Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #1

2023/02/26

Name: ____
Student Number: ____________________________________

1. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

   "$\Leftarrow$": Let U_1, U_2 be two subspaces of V. If $U_1 \subseteq U_2$, then we $U_1 \cup U_2 = U_2$ is a subspace. If $U_2 \subseteq U_1$, then $U_1 \cup U_2 = U_1$ is a subspace.
   "$\Rightarrow$": Let U_1, U_2 be subspaces of V, we assume $U_1 \cup U_2$ is a subspace. If $U_1 \subseteq U_2$, then we are done.
   If $U_1 \nsubseteq U_2$, then we need to show $U_2 \subseteq U_1$.
   Take $x \in U_2$, since $U_1 \nsubseteq U_2$, then there must exist some vector in U_1 that is not in U_2, denote it as y, so $y \in U_1, y \notin U_2$. Since $U_1 \cup U_2$ is a subspace, $x + y \in U_1 \cup U_2$. Since $x \in U_2 \subseteq U_1 \cup U_2$ and $y \in U_1 \subseteq U_1 \cup U_2$, thus we have either $x + y \in U_1$ or $x + y \in U_2$.
   If $x + y \in U_2$, then since $x \in U_2$ and U_2 is a subspace, we have $y = (x + y) - x \in U_2$, which contradicts $y \notin U_2$. Thus $x + y \in U_1$. But since $y \in U_1$ and U_1 is a subspacce, we have $x = (x + y) - y \in U_1$, therefore $U_2 \subseteq U_1$.

2. Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f dx = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

Let $U = \{f \in {}^{\mathbf{[0,1]}} : f$ is continuous and $\int_0^1 f dx = b\}$, recall that the zero element in $\mathbf{R}^{[0,1]}$ is the "zero function" $z : [0,1] \to \mathbf{R}$ defined by $z(x) = 0$ for all $x \in [0,1]$.
"$\Rightarrow$": If U is a subspace of $\mathbf{R}^{[0,1]}$, then the zero element is in U, that is z is continuous and $\int_0^1 z dx = b \Rightarrow b = \int_0^1 0 dx = 0$.
"$\Leftarrow$": suppose $b = 0$, so the set $U = \{f \in {}^{\mathbf{[0,1]}} : f$ is continuous and $\int_0^1 f dx = 0\}$. We want to show U is a subspace.

1. the zero function z is continuous and $\int_0^1 z dx = \int_0^1 0 dx = 0$, so $z \in U$.
2. $\forall c \in \mathbf{R}$, $f, g \in U$, we know $f + g, cf$ are continuous functions on $[0,1]$. And $\int_0^1 f + g dx = \int_0^1 f dx + \int_0^1 g dx = 0 + 0 = 0$, $\int_0^1 (cf) dx = c \int_0^1 f dx = c\dot{0} = 0$. Therefore $f + g, cf \in U$.

Thus U is a subspace.