LaTeX Workshop (Basic)

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1 Exercise 1: Mathematical Environment

1.1 Question 1

If f is a continuous, odd function on $[-1, 1]$, then $\int_{-1}^{1} f(x)dx = 0$

1.2 Question 2

The sine rule states that given a triangle ABC with side lengths a, b and c, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

1.3 Question 3

Let A be a finite set of size m where $m \geq 1$ and let $a \in A$. Then $|A \backslash \{a\}| + 1 = m$.

2 Exercise 2: DisplayMath Environment

2.1 Question 1

Give an angle $\theta \in [0, 2\pi]$, we have that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

2.2 Question 2

Here is an example of a differential equation in terms of x and y,

$$x^2 y \frac{dy}{dx} + xy = 0, \ y = 1 \text{ when } x = 0.$$

2.3 Question 3

Give a function $f(x)$ whose domain $I \subset \mathbb{R}$ contains points arbitrarily close to a. We say that

$$\lim_{x \to a} f(x) = L$$

if $\forall \epsilon > 0, \exists \delta > 0$ such that

$$0 < |x - a| < L \implies |f(x) - L| < \epsilon$$

3 Exercise 3: Align Environment and Delimiters

3.1 Question 1

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors with the same initial point and whose terminal endpoints do not lie along a line, then:

$$\begin{aligned}
(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} - \mathbf{c}) &= \mathbf{a} \times (\mathbf{a} - \mathbf{c}) + (-\mathbf{b}) \times (\mathbf{a} - \mathbf{c}) \\
&= \mathbf{a} \times \mathbf{a} + \mathbf{a} \times (-\mathbf{c}) + (-\mathbf{b}) \times \mathbf{a} + (-\mathbf{b}) \times (-\mathbf{c}) \\
&= -(\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{c}) \\
&= (\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})
\end{aligned}$$

3.2 Question 2

Using L'H Rule, the limit may be evaluated as follows:

$$\begin{aligned}
\lim_{x \to 0} (\frac{1}{\sin x} - \frac{1}{x}) &= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \\
&= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \\
&= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} \\
&= \frac{\sin 0}{2 \cos 0 - 0 \sin 0} \\
&= 0
\end{aligned}$$

3.3 Question 3

The triangle inequality for the euclidean norm of $\mathbb{R}^n$ shows that

$$|| \sum_{i=1}^{m} a_i || \leq \sum_{i=1}^{m} ||a_i||$$

Use this to prove that

$$\left(\int_X \left| \int_Y f(x,y) dy \right|^2 dx \right)^{\frac{1}{2}} \leq \int_Y \left(\int_X |f(x,y)|^2 dx \right)^{\frac{1}{2}} dy$$

4 Exercise 4: Array Environment

4.1 Question 1

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -3 & 2 & -2a \\ 0 & 0 & 6-2a & 3b+2a \end{array}\right)$$

4.2 Question 2

$$f(x) := \begin{cases} 1 & \text{if } x \in 0, 1; \\ \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ with } p, q \in \mathbb{Z}_{>0} \text{ relatively prime;} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

4.3 Question 3

$$\begin{cases} a + 2b + c = 0 \\ b + c = 1 \\ a + c = 1 \\ a + b + c = 1 \end{cases} \implies \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) \implies \mathbf{Ax{=}b}$$

5 Post Workshop Exercise: Mathematical Notations and Environment

5.1 Question 1

Let $f(x, y)$ be a differentiable, two-variable function and let (a, b) be a point in $\mathbb{R}^2$. If the gradient of the function at (a, b), $\nabla f(a, b)$ is not equal to 0, then $\nabla f(a, b)$ is orthogonal to the level curve of f that contains (a, b).

5.2 Question 2

Let X be compact, Hausdroff topological space that has more than one point. Show that there is a non-constant, continuous function $f : X \to \mathbb{R}^+$.

5.3 Question 3

A real value function f is continuous at x if:

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall y \in \mathbb{R}, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

5.4 Question 4

The **set difference** between two sets S and T is written as $S \backslash T$, and means the set that consists of the elements of S which are not the elements of T, that is:

$$x \in S \backslash T \implies x \in S \wedge x \notin T.$$

5.5 Question 5

The **Cauchy-Schwartz Inequality** states that

$$(a_1b_1 + a_2b_2 + ... + a_nb_n) \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2)$$

for any real numbers $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$. Use this to prove the inequality:

$$(a_1 + ... + a_n)^2 \leq n(a_1^2 + ... + a_n^2).$$

5.6 Question 6

Given that $1 + z + z^2 + ... + z^n = \frac{1-z^{n+1}}{1-z}$ where $z \neq 1$, prove the Lagrange's trigonometric identity:

$$\sum_{n=1}^{N} \cos(n\theta) = -\frac{1}{2} + \frac{\sin((N+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}$$

5.7 Question 7

The P.D.F. of normal distribution is given by

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and the C.D.F. of normal distribution is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\mu^2}{2}} du.$$

5.8 Question 8

Let $r \in \mathbb{R}_{>0}$ with $r \neq 1$. Let $\theta \in (0, \pi)$. Determine the value of the integral

$$\frac{1}{2\pi i} \int_{C(0,r)} \frac{1-\zeta^2}{1-2\zeta\cos\theta + \zeta^2} d\zeta$$

in terms of r and θ.

5.9 Question 9

The real number ln2 can be expressed as the following infinite sum,

$$\ln 2 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \frac{1}{1\times 2} + \frac{1}{3\times 4} + \frac{1}{5\times 6} + \dots.$$

5.10 Question 10

This is an example of a very complicated triple integral,

$$\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \int_{\sqrt{x^2+y^2-2x+1}}^{\sqrt{2-y^2-x^2+2x}} z^2 z dz dy dx.$$

6 Post Workshop Exercise: Alignment and Delimiters

6.1 Question 1

Let $m, n \in \mathbb{N}$ such that A is an $m \times n$ matrix and B is an $n \times m$ matrix. Then,

$$\begin{aligned}
\mathrm{Tr}(AB) &= \sum_{i=1}^{n} (AB)_{i,i} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{i,k} b_{k,i} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{k,i} a_{i,k} \\
&= \sum_{k=1}^{n} (BA)_{k,k} \\
&= \mathrm{Tr}(BA)
\end{aligned}$$

6.2 Question 2

$$\begin{aligned}
&\lambda \text{ is an eigenvalue of } A \\
&\Leftrightarrow Au = \lambda u \text{ for some nonzero column vector } u \text{ in } \mathbb{R}^n \\
&\Leftrightarrow \lambda u - Au = 0 \text{ for some nonzero column vector } u \text{ in } \mathbb{R}^n \\
&\Leftrightarrow (\lambda I - A)u = 0 \text{ for some nonzero column vector } u \text{ in } \mathbb{R}^n \\
&\Leftrightarrow \text{ the linear system } (\lambda I - A)x = 0 \text{ has non-trivial solutions} \\
&\Leftrightarrow \det(\lambda I - A) = 0.
\end{aligned}$$

6.3 Question 3

Let $\Omega \in \mathbb{R}$ and let f and g be two real-valued measurable functions on $\mathbb{R}$. Show that

$$\int_{\Omega} |f(x)g(x)| dx \leq \left(\int_{\Omega} |f(x)|^p dx \right)^{1/p} \left(\int_{\Omega} |g(x)|^q dx \right)^{1/q}$$

for any $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$.

6.4 Question 4

Using **De Moivre's Formula**:

$$\sin x = \frac{(\cos \frac{x}{n} + i \sin \frac{x}{n})^n - (\cos \frac{x}{n} - i \sin \frac{x}{n})^n}{2i},$$

we verify that

$$\begin{aligned}\sin x &= x\prod_{n-1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)\\ &= x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\dots\end{aligned}$$

6.5 Question 5

$$\begin{aligned}\int\frac{dx}{x^2-a^2} &= -\frac{1}{a}\text{arccoth}\frac{x}{a}+C\\ &= -\frac{1}{a}\left(\frac{1}{2}\ln\left(\frac{x+a}{x-a}\right)\right)+C\\ &= -\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right)+C\text{by simplifying}\\ &= \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+C\end{aligned}$$

6.6 Question 6

Let $f(x)=\frac{1}{x}$. For every $a\in\mathbb{R}\backslash\{0\}$,

$$\begin{aligned}f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\ &= \lim_{x\to a}\frac{\frac{1}{x}-\frac{1}{a}}{x-a}\\ &= \lim_{x\to a}\frac{\frac{a-x}{ax}}{x-a}\\ &= \lim_{x\to a}\frac{-1}{ax}\\ &= -\frac{1}{a^2}.\end{aligned}$$

6.7 Question 7

By the **Products of Sins of Pi**, we have:

$$\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)=\frac{n}{2^{n-1}}$$

Therefore, we have

$$\begin{aligned}\ln\left(\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)\right) &= \sum_{k=1}^{n-1}\ln\left(\sin\left(\frac{k\pi}{n}\right)\right)\\ &= \ln\left(\frac{n}{2^{n-1}}\right)\\ &= \ln n-(n-1)\ln 2\end{aligned}$$

7 Post Workshop Exercise: Array Environment

7.1 Question 1

Given a natural number n,

$$\sum_{j=0}^{m} \left(\begin{array}{c} n \\ k \end{array} \right) = \left(\begin{array}{c} n+m+1 \\ n+1 \end{array} \right) = \left(\begin{array}{c} n+m+1 \\ m \end{array} \right)$$

7.2 Question 2

Let B_n the Bell number for $n \in \mathbb{Z}_{\leq 0}$. Then,

$$B_{n+1} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) B_k$$

where $\left(\begin{array}{c} n \\ k \end{array} \right)$ are binomial coefficients.

7.3 Question 3

Let

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$$

be a space transformation. Then, the *Jacobian* of T is the following 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \left| \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{array} \right|.$$

7.4 Question 4

Let function $f : \mathbb{R} \to \mathbb{R}$ be a function such that each of its first-order partial derivatives exist on $\mathbb{R}^n$. Then the Jacobian matrix of f is defined to be

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^T f_1 \\ \vdots \\ \nabla^T f_n \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

7.5 Question 5

The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \left\{ \begin{array}{ll} \exp(-\frac{1}{1-x^2}) & \text{if } x \in (-1,1) \\ 0 & \text{otherwise} \end{array} \right.$$

is smooth and compactly supported.

7.6 Question 6

Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be the least squares solution to the equation.Then we have:

$$
\begin{aligned}
& A^T A v = A^T b \\
\implies & \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
\implies & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/14 \\ 13/7 \end{pmatrix}
\end{aligned}
$$