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Determinants (行列式)

4.3

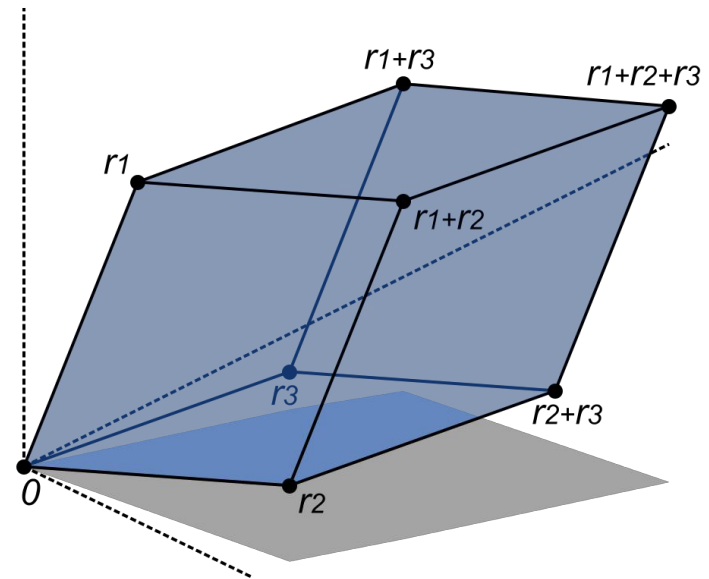
FORMULAS FOR THE DETERMINANT

Formula from pivots

A Big formula

Expansion Rule

*Calculations (Some
Tricks)*



I. Formula from Pivots (行列式的计算公式：主元)

Theorem 1 *If A is invertible, then $PA = LDU$ and $|P| = \pm 1$. The product rule gives*

$$|A| = \pm |L| |D| |U| = \pm |D| = \pm (\text{product of the pivots})$$

The sign ± 1 depends on whether the number of row exchanges is even (偶) or odd (奇). (+1 for even; -1 for odd)

For example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & (ad - bc)/a \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix},$$

the product of the pivots is $ad - bc = |A| = |D|$.

If there is a row exchange, then

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}, \text{ and } |A| = -|D|.$$

Example 1 The $-1, 2, -1$ second difference matrix

$$\mathbf{A}_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} \frac{2}{1} & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{U}}$$

Its determinant is the product of its pivots.

$$|\mathbf{A}_4| = \left(\frac{2}{1}\right) \left(\frac{3}{2}\right) \left(\frac{4}{3}\right) \left(\frac{5}{4}\right) = 5.$$

In general, for \mathbf{A}_n in this pattern, we have $|\mathbf{A}_n| = n + 1$.

II. A Big formula – An equivalent definition (行列式的等价定义)

For $n = 2$, we have

$$\begin{aligned}
 \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a+0 & 0+b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \\
 &= \begin{vmatrix} a & 0 \\ c+0 & 0+d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c+0 & 0+d \end{vmatrix} \\
 &= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} \\
 &= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = ad - bc.
 \end{aligned}$$

To get nonzero terms: Suppose 1st row has a nonzero term in column α , 2nd row is nonzero in column β , ..., and finally the n -th row in column v .

Then the column numbers α, β, \dots, v are all different.

They are a reordering, or *permutation* (排列), of the numbers 1, 2, ..., n .

For $n = 3$, we have

$$\begin{aligned}
 |\mathbf{A}| &= \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} \\
 &\quad + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} \\
 &= a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} \\
 &\quad + a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ a_{21} & & \\ & & 1 \end{vmatrix} + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} \\
 &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.
 \end{aligned}$$

There are $n!$ ways to permute numbers $1, 2, \dots, n$.

Column numbers: $\alpha, \beta, \nu = (1, 2, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1), (2, 1, 3), (1, 3, 2)$.

$$|A|$$

$$\begin{aligned}
 &= a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix} \\
 &\quad + a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} \\
 &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.
 \end{aligned}$$

- Every term is a product of $n = 3$ entries a_{ij} , with *each row and column represented once*.
- If the columns come in the order (α, \dots, ν) , that term is the product $a_{1\alpha} \dots a_{n\nu}$ times the determinant of a permutation matrix \mathbf{P} .
- The determinant of the whole matrix \mathbf{A} is the sum of these $n!$ terms.

*This leads a **'big formula'** for computing the determinant of*

$$\mathbf{A} = [a_{ij}]_{n \times n}.$$

Theorem 2 (Big formula)

$$|\mathbf{A}| = \sum_{\text{all } \mathbf{P}'_s} (a_{1\alpha} a_{2\beta} \cdots a_{nv}) |\mathbf{P}|.$$

Notes:

$|\mathbf{P}| = 1$ or -1 for an *even* or *odd* number of row exchanges.

For example,

$$\mathbf{P} = \begin{bmatrix} & 1 & & \\ & & 1 & \\ 1 & & & \end{bmatrix}, \begin{bmatrix} & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix},$$

with determinants equal to 1, 1, -1 , respectively.

Equivalently, it depends on the permutation of the n numbers being an even permutation (偶排列) or odd permutation (奇排列).

~~(2,3,1)~~, ~~(3,1,2)~~: even; ~~(1,3,2)~~: odd.

(1,3,2) requires one exchange to recover (1,2,3);

(2,3,1), (3,1,2) requires two exchanges to recover (1,2,3).

Example 2 Let $f(x) = \begin{vmatrix} x & 1 & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 2x & 1 \end{vmatrix}$.

Find the coefficient of x^3 .

Solution

$$f(x) = \begin{vmatrix} x & 1 & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 2x & 1 \end{vmatrix}$$

The coefficient of x^3 is -1 .

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} a_{11}a_{22}a_{33}a_{44} = x^3,$$

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} a_{11}a_{22}a_{34}a_{43} = -2x^3.$$

III. Expansion of $\det A$ in cofactors (使用代数余子式展开行列式)

No row or column can be used twice in the same term.

For $n = 3$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

Theorem 3 *The determinant of A is a combination of any row i times its cofactors:*

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}.$$

The cofactor C_{ij} is the determinant of M_{ij} with the correct sign:

$$C_{ij} = (-1)^{i+j} |M_{ij}|.$$

The submatrix M_{ij} is formed by *throwing away row i and column j .*

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \quad M_{23} = \begin{bmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix},$$

$$C_{23} = (-1)^{2+3} |M_{23}| = -|M_{23}|.$$

注意：一个元素的代数余子式只与该元素所处位置有关，而与该元素等于多少无关。

思考 设 $D = \begin{vmatrix} 3 & -5 & 2 & 1 \\ 1 & 1 & 0 & -5 \\ -1 & 3 & 1 & 3 \\ 2 & -4 & -1 & -3 \end{vmatrix}$, 求 $\frac{2C_{11} - 4C_{12} - C_{13} - 3C_{14}}{\text{与该元素为多少无关}}.$

结论：某一行元素依次乘以另一行元素的代数余子式再求和，其结果等于0.

Example 1 (Continued) The $-1, 2, -1$ second difference matrix (4×4)

$$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

$$C_{11} = |A_3| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}, \quad C_{12} = (-1)^{1+2} \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = |A_2|.$$

$$|A_4| = 2C_{11} - C_{12} = 2|A_3| - |A_2| = 2(4) - 3 = 5.$$

The same idea applies to every A_n : $|A_n| = 2|A_{n-1}| - |A_{n-2}|.$

By recursion (递推), we can get $|A_n| = 2(n) - (n-1) = n+1.$

IV. Some Tricks for Computing Determinants

例1 计算 $D = \begin{vmatrix} a_1 + \lambda & a_2 & \cdots & a_n \\ a_1 & a_2 + \lambda & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n + \lambda \end{vmatrix}.$

解1 D 的每个列写成两个子列之和:

$$D = \begin{vmatrix} a_1 + \lambda & a_2 + 0 & \cdots & a_n + 0 \\ a_1 + 0 & a_2 + \lambda & \cdots & a_n + 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 + 0 & a_2 + 0 & \cdots & a_n + \lambda \end{vmatrix},$$

D 可以分解为 2^n 个行列式的和.

这些行列式中只有 $n + 1$ 个非零, 它们的和为

$$D = \begin{vmatrix} a_1 & & & \\ a_1 & \lambda & & \\ \vdots & & \ddots & \\ a_1 & & & \lambda \end{vmatrix} + \begin{vmatrix} \lambda & a_2 & & \\ & a_2 & & \\ \vdots & & \ddots & \\ a_2 & & & \lambda \end{vmatrix} + \cdots$$

$$+ \begin{vmatrix} \lambda & & & a_n \\ & \lambda & & a_n \\ & & \ddots & \vdots \\ & & & a_n \end{vmatrix} + \begin{vmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{vmatrix},$$

分拆法

故

$$D = \lambda^{n-1} (a_1 + a_2 + \cdots + a_n + \lambda).$$

解2 将 D 的每列都加到第一列, 得

$$\begin{aligned}
 D &= \begin{vmatrix} a_1 + a_2 + \cdots + a_n + \lambda & a_2 & \cdots & a_n \\ a_1 + a_2 + \cdots + a_n + \lambda & a_2 + \lambda & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 + a_2 + \cdots + a_n + \lambda & a_2 & \cdots & a_n + \lambda \end{vmatrix} \\
 &= (a_1 + a_2 + \cdots + a_n + \lambda) \begin{vmatrix} 1 & a_2 & \cdots & a_n \\ 1 & a_2 + \lambda & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ 1 & a_2 & \cdots & a_n + \lambda \end{vmatrix}.
 \end{aligned}$$

将右端行列式的第 i 列减去第一列的 a_i 倍 ($2 \leq i \leq n$), 得

$$D = (a_1 + a_2 + \cdots + a_n + \lambda) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & \lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= (a_1 + a_2 + \cdots + a_n + \lambda) \lambda^{n-1}.$$

三角化法

解3 先把第一行乘以 (-1) 加到以下各行,
再把后面各列加到第一列.

$$\begin{aligned}
 D &= \begin{vmatrix} a_1 + \lambda & a_2 & \cdots & a_n \\ a_1 & a_2 + \lambda & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n + \lambda \end{vmatrix} = \begin{vmatrix} a_1 + \lambda & a_2 & \cdots & a_n \\ -\lambda & \lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -\lambda & 0 & \cdots & \lambda \end{vmatrix} \\
 &= \begin{vmatrix} a_1 + a_2 + \cdots + a_n + \lambda & a_2 & \cdots & a_n \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda \end{vmatrix} \\
 &= (a_1 + a_2 + \cdots + a_n + \lambda) \lambda^{n-1}.
 \end{aligned}$$

三角化法

解4 先加上一行一列, 化为一个 $n+1$ 阶行列式:

$$\begin{aligned}
 D &= \begin{vmatrix} a_1 + \lambda & a_2 & \cdots & a_n \\ a_1 & a_2 + \lambda & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n + \lambda \end{vmatrix} = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & a_1 + \lambda & a_2 & \cdots & a_n \\ 0 & a_1 & a_2 + \lambda & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_1 & a_2 & \cdots & a_n + \lambda \end{vmatrix}_{n+1} \\
 &= \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{n+1}.
 \end{aligned}$$

当 $\lambda \neq 0$ 时, 把后 n 列的 $1/\lambda$ 倍都加到第一列, 得

$$D = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{n+1} = \begin{vmatrix} 1 + \sum_{i=1}^n \frac{a_i}{\lambda} & a_1 & a_2 & \cdots & a_n \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{n+1}$$

$$= \left(1 + \sum_{i=1}^n \frac{a_i}{\lambda} \right) \lambda^n = (a_1 + a_2 + \cdots + a_n + \lambda) \lambda^{n-1}.$$

当 $\lambda = 0$ 时, 上式显然成立.

加边法

All roads lead to Rome

例2 证明Vandermonde行列式

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq j < i \leq n} (x_i - x_j).$$

证 对行列式的阶数 n 用数学归纳法. 因为

$$D_2 = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1 = \prod_{1 \leq j < i \leq 2} (x_i - x_j),$$

所以 $n=2$ 时, 等式成立.

假设等式对于 $n-1$ 阶 **Vandermonde** 行列式成立.

从第 n 行开始, 每行减去上一行的 x_1 倍, 有

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ 0 & x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_n(x_n - x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_2^{n-3}(x_2 - x_1) & x_3^{n-3}(x_3 - x_1) & \cdots & x_n^{n-3}(x_n - x_1) \\ 0 & x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

按第一列展开, 并提出每列的公因子, 就有

$$D_n = (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}.$$

$n-1$ 阶Vandermonde行列式

因此由归纳法假设得

$$\begin{aligned} D_n &= (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \prod_{2 \leq j < i \leq n} (x_i - x_j) \\ &= \prod_{1 \leq j < i \leq n} (x_i - x_j). \end{aligned}$$

归纳法

应用 设曲线 $y = a_0 + a_1x + a_2x^2 + a_3x^3$ 通过四点 $(1, 3)$ 、 $(2, 4)$ 、 $(3, 3)$ 、 $(4, -3)$, 求系数 a_0, a_1, a_2, a_3 .

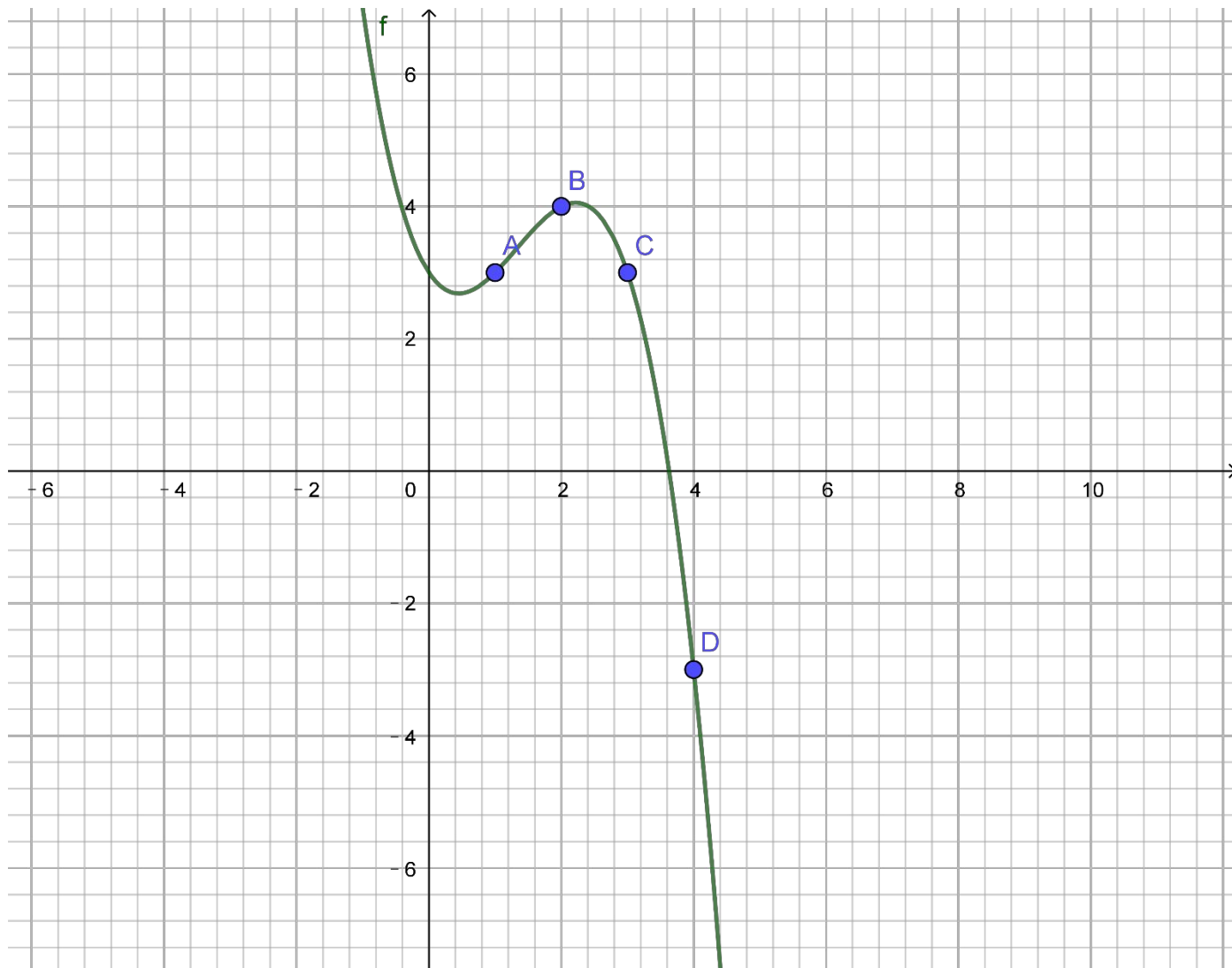
解 把四个点的坐标代入曲线方程, 得线性方程组

$$\begin{cases} a_0 + a_1 + a_2 + a_3 = 3, \\ a_0 + 2a_1 + 4a_2 + 8a_3 = 4, \\ a_0 + 3a_1 + 9a_2 + 27a_3 = 3, \\ a_0 + 4a_1 + 16a_2 + 64a_3 = -3. \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{vmatrix} \quad \begin{array}{l} \text{Vandermonde 行列式} \\ = (4-1)(3-1)(2-1)(4-2)(3-2)(4-3) = 12 \neq 0, \\ \text{所以方程有唯一解} \end{array}$$

$$\begin{aligned} a_0 &= 3, & a_1 &= -\frac{3}{2}, \\ a_2 &= 2, & a_3 &= -\frac{1}{2} \end{aligned}$$

$$\text{即曲线方程为 } y = 3 - \frac{3}{2}x + 2x^2 - \frac{1}{2}x^3.$$



一般地，过 $n + 1$ 个 x 坐标不同的点 (x_i, y_i) , $i = 1, \dots, n + 1$, 可唯一地确定一个 n 次曲线的方程 $y = a_0 + a_1x + \dots + a_nx^n$

例3 计算

$$D_n = \begin{vmatrix} \alpha + \beta & \alpha\beta & & & \\ 1 & \alpha + \beta & \alpha\beta & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha + \beta & \alpha\beta \\ & & & 1 & \alpha + \beta \end{vmatrix}.$$

解 按照第一行展开, 得

$$D_n = (\alpha + \beta) \begin{vmatrix} \alpha\beta & & & \\ 1 & \alpha + \beta & \alpha\beta & \\ & \ddots & \ddots & \ddots \\ & & 1 & \alpha + \beta \end{vmatrix} - \alpha\beta \begin{vmatrix} 1 & \alpha\beta & & \\ 0 & \alpha + \beta & \alpha\beta & \\ & \ddots & \ddots & \ddots \\ & & 1 & \alpha + \beta \end{vmatrix}.$$

再把第二个行列式按照第一列展开, 得

$$D_n = (\alpha + \beta)D_{n-1} - \alpha\beta D_{n-2},$$

$$\begin{aligned}\text{从而 } D_n - \alpha D_{n-1} &= \beta(D_{n-1} - \alpha D_{n-2}) \\ &= \beta^2(D_{n-2} - \alpha D_{n-3}) = \cdots = \beta^{n-2}(D_2 - \alpha D_1),\end{aligned}$$

$$\text{因 } D_1 = \alpha + \beta, D_2 = \alpha^2 + \alpha\beta + \beta^2, \text{ 故}$$

$$D_n - \alpha D_{n-1} = \beta^n.$$

于是

$$\begin{aligned}D_n &= \alpha D_{n-1} + \beta^n = \alpha(\alpha D_{n-2} + \beta^{n-1}) + \beta^n \\ &= \alpha^2 D_{n-2} + \alpha\beta^{n-1} + \beta^n = \alpha^3 D_{n-3} + \alpha^2 \beta^{n-2} + \alpha\beta^{n-1} + \beta^n \\ &= \cdots = \alpha^{n-1} D_1 + \alpha^{n-2} \beta^2 + \cdots + \alpha\beta^{n-1} + \beta^n \\ &= \alpha^n + \alpha^{n-1} \beta + \alpha^{n-2} \beta^2 + \cdots + \alpha\beta^{n-1} + \beta^n.\end{aligned}$$

递推法

或者由

$$D_n - \alpha D_{n-1} = \beta^n.$$

类似可得出

$$D_n - \beta D_{n-1} = \alpha^n. \quad \alpha, \beta \text{ equal.}$$

联立两式，可解得

$$\begin{aligned} D_n &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}. \\ &= \alpha^n + \alpha^{n-1}\beta + \alpha^{n-2}\beta^2 + \cdots + \alpha\beta^{n-1} + \beta^n. \end{aligned}$$

当 $n = 1, 2$ 时, 上式也成立.

递推法

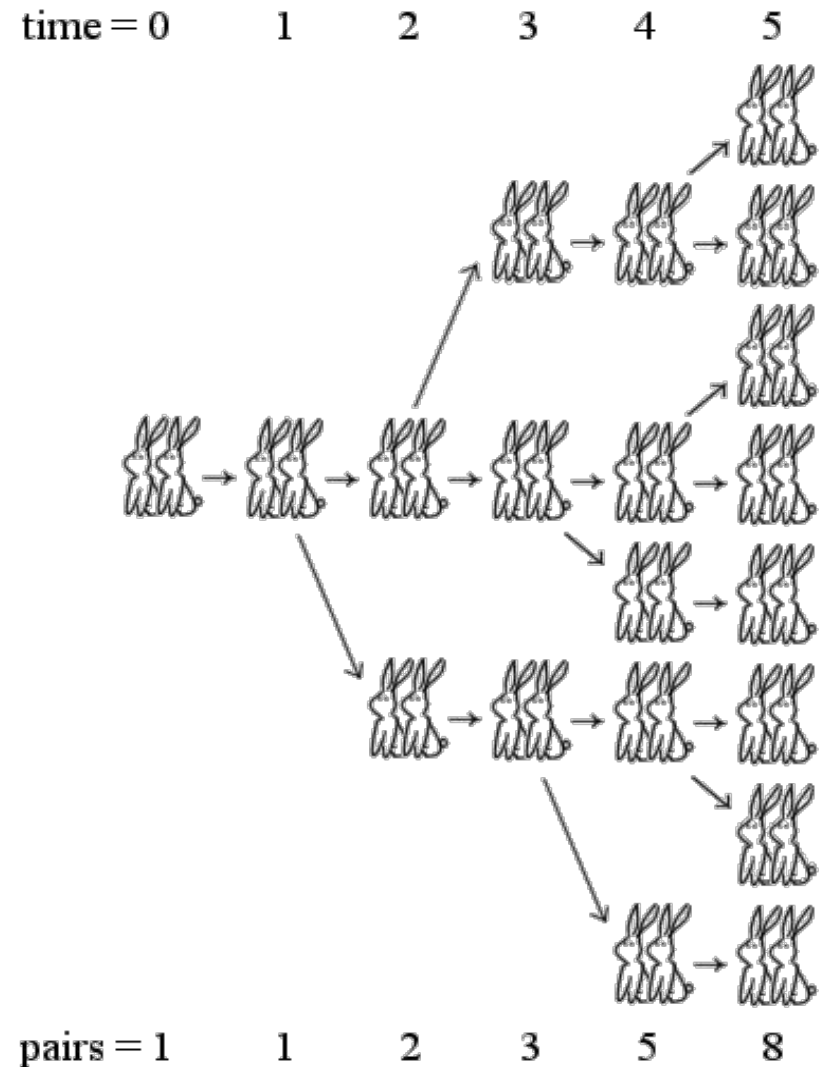
拓展应用: Fibonacci数列



Fibonacci gave this sequence as an answer to the following mathematical puzzle:

A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

The answer is the sequence **1, 1, 2, 3, 5, 8, 13, 21, ...**, as illustrated below:



Fibonacci数列: 1, 2, 3, 5, 8, 13, 21, 35, ...

满足: $F_n = F_{n-1} + F_{n-2} (n \geq 3), \quad F_1 = 1, \quad F_2 = 2.$

(1) 证明: Fibonacci数列的通项可由下述行列式表示:

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix}.$$

(2) 求Fibonacci数列的通项公式.

(1) 证

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix}.$$

把上述行列式按第1列展开, 得

$$\begin{aligned} F_n &= F_{n-1} + 1 \cdot (-1)^{2+1} (-1) F_{n-2} \\ &= F_{n-1} + F_{n-2}. \quad (n \geq 3) \end{aligned}$$

(2) 解

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix}.$$

$$D_n = \begin{vmatrix} \alpha + \beta & \alpha\beta & & & \\ & 1 & \alpha + \beta & \alpha\beta & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & \alpha + \beta & \alpha\beta \\ & & & & 1 & \alpha + \beta \end{vmatrix}.$$

令 $\alpha + \beta = 1$, $\alpha\beta = -1$, 则 α, β 是方程 $x^2 - x - 1 = 0$ 的两个根: $\alpha = \frac{1 + \sqrt{5}}{2}$, $\beta = \frac{1 - \sqrt{5}}{2}$.

于是

$$F_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].$$

Fibonacci数列的通项公式

利用**递推法**进行计算还可推广到：

一般的 n 阶三对角行列式

$$D_n = \begin{vmatrix} a & b & & & \\ c & a & b & & \\ & c & a & b & \\ & & \ddots & \ddots & \ddots \\ & & c & \ddots & \ddots & a & b \\ & & & \ddots & & c & a \end{vmatrix}$$

$$D_n = \begin{vmatrix} \alpha + \beta & \alpha\beta & & & \\ 1 & \alpha + \beta & \alpha\beta & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha + \beta & \alpha\beta \\ & & & 1 & \alpha + \beta \end{vmatrix}.$$

例4 计算 $D = \begin{vmatrix} a_0 & 1 & 2 & \cdots & n \\ 1 & a_1 & 0 & \cdots & 0 \\ 2 & 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n & 0 & 0 & \cdots & a_n \end{vmatrix}.$

爪形行列式
(箭头形行列式)

解 按照 a_1, a_2, \dots, a_n 中 0 的个数分别进行讨论.

情况1:

a_1, a_2, \dots, a_n 中有两个或两个以上为 0.

由于行列式有两行成比例, 所以行列式的值为 0.

解(续) 情况2: a_1, a_2, \dots, a_n 中只有一个为0.

设 $a_k = 0$, 而 $a_i \neq 0, i \neq k$.

$$\begin{aligned}
 D &= \begin{vmatrix} a_0 & 1 & \cdots & k & \cdots & n \\ 1 & a_1 & & & & \\ \vdots & & \ddots & & & \\ k & & & 0 & & \\ \vdots & & & & \ddots & \\ n & & & & & a_n \end{vmatrix} = (-1)^{k+1+1} k \begin{vmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n \\ a_1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & 0 & & & & \\ & & & a_{k-1} & 0 & & & \\ & & & & & a_{k+1} & \ddots & \\ & & & & & & \ddots & a_n \end{vmatrix} \\
 &= (-1)^{k+2} (-1)^{1+k} k^2 \begin{vmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_{k-1} & & \\ & & & a_{k+1} & \\ & & & & \ddots & \\ & & & & & a_n \end{vmatrix} = -k^2 a_1 \cdots a_{k-1} a_{k+1} \cdots a_n.
 \end{aligned}$$

解(续) 情况3: a_1, a_2, \dots, a_n 全不为0.

$$D = \begin{vmatrix} a_0 & 1 & 2 & \cdots & n \\ 1 & a_1 & 0 & \cdots & 0 \\ 2 & 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n & 0 & 0 & \cdots & a_n \end{vmatrix} = a_1 a_2 \cdots a_n \begin{vmatrix} a_0 & 1 & 2 & \cdots & n \\ \frac{1}{a_1} & 1 & 0 & \cdots & 0 \\ \frac{2}{a_2} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{n}{a_n} & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= a_1 a_2 \cdots a_n \begin{vmatrix} a_0 - \frac{1}{a_1} - \frac{2^2}{a_2} - \cdots - \frac{n^2}{a_n} & 0 & 0 & \cdots & 0 \\ \frac{1}{a_1} & 1 & 0 & \cdots & 0 \\ \frac{2}{a_2} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{n}{a_n} & 0 & 0 & \cdots & 1 \end{vmatrix} = a_1 a_2 \cdots a_n \left(a_0 - \frac{1}{a_1} - \frac{2^2}{a_2} - \cdots - \frac{n^2}{a_n} \right).$$

例5 设 $\mathbf{A} = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$, 计算 \mathbf{A} 的行列式.

解 由于

$$\mathbf{A}^T \mathbf{A} = \begin{bmatrix} a^2 + b^2 + c^2 + d^2 & & & \\ & a^2 + b^2 + c^2 + d^2 & & \\ & & a^2 + b^2 + c^2 + d^2 & \\ & & & a^2 + b^2 + c^2 + d^2 \end{bmatrix}$$

所以

$$|\mathbf{A}|^2 = |\mathbf{A}^T| |\mathbf{A}| = |\mathbf{A}^T \mathbf{A}| = (a^2 + b^2 + c^2 + d^2)^4$$

因此

$$|\mathbf{A}| = \pm (a^2 + b^2 + c^2 + d^2)^2$$

但由于 \mathbf{A} 的主对角元全是 a , 行列式 $|\mathbf{A}|$ 中 a^4 项符号为“+”, 故

$$|\mathbf{A}| = (a^2 + b^2 + c^2 + d^2)^2.$$

观察 下面行列式的元素有何特点？

$$D = \begin{vmatrix} \alpha & \beta & 0 & 0 & 0 \\ \beta & \alpha & 0 & 0 & 0 \\ x_1 & y_1 & b & c & c \\ x_2 & y_2 & c & b & c \\ x_3 & y_3 & c & c & b \end{vmatrix}.$$

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定义 在 n 阶行列式 D 中, 任取 k 行 i_1, i_2, \dots, i_k 与 k 列 j_1, j_2, \dots, j_k , $k \leq n$, 将这些行与列交叉处的元素按原来相对位置构成的 k 阶子矩阵, 记为 N . 其行列式, 称为 D 的一个 **k 阶子式(the minor of the k -th order)**, 记为 $|N|$.

划去这些行和列后所剩下的元素依原次序构成的一个 $n-k$ 阶子矩阵, 记为 M .

称

$$C = (-1)^{i_1+i_2+\dots+i_k+j_1+j_2+\dots+j_k} |M|$$

为 $|N|$ 的**代数余子式(cofactor)**.

例如, $D = \begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ -3 & 3 & -7 & 9 & -5 \\ 2 & 0 & 4 & -2 & 1 \\ 3 & -5 & 7 & -14 & 6 \\ 4 & -4 & 10 & -10 & 2 \end{vmatrix},$

二阶子式

$$\begin{vmatrix} -1 & 2 \\ -5 & 7 \end{vmatrix},$$

其代数余子式

$$C = (-1)^{1+4+2+3} \begin{vmatrix} -3 & 9 & -5 \\ 2 & -2 & 1 \\ 4 & -10 & 2 \end{vmatrix}.$$

定理 (Laplace) 在 n 阶行列式 D 中, 任取 k 行(列), 由这 k 行(列)所组成的一切 k 阶子式与它们的代数余子式的乘积的和等于行列式 D .

例如,

$$D = \begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ -3 & 3 & -7 & 9 & -5 \\ 2 & 0 & 4 & -2 & 1 \\ 3 & -5 & 7 & -14 & 6 \\ 4 & -4 & 10 & -10 & 2 \end{vmatrix},$$

取定 1, 3, 4 行后, D 可以表示为 $C_5^3 = 10$ 个乘积的和.

例6 计算 $D = \begin{vmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 3 & 1 \end{vmatrix}$.

在行列式中取定第一、二行, 得到6个子式:

$$\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 4 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}.$$

它们对应的代数余子式为

$$(-1)^{(1+2)+(1+2)} \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix},$$

$$(-1)^{(1+2)+(1+3)} \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = - \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix}, \dots\dots$$

根据Laplace定理, 得

$$\begin{aligned} D &= (-1) \times (-8) - 2 \times (-3) + 1 \times (-1) \\ &\quad + 5 \times 1 - 6 \times 3 + (-7) \times 1 \\ &= -7. \end{aligned}$$

例7 设

$$D = \begin{vmatrix} \boxed{\begin{matrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{matrix}} & \mathbf{0} \\ c_{11} & \cdots & c_{1k} & \boxed{\begin{matrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{matrix}} \\ \vdots & & \vdots & \vdots \\ c_{n1} & \cdots & c_{nk} & \end{vmatrix},$$

分块法

$$D_1 = \det[a_{ij}] = \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}, \quad D_2 = \det[b_{ij}] = \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix},$$

证明 $D = D_1 D_2$.

思考

设

$$D = \begin{vmatrix} c_{11} & \cdots & c_{1n} & a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{k1} & \cdots & c_{kn} & a_{k1} & \cdots & a_{kk} \\ b_{11} & \cdots & b_{1n} & & & \\ \vdots & \ddots & \vdots & & & \\ b_{n1} & \cdots & b_{nn} & & & \end{vmatrix},$$

$$D_1 = \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}, \quad D_2 = \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix},$$

那么 $D = -D_1 D_2$?

$$D = (-1)^{kn} D_1 D_2.$$



例8 证明

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ 0 & 0 & 0 & 0 & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{vmatrix} = 0$$

证 选定第4, 5, 6行, 在这三行中再任意选定三列一共可以构成 $C_6^3 = 20$ 个不同的三阶行列式, 但每个行列式中至少有一列的元素全部为0, 故 $D = 0$.

例9 设 A, B, C, D 都是 n 阶矩阵, $|A| \neq 0, AC = CA$, 证明:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$$

证 做分块初等行变换化矩阵为上三角分块阵,

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & -CA^{-1}B + D \end{bmatrix}$$

两边再取行列式

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| |-CA^{-1}B + D|$$

$$= |-ACA^{-1}B + AD| = |-CB + AD| = |AD - CB|$$

这里用到了 $AC = CA$.

例10 已知 A, B 为 n 阶矩阵, 证明 $\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A+B||A-B|$.

证 利用左乘和右乘块初等矩阵构造出 $A+B, A-B$ 和上三角块阵. 如第1行加到第2行; 再第2列乘 $(-I)$ 加到第1列.

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} A & B \\ A+B & A+B \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} &= \begin{bmatrix} A & B \\ A+B & A+B \end{bmatrix} \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \\ &= \begin{bmatrix} A-B & B \\ 0 & A+B \end{bmatrix} \end{aligned}$$

上式两边取行列式

$$\text{因为 } \begin{vmatrix} I & 0 \\ I & I \end{vmatrix} = \begin{vmatrix} I & 0 \\ -I & I \end{vmatrix} = 1, \quad \text{所以 } \begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A+B||A-B|.$$

设 A, B 均为 n 阶矩阵, 证明

$$|I_n - AB| = |I_n - BA|.$$

(推论: $I_n - AB$ 可逆当且仅当 $I_n - BA$ 可逆.)

提示:

从分块矩阵 $\begin{bmatrix} I_n & A \\ B & I_n \end{bmatrix}$ 出发,

例11 (抽象型行列式)

设4阶矩阵 $A = [\alpha, \gamma_1, \gamma_2, \gamma_3]$, $B = [\beta, \gamma_1, \gamma_2, \gamma_3]$,
其中 $\alpha, \beta, \gamma_1, \gamma_2, \gamma_3$ 是4维列向量, 且 $|A| = 3$, $|B| = -1$,
求 $|A + 2B|$.

解 $A + 2B = [\alpha + 2\beta, 3\gamma_1, 3\gamma_2, 3\gamma_3].$

因此

$$|A + 2B| = |\alpha + 2\beta, 3\gamma_1, 3\gamma_2, 3\gamma_3|$$

$$= 3^3 |\alpha + 2\beta, \gamma_1, \gamma_2, \gamma_3| \quad \text{因子能提}$$

$$= 3^3 (|\alpha, \gamma_1, \gamma_2, \gamma_3| + 2|\beta, \gamma_1, \gamma_2, \gamma_3|) \quad \text{行列可拆}$$

$$= 27.$$

例12 (抽象型行列式)

已知 A 是3阶矩阵, $\alpha_1, \alpha_2, \alpha_3$ 是3维线性无关的列向量, 若 $A\alpha_1 = \alpha_1 + \alpha_2$, $A\alpha_2 = \alpha_2 + \alpha_3$, $A\alpha_3 = \alpha_3 + \alpha_1$, 求 $|A|$.

解 $A[\alpha_1, \alpha_2, \alpha_3] = [\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_1].$

$$\text{即 } A[\alpha_1, \alpha_2, \alpha_3] = [\alpha_1, \alpha_2, \alpha_3] \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

记 $P = [\alpha_1, \alpha_2, \alpha_3]$, 则 P 可逆, 从而

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2.$$

Key words:

Formula from pivots

The Big Formula

Expansion

Calculations (Tricks)

Homework

See Blackboard

