Step-1

We have to find an orthonormal set q_1, q_2, q_3 for which q_1, q_2 spans the column space of

$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$, which fundamental subspace contains q_3, and we have to find the least- squares solution of $Ax = b$ if $b = \begin{bmatrix} 1 & 2 & 7 \end{bmatrix}^T$.

Step-2

Given $A = \begin{bmatrix} a & b \end{bmatrix}$ where

$a = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

Step-3

Now

$$q_1 = \frac{a}{\|a\|}$$
$$= \frac{1}{\sqrt{1^2 + 2^2 + (-2)^2}} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$
$$= \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

Step-4

$q_2 = \frac{\beta}{\|\beta\|}$ where $\beta = b - \left(q_1^T b\right) q_1$

Now

$$q_1^T b = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$
$$= \frac{1-2-8}{3}$$
$$= -3$$

Step-5

Therefore $\beta = b - \left(q_1^T b\right) q_1$

$$\beta = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\|\beta\| = \sqrt{4+1+4}$$
$$= 3$$

Step-6

Therefore

$$q_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

Since q_1, q_2, q_3 are orthonormal this implies $q_1^T q_3 = 0$ and $q_2^T q_3 = 0$ â€¦â€¦ (1)

Step-7

Let $q_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$q_1^T q_3 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \frac{1}{3}x + \frac{2}{3}y - \frac{2}{3}z$$

Step-8

$$q_2^T q_3 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \frac{2}{3}x + \frac{1}{3}y + \frac{2}{3}z$$

By (1),

$$\frac{1}{3}x + \frac{2}{3}y - \frac{2}{3}z = 0$$

And $\frac{2}{3}x + \frac{1}{3}y + \frac{2}{3}z = 0$

By using cross product,

$$x = \frac{2}{3}, y = -\frac{2}{3}, z = -\frac{1}{3}$$

Or $x = -\frac{2}{3}, y = \frac{2}{3}, z = \frac{1}{3}$

Therefore $q_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

Step-9

Hence required orthonormal set

$$= \{q_1, q_2, q_3\} = \left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \right\}$$

By the method of least squares,

$$\hat{x} = \begin{bmatrix} q_1^T b \\ q_2^T b \end{bmatrix}$$

Step-10

Now

$$b = \begin{bmatrix}1 & 2 & 7\end{bmatrix}^T$$
$$= \begin{bmatrix}1\\2\\7\end{bmatrix}$$

Step-11

$$q_1^T b = \begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{-2}{3}\end{bmatrix}\begin{bmatrix}1\\2\\7\end{bmatrix}$$
$$= \frac{1+4-14}{3}$$
$$= -3$$

Step-12

$$q_2^T b = \begin{bmatrix}\frac{2}{3} & \frac{1}{3} & \frac{2}{3}\end{bmatrix}\begin{bmatrix}1\\2\\7\end{bmatrix}$$
$$= \frac{2+2+14}{3}$$
$$= 6$$

The least squares solution is $\hat{x} = \begin{bmatrix}q_1^T b\\ q_2^T b\end{bmatrix} = \begin{bmatrix}-3\\ 6\end{bmatrix}$