Step-1

Given that $A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$

To find the eigen values of *A,* we consider $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda) - 0 = 0$$
$$\Rightarrow \lambda = 2, 4$$

The eigen values are $\lambda = 2$ and 4

Step-2

To find the eigen vector corresponding to $\lambda = 2$, we solve $(A - 2I)x = 0$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Using the row operation $R_2 \to 2R_2 - R_1$, $R_1 / 2$ on the coefficient matrix, we get $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This is the reduced matrix and so, rewriting the homogeneous equations from this, we get

$$x_1 = 0$$

So, any real number $k = x_2$ satisfies the system.

Therefore, putting $k = 1$, the solution set $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the eigen vector of $\lambda = 2$

Step-3

Similarly, for $\lambda = 4$, we solve $(A - 4I)x = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using $R_2 \leftrightarrow R_1$ on the coefficient matrix, we get $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This is the reduced matrix and so, the homogeneous equation is $x_1 - 2x_2 = 0$

Putting x_2= 1, we get x_1= 2

So, $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 4$

Step-4

While the eigen vectors corresponding to distinct eigen values are linearly independent, and so, the matrix *S* whose columns are the eigen vectors of distinct eigen values is non singular and so invertible.

Writing $S = [X_1 X_2] = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$,we get $S^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ such that

$$\begin{aligned} S^{-1}AS &= \frac{1}{2}\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$= \Lambda$ the diagonal matrix. â€¦â€¦ (1)

Step-5

Similarly, writing $S = [X_2 X_1] = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, we get $S^{-1}AS = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} = \Lambda$ the diagonal matrix.

â€¦â€¦ (2)

(1) and (2) are the two ways in which we can diagonalize the given matrix *A*.

Step-6

We can write $A = S\Lambda S^{-1}$

Applying the n^{th} powers on both sides, we get $A^n = S\Lambda^n S^{-1}$ where Λ^n is the diagonal matrix whose diagonal entries are the n^{th} powers of the eigen values of *A*.

Putting n = -1, we get $A^{-1} = S\begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix}S^{-1}$

Step-7

Using $\lambda_1 = 2, \lambda_2 = 4$, we get $S^{-1}A^{-1}S = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} = \Lambda$ and so, $A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ and using $\lambda_2 = 2, \lambda_1 = 4$, we get $A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are the two ways possible.