DEFORMATIONS IN POSITIVE CHARACTERISTIC

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Introduction

In class we went over the proof of the main theorem of Lurie's DAG-X [3], namely the correspondence between formal moduli problems over k and dg Lie k-algebras, where k is a field of characteristic 0. More precisely, we have the following theorem.

Theorem (Lurie [3], Pridham)**.** *Let k be a field of characteristic 0. Let Moduli_k be the ∞-category of formal moduli problems over k, and Lie_k the ∞-category of dg Lie k-algebras. Then there is an equivalence of ∞-categories*

$$\mathrm{Moduli}_k \simeq \mathrm{Lie}_k.$$

One might ask if we can easily transport this theory to the case where k has characteristic $p > 0$. In the definition of Moduli_k for a characteristic 0 field, we took as our small algebra objects the augmented commutative dg-k-algebras A such that

- $\mathrm{H}^n(A) = 0$ for $n > 0$ and $n \ll 0$;
- $\mathrm{H}^n(A)$ is a finite-dimensional k-vector space for each n;
- $\mathrm{H}^0(A)$ is a local ring with maximal ideal $\mathfrak{m}$, and the canonical map $k \to \mathrm{H}^0(A)/\mathfrak{m}$ is an isomorphism.

In characteristic 0, the ∞-category $\mathrm{CAlg}_k^{\mathrm{dg}}$ of commutative dg-k-algebras is equivalent to the ∞-category $\mathrm{CAlg}_k^{\mathbf{E}_\infty}$ of $\mathbf{E}_\infty$-k-algebras, and the ∞-category CAlg_k^{Δ} of simplicial commutative k-algebras sits inside the connective $\mathbf{E}_\infty$-k-algebras. More specifically, we have functors

$$\mathrm{CAlg}_k^{\Delta} \xrightarrow{\phi} \mathrm{CAlg}_k^{\mathrm{dg}} \xrightarrow{\psi} \mathrm{CAlg}_k^{\mathbf{E}_\infty},$$

where ψ is an equivalence of ∞-categories and ϕ is fully faithful with essential image the connective objects in $\mathrm{CAlg}_k^{\mathbf{E}_\infty}$. However, when k has characteristic $p > 0$, the functors above are no longer fully faithful, and the theory is hard to work with. So in positive characteristic, the approaches to "derived" algebraic geometry diverge into the world of simplicial commutative rings and the world of (connective) $\mathbf{E}_\infty$-rings.

Very recently Lukas Brantner and Akhil Mathew have published a paper [1], in which they extend the above theorem to fields of arbitrary characteristic. They prove results for formal moduli functors in both the simplicial commutative ring world and the $\mathbf{E}_\infty$-ring world. In this overview, we will go over the abstract framework Brantner and Mathew use, with the idea that we can apply it to both CAlg_k^{Δ} (simplicial commutative k-algebras) or $\mathrm{CAlg}_k^{\geqslant 0,\mathbf{E}_\infty}$ (connective $\mathbf{E}_\infty$-k-algebras). With this in mind, we let CAlg_k denote either of these two ∞-categories, making explicit which one we need when necessary.

The idea of Brantner and Mathew is to replace the ∞-category Lie_k of dg Lie algebras over k with the ∞-category Lie_k^{π} of *(spectral or non-spectral, respectively) partition Lie algebras* over k. They then prove a generalization of the above theorem, stated as follows.

Main theorem. *Let k be a field of arbitrary characteristic. Then there is an equivalence of ∞-categories*

$$\mathrm{Moduli}_k^{\mathrm{CAlg}} \simeq \mathrm{Lie}_k^{\pi},$$

via the tangent fibre functor $X \mapsto \mathbb{T}_X$, where $\mathrm{Moduli}_k^{\mathrm{CAlg}}$ is the ∞-category of formal moduli problems over k defined using CAlg.

In this paper, we will consider fields of both zero and positive characteristic; Brantner and Mathew prove results for certain rings of mixed characteristic, which we will not go into here.

Let us briefly recall how we constructed the equivalence in the Lurie-Pridham theorem. We defined a functor

$$\Psi : \mathrm{Lie}_k \to \mathrm{Moduli}_k$$

as follows. For $\mathfrak{g}_\bullet \in \mathrm{Lie}_k$, we have the (cohomological) Chevalley-Eilenberg complex $C^\bullet_{\mathrm{Lie}}(\mathfrak{g}_\bullet)$ of $\mathfrak{g}_\bullet$. This defines a contravariant functor to augmented dg-k-algebras

$$C^\bullet_{\mathrm{Lie}} : \mathrm{Lie}_k \to (\mathrm{AugCAlg}_k^{\mathrm{dg}})^{\mathrm{op}}.$$

Next, we have the formal spectrum functor (represented by $A \in \mathrm{AugCAlg}_k^{\mathrm{dg}}$)

$$\mathrm{Spf}\, A : \mathrm{AugCAlg}_k^{\mathrm{dg}} \to \mathcal{S},$$

$$B \mapsto \mathrm{Map}_{\mathrm{AugCAlg}_k^{\mathrm{dg}}}(A, B),$$

where Map is the ∞-categorical mapping space. This gives us a formal moduli problem, and we thus define

$$\Psi(\mathfrak{g}_\bullet) = \mathrm{Spf}\, C^\bullet_{\mathrm{Lie}}(\mathfrak{g}_\bullet).$$

To construct an inverse functor

$$\Phi : \mathrm{Moduli}_k \to \mathrm{Lie}_k,$$

we will need to use the Koszul duality functor $\mathbb{D}$, which we can obtain as an adjoint of $C^\bullet_{\mathrm{Lie}}$. The main effort in proving the equivalence in the Lurie-Pridham theorem is showing that this Koszul duality functor restricts to an equivalence on certain "nice" subclasses of objects, and takes "nice" pullbacks to pushouts. In this overview, we will attempt to cover Brantner and Mathew's argument for the analog of this equivalence over "nice" objects in the case of an arbitrary field.

Preliminaries

Let k be a field and $\mathcal{S}$ the ∞-category of spaces.

Definition. An $A \in \mathrm{CAlg}_k$ is called *Artinian* if $\pi_0 A$ is a local Artinian ring with residue field k, and $\pi_* A$ is a finite-dimensional k-vector space. The ∞-category of Artinian $A \in \mathrm{CAlg}$ will be denoted $\mathrm{CAlg}_k^{\mathrm{Art}}$.

Definition. A functor $X : \mathrm{CAlg}_k^{\mathrm{Art}} \to \mathcal{S}$ is a *derived formal moduli problem over* k if $X(k)$ is contractible and the following condition is satisfied. For each pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

in $\mathrm{CAlg}_k^{\mathrm{Art}}$ such that the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective, the diagram

$$\begin{array}{ccc} X(A) & \longrightarrow & X(A_0) \\ \downarrow & & \downarrow \\ X(A_1) & \longrightarrow & X(A_{01}) \end{array}$$

is a pullback diagram in $\mathcal{S}$. The ∞-category of such derived formal moduli problems will be denoted $\mathrm{Moduli}_k^{\mathrm{CAlg}}$.

So how exactly do we construct the ∞-category Lie_k^π of partition Lie k-algebras? The process is pretty involved, and requires formalizing the so-called *partition Lie algebra monad*. We will take the construction of the partition Lie algebra monad $T^\pi_{\mathrm{Lie}_k}$ as a sort of a black box. We then define

$$\mathrm{Lie}_k^\pi := \mathrm{Alg}_{T^\pi_{\mathrm{Lie}_k}},$$

i.e. Lie_k^π is the ∞-category of $T^\pi_{\mathrm{Lie}_k}$-algebras.

We start with the definition of a monad, which we talk about informally. For a precise definition, see [4, Section 4.7].

Definition. Let Mod_k be the ∞-category of chain complexes of k-vector spaces. A *monad* T on Mod_k is a functor $T : \mathrm{Mod}_k \to \mathrm{Mod}_k$ with natural transformations $\mathrm{id} \to T$, $T \circ T \to T$, and some coherence data. Put another way, a monad T on an ∞-category $\mathcal{C}$ is an algebra object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, which works since $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ has a monoidal structure given by composition of endofunctors. We can form an ∞-category of Alg_T of T-algebras, which can be thought of as complexes $M \in \mathrm{Mod}_k$ with natural transformations $T(M) \to M$, and some coherence data.

We will also need to describe an adjoint pair, cot and sqz. An ∞-operad is more or less an ∞-category equipped with morphism spaces that *take in more than one input*, and some compatibility conditions. For a detailed construction, see [4]. Given an ∞-operad $\mathcal{O}$ in Mod_k, we can construct the ∞-category of $\mathcal{O}$-algebras in Mod_k, denoted $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_k)$. Suppose we have $\mathcal{O}(0) = 0$ and $\mathcal{O}(1) \simeq k$; then if we restrict along the map from $\mathcal{O}$ to the trivial operad, this gives us a *square-zero* functor,

$$\mathrm{sqz} : \mathrm{Mod}_k \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_k),$$
$$V \mapsto (V \text{ with an } \mathcal{O}\text{-algebra structure having trivial operadic maps}).$$

There exists a left adjoint to sqz, the *cotangent fibre* functor,

$$\mathrm{cot} : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_k) \to \mathrm{Mod}_k.$$

We can take $\mathcal{O}$ to be the nonunital $\mathbf{E}_\infty$-operad to recover $\mathbf{E}_\infty$-k-algebras. Simplicial commutative k-algebras are not algebras over an ∞-operad, but we can still recover a similar construction.

Proposition. *There is a unique monad called the* partition Lie algebra monad $T^{\pi}_{\mathrm{Lie}_k}$ *on* Mod_k *which preserves sifted colimits and* $T^{\pi}_{\mathrm{Lie}_k}(V) = \mathrm{cot}(\mathrm{sqz}(V^\vee))^\vee$ *for all* V *fd.*

This is very abstract, so let us see some properties of this monad. First, we define the following.

Definition. The *n-th partition complex* $|\Pi_n|$ (for $n \geqslant 1$) is the genuine Σ_n-space constructed as the geometric realization of the simplicial Σ_n-set

$$\Pi_n := \mathrm{N}_\bullet(P),$$

the nerve of the poset P of partitions of $\{1, \dots, n\}$ minus the partitions

$$\boxed{1} \cdots \boxed{n} \quad \text{and} \quad \boxed{1 \cdots n}.$$

Now given a truncated $V \in \mathrm{Mod}_k^{\leqslant N}$ (in the spectral case), we have

$$T^{\pi}_{\mathrm{Lie}_k}(V) \simeq \bigoplus\nolimits_n \left(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond; k) \otimes V^{\otimes n}\right)^{h\Sigma_n},$$

where $\widetilde{C}^\bullet(X; k)$ are the k-valued cosimplices on X, $X^\diamond := S^0 * X$ is the unreduced suspension, and $(-)^{h\Sigma_n}$ takes homotopy fixed points. In the non-spectral (i.e., simplicial commutative k-algabras), for $V \simeq \mathrm{Tot}(V^\bullet) \in \mathrm{Mod}_k^{\leqslant 0}$ (i.e., V represented by a cosimplicial k-vector space $V^\bullet$), we have

$$T^{\pi}_{\mathrm{Lie}_k}(V) \simeq \bigoplus\nolimits_n \mathrm{Tot}\left(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond; k) \otimes (V^\bullet)^{\otimes n}\right)^{\Sigma_n},$$

where $(-)^{\Sigma_n}$ takes strict fixed points.

We will no longer reference this construction explicitly unless needed, and instead use a more general approach that will realize $T^{\pi}_{\mathrm{Lie}_k}$ as a certain monad $T^\vee$.

1. Main part

In characteristic 0, the main part of the proof relied on proving that the Koszul duality functor

$$\mathbb{D} : (\mathrm{AugCAlg}_k^{\mathrm{dg}})^{\mathrm{op}} \to \mathrm{Lie}_k$$

restricts to an equivalence on a subcollection of $\mathrm{AugCAlg}_k^{\mathrm{dg}}$, and takes pullbacks to pushouts. Let us outline this more precisely. We identify dg-k-algebras with $\mathbf{E}_\infty$-k-algebras, and denote by $\mathrm{AugCAlg}_k$ the ∞-category of augmented connective $\mathbf{E}_\infty$-k-algebras (resp. augmented simplicial commutative k-algebras).

Definition. An $A \in \mathrm{AugCAlg}_k$ is called *complete local Noetherian* if

- $\pi_0 A$ is Noetherian, and complete with respect to the augmentation ideal;
- $\pi_i A$ is a finitely-generated $\pi_0 A$-module for each i.

The ∞-category of complete local Noetherian $\mathbf{E}_\infty$-k-algebras (resp. simplicial commutative k-algebras) will be denoted cNCAlg_k.

Theorem (Lurie-Pridham)**.** *For k of characteristic 0, the functor $\mathbb{D}$ restricts to an equivalence*

$$\mathbb{D} : (\mathrm{cNCAlg}_k)^{\mathrm{op}} \simeq \mathrm{Lie}_k^{<0,\mathrm{fd}},$$

where $\mathrm{Lie}_k^{<0,\mathrm{fd}}$ is the ∞-category of dg Lie k-algebras $\mathfrak{g}_\bullet$ with $\mathfrak{g}_n = 0$ for $n \geqslant 0$ and $\pi_i \mathfrak{g}_\bullet$ finite-dimensional for all i. Moreover, if

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

is a pullback in cNCAlg_k such that $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjections, then

$$\begin{tikzcd} \mathbb{D}(A_{01}) \arrow[r] \arrow[d] & \mathbb{D}(A_0) \arrow[d] \\ \mathbb{D}(A_1) \arrow[r] & \mathbb{D}(A) \end{tikzcd}$$

is a pushout in $\mathrm{Lie}_k^{<0,\mathrm{fd}}$.

Now we wish to transport this result to the case where k has characteristic $p > 0$. To do this, Brantner and Mathew use Lurie's ∞-categorical version of the Barr-Beck comonadicity theorem. We will need to set up a lot of technical definitions to get going. We begin with the definition of a monadic adjunction.

Definition. Let $\mathcal{C}$ be an ∞-category, and let T be an algebra object of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, i.e., a monad on $\mathcal{C}$. Associated to any such T is the ∞-category $\mathrm{Alg}_T(\mathcal{C})$ of T-algebras in $\mathcal{C}$. If an adjunction arises as the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} = \mathrm{Alg}_T(\mathcal{C}), \arrow[l, bend left, "G"] \end{tikzcd}$$

we say that $(F \dashv G)$ is a *monadic adjunction*.

The "Barr-Beck-Lurie theorem" gives necessary and sufficient conditions for an adjunction of ∞-categories to be monadic, essentially giving conditions for "promoting" $G \circ F$ to a monad on $\mathcal{C}$.

Theorem (Barr-Beck-Lurie, [4])**.** *An adjunction $(F \dashv G)$ of ∞-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

is monadic if and only if both

- *$f \in \mathrm{Mor}(\mathcal{D})$ an equivalence $\iff$ $G(f) \in \mathrm{Mor}(\mathcal{C})$ is an equivalence;*
- *Any G-split simplicial object $X_\bullet$ of $\mathcal{D}$ admits a a colimit in $\mathcal{D}$ which is preserved by G.*

Dually, we have the notion of comonadicity, for which we have the theorem

Theorem (Barr-Beck-Lurie, [4])**.** *An adjunction $(F \dashv G)$ of ∞-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

is comonadic if and only if both

- $f \in \mathrm{Mor}(\mathcal{C})$ *an equivalence* $\iff$ $F(f) \in \mathrm{Mor}(\mathcal{D})$ *is an equivalence;*
- *Any F-split cosimplicial object $X^\bullet$ of $\mathcal{C}$ admits a a limit in $\mathcal{C}$ which is preserved by F.*

For the definition of a G-split (resp. F-split) object, see [4, Section 4.7.2].

Now we want to formalize the notion of the cotangent fibre above, and then use that cotangent functor to prove the results we need for the Koszul duality functor. In the characteristic 0 case, we could take any augmented dg-k-algebra A and equip $\mathrm{cot}(A)^\vee[-1]$ with a dg Lie algebra structure, which gave us the Koszul duality functor $\mathbb{D}(A)$"="$\mathrm{cot}(A)^\vee[-1]$ + dg structure. To do this in a more general setting, we need to assume that our monad T admits an augmentation to the identity monad. We can use the language of adjunctions to talk about the properties of monads we want, and we begin with the following definition. Let $\mathrm{Mod}_k^{\geqslant 0}$ be the ∞-subcategory of connective objects of Mod_k.

Definition. A pair of adjunctions (free $\dashv$ forget) and (cot $\dashv$ sqz),

$$\begin{tikzcd} \mathrm{Mod}_k^{\geqslant 0} \arrow[r, bend left, "\text{free}"] & \mathcal{C} \arrow[l, bend left, "\text{forget}"] \arrow[r, bend left, "\text{cot}"] & \mathrm{Mod}_k^{\geqslant 0}, \arrow[l, bend left, "\text{sqz}"] \end{tikzcd}$$

is called an *augmented monadic adjunction* if

- forget $\circ$ sqz $\circ$ cot $\circ$ free $=$ identity;
- $\mathcal{C}$ pointed (has a zero object) and presentable (has small colimits and has a small subcollection of objects which generate all objects under colimits);
- (free $\dashv$ forget) is a monadic adjuction, with forget preserving sifted colimits.

The idea is that we can identify $\mathcal{C}$ in the above definition with Alg_T for $T = \text{forget} \circ \text{free}$, which has a natural augmentation to the identity as given by the first bullet point in the definition. In our situation, we want to build $\mathcal{C}$ as the ∞-category $\mathrm{AugCAlg}_k$. To do this, we need to choose the right adjunction (free $\dashv$ forget). We can do this by

- free(A) is the free $\mathbf{E}_\infty$-k-algebra/simpicial commutative k-algebra on A;
- forget(A) is the augmentation ideal of A.

To complete a proof of an analog of the Lurie-Pridham theorem in arbitrary characteristic, we will need to find a full ∞-subcategory $\mathcal{C}_{\mathrm{afp}}$ of $\mathcal{C}$ in the definition of an augmented monadic adjunction above which has the following desired properties.

- The adjunction (cot $\dashv$ sqz) restricts to a comonadic adjuction

$$\begin{tikzcd} \mathcal{C}_{\mathrm{afp}} \arrow[r, bend left, "\text{cot}"] & \mathrm{Mod}_k^{\geqslant 0,\mathrm{fd}}, \arrow[l, bend left, "\text{sqz}"] \end{tikzcd} \tag{$*$}$$

  where $\mathrm{Mod}_k^{\geqslant 0,\mathrm{fd}}$ is the ∞-subcategory of $\mathrm{Mod}_k^{\geqslant 0}$ consisting of $M \in \mathrm{Mod}_k^{\geqslant 0}$ with $\pi_i M$ finite-dimensional for all i.
- The monad on $\mathrm{Mod}_k^{\leqslant 0,\mathrm{fd}}$ (coconnective objects of Mod_k with finite-dimensional homotopy groups) defined by $M \mapsto \mathrm{cot}(\mathrm{sqz}(M^\vee))^\vee$ extends uniquely to a monad $T^\vee$ on Mod_k which preserves sifted colimits.
- If $\mathbb{D} : \mathcal{C}_{\mathrm{afp}}^{\mathrm{op}} \to \mathrm{Alg}_{T^\vee}$ is defined by $A \mapsto \mathrm{cot}(A)^\vee$, then for the cospan

$$\begin{tikzcd} & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

in $\mathcal{C}_{\text{afp}}$ such that the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective, the pullback

$$\begin{tikzcd} A_0 \times_{A_{01}} A_1 \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

exists in $\mathcal{C}_{\text{afp}}$ and

$$\begin{tikzcd} \mathbb{D}(A_{01}) \arrow[r] \arrow[d] & \mathbb{D}(A_0) \arrow[d] \\ \mathbb{D}(A_1) \arrow[r] & \mathbb{D}(A_0 \times_{A_{01}} A_1) \end{tikzcd}$$

is a pushout in $\mathrm{Alg}_{T^\vee}$.

The idea that Brantner and Mathew want to use bar-cobar duality to give equivalences

$$\mathcal{C}_{\text{afp}} \simeq \mathrm{coAlg}_T(\mathrm{Mod}_k^{\geqslant 0,\text{fd}}) \simeq \mathrm{Alg}_{T^\vee}(\mathrm{Mod}_k^{\leqslant 0,\text{fd}})^{\text{op}},$$

where $T = \mathrm{sqz} \circ \mathrm{cot}$ is the comonad on $\mathrm{Mod}_k^{\geqslant 0,\text{fd}}$ induced by the adjunction $(*)$.

Now we will describe a filtered version of an augmented monadic adjunction, which will be the right method to construct $\mathcal{C}_{\text{afp}}$. We first define categories of filtered and graded objects.

Definition. For a (presentable, stable) ∞-category $\mathcal{C}$, we define

$$\mathrm{Fil}(\mathcal{C}) := \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geqslant 1})^{\text{op}}, \mathcal{C})$$

(for $\mathbf{Z}_{\geqslant 1}$ considered as a poset) to be the ∞-category of *filtered objects of* $\mathcal{C}$, i.e., sequences

$$F^*X = \{\cdots \to F^iX \to F^{i-1}X \to \cdots \to F^1X\}$$

in $\mathcal{C}$. The functor $(-)^1 : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$ defined by $F^*X \mapsto F^1X$ has a left adjoint $(-)_1 : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$, with

$$(X)_1 = \{\cdots \to 0 \to \cdots \to 0 \to X\}.$$

We define

$$\mathrm{Gr}(\mathcal{C}) := \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geqslant 1}^{\text{disc}}), \mathcal{C}),$$

where $\mathbf{Z}_{\geqslant 1}^{\text{disc}}$ is $\mathbf{Z}_{\geqslant 1}$ considered as a discrete category (only identity morphisms). We call this the ∞-category of *graded objects of* $\mathcal{C}$, i.e., objects $X_* = \{X_i\}_{i \geqslant 1}$. We have a functor $\mathrm{Gr}(\mathcal{C}) \to \mathcal{C}$ given by

$$X_* \mapsto \textstyle\bigoplus_{i \geqslant 1} X_i.$$

Moreover, we have a functor $\mathrm{Gr} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$ given by

$$F^*X \mapsto \mathrm{Gr}(F^*X)_* := \{F^iX/F^{i+1}X\}_{i \geqslant 1}.$$

Definition. A *filtered augmented monadic adjunction* is the following data.

- A diagram of left adjoints

$$\begin{tikzcd} \mathrm{Mod}_k^{\geqslant 0} \arrow[r, "\text{free}"] \arrow[d, "(-)_1"'] & \mathcal{C} \arrow[r, "\text{cot}"] \arrow[d, "\text{adic}"'] & \mathrm{Mod}_k^{\geqslant 0} \arrow[d, "(-)_1"] \\ \mathrm{FilMod}_k^{\geqslant 0} \arrow[r, "\text{free}"] \arrow[d, "\mathrm{Gr}"'] & \mathcal{C}^{\mathrm{Fil}} \arrow[r, "\text{cot}"] \arrow[d] & \mathrm{FilMod}_k^{\geqslant 0} \arrow[d, "\mathrm{Gr}"] \\ \mathrm{GrMod}_k^{\geqslant 0} \arrow[r, "\text{free}"] & \mathcal{C}^{\mathrm{Gr}} \arrow[r, "\text{cot}"] & \mathrm{GrMod}_k^{\geqslant 0}, \end{tikzcd}$$

  where

$$\mathrm{FilMod}_k^{\geqslant 0} := \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geqslant 1})^{\text{op}}, \mathrm{Mod}_k^{\geqslant 0}) \quad \text{and} \quad \mathrm{GrMod}_k^{\geqslant 0} := \mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geqslant 1}^{\text{disc}}), \mathrm{Mod}_k^{\geqslant 0})$$

as above and adic is an abstraction of the idea of associating to an augmented commutative ring with augmentation ideal $\mathfrak{m}$ to its $\mathfrak{m}$-adic filtration (remember we want to think of $\mathcal{C}_{\mathrm{afp}}$ as cNCAlg_k or similiar), and $\mathcal{C}^{\mathrm{Fil}} \to \mathcal{C}^{\mathrm{Gr}}$ is like taking the associated grading, similiar to the functor Gr.

- The horizontal compositions are the identity, and the vertical adjuctions are augmented monadic in the sense of the previous defintion (the (free $\dashv$ cot) at each level are monadic and the right adjoint preserves sifted colimits).
- The right adjoints $(-)^1$ to both $(-)_1$ and adic give commutative squares on the top row.
- The right adjoints forget and sqz of free and forget give commutative squares on the bottom row.
- The functor $F = \mathrm{forget} \circ \mathrm{free}$ on $\mathrm{GrMod}_k^{\geqslant 0}$ is admissible, i.e., both
  - if $X_* \in \mathrm{Gr}^{\mathrm{pfd}}\mathrm{Mod}_k^{\geqslant 0}$ ($X_i \in \mathrm{Mod}_k^{\geqslant 0,\mathrm{fd}}$ for all $i \geqslant 1$), then $F(X_*) \in \mathrm{Gr}^{\mathrm{pfd}}\mathrm{Mod}_k^{\geqslant 0}$ and
  - if $X^\bullet$ is a cosimplicial object of $\mathrm{Gr}^{\mathrm{pfd}}\mathrm{Mod}_k^{\geqslant 0}$ such that the totalization (computed in GrMod_k) $\mathrm{Tot}(X^\bullet) \in \mathrm{Gr}^{\mathrm{pfd}}\mathrm{Mod}_k^{\geqslant 0}$, then $F(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot}(F(X^\bullet))$ is an equivalence.
- For $A \in \mathcal{C}^{\mathrm{Gr}}$, there is a (functorial) tower $\{A^{(i)}\}_{i\geqslant 1}$ in $\mathcal{C}^{\mathrm{Gr}}$ and compatible maps $A \to A^{(i)}$ such that
  - there is a natural isomorphism $A^{(1)} \simeq \mathrm{sqz}(\mathrm{cot}(A))$;
  - there is a natural isomorphism $\mathrm{forget}(A^{(i)})/\mathrm{forget}(A^{(i+1)}) \simeq G_i(\mathrm{cot}(A))$ for an admissible functor G_i on $\mathrm{GrMod}_k^{\geqslant 0}$, and G_i is i-increasing for $i > 1$ (see [1, Definition 2.12]);
  - there is an equivalence on graded components of (internal) degree $\leqslant i$ induced by the map $\mathrm{forget}(A) \to \mathrm{forget}(A^{(i)})$ in $\mathrm{GrMod}_k^{\geqslant 0}$.
- Let $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$ be the ∞-subcategory of $\mathcal{C}^{\mathrm{Gr}}$ of the $A \in \mathcal{C}^{\mathrm{Gr}}$ with $\mathrm{cot}(A) \in \mathrm{Gr}^{\mathrm{fd}}\mathrm{Mod}_k^{\geqslant 0}$ ($X_* \in \mathrm{GrMod}_k^{\geqslant 0}$ such that $\bigoplus_{i\geqslant 1} X_i \in \mathrm{Mod}_k^{\geqslant 0,\mathrm{fd}}$ for all $i \geqslant 1$). Then if

$$\begin{tikzcd} & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

  is a cospan in $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$ with the maps $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ surjective, then the pullback $A_0 \times_{A_{01}} A_1 \in \mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$.
- For any $M \in \mathrm{Gr}^{\mathrm{fd}}\mathrm{Mod}_k^{\geqslant 0}$, we have $\mathrm{sqz}(M) \in \mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$.
- Let $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ be the full ∞-subcategory of $\mathcal{C}^{\mathrm{Fil}}$ of the complete A such that $\mathrm{Gr}(A)_* \in \mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$. Then if $A \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$, we have that $\mathrm{cot}(A)$ and $\mathrm{adic}((A)^1)$ are complete.

We can think of $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$ and $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ as in the following example, when $\mathcal{C} = \mathrm{AugCAlg}_k^{\Delta}$. Then with $\mathcal{C}^{\mathrm{Fil}}$ the ∞-category of augmented filtered simplicial commutative k-algebras and $\mathcal{C}^{\mathrm{Gr}}$ the ∞-category of augmented graded simplicial commutative k-algebras, we have that

- $A \in \mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$ means $A \in \mathcal{C}^{\mathrm{Gr}}$ such that $\pi_0 A$ is Noetherian and $\pi_i A$ is a finitely-generated $\pi_0 A$-module for all $i \geqslant 0$;
- $A \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ means $A \in \mathcal{C}^{\mathrm{Fil}}$ such that $\mathrm{Gr}(A)_* \in \mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$.

We know state the main result of Brantner and Mathew.

Theorem (Brantner-Mathew)**.** *Let us suppose we are given the data of an augmented monadic adjunction as in the above definition, and let $\mathcal{C}_{\mathrm{afp}}$ be the ∞-subcategory of $\mathcal{C}$ consisting of the $A \in \mathcal{C}$ such that* $\mathrm{adic}(A) \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$. *Then*

(1) *the adjunction* (cot $\dashv$ sqz) *restricts to a comonadic adjunction*

$$\begin{tikzcd} \mathcal{C}_{\mathrm{afp}} \arrow[r, bend left, "\mathrm{cot}"] & \mathrm{Mod}_k^{\geqslant 0,\mathrm{fd}} \arrow[l, bend left, "\mathrm{sqz}"] \end{tikzcd};$$

(2) *the monad* $M \mapsto \mathrm{cot}(\mathrm{sqz}(M^\vee))^\vee$ *on* $\mathrm{Mod}_k^{\leqslant 0,\mathrm{fd}}$ *extends uniquely to a monad* $T^\vee$ *on* Mod_k *which preserves sifted colimits;*

(3) *the fully-faithfull embedding* $\mathbb{D} : \mathcal{C}_{\mathrm{afp}}^{\mathrm{op}} \to \mathrm{Alg}_{T^\vee}$ *defined by* $A \mapsto \mathrm{cot}(A)^\vee$ *satisfies the following. If* $A_0 \to A_{01} \leftarrow A_1$ *are morphisms in* $\mathcal{C}_{\mathrm{afp}}$ *such that the maps* $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ *are surjective, then the pullback* $A_0 \times_{A_{01}} A_1 \in \mathcal{C}_{\mathrm{afp}}$ *and*

$$\begin{tikzcd} \mathbb{D}(A_{01}) \arrow[r] \arrow[d] & \mathbb{D}(A_0) \arrow[d] \\ \mathbb{D}(A_1) \arrow[r] & \mathbb{D}(A_0 \times_{A_{01}} A_1) \end{tikzcd}$$

is a pushout in $\mathrm{Alg}_{T^\vee}$.

As a result of this theorem, we get the following. Let $\mathcal{C}_{\mathrm{Art}}$ be the smallest full ∞-subcategory of $\mathcal{C}_{\mathrm{afp}}$ such that the terminal object of $\mathcal{C}_{\mathrm{afp}}$ is in $\mathcal{C}_{\mathrm{Art}}$ and for any $A \in \mathcal{C}_{\mathrm{Art}}$ equipped with a morphism $A \to \mathrm{sqz}(k[n])$ for some $n > 0$, the pullback $A \times_{\mathrm{sqz}(k[n])} * \in \mathcal{C}_{\mathrm{Art}}$ (analogous to Artinian algebras). We define the ∞-category $\mathrm{Moduli}_{\mathcal{C}}$ of $\mathcal{C}$-*based formal moduli problems* as the ∞-subcategory of $\mathrm{Fun}(\mathcal{C}_{\mathrm{Art}}, \mathcal{S})$ consisting of $X : \mathcal{C}_{\mathrm{Art}} \to \mathcal{S}$ such that $X(*)$ is contractible, and, for $A \in \mathcal{C}_{\mathrm{Art}}$ equipped with a map $A \to \mathrm{sqz}(k[n])$ for some $n > 0$, we have

$$X(A \times_{\mathrm{sqz}(k[n])} *) \simeq X(A) \times_{X(\mathrm{sqz}(k[n]))} X(*).$$

Theorem (Brantner-Mathew)**.** *Again given the data of a filtered augmented monadic adjunction as in the previous theorem and above definition, we have an equivalence of* ∞*-categories*

$$\mathrm{Alg}_{T^\vee} \simeq \mathrm{Moduli}_{\mathcal{C}},$$

given by $\mathfrak{g} \mapsto (R \mapsto \mathrm{Map}_{\mathrm{Alg}_{T^\vee}}(\mathbb{D}(R), \mathfrak{g}))$*; moreover this equivalence is such that the composition*

$$\mathrm{Moduli}_{\mathcal{C}} \to \mathrm{Alg}_{T^\vee} \to \mathrm{Mod}_k$$

is equivalent to the tangent fibre functor $X \mapsto \mathbb{T}_X$.

Taking $\mathcal{C} = \mathrm{CAlg}_k^{\Delta}$ or $\mathcal{C} = \mathrm{CAlg}_k^{\mathbf{E}_\infty}$ gives us the desired result for fields k of zero and positive characteristic, both on the derived and spectral sides.

We will review only the proof of the claim (3) of the first theorem in this section. Hence we will take the existence of the monad $T^\vee$ for granted, and that the $(\mathrm{cot} \dashv \mathrm{sqz})$ adjunction restricts to a comonadic adjunction. The proof that this adjunctions restricts to a comonadic adjunction on a certain class of objects essentially uses a Bar construction argument.

Now to prove (3). Suppose

$$\begin{tikzcd} & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

are morphisms in $\mathcal{C}_{\mathrm{afp}}$ such that the maps

$$\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$$

are surjective. We have that the pullback $A_0 \times_{A_{01}} A_1 \in \mathcal{C}_{\mathrm{afp}}$, as shown in the following proposition.

Proposition (see [1], Remark 4.46)**.** *The pullback* $A_0 \times_{A_{01}} A_1 \in \mathcal{C}_{\mathrm{afp}}$.

Proof. We have $\mathrm{adic}(A_0), \mathrm{adic}(A_1), \mathrm{adic}(A_{01}) \in \mathcal{C}_{\mathrm{afp}}^{\mathrm{Fil}}$, and the maps $\pi_0\mathrm{adic}(A_0) \to \pi_0\mathrm{adic}(A_{01}) \leftarrow \pi_0\mathrm{adic}(A_1)$ are surjective. By the assumptions in the definition of a filtered augmented monadic adjunction, namely that pullbacks exist in $\mathcal{C}_{\mathrm{afp}}^{\mathrm{Gr}}$ if the cospan considered induces surjections on π_0's, we have that

$$\mathrm{Gr}(\mathrm{adic}(A_0)) \times_{\mathrm{Gr}(\mathrm{adic}(A_{01}))} \mathrm{Gr}(\mathrm{adic}(A_1)) \in \mathcal{C}_{\mathrm{afp}}^{\mathrm{Gr}};$$

now we have a natural map

$$\mathrm{Gr}(\mathrm{adic}(A_0) \times_{\mathrm{adic}(A_{01})} \mathrm{adic}(A_1)) \to \mathrm{Gr}(\mathrm{adic}(A_0)) \times_{\mathrm{Gr}(\mathrm{adic}(A_{01}))} \mathrm{Gr}(\mathrm{adic}(A_1)),$$

which induces an equivalence after applying forget (right adjoints give commutative squares). We then have that

$$\mathrm{adic}(A_0) \times_{\mathrm{adic}(A_{01})} \mathrm{adic}(A_1) \in \mathcal{C}_{\mathrm{afp}}^{\mathrm{Fil}},$$

as it is complete. Now if $A \in \mathcal{C}_{\mathrm{afp}}^{\mathrm{Fil}}$, then $(A)^1 \in \mathcal{C}_{\mathrm{afp}}$ since $\cot((A)^1) \simeq (\cot(A))^1 \in \mathrm{Mod}_k^{\geqslant 0,\mathrm{fd}}$ (as $\cot(A)$ is complete by the assumptions in the definition and $\mathrm{Gr}(\cot(A)) \in \mathrm{Gr}^{\mathrm{fd}}\mathrm{Mod}_k^{\geqslant 0}$ by definition). So applying $(-)^1$ gives $A_0 \times_{A_{01}} A_1 \in \mathcal{C}_{\mathrm{afp}}$, and we are done. □

Our claim is that the natural map

$$\eta : \mathbb{D}(A_0) \sqcup_{\mathbb{D}(A_{01})} \mathbb{D}(A_1) \to \mathbb{D}(A_0 \times_{A_{01}} A_1)$$

is an equivalence.

First, we show that in the case where A_0, A_1, and A_{01} are square-zero extensions, that η is an equivalence. Suppose $V_0, V_1, V_{01} \in \mathrm{Mod}_k^{\geqslant 0,\mathrm{fd}}$, and suppose we have maps $V_0 \to V_{01} \leftarrow V_1$ inducing surjections $\pi_0 V_0 \to \pi_0 V_{01} \leftarrow \pi_0 V_1$. Then if we let $A_0 = \mathrm{sqz}(V_0)$, $A_1 = \mathrm{sqz}(V_1)$, and $A_{01} = \mathrm{sqz}(V_{01})$, we have that η is an equivalence. This is since the left hand side is a pushout of the free $T^\vee$-algebras $V_0^\vee$, $V_1^\vee$, and $V_{01}^\vee$, and the right hand side is a the free $T^\vee$-algebra on $(V_0 \times_{V_{01}} V_1)^\vee$, and free commutes with pullbacks and duals (and in general all sifted colimits).

Now to prove the more general case, Brantner and Mathew use cobar resolutions of A_0, A_1, and A_{01} to reduce to the case above where everything is a square-zero extension. We have a canonical cobar resolution $X_A^\bullet$ of any $A \in \mathcal{C}_{\mathrm{afp}}$ given by

$$X_A^0 = \mathrm{sqz}(\cot(A)),$$
$$X_A^1 = \mathrm{sqz}(\cot(\mathrm{sqz}(\cot(A)))),$$

etc. Now let $X_{A_0}^\bullet$, $X_{A_1}^\bullet$, and $X_{A_{01}}^\bullet$ be the associated canonical cobar resolutions of A_0, A_1, and A_{01} respectively; these are cosimplicial objects of $\mathcal{C}_{\mathrm{afp}}$. We have natural maps

$$\begin{tikzcd} & X_{A_0}^\bullet \arrow[d] \\ X_{A_1}^\bullet \arrow[r] & X_{A_{01}}^\bullet, \end{tikzcd}$$

which in turn induce surjections

$$\pi_0 X_{A_0}^i \to \pi_0 X_{A_{01}}^i \leftarrow \pi_0 X_{A_1}^i$$

for each i. Hence we can form the cosimplicial object $X^\bullet$ of $\mathcal{C}_{\mathrm{afp}}$ by the pullback of cosimplicial objects,

$$X^\bullet := X_{A_0}^\bullet \times_{X_{A_{01}}^\bullet} X_{A_1}^\bullet.$$

By the comonadicity of the adjunction in (1), for $A \in \mathcal{C}_{\mathrm{afp}}$,

$$A \simeq \mathrm{Tot}(X_A^\bullet).$$

So we have $\mathrm{Tot}(X^\bullet) \simeq A_0 \times_{A_{01}} A_1$, and hence we have the equivalence

$$\mathbb{D}(X_{A_0}^i) \sqcup_{\mathbb{D}(X_{A_{01}}^i)} \mathbb{D}(X_{A_1}^i) \simeq \mathbb{D}(X^i)$$

since each component is a square-zero extension and we already proved that case above.

To complete the proof, i.e., to show that η is an equivalence, we need to verify that for any $A \in \mathcal{C}_{\mathrm{afp}}$,

$$|\mathbb{D}(X_A^\bullet)| \simeq \mathbb{D}(A), \tag{$**$}$$

and in addition

$$|\mathbb{D}(X^\bullet)| \simeq \mathbb{D}(A_0 \times_{A_{01}} A_1). \tag{$***$}$$

As $T^\vee$ preserved sifted colimits, we can compute geometric realizations $|-|$ in Mod_k.

The claim $(**)$ follows from applying $(-)^\vee$ to the comonadic adjunction in (1). To show $(***)$, we begin by forming the cosimplicial object

$$\widetilde{X}^\bullet := \mathrm{adic}(X_{A_0}^\bullet) \times_{\mathrm{adic}(X_{A_{01}}^\bullet)} \mathrm{adic}(X_{A_1}^\bullet).$$

We know that $\widetilde{X}^\bullet$ is a lift of $X^\bullet$ to $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$, as shown in the proof of the above proposition. Next, we have the following lemmas.

Lemma. $\mathrm{Tot}(\mathrm{Gr}(\widetilde{X}^\bullet)) \in \mathrm{Gr}^{\mathrm{pfd}}\mathrm{Mod}_k^{\geqslant 0}$.

Proof. By construction, the cosimplicial objects $\cot(X^\bullet_{A_0})$, $\cot(X^\bullet_{A_1})$, and $\cot(X^\bullet_{A_{01}})$ are split in $\mathrm{Mod}_k^{\geqslant 0,\mathrm{fd}}$. As $\mathrm{Gr}(\mathrm{adic}(A)) \simeq \mathrm{free}([\cot(A)]_1)$ (see [1, Proposition 4.26]) for $A \in \mathcal{C}$, we have that $\mathrm{Gr}(\mathrm{adic}(X^\bullet_{A_0}))$, $\mathrm{Gr}(\mathrm{adic}(X^\bullet_{A_1})$, and $\mathrm{Gr}(\mathrm{adic}(X^\bullet_{A_{01}})$ have splittings as well, and hence admit totalizations in $\mathrm{Gr}^{\mathrm{pfd}}\mathrm{Mod}_k^{\geqslant 0}$. The maps between these objects are component-wise surjective on the π_0's, hence we conclude that $\mathrm{Gr}(\widetilde{X}^\bullet)$ also has a totalization in $\mathrm{Gr}^{\mathrm{pfd}}\mathrm{Mod}_k^{\geqslant 0}$. □

Lemma. $\mathrm{Tot}(\widetilde{X}^\bullet) \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$.

Proof. By Proposition 4.26 of [1], we have that the maps

$$\mathrm{adic}(A_0) \to \mathrm{Tot}(\mathrm{adic}(X^\bullet_{A_0}),$$

(resp. for A_1 and A_{01}), induce equivalences after applying Gr. Hence

$$\mathrm{Tot}(\widetilde{X}^\bullet) \simeq \mathrm{Tot}(\mathrm{adic}(X^\bullet_{A_0})) \times_{\mathrm{Tot}(\mathrm{adic}(X^\bullet_{A_{01}}))} \mathrm{Tot}(\mathrm{adic}(X^\bullet_{A_1})) \simeq \mathrm{adic}(A_0) \times_{\mathrm{adic}(A_{01})} \mathrm{adic}(A_1),$$

which is in $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ by the proposition proved before. □

We use Proposition 4.49 of [1], which states that under the condition that we have a lift $\widetilde{X}^\bullet$ of $X^\bullet$ with $\mathrm{Tot}(\widetilde{X}^\bullet) \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$, together with some other conditions (which are satisified in the present situation), the totalization $\mathrm{Tot}(X^\bullet)$ exists in $\mathcal{C}_{\mathrm{afp}}$, and we have an equivalence

$$\cot(\mathrm{Tot}(X^\bullet)) \simeq \mathrm{Tot}(\cot(X^\bullet)).$$

From this, we have the equivalences

$$\cot(A_0 \times_{A_{01}} A_1) \simeq \cot(\mathrm{Tot}(X^\bullet)) \simeq \mathrm{Tot}(\cot(X^\bullet)).$$

Since $\cot(X^\bullet) \in \mathrm{Mod}_k^{\geqslant 0,\mathrm{fd}}$, it is equivalent to the double dual of itself, hence applying duality we get that the natural map

$$|\mathbb{D}(X^\bullet)| \to \mathbb{D}(A_0 \times_{A_{01}} A_1)$$

is an equivalence. Therefore we have proved $(**)$ and $(***)$, so η is an equivalence.

Thank you for reading, and please forgive for sloppiness in some parts.

References

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