Step-1

Let A be the matrix with following values:

Eigen values: $\lambda = (1,3)$

Eigen vectors:

$$v_1 = (5,2)$$
$$v_2 = (2,1)$$

Step-2

Find the solutions to the following:

$$u_{k+1} = Au_k$$
$$du/dt = Au$$

Initial value: $u(0) = (9,4)$

Step-3

Recall that the solution to a difference equation $u_{k+1} = Au_k$ is $u_k = A^k u_0$. Also recall that $A^k = S\Lambda^k S^{-1}$.

Firstly compute $A^k = S\Lambda^k S^{-1}$. Eigen vector matrix and Eigen value matrix is as follows:

$$S = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$
$$S^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$
$$\Lambda^k = \begin{bmatrix} 1^k & 0 \\ 0 & 3^k \end{bmatrix}$$

Step-4

Now, do the following calculations to get A^k:

$$
\begin{aligned}
A^k &= S\Lambda^k S^{-1} \\
&= \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 2(3)^k \\ 2 & (3)^k \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 5-4(3)^k & -10+10(3)^k \\ 2-2(3)^k & -4+5(3)^k \end{bmatrix}
\end{aligned}
$$

Step-5

Now, do the following calculations to get $u_k = A^k u_0$

$$
\begin{aligned}
u_k &= A^k u_0 \\
&= \begin{bmatrix} 5-4(3)^k & -10+10(3)^k \\ 2-2(3)^k & -4+5(3)^k \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 5+4(3)^k \\ 2+2(3)^k \end{bmatrix}
\end{aligned}
$$

Step-6

Therefore, solution to a difference equation $u_{k+1} = Au_k$ is:

$$
\boxed{u_k = \begin{bmatrix} 5+4(3)^k \\ 2+2(3)^k \end{bmatrix}}
$$

Step-7

Recall that the solution to a differential equation $du/dt = Au$ is $u(t) = e^{At}u_0$. Also recall that $e^{At} = Se^{\Lambda t}S^{-1}$.

Firstly compute $e^{At} = Se^{\Lambda t}S^{-1}$. Eigen vector matrix and Eigen value matrix are already defined. Now, do the following calculations to get e^{At}:

$$
\begin{aligned}
e^{At} &= Se^{\Lambda t}S^{-1} \\
&= \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 5e^t & 2e^{3t} \\ 2e^t & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 5e^t - 4e^{3t} & -10e^t + 10e^{3t} \\ 2e^t - 2e^{3t} & -4e^t + 5e^{3t} \end{bmatrix}
\end{aligned}
$$

Step-8

Now, do the following calculations to get $u(t) = e^{At}u_0$.

$$
\begin{aligned}
u(t) &= e^{At}u_0 \\
&= \begin{bmatrix} 5e^t - 4e^{3t} & -10e^t + 10e^{3t} \\ 2e^t - 2e^{3t} & -4e^t + 5e^{3t} \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 5e^t + 4e^{3t} \\ 2e^t + 2e^{3t} \end{bmatrix}
\end{aligned}
$$

Step-9

Therefore, solution to a differential equation $du/dt = Au$ is:

$$
\boxed{u(t) = \begin{bmatrix} 5e^t + 4e^{3t} \\ 2e^t + 2e^{3t} \end{bmatrix}}
$$