1. Label the following statements as **True** or **False. Along with your answer, provide an informal proof, counterexample, or other explanation.**

    1. The sum of two positive operators on a finite-dimensional complex inner product space is positive.

       True. Let A, B be two positive operators, then $(A+B)^* = A+B$, $\langle (A+B)x, y\rangle = \langle Ax, y\rangle + \langle Bx, y\rangle \geqslant 0$ for all $v \in V$, then $A + B$ is positive.

    2. Let V be a 5-dimensional vector space and $T \in \mathcal{L}(V)$. Then there exists a 3-dimensional subspace U of V invariant under T.

       True. If $T_{\mathbf{C}}$ has a complex eigenvalue, then T has a two dimensional invariant subspace U_1. And since T must be at least one real eigenvalue, then T has a one-dimensional invariant subspace U_2, $U = U_1 \oplus U_2$. If $T_{\mathbf{C}}$ doesn't have a complex eigenvalue, then all eigenvalues of $T_{\mathbf{C}}$ are real, and T has 5 real eigenvalues. Assume the Jordan basis of T is $u_1, \cdots, u_5$, $U = \operatorname{span}\{u_3, u_4, u_5\}$.

    3. Any polynomial of degree n with leading coefficients $(-1)^n$ is the characteristic polynomial of some linear operators.

       True.

       $$\begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & 0 & & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 \quad 0 & -a_{n-1} \end{bmatrix}$$

    4. If x, y and z are vectors in an inner product space such that $\langle x, y\rangle = \langle x, z\rangle$, then $y = z$.

       False. $\langle x, y\rangle = \langle x, z\rangle \Rightarrow \langle x, y - z\rangle = 0 \Rightarrow x \perp (y - z)$.

    5. Every normal operator is diagonalizable.

       False. $T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is normal since $T^*T = TT^*$, but T is not diagonalizable.

2. Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is defined by

   $$T(x_1, x_2, x_3) = (2x_1, x_2 - x_3, x_2 + x_3).$$

    1. Determine the eigenspace of T corresponding to each eigenvalue.

       Sol: $T(x_1, x_2, x_3) = (2x_1, x_2 - x_3, x_2 + x_3) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$, $|\lambda I - A| =$

$(\lambda-2)(\lambda-(1+i))(\lambda-(1-i))$, so the eigenvalues are $2, 1+i, 1-i$, and the corresponding eigenvectors are $k_1\begin{bmatrix}1\\0\\0\end{bmatrix}$, $k_2\begin{bmatrix}0\\1\\i\end{bmatrix}$, $k_3\begin{bmatrix}0\\1\\-i\end{bmatrix}$.

2. Find the Jordan form and a Jordan basis of T.

   Sol: The Jordan form is $\begin{bmatrix}2 & & \\ & 1+i & \\ & & 1-i\end{bmatrix}$, the Jordan basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\i\end{bmatrix}$, $\begin{bmatrix}0\\1\\-i\end{bmatrix}$.

3. Find the minimal polynomial of T.

   Sol: The minimal polynomial is $(\lambda-2)(\lambda-(1+i))(\lambda-(1-i))$.

4. Find the trace of T, trace T.

   Sol: $\operatorname{Tr}(T)=4$.

5. Find the determinant of T, $\det T$.

   Sol: $\det T=4$.

3. Suppose V is a finite-dimensional inner product space, $T\in\mathcal{L}(V)$ is normal, and U is a subspace of V that is invariant under T. Show that $U^\perp$ is invariant under T.

   Sol: See textbook 9.30 (a).

4. Let T be a linear operator on a finite-dimensional vector space V, and let v be a nonzero vector in V. The subspace

$$U=\operatorname{span}(\{v, Tv, T^2v, \cdots\})$$

is called the T-cyclic subspace of V generated by v.

1. Show that U is a finite-dimensional invariant subspace of V.

   Sol: Since V is finite-dimensional, U is also finite-dimensional, $U=\operatorname{span}(\{v, Tv, \cdots, T^{k-2}v\})\neq\operatorname{span}(\{v, Tv, \cdots, T^{k-2}v\})$.

   $\forall u=a_0v+\cdots+a_{k-1}T^{k-1}v\in U$, then $Tu=a_0Tv+\cdots+a_{k-1}T^kv\in\operatorname{span}(\{v, Tv, \cdots,\})=U$, then U is an invariant subspace of V.

2. Let $k=\dim U$. Show that $\{v, Tv, T^2v, \cdots, T^{k-1}v\}$ is a basis for U.

   Sol: We can show $v, Tv, \cdots, T^{k-1}v$ is linearly independent.

   According to 2.21, $\exists l$, s.t. $T^lv\in\operatorname{span}\{v, Tv, \cdots, T^{l-1}v\}$, then $T^lv=b_0v+\cdots+b_{l-1}T^{l-1}v$.

   $\forall m>l, T^mv=T^{m-l}T^lv=T^{m-l}(b_0v+\cdots+b_{l-1}T^{l-1}v)=b_0T^{m-l}v+\cdots+b_{l-1}T^{m-1}v=\in\operatorname{span}\{v,\cdots,T^{l-1}v\}$, then $U=\operatorname{span}\{v,\cdots,T^{l-1}v\}$, which contradicts to $\dim U=k$.

3. If $a_0v + a_1Tv + a_2T^2v + \cdots + a_{k-1}T^{k-1}(v) + T^k(v) = 0$, show that the characteristic polynomial of $T|_U$ is

$$f(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k).$$

Sol: $a_0v + \cdots + T^kv = 0 \Rightarrow (a_0I + \cdots + T^k)u = 0$, $\forall u \in U$, then $a_0I + a_1T|_U + \cdots + (T|_U)^k = 0$.

Let $p(\lambda)$ be the minimal polynomial of $T|_U$, $q(\lambda) = a_0 + a_1\lambda + \cdots + \lambda^k$, then $p(\lambda)|q(\lambda)$.

If $\deg p(\lambda) < k$, then $(b_0I + \cdots + b_{l-1}(T|_U)^{l-1} + (T|_U)^l)v = 0 \Rightarrow v, Tv, \cdots, T^lv$ is linearly dependent, which is a contradiction!

Thus $\deg p(\lambda) = k$, $p(\lambda) = q(\lambda)$, the characteristic polynomial $p(\lambda) = q(\lambda)$.

4. Let $g(t)$ be the characteristic polynomial of T, show that $g(T) = 0$, where 0 is the zero operator. That is, T "satisfies" its characteristic equation.

Sol: Assume $g(T) \neq 0$, $\exists u \neq 0$, s.t. $g(T)u \neq 0$. Let $U = \text{span}(\{u, Tu, \cdots\})$, $p(\lambda)$ be the characteristic polynomial of $T|_U$. According to 3, we have $p(T)u = 0$. Since $p(\lambda)|g(\lambda)$, $g(\lambda) = p(\lambda)q(\lambda) \Rightarrow g(T)u = q(T)p(T)u = 0$, which is a contradiction!

5. If $\mathbf{F} = \mathbf{C}$, show that T is an isometry if and only if T is normal and $|\lambda| = 1$ for every eigenvalue λ of T.

Sol: "$\Rightarrow$" T is isometry, then $T^*T = TT^* = I$, T is normal. According to 7.43, we have $|\lambda| = 1$ for every eigenvalue.

"$\Leftarrow$" Since T is normal, according to 7.24, V has an orthonormal basis consisting of all eigenvectors of T. Since $|\lambda| = 1$ for every eigenvalue, by 7.43, T is an isometry.

6. Let $\mathcal{P}_2(\mathbf{R})$ and $\mathcal{P}_1(\mathbf{R})$ be the polynomial spaces with inner products defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx, \quad f, g \in \mathcal{P}_2(\mathbf{R})$$

Let $T : \mathcal{P}_2(\mathbf{R}) \to \mathcal{P}_1(\mathbf{R})$ be the linear operator defined by $T(f(x)) = f'(x)$.

1. Find orthonormal bases $\{v_1, v_2, v_3\}$ for $\mathcal{P}_2(\mathbf{R})$ and $\{u_1, u_2\}$ for $\mathcal{P}_1(\mathbf{R})$.

   Sol: $1, x, x^2$ is a basis of $\mathcal{P}_2(\mathbf{R})$, using Gram-Schmidt process, we have

$$v_1 = \sqrt{\frac{8}{45}}\Big(x^2 - \frac{1}{3}\Big), \quad v_2 = \sqrt{\frac{3}{2}}x, \quad v_3 = \frac{1}{\sqrt{2}},$$

   so $\{v_1, v_2, v_3\}$ is an orthonormal basis for $\mathcal{P}_2(\mathbf{R})$ and let $u_1 = v_2$, $u_2 = v_3$,$\{u_1, u_2\}$ is an orthonormal basis for $\mathcal{P}_1(\mathbf{R})$.

2. Find $p \in \mathcal{P}_1(\mathbf{R})$ that makes

$$\int_{-1}^{1} |x^5 - p(x)|^2 dx$$

as small as possible.

Sol: $p(x) = \frac{3}{7}x$.

3. Find the singular values σ_1, σ_2 of T such that $T(v_i) = \sigma_i u_i$, $i = 1, 2$, and $T(v_3) = 0$.

   Sol: $Tv_3 = 0, Tv_1 = \sqrt{\frac{3}{2}} = \sqrt{3}u_1, Tv_2 = \frac{3\sqrt{5}}{\sqrt{2}}x = \sqrt{15}u_2$, then $\mathcal{M}(T; v_1, v_2, v_3; u_1, u_2) = \begin{bmatrix} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$, denoted it as A.

   $A^T A = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the eigenvalues of $A^T A$ are $15, 3, 0$, so the singular values are $\sigma_1 = \sqrt{15}, \sigma_2 = \sqrt{3}$.

7. Let V be a real inner product space. A function $f : V \to V$ is called a rigid motion if

$$\|f(x) - f(y)\| = \|x - y\|$$

for all $x, y \in V$. For example, any **isometry** on a finite-dimensional real inner product space is a rigid motion. Another class of rigid motions is the translations. A function $g : V \to V$, where V is a real inner product space, is called a translation if there exists a vector $v_0 \in V$ such that $g(v) = v + v_0$ for all $v \in V$. Let $f : V \to V$ be a rigid motion on a finite-dimensional real inner product space V, show that there exists a unique isometry T on V and a unique translation g on V such that $f = g \circ T$.

Sol: Let $g(x) = f(x) + f(0)$, $g^{-1}(x) = f(x) - f(0)$. We want to show $g^{-1} \circ f$ is an isometry, i.e. $g^{-1} \circ f$ is linear and $\|g^{-1} \circ f(x)\| = \|x\|$ holds for amy $x \in V$.

Step 1: prove $g^{-1} \circ f$ is linear.

Firstly, prove $g^{-1} \circ f(x + y) = g^{-1} \circ f(x) + g^{-1} \circ f(y)$ holds for any $x, y \in V$. WTS: $f(x + y) - f(0) = f(x) - f(0) + f(y) - f(0)$, i.e. $f(x + y) - f(y) = f(x) - f(0)$. $\|f(x + y) - f(y)\| = \|x\| = \|f(x) - f(0)\|$, so we only need to prove that the directions of $f(x + y) - f(y)$ and $f(x) - f(0)$ are same. Since

$$\begin{aligned}
&\langle f(x) - f(0), f(x + y) - f(y)\rangle = \langle f(x), f(x + y)\rangle - \langle f(x), f(y)\rangle - \langle f(0), f(x + y)\rangle + \langle f(0), f(y)\rangle \\
&= \tfrac{1}{4}(\|f(x) + f(x + y)\|^2 - \|f(x) - f(x + y)\|^2) - \tfrac{1}{4}(\|f(x) + f(y)\|^2 - \|f(x) - f(y)\|^2) \\
&\quad -\tfrac{1}{4}(\|f(0) + f(x + y)\|^2 - \|f(0) - f(x + y)\|^2) + \tfrac{1}{4}(\|f(0) + f(y)\|^2 - \|f(0) - f(y)\|^2) \\
&= \tfrac{1}{2}\langle f(x) - f(0), f(x + y) - f(y)\rangle + \tfrac{1}{2}\|x\|^2
\end{aligned}$$

then $\langle f(x) - f(0), f(x + y) - f(y)\rangle = \|x\|^2 = \|f(x) - f(0)\| \cdot \|f(x + y) - f(y)\|$, i.e. the Cauchy-Schwarz inequality takes "=" ! So the directions of $f(x+y)-f(y)$ and $f(x)-f(0)$ are the same, and according to their norms are also the same, we have $f(x) - f(0) = f(x + y) - f(y)$, i.e. $g^{-1} \circ f(x + y) = g^{-1} \circ f(x) + g^{-1} \circ f(y)$ holds for any $x, y \in V$.

Secondly, prove $g^{-1} \circ f(\lambda x) = \lambda g^{-1} \circ f(x)$ holds for any $x \in V$, $\lambda \in \mathbb{R}$.

WTS: $f(\lambda x) - f(0) = \lambda(f(x) - f(0))$. Since

$$\|f(\lambda x) - f(0)\| = |\lambda| \cdot \|x\| = |\lambda| \cdot \|f(x) - f(0)\| = \|\lambda(f(x) - f(0))\|.$$

Similarly, we only need to prove the directions of $f(\lambda x) - f(0)$ and $\lambda(f(x) - f(0))$ are the same.

$$\begin{aligned}
&\langle f(\lambda x) - f(0), \lambda(f(x) - f(0))\rangle = \lambda\Big(\langle f(\lambda x), f(x)\rangle - \langle f(\lambda x), f(0)\rangle - \langle f(0), f(x)\rangle + \langle f(0), f(0)\rangle\Big) \\
&= \lambda\Big(\tfrac{1}{4}(\|f(\lambda x) + f(x)\|^2 - \|f(\lambda x) - f(x)\|^2) - \tfrac{1}{4}(\|f(\lambda x) + f(0)\|^2 - \|f(\lambda x) - f(0)\|^2) \\
&\quad -\tfrac{1}{4}(\|f(0) + f(x)\|^2 - \|f(0) - f(x)\|^2) + \|f(0)\|^2\Big) \\
&= \lambda\Big(\tfrac{1}{2}\langle f(\lambda x) - f(0), f(x) - f(0)\rangle + \tfrac{1}{2}\lambda\|x\|^2
\end{aligned}$$

then $\langle f(\lambda x) - f(0), \lambda(f(x) - f(0))\rangle = \lambda^2\|x\|^2 = \|f(\lambda x) - f(0)\| \cdot \|\lambda(f(x) - f(0))\|$.i.e. the Cauchy-Schwarz inequality takes "=" !, so we have $g^{-1} \circ f(\lambda x) = \lambda g^{-1} \circ f(x)$ for all $x \in V$, $\lambda \in \mathbb{R}$.

Step 2: prove $g^{-1} \circ f$ is isometry.

$\forall x \in V$, $\|g^{-1} \circ f\| = \|f(x) - f(0)\| = \|x - 0\| = \|x\|$, so it's isometry.

Let $g^{-1} \circ f = T$, we have $f = g \circ T$, where g is a translation, T is an isometry, so we have finished the proof of the existence of g and T.

Step 3: show the uniqueness.

Assume $f = g_1 \circ T_1 = g_2 \circ T_2$, where $g_1(v) = v + v_1$, $g_2(v) = v + v_2$ are two translation, T_1, T_2 are ismoetry. We have

$$g_2^{-1} \circ g_1 = T_2 \circ T_1^{-1}, \quad g_2^{-1} \circ g_1(v) = g_2^{-1}(v + v_1) = v + v_1 - v_2$$

Since $T_2 \circ T_1^{-1}$ is linear, $g_2^{-1} \circ g_1$ is also lienar, then $g_2^{-1} \circ g_1(0) = 0 + v_1 - v_2 = 0 \Rightarrow v_1 = v_2 \Rightarrow g_1 = g_2$, so $T_2 \circ T_1^{-1} = g_2^{-1} \circ g_1 = I \Rightarrow T_1 = T_2$.