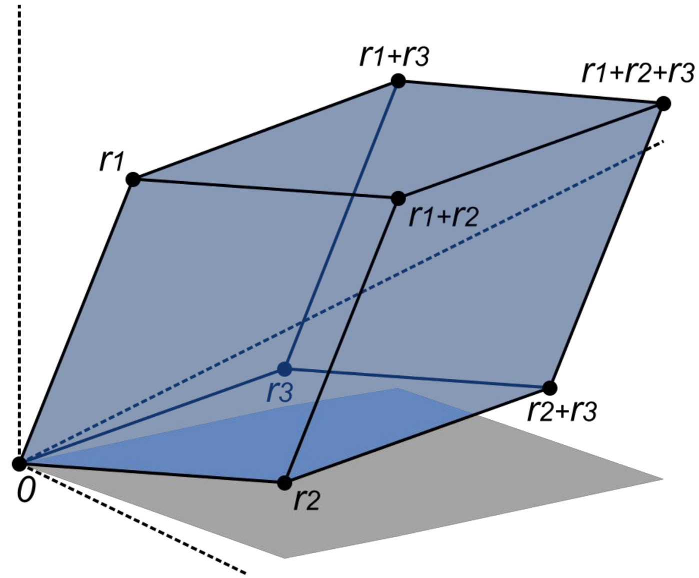
4 Determinants（行列式）

4.1-2

DETERMINANTS AND PROPERTIES

- Introduction
- Definition
- Properties
- Calculations



I. Introduction

Using elimination to solve the system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then the solution is

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{\boxed{a_{11}a_{22} - a_{12}a_{21}}}, \qquad x_2 = \frac{a_{11}b_2 - b_1a_{21}}{\boxed{a_{11}a_{22} - a_{12}a_{21}}}.$$

The denominator can be determined by the 4 numbers.

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

is the determinant of this 2 by 2 coefficient matrix.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1,} \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2,} \end{cases}$$

$$D = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} \qquad D = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & \boxed{a_{22}} \end{vmatrix}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \qquad D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

$$x_1 = \frac{\boxed{b_1 a_{22} - a_{12} b_2}}{a_{11}a_{22} - a_{12}a_{21}}, \qquad x_2 = \frac{\boxed{a_{11} b_2 - b_1 a_{21}}}{a_{11}a_{22} - a_{12}a_{21}}.$$

So the solution is $x_1 = \dfrac{D_1}{D}, \quad x_2 = \dfrac{D_2}{D}.$

For a system of linear equations in 3 unknowns:

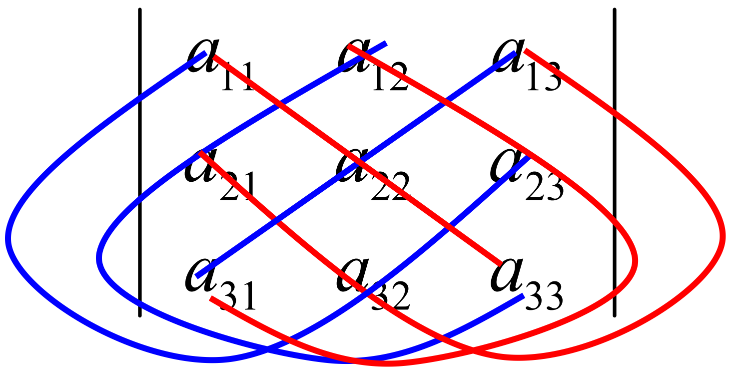
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3, \end{cases}$$

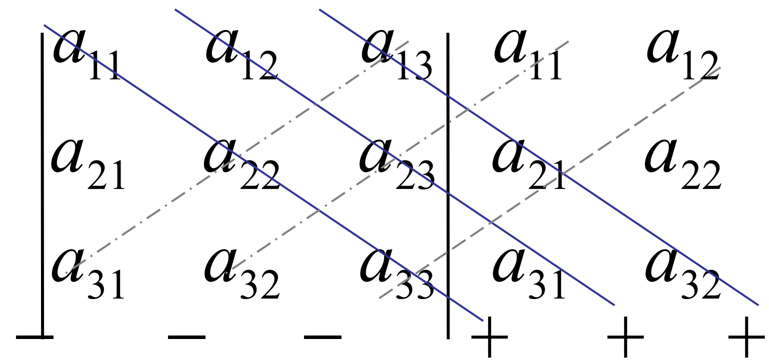
By eliminating x_2, x_3,

$$(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32})x_1$$
$$= b_1a_{22}a_{33} + b_3a_{12}a_{23} + b_2a_{13}a_{32} - b_3a_{13}a_{22} - b_2a_{12}a_{33} - b_1a_{23}a_{32}.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

(1) 对角线法则 (又称 沙路法, *Sarrus' rule*)





$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

- **The determinant is a number.**

- **6 terms (+ or −)**

For example, the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

has determinant $|\boldsymbol{A}|$
$= (1 + 4 + 3) - (6 + 2 + 1) = -1.$

(2) 展开法则 (*expansion rule*)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underline{a_{11}a_{22}a_{33}} + \underline{a_{12}a_{23}a_{31}} + \underline{a_{13}a_{21}a_{32}} - \underline{a_{11}a_{23}a_{32}} - \underline{a_{12}a_{21}a_{33}} - \underline{a_{13}a_{22}a_{31}}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Use determinants to solve the system of linear equations:

$$\begin{cases} a_{11}x_1 + \boxed{a_{12}}x_2 + \boxed{a_{13}}x_3 = \boxed{b_1}, \\ a_{21}x_1 + \boxed{a_{22}}x_2 + \boxed{a_{23}}x_3 = \boxed{b_2}, \\ a_{31}x_1 + \boxed{a_{32}}x_2 + \boxed{a_{33}}x_3 = \boxed{b_3}, \end{cases}$$

If the determinant $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,$

Let $D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}.$

the system has unique solution.

That is, $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\quad D = \begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ \boxed{a_{21}} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix}$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix},$$

Then the solution to this system is

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}.$$

II. Definition and expansion (行列式的定义与展开法则)

Definition 1 We now study *the determinant of a **square** matrix*. For

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the **determinant** (行列式) of $\boldsymbol{A}$ is defined as $ad - bc$, denoted by $|\boldsymbol{A}|$ or $\det(\boldsymbol{A})$.

For a 3×3 matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

the determinant $|\boldsymbol{A}|$ equals

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23}.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Let $\boldsymbol{A}$ be a matrix of degree n:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}]_{n\times n}$$

For $1 \le i, j \le n$, let $\boldsymbol{M}_{ij}$ be the $(n-1)\times(n-1)$ matrix resulted from deleting the i-th row and j-th column of $\boldsymbol{A}$.

Definition 2 Define the **determinant** of a matrix $\boldsymbol{A}$ of degree n as

$$|\boldsymbol{A}| = a_{11}|\boldsymbol{M}_{11}| - a_{12}|\boldsymbol{M}_{12}| + \cdots + (-1)^{1+n}a_{1n}|\boldsymbol{M}_{1n}|.$$

We make an observation. If $|\boldsymbol{A}|$ is of the form

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} \quad \textbf{\textit{or}} \quad \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \textbf{\textit{or}} \quad \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

then $|\boldsymbol{A}| = a_{11}a_{22}\ldots a_{nn}$.

If $\boldsymbol{A}$ is an *upper triangular* matrix or *lower triangular* matrix (or *diagonal matrix*), then $|\boldsymbol{A}|$ equals the product of diagonal entries of the matrix.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & 0 & \cdots & 0 \end{vmatrix} = ?$$

(reverse-triangular matrix)

Matrix & Determinant

Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

	Matrix A	Determinant $\lvert A\rvert$
外观	行数为m，列数为n	行数与列数相等
	中括号	竖线
本质	表示mn个数	表示1个数

Expansion of det$A$ in cofactors (A的行列式用代数余子式展开)

The definition of determinant is expanded along **row 1**. Actually it can be extended along ***any row, or any column***, resulting in same value of the determinant.

Theorem 1 *The determinant of* ***A*** *can be calculated by expanding along* ***row i***,

$$|\boldsymbol{A}| = (-1)^{i+1} a_{i1} |\boldsymbol{M}_{i1}| + (-1)^{i+2} a_{i2} |\boldsymbol{M}_{i2}| + \cdots + (-1)^{i+n} a_{in} |\boldsymbol{M}_{in}|,$$

and by expanding along ***column j***,

$$|\boldsymbol{A}| = (-1)^{1+j} a_{1j} |\boldsymbol{M}_{1j}| + (-1)^{2+j} a_{2j} |\boldsymbol{M}_{2j}| + \cdots + (-1)^{n+j} a_{nj} |\boldsymbol{M}_{nj}|.$$

Note: The determinant of the submatrix $\boldsymbol{M}_{ij}$ with the correct sign is also called the **cofactor** (代数余子式), denoted by $C_{ij} = (-1)^{i+j} |\boldsymbol{M}_{ij}|$.

Pay attention to the sign!

For example,

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \quad \boxed{\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}}$$

Example 1 Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 2 & 3 \end{bmatrix}$.

Notice that $a_{12} = a_{14} = 0$. The determinant

$$|\boldsymbol{A}| = |\boldsymbol{M}_{11}| + 3|\boldsymbol{M}_{13}|$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix} + 3 \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 3 \end{vmatrix}$$

$$= (-2 - 3) + 3 \times (3) = 4.$$

Method: To find the value of determinant, choose a row or a column which has most entries equal to 0 to expand.

Exercise:

Please calculate

$$D = \begin{vmatrix} 5 & 3 & -1 & 2 & 0 \\ 1 & 7 & 2 & 5 & 2 \\ 0 & -2 & 3 & 0 & 0 \\ 0 & -4 & -1 & 0 & 0 \\ 0 & 2 & 3 & 5 & 0 \end{vmatrix}.$$

$$D = \begin{vmatrix} 5 & 3 & -1 & 2 & 0 \\ 1 & 7 & 2 & 5 & 2 \\ 0 & -2 & 3 & 0 & 0 \\ 0 & -4 & -1 & 0 & 0 \\ 0 & 2 & 3 & 5 & 0 \end{vmatrix}.$$

Corollary

If a matrix $\boldsymbol{A}$ *has a zero-row or a zero-column, then* $|\boldsymbol{A}| = 0.$

Theorem 2

A permutation matrix has determinant 1 *or* -1.

Recall that for a permutation matrix (若干初等对换矩阵的乘积), each row and each column has exactly one non-zero entry, which is 1.

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad |\boldsymbol{I}| = 1.$$

$$\boldsymbol{P}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{P}_1 = \boldsymbol{P}_{13}. \qquad |\boldsymbol{P}_1| = -1.$$

$$\boldsymbol{P}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{P}_2 = \boldsymbol{P}_{23}\boldsymbol{P}_{12}. \qquad |\boldsymbol{P}_2| = 1.$$

III. Properties of Determinants (行列式的性质)

Let $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$,

then $|\boldsymbol{AB}| = \begin{vmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{vmatrix}$

$$= (ax+bz)(cy+dw) - (ay+bw)(cx+dz)$$

$$= adxw + bcyz - adyz - bcxw.$$

On the other hand, we have

$$|\boldsymbol{A}||\boldsymbol{B}| = (ad-bc)(xw-yz)$$

$$= adxw + bcyz - adyz - bcxw.$$

Thus $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$.

This is actually true for the general case (not only for the degree 2 case).

Theorem 3：Product (矩阵乘积的行列式)

$|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are both n by n matrix.

- Recall that for a square matrix $\boldsymbol{A}$,

either $\boldsymbol{A} = \boldsymbol{LU}$ (LU factorization without permutations),

or $\boldsymbol{PA} = \boldsymbol{LU}$ (LU factorization with permutations).

By Theorem 2, either $|\boldsymbol{A}| = |\boldsymbol{L}||\boldsymbol{U}|$, or $-|\boldsymbol{A}| = |\boldsymbol{L}||\boldsymbol{U}|$.

Then $\boldsymbol{A}$ is invertible if and only if $\boldsymbol{U}$ is invertible.

(*determinant* = $\pm$ *product of the pivots*)

Theorem 4

A matrix $\boldsymbol{A}$ is invertible if and only if $|\boldsymbol{A}| \neq 0$.

If $\boldsymbol{A}$ is invertible, then $|\boldsymbol{A}^{-1}| = 1/|\boldsymbol{A}|$.

Remark $\quad |\boldsymbol{A}^k| = |\boldsymbol{A}|^k$.

Exercise Compute:

$$(1)\ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$(2)\ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} =$$

$$(3)\ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} =$$

Conclusion:
The determinant of an elementary matrix is not zero.
(In fact, every elementary matrix is invertible).

Let

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad D^{\mathrm{T}}=\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix},$$

Obviously, $(D^{\mathrm{T}})^{\mathrm{T}}=D$.

Actually, we also have $D^{\mathrm{T}}=D$.

Property 1 The transpose of $\boldsymbol{A}$ has the same determinant as $\boldsymbol{A}$ itself. (行列式与它的转置行列式相等.)

(注　行列式中行与列具有同等的地位, 因此行列式的性质, 凡是对行成立的对列也同样成立.)

- If $\boldsymbol{Q}$ is an orthogonal matrix (i.e. $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}=\boldsymbol{I}$), then $|\boldsymbol{Q}|$ is either 1 or -1.

Property 2 The determinant changes sign when two rows (or columns) are exchanged. (互换行列式的两行(列)，行列式变号.)

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad D_1 = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tn} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

$$D = -D_1.$$

Exercise

$$\begin{vmatrix} 6 & 7 & 7 & 7 & 6 \\ 6 & 7 & 7 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 7 & 2 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = ?$$

$$\begin{vmatrix} 5 & 5 & 10 & 10 & 10 \\ 1 & 1 & 2 & 2 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 4 & 3 & 6 & 1 & 0 \\ 1 & 4 & 2 & 3 & 1 \end{vmatrix} = ?$$

Corollary If two rows (or columns) of $\boldsymbol{A}$ are equal, then $|\boldsymbol{A}| = 0$.

(如果行列式有两行(列)完全相同,则此行列式为零.)

Property 3 ***Scalar multiplication*****:** 行列式的某一行(列)中所有的元素都乘以同一数 k, 等于用数 k 乘此行列式.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Be careful!

$|k\boldsymbol{A}| = k|\boldsymbol{A}|$??

$|k\boldsymbol{A}| = k^n|\boldsymbol{A}|$

Property 4 ***Vector addition*****:** 若行列式的某一列(行)的元素都是两数之和:

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i}+a'_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i}+a'_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni}+a'_{ni} & \cdots & a_{nn} \end{vmatrix},$$

则 D 等于下列两个行列式之和:

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & \cdots & a'_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a'_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a'_{ni} & \cdots & a_{nn} \end{vmatrix}.$$

Exercise: **True or false?**

$$\begin{vmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix}.$$

Property 5 倍加变换： 将行列式的某一列(行)的各元素乘以同一数然后加到另一列(行)对应的元素上去，行列式不变.

$$
\begin{vmatrix}
a_{11} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2i} & \cdots & a_{2j} & \cdots & a_{2j} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{n1} & \cdots & a_{ni} & \cdots & a_{nj} & \cdots & a_{nj}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
a_{11} & \cdots & a_{1i} + ka_{1j} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2i} + ka_{2j} & \cdots & a_{2j} & \cdots & a_{2j} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{n1} & \cdots & a_{ni} + ka_{nj} & \cdots & a_{nj} & \cdots & a_{nj}
\end{vmatrix}.
$$

Summary of

The properties of Determinant

(可用于计算)

转置不改

换行反号

因子能提

行列可拆

倍加不变

We consider the effects of elementary operations on determinants.

Example 2 Compute the determinant of the matrix $\boldsymbol{A}$, where

$$\boldsymbol{A}=\begin{bmatrix}1&-4&2\\-2&8&-9\\-1&7&0\end{bmatrix}.$$

Solution *The strategy is to reduce* $\boldsymbol{A}$ *to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries.*

The first two row replacements in column 1 do not change the determinant: (倍加不变)

$$|\boldsymbol{A}|=\begin{vmatrix}1&-4&2\\-2&8&-9\\-1&7&0\end{vmatrix}=\begin{vmatrix}1&-4&2\\0&0&-5\\-1&7&0\end{vmatrix}=\begin{vmatrix}1&-4&2\\0&0&-5\\0&3&2\end{vmatrix}$$

An interchange of rows 2 and 3 reverses the sign of the determinant (换行反号), so

$$|\boldsymbol{A}|=-\begin{vmatrix}1&-4&2\\0&3&2\\0&0&-5\end{vmatrix}=-(1)(3)(-5)=15.$$

多种方法可以根据需要进行选择.

Example 3 Find the determinant of the matrix $\boldsymbol{A}$, where

$$\boldsymbol{A}=\begin{bmatrix}1&1&3&0\\1&1&4&5\\0&1&0&1\\3&0&10&3\end{bmatrix}.$$

Solution

$$\boldsymbol{A}\ \rightarrow \boldsymbol{B}=\begin{bmatrix}1&1&3&0\\0&0&1&5\\0&1&0&1\\0&-3&1&3\end{bmatrix},$$

Thus

$$|\boldsymbol{A}|=|\boldsymbol{B}|=\begin{vmatrix}0&1&5\\1&0&1\\-3&1&3\end{vmatrix}=\begin{vmatrix}0&1&5\\1&0&1\\0&1&6\end{vmatrix}=-\begin{vmatrix}1&5\\1&6\end{vmatrix}=-1.$$

Remark 任何行列式总可以利用三种行变换把行列式化为上三角形行列式或下三角行列式.

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三角化法
(Using elementary operations to find determinants)

Exercise

$$D=\begin{vmatrix} 4 & 1 & 2 & 4 \\ 1 & 2 & 0 & 2 \\ 10 & 5 & 2 & 0 \\ 1 & 1 & 1 & 7 \end{vmatrix}$$

Example 4 Find the determinant:

$$D_n=\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{vmatrix}_{n\times n}.$$

Solution

$$D_n=\begin{vmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{vmatrix}$$

D_n的每行元素之和均为$a+(n-1)b$
把各列加到第1列

$$D_n=\begin{vmatrix} a+(n-1)b & b & \cdots & b \\ a+(n-1)b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ a+(n-1)b & b & \cdots & a \end{vmatrix}$$

提出公因子$a+(n-1)b$

$$=[a+(n-1)b]\begin{vmatrix} 1 & b & \cdots & b \\ 1 & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ 1 & b & \cdots & a \end{vmatrix}$$

将第1行乘(−1)加到其余各行，化为上三角行列式

$$=[a+(n-1)b]\begin{vmatrix} 1 & b & \cdots & b \\ & a-b & \cdots & 0 \\ & & \ddots & \vdots \\ & & & a-b \end{vmatrix}$$

$$=[a+(n-1)b](a-b)^{n-1}$$

Key words:

Definition (Expansion)

Properties

Using elementary operations to find determinants

Homework

See Blackboard