Linear Algebra-A

Assignments - Week 12

Supplementary Problem Set

1. Let $\boldsymbol{A}$ be a square matrix of order n $(n \geq 2)$, with $\boldsymbol{A}^*$ as its adjoint matrix. Please prove the following statement about the rank of $\boldsymbol{A}^*$:

$$\operatorname{rank}(\boldsymbol{A}^*) = \begin{cases} n, & \text{if } \operatorname{rank}(\boldsymbol{A}) = n, \\ 1, & \text{if } \operatorname{rank}(\boldsymbol{A}) = n-1, \\ 0, & \text{if } \operatorname{rank}(\boldsymbol{A}) < n-1. \end{cases}$$

2. (1) Let $\boldsymbol{A} = \begin{bmatrix} 0 & -2 & -2 \\ 2 & -4 & -2 \\ -2 & 2 & 0 \end{bmatrix}$, find all the eigenvalues and eigenvectors of $\boldsymbol{A}$. Is $\boldsymbol{A}$ diagonalizable? If so, write it as $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix.

   (2) The matrix $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 1 \\ x & 4 & y \\ -3 & -3 & 5 \end{bmatrix}$ has 3 linearly independent eigenvectors, and $\lambda = 2$ is an eigenvalue of multiplicity 2 (i.e., its algebraic multiplicity = 2). Find an invertible matrix $\boldsymbol{P}$, such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

3. Let $\lambda_1, \lambda_2, \cdots \lambda_m$ be m *distinct* eigenvalues of an $n \times n$ matrix $\boldsymbol{A}$. The vectors $\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_{r_i}}$ are *independent* eigenvectors corresponding to λ_i $(i = 1,2,\cdots,m)$. Let $\Phi_i = \{\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_{r_i}}\}$ $(i = 1,2,\cdots,m)$. Please show that the set of vectors $\bigcup_{i=1}^{m} \Phi_i$ (with $r_1 + r_2 + \cdots r_m$ vectors totally) is linearly independent.
   注：我们在课上已经证明了属于不同特征值的特征向量是线性无关的。更进一步，本命题是想请大家证明：如果每个特征值都有一个或多个线性无关的特征向量，那么把它们全部并在一起得到的向量组也一定是线性无关的。(这也是我们在将一个矩阵$\boldsymbol{A}$进行对角化时去构造$\boldsymbol{S}$的依据。)

4. (1) Let $\boldsymbol{A}, \boldsymbol{B}$ be square matrices of order n. Please show that if $\lambda_1 (\lambda_1 \neq 0)$ is an eigenvalue of $\boldsymbol{AB}$, then λ_1 is also an eigenvalue of $\boldsymbol{BA}$.
   注：如果$\boldsymbol{A}, \boldsymbol{B}$是同阶方阵，则$\boldsymbol{AB}$的特征值也是$\boldsymbol{BA}$的特征值. 从特征值和特征向量的定义来证明即可. 这个命题中不涉及特征值的重数.

$AB x_1 = \lambda_1 x_1 \Rightarrow BA(Bx_1) = \lambda_1 (Bx_1)$

$\checkmark$

(2) Generally, let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ be an $n \times m$ matrix, and $m \geq n$, then $\boldsymbol{AB}$ and $\boldsymbol{BA}$ have same nonzero eigenvalues. To prove this, please show that $|\lambda \boldsymbol{I} - \boldsymbol{AB}| = \lambda^{m-n}|\lambda \boldsymbol{I} - \boldsymbol{BA}|$.

注： 1、这个定理即：如果$\boldsymbol{A}, \boldsymbol{B}$分别是$m \times n$和$n \times m$的矩阵，则$\boldsymbol{AB}$和$\boldsymbol{BA}$的非零特征值相同．由此可将 3(1)中的结论进一步加强：如果$\boldsymbol{A}, \boldsymbol{B}$是同阶方阵，则$\boldsymbol{AB}$的特征值和$\boldsymbol{BA}$的特征值及其代数重数完全相同．

2、在行列式一章的补充题中曾经证明了这个结论：Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices. Please prove that

$$|\boldsymbol{I}_n - \boldsymbol{AB}| = |\boldsymbol{I}_n - \boldsymbol{BA}|.$$

本周的这个定理可以看作是之前补充题的升级版．

5. ***True or false***: If true, please give a proof. Otherwise, please give a counterexample.
   (1) An idempotent matrix (幂等矩阵，$\boldsymbol{A}^2 = \boldsymbol{A}$) is always diagonalizable. （If this is true, then as an example, a projection matrix is always diagonalizable.）
   注：此题与书上 5.2 节的 38 题类似，可以从空间的维数角度，或是矩阵的 rank 不等式来进行证明．
   (2) If $\boldsymbol{A}^2 = \boldsymbol{I}$ (called "involutory matrix", 对合矩阵), then $\boldsymbol{A}$ is always diagonalizable.
   (3) A 2×2 rotation matrix is always diagonalizable.
   (4) A rank-1 matrix (秩为 1 的矩阵) is always diagonalizable.

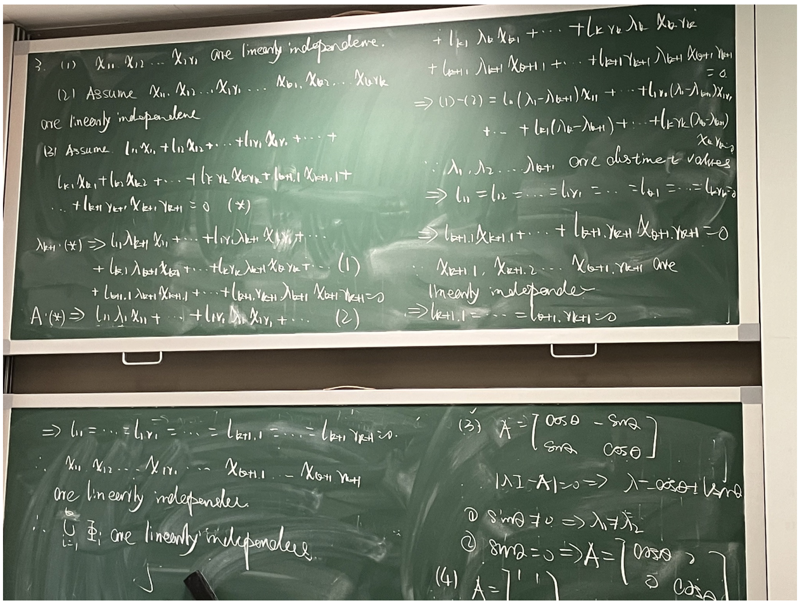
(1) $A^2 = A$. $\text{rank}(A) + \text{rank}(A-I) = n$ ✓

$\Rightarrow [n - \text{rank}(A)] + [n - \text{rank}(A-I)] = n$

$\Rightarrow$ A has n linearly independent eigenvectors

(2) $A^2 = I \rightarrow (A+I)(A-I) = 0$ $\text{rank}(A+I) + \text{rank}(A-I) = n$ ✓

(4) X $\boxed{A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}}$



Q3 answer