Step-1

Consider the system of equations:

$$\begin{aligned} x+3y+5z&=1 \\ 2x+6y+9z&=5\text{and} \\ -x-3y+3z&=5 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Step-2

Now consider the system,

$$\begin{aligned} x+3y+5z&=1 \\ 2x+6y+9z&=5 \\ -x-3y+3z&=5 \end{aligned}$$

$$A=\begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix},$$

$$x=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$b=\begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

Step-3

Given that;

$$Ax=b$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

$$\xrightarrow[R_3+R_1]{R_2-2R_1,}\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

Step-4

$$\xrightarrow{\frac{1}{3}R_2,\frac{1}{6}R_3}\begin{bmatrix}1&3&3\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\xrightarrow{R_3-R_2,R_1-3R_2}\begin{bmatrix}1&3&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

Step-5

This is in the form $Rx=b'$

R is in the reduced row echelon form.

First, three columns are pivot columns, thus x, z are pivot variables and y is the variable.

$$\begin{aligned}x+3y+3z&=1\\z&=1\end{aligned}$$

Step-6

Now

$$\begin{aligned}x&=1-3y-3z\\&=-3y-2\end{aligned}$$

Thus the complete solution of the system is;

$$\boxed{\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-3y-2\\y\\1\end{bmatrix}=y\begin{bmatrix}-3\\1\\0\end{bmatrix}+\begin{bmatrix}-2\\0\\1\end{bmatrix}}$$

Step-7

Now consider the system:

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\underline{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{R_3 - R_2} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Step-8

This is in the row echelon form, first and third are pivot columns *x, z* are pivot variables *y, t* are free variables.

Step-9

$$\begin{aligned} x + 3y + z + 2t &= 1 \\ 2z + 4t &= 1 \end{aligned}$$

$$\underline{R_3 - R_2} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2z &= 1 - 4t \\ z &= \frac{1-4t}{2} \\ &= \frac{1}{2} - 2t \end{aligned}$$

Step-10

Therefore,

$$\begin{aligned} x + 3y + \frac{1-4t}{2} + 2t &= 1 \\ 2x + 6y &= 1 \\ x &= -3y + \frac{1}{2} \end{aligned}$$

Step-11

Therefore,

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -3y + \frac{1}{2} \\ y \\ -2t + \frac{1}{2} \\ t \end{bmatrix}$$

Hence the complete solution of the system is;

$$\boxed{\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}$$