Step-1

Consider the matrices,

$A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}$, and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

The objective is to determine the echelon form U, the free variables, and special solutions.

The system $Ax = b$ is consistent when b satisfies $b_2 =$-------, and finds the complete solution.

Step-2

Consider the system,

$$Ax = b$$

$$\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Apply row operation, $R_2 \to R_2 - 2R_1$.

$$\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \end{bmatrix} \qquad \text{......(1)}$$

Hence, the echelon form of matrix A is,

$$U = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-3

Consider the system,

$$Ax = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply row operation,

Apply row operation, $R_2 \to R_2 - 2R_1$.

$$\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equations form is,

$$x_2 + 3x_4 = 0$$

Put, $x_4 = k_1, x_3 = k_2, x_1 = k_3$.

$$x_2 = -3k_1$$

Hence, the free variables are, $\boxed{x_1, x_3, \text{ and } x_4}$.

Step-4

So, the solution is,

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} k_3 \\ -3k_1 \\ k_2 \\ k_1 \end{bmatrix} \\ &= \begin{bmatrix} 0+0+k_3+0 \\ -3k_1+0+0+0 \\ 0+k_2+0+0 \\ k_1+0+0+0 \end{bmatrix} \\ &= k_1 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Therefore, the special solutions are, $\boxed{(0,-3,0,1),(0,0,1,0),(1,0,0,0)}$.

Step-5

The system is consistent if,

$b_2 - 2b_1 = 0.$ From system (1).

Therefore,

$b_2 = 2b_1$

Step-6

Consider the non-homogeneous system,

$Ax = \begin{bmatrix} b_1 \\ 2b_1 \end{bmatrix}$ Since, $b_2 = 2b_1$.

$$\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ 2b_1 \end{bmatrix}$$

Step-7

Apply row operation, $R_2 \to R_2 - 2R_1$.

$$\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_2 + 3x_4 &= b_1 \\ x_2 &= b_1 - 3x_4 \end{aligned}$$

Put, $x_4 = k_1, x_3 = k_2, x_1 = k_3$.

$x_2 = 1 - 3k_1$

Therefore, the solution is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_3 \\ b_1 - 3k_1 \\ k_2 \\ k_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} k_3 \\ -3k_1 \\ k_2 \\ k_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-8

Therefore, the complete solution is,

$$x_{\text{complete}} = x_{\text{particular}} + x_{\text{nullspace}}$$
$$x = x_p + x_n$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$