Step-1

Given that $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $AB = BA$.

Also given that $B = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$

We have to show that *B* is also diagonal.

Step-2

Since $AB = BA$

So

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1(\mathbf{a})+0(\mathbf{c}) & 1(\mathbf{b})+0(\mathbf{d}) \\ 0(\mathbf{a})+2(\mathbf{c}) & 0(\mathbf{b})+2(\mathbf{d}) \end{bmatrix} = \begin{bmatrix} \mathbf{a}(1)+\mathbf{b}(0) & \mathbf{a}(0)+\mathbf{b}(2) \\ \mathbf{c}(1)+\mathbf{d}(0) & \mathbf{c}(0)+\mathbf{d}(2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ 2\mathbf{c} & 2\mathbf{d} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & 2\mathbf{b} \\ \mathbf{c} & 2\mathbf{d} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix}$$

Step-3

From this, we get $\mathbf{b} = 2\mathbf{b}$ and $2\mathbf{c} = \mathbf{c}$

$\Rightarrow \mathbf{b} = 0$, $\mathbf{c} = 0$

Hence $B = \begin{bmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{d} \end{bmatrix}$

Hence *B* is also diagonal.

Step-4

Let λ be the eigenvalue of *A*.

Then $Ax = \lambda x$

Multiply both sides with B, we get

$$
\begin{aligned}
&BAx = B\lambda x \\
&\Rightarrow ABx = B\lambda x \qquad (\text{Since } AB = BA) \\
&\qquad\quad = \lambda Bx
\end{aligned}
$$

Thus x and Bx are eigenvectors of A with same λ.

Step-5

But the eigenvectors of A are distinct.

$\Rightarrow Bx$ must be multiple of x

$\Rightarrow x$ is an eigenvector of B as well as x is an eigen vector of A.

Therefore A and B have same eigenvectors but different eigenvalues.

Here we are getting the value of B.

$\Rightarrow \mathbf{b} = 0$, $\mathbf{c} = 0$

Hence B forms a two dimensional subspace of matrix space.

Since B has two nonzero columns.

So the rank of B is 2.