Step-1

(a)

The objective is to provide examples of matrices A and B for which $A+B$ is not invertible although A and B are invertible.

Step-2

Assume that,

$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, and $B=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$.

A matrix is invertible when its determinant is not equal to zero.

$$\begin{aligned}\det A &= 1-0\\ &= 1\\ \det B &= (-1)(-1)-0\\ &= 1\end{aligned}$$

Therefore, the matrices A and B are invertible.

Step-3

Addition of matrices A and B is,

$$\begin{aligned}A+B &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\\ &= \begin{bmatrix}1-1 & 0+0\\ 0+0 & 1-1\end{bmatrix}\\ &= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{aligned}$$

The determinant of $A+B$ is,

$$\begin{aligned}\det(A+B) &= 0-0\\ &= 0\end{aligned}$$

Hence $A+B$ is not invertible although A and B are invertible.

Step-4

(b)

The objective is to provide examples of matrices A and B for which $A+B$ is invertible although A and B are not invertible.

Step-5

Assume that,

$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, and $B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$.

Addition of matrices A and B is,

$$\begin{aligned} A+B &= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \\ &= \begin{bmatrix}1+0 & 0+0\\ 0+0 & 0+1\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \end{aligned}$$

The determinant of $A+B$ is,

$$\begin{aligned} \det(A+B) &= 1-0 \\ &= 1 \\ &\neq 0 \end{aligned}$$

Therefore $A+B$ is invertible.

Step-6

The determinant of the matrix A is,

$$\begin{aligned} \det A &= (1)(0)-(0)(0) \\ &= 0 \\ \det B &= (0)(1)-(0)(0) \\ &= 0 \end{aligned}$$

Therefore, the matrices A and B are not invertible.

Hence, $A+B$ is invertible although A and B are not invertible.

Step-7

(c)

The objective is to provide examples of matrices A and B for which A and B are invertible and $A+B$ is also invertible.

Step-8

Assume that,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The determinant of matrix A is,

$$\begin{aligned} \det A &= 1 - 0 \\ &= 1 \\ &\neq 0 \end{aligned}$$

Step-9

The determinant of matrix B is,

$$\begin{aligned} \det B &= 0 - (1) - 1 \\ &= 1 \\ &\neq 0 \end{aligned}$$

Addition of matrices A and B is,

$$\begin{aligned} A + B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1+0 & 0+1 \\ 0-1 & 1+0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

The determinant of matrix $A+B$ is,

$$\begin{aligned} \det(A+B) &= (1)(1) - (1)(-1) \\ &= 1 + 1 \\ &= 2 \\ &\neq 0 \end{aligned}$$

Hence, the matrices A, B, and $A+B$ are invertible.

Step-10

Consider,

$$A^{-1}(A+B)B^{-1} = B^{-1} + A^{-1}$$

Sum of two invertible matrices is also invertible.

Therefore, $C = B^{-1} + A^{-1}$ is invertible.

The inverse of C is,

$$\begin{aligned}
\left(B^{-1}+A^{-1}\right)^{-1} &= \left(A^{-1}\left(A+B\right)B^{-1}\right)^{-1} \\
&= \left(B^{-1}\right)^{-1}\left(A+B\right)^{-1}\left(A^{-1}\right)^{-1} && \text{Since, } \left(ABC\right)^{-1} = C^{-1}B^{-1}A^{-1}. \\
C^{-1} &= \boxed{B\left(A+B\right)^{-1}A} && \text{Since, } \left(A^{-1}\right)^{-1} = A.
\end{aligned}$$

Substitute, A, and B values in $C,$ obtained as,

$$\begin{aligned}
C &= B^{-1} + A^{-1} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

The determinant of C is,

$$\begin{aligned}
\det C &= 1+1 \\
&= 2 \\
&\neq 0
\end{aligned}$$

Hence, C is invertible.