Step-1

A real symmetric matrix A is given by

$$A^H = A$$

We know that the entries of a real symmetric matrix are symmetric to the diagonal.

$$a_{ij} = a_{ji}$$

Consider an n by n symmetric matrix, then there are n entries on the diagonal and $(n-1)+\cdots+1$ entries above the diagonal that can be chosen arbitrary.

Since, other $1+\cdots+(n-1)$ entries below the diagonal are determined by the symmetry of the matrix.

So, there are only

$$n+(n-1)+\cdots+1=\frac{n(n+1)}{2}$$

degrees of freedom in the selection of the n^2 entries in an n by n real symmetric matrix S.

Therefore, dimension of an n by n symmetric matrix is $\boxed{\frac{n(n+1)}{2}}$.

Step-2

If A is a real symmetric matrix and Q is an orthogonal matrix, then we can write:

$$\begin{aligned} A &= A^H \\ &= Q\Lambda Q^H \\ &= \begin{bmatrix} q_1 & \cdots & q_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1{}^H \\ \vdots \\ q_n{}^H \end{bmatrix} \\ &= q_1 q_1{}^H \lambda_1 + \cdots + q_n q_n{}^H \lambda_n \end{aligned}$$

Here matrix $q_n q_n{}^H$ is the projection matrix onto q_n, every symmetric matrix A is a combination of n projections matrices.

If matrix A changes then the projections also changes.

Since the dimension exceeds n, there is no basis of n fixed projection matrices, in the space S of symmetric matrices.