring with residue class field k. The fundamental spectral sequence when $B = k$ is

(11.1) $$E^2_{pq} = H_{p+q}(S^k_q \underline{L} D_{k/A}) \Longrightarrow \mathrm{Tor}^A_{p+q}(k,k)$$

and it has an edge homomorphism $\mathrm{Tor}^A_n(k,k) \to D_n(k/A)$ which annihilates the decomposable elements of $\mathrm{Tor}^A_*(k,k)$. (6.15).

Theorem 11.2: If the characteristic of k is zero, then all the differentials in the spectral sequence 11.1 are zero, hence $D_*(k/A)$ is isomorphic to the indecomposable space of $\mathrm{Tor}^A_*(k,k)$.

Proof: By a result of Assmus [ ] $\mathrm{Tor}^A_*(k,k)$ is a commutative Hopf algebra, hence by a theorem of Borel (see [ ]) $\mathrm{Tor}^A_*(k,k)$ is isomorphic to a tensor product of an exterior algebra with odd dimensional generators and a polynomial ring with even dimensional generators. In other words, if $W_* \subset \mathrm{Tor}^A_*(k,k)$ is a complementary subspace to the decomposable elements, then $\tilde{S}W_* \simeq \mathrm{Tor}^A_*(k,k)$ where $\tilde{S}$ is the skew-commutative symmetric algebra functor (7.42). By 7.43 $E^2_{*q} \simeq \tilde{S}_q D_*(k/A)$. Abbreviating $\mathrm{Tor}^A_n(k,k)$ and $D_n(k/A)$ by T_n and D_n respectively we show by induction on n that the composition $W_q \to T_q \to D_q$ is an isomorphism for $q < n$ and surjective for $q = n$. For $n = 1$ this is clear since all three are m_A/m_A^2. Assuming the assertion is true for n it follows that all differentials issuing from $E^2_{p1} = D_p$ are zero for $p < n$, hence as E^2 is generated by $E^2_{*,1}$ as an algebra and as the differentials are derivations, we have that all differentials issuing from E^r_{pq} are zero for $p < n$. Hence $E^\infty_{pq} = E^2_{pq}/B^\infty_{pq}$ for $p < n$, so

$$\operatorname{gr} T_n = \bigoplus_{p=0}^{n-1} E^\infty_{p,n-p} = \bigoplus_{p=0}^{n-2} E^2_{p,n-p}/B^\infty_{p,n-p} \oplus D_n .$$

As $W_q \simeq D_q$ for $q < n$, we have $\dim E^2_{pq} = \dim(\tilde{S}_q D_*)_{p+q} = \dim(\tilde{S}_q W_*)_{p+q}$ for $p \leq n-2$. But

$$\dim T_n = \dim \operatorname{gr} T_n = \sum_{p=0}^{n-2} \dim(E^2_{p,n-p}/B^\infty_{p,n-p}) + \dim D_n$$

$$\leq \sum_{p=0}^{n-2} \dim(\tilde{S}_{n-p} W_*)_n + \dim W_n = \dim T_n$$

which shows that $B^\infty_{p,n-p} = 0$ for $0 \leq p \leq n-2$ and $\dim W_n = \dim$ I The latter equality shows that the surjective map $W_n \to D_n$ is an isomorphism. The former equality shows that all differentials issuing from $E_{n,1}$ are zero, and hence the edge homomorphism $T_{n+1} \to D_{n+1}$ is surjective. But we know that the edge homomorphism kills decomposable elements, hence $W_{n+1} \to D_{n+1}$ is surjective and the induction is complete. Q.E.D.

11.3. If the characteristic of k is $p > 0$, then the above arguments work in dimensions $< 2p$. In effect $H_*(S^k_q \underline{L} D_{k/\Lambda})$ $\tilde{S}_q(D_*)$ for $q < p$ because the proof of 7.43 used only the complete reducibility of the representation of the symmetric group Σ_q which holds in characteristic p if $q < p$. Also Borel's theorem in characteristic p shows that $\operatorname{Tor}^A_*(k,k)$ is a tensor product of monogenic algebras $k[x]/(x^m)$ where $m = 2$ if p and degree x are odd and $m = p^a, 1 \leq a \leq \infty$ otherwise.

(Actually only $m = p$ can occur when the degree of x is even because Tor_* is the homology of simplicial k-algebra, hence admits a canonical system of divided power operations for elements of degree ≥ 2, so that $x^2 = 0$ for all x of odd degree ≥ 1 and $x^p = 0$ for all x of even degree ≥ 2.) Thus $\mathrm{Tor}_* \simeq \widetilde{S}W_*$ in dimensions $< 2p$. Therefore we have the following.

Theorem 11.4: If the characteristic of k is $p > 0$, then $D_q(k/A)$ is isomorphic to the indecomposable quotient space of $\mathrm{Tor}_q^A(k,k)$ if $q < 2p$.

Remark 11.5: This is false for $q = 2p$ if A is not regular since the kernel of the map $\mathrm{Tor}_{2p} \to D_{2p}$ will contain the p-th divided powers of elements of Tor_2 and these are not decomposable elements of Tor_{2p}.

11.6. For the rest of this section we suppose that k is of characteristic zero. The dual of the Hopf algebra $\mathrm{Tor}_*^A(k,k)$ may be shown to be $\mathrm{Ext}_A^*(k,k)$ with the Yoneda product. Ext^* is cocommutative hence by a theorem of Milnor and Moore [ ] it is the universal enveloping algebra of its (skew-)Lie algebra of primitive elements. Under duality the primitive subspace of Ext^* corresponds to the indecomposable quotient space of Tor_*. As $D^*(k/A)$ is the dual of $D_*(k/A)$ we have

Theorem 11.7: If k is of characteristic zero, then $D^*(k/A)$ is canonically isomorphic to the subspace of primitive elements in $\mathrm{Ext}_A^*(k,k)$. In particular $D^*(k/A)$ has a natural (skew-) Lie algebra structure and there is a canonical Hopf algebra isomorphism

$$\boxed{U\{D^*(k/A)\} \simeq \mathrm{Ext}_A^*(k,k)}$$

where U denotes the universal enveloping algebra. Moreover there is a canonical algebra isomorphism

$$\tilde{S}\{D_*(k/A)\} \simeq \mathrm{Tor}_*^A(k,k) \ .$$

The last assertion follows from the Poincare-Birkhoff-Witt theorem in characteristic 0, which when $\underline{g}$ is a Lie algebra gives a canonical coalgebra isomorphism $\exp\colon \tilde{S}(\underline{g}) \to U(\underline{g})$ and hence a canonical algebra isomorphism ${}^t(\exp)\colon U(\underline{g})' \to \tilde{S}(\underline{g})' \simeq \tilde{S}(\underline{g}')$.

11.8. The calculation of $D^*(k/A)$ with its Lie algebra structure is therefore equivalent to the calculation of $\mathrm{Ext}_A^*(k,k)$ with its Hopf algebra structure. If we assume A is complete, which doesn't change any of the groups involved, then by Cohen A has a field of representatives, so we may view A as a k-algebra and use a continuous form of the cobar construction to calculate Ext^* as follows.

Let $m = m_A$ be the maximal ideal of A. As A is complete it is a linearly compact vector space over k with topological dual

$$m' = \varinjlim_{n} \mathrm{Hom}_k(m/m^n, k) \ .$$

The topological dual of the complex $m[1]$, which is m in dimension 1 and zero elsewhere is $m'[-1]$. Let d be the unique degree -1 derivation of the tensor algebra $T(m'[-1])$ such that $d: m'[-1] \to m'[-1] \otimes m'[-1]$ is the transpose of the multiplication map $m \otimes m \to m$. The associativity of multiplication implies $d^2 = 0$, and the commutativity implies that d is a coderivation of $T(m'[-1])$ when this is endowed with the Hopf algebra structure in which $m'[-1]$ is primitive. Thus $T(m'[-1])$ together with d is a differential graded Hopf algebra. Its space of primitive elements is the differential graded Lie algebra $L(m'[-1])$ together with the induced differential, where L is the free (skew-) Lie algebra functor on graded vector spaces.

We recall the standard convention $H^q(K) = H_{-q}K$ for any complex K .

Proposition 11.9: There is a canonical Hopf algebra isomorphism

$$\boxed{\mathrm{Ext}_A^*(k,k) \simeq H^*\{T(m'[-1]),d\}}$$

and hence a canonical Lie algebra isomorphism

$$D^*(k/A) \simeq H^*\{L(m'[-1],d)\}$$

Proof: The second formula follows from the first using 11.7 and the fact that in characteristic zero the primitive element and homology functors for differential graded Hopf algebras commute (see [ ]). The first statement is presumably well known (compare [4] 1958-59, p.15-09 for the case of graded algebras) so we shall present only an outline of its justification. Let A' be the continuous k-dual of A and make $A' \otimes T(m'[-1])$ into a differential graded right $T(m'[-1])$ module by defining $d\colon A' \to A'\otimes m'[-1]$ to be dual to the multiplication $A\otimes m \to A$. $I^{\cdot} = A'\otimes T(m'[-1])$ is a resolution of k, since it is the cobar construction of the coalgebra A'. Moreover A' is an injective A module, in fact, it is an injective hull of k, so one sees that $I^{\cdot}$ is an injective resolution of k as an A module, whence there is canonical k module isomorphism $\mathrm{Ext}_A^*(k,k) = H^*\{\mathrm{Hom}_A(k,I^{\cdot})\} = H^*(T(m'[-1]),d)$.

To see this isomorphism is compatible with products we recall that the Yondeda product of $u \in \mathrm{Ext}_A^p(k,k)$ and $v \in \mathrm{Ext}_A^q(k,k)$, denoted $u\cdot v \in \mathrm{Ext}_A^{p+q}(k,k)$, is the composition $v\circ u$ when u and v are thought of as degree p and q maps from k to k in the derived category $D(A)$. If we represent u and v by maps $\tilde{u}\colon k[p] \to I^{\cdot}$, $\tilde{v}\colon k[q] \to I^{\cdot}$ then to calculate $v\circ u$ we must choose a map φ in the diagram

$$\begin{tikzcd}
 & & & & & & k \arrow[d, "\tilde{u}"'] & \\
 & & & k \arrow[dr] \arrow[ddr, "\tilde{v}"'] & & & & \\
 & & & & I^0 \arrow[r] \arrow[dd, "\varphi"] & \cdots \arrow[r] & I^p \arrow[r] \arrow[dd, "\varphi"] & \cdots \\
 & k \arrow[dr] & & & & & & \\
 {} \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & \cdots \arrow[r] & I^q \arrow[r] & \cdots \arrow[r] & I^{p+q} \arrow[r] & \cdots
\end{tikzcd}$$

whence $v\circ u$ is represented by $\varphi\circ\tilde{u}\colon k[p+q] \to I^{\cdot}$. In the situation at hand $\tilde{v}(\lambda) = \lambda\tilde{v}(1)$ where $\tilde{v}(1) \in T(m'[-1])^q$ so we may take φ to be $\varphi(\alpha) = \alpha\cdot\tilde{v}(1)$, where we use the right

$T(m'[-1])$ module structure of $I^{\cdot}$. Then $(\varphi\circ\tilde{u})(1) = \tilde{u}(1)\cdot\tilde{v}(1)$, i.e. the Yoneda product $u \cdot v \in \mathrm{Ext}^{p+q}$ corresponds to the product of u and v in the algebra $H^*(T(m'[-1]),d)$.

Finally we must show the isomorphism is compatible with the coalgebra structure on Ext^* , which we recall comes via duality from the algebra structure on Tor_* . The tensor product of the natural coalgebra structures on A' and $T(m'[-1])$ is a coalgebra structure on $I^{\cdot}$ for which d is a coderivation (see [ ] for details). In the special case where A is finite dimensional over k , it follows that the dual of $I^{\cdot}$ is a free differential graded algebra resolution of k (in fact its the bar resolution $B(A)$) , hence Tor_* is algebra-isomorphic to $H_*(k\otimes_A(I^{\cdot})')$ and Ext^* is coalgebra-isomorphic to $H^*(\mathrm{Hom}_A(k,I^{\cdot})) = H^*(T(m'[-1]),d)$. The general case then follows from the special case by setting $A_n = A/m^n$ and letting $n \to \infty$. Q.E.D.

The map $A \mapsto D^*(k/A)$ is a contravariant functor from the category of local noetherian rings with residue field k to the category of (skew-graded) Lie algebras over k which transforms direct products into direct sums in virtue of the following Kunneth-type formula:

Theorem 11.10: If A and B are local noetherian rings with residue field k (of char. 0), then the canonical homomorphism of Lie algebras

$$D^*(k/A) \vee D^*(k/B) \simeq D^*(k/A\times_k B)$$

is an isomorphism, where $\vee$ denotes direct sum in the category of Lie algebras.

Proof: If R and S are two non-commutative augmented k-algebras, then their direct sum in this category is given by the formula

$$(11.11) \qquad R \vee S = \oplus_{\alpha} (R \vee S)_{\alpha}$$

where α runs over all words in the free monoid generated by two letters a and b, where

$$(11.12) \qquad (R \vee S)_{(\underbrace{a\ldots a}_{p}\ \underbrace{b\ldots b}_{q}\ldots)} = \underbrace{\bar{R}\otimes\ldots\otimes\bar{R}}_{p}\otimes\underbrace{\bar{S}\otimes\ldots\otimes\bar{S}}_{q}\otimes\ldots$$

and where $\bar{R}$ and $\bar{S}$ denote the augmentation ideals of R and S respectively. If R and S are differential graded algebras and $R \vee S$ is endowed with the differential which is the unique derivation of $R \vee S$ coinciding on $\bar{R} \simeq (R \vee S)_a$ and $\bar{S} \simeq (R \vee S)_b$ with the differentials of R and S respectively, then the decomposition 11.11 is compatible with the differentials and shows that

$$H(R \vee S) \simeq H(R) \vee H(S) .$$

If $C = A \times_k B$, then $m_C = m_A \oplus m_B$ and there is an isomorphism of differential graded algebras

$$T(m_A^![-1]) \vee T(m_B^![-1]) \simeq T(m_C^![-1])$$

and so an isomorphism of homology algebras, which by 11.9 gives isomorphisms

$$\begin{array}{ccc} \mathrm{Ext}^*_A(k,k)\vee\mathrm{Ext}^*_B(k,k) & \simeq & \mathrm{Ext}^*_C(k,k) \\ \uparrow\wr & & \uparrow\wr \\ U(D^*(k/A)\vee D^*(k/B)) & \simeq & U(D^*(k/C)) \end{array} .$$

Taking primitive elements, the theorem follows.

Remark 11.13: If we consider the category of local noetherian k algebras with residue field k which are essentially of finite type (resp. complete), then $C = (A\otimes_k B)_m$ (resp. $C = A\hat{\otimes}_k B$) is the direct sum in this category and we have the following formula

$$D^*(k/C) \simeq D^*(k/A) \times D^*(k/B) \quad ,$$

which follows easily from 4.9 and the compatibility of D^* with flat base change (4.9).

Remark 11.14. It is possible to write down a formula analogous to 11.11 and 11.12 for the direct sum of skew Lie algebras. For the formula see any account of the Hilton-Milnor theorem in algebraic topology.

Examples: 11.15. Suppose $A = k\oplus m$ where $m^2 = 0$. Then the cobar construction has zero differentials so $\mathrm{Ext}^*_A(k,k) = T(m'[-1])$ and $D^*(k/A) = L(m'[-1])$. If $\dim \underline{m} = 1$ and x is a basis for $\underline{m}'$, then $D^1(k/A) = kx$, $D^2(k/A) = kx^2$ where $x^2 = \frac{1}{2}[x,x]$ and all other $D^q = 0$. If $\dim \underline{m} = 2$ and x,y is a basis for m', then $D^1 = kx \oplus ky$, $D^2 = kx^2 \oplus k[x,y] \oplus ky^2$, $D^3 = k[x,[x,y]] \oplus k[y,[x,y]]$ etc.

11.16. Suppose C is the complete local ring at the intersection of a p plane and a q plane meeting transversally. Then $C = A \times_k B$ where A and B are power series rings in p and q variables. $D^1(k/A) = (m_C/m_C^2)' = (m_A/m_A^2)' \oplus (m_B/m_B^2)'$ so if $x_1,\ldots,x_p$ is a basis for m_A/m_A^2 and $y_1,\ldots,y_q$ is a basis for $(m_B/m_B^2)'$, then $D^*(k/C)$ is the quotient of the free skew Lie algebra generated by the degree 1 elements $x_1,\ldots,x_p,y_1,\ldots,y_q$ by the ideal generated by the relations

$$[x_i,x_j] = 0$$

$$[y_i,y_j] = 0$$