$\sum \frac{}{n!}$, $\frac{}{(2n+1)!}$, $\frac{}{(2n)!}$, $\frac{}{2n+1}$

e^x $\sin x$ $\cos x$ $\tan x$

$\begin{cases} x\to 0 \\ x\to\infty \Rightarrow t=\frac{1}{x}\to 0 \\ x\to a\neq 0 \Rightarrow x=a\text{处展开} \end{cases}$

Calculus II 第十章quiz 3

考点一:用泰勒级数求极限. $x=\frac{1}{n}$

1.(2022年期末) Use Taylor series to evaluate $\lim\limits_{n\to\infty}(n^3\sin\frac{2}{n}-2n^2)$. $=\lim\limits_{n\to\infty}\left(n^3\left(\frac{2}{n}-\frac{(2/n)^3}{3!}+\frac{(2/n)^5}{5!}\right)-2n^2\right)=-\frac{4}{3}$

$=\lim\limits_{x\to 0}\frac{\sin 2x-2x}{x^3}=-\frac{4}{3}$

2.(2021年期中) Find $\lim\limits_{n\to\infty}((n^2-n)e^{\frac{1}{n}}-\sqrt{n^4+1})$.

$=\lim\limits_{n\to\infty}\left((n^2-n)\left(1+\frac{1}{n}+\frac{(1/n)^2}{2!}\right)-\sqrt{n^4+1}\right)=-\frac{1}{2}$

3.(2020年期末) Find the real numbers $a, b(b\neq 0)$, which satisfy $\lim\limits_{x\to 0}\frac{\cos(\sin x)-\sqrt{1-x^2}}{x^a}=b$. $a=4,\ b=\frac{1}{3}$

4. Using Taylor series to compute the following series. $=\lim\limits_{x\to\infty}x\left(\sqrt[6]{1+\frac{1}{x}}-\sqrt[6]{1-\frac{1}{x}}\right)$

(1) $\lim\limits_{x\to 1}\frac{x^2-1}{\ln x}=\lim\limits_{x\to 1}\frac{(x+1)(x-1)}{\ln(1+(x-1))}=2$ (2) $\lim\limits_{x\to\infty}(\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5})=\lim\limits_{x\to\infty}x\left(1+\frac{1}{6x}-1+\frac{1}{6x}\right)=\frac{1}{3}$

(3) $\lim\limits_{x\to\infty}x^{\frac{3}{2}}(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x})$ $=-\frac{1}{4}$ (4) $\lim\limits_{x\to\infty}[x-x^2\ln(1+\frac{1}{x})]$

$=\lim\limits_{x\to\infty}\left(x-x^2\left(\frac{1}{x}-\frac{(1/x)^2}{2}+\frac{(1/x)^3}{3}+\cdots\right)\right)=\frac{1}{2}$

考点二:求函数的泰勒展开式.

1.(2021年期末) Find the Maclaurin series for $f(x)=\int_0^{x^2}\frac{1}{1-t}\,dt$, $-1<x<1$.

$f'(x)=\frac{2x}{1-x^2}$ $=\int_0^{x^2}\sum\limits_{n=0}^{\infty}t^n\,dt=\left.\sum\limits_{n=0}^{\infty}\frac{t^{n+1}}{n+1}\right]_0^{x^2}=\sum\limits_{n=0}^{\infty}\frac{x^{2n+2}}{n+1}$

2. (2019年期中) Find the Taylor series for $f(x)=\ln(x+\sqrt{x^2+1})$ at $x=0$. $f(x)=x+\sum\limits_{k=1}^{\infty}\frac{1}{2k+1}\binom{-\frac{1}{2}}{k}x^{2k+1}$

在0处展开

3. (2018年期中) Find the Maclaurin series for the function $f(x)=\frac{1}{(2-x)^2}$. $=\left(\frac{1}{2-x}\right)'=\left(\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}}\right)'=\left(\frac{1}{2}\sum\limits_{n=0}^{\infty}\left(\frac{x}{2}\right)^n\right)'$ $=\sum\limits_{n=0}^{\infty}\frac{nx^{n-1}}{2^{n+1}}$

4. Find the Maclaurin series for the function $f(x)=\tan^{-1}(\frac{1+x}{1-x})$.

$f'(x)=\frac{1}{1+x^2}=\sum\limits_{n=0}^{\infty}(-1)^n x^{2n}\Rightarrow f(x)=\sum\limits_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}+\frac{\pi}{4}$

考点三:求泰勒级数的和函数.

1.(2021年期中) For the power series $f(x)=\sum\limits_{n=1}^{\infty}\frac{n+2}{n(n+1)}x^n$, $=\sum\left(\frac{1}{n+1}+\frac{2}{n(n+1)}\right)x^n$

$=\sum\limits_{n=0}^{\infty}\frac{x^n}{n+1}+\sum\limits_{n=0}^{\infty}\frac{2x^n}{n(n+1)}$

$\frac{x^{n+1}}{n(n+1)}$ $=\frac{1}{x}\int\left(\int\sum\limits_{n=1}^{\infty}x^n dx\right)dx$ $\frac{x^n}{n}$ $=\frac{1}{x}\int\int\frac{1}{1-x}dx\,dx$

(1) For what values of x does the power series converge?

(2) Find the sum of the series within the interval of convergence.

$\frac{1}{x}\sum\limits_{n=0}^{\infty}\int x^n dx$ $=\frac{1}{x}\int\frac{1}{1-x}dx$ $=\frac{-\ln(1-x)}{x}$

$2\sum\frac{1}{n}x^n-\sum\frac{x^n}{n+1}$

$\int\sum x^{n-1}dx$

2. Find the sum function for the following series $\sum\limits_{n=0}^{\infty}\frac{n^2+1}{2^n n!}x^n$.

$\sum\limits_{n=0}^{\infty}\frac{n^2+1}{2^n\cdot n!}x^n=\sum\limits_{n=0}^{\infty}\frac{n^2-n}{2^n\cdot n!}x^n+\sum\limits_{n=0}^{\infty}\frac{n}{2^n\cdot n!}x^n+\sum\limits_{n=0}^{\infty}\frac{1}{2^n\cdot n!}x^n$

$=\left(\frac{x}{2}\right)^2\sum\limits_{n=2}^{\infty}\frac{1}{(n-2)!}\left(\frac{x}{2}\right)^{n-2}+\frac{x}{2}\sum\limits_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{x}{2}\right)^{n-1}+\sum\limits_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x}{2}\right)^n$

$=\left(\frac{x}{2}\right)^2e^{\frac{x}{2}}+\frac{x}{2}e^{\frac{x}{2}}+e^{\frac{x}{2}}$