## Step-1

Given that for any A and b, one and only one of the following systems has a solution:

(i) 
$$Ax = b$$

(ii) 
$$A^T y = 0, y^T b \neq 0$$

And b in the column space  $\mathbf{C}(A)$  or there is a y in  $\mathbf{N}(A^T)$  such that  $y^T b \neq 0$ 

We have to show that it is contradictory for (i) and (ii) both to have solution.

## Step-2

Given b is in the column space C(A) implies Ax = b  $\hat{a} \in \hat{a} \in A$ 

And y belongs to  $\mathbf{N}(A^T)$  means  $y^T A = 0$   $\hat{\mathbf{a}} \in \hat{\mathbf{a}} \in \hat{\mathbf{a}} \in [0, \infty)$ .

Then we have

$$y^{T}b = y^{T} (Ax) \text{ (by (1))}$$
$$= (y^{T}A)x$$
$$= 0x \text{ (by (2))}$$
$$= 0$$

This contradicts to  $y^T b \neq 0$ .

Therefore only one of the systems (i) and (ii) has a solution.