Step-1

The objective is to find that under what conditions on b_1, b_2, b_3, b_4 , the following systems are solvable, and we have to find x for the systems:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Step-2

Consider the following system,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Consider

$$A_{1} = \begin{bmatrix} 1 & 2 & b_{1} \\ 2 & 4 & b_{2} \\ 2 & 5 & b_{3} \\ 3 & 9 & b_{4} \end{bmatrix}$$

$$R_{2} - 2R_{1}, \begin{bmatrix} 1 & 2 & b_{1} \\ 0 & 0 & b_{2} - 2b_{1} \\ 0 & 1 & b_{3} - 2b_{1} \\ 0 & 3 & b_{4} - 3b_{1} \end{bmatrix}$$

Step-3

Simplify further:

$$\underbrace{R_2 \leftrightarrow R_3}_{ \begin{array}{cccc} R_2 \leftrightarrow R_3 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \\ \end{array}}_{ \begin{array}{cccc} \end{array}$$

$$\begin{bmatrix} R_1 - 2R_2, \\ 0 & 1 & b_3 - 2b_1 \\ R_4 - 3R_2 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & b_4 - 3b_3 + 3b_1 \end{bmatrix}$$

The system is consistent if

$$b_2 - 2b_1 = 0$$
$$b_4 - 3b_3 + 3b_1 = 0$$

Step-4

Next, we consider the following system,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{bmatrix}$$

Step-5

$$R_1 - 2R_3, \begin{bmatrix} 1 & 0 & 1 & 5b_1 - 2b_3 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_4 - 3b_3 + 3b_1 \end{bmatrix}$$

$$\frac{R_{23}}{0} \begin{bmatrix}
1 & 0 & 1 & (5b_1 - 2b_3) \\
0 & 1 & 1 & (b_3 - 2b_1) \\
0 & 0 & 0 & b_2 - 2b_1 \\
0 & 0 & 0 & b_4 - 3b_3 + 3b_1
\end{bmatrix}$$

There are two pivot columns. Therefore there are two free columns.

Step-6

Therefore, the system is consistent if $b_3 - 2b_1 = 0$, and $b_4 - 3b_3 + 3b_1 = 0$.

Step-7

Or

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix}$$
 is the general solution of the first system.

For the solution of the second system,

$$x_1 + x_3 = 5b_1 - 2b_3$$

$$x_2 + x_3 = b_3 - 2b_1$$

$$b_2 - 2b_1 = 0, b_4 - 3b_3 + 3b_1 = 0$$

This implies:

$$x_1 = -x_3 + 5b_1 - 2b_3$$

$$x_2 = -x_3 + b_3 - 2b_1$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 + 5b_1 - 2b_3 \\ -x_3 + b_3 - 2b_1 \\ x_3 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix}$$

This is the general solution of the second system.