Step-1

Given that the vector f is odd if $f_{n-j} = -f_{-j}$ (that is for $n = 4$, $f_0 = 0,\ f_2 = 0,\ f_3 = -f_1$)

For the 4 by 4 matrix, we have to write out the formulas for c_0, c_1, c_2, c_3 and we have to verify that if f is odd, then c is odd.

Step-2

Let

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

$$f = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Step-3

Fourier matrix

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$\left(\begin{aligned} &\text{since } i^2 = -1, i^3 = -i, \\ &i^4 = 1, i^6 = i^4 i^2 = -1, \\ &i^9 = i^6 i^3 = i \end{aligned} \right)$$

Step-4

Now

$F_4 c = y$

$\Rightarrow c = F_4^{-1} y$

But $F_n^{-1} = \frac{\overline{F_n}}{n} y, (n \text{ is the order of } F_n)$

Therefore $c = \frac{\overline{F_4}}{4} y$

Step-5

$$\Rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} f_0 + f_1 + f_2 + f_3 \\ f_0 - if_1 - f_2 + if_3 \\ f_0 - f_1 + f_2 - f_3 \\ f_0 + if_1 - f_2 - if_3 \end{bmatrix}$$

Step-6

Therefore

$$c_0 = \frac{f_0 + f_1 + f_2 + f_3}{4}$$

$$c_1 = \frac{f_0 - if_1 - f_2 + if_3}{4}$$

$$c_2 = \frac{f_0 - f_1 + f_2 - f_3}{4}$$

$$c_3 = \frac{f_0 + if_1 - f_2 - if_3}{4}$$

Step-7

Given $f_0 = 0,\ f_2 = 0,\ f_3 = -f_1$

Therefore

$$c_0 = \frac{0 + f_1 + 0 - f_1}{4} = 0$$
$$c_1 = \frac{0 - if_1 + 0 - if_1}{4} = \frac{-if_1}{2}$$
$$c_2 = \frac{0 - f_1 + 0 + f_1}{4} = 0$$
$$c_3 = \frac{0 + if_1 + 0 + if_1}{4} = \frac{if_1}{2}$$

Therefore $c_0 = 0, c_2 = 0, c_3 = -c_1$

So c is also odd.

So if f is odd then c is also odd.