

Step-1

Given matrix is $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$.

We have to find the norm and condition number from the square roots of $\lambda_{\max}(A^T A)$ and $\lambda_{\min}(A^T A)$ of the given matrix.

Step-2

We have $\lambda_{\max}(A^T A) = \|A\|^2$ (1)

So, the root of $\lambda_{\max}(A^T A)$ is nothing but the norm of the matrix A .

Also, $\lambda_{\min}(A^T A) = \frac{1}{\|A^{-1}\|^2}$

In other words,

The reciprocal of the square root of $\lambda_{\min}(A^T A)$ is nothing but the norm of A^{-1} .

Further, the **conditional number** of A is

$$\begin{aligned} c &= \|A\| \|A^{-1}\| \\ &= \sqrt{\lambda_{\max}(A^T A) \times \frac{1}{\lambda_{\min}(A^T A)}} \end{aligned} \quad (2)$$

Step-3

Let $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

So, $A^T = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ and $A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ are the positive definite matrices.

So, we can proceed with the details of the positive definite matrix.

The characteristic equation of $A^T A$ is

$$\begin{aligned}
& \left| A^T A - \lambda I \right| = 0 \\
& \Rightarrow \begin{vmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = 0 \\
& \Rightarrow (4-\lambda)(4-\lambda) = 0 \\
& \Rightarrow \lambda = 4, 4
\end{aligned}$$

So, the eigenvalues are 4, 4.

Step-4

Now $\lambda_{\max}(A^T A) = 4$, and so, $\|A\| = 2$ by (1)

And $\lambda_{\min}(A^T A) = 4$

In view of (2) above, we get $c = \sqrt{4 \times \frac{1}{4}}$

Since the norm of a vector is non negative, we consider the positive square root of this and thus, $c = 1$.

Hence the norm of the given matrix $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ is 4 and the condition is $\boxed{c=1}$.

Step-5

Given that $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

So, $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

The characteristic equation of $A^T A$ is

$$\begin{aligned}
& \left| A^T A - \lambda I \right| = 0 \\
& \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \\
& \Rightarrow (1-\lambda)(1-\lambda) - 1 = 0 \\
& \Rightarrow \lambda^2 - 2\lambda = 0 \\
& \Rightarrow \lambda = 0, 2
\end{aligned}$$

So, the eigenvalues are 0, 2

Step-6

Now we have $\lambda_{\max}(A^T A) = 2$,

So by (1), we get $\|A\| = \sqrt{2}$

And $\lambda_{\min}(A^T A) = 0$

In view of (2) above, we get $c = \sqrt{4 \times \frac{1}{0}}$

This is an infinite number and so, a singular value.

Hence the norm of the given matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is $\sqrt{2}$ and the condition number is an infinite number.

Step-7

Given that $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

So, $A^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ a positive definite matrix and so, we can proceed with the details of the positive definite matrix.

The characteristic equation of $A^T A$ is

$$\begin{aligned} |A^T A - \lambda I| &= 0 \\ \Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} &= 0 \\ \Rightarrow (2-\lambda)(2-\lambda) &= 0 \\ \Rightarrow \lambda &= 2, 2 \end{aligned}$$

So, the eigenvalues are 2, 2.

Step-8

Now $\lambda_{\max}(A^T A) = 2$ and so, $\|A\| = \sqrt{2}$ by (1)

And $\lambda_{\min}(A^T A) = 2$

In view of (2) above, we get $c = \sqrt{2 \times \frac{1}{2}}$

Since the norm of a vector is non negative, we consider the positive square root of this and thus, $c = 1$.

Hence the norm of the given matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is $\sqrt{2}$ and the condition number is $c = 1$.