

Step-1

Given

$$S_1 = |3|, \quad S_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}, \quad S_3 = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

We need to compute the determinants of S_1, S_2, S_3 of these 1, 3, 1 tridiagonal matrices.

Step-2

Now

$$\begin{aligned} S_1 &= |3| \\ &= 3 \end{aligned}$$

$$\begin{aligned} S_2 &= \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \\ &= 9 - 1 \\ &= 8 \end{aligned}$$

Step-3

Then

$$\begin{aligned} S_3 &= \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} \\ &= 3 \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} \text{ expanding I row} \\ &= 3S_2 - 3 \\ &= 3S_2 - S_1 \\ &= 24 - 3 \\ &= 21 \end{aligned}$$

Step-4

So, we can guess that

$$\begin{aligned}
S_4 &= 3S_3 - S_2 \\
&= 3(21) - 8 \\
&= 63 - 8 \\
&= \boxed{55}
\end{aligned}$$

Step-5

Calculating S_4 directly we get

$$\begin{aligned}
S_4 &= \begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix} \\
&= 3 \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}
\end{aligned}$$

Step-6

On solving

$$\begin{aligned}
&= 3.S_3 - \left[\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} \right] \\
&= 3(21) - [9 - 1] \\
&= 63 - 8 \\
&= \boxed{55}
\end{aligned}$$