Characteristic and Minimal Polynomials

Lecture 25

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Characteristic and Minimal Polynomials

1. The Cayley-Hamilton Theorem
2. The Minimal Polynomial
3. Homework Assignment 25

$T \in \mathcal{L}(\mathbb{C}^5)$. $T(e_1, e_2, e_3, e_4, e_5) = (e_1\ e_2\ e_3\ e_4\ e_5)\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}}_{A}$

$|A - \lambda I| = 0 \Rightarrow -\lambda^5 + 6\lambda - 3 = 0$

The Cayley-Hamilton Theorem

The next definition associates a polynomial with each operator on V if $\mathbb{F}=\mathbb{C}$. For $\mathbb{F}=\mathbb{R}$, the definition will be given in the next Chapter.

8.34 **Definition** ***characteristic polynomial***

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T, with multiplicities $d_1, \ldots, d_m$. The polynomial

$$p(z)=(z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}$$

$d_1+\cdots+d_m=n=\dim V$

$\dim G(\lambda_1, T) = d_1$

is called the ***characteristic polynomial*** of T.

$T(z_1, z_2, z_3)=(z_2, z_3, 0) \rightarrow p(z)=z^3$

$\dim G(0,T)=3$. $\dim E(0,T)=$

Example

Suppose $T \in \mathscr{L}(\mathbb{C}^3)$ is defined as in Example 8.25. Because the eigenvalues of T are 6, with multiplicity 2, and 7, with multiplicity 1, we see that characteristic polynomial of T is $(z-6)^2(z-7)$.

Degree and zeros of characteristic polynomial

8.36 Degree and zeros of characteristic polynomial

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then

(a) the characteristic polynomial of T has degree $\dim V$;

(b) the zeros of the characteristic polynomial of T are the eigenvalues of T.

Proof.

Clearly part (a) follows from 8.26 and part (b) follows from the definition of the characteristic polynomial. $\square$

Cayley-Hamilton Theorem

$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

$q(z) = (z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}$

$\Rightarrow q(T) = (T-\lambda_1 I)^{d_1} \cdots (T-\lambda_m I)^{d_m} = 0$

8.21

$V = u_1 + \cdots + u_m$

$u_j \in G(\lambda_j, T)$

8.37 Cayley–Hamilton Theorem

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let q denote the characteristic polynomial of T. Then $q(T) = 0$.

$(T-\lambda_j I)|_{G(\lambda_j,T)}$ nilpotent $\longrightarrow (T-\lambda_j I)^{d_j}(u_j) = 0$

$\dim G(\lambda_j, T) = d_j$

Proof.

Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the distinct eigenvalues of the operator T, and let $d_1, d_2, \cdots, d_m$ be the dimensions of the corresponding generalized eigenspaces $G(\lambda_1, T), \cdots, G(\lambda_m, T)$. For each $j \in \{1, \cdots, m\}$, we know that $(T-\lambda_j I)|_{G(\lambda_j,T)}$ is nilpotent. Thus we have $(T-\lambda_j I)^{d_j}|_{G(\lambda_j,T)} = 0$ (by 8.18). We can show that $q(T)|_{G(\lambda_j,T)} = 0$, and together with 8.21, we conclude that $q(T) = 0$. □

The Minimal Polynomial

In this subsection we introduce another important polynomial associated with each operator. We begin with the following definition.

8.38 Definition ***monic polynomial***

A ***monic polynomial*** is a polynomial whose highest-degree coefficient equals 1.

Example

The polynomial $2+9z^2+z^7$ is a monic polynomial of degree 7.

Uniqueness

$p_1 = p_2 \Rightarrow \deg(p_1 - p_2) < m$ X

8.40 Minimal polynomial

$T \in \mathcal{L}(V)$, monic polynomial $p(T)=0$

$p(z)$ existence + uniqueness

Suppose $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial p of smallest degree such that $p(T) = 0$.

m smallest. $I, T, T^2, \dots, T^m$ linearly dependent

$T^m + a_{m-1}T^{m-1} + \dots + a_1 T + a_0 I = 0$ $\quad p(z) = z^m + a_{m-1}z^{m-1} + \dots + a_1 z + a_0.$

Existence

Minimal Polynomial

The last result justifies the following definition.

8.43 Definition ***minimal polynomial***

Suppose $T \in \mathcal{L}(V)$. Then the ***minimal polynomial*** of T is the unique monic polynomial p of smallest degree such that $p(T) = 0$.

e.g. $\begin{bmatrix} \boxed{\begin{matrix}\lambda & 1 & \\ & \lambda & 1 \\ & & \lambda\end{matrix}} & & \\ & \boxed{\begin{matrix}\lambda & 1 \\ & \lambda\end{matrix}} & \\ & & \lambda \end{bmatrix}$ $\quad q(z) = (z-\lambda)^6$

$p(z) = (z-\lambda)^3$

The next result completely characterizes the polynomials that when applied to an operator give the 0 operator.

8.46 $q(T) = 0$ **implies** q **is a multiple of the minimal polynomial**

Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T.

$q(z) = (z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}$ $\quad 1 \leq p_k \leq d_k$

$p(z) = (z-\lambda_1)^{p_1} \cdots (z-\lambda_m)^{p_m}$

Characteristic Polynomial and Minimal Polynomial

The next result is stated only for complex vector spaces, because we have not yet defined the characteristic polynomial when $\mathbb{F} = \mathbb{R}$. However, the result also holds for real vector spaces, as we will see in the next chapter.

8.48 Characteristic polynomial is a multiple of minimal polynomial

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T.

Zeros

We know that the zeros of the characteristic polynomial of T are the eigenvalues of T. Now we show that the minimal polynomial has the same zeros.

8.49 Eigenvalues are the zeros of the minimal polynomial

Let $T \in \mathcal{L}(V)$. Then the zeros of the minimal polynomial of T are precisely the eigenvalues of T.

Examples

Example

Find the minimal polynomial of the operator $T \in \mathscr{L}(\mathbb{C}^3)$ in Example 8.30.

Example

Find the minimal polynomial of the operator $T \in \mathscr{L}(\mathbb{C}^3)$ defined by $T(z_1, z_2, z_3) = (6z_1, 6z_2, 7z_3)$.

Example

What are the eigenvalues of the operator in Example 8.45?

Homework Assignment 25

8.C: ~~3~~, ~~4~~, ~~6~~, 8, 10, 12, 14, 15, 18, 20.