

## Step-1

$$\text{Given } a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

We have to express the Gram-Schmidt orthogonalization of  $a_1, a_2$  as  $A = QR$ , and we have to find the shapes of  $A, Q, R$  if given  $n$  vectors  $a_i$  with  $m$  components.

## Step-2

$$\begin{aligned} q_1 &= \frac{a_1}{\|a_1\|} \\ &= \frac{1}{\sqrt{1^2 + 2^2 + 2^2}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \end{aligned}$$

## Step-3

$$\begin{aligned} q_2 &= \frac{\beta}{\|\beta\|} \text{ where} \\ \beta &= a_2 - (q_1^T a_2) q_1 \end{aligned}$$

## Step-4

$$\begin{aligned} q_1^T a_2 &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\ &= \frac{1+6+2}{3} \\ &= 3 \end{aligned}$$

## Step-5

$$\begin{aligned}(q_1^T a_2) q_1 &= 3 \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\end{aligned}$$

### Step-6

$$\begin{aligned}\beta &= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\|\beta\| &= \sqrt{0+1+1} \\ &= \sqrt{2}\end{aligned}$$

### Step-7

Therefore

$$\begin{aligned}q_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\end{aligned}$$

### Step-8

$$\begin{aligned}q_1^T a_1 &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ &= \frac{1+4+4}{3} \\ &= 3\end{aligned}$$

### Step-9

$$q_1^T a_2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{1+6+2}{3}$$

$$= 3$$

$$q_2^T a_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{0+3-1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

## Step-10

$$A = [a_1 \ a_2]$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= [q_1 \ q_2] \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$= QR$$

$$\text{Therefore } A = QR = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$

## Step-11

In the above problem, there are two vectors with three components, then we have  $A$  has a matrix of order 3 by 2,  $Q$  is also a matrix of order 3 by 2, and  $R$  is a matrix of order 2 by 2.

In the same way, if there are  $n$  vectors  $a_i$  with  $m$  components then  $A$  has a matrix of order  $m$  by  $n$ ,  $Q$  is also a matrix of order  $m$  by  $n$ , and  $R$  is a matrix of order  $n$  by  $n$ .