Step-1

We need to choose the value of *, so that the matrix* $R = PAP^{-1}$ *will be triangular.*

Consider

$$\begin{aligned}PA &= \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & -1 \\ 3 & 5\end{bmatrix} \\ &= \begin{bmatrix}\cos\theta - 3\sin\theta & -\cos\theta - 5\sin\theta \\ \sin\theta + 3\cos\theta & -\sin\theta + 5\cos\theta\end{bmatrix}\end{aligned}$$

Step-2

Now, when $P = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$*, we have* $P^{-1} = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$.

Thus,

$$\begin{aligned}R &= PAP^{-1} \\ &= \begin{bmatrix}\cos\theta - 3\sin\theta & -\cos\theta - 5\sin\theta \\ \sin\theta + 3\cos\theta & -\sin\theta + 5\cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} \\ &= \begin{bmatrix}\cos^2\theta - 2\sin\theta\cos\theta + 5\sin^2\theta & -4\sin\theta\cos\theta - 3\sin^2\theta - \cos^2\theta \\ -4\sin\theta\cos\theta + 3\cos^2\theta + \sin^2\theta & \sin^2\theta + 2\sin\theta\cos\theta + 5\cos^2\theta\end{bmatrix}\end{aligned}$$

Step-3

Since, R has to be triangular, we want $-4\sin\theta\cos\theta + 3\cos^2\theta + \sin^2\theta = 0$.

Consider

$$\begin{aligned}-4\sin\theta\cos\theta + 3\cos^2\theta + \sin^2\theta &= 0 \\ -4\sin\theta\sqrt{1-\sin^2\theta} + 2\cos^2\theta + 1 &= 0 \\ -4\sin\theta\sqrt{1-\sin^2\theta} + 2\left(1-\sin^2\theta\right) + 1 &= 0 \\ 3 - 2\sin^2\theta - 4\sin\theta\sqrt{1-\sin^2\theta} &= 0\end{aligned}$$

Step-4

Let $x = \sin\theta$*. Therefore,*

$$3-2x^2-4x\sqrt{1-x^2}=0$$

$$3-2x^2=4x\sqrt{1-x^2}$$

$$\frac{3-2x^2}{4x}=\sqrt{1-x^2}$$

$$\frac{9-12x^2+4x^4}{16x^2}=1-x^2$$

Cross multiply and simplify:

$$\begin{aligned}9-12x^2+4x^4&=16x^2\left(1-x^2\right)\\&=16x^2-16x^4\end{aligned}$$

$$20x^4-28x^2+9=0$$

Step-5

The equation $20x^4-28x^2+9=0$ *is quadratic in* x^2. *Its roots are given by,*

$$\begin{aligned}x^2&=\frac{28\pm\sqrt{784-720}}{40}\\&=\frac{28\pm 8}{40}\\&=\frac{9}{10}\text{ or }\frac{1}{2}\end{aligned}$$

Step-6

Therefore, $\sin^2\theta=\frac{9}{10}\text{ or }\frac{1}{2}$. *Therefore,* $\cos^2\theta=\frac{1}{10}\text{ or }\frac{1}{2}$ *respectively.*

Let $\sin\theta=\frac{1}{\sqrt{2}}$ *and* $\cos\theta=\frac{1}{\sqrt{2}}$. *Thus,* $\theta=45°$.

For this value of , we get

$$\begin{aligned}R&=\begin{bmatrix}\frac{1}{2}-2\left(\frac{1}{2}\right)+5\left(\frac{1}{2}\right) & -4\left(\frac{1}{2}\right)-3\left(\frac{1}{2}\right)-\frac{1}{2}\\ -4\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)+\frac{1}{2} & \frac{1}{2}+2\left(\frac{1}{2}\right)+5\left(\frac{1}{2}\right)\end{bmatrix}\\&=\begin{bmatrix}2 & -4\\ 0 & 4\end{bmatrix}\end{aligned}$$

Step-7

The eigenvalues of R can be obtained by solving $\det(R - \lambda I) = 0$.

This gives

$$\begin{vmatrix} 2-\lambda & -4 \\ 0 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(4-\lambda) = 0$$

Therefore, $\lambda = 2$ *and* $\lambda = 4$.