Step-1

The objective is to find the projection matrix that projects the $x-y$ plane onto the line $x+y=0$.

The vector that satisfies the vector equation $x+y=0$ is:

$x=-y$.

Take $y=1$, then $x=-1$.

So all the points on the line are in the column space of the matrix $a=(-1,1)=\begin{bmatrix}-1\\1\end{bmatrix}$.

The matrix that projects onto the line through $a=(-1,1)$ is:

$$P=\frac{aa^T}{a^Ta}.$$

Step-2

Determine aa^T and a^Ta.

$$\begin{aligned}aa^T&=\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}^T\\&=\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1&1\end{bmatrix}\\&=\begin{bmatrix}(-1)(-1)&(-1)(1)\\(-1)(1)&(1)(1)\end{bmatrix}\\&=\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\end{aligned}$$

$$\begin{aligned}a^Ta&=\begin{bmatrix}-1\\1\end{bmatrix}^T\begin{bmatrix}-1\\1\end{bmatrix}\\&=\begin{bmatrix}-1&1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}\\&=(-1)^2+1^2\\&=2\end{aligned}$$

Step-3

Substitute the values in $P = \frac{aa^T}{a^T a}$.

$$P = \frac{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}{2}$$
$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore, the matrix that projects onto the line $x + y = 0$ is $\boxed{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}}$.