4 Suppose $T \in \mathcal{L}(V)$ and $\alpha, \beta \in \mathbf{F}$ with $\alpha \neq \beta$. Prove that

$$G(\alpha, T) \cap G(\beta, T) = \{0\}.$$

$G(\alpha,T) = \operatorname{null}(T-\alpha I)^{\dim V}$ $\quad G(\beta,T) = \operatorname{null}(T-\beta I)^{\dim V}$ $\quad \dim V = n$

$\forall\, v \in G(\alpha,T) \cap G(\beta,T)$, $\alpha \neq \beta$

If $v \neq 0$, by 8.13 v, v is linearly independent $\ast\!\!\to$

so $v = 0$ i.e. $G(\alpha,T) \cap G(\beta,T) = \{0\}$.

5 Suppose $T \in \mathcal{L}(V)$, m is a positive integer, and $v \in V$ is such that $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that

$$v, Tv, T^2 v, \ldots, T^{m-1} v$$

is linearly independent.

Suppose $a_0 v + a_1 Tv + a_2 T^2 v + \cdots + a_{m-1} T^{m-1} v = 0$. $(\ast)$ $\quad a_0, \cdots, a_{m-1} \in \mathbb{F}$

$T^{m-1}(a_0 v + a_1 Tv + a_2 T^2 v + \cdots + a_{m-1} T^{m-1} v) = 0$

$\Rightarrow a_0 T^{m-1} v + a_1 T^m v + \cdots + a_{m-1} T^{2(m-1)} v = 0$ $\quad (T^k v = 0,\ k > m-1)$

$\Rightarrow a_0 T^{m-1} v = 0$ $\quad (T^{m-1} v \neq 0)$

$\Rightarrow a_0 = 0$

so $(\ast) \Rightarrow a_1 Tv + a_2 T^2 v + \cdots + a_{m-1} T^{m-1} v = 0$

$\Rightarrow T^{m-2}(a_1 Tv + \cdots + a_{m-1} T^{m-1} v) = 0$

$\Rightarrow a_1 T^{m-1} v + \cdots + a_{m-1} T^{2m-3} v = 0$

$\Rightarrow a_1 T^{m-1} v = 0 \Rightarrow a_1 = 0$

Similarly, we have $a_2 = \cdots = a_{m-1} = 0$. so $v, Tv, \cdots, T^{m-1}v$ is linearly independent.

6 Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is defined by $T(z_1, z_2, z_3) = (z_2, z_3, 0)$. Prove that T has no square root. More precisely, prove that there does not exist $S \in \mathcal{L}(\mathbf{C}^3)$ such that $S^2 = T$. 反证法

Suppose T has a square root $S \in \mathcal{L}(\mathbb{C}^3)$, $T = S^2$

$V = \text{null } T^3 = \text{null } S^6 = \text{null } S^3 = \text{null } ST \subset \text{null } S^2T = \text{null } T^2$
$(T^3 = 0)$

$T^2(z_1, z_2, z_3) = (z_3, 0, 0)$, $\underline{\text{null } T^2 = \{(z_1, z_2, 0) : z_1, z_2 \in \mathbb{C}\}} \neq V$ $\divideontimes$

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10 Suppose that $T \in \mathcal{L}(V)$ is not nilpotent. Let $\underline{n = \dim V}$. Show that $V = \text{null } T^{n-1} \oplus \text{range } T^{n-1}$.

$T \in \mathcal{L}(V)$ is not nilpotent, then $\dim \text{null } T^{n-1} < n$

And $\text{null } T^{n-1} = \text{null } T^n$, $\xrightarrow{\text{Prp } 8.5}$ $V = \text{null } T^n \oplus \text{range } T^n$

$\Rightarrow V = \text{null } T^{n-1} \oplus \text{range } T^n$

If $\text{null } T = \{0\}$. $\text{null } T^{n-1} = \{0\}$
$\text{range } T^{n-1} = V$, we are done

$\text{null } T \neq \{0\}$

If $\text{null } T^{n-1} \subsetneq \text{null } T^n$, $\text{null } T^n = V \Rightarrow T$ is nilpotent $\times$
so $\text{null } T^{n-1} = \text{null } T^n$

And $\text{range } T^n \subset \text{range } T^{n-1}$, by Fundamental Theorem of Linear Maps,

$\dim \text{range } T^n = \dim V - \dim \text{null } T^n = \dim V - \dim \text{null } T^{n-1} = \dim \text{range } T^{n-1}$

$\Rightarrow \text{range } T^n = \text{range } T^{n-1}$

so $V = \text{null } T^{n-1} \oplus \text{range } T^{n-1}$

12 Suppose $N \in \mathcal{L}(V)$ and there exists a basis of V with respect to which N has an upper-triangular matrix with only 0's on the diagonal. Prove that N is nilpotent.

Suppose $v_1, \cdots, v_n$ is a basis of V such that

$$N(v_1,\cdots,v_n)=(v_1,\cdots,v_n)\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & 0 & a_{23} \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{n-1,n} \\ & & & 0 \end{pmatrix}$$

so $Nv_1=0$. $Nv_2=a_{12}v_1\in \operatorname{span}(v_1) \Rightarrow N^2v_2=N(a_{12}v_1)=0$

Similarly, $Nv_3\in\operatorname{span}(v_1,v_2)$, $N^3v_3=0$

Continuing this process, we have $N^nv_n=0$. $v_n\in\operatorname{span}(v_1,\cdots,v_{n-1})$.

$\forall v\in V$, $v=k_1v_1+\cdots+k_nv_n$

$$N^nv=N^n(k_1v_1+\cdots+k_nv_n)=k_1N^nv_1+\cdots+k_nN^nv_n=0$$

$\Rightarrow N^n=0 \Rightarrow N$ is nilpotent.

13 Suppose V is an inner product space and $N\in\mathcal{L}(V)$ is normal and nilpotent. Prove that $N=0$.

N is normal $\Leftrightarrow N^*N=NN^*$

N is nilpotent $\Leftrightarrow \exists k\in\mathbb{Z}^+$, s.t. $N^k=0$.

$\mathbb{F}=\mathbb{C}$. N is normal spectral theorem

$\mathbb{F}=\mathbb{R}$. N is normal $\not\Rightarrow N$ is self-adjoint

we cannot use spectral theorem.

$\Rightarrow N^*N$ is normal，无论$\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$，都可以对N^*N使用spectral theorem.

N^*N is self-adjoint. by Spectral theorem, $\exists$ an orthonormal $\underbrace{\text{basis}}_{v_1,\cdots,v_n}$ of V consisting of all eigenvectors of V.

$$N^*N(v_1,\cdots,v_n)=(v_1,\cdots,v_n)\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

N is nilpotent $\Rightarrow N^*$ is nilpotent $\Rightarrow N^*N$ is nilpotent $\Rightarrow \lambda_1=\cdots=\lambda_n=0$

so $(N^*N)^n(v_1,\cdots,v_n)=(v_1,\cdots,v_n)\begin{pmatrix}\lambda_1^n & & \\ & \ddots & \\ & & \lambda_n^n\end{pmatrix}=(v_1,\cdots,v_n)\begin{pmatrix}0 & & \\ & \ddots & \\ & & 0\end{pmatrix}$

$\forall v\in V$. $N^*Nv=0$

$\|Nv\|^2 = \langle Nv, Nv\rangle = \langle v, N^*Nv\rangle = 0 \Rightarrow \|Nv\| = 0 \Rightarrow N = 0.$

15 Suppose $N \in \mathcal{L}(V)$ is such that $\operatorname{null} N^{\dim V - 1} \neq \operatorname{null} N^{\dim V}$. Prove that N is nilpotent and that

n

$$\dim \operatorname{null} N^j = j$$

for every integer j with $0 \le j \le \dim V$.

考虑 $\operatorname{null} N = \{0\}$ or $\operatorname{null} N \neq \{0\}$

If N is invertible. $\operatorname{null} N^{n-1} = \operatorname{null} N^n = \{0\}$ $\Rightarrow\!\!\!\!\times$

so N is not invertible

$j=0$: $N^0 = I \Rightarrow \operatorname{null} N^0 = \{0\} \Rightarrow \dim \operatorname{null} N^0 = 0.$

$j=1$: N is not invertible $\Rightarrow \dim \operatorname{null} N \geq 1$

If $\dim \operatorname{null} N > 1 \Rightarrow \dim \operatorname{null} N^{n-1} \geq n$

$\Rightarrow \dim \operatorname{null} N^{n-1} = \dim \operatorname{null} N^n$ $\Rightarrow\!\!\!\!\times$

so $\dim \operatorname{null} N = 1$

Similarly, we have $\dim \operatorname{null} N^j = j$. $\forall j = 2, \cdots, n$.