Step-1

Consider two vector spaces $\mathbf{R}^n$ and $\mathbf{R}^m$.

Let A be an m by n matrix. Thus, the matrix A acts on the vectors of $\mathbf{R}^n$ and produces a vector from $\mathbf{R}^m$.

Let the matrix A is of the range r.

Step-2

Suppose $\mathbf{V}$ is the row space of the matrix A and $\mathbf{W}$ is the nullspace of the matrix A. Therefore, it is clear that $\mathbf{V} \cap \mathbf{W} = \{0\}$.

Thus, we should get $\dim \mathbf{V} + \dim \mathbf{W} = \dim(\mathbf{V} + \mathbf{W})$.

Step-3

Since rank of the matrix A is r, there are r independent vectors in the row space of A.

Thus, the dimension of $\mathbf{V}$ is r.

Naturally, there are r independent columns of A. Thus, the dimension of $\mathbf{W}$ is $n-r$.

Step-4

Therefore, we get

$$\begin{aligned}\dim \mathbf{V} + \dim \mathbf{W} &= r + (n-r) \\ &= n\end{aligned}$$

Step-5

We know that row space and nullspace are the orthogonal complements of each other.

Therefore, $\mathbf{V} + \mathbf{W}$ is equal to the entire vector space $\mathbf{R}^n$.

Therefore,

$$\begin{aligned}\dim(\mathbf{V} + \mathbf{W}) &= \dim \mathbf{R}^n \\ &= n\end{aligned}$$

Step-6

Thus, we get $\boxed{\dim \mathbf{V} + \dim \mathbf{W} = \dim(\mathbf{V} + \mathbf{W})}$.