Step-1

(a) Let us show that the $p+q$ number of vectors $x_1, x_2, ..., x_p$ and $y_1, y_2, ..., y_q$ are linearly independent. We will show this by contradiction.

Suppose, if possible, there exists a non zero linear combination of these vectors, which produces the zero vector.

Therefore, we can write the following:

$$\begin{aligned} z &= a_1x_1 + ... + a_px_p \\ &= b_1C_1y + ... + b_qCy_q \end{aligned}$$

Step-2

Now consider the following:

$$\begin{aligned} z^T Az &= \left(a_1x_1 + ... + a_px_p\right) A \begin{bmatrix} a_1x_1 \\ \vdots \\ a_px_p \end{bmatrix} \\ &= \left(a_1x_1 + ... + a_px_p\right) \begin{bmatrix} a_1Ax_1 \\ \vdots \\ a_pAx_p \end{bmatrix} \\ &= \left(a_1x_1 + ... + a_px_p\right) \begin{bmatrix} a_1\lambda_1x_1 \\ \vdots \\ a_p\lambda_px_p \end{bmatrix} \\ &= \lambda_1a_1^2 + ... + \lambda_pa_p^2 \end{aligned}$$

The last equality is true because the vectors $x_1, x_2, ..., x_p$ are orthonormal vectors.

Step-3

Similarly, consider the following:

$$\begin{aligned} z^T Az &= \left(b_1C^Tx_1 + ... + b_pC^Tx_p\right) A \begin{bmatrix} b_1Cx_1 \\ \vdots \\ b_pCx_p \end{bmatrix} \\ &= \left(b_1C^Tx_1 + ... + b_pC^Tx_p\right) \begin{bmatrix} b_1ACx_1 \\ \vdots \\ b_pACx_p \end{bmatrix} \\ &= \mu_1b_1^2 + ... + \mu_pb_p^2 \end{aligned}$$

Step-4

Now the eigenvalues ?$_i$ are assumed to be positive and the eigenvalues $\hat{A}\mu_i$ are assumed to be negative. Therefore, we get,

$$\boxed{\begin{array}{l}\lambda_1 a_1^2+\ldots+\lambda_p a_p^2 \geq 0 \\ \mu_1 b_1^2+\ldots+\mu_p b_p^2 \leq 0\end{array}}$$

Step-5

(b) But, $\lambda_1 a_1^2+\ldots+\lambda_p a_p^2=\mu_1 b_1^2+\ldots+\mu_p b_p^2$. Therefore, each must be equal to be zero. This is true only if all *a*â€™s are zero and all *b*â€™s are zero.

Therefore, the vectors $x_1, x_2, \ldots, x_p$ and $y_1, y_2, \ldots, y_q$ are linearly independent. If A is assumed to be an n by n matrix, then the number of its eigenvectors cannot exceed n. Therefore, we have

$$\boxed{p+q \leq n}$$

Step-6

(c) Under the assumption that there is no zero eigenvalue, we can say that A has $n-p$ negative eigenvalues and $C^{\mathrm{T}}AC$ has $n-q$ positive eigenvalues.

Arguing as above, we can say that $(n-p)+(n-q) \leq n$. This gives the following:

$$\begin{aligned}(n-p)+(n-q) &\leq n \\ 2n-p-q &\leq n \\ n &\leq p+q\end{aligned}$$

Thus, we have $p+q \leq n$ and $p+q \geq n$. This shows that $p+q=n$. This further gives $p=n-q$.

Recall that we have assumed that A has p positive eigenvalues and $C^{\mathrm{T}}AC$ has q negative eigenvalues. Also, we have assumed that there are no negative eigenvalues. Therefore, $C^{\mathrm{T}}AC$ has $n-q$ positive eigenvalues. Now we have shown that $p=n-q$. Therefore, the number of positive eigenvalues of A is equal to the number of positive eigenvalue of $C^{\mathrm{T}}AC$. By the same argument, the number of negative eigenvalues of A is equal to the number of negative eigenvalue of $C^{\mathrm{T}}AC$.

Thus, law of inertia is proved algebraically.