8.D 2, 3, 4, 5, 6, 7.

2 Find the characteristic polynomial and the minimal polynomial of the operator N in Example 8.54. $\mathbb{C}^6$

$N(z_1, z_2, z_3, z_4, z_5, z_6) = (0, z_1, z_2, 0, z_4, 0)$

$N^2(z_1, z_2, z_3, z_4, z_5, z_6) = (0, 0, z_1, 0, 0, 0)$

$N^3(z_1, z_2, z_3, z_4, z_5, z_6) = (0, 0, 0, 0, 0, 0)$

so N is nilpotent $\Rightarrow$ all eigenvalues of N are 0

$\Rightarrow$ the characteristic polynomial is $q(z) = z^{\dim V} = z^6$

The minimal polynomial is $p(z) = z^3$.

3 Suppose $N \in \mathcal{L}(V)$ is nilpotent. Prove that the minimal polynomial of N is z^{m+1}, where m is the length of the longest consecutive string of 1's that appears on the line directly above the diagonal in the matrix of N with respect to any Jordan basis for N.

$$J = \begin{pmatrix} J_1(0) & & & \\ & J_2(0) & & \\ & & \ddots & \\ & & & J_s(0) \end{pmatrix}$$

$\operatorname{rank} J_1(0) \le \operatorname{rank} J_2(0) \le \cdots \le \operatorname{rank} J_s(0)$

$$J_s(0) = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}_{(m+1)\times(m+1)}$$

$J^k = 0 \Leftrightarrow J_s^k(0) = 0 \Leftrightarrow k \geq m+1$ i.e. $J^k=0$ 的最小的 $k = m+1$

so the minimal polynomial of N is z^{m+1}

4 Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V that is a Jordan basis for T. Describe the matrix of T with respect to the basis $v_n, \dots, v_1$ obtained by reversing the order of the v's.

$$\mathcal{M}(T; v_1, \dots, v_n) = \begin{pmatrix} \lambda_1 & 1 & & & & \\ & \ddots & 1 & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & & & & \lambda_m & 1 \\ & & & & & \ddots \; 1 \\ & & & & & \lambda_m \end{pmatrix} = J_1$$

$$\mathcal{M}(T; v_n, \dots, v_1) = \begin{pmatrix} \lambda_m & & & & & \\ 1 & \ddots & & & & \\ & 1 & \lambda_m & & & \\ & & & \ddots & & \\ & & & & \lambda_1 & \\ & & & & 1 & \ddots \\ & & & & & 1 \;\; \lambda_1 \end{pmatrix}$$

$$(v_n, \dots, v_1) = (v_1, \dots, v_n)\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \stackrel{\triangle}{=} P \qquad P^{-1} = P$$

$$T(v_n, \dots, v_1) = T(v_1, \dots, v_n)P$$

$$= (v_1, \dots, v_n) J_1 P$$

$$= (v_n, \dots, v_1) \underline{P^{-1} J_1 P}$$

5 Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V that is a Jordan basis for T. Describe the matrix of T^2 with respect to this basis.

$$\mathcal{M}(T; v_1, \dots, v_n) = \begin{pmatrix} J(\lambda_1) & & \\ & \ddots & \\ & & J(\lambda_m) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 & & & & \\ & \ddots & 1 & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & & & & \lambda_m & 1 \\ & & & & & \ddots \; 1 \\ & & & & & \lambda_m \end{pmatrix}$$

$$\mathcal{M}(T^2; v_1, \dots, v_n) = \begin{pmatrix} J(\lambda_1) & & \\ & \ddots & \\ & & J(\lambda_m) \end{pmatrix}^2 = \begin{pmatrix} J^2(\lambda_1) & & \\ & \ddots & \\ & & J^2(\lambda_m) \end{pmatrix}$$

$$J^2(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}^2 = \begin{pmatrix} \lambda_i^2 & 2\lambda_i & 1 & 0 & \cdots & 0 \\ & \lambda_i^2 & 2\lambda_i & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & 1 \\ & & & & \ddots & 2\lambda_i \\ & & & & & \lambda_i^2 \end{pmatrix}$$

$$= \left(\lambda_i I + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}\right)^2 = \lambda_i^2 I + 2\lambda_i \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & \\ & 0 & \ddots & \ddots \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^2$$

6 Suppose $N \in \mathcal{L}(V)$ is nilpotent and $v_1, \ldots, v_n$ and $m_1, \ldots, m_n$ are as in 8.55. Prove that $N^{m_1}v_1, \ldots, N^{m_n}v_n$ is a basis of null N.
[*The exercise above implies that n, which equals* dim null N, *depends only on N and not on the specific Jordan basis chosen for N.*]

Take $v_1, \cdots, v_l \in \operatorname{range} N$.

$$\left.\begin{array}{llll} v_1, & Nv_1, & \cdots, & N^{m_1-1}v_1 \\ v_2, & Nv_2, & \cdots, & N^{m_2-1}v_2 \\ & \vdots & & \\ v_l & Nv_l & \cdots & N^{m_l-1}v_l \end{array}\right\} \text{a basis of range } N.$$

$\Downarrow$ 扩充

$v_1, \cdots, v_l \in \operatorname{range} N$, then $\exists\, u_1, \cdots, u_l \in V$, s.t. $Nu_l = v_l$

$$\begin{array}{ll} u_1 & \\ u_2 & \\ \vdots & \\ u_l & \end{array} \boxed{\begin{array}{llll} Nu_1 & N^2u_1 & \cdots & N^{m_1}u_1 \\ Nu_2 & N^2u_2 & \cdots & N^{m_2}u_2 \\ & \vdots & & \\ Nu_l & N^2u_l & \cdots & N^{m_l}u_l \end{array}} \to \text{basis of range } N$$

$\Downarrow$ 扩充

$$\boxed{\begin{array}{lllll} u_1 & Nu_1 & N^2u_1 & \cdots & N^{m_1}u_1 \\ u_2 & Nu_2 & N^2u_2 & \cdots & N^{m_2}u_2 \\ \vdots & & \vdots & & \\ u_l & Nu_l & N^2u_l & \cdots & N^{m_l}u_l \\ & & & & u_{l+1} \\ & & & & \vdots \\ & & & & u_{l+p} \end{array}} \to \text{basis of } V$$

(inner box $\to$ basis of range N)

$\downarrow$

$\dim \operatorname{null} N = l+p$

$N^{m_1}u_1, \cdots, N^{m_l}u_l \in \operatorname{null} N$

$Nu_{l+1} = \cdots = Nu_{l+p} = 0 \Rightarrow u_{l+1}, \cdots, u_{l+p} \in \operatorname{null} N$

$N^{m_1}u_1, \cdots, N^{m_l}u_l, u_{l+1} \cdots u_{l+p}$ basis的一部分 故线性无关

$\Downarrow$

null N的基

7 Suppose $p, q \in \mathcal{P}(\mathbf{C})$ are monic polynomials with the same zeros and q is a polynomial multiple of p. Prove that there exists $T \in \mathcal{L}(C^{\deg q})$ such that the characteristic polynomial of T is q and the minimal polynomial of T is p.

Suppose $p(z) = (z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}$, $q(z) = (z-\lambda_1)^{k_1} \cdots (z-\lambda_m)^{k_m}$ $k_i \geq d_i$. $\forall i = 1, \cdots, m$

There ~~exists~~ exists a basis of V s.t.

$$\mathcal{M}(T; v_1, \cdots, v_n) = \begin{pmatrix} J(\lambda_1) & & \\ & \ddots & \\ & & J(\lambda_m) \end{pmatrix}$$

$$J(\lambda_i) = \underbrace{\begin{pmatrix} \boxed{\overbrace{\begin{matrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{matrix}}^{d_i \times d_i}} & \\ & \begin{matrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_i \end{matrix} \end{pmatrix}}_{k_i \times k_i}$$ where 1 appears $d_i - 1$ times.

minimal polynomial: $(z-\lambda_i)^{d_i} \rightarrow \exists$ d_i 阶 Jordan 块