考试科目: 线 性 代 数 A　　　　**开课单位:** 数 学 系

考试时长: 120 分钟

题 号	1	2	3	4	5	6	7	8
分 值	15 分	25 分	10 分	16 分	10 分	6 分	16 分	12 分

本试卷共 (8) 大题, 满分 (110) 分. 请将所有答案写在答题本上.

This exam includes 8 questions and the score is 110 in total. Write all your answers on the examination book.

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1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Let A be an $m \times n$ real matrix and b be a column vector in $\mathbb{R}^m$. Which of the following statements is correct? (　　)

(A) If $Ax = b$ has infinite many solutions, then $Ax = 0$ has a nonzero solution.

(B) If the system $Ax = 0$ has only zero solution, then $Ax = b$ has one and only one solution.

(C) If the rank of A is n, then the system $Ax = b$ must have a solution.

(D) If A is a square matrix (i.e., $m = n$), then the system $Ax = b$ is consistent if and only if A is invertible.

设 A 为 $m \times n$ 实矩阵, b 是 $\mathbb{R}^m$ 中的列向量. 下列陈述中哪个是正确的? (　　)

(A) 如果 $Ax = b$ 有无穷多个解, 则 $Ax = 0$ 有非零解.

(B) 如果方程组 $Ax = 0$ 只有零解, 则 $Ax = b$ 有且仅有一个解.

(C) 如果 A 的秩为 n, 则方程组 $Ax = b$ 必有解.

可以A不满秩, b给得好

(D) 如果 A 是方阵 (即 $m = n$), 则方程组 $Ax = b$ 是相容的当且仅当 A 可逆.

(2) Suppose A is an $m \times n$ matrix, B is an $n \times m$ matrix, and I is the $m \times m$ identity matrix. If $AB = I$, then (　　) C

full row rank　　full column rank

(A) the column vectors of A are linearly independent, and the row vectors of B are linearly independent.

(B) the column vectors of A are linearly independent, and the column vectors of B are linearly independent.

B full column rank

(C) the row vectors of A are linearly independent, and the column vectors of B are linearly independent.

(D) the row vectors of A are linearly independent, and the row vectors of B are linearly independent.

设 A 为 $m \times n$ 型矩阵, B 为 $n \times m$ 型矩阵, I 为 m 阶单位矩阵. 若 $AB = I$, 则 (　　)

(A) A 的列向量组线性无关, B 的行向量组线性无关.

A $\boxed{\text{full row rank} \Leftrightarrow AC = I}$

$\Leftrightarrow A x_i = e_i, \forall i$

$\underset{m\times n}{A}[x_1, x_2, \cdots, x_m] = I_m$.

$\Leftrightarrow Ax = b$ 有解 $\forall b \in R^m$

(B) A 的列向量组线性无关, B 的列向量组线性无关.

(C) A 的行向量组线性无关, B 的列向量组线性无关.

(D) A 的行向量组线性无关, B 的行向量组线性无关.

(3) Let A be a 3×3 matrix, and let B be the matrix formed by adding the second column of A to its first column. Suppose that after exchanging the second and third rows of B, the resulting matrix is the 3×3 identity matrix. Let $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Then $A = (\quad)$

(A) P_1P_2.

(B) $P_1^{-1}P_2$.

(C) P_2P_1.

(D) $P_2P_1^{-1}$.

设 A 为 3 阶方阵, 将 A 的第二列加到第一列得矩阵 B. 假设交换 B 的第二行与第三行可以得到 3 阶单位矩阵. 记 $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. 则 $A = (\quad)$

(A) P_1P_2.

(B) $P_1^{-1}P_2$.

(C) P_2P_1.

(D) $P_2P_1^{-1}$.

$AP_1 = B$　　$P_2AP_1 = I$

$P_2B = I$　　$A = P_2^{-1}P_1^{-1}$

$\boxed{P_2^{-1} = P_2}$

Permutation inverse equals permutation transpose

(4) Let $A = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ 2 & -4 & t \end{bmatrix}$, where $t \in \mathbb{R}$. Suppose $\operatorname{rank}(A) = 2$. Then (D)

(A) $t = -6$.　　$\rightarrow \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & t \end{bmatrix}$

(B) $t = 6$.

(C) $t \neq 0$.

(D) t can be any real number.

t can be eliminated

设 $A = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ 2 & -4 & t \end{bmatrix}$, 其中 $t \in \mathbb{R}$. 假设 $\operatorname{rank}(A) = 2$. 则 $(\quad)$

(A) $t = -6$.

(B) $t = 6$.

(C) $t \neq 0$.

(D) t 可以是任意实数.

(5) Which of the following statements is incorrect? $(\quad)$

(A) For any matrix A, $\operatorname{rank}(A) = \dim C(A)$.

(B) If $v_1, \cdots, v_m$ are pairwise orthogonal nonzero vectors, then the vectors $v_1, \cdots, v_m$ are linear independent.

(C) If A is an upper triangular $n \times n$ matrix such that $A^2 = 0$, then $A = 0$.

(D) Let A, B be $n \times n$ matrices such that AB is invertible. Then both A and B are invertible.

下列哪个论断是错误的？(C)

(A) 对于任意矩阵 A, $\mathrm{rank}(A) = \dim C(A)$. ✓

(B) 如果 $v_1, \cdots, v_m$ 是一组两两正交的非零向量, 则向量组 $v_1, \cdots, v_m$ 线性无关.

(C) 如果 A 是 $n \times n$ 上三角矩阵且 $A^2 = 0$, 则 $A \neq 0$. ✗

(D) 设 A, B 为 $n \times n$ 矩阵且 AB 可逆. 则 A 和 B 都可逆.

$(AB)C = I$　　　$D(AB) = I$

full row rank $\Longrightarrow A(BC) = I$　　　$(DA)B = I$

full row rank

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let A, B be invertible $n \times n$ matrices. Then the inverse of the block matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ is $\underline{\begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix}}$

设 A, B 均为 $n \times n$ 可逆矩阵. 则分块矩阵 $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ 的逆为 __________

(2) Suppose A is a 3×4 matrix and $\dim N(A) = 2$. Then $\dim N(A^T) =$ __________

设 A 为 3×4 矩阵且 $\dim N(A) = 2$. 则 $\dim N(A^T) =$ __________

(3) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$

设 $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. 则 $A^{-1} =$ __________

(4) Let u, v be vectors in $\mathbb{R}^n$ such that $\|u\| = 3$, $\|v\| = 4$ and $u^T v = -3$.

Then $\|2u + 3v\| =$ 12

设 u, v 为 $\mathbb{R}^n$ 中的向量, 满足 $\|u\| = 3$, $\|v\| = 4$ 以及 $u^T v = -3$. 则 $\|2u+3v\| =$ __________

(5) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

Then the least squares solution to $Ax = b$ is $\hat{x} =$ __________

设 $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

则 $Ax = b$ 的最小二乘解是 $\hat{x} =$ __________

3. (10 points) Find the LU factorization of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

求矩阵 $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ 的 LU 分解.

4. (16 points) Let $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

Please give a basis for each of the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$ and $N(A^T)$, respectively.

(16 分) 设 $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

请对四个基本子空间 $C(A)$, $N(A)$, $C(A^T)$ 和 $N(A^T)$ 分别给出各自的一组基.

5. (10 points) Let $E = \{u_1, u_2, u_3\}$ and $F = \{v_1, v_2\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}.$$

Find the matrix A representing T with respect to the ordered bases E and F.

(10 分) 设 $E = \{u_1, u_2, u_3\}$, $F = \{v_1, v_2\}$, 其中

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

定义线性变换 $T : \mathbb{R}^3 \to \mathbb{R}^2$ 如下

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}.$$

求 T 在 E 和 F 这两组有序基下的矩阵表示 A.

6. (6 points) Let A, B be $n\times n$ matrices. Suppose A and B are both symmetric. Is AB necessarily symmetric? If yes, please give a proof. Otherwise please give a counterexample.

(6 分) 设 A, B 均为 $n\times n$ 矩阵. 假设 A 和 B 都是对称矩阵. AB 是否一定是对称矩阵? 若是, 请给出证明. 否则请给出一个反例.

不一定. eg. $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

7. (16 points) The following two questions are independent:

(a) Let A be the 2×2 matrix such that the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto Av$ rotates every vector in $\mathbb{R}^2$ through 60° counter-clockwise (about the origin).
Find A and A^{2020}.

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$A^{2020} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

(b) Three planes Π_1, Π_2, Π_3 in the space $\mathbb{R}^3$ are given by the equations

$$\Pi_1 : \ x+y+z=0\,,$$
$$\Pi_2 : \ 2x-y+4z=0\,,$$
$$\Pi_3 : \ -x+2y-z=0\,.$$

Determine a matrix representative (in the standard basis of $\mathbb{R}^3$) of a linear transformation taking the xy plane to Π_1, the yz plane to Π_2 and the zx plane to Π_3.

(16 分) 以下两个小题是相互独立的:

(a) 设 A 是 2×2 矩阵使得线性变换 $\mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto Av$ 把 $\mathbb{R}^2$ 中每个向量 (绕原点) 逆时针转动 60°.
求 A 和 A^{2020}.

(b) 在空间 $\mathbb{R}^3$ 中由以下方程给出三个平面 Π_1, Π_2, Π_3:

$$\Pi_1 : \ x+y+z=0\,,$$
$$\Pi_2 : \ 2x-y+4z=0\,,$$
$$\Pi_3 : \ -x+2y-z=0\,.$$

$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(e_1) \subset \Pi_1 \cap \Pi_3$

求一个矩阵, 使它 (在 $\mathbb{R}^3$ 的标准基下) 表示的线性变换将 xy 平面映射成 Π_1, 将 yz 平面映射成 Π_2 并将 zx 平面映射成 Π_3.

8. (12 points) Let A be a 3×3 matrix such that $\mathrm{rank}(A) = 2$ and $A^3 = 0$.

(a) Prove that $\mathrm{rank}(A^2) = 1$.

(b) Let $\alpha_1 \in \mathbb{R}^3$ be a nonzero vector such that $A\alpha_1 = 0$. Prove that there exist vectors α_2, α_3 such that $A\alpha_2 = \alpha_1$, $A^2\alpha_3 = \alpha_1$.

(c) For any vectors α_2, α_3 described as above, show that $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.

(*In this problem, you are allowed to assume the statements of some questions to answer subsequent questions.*)

$\Leftrightarrow C(A) \subset N(A^2)$

$A^2A=0$

$\dim C(A)=2$

$\dim N(A^2)=3-\dim C(A^2)=2$

(12 分) 设 A 是 3×3 矩阵, 它满足 $\operatorname{rank}(A)=2$ 及 $A^3=0$.

(a) 证明 $\operatorname{rank}(A^2)=1$.

$\alpha_1 \in N(A) \subset N(A^2) = C(A)$

$\alpha_1 \in C(A^2)$

(b) 设 $\alpha_1\in\mathbb{R}^3$ 是满足 $A\alpha_1=0$ 的非零向量. 证明: 存在向量 α_2, α_3 使得 $A\alpha_2=\alpha_1$, $A^2\alpha_3=\alpha_1$.

$\alpha_1 \in N(A) = C(A^2)$

(c) 证明: 对于任意满足上述条件的向量 α_2,α_3, 向量组 $\alpha_1,\alpha_2,\alpha_3$ 线性无关.

$AA^2=0$

$C(A^2)\subset N(A)$

$\dim C(A^2)=1$

$\dim N(A)=3-2=1$

$C(A^2)=N(A)$

左乘矩阵

消去零向量

(本题中, 允许承认前面小题的结果来用于后续问题的解答.)

$A\alpha_1=0.\quad A\alpha_2=\alpha_1\quad A^2\alpha_3=\alpha_1$

$c_1\alpha_1+c_2\alpha_2+c_3\alpha_3=0$

$A^2(c_1\alpha_1+c_2\alpha_2+c_3\alpha_3)=0 \Rightarrow c_3=0$

$c_1\alpha_1+c_2\alpha_2=0$

$c_1\underbrace{A\alpha_1}_{0}+c_2A\alpha_2=0 \Rightarrow c_2=0 \Rightarrow c_1=0$