Step-1

Given $a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, and $b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$

We have to solve the least squares problem $Ax = b$ by using $A = QR$

Step-2

$$q_1 = \frac{a_1}{\|a_1\|}$$
$$= \frac{1}{\sqrt{1^2 + 2^2 + 2^2}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

Step-3

$q_2 = \dfrac{\beta}{\|\beta\|}$ where

$$\beta = a_2 - \left(q_1^T a_2\right) q_1$$

Step-4

$$q_1^T a_2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$
$$= \frac{1+6+2}{3}$$
$$= 3$$

Step-5

$$\left(q_1^T a_2\right) q_1 = 3\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Step-6

$$\beta = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\|\beta\| = \sqrt{0+1+1}$$
$$= \sqrt{2}$$

Step-7

Therefore

$$q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Step-8

$$q_1^T a_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$
$$= \frac{1+4+4}{3}$$
$$= 3$$

Step-9

$$q_1^T a_2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$
$$= \frac{1+6+2}{3}$$
$$= 3$$

$$q_2^T a_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$
$$= \frac{0+3-1}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}}$$
$$= \sqrt{2}$$

Step-10

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$
$$= QR$$

Therefore $A = QR = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$

Step-11

By the method of least squares,

$R\,\hat{x} = Q^T b$ where $b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$

And $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$

Step-12

Now

$$R\hat{x} = Q^T b$$

$$\Rightarrow \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3C + 3D \\ \sqrt{2}D \end{bmatrix} = \begin{bmatrix} 5/3 \\ 0 \end{bmatrix}$$

Step-13

$$\Rightarrow 3C + 3D = \frac{5}{3} \text{ and } \sqrt{2}D = 0$$

$$\Rightarrow D = 0, 3C = \frac{5}{3}$$

Therefore $C = \frac{5}{9}, D = 0$

Hence the solution of the system $Ax = b$ is $\hat{x} = \begin{bmatrix} 5/9 \\ 0 \end{bmatrix}$