Step-1

We have to find what matrix represents $\frac{d^2}{dt^2}$ on the space $\mathbf{P}_3$.

For this we have construct a 4 by 4 matrix from the standard basis $1, t, t^2, t^3$.

Step-2

Now

$$\begin{aligned}\frac{d^2}{dt^2}(1) &= \frac{d}{dt}\left(\frac{d}{dt}1\right)\\ &= \frac{d}{dt}(0)\\ &= 0\\ &= 0.1 + 0.t + 0.t^2 + 0.t^3\end{aligned}$$

Step-3

And

$$\begin{aligned}\frac{d^2}{dt^2}(t) &= \frac{d}{dt}\left(\frac{d}{dt}(t)\right)\\ &= \frac{d}{dt}(1)\\ &= 0\\ &= 0.1 + 0.t + 0.t^2 + 0.t^3\end{aligned}$$

Step-4

And

$$\begin{aligned}\frac{d^2}{dt^2}(t^2) &= \frac{d}{dt}\left(\frac{d}{dt}(t^2)\right)\\ &= \frac{d}{dt}(2t)\\ &= 2\\ &= 2.1 + 0.t + 0.t^2 + 0.t^3\end{aligned}$$

Step-5

And

$$\begin{aligned}\frac{d^2}{dt^2}\left(t^3\right) &= \frac{d}{dt}\left(\frac{d}{dt}\left(t^3\right)\right)\\ &= \frac{d}{dt}\left(3t^2\right)\\ &= 6t\\ &= 0.1 + 6.t + 0.t^2 + 0.t^3\end{aligned}$$

Step-6

We can write this in matrix from as follows $M = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Hence $\boxed{M = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$ is the matrix that represents $\frac{d^2}{dt^2}$ from the standard basis $1, t, t^2, t^3$.

Step-7

We have to find the null space of *M*.

For this, we have to solve $Mx = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x_3 = 0,\ 6x_4 = 0$$

$$\Rightarrow \quad x_3 = 0,\ x_4 = 0$$

Step-8

Here x_1, x_2 are free variables.

So choose $x_1 = s, x_2 = t$, where *s, t* are parameters.

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 0 \\ 0 \end{bmatrix}$$
$$= s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Step-9

The basis of null space of $\boxed{M = \{(1,0,0,0), (0,1,0,0)\}}$.

Step-10

We get that the null space is spanned by $(1,0,0,0), (0,1,0,0)$, which gives linear P_1, second derivative of linear functions are zero.

We get the 3, 4 columns of *M* are independent

Hence the column space is same as null space because the second derivatives of cubic polynomials are linear.