Determinant

Lecture 30

Dept. of Math., SUSTech

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$T \in \mathcal{L}(V)$. $\dim V = n$.

$\lambda_1, \cdots, \lambda_n$ eigenvalues

$\det T = \lambda_1 \cdots \lambda_n$

$\mathcal{M}(T, (v_1, v_2, \cdots, v_n)) = A$

$\det A = \det T$

$\mathcal{M}(T, (u_1, u_2, \cdots, u_n)) = B$

$B = M^{-1} A M$ $\quad \det(AB) = \det(BA)$

$\det(B) = \det(\underbrace{M^{-1}}\underbrace{AM})$

$= \det(AM \cdot M^{-1}) = \det A$

$$T(w_1, \cdots, w_n) = (w_1, \cdots, w_n) \underbrace{\begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D}$$

$\det D = \lambda_1 \cdots \lambda_n = \det T$

5.27 complex vector space V

$\dim V = n$

$\det A = \det B = \det D$

Determinant

Determinant of an Operator

10.20 **Definition** ***determinant of an operator***, $\det T$

Suppose $T \in \mathcal{L}(V)$.

- If $\mathbf{F} = \mathbf{C}$, then the ***determinant*** of T is the product of the eigenvalues of T, with each eigenvalue repeated according to its multiplicity.
- If $\mathbf{F} = \mathbf{R}$, then the ***determinant*** of T is the product of the eigenvalues of $T_{\mathbf{C}}$, with each eigenvalue repeated according to its multiplicity.

The determinant of T is denoted by $\det T$.

Determinant and Characteristic Polynomial

10.22 Determinant and characteristic polynomial

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then $\det T$ equals $(-1)^n$ times the constant term of the characteristic polynomial of T.

Characteristic polynomial of T

10.23 Characteristic polynomial, trace, and determinant

Suppose $T \in \mathcal{L}(V)$. Then the characteristic polynomial of T can be written as

$$q(z) = z^n - (\operatorname{trace} T)z^{n-1} + \cdots + (-1)^n(\det T).$$

Determinant of Matrix

10.24 Invertible is equivalent to nonzero determinant

An operator on V is invertible if and only if its determinant is nonzero.

10.25 Characteristic polynomial of T equals $\det(zI - T)$

Suppose $T \in \mathcal{L}(V)$. Then the characteristic polynomial of T equals $\det(zI - T)$.

$A \Leftrightarrow \det(\lambda I - A)$

$T \Leftrightarrow \det(zI - T)$

Permutation

10.27 **Definition** ***permutation,*** perm n

- A ***permutation*** of $(1, \ldots, n)$ is a list $(m_1, \ldots, m_n)$ that contains each of the numbers $1, \ldots, n$ exactly once.
- The set of all permutations of $(1, \ldots, n)$ is denoted perm n.

Sign of a permutation

10.30 **Definition** ***sign of a permutation***

- The ***sign*** of a permutation $(m_1, \ldots, m_n)$ is defined to be 1 if the number of pairs of integers (j, k) with $1 \le j < k \le n$ such that j appears after k in the list $(m_1, \ldots, m_n)$ is even and -1 if the number of such pairs is odd.

- In other words, the sign of a permutation equals 1 if the natural order has been changed an even number of times and equals -1 if the natural order has been changed an odd number of times.

Properties of Determinant

10.32 Interchanging two entries in a permutation

Interchanging two entries in a permutation multiplies the sign of the permutation by -1.

Determinant of a matrix

10.33 **Definition** ***determinant of a matrix,*** det A "Big Formula"

Suppose A is an n-by-n matrix

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \dots & A_{n,n} \end{pmatrix}.$$

The ***determinant*** of A, denoted det A, is defined by

$$\det A = \sum_{(m_1,\dots,m_n)\in \operatorname{perm} n} \big(\operatorname{sign}(m_1,\dots,m_n)\big) A_{m_1,1} \cdots A_{m_n,n}.$$

Properties of Determinant

10.36 Interchanging two columns in a matrix

Suppose A is a square matrix and B is the matrix obtained from A by interchanging two columns. Then

$$\det A = -\det B.$$

10.37 Matrices with two equal columns

If A is a square matrix that has two equal columns, then $\det A = 0$.

Permuting the columns of a matrix

10.38 Permuting the columns of a matrix

Suppose $A = \begin{pmatrix} A_{\cdot,1} & \dots & A_{\cdot,n} \end{pmatrix}$ is an n-by-n matrix and $(m_1, \dots, m_n)$ is a permutation. Then

$$\det \begin{pmatrix} A_{\cdot,m_1} & \dots & A_{\cdot,m_n} \end{pmatrix} = \big(\operatorname{sign}(m_1, \dots, m_n)\big) \det A.$$

Properties of Determinant

10.39 Determinant is a linear function of each column

Suppose k, n are positive integers with $1 \leq k \leq n$. Fix n-by-1 matrices $A_{\cdot,1}, \ldots, A_{\cdot,n}$ except $A_{\cdot,k}$. Then the function that takes an n-by-1 column vector $A_{\cdot,k}$ to

$$\det\begin{pmatrix} A_{\cdot,1} & \ldots & A_{\cdot,k} & \ldots & A_{\cdot,n} \end{pmatrix}$$

is a linear map from the vector space of n-by-1 matrices with entries in $\mathbf{F}$ to $\mathbf{F}$.

10.40 Determinant is multiplicative

Suppose A and B are square matrices of the same size. Then

$$\det(AB) = \det(BA) = (\det A)(\det B).$$

Properties of Determinant

Now we can prove that the determinant of the matrix of an operator is independent of the basis with respect to which the matrix is computed. Note the similarity of the proof of the analogous result about the trace (see 10.15).

10.41 Determinant of matrix of operator does not depend on basis

Let $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then

$$\det \mathcal{M}\big(T, (u_1, \ldots, u_n)\big) = \det \mathcal{M}\big(T, (v_1, \ldots, v_n)\big).$$

The result below states that the determinant of an operator equals the determinant of the matrix of the operator.

10.42 Determinant of an operator equals determinant of its matrix

Suppose $T \in \mathcal{L}(V)$. Then $\det T = \det \mathcal{M}(T)$.

Properties of Determinant

10.44 Determinant is multiplicative

Suppose $S, T \in \mathcal{L}(V)$. Then

$$\det(ST) = \det(TS) = (\det S)(\det T).$$

Isometries have determinant with absolute value 1.

Theorem $S \in \mathcal{L}(V)$ isometry

Suppose V is an inner product space and $S \in \mathscr{L}(V)$ is an isometry. Then $|\det S| = 1$.

$T \in \mathcal{L}(V)$

$$T = S\sqrt{T^*T}$$

$$\det T = \det S \cdot \det\sqrt{T^*T}$$

$$|\det T| = \underbrace{|\det S|}_{1} \cdot |\det\sqrt{T^*T}|$$

The sign of the determinant

Most applied mathematicians agree that determinants should rarely be used in serious numeric calculations.

$T \in \mathcal{L}(V)$ invertible

$\det T \gtrless 0$ V real inner product space

$\lambda_1 \cdots \lambda_n$ S isometry

10.47 $|\det T| = \det \sqrt{T^*T}$

Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Then

$$|\det T| = \det \sqrt{T^*T}.$$

10.48 **Definition** ***box***

A ***box*** in $\mathbf{R}^n$ is a set of the form

$$\{(y_1, \ldots, y_n) \in \mathbf{R}^n : x_j < y_j < x_j + r_j \text{ for } j = 1, \ldots, n\},$$

where $r_1, \ldots, r_n$ are positive numbers and $(x_1, \ldots, x_n) \in \mathbf{R}^n$. The numbers $r_1, \ldots, r_n$ are called the ***side lengths*** of the box.

Volume of a Box

Volume of a Box

10.49 Definition ***volume of a box***

The ***volume*** of a box B in $\mathbf{R}^n$ with side lengths $r_1, \dots, r_n$ is defined to be $r_1 \cdots r_n$ and is denoted by volume B.

Volume

10.50 Definition ***volume***

Suppose $\Omega \subset \mathbf{R}^n$. Then the ***volume*** of Ω, denoted volume Ω, is defined to be the infimum of

$$\text{volume } B_1 + \text{volume } B_2 + \cdots,$$

where the infimum is taken over all sequences $B_1, B_2, \ldots$ of boxes in $\mathbf{R}^n$ whose union contains Ω.

Integrals

10.51 **Notation** $T(\Omega)$

For T a function defined on a set Ω, define $T(\Omega)$ by

$$T(\Omega) = \{Tx : x \in \Omega\}.$$

We begin by looking at positive operators.

10.52 Positive operators change volume by factor of determinant

Suppose $T \in \mathcal{L}(\mathbf{R}^n)$ is a positive operator and $\Omega \subset \mathbf{R}^n$. Then

$$\text{volume } T(\Omega) = (\det T)(\text{volume } \Omega).$$

$\sigma : \Omega \to \mathbb{R}^n$ function

open subset

$f(x) - f(x_0) \approx f'(x_0)(x - x_0)$

$T(x) = \sigma'(x)$

$\sigma(x) = (\sigma_1(x), \cdots, \sigma_n(x))$

$\det T = \det m(\sigma'(x))$

Change of variables in an integral

Our next tool is the following result, which states that isometries do not change volume.

10.53 An isometry does not change volume

Suppose $S \in \mathcal{L}(\mathbf{R}^n)$ is an isometry and $\Omega \subset \mathbf{R}^n$. Then

$$\text{volume } S(\Omega) = \text{volume } \Omega.$$

Now we can prove that an operator $T \in \mathscr{L}(\mathbb{R}^n)$ changes volume by a factor of $|\det T|$.

10.54 T changes volume by factor of $|\det T|$

Suppose $T \in \mathcal{L}(\mathbf{R}^n)$ and $\Omega \subset \mathbf{R}^n$. Then

$$\text{volume } T(\Omega) = |\det T|(\text{volume } \Omega).$$

Integral: Definition

10.55 Definition ***integral,*** $\int_\Omega f$

If $\Omega \subset \mathbf{R}^n$ and f is a real-valued function on Ω, then the ***integral*** of f over Ω, denoted $\int_\Omega f$ or $\int_\Omega f(x)\,dx$, is defined by breaking Ω into pieces small enough that f is almost constant on each piece. On each piece, multiply the (almost constant) value of f by the volume of the piece, then add up these numbers for all the pieces, getting an approximation to the integral that becomes more accurate as Ω is divided into finer pieces.

$$\mathcal{M}(\sigma'(x)) = \begin{bmatrix} D_1\sigma_1 & D_2\sigma_1 & \cdots & D_n\sigma_1 \\ D_1\sigma_2 & D_2\sigma_2 & \cdots & D_n\sigma_2 \\ \vdots & & & \\ D_1\sigma_n & D_2\sigma_n & & D_n\sigma_n \end{bmatrix}$$

Integral: Derivative

10.56 Definition ***differentiable, derivative,*** $\sigma'(x)$

Suppose Ω is an open subset of $\mathbf{R}^n$ and σ is a function from Ω to $\mathbf{R}^n$. For $x \in \Omega$, the function σ is called ***differentiable*** at x if there exists an operator $T \in \mathcal{L}(\mathbf{R}^n)$ such that

$$\lim_{y \to 0} \frac{\|\sigma(x+y) - \sigma(x) - Ty\|}{\|y\|} = 0.$$

If σ is differentiable at x, then the unique operator $T \in \mathcal{L}(\mathbf{R}^n)$ satisfying the equation above is called the ***derivative*** of σ at x and is denoted by $\sigma'(x)$.

Change of variables in an integral

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当然, 这些事实是需要证明的. 但是, 这些结论的详细证明, 实在过于冗长繁琐, 不论是对教还是学, 都是沉重的负担. 因此, 我们有意识地舍弃严格的证明, 而采用一种比较温和的、不甚严格的处理. 这样做, 既不会掩盖二重积分换元的思想本质, 由于我们的习题中常见的那些区域相当简单, 这种处理方式也足以够用.

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摘自《数学分析教程》(常庚哲、史济怀, 第一版, 第二册第187页)

The Area Formula

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map and $n \leq m$. Then for each L^n-measurable set $A \subseteq \mathbb{R}^n$,

$$\int_A Jf(x)dx = \int_{\mathbb{R}^m} H^0(A \cap f^{-1}(y))dH^n(y),$$

where $Jf(x)$ is Jacobian of $f(x)$.

Advanced Mathematics, Volume 1,

Geometric Measure Theory–An introduction, Fanghua Lin and Xiaoping Yang, 2001.

Prof. Lin

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Education

Ph.D., Mathematics, University of Minnesota, USA, 1985.

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Research Interests

My research interests are in nonlinear partial differential equations, geometric measure theory and geometric & applied analysis. Recent researches are concentrated mainly on the analysis of classical and complex fluids including liquid crystals, the theory of homogenizations and geometric variational problems.

Change of variables in an integral

10.58 Change of variables in an integral

Suppose Ω is an open subset of $\mathbf{R}^n$ and $\sigma : \Omega \to \mathbf{R}^n$ is differentiable at every point of Ω. If f is a real-valued function defined on $\sigma(\Omega)$, then

$$\int_{\sigma(\Omega)} f(y)\, dy = \int_{\Omega} f\big(\sigma(x)\big) |\det \sigma'(x)|\, dx.$$

Example: Polar Coordinates

10.59 Example ***polar coordinates***

Define $\sigma : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\sigma(r, \theta) = (r \cos\theta, r \sin\theta),$$

where we have used r, θ as the coordinates instead of x_1, x_2 for reasons that will be obvious to everyone familiar with polar coordinates (and will be a mystery to everyone else). For this choice of σ, the matrix of partial derivatives corresponding to 10.57 is

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix},$$

as you should verify. The determinant of the matrix above equals r, thus explaining why a factor of r is needed when computing an integral in polar coordinates.

For example, note the extra factor of r in the following familiar formula involving integrating a function f over a disk in $\mathbf{R}^2$:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y)\, dy\, dx = \int_{0}^{2\pi} \int_{0}^{1} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

Example: Spherical Coordinates

10.60 **Example** *spherical coordinates*

Define $\sigma : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$\sigma(\rho, \varphi, \theta) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi),$$

where we have used ρ, θ, φ as the coordinates instead of x_1, x_2, x_3 for reasons that will be obvious to everyone familiar with spherical coordinates (and will be a mystery to everyone else). For this choice of σ, the matrix of partial derivatives corresponding to 10.57 is

$$\begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix},$$

as you should verify. The determinant of the matrix above equals $\rho^2 \sin\varphi$, thus explaining why a factor of $\rho^2 \sin\varphi$ is needed when computing an integral in spherical coordinates.

For example, note the extra factor of $\rho^2 \sin\varphi$ in the following familiar formula involving integrating a function f over a ball in $\mathbf{R}^3$:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x, y, z)\, dz\, dy\, dx$$

$$= \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{1} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Homework Assignment 30

10.B: 1, 2, 3, 7, 8.