Step-1

The objective is to determine the singular value decomposition of a matrix.

Consider the matrix.

$$A=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$$

Then,

$$A^T=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$$

Therefore, AA^T, A^TA are given by,

$$\begin{aligned}AA^T&=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\\ &=\begin{bmatrix}1+1 & 1+0\\ 1+0 & 1+0\end{bmatrix}\\ &=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\\ A^TA&=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\\ &=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\end{aligned}$$

Step-2

The Eigen values are given by,

$$\det\left(AA^T-\lambda I\right)=0$$

$$\begin{vmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix}=0$$

$$(2-\lambda)(1-\lambda)-1=0$$

$$\lambda^2-3\lambda+1=0$$

$$\begin{aligned}\lambda&=\frac{3\pm\sqrt{9-4}}{2}\\ &=\frac{3\pm\sqrt{5}}{2}\end{aligned}$$

Step-3

The Eigen vector corresponding to the Eigen value $\lambda = \frac{3+\sqrt{5}}{2}$:

$$\left(AA^T - \frac{3+\sqrt{5}}{2}I\right)\mathrm{X} = 0$$

$$\begin{pmatrix} 2-\frac{3+\sqrt{5}}{2} & 1 \\ 1 & 1-\frac{3+\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left(\frac{1-\sqrt{5}}{2}\right)x + y = 0,\ \ x - y\left(\frac{1+\sqrt{5}}{2}\right) = 0$$

Therefore, Eigen vector corresponding to the Eigen value $\lambda = \frac{3+\sqrt{5}}{2}$ is

$$u_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

Step-4

The Eigen vector corresponding to the Eigen value $\lambda = \frac{3-\sqrt{5}}{2}$:

$$\left(AA^T - \frac{3-\sqrt{5}}{2}I\right)\mathrm{X} = 0$$

$$\begin{pmatrix} 2-\frac{3-\sqrt{5}}{2} & 1 \\ 1 & 1-\frac{3-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left(\frac{1+\sqrt{5}}{2}\right)x + y = 0, x - \left(\frac{1-\sqrt{5}}{2}\right)y = 0$$

Therefore, Eigen vector corresponding to the Eigen value $\lambda = \frac{3-\sqrt{5}}{2}$ is,

$$u_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

Step-5

The unit eigenvectors of AA^T are given by,

$$\widehat{u_1} = \begin{bmatrix} \dfrac{\frac{1}{2}\left(\sqrt{5}+1\right)}{\sqrt{\frac{1}{4}\left(5+1+2\sqrt{5}\right)+1}} \\ \dfrac{1}{\sqrt{\frac{1}{4}\left(5+1+2\sqrt{5}\right)+1}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\sqrt{5}+1}{\sqrt{10+2\sqrt{5}}} \\ \dfrac{1}{\sqrt{10+2\sqrt{5}}} \end{bmatrix}$$

$$\widehat{u_2} = \begin{bmatrix} \dfrac{\frac{1}{2}\left(1-\sqrt{5}\right)}{\sqrt{\frac{1}{4}\left(1+5-2\sqrt{5}\right)+1}} \\ \dfrac{1}{\sqrt{\frac{1}{4}\left(1+5-2\sqrt{5}\right)+1}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}} \\ \dfrac{1}{\sqrt{10-2\sqrt{5}}} \end{bmatrix}$$

Step-6

Eigen values of AA^T are $\lambda = \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$, so consider

$$\sigma_1^2 = \frac{3+\sqrt{5}}{2}, \sigma_2^2 = \frac{3-\sqrt{5}}{2}$$

Consider the positive square roots of σ_1^2, σ_2^2.

Thus

$$\sigma_1 = \frac{1+\sqrt{5}}{2}, \sigma_2 = \frac{\sqrt{5}-1}{2}$$

Since, $A = A^T$ and one required that $Av_2 = \sigma_2 u_2$, so the unit eigenvectors of $A^T A$ are,

$$\begin{aligned}\widehat{u_1} &= \widehat{v_1}\\ &= \begin{bmatrix}\frac{\sqrt{5}+1}{\sqrt{10+2\sqrt{5}}}\\ \frac{1}{\sqrt{10+2\sqrt{5}}}\end{bmatrix}\\ \widehat{v_2} &= -\widehat{u_2}\\ &= \begin{bmatrix}\frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}}\\ -\frac{1}{\sqrt{10-2\sqrt{5}}}\end{bmatrix}\end{aligned}$$

Step-7

Hence, singular value of decomposition of the given matrix A is,

$$\begin{aligned}A &= U\Sigma V^T\\ &= \left(\widehat{u_1}, \ \widehat{u_2}\right)\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\left(\widehat{v_1}, \ \widehat{v_2}\right)^T\\ &= \begin{pmatrix}\frac{\sqrt{5}+1}{\sqrt{10+2\sqrt{5}}} & \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}}\\ \frac{1}{\sqrt{10+2\sqrt{5}}} & \frac{1}{\sqrt{10-2\sqrt{5}}}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{5}+1}{2} & 0\\ 0 & \frac{\sqrt{5}-1}{2}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{5}+1}{\sqrt{10+2\sqrt{5}}} & \frac{1}{\sqrt{10+2\sqrt{5}}}\\ \frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}} & -\frac{1}{\sqrt{10-2\sqrt{5}}}\end{pmatrix}\end{aligned}$$