Step-1

Given that the basis for the space of all 2 by 2 matrices contains the four vectors

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

We have to find the matrix A with respect to this basis for the linear transformation of transposing.

Step-2

Let $e_1=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},\ e_2=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\ e_3=\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix},\ e_4=\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Here T is a linear transformation of transposing.

Then

$$\begin{aligned}T(e_1)&=T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)\\&=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\\&=e_1\\&=1.e_1+0.e_2+0.e_3+0.e_4\end{aligned}$$

Step-3

And

$$\begin{aligned}T(e_2)&=T\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)\\&=\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\\&=e_3\\&=0.e_1+0.e_2+1.e_3+0.e_4\end{aligned}$$

Step-4

And

$$
\begin{aligned}
T(e_3) &= T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= e_2 \\
&= 0.e_1 + 1.e_2 + 0.e_3 + 0.e_4
\end{aligned}
$$

Step-5

And

$$
\begin{aligned}
T(e_4) &= T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= e_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= e_4 \\
&= 0.e_1 + 0.e_2 + 0.e_3 + 1.e_4
\end{aligned}
$$

Therefore, the matrix of T under the basis $\{e_1, e_2, e_3, e_4\}$ is $\boxed{A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$.

Step-6

We have $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Therefore,

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Hence $\boxed{A^2 = I}$