Step-1

(a)

The objective is to construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.

Consider the system of equation as shown below:

$$\begin{aligned} 4y+5z&=1 \\ 2z&=2 \\ x+y+z&=3 \end{aligned}$$

Write the above system of equations in matrix form:

$A=\begin{pmatrix} 0 & 4 & 5 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Augmented matrix of the above system is $[A\,|\,B]=\left[\begin{array}{ccc|c} 0 & 4 & 5 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right]$.

Step-2

Consider the matrix $\left[\begin{array}{ccc|c} 0 & 4 & 5 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 0 & 4 & 5 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 4 & 5 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 4 & 5 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

Here, we used only two row exchanges to get the triangular matrix.

Now the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 2 \end{pmatrix}$ is in triangular form.

Now write the equivalent system of equations as shown below

$$\begin{aligned} 3z+y+z&=3 \\ 4y+5z&=1 \\ 2z&=2 \end{aligned}$$

Now use back substitution to find the solution.

$$2z=2 \Rightarrow z=1$$

Substitute $z=1$ in $4y+5z=1$.

$$\begin{aligned} 4y+5(1)&=1 \\ 4y+5&=1 \end{aligned}$$

$$\begin{aligned} 4y&=1-5 \\ 4y&=-4 \\ y&=-1 \end{aligned}$$

Substitute $z=1$, $y=-1$ in $3z+y+z=3$, then

$$\begin{aligned} 3z+(-1)+(1)&=3 \\ 3z&=3 \\ z&=1 \end{aligned}$$

Step-3

(b)

Consider the system of equations as shown below:

$$\begin{aligned} x+y+2z&=1 \\ x+z&=2 \\ 2y+z&=3 \end{aligned}$$

Write the above system of equations in matrix form:

$A=\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$, $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Augmented matrix of the above system is $[A \mid B]=\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{array}\right]$.

Consider the matrix $\left[\begin{array}{ccc|c}1 & 1 & 2 & 1\\ 1 & 0 & 1 & 2\\ 0 & 2 & 1 & 3\end{array}\right]$.

$$\left[\begin{array}{ccc|c}1 & 1 & 2 & 1\\ 1 & 0 & 1 & 2\\ 0 & 2 & 1 & 3\end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c}1 & 1 & 2 & 1\\ 0 & 2 & 1 & 3\\ 1 & 0 & 1 & 2\end{array}\right]$$

Notice that if we change any row operations, then cannot get the triangular form.

Now use row operations to get the triangular form.

$$\left[\begin{array}{ccc|c}1 & 1 & 2 & 1\\ 1 & 0 & 1 & 2\\ 0 & 2 & 1 & 3\end{array}\right] \xrightarrow{R_2 : R_2 - R_1} \left[\begin{array}{ccc|c}1 & 1 & 2 & 1\\ 0 & -1 & -1 & 1\\ 0 & 2 & 1 & 3\end{array}\right]$$
$$\xrightarrow{R_3 : R_3 + 2R_2} \left[\begin{array}{ccc|c}1 & 1 & 2 & 1\\ 0 & -1 & -1 & 1\\ 0 & 0 & -1 & 5\end{array}\right]$$

Step-4

Now the matrix $\begin{pmatrix}1 & 1 & 2\\ 0 & -1 & -1\\ 0 & 0 & -1\end{pmatrix}$ is in triangular form.

Now write the equivalent system of equations as shown below

$$\begin{aligned} x + y + 2z &= 1\\ -y - z &= 1\\ -z &= 5 \end{aligned}$$

Now use back substitution to find the solution.

From the third equation $z = -5$.

Substitute $z = -5$ in $-y - z = 1$.

$$\begin{aligned} -y - (-5) &= 1\\ -y + 5 &= 1 \end{aligned}$$

$$\begin{aligned} -y &= 1 - 5\\ -y &= -4\\ y &= 4 \end{aligned}$$

Now substitute $z=-5$, $y=4$ in $x+y+2z=1$

$$
\begin{aligned}
x+y+2z&=1\\
x+(4)+2(-5)&=1
\end{aligned}
$$

$$
\begin{aligned}
x+4-10&=1\\
x-6&=1\\
x&=1+6\\
x&=7
\end{aligned}
$$

Therefore, solution of the system of equation is $\boxed{x=7, y=4 \text{ and } z=-5}$.