本试卷共 (7) 大题, 满分 (100) 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

This test includes 7 questions. Write ***all your answers*** on the examination book.

Please put away all books, calculators, cell phones and other devices. Please write carefully and clearly in complete sentences. Your explanations are your only representative when your work is being graded.

Unless otherwise noted, vector spaces are over $\mathbb{F}$ and with finite dimensions, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

1. (30 points, 6 points each) Label the following statements as **True** or **False**. **Along with your answer, provide an informal proof, counterexample, or other explanation.**

   (a) Suppose $T \in \mathcal{L}(V, W)$ and $v_1, v_2, \cdots, v_n$ is linearly independent in V, then $Tv_1, Tv_2, \cdots, Tv_n$ is linearly independent.

   (b) There exists a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that range T = null T.

   (c) Suppose $T \in \mathcal{L}(V)$ and u, v are eigenvectors of T such that $u + v$ is also an eigenvector of T. Then u and v are eigenvectors of T corresponding to the same eigenvalue.

   (d) Suppose U is a subspace of V such that V/U is finite dimensional. Then V is isomorphic to $U \times (V/U)$.

   (e) Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

2. (10 points) Let V and W be vector spaces, and suppose that V has a finite basis $v_1, v_2, \cdots, v_n$. If $S, T \in \mathcal{L}(V, W)$ and $S(v_i) = T(v_i)$ for $i = 1, 2, \cdots, n$. Show that $S = T$.

3. (10 points) Let $\mathbb{R}^{3\times3}$ be the vector space of 3×3 real matrices. Let U_1 be the subset consisting of all skew-symmetric matrices in $\mathbb{R}^{3\times3}$ and U_2 be the subset consisting of all symmetric matrices in $\mathbb{R}^{3\times3}$.

   (a) Show that U_1 and U_2 are subspaces of $\mathbb{R}^{3\times3}$.

   (b) Prove that $\mathbb{R}^{3\times3} = U_1 \oplus U_2$.

4. (10 points) State and prove the **Fundamental Theorem of Linear Maps**.

5. (20 points) Recall that $\mathcal{P}_4(\mathbb{R})$ is the set of polynomials with real coefficients and degree less or equal to 4. Let
$$V = \{f(x) \in \mathcal{P}_4(\mathbb{R}) \mid f(2) = 0 \text{ and } f(1) = f(-1)\},$$
   i.e., it is the set of polynomials with degree ≤ 4 such that
$$f(2) = 0 \text{ and } f(1) = f(-1).$$

   (a) Show that V is a subspace of $\mathcal{P}_4(\mathbb{R})$.

(b) Show that $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_4x^4 \in V$ if and only if

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \in N(A), \ A = \begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 0 & 2 & 0 & 2 & 0 \end{bmatrix}.$$

Here $N(A)$ denotes the nullspace of A.

(c) Find a basis for V and hence find the dimension.

6. (10 points) Prove that a nonempty subset A of V is an affine subset of V if and only if $\lambda v + (1-\lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$.

7. (10 points) Let $V = \mathbb{R}^{n\times n}$. Define

$$\langle A, B\rangle = \text{tr }(B^TA), \text{ for } A, B \in V.$$

Recall that the trace of a matrix A is defined by

$$\text{tr }(A) = \sum_{i=1}^{n} a_{ii}.$$

Show that the above definition defines an inner product on V.

8. (Bonus) Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Show that if $T^2 = I$, then T is diagonalizable.