Step-1

Consider the matrix $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$.

The objective is to find all eigenvalues and eigenvectors of A and write two different diagonalizing matrices S.

Step-2

Find the eigenvalues for the matrix A as,

For that, the characteristic polynomial of A is,

$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{vmatrix} R_1 \to R_1+R_2+R_3\\ &= \begin{vmatrix}3-\lambda & 3-\lambda & 3-\lambda\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{vmatrix}\\ &= (3-\lambda)\begin{vmatrix}1 & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{vmatrix} \begin{matrix}R_2 \to R_2-R_1,\\ R_3 \to R_3-R_1\end{matrix}\\ &= (3-\lambda)\begin{vmatrix}1 & 1 & 1\\ 0 & -\lambda & 0\\ 0 & 0 & -\lambda\end{vmatrix}\end{aligned}$$

$$=(3-\lambda)(1)(-\lambda)(-\lambda)$$

Find the eigenvalues by equating the characteristic polynomial to zero.

That is,

$$(3-\lambda)(1)(-\lambda)(-\lambda)=0$$

Therefore, the eigenvalues are $\lambda_1=0, \lambda_2=0,$ and $\lambda_3=3$.

Step-3

Find the eigenvector X corresponding to the eigenvalue $\lambda_1=0$ as,

$$\begin{aligned}(A-\lambda_1 I)x &= 0\\ (A-0(I))x &= 0 \qquad \text{since } \lambda_1 = 0\\ Ax &= 0\end{aligned}$$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = 0$$

By applying the row operation on the coefficient matrix, the system reduces to $\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = 0$

From the above obtain the system,

$x_1 + x_2 + x_3 = 0$ â€¦â€¦ (1)

Supposing x_2, x_3 are free variables and letting $x_2 = k_1, x_3 = k_2$, obtain $x_1 = -k_1 - k_2$

So, the solution set of $(A-\lambda_1 I)x = 0$ is

$$\begin{aligned}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} &= \begin{bmatrix}-k_1 - k_2\\ k_1\\ k_2\end{bmatrix}\\ &= k_1\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix} + k_2\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\end{aligned}$$

So, the eigen vectors corresponding to the repeated eigen values $\lambda_1 = 0, \lambda_2 = 0$ are

$$\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

Step-4

Find the eigenvector X corresponding to the eigenvalue $\lambda_3 = 3$ as,

$$(A-\lambda_3 I)x=0$$

$$(A-3(I))x=0 \qquad \text{since } \lambda_3=3$$

$$\left(\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

Applying the row operations on the coefficient matrix,

$$\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}\sim\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}\begin{matrix}R_2\to 2R_2+R_1,\\R_3\to 2R_3+R_1\end{matrix}$$

$$\sim\begin{bmatrix}-2&1&1\\0&-3&3\\0&3&-3\end{bmatrix}\begin{matrix}R_3\to R_3+R_2\\R_2/-3\end{matrix}$$

$$\sim\begin{bmatrix}-2&1&1\\0&1&-1\\0&0&0\end{bmatrix}$$

Step-5

This is the reduced matrix and so, we rewrite the homogeneous equations from this as,

$$\left.\begin{matrix}-2x_1+x_2+x_3=0\\x_2-x_3=0\end{matrix}\right\}$$ â€¦â€¦ (2)

$\Rightarrow x_2=x_3,$

$\Rightarrow x_1=x_3$

Using $x_3=1,$ obtain the solution set $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ is the eigen vector corresponding to $\lambda_3=3$

Step-6

While the number of eigen values is equal to the number of eigen vectors of A, we say that A is diagonalizable.

Using the eigen vectors as the columns of the matrix *S*, we see that the diagonalization of the given matrix is $A = S\Lambda S^{-1}$ where $S = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, S^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ â€¦â€¦ (3)

Step-7

On the other hand, we try for the other diagonalization.

Let us consider (1) and write $x_3 = -x_1 - x_2$

Then the solution set is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = m\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + n\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ where $x_1 = m, x_2 = n$ are the parameters.

So, the other possible eigen vectors corresponding to $\lambda_1 = 0, \lambda_2 = 0$ are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Similarly, considering (2), the eigen vector corresponding to $\lambda_3 = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Using these eigen vectors as the columns of *S*, we get $A = S\Lambda S^{-1}$ where $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $S^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ â€¦â€¦ (4)

Observe that (3) and (4) are the different diagonalizations for the same matrix.