Step-1

The 8 by 8 Hilbert Matrix is

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} \\ \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} \end{bmatrix}$$

We have to compute $\lambda_{\max}$ and $\lambda_{\min}$ for the given Hilbert matrix.

Step-2

If A is a 8×8 Hilbert matrix, then its characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 0.5 & 0.33 & 0.25 & 0.2 & 0.166 & 0.143 & 0.125 \\ 0.5 & 0.33-\lambda & 0.25 & 0.2 & 0.166 & 0.143 & 0.125 & 0.111 \\ 0.33 & 0.25 & 0.2-\lambda & 0.166 & 0.143 & 0.125 & 0.111 & 0.1 \\ 0.25 & 0.2 & 0.166 & 0.143-\lambda & 0.125 & 0.111 & 0.1 & 0.091 \\ 0.2 & 0.166 & 0.143 & 0.125 & 0.111-\lambda & 0.1 & 0.091 & 0.083 \\ 0.166 & 0.143 & 0.125 & 0.111 & 0.1 & 0.091-\lambda & 0.083 & 0.077 \\ 0.143 & 0.125 & 0.111 & 0.1 & 0.091 & 0.083 & 0.077-\lambda & 0.071 \\ 0.125 & 0.111 & 0.1 & 0.091 & 0.083 & 0.077 & 0.0714 & 0.06-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} \Rightarrow 4.1\times10^{-9} &- 1.2156\times10^{-8}\lambda - 10^{-10}\lambda^2 + 1.81\times10^{-8}\lambda^3 \\ &+ 0.0000202315\lambda^4 - 0.0141032315\lambda^5 + 0.55090732\lambda^6 \\ &- 2.0212\lambda^7 + \lambda^8 = 0 \end{aligned}$$

Step-3

Solving this equation, we get $\lambda = 0.2982, 1.6952$ and the remaining are the complex eigenvalues

$$\begin{aligned}\lambda &= 0.04465 \pm 0.022i \\ &= -0.0348 \pm 0.022i \\ &= 0.00404 \pm 0.044i\end{aligned}$$

We know that if λ is the eigenvalue and x the corresponding eigenvector of A, then

$Ax = \lambda x$ where λx is the column vector which can be comfortably seen as b.

Given that $\|b\| = 1$

So, $x = A^{-1}b$ and $\|x\| = \|A^{-1}b\|$

$\leq \|A^{-1}\|\|b\|$

$\leq \|A^{-1}\|$

$$\leq \frac{1}{\lambda_{\min}(A)} \left(\text{Since } \lambda_{\min}(A) = \frac{1}{\|A^{-1}\|} \right)$$

Therefore, $\frac{1}{\lambda_{\min}(A)}$ is maximum value of $\|x\|$.

Step-4

Given that b is rounded off to less than 10^{-16} error.

So, the largest error caused in $\|x\|$ is given by

$\|\delta x\| \leq \|A^{-1}\|\|\delta b\|$

$$\leq \frac{1}{\sqrt{\lambda_{\min}(A^T A)}} \times 10^{-16}$$

$$\leq \frac{1}{\lambda_{\min}(A)} \times 10^{-16}$$

So the maximum possible error in x is $\boxed{\frac{1}{\lambda_{\min}(A)} \times 10^{-16}}$.