

## Step-1

We have to solve the system  $F_4 c = y$ , where

$$c_0 + c_1 + c_2 + c_3 = 2$$

$$c_0 + i c_1 + i^2 c_2 + i^3 c_3 = 0$$

$$c_0 + i^2 c_1 + i^4 c_2 + i^6 c_3 = -2$$

$$c_0 + i^3 c_1 + i^6 c_2 + i^9 c_3 = 0$$

And we have to verify that  $c_0 + c_1 e^{ix} + c_2 e^{2ix} + c_3 e^{3ix}$  takes the values 2, 0, -2, 0 at the points  $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

## Step-2

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Let

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

And

## Step-3

Fourier matrix

Fourier matrix

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$\begin{pmatrix} \text{since } i^2 = -1, i^3 = -i \\ i^4 = 1, i^6 = i^4 i^2 = -1 \\ i^9 = i^6 i^3 = i \end{pmatrix}$$

## Step-4

Now

$$F_4 c = y$$

$$\Rightarrow c = F_4^{-1} y$$

$$\text{But } F_n^{-1} = \frac{\overline{F_n}}{n} y, (n \text{ is the order of } F_n)$$

$$\text{Therefore } c = \frac{\overline{F_4}}{4} y$$

## Step-5

$$\begin{aligned} \Rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} &= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 0 \\ 4 \\ 0 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore the solution of the system  $F_4 c = y$  is  $c = (c_0, c_1, c_2, c_3) = (0, 1, 0, 1)$

## Step-6

$$\text{At } x=0, c_0 + c_1 e^{ix} + c_2 e^{2ix} + c_3 e^{3ix} = 2$$

$$\Rightarrow c_0 + c_1 + c_2 + c_3 = 2 \text{ (since } e^0 = 1)$$

$$\Rightarrow 0 + 1 + 0 + 1 = 2$$

$$\Rightarrow \boxed{2=2}$$

## Step-7

$$\text{At } x = \pi/2, c_0 + c_1 e^{i\pi/2} + c_2 e^{2i\pi/2} + c_3 e^{3i\pi/2} = 0$$

$$\Rightarrow c_0 + ic_1 - ic_2 - ic_3 = 0 \text{ (since } e^{i\pi/2} = i, e^{2i\pi/2} = -1, e^{3i\pi/2} = -i)$$

$$\Rightarrow 0 + i + 0 - i = 0$$

$$\Rightarrow \boxed{0=0}$$

$$\text{At } x = \pi, c_0 + c_1 e^{i\pi} + c_2 e^{2i\pi} + c_3 e^{3i\pi} = -2$$

$$\Rightarrow c_0 - ic_1 + ic_2 - ic_3 = -2 \text{ (since } e^{i\pi} = -1, e^{2i\pi} = 1, e^{3i\pi} = -1)$$

$$\Rightarrow 0 - 1(1) + 0 - 1(1) = -2$$

$$\Rightarrow \boxed{-2=-2}$$

## Step-8

$$\text{At } x = 3\pi/2, c_0 + c_1 e^{3i\pi/2} + c_2 e^{6i\pi/2} + c_3 e^{9i\pi/2} = 0$$

$$\Rightarrow c_0 - ic_1 - ic_2 + ic_3 = 0 \text{ (since } e^{i\pi} = -1, e^{2i\pi} = 1, e^{3i\pi} = -1)$$

$$\Rightarrow 0 - i(1) + 0 + i(1) = 0$$

$$\Rightarrow \boxed{0=0}$$

Therefore the values of  $c_0 + c_1 e^{ix} + c_2 e^{2ix} + c_3 e^{3ix}$  at  $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$  are 2, 0, -2, 0.