Step-1

(a) $\square^4$ is a vector space. S be a subspace of vectors for which $x_1 = 2x_4$

That means there is a dependence of 1st and 4th entries of the vector.

Consequently, the 2nd and 3rd entries are free variables.

In other words, 2nd and 3rd may be any real numbers.

We know that any real number in 2nd place of the vector in $\square^4$ can be spanned by the standard vector $(0,1,0,0)$.

Similarly, any real number occupying 3rd place is spanned by $(0,0,1,0)$

Putting $x_1 = 1$ we get $x_4 = 2$ while other entries are zero in that vector.

Therefore the basis of the subspace satisfying $x_1 = 2x_4$ is $\{(2,0,0,1), (0,1,0,0), (0,0,1,0)\}$

Step-2

(b) S be the subspace of $\mathbb{R}^4$, the vectors for which $x_1 + x_2 + x_3 = 0$ and $x_3 + x_4 = 0$

$x_3 = -x_4$

Using this in the first condition, we get $x_1 + x_2 - x_4 = 0$

$\Rightarrow \; x_1 = -x_2 + x_4$

Thus $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix}$

$= x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ spans S and is linearly independent

Therefore, $\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}\right\}$ is the basis for the subspace S satisfied by $x_1+x_2+x_3=0$ and $x_3+x_4=0$

Step-3

(c) $(1,1,1,1)=(2,3,4,5)-(1,2,3,4)$

That is one vector is a linear combination of other vectors.

Therefore, the given vectors are linearly dependent.

So, we choose any vectors from this given set which are independent and thus form a basis to the subspace of $\square^4$

$(1,1,1,1),(1,2,3,4)$ cannot be written as the scalar multiples of each other.

So, these are linearly independent.

Therefore, they form a basis to the subspace of $\square^4$

This subspace can be written as $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+t\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$

$x_1=1+t, x_2=1+2t, x_3 1+3t, x_4=1+4t$

$\Rightarrow x_2=2x_1-1, x_3=3x_1-2, x_4=4x_1-3$