Step-1

Consider the beetle matrix:

$$A=\begin{bmatrix}0 & 0 & 6\\ \frac{1}{2} & 0 & 0\\ 0 & \frac{1}{3} & 0\end{bmatrix}$$

Let x_k represents the number of beetles that are *k* years old.

So, we get the following equation from the matrix *A*.

$$x_1=6x_3$$
$$x_2=\frac{1}{2}x_1$$
$$x_3=\frac{1}{3}x_2$$

Step-2

To find the distribution of 3,000 beetles for six years, we have to compute

$$A^k\begin{bmatrix}3,000\\ 3,000\\ 3,000\end{bmatrix}$$

For $k=1,2,3,4,5,\text{ and }6$.

For $k=1$, we have

$$A^1\begin{bmatrix}3,000\\ 3,000\\ 3,000\end{bmatrix}=\begin{bmatrix}0 & 0 & 6\\ \frac{1}{2} & 0 & 0\\ 0 & \frac{1}{3} & 0\end{bmatrix}\begin{bmatrix}3,000\\ 3,000\\ 3,000\end{bmatrix}$$
$$=\begin{bmatrix}18,000\\ 1,500\\ 1,000\end{bmatrix}$$

Step-3

Therefore, the population of the beetles for first year is $\boxed{\begin{bmatrix} 18,000 \\ 1,500 \\ 1,000 \end{bmatrix}}$.

Step-4

For $k=2$, we have

$$
\begin{aligned}
A^2\begin{bmatrix} 3,000 \\ 3,000 \\ 3,000 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}^2 \begin{bmatrix} 3,000 \\ 3,000 \\ 3,000 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ \frac{1}{6} & 0 & 0 \end{bmatrix} \begin{bmatrix} 3,000 \\ 3,000 \\ 3,000 \end{bmatrix} \\
&= \begin{bmatrix} 6,000 \\ 9,000 \\ 500 \end{bmatrix}
\end{aligned}
$$

Therefore, the population of the beetles for second year is $\boxed{\begin{bmatrix} 6,000 \\ 9,000 \\ 500 \end{bmatrix}}$.

For $k=3$, we have

$$A^3\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}=\begin{bmatrix}0&0&6\\\frac{1}{2}&0&0\\0&\frac{1}{3}&0\end{bmatrix}^3\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$
$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$
$$=\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$

Therefore, the population of the beetles for third year is $\boxed{\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}}$.

Step-5

For $k=4$, we have

$$A^4\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}=\begin{bmatrix}0&0&6\\\frac{1}{2}&0&0\\0&\frac{1}{3}&0\end{bmatrix}^4\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$
$$=\begin{bmatrix}0&0&6\\\frac{1}{2}&0&0\\0&\frac{1}{3}&0\end{bmatrix}\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$
$$=\begin{bmatrix}18,000\\1,500\\1,000\end{bmatrix}$$

Therefore, the population of the beetles for fourth year is $\boxed{\begin{bmatrix}18,000\\1,500\\1,000\end{bmatrix}}$.

Step-6

For $k=5$, we have

$$A^5\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}=\begin{bmatrix}0&0&6\\\frac{1}{2}&0&0\\0&\frac{1}{3}&0\end{bmatrix}^5\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$
$$=\begin{bmatrix}0&2&0\\0&0&3\\\frac{1}{6}&0&0\end{bmatrix}\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$
$$=\begin{bmatrix}6,000\\9,000\\500\end{bmatrix}$$

Therefore, the population of the beetles for fifth year is $\boxed{\begin{bmatrix}6,000\\9,000\\500\end{bmatrix}}$.

For $k=6$, we have

$$A^6\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}=\begin{bmatrix}0&0&6\\\frac{1}{2}&0&0\\0&\frac{1}{3}&0\end{bmatrix}^6\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$
$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$
$$=\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$

Therefore, the population of the beetles for sixth year is $\boxed{\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}}$.

Step-7

By observing the above distributions, we have seen that after every 3 years we got the initial population.

$$A^3\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}=\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$
$$A^6\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}=\begin{bmatrix}3,000\\3,000\\3,000\end{bmatrix}$$

This only possible if $A^3 = I$.

Thus, $\boxed{A^3 = I}$.