

Step-1

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We observe that the entries below the principal diagonal are zero.

So, this is the reduced form.

We write the homogeneous equations using this.

$$Ax = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.,

$$\Rightarrow \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_2 = 0, x_3 = 0$$

From this, we can say that any real value $k = x_1$ can satisfy the system.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the solution set is

Therefore, the null space of the given matrix is spanned by $\{(k, 0, 0)\}$

In other words, the basis of the null space of A is $N(A) = \{(1, 0, 0)\}$

Further, dimension of $N(A) = 1$

Step-2

While the dimension of the given matrix is 3 and we have $\dim C(A) + \dim N(A) = 3$

Therefore, dimension of $C(A) = 2$

The standard basis that spans the column space is $C(A) = \{(1, 0, 0), (0, 1, 0)\}$

Step-3

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Similarly,

Basis for column space of A^T or the row space of $A = \{(0,1,0), (0,0,1)\}$ and its dimension is 2

Basis for null space of $A^T = \{(1,0,0)\}$ and its dimension is 1

Therefore,

$C(A) = \{(1,0,0), (0,1,0)\}, \dim C(A) = 2$
$C(A^T) = \{(0,1,0), (0,0,1)\}, \dim C(A^T) = 2$
$N(A) = \{(1,0,0)\}, \dim N(A) = 1$
$N(A^T) = \{(1,0,0)\}, \dim N(A^T) = 1$

Step-4

On the other hand,

$$I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_2 \rightarrow R_2 - R_3$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_1 \rightarrow R_1 - R_2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The standard basis that spans column space $C(A) = \{(1,0,0), (0,1,0), (0,0,1)\}$ and its dimension is 3

The basis for row space $C(A^T) = \{(1,0,0), (0,1,0), (0,0,1)\}$ and its dimension is 3

Step-5

We write the homogeneous equations using this.

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.,

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = 0, x_2 = 0 \text{ and } x_3 = 0$$

The basis of the null space of A is empty. i.e. $N(A) = \{(0, 0, 0)\}$ and its dimension is 0

The basis of the null space of A is empty. i.e. $N(A^T) = \{(0, 0, 0)\}$ and its dimension is 0

Therefore,

$C(A) = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}, \dim C(A) = 3$ $C(A^T) = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}, \dim C(A^T) = 3$ $N(A) = \{(0, 0, 0)\}, \dim N(A) = 0$ $N(A^T) = \{(0, 0, 0)\}, \dim N(A^T) = 0$
