

## Step-1

To find the closest straight line to the parabola  $y = x^2$  over  $-1 \leq x \leq 1$

Let the closest straight line to the parabola  $y = x^2$  over  $-1 \leq x \leq 1$  be  $y = C + Dx$

Then by the least squares, the equation  $A^T A \hat{x} = A^T b$  is

$$\begin{bmatrix} (1,1) & (1,x) \\ (x,1) & (x,x) \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} (1,x^2) \\ (x,x^2) \end{bmatrix}$$

â€ˆâ€ˆâ€ˆ (1)

## Step-2

Now, calculate inner products that are used in formula;

$$\begin{aligned} (1,1) &= \int_{-1}^1 1 \cdot 1 \, dx \\ &= [x]_{-1}^1 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

## Step-3

Again, calculate other inner product

$$\begin{aligned} (x,1) &= \int_{-1}^1 x \cdot 1 \, dx \\ &= \left[ \frac{x^2}{2} \right]_{-1}^1 \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

Similarly  $(1,x) = 0$

$$\begin{aligned}
 (x, x) &= \int_{-1}^1 x^2 dx \\
 &= \left[ \frac{x^3}{3} \right]_{-1}^1 \\
 &= \frac{1}{3} + \frac{1}{3} \\
 &= \frac{2}{3}
 \end{aligned}$$

### Step-4

Now, calculate inner product on R.H.S;

$$\begin{aligned}
 (1, x^2) &= \int_{-1}^1 1 \cdot x^2 dx \\
 &= \left[ \frac{x^3}{3} \right]_{-1}^1 \\
 &= \frac{1}{3} + \frac{1}{3} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 (x, x^2) &= \int_{-1}^1 x^3 dx \\
 &= \left[ \frac{x^4}{4} \right]_{-1}^1 \\
 &= \frac{1}{4} - \frac{1}{4} \\
 &= 0
 \end{aligned}$$

### Step-5

Substitute inner product values obtained above in (1), and get;

$$\begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$$

This implies;

$$2C = \frac{2}{3}$$

$$\frac{2}{3}D = 0$$

Thus,

$$C = \frac{1}{3}$$

$$D = 0$$

### Step-6

Therefore,

$$y = C + Dx$$

$$= \frac{1}{3} + 0x$$

$$= \frac{1}{3}$$

Hence the closest straight line to the parabola is  $y = \frac{1}{3}$