1. Label the following statements as **True** or **False. Along with your answer, provide an informal proof, counterexample, or other explanation.**

    1. The empty set is a subspace of every vector space.

       False. Since 0 is not in the empty set.

    2. The dimension of $P_n(\mathbf{F})$ is n.

       False. $\dim P_n(\mathbf{F}) = n + 1$.

    3. Let u, v, w be distinct vectors of a vector space V. If u, v, w is a basis for V, then $u + v + w, v + w, w$ is also a basis for V.

       True. $u + v + w, v + w, w$ is linearly independent and $\dim V = 3 =$ the length of $u + v + w, v + w, w$, so $u + v + w, v + w, w$ is a basis for V.

    4. $P_n(\mathbf{F})$ is isomorphic to $P_m(\mathbf{F})$ if and only if $n = m$.

       True. Since $\dim P_n(\mathbf{F}) = n + 1$, $\dim P_m(\mathbf{F}) = m + 1$, $P_n(\mathbf{F})$ is isomorphic to $P_m(\mathbf{F}) \iff \dim P_n(\mathbf{F}) = \dim P_m(\mathbf{F}) \iff n + 1 = m + 1 \iff n = m$.

    5. If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V , then $U = \{0\}$ or $U = V$.

       True. Assume U is a subspace of V, $U \neq \{0\}$, $U \neq V$, $u_1, \cdots, u_k$ is a basis of U, then extend it to a basis of V: $u_1, \cdots, u_k, u_{k+1} \cdots, u_n$. Define $T \in \mathcal{L}(V)$, $Tu_1 = u_{k+1}, Tu_i = 0 (i \geqslant 2)$, we have U isn't invariant under T.

2. Show that the set of solutions V, to the system of linear equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \end{cases}$$

   is a subspace of $\mathbf{R}^3$. Find a basis for this subspace, V.

   Sol: Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \end{bmatrix}$, $X = (x_1, x_2, x_3)^T$ be a solution of the linear equations, then $AX = 0$, so we have $\mathbf{0}$ is a solution. If X_1, X_2 are two solutions, $\forall k_1, k_2 \in \mathbf{R}$, $A(k_1X_1 + k_2X_2) = k_1AX_1 + k_2AX_2 = 0 \Rightarrow k_1X_1 + k_2X_2 \in V$, thus V is a subspace of $\mathbf{R}^3$.

   And rank $A = 2$, so $\dim V = 3 - 2 = 1$, $(1, 1, 1)^T \in V$ which is a basis of V.

3. State and prove the **Fundamental Theorem of Linear Maps**.

4. Suppose V is a finite-dimensional inner product space and $T \in \mathcal{L}(V)$.

1. State the definition of invariant subspace.

   Sol: If U is a subspace of V, $\forall u \in U$, we have $Tu \in U$, we say U is invariant under T.

2. Let U be a subspace of V. Show that U and $U^\perp$ are invariant under T if and only if $P_U T = T P_U$.

   Sol: "$\Rightarrow$" $\forall v \in V$, v can be written as $v = u_1 + u_2$, $u_1 \in U$, $u_2 \in U^\perp$. Since U is invariant under T, $Tu_1 \in U$. Since $U^\perp$ is invariant under T, $Tu_2 \in U^\perp$. We have $P_U T v = P_U T(u_1 + u_2) = P_U T u_1 + P_U T u_2 = Tu_1 = T P_U (u_1 + u_2) = T P_U v$ holds for all $v \in V$, then $P_U T = T P_U$.

   "$\Leftarrow$" $\forall u_1 \in U$, since $P_U T = T P_U$, $P_U T u_1 = T P_U u_1 = T u_1 \Rightarrow T u_1 \in U$, so U is invariant under T.

   And $\forall u_2 \in U^\perp$, since $P_U T = T P_U$, $P_U T u_2 = T P_U u_2 = 0 \Rightarrow T u_2 \in U^\perp$, $U^\perp$ is invariant under T.

5. Let T be the linear operator on $\mathbf{R}^3$ defined by

$$T(x_1, x_2, x_3) = (-3x_1 + 3x_2 - 2x_3, -7x_1 + 6x_2 - 3x_3, x_1 - x_2 + 2x_3).$$

1. Determine the eigenspace of T corresponding to each eigenvalue.

   Sol:

$$T(x_1, x_2, x_3) = (-3x_1 + 3x_2 - 2x_3, -7x_1 + 6x_2 - 3x_3, x_1 - x_2 + 2x_3) = \begin{bmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

   Let $A = \begin{bmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{bmatrix}$, then $\mathcal{M}(T; e_1, e_2, e_3) = A$. The characteristic polynomial is $\det(\lambda I - A) = (\lambda - 2)^2(\lambda - 1)$.

   Solving $(2I - A)X = 0$, we have $X_1 = k \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $k \in \mathbf{R}$.

   Solving $(I - A)X = 0$, we have $X_2 = l \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $l \in \mathbf{R}$.

   Then the eigenspaces are $E(2, T) = \operatorname{span}\{(1, 1, -1)^T\}$, $E(1, T) = \operatorname{span}\{(1, 2, 1)^T\}$.

2. Find the Jordan form and a Jordan basis of T.

   Sol: Since $\dim E(2, T) = 1$, $\dim E(1, T) = 1$, the Jordan form is

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assume the Jordan basis is η_1, η_2, η_3, then $\mathcal{M}(T; \eta_1, \eta_2, \eta_3) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $T\eta_1 = 2\eta_1$, $T\eta_2 = \eta_1 + 2\eta_2$, $T\eta_3 = \eta_3$.

Take $\eta_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\eta_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, we can solve $(T - 2I)\eta_2 = \eta_1$, $\eta_2 = k \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $k \in \mathbf{R}$, simply, we take $k = 0$, then $\eta_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Let $P = (\eta_1, \eta_2, \eta_3)$, $P^{-1}AP = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3. Find the minimal polynomial of T.

   Sol: The minimal polynomial is $(\lambda - 2)^2(\lambda - 1)$.

4. Compute trace T and $\det T$.

   Sol: $\mathrm{Tr}(T) = 5$, $\det T = 4$.

6. Suppose T is a linear operator defined on $\mathbf{R}^4$ with $T^2 = -I$.

   1. Show that the only eigenvalues of $T_{\mathbf{C}}$ are i and $-i$, where $T_{\mathbf{C}}$ is the complexification of T.

      Sol: Since $T_{\mathbf{C}}^2 = -I$, let λ be the eigenvalues of T, then $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$.

   2. Show that v is an eigenvector of $T_{\mathbf{C}}$ with respect to i if and only if $\bar{v}$ is an eigenvector of TC with respect to $-i$, and hence show that there is a basis consisting of complex eigenvectors of $T_{\mathbf{C}}$ of the form $v_1, v_2, \bar{v}_1, \bar{v}_2$.

      Sol: $\forall v = u_1 + iu_2$, $u_1, u_2 \in \mathbf{R}^4$, $T_{\mathbf{C}}v = iv \iff Tu_1 + iTu_2 = -u_2 + iu_1 \iff Tu_1 = -u_2, Tu_2 = u_1 \iff T_{\mathbf{C}}(u_1 - iu_2) = Tu_1 - iTu_2 = -u_2 - iu_1 = -i(u_1 - iu_2) = -i\bar{v}$. The minimal polynomial of $T_{\mathbf{C}}$ is $\lambda^2 + 1 \Rightarrow T_{\mathbf{C}}$ is diagonalizable and the multiplicities of $\pm i$ are equal, then i has two linearly independent eigenvectors, denoted as v_1, v_2, $\bar{v}_1, \bar{v}_2$ are two linearly independent eigenvectors of $-i$, so $v_1, v_2, \bar{v}_1, \bar{v}_2$ is basis.

   3. Show that there is a basis of $\mathbf{R}^4$ with respect to which T has the following matrix representation

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Sol: $v_1 = u_1 + iu_2$, $v_2 = u_3 + iu_4$, $Tu_1 = u_2$, $Tu_2 = -u_1$, $Tu_3 = u_4$, $Tu_4 = -u_3$, then

$$\mathcal{M}(T; u_1, u_2, u_3, u_4) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

7. **(Jordan-Chevalley Decomposition)** Let T be a linear operator on a finite-dimensional complex vector space V , and let $\lambda_1, \lambda_2, \cdots \lambda_k$ be the distinct eigenvalues of T. Let $S : V \to V$ be the mapping defined by

$$S(x) = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k,$$

where, for each i, v_i is the unique vector in $G(\lambda_i, T)$ such that $x = v_1 + v_2 + \cdots + v_k$.

1. Prove that S is a diagonalizable linear operator on V.

   Sol: Let

$$\begin{aligned}
&(T-\lambda_1 I)^{m_{11}-1}v_{11}, \cdots, v_{11}, (T-\lambda_1 I)^{m_{12}-1}v_{12}, \cdots, v_{12}, \cdots, (T-\lambda_1 I)^{m_{1l_1}-1}v_{1l_1}, \cdots, v_{1l_1} \\
&(T-\lambda_2 I)^{m_{21}-1}v_{21}, \cdots, v_{21}, (T-\lambda_2 I)^{m_{22}-1}v_{22}, \cdots, v_{22}, \cdots, (T-\lambda_1 I)^{m_{2l_2}-1}v_{2l_2}, \cdots, v_{2l_2} \\
&\ddots \\
&(T-\lambda_k I)^{m_{k1}-1}v_{k1}, \cdots, v_{k1}, (T-\lambda_1 I)^{m_{k2}-1}v_{k2}, \cdots, v_{k2}, \cdots, (T-\lambda_1 I)^{m_{kl_k}-1}v_{kl_k}, \cdots, v_{kl_k}
\end{aligned}$$

   be the Jordan basis of T. We have $(T - \lambda_i I)^p v_{ij} \in G(\lambda_i, T)$.
   $\forall v \in G(\lambda_i, T)$, $v = 0 + 0 + \cdots + 0 + v + 0 + \cdots + 0$,

$$S(v) = \lambda_1 \cdot 0 + \cdots + \lambda_{i-1} \cdot 0 + \lambda_i v + \lambda_{i+1} \cdot 0 + \cdots + \lambda_k \cdot 0 = \lambda_i v \Rightarrow v \text{ is an eigenvector of } S \text{ w.r.t. } \lambda_i$$

   so all the Jordan basis are the eigenvectors of S, S is diagonalizable.

2. Let $N = T - S$. Prove that N is nilpotent and commutes with S, that is $SN = NS$.

   Sol:

$$\begin{aligned}
&\mathcal{M}(T; (T-\lambda_1 I)^{m_{11}-1}v_{11}, \cdots, v_{kl_k}) \\
= &\begin{bmatrix} J_{m_{11}}(\lambda_1) & & & & & & & \\ & J_{m_{12}}(\lambda_1) & & & & & & \\ & & \ddots & & & & & \\ & & & J_{m_{1l_1}}(\lambda_1) & & & & \\ & & & & J_{m_{21}}(\lambda_2) & & & \\ & & & & & J_{m_{22}}(\lambda_2) & & \\ & & & & & & \ddots & \\ & & & & & & & J_{m_{kl_k}}(\lambda_k) \end{bmatrix}
\end{aligned}$$

$$
\mathcal{M}(S;(T-\lambda_1 I)^{m_{11}-1}v_{11},\cdots,v_{kl_k})
$$

$$
=\begin{bmatrix}
\lambda_1 I_{m_{11}} & & & & & & & \\
& \lambda_1 I_{m_{12}} & & & & & & \\
& & \ddots & & & & & \\
& & & \lambda_1 I_{m_{1l_1}} & & & & \\
& & & & \lambda_2 I_{m_{21}} & & & \\
& & & & & \lambda_2 I_{m_{22}} & & \\
& & & & & & \ddots & \\
& & & & & & & \lambda_k I_{m_{kl_k}}
\end{bmatrix}
$$

$$
\mathcal{M}(N;(T-\lambda_1 I)^{m_{11}-1}v_{11},\cdots,v_{kl_k})
$$

$$
=\mathcal{M}(T;(T-\lambda_1 I)^{m_{11}-1}v_{11},\cdots,v_{kl_k})-\mathcal{M}(S;(T-\lambda_1 I)^{m_{11}-1}v_{11},\cdots,v_{kl_k})
$$

$$
=\begin{bmatrix}
J_{m_{11}}(0) & & & & & & & \\
& J_{m_{12}}(0) & & & & & & \\
& & \ddots & & & & & \\
& & & J_{m_{1l_1}}(0) & & & & \\
& & & & J_{m_{21}}(0) & & & \\
& & & & & J_{m_{22}}(0) & & \\
& & & & & & \ddots & \\
& & & & & & & J_{m_{kl_k}}(0)
\end{bmatrix}
$$

Since $J_p(0)$ is nilpotent, $\mathcal{M}(N; (T-\lambda_1 I)^{m_{11}-1}v_{11}, \cdots, v_{kl_k})$ is nilpotent, N is nilpotent.

$$\begin{aligned}
&\mathcal{M}(SN; (T-\lambda_1 I)^{m_{11}-1}v_{11}, \cdots, v_{kl_k}) \\
&= \mathcal{M}(S; (T-\lambda_1 I)^{m_{11}-1}v_{11}, \cdots, v_{kl_k})\mathcal{M}(N; (T-\lambda_1 I)^{m_{11}-1}v_{11}, \cdots, v_{kl_k}) \\
&= \begin{bmatrix} \lambda_1 I_{m_{11}} J_{m_{11}}(0) & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 I_{m_{1l_1}} J_{m_{1l_1}}(0) & & & & \\ & & & \ddots & & & \\ & & & & \lambda_2 I_{m_{2l_2}} J_{m_{2l_2}}(0) & & \\ & & & & & \ddots & \\ & & & & & & \lambda_k I_{m_{kl_k}} J_{m_{kl_k}}(0) \end{bmatrix} \\
&= \begin{bmatrix} J_{m_{11}}(0)\lambda_1 I_{m_{11}} & & & & & & \\ & \ddots & & & & & \\ & & J_{m_{1l_1}}(0)\lambda_1 I_{m_{1l_1}} & & & & \\ & & & \ddots & & & \\ & & & & J_{m_{2l_2}}(0)\lambda_2 I_{m_{2l_2}} & & \\ & & & & & \ddots & \\ & & & & & & J_{m_{kl_k}}(0)\lambda_k I_{m_{kl_k}} \end{bmatrix} \\
&= \mathcal{M}(N; (T-\lambda_1 I)^{m_{11}-1}v_{11}, \cdots, v_{kl_k})\mathcal{M}(S; (T-\lambda_1 I)^{m_{11}-1}v_{11}, \cdots, v_{kl_k}) \\
&= \mathcal{M}(NS; (T-\lambda_1 I)^{m_{11}-1}v_{11}, \cdots, v_{kl_k})
\end{aligned}$$

so $SN = NS$.