Step-1

Consider the matrix equation,

$$u\begin{pmatrix}1\\2\\3\end{pmatrix}+v\begin{pmatrix}1\\0\\1\end{pmatrix}+w\begin{pmatrix}1\\3\\4\end{pmatrix}=\begin{pmatrix}2\\5\\7\end{pmatrix} \qquad \text{......} \ (1)$$

The objective is to find the solution (u,v,w) for the above system other than $(1,0,1)$.

Step-2

The matrix equation (1) has a solution, when the linear combination of columns in left hand side of equation (1) satisfies the right hand side part of the equation (1).

The system of equations as,

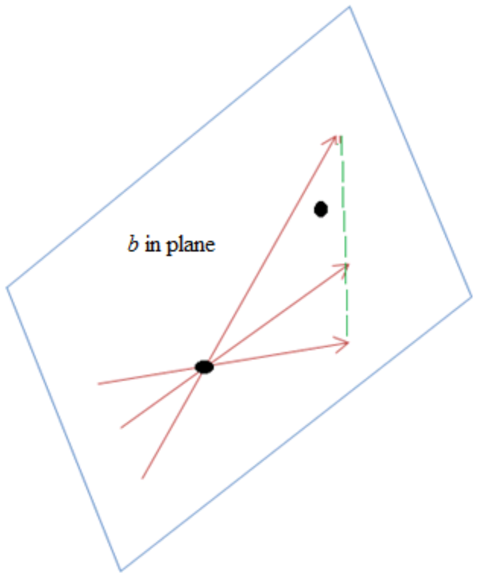
$$\begin{aligned} u+v+w&=2,\\ 2u+\ 3w&=5,\\ 3u+v+4w&=7. \end{aligned}$$

Add first two equations and subtract from the third equation, obtained as $0=0$.

Thus, the equations are consistent.

Step-3

The system has infinitely many solutions shown in the following diagram



Step-4

Now find the solution by Gaussian elimination process.

Consider the system,

$$\begin{aligned} u+v+w&=2,\\ 2u+\ \ 3w&=5,\\ 3u+v+4w&=7. \end{aligned}$$

Subtract two times the first equation from second, and subtract 3 times the first equation from third, obtained as follows:

$$\begin{aligned} u+v+w&=2,\\ -2v+w&=1,\\ -2v+w&=1. \end{aligned}$$

Subtract -1 times the second equation from third equation, obtained as,

$$\begin{aligned} u+v+w&=2,\\ -2v+w&=1,\\ 0&=0. \end{aligned}$$

Put,

$$w=k.$$

$$v = \frac{k-1}{2}$$
$$u = 2 - \frac{k-1}{2} - k$$
$$= \frac{5-3k}{2}$$

So, the solution is,

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{5-3k}{2} \\ \frac{k-1}{2} \\ k \end{bmatrix}$$

These are infinitely many solutions.

Take $k = -1,$ obtained the solution as follows:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{5-3(-1)}{2} \\ \frac{-1-1}{2} \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

Step-5

That is, $4(\text{column1}) + (-1)(\text{column2}) + (-1)(\text{column3}) = b$

Hence, the solution is, $\boxed{(u, v, w) = (4, -1, -1)}$.