Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #7

2023/04/06

Name: ______________________________

Student Number: ______________________________

1. Suppose $T \in \mathcal{L}(V)$ and there exists a positive integer n such that $T^n = 0$. Prove that $I - T$ is invertible and that $(I-T)^{-1} = I + T + \cdots + T^{n-1}$.

   设 $T \in \mathcal{L}(V)$，且存在一个正整数 n 使得 $T^n = 0$. 证明 $I - T$ 可逆且 $(I-T)^{-1} = I + T + \cdots + T^{n-1}$.

   Proof. We have

$$\begin{aligned}(I-T)(I+T+\cdots+T^{n-1}) &= I + T + \cdots + T^{n-1} - T - T^2 - \cdots - T^{n-1} - T^n \\ &= I - T^n \\ &= I\end{aligned}$$

and

$$\begin{aligned}(I+T+\cdots+T^{n-1})(I-T) &= I + T + \cdots + T^{n-1} - T - T^2 - \cdots - T^{n-1} - T^n \\ &= I - T^n \\ &= I\end{aligned}$$

Therefore $I - T$ is invertible and $(I-T)^{-1} = I + T + \cdots + T^{n-1}$.

□

2. Suppose V is a finite dimensional vector space over $\mathbb{C}$, $\mathcal{A}, \mathcal{B} \in \mathcal{L}(V)$, $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$. Prove that $\mathcal{A}, \mathcal{B}$ have some eigenvectors in common.

   设 V 是复数域上的有限维向量空间, $\mathcal{A}, \mathcal{B} \in \mathcal{L}(V)$, $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$. 证明 $\mathcal{A}, \mathcal{B}$ 有公共的特征向量.

   Proof. Since V is a finite dimensional vector space over $\mathbb{C}$, $\mathcal{A} \in \mathcal{L}(V)$, then $\mathcal{A}$ has an eigenvalue, we denoted it as λ_1.

   Let $E_{\lambda_1} = \{\xi \in V : \mathcal{A}\xi = \lambda_1 \xi\}$ be the corresponding eigenspace of $\mathcal{A}$, $\forall \xi \in V_{\lambda_1}$, we have

$$\mathcal{A}\mathcal{B}\xi = \mathcal{B}\mathcal{A}\xi = \lambda_1 \mathcal{B}\xi \Rightarrow \mathcal{B}\xi \in V_{\lambda_1}$$

   so V_{λ_1} is an invariant subspace of $\mathcal{B}$.

   Consider $\mathcal{B}|_{V_{\lambda_1}}$, $\mathcal{B}|_{V_{\lambda_1}} \in \mathcal{L}(V_{\lambda_1})$, so $\mathcal{B}|_{V_{\lambda_1}}$ has an eigenvalue, denoted as λ_2, and corresponding eigenvector η, i.e. $\mathcal{B}|_{V_{\lambda_1}}\eta = \lambda_2 \eta$, then λ_2 is also an eigenvalue of $\mathcal{B}$, η is also the corresponding eigenvector of $\mathcal{B}$, i.e. $\mathcal{B}\eta = \lambda_2 \eta$.

   And since $\eta \in V_{\lambda_1}$, then η is also an eigenvector of $\mathcal{A}$, thus η is an common eigenvector of $\mathcal{A}$ and $\mathcal{B}$. $\square$