

Step-1

We need to construct a matrix with the required property and need to explain why we can't.

(a)

Given a column space contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ row space contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The required matrix is

The dimension of this matrix is,

Apply $R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Apply $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The matrix has two pivots of columns, so $\dim(\text{col } A) = 2$.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

So, basis for column space is

Step-2

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transpose of matrix is

So, 1 and 3 columns are independent and the dimension is,

$$\begin{aligned} \dim(\text{col } A^T) &= \dim(\text{row } (A)) \\ &= 2 \end{aligned}$$

The basis for row space is,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, row space contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

Therefore, the required matrix is $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Step-3

(b)

Given a column space has basis $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, null space has basis $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

We know that $\dim(N(A)) + \dim(\text{col}(A)) = \text{number of columns}$

i.e. $\dim(\text{col } A) = 1$, $\dim(N(A)) = 1$ and the number of columns = 3

So,

$$1 + 1 \neq 3$$

$$2 \neq 3$$

Therefore, it's impossible.

Step-4

(c)

Given the dimension of null space = 1 + dimension of left null space.

The required matrix is $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Let's write in $AX = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} \\ = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, the basis for null space is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and the dimension of null space is 1.

$$\text{i.e. } \dim(\text{Nul}(A)) = 1$$

Step-5

The transpose of the matrix is $A^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $x \in \text{Nul}(A^T)$ write in $A^T X = 0$

$$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = 0$$

The dimension of left null space is 0.

$$\text{i.e. } \dim(\text{Nul}(A^T)) = 0$$

Therefore,

$$\dim(\text{Nul}(A)) = 1 + \dim(\text{Nul}(A^T))$$

$$1 = 1 + 0$$

$$1 = 1$$

Therefore, the dimension of null space = 1 + dimension of left null space.

Therefore, the required matrix is $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Step-6

(d)

Given left null space contains $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, row space contains $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

The required matrix is $A = \begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$

The transpose of the matrix is $A^T = \begin{bmatrix} -9 & 3 \\ -3 & 1 \end{bmatrix}$

Let's write in $A^T X = 0$

$$\Rightarrow -9x_1 + 3x_2 = 0$$

$$-3x_1 + x_2 = 0$$

$$\Rightarrow 3x_1 = x_2$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_1 \end{bmatrix} \\ = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

So, left null space contains $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\dim(N(A^T)) = 1$.

The second row is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. So, the row space contains $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Therefore, the required matrix is $A = \begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$.

Step-7

(e)

Given row space = column space, nullspace \neq leftnullspace

Let $A \in \mathbb{R}^{m \times n}$ consider m by n matrix.

We need to construct the matrix with these properties.

Given row space = column space

The dimension of row space is r and dimension of column space is r .

It requires $m = n$.

Then $m - r = n - r$

Step-8

We know that $\dim(N(A)) + \dim(\text{col}(A)) = \text{number of columns}$ and $\dim(N(A^T)) + \dim(\text{col}(A^T)) = \text{number of columns}$

$$\text{i.e. } \dim(N(A)) + \dim(C(A)) = \dim(N(A^T)) + \dim(C(A^T))$$

$$\Rightarrow \dim(N(A)) + r = \dim(N(A^T)) + r \quad (\text{Since } \dim C(A) = \dim C(A^T) = r)$$

$$\Rightarrow \dim(N(A)) = \dim(N(A^T))$$

But given that $\text{nullspace} \neq \text{leftnullspace}$

Therefore, it is impossible.