Step-1

Given that $A=\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ is Hermitian (complex b)

(a)

Need to find the pivots and determinant of the Hermitian.

$A=\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$

Apply this row operation $R_2 \to R_2 - \frac{\bar{b}}{a} R_1$ and get;

$$\begin{pmatrix} a & b \\ 0 & c-\frac{|b|^2}{a} \end{pmatrix}$$

So the pivots of A are a and $c-\frac{|b|^2}{a}$

Step-2

The determinant is,

$$\det A = a\left(c-\frac{|b|^2}{a}\right)$$

$$= ac-|b|^2$$

Therefore, the pivots of A are $\boxed{a \text{ and } c-\frac{|b|^2}{a}}$ and the determinant is $\boxed{ac-|b|^2}$.

Step-3

(b).

Need to complete the square for $x^H Ax$.

Given $x^H = \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \end{pmatrix}$ can be complex.

So,

$$x^H Ax = \begin{pmatrix}\overline{x}_1 & \overline{x}_2\end{pmatrix}\begin{pmatrix}a & b\\ b & c\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$= \begin{pmatrix}a\overline{x}_1 + b\overline{x}_2 & b\overline{x}_1 + c\overline{x}_2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$= \left(a\overline{x}_1 + b\overline{x}_2\right)x_1 + \left(b\overline{x}_1 + c\overline{x}_2\right)x_2$$

$$= a\overline{x}_1 x_1 + b\overline{x}_2 x_1 + b\overline{x}_1 x_2 + c\overline{x}_2 x_2$$

$$= a\overline{x}_1 x_1 + b\left(\overline{x}_2 x_1 + \overline{x}_1 x_2\right) + c\overline{x}_2 x_2$$

$$= a\left|\overline{x}_1\right|^2 + b.2\,\mathrm{Re}\,\overline{x}_1 x_2 + c\left|\overline{x}_2\right|^2$$

Therefore, the value of $x^H Ax$ is $\boxed{a\left|\overline{x}_1\right|^2 + b.2\,\mathrm{Re}\,\overline{x}_1 x_2 + c\left|\overline{x}_2\right|^2}$.

Step-4

Now, complete the square by using pivots a and $c - \frac{|b|^2}{a}$.

Therefore

$$a\left|\overline{x}_1\right|^2 + 2\,\mathrm{Re}\,b\overline{x}_1 x_2 + c\left|\overline{x}_2\right|^2$$

$$= a\left|x_1 + \left(\frac{b}{a}\right)x_2\right|^2 + \left(c - \frac{|b|^2}{a}\right)\left|\overline{x}_2\right|^2.$$

Therefore, $$a\left|\overline{x}_1\right|^2 + b.2\,\mathrm{Re}\,\overline{x}_1 x_2 + c\left|\overline{x}_2\right|^2 == a\left|x_1 + \left(\frac{b}{a}\right)x_2\right|^2 + \boxed{\left(c - \frac{|b|^2}{a}\right)}\left|\overline{x}_2\right|^2.$$

Step-5

(c).

As $x^H Ax$ is a sum of squares, so the pivots are greater than zero.

$a > 0$ and $c - \frac{|b|^2}{a} > 0$

This implies;

$a > 0$ and $ac - |b|^2 > 0$.

$a > 0$ and $ac > |b|^2$.

Now these facts ensure that $\boxed{A \text{ is positive definite}}$.

Step-6

(d).

Let $B = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 4+i \\ 4-i & 6 \end{pmatrix}$.

$$\begin{aligned}\det B &= 2 - (1-i)(1+i) \\ &= 2 - (1+1) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

Therefore, B is positive semi definite

Hence, B is $\boxed{\text{not}}$ positive definite.

Step-7

Now consider,

Step-8

$C = \begin{pmatrix} 3 & 4+i \\ 4-i & 6 \end{pmatrix}$

Thus,

$\det C = 18 - (4-i)(4+i)$

$= 18 - (16+1)$

$= 18 - 17$

$= 1$

Therefore, $\boxed{C \text{ is positive definite.}}$

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