

## Step-1

Let the vector  $x = (1, 1, \dots, 1)$ .

The Rayleigh quotient is given by,

$$R(x) = \frac{x^T A x}{x^T x}$$

## Step-2

We have,

$$\begin{aligned} x^T x &= (1, 1, \dots, 1) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\ &= 1 + 1 + \dots + 1 \\ &= n \end{aligned}$$

## Step-3

Also, we get

$$\begin{aligned} x^T A x &= (1, 1, \dots, 1) \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\ &= (1, 1, \dots, 1) \begin{bmatrix} a_{11} + a_{12} + \cdots + a_{1n} \\ a_{21} + a_{22} + \cdots + a_{2n} \\ \vdots \\ a_{n1} + a_{n2} + \cdots + a_{nn} \end{bmatrix} \\ &= a_{11} + a_{12} + \cdots + a_{1n} + a_{21} + a_{22} + \cdots + a_{2n} + \cdots + a_{n1} + a_{n2} + \cdots + a_{nn} \end{aligned}$$

Therefore,  $x^T A x$  is equal to the sum of all the entries of the matrix  $A$ .

## Step-4

Thus, we get the following:

$$R(x) = \frac{x^T Ax}{x^T x} = \frac{a_{11} + a_{12} + \dots + a_{1n} + a_{21} + a_{22} + \dots + a_{2n} + \dots + a_{n1} + a_{n2} + \dots + a_{nn}}{n}$$

We know that Rayleigh's quotient always lies between the smallest eigenvalue  $\hat{\lambda}_1$  and the largest eigenvalue  $\hat{\lambda}_n$ .

Thus,

$$\hat{\lambda}_1 \leq \frac{a_{11} + a_{12} + \dots + a_{1n} + a_{21} + a_{22} + \dots + a_{2n} + \dots + a_{n1} + a_{n2} + \dots + a_{nn}}{n} \leq \hat{\lambda}_n$$

This gives,

$$n\hat{\lambda}_1 \leq a_{11} + a_{12} + \dots + a_{1n} + a_{21} + a_{22} + \dots + a_{2n} + \dots + a_{n1} + a_{n2} + \dots + a_{nn} \leq n\hat{\lambda}_n.$$

Therefore, the relation is:  $n\hat{\lambda}_1 \leq \boxed{\text{sum of all the entries } a_{ij}} \leq n\hat{\lambda}_n$ .