Step-1

The eigen values of $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix}$ are $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 1 \\ 3 & 6-\lambda & 3 \\ 1 & 8 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 11\lambda^2 = 0$$
$$\Rightarrow \lambda^2(-\lambda + 11) = 0$$
$$\Rightarrow \lambda = 0, 0, 11$$

Step-2

If exchange row1 and 2 and columns1 and 2 of A, then we get

$$PAP = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}$$

The eigen values are

$$\begin{vmatrix} 6-\lambda & 3 & 3 \\ 2 & 1-\lambda & 1 \\ 8 & 4 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 = 0$$
$$\Rightarrow \lambda^2(\lambda - 11) = 0$$
$$\Rightarrow \lambda = 0, 0, 11$$

Step-3

Therefore when P exchange rows 1 and 2, columns 1 and 2, the eigen vector of A for $\lambda = 11$

Taking $(A - \lambda I)x = 0$,

$$\begin{bmatrix} 1-11 & 2 & 1 \\ 3 & 6-11 & 3 \\ 4 & 8 & 4-11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -10 & 2 & 1 \\ 3 & -5 & 3 \\ 4 & 8 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-4

Applying the row operations on this,

$$R_2 \to 10R_2 + 3R_1, R_3 \to 10R_3 + 4R_1 \Rightarrow \begin{bmatrix} -10 & 2 & 1 \\ 0 & -44 & 33 \\ 0 & 88 & -66 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2, R_2 / -11 \Rightarrow \begin{bmatrix} -10 & 2 & 1 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is the reduced matrix and so, we rewriting the homogeneous equations, we get

$$4x_2 - 3x_3 = 0$$
$$10x_1 - 2x_2 - x_3 = 0$$

$x_2 = \frac{3}{4}x_3$ from the first equation

Using this in the second equation, we get $x_1 = \frac{1}{4}x_3$

Putting $x_3 = 4$, the solution set is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ is the eigen vector corresponding to the eigen value 11.

Step-5

Similarly, the eigen vector corresponding to the eigen value 11 for the matrix *PAP* is also $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$