Step-1

Given that add the extra column b and reduce A to echelon form, A combination of the rows A has produced the zero row.

We need to find the combination, need to find the null space basis vector and left null space basis vector.

(a)

Given $\begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix}$

Apply $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 4R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & -2 & b_3 - 4b_1 \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

Therefore, $(\text{row } 3) - (\text{row } 2) - (\text{row } 1) = 0$

Therefore, the combination is $\boxed{(\text{row } 3) - (\text{row } 2) - (\text{row } 1) = 0}$.

Step-2

So, the combination of rows $(\text{row } 3) - (\text{row } 2) - (\text{row } 1)$ gives zero.

Therefore $\boxed{(1, -1, -1)}$ is in the left null space of the given matrix

The transpose of the matrix is $A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$

In other way, we write the homogeneous equations using this.

$Ax = 0$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 3x_2 + 4x_3 = 0$$
$$-2x_2 - 2x_3 = 0$$
$$\Rightarrow x_2 = -x_3$$
$$x_1 = -3x_2 - 4x_3 \qquad \text{(Substitue } x_2 \text{ value)}$$
$$= -x_3$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ -x_3 \end{bmatrix}$$
$$= x_3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Therefore, the vectors $\boxed{(1,-1,-1)}$ are in null space.

Step-3

(b)

Given $\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$

Apply $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 1 & b_4 - 2b_1 \end{bmatrix}$$

Apply$R_4 \to R_4 + R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 0 & -4b_1 + b_2 + b_4 \end{bmatrix}$$

Therefore, $(\text{row } 3) - 2(\text{row } 1) = 0 \text{ and} (\text{row} 4) + (\text{row} 2) - 4(\text{row} 1) = 0$

Therefore, the combination is $\boxed{(\text{row } 3) - 2(\text{row } 1) = 0 \text{ and} (\text{row} 4) + (\text{row} 2) - 4(\text{row} 1) = 0}$.

Step-4

So, the combinations of rows $(\text{row } 3) - 2(\text{row } 1) = 0 \text{ and} (\text{row} 4) + (\text{row} 2) - 4(\text{row} 1)$ give zero.

Therefore $\boxed{\{(-2,0,1,0),(-4,1,0,1)\}}$ is in the left null space of the given matrix

In other way, the transpose of the matrix is $A^T = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 3 & 4 & 5 \end{bmatrix}$

Step-5

In other way, we write the homogeneous equations using this.

$A^T x = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$-2x_2 + x_4 = 0$$
$$\Rightarrow x_4 = x_2$$
$$x_1 = -2x_2 - 2x_3 - 2x_4 \qquad (\text{Substitue } x_4 \text{ value})$$
$$= -2x_2 - 2x_3 - 2x_2$$
$$= -4x_2 - 2x_3$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4x_2 - 2x_3 \\ x_2 \\ x_3 \\ x_2 \end{bmatrix}$$
$$= x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore $\boxed{\{(-2,0,1,0),(-4,1,0,1)\}}$ is in the left null space of the given matrix