Step-1

Consider that **S** contains only the vectors

$(1,5,1)$ and $(2,2,2)$.

Note that **S** is not a subspace.

The objective is to fill the blank â€œ$\mathbf{S}^{\perp}$ is the nullspace of the matrix $A =$ ________ .â€

Step-2

The set $\mathbf{S}^{\perp}$ is defined as,

$\mathbf{S}^{\perp} = \{x / y \cdot x = 0 \text{ for all } y \in \mathbf{S}\}$.

Here, **S** contains only the vectors $(1,5,1)$ and $(2,2,2)$.

Therefore, $\mathbf{S}^{\perp}$ can be written as,

$\mathbf{S}^{\perp} = \{x / (1,5,1) \cdot x = 0 \text{ and } (2,2,2) \cdot x = 0\}$.

The equations $(1,5,1) \cdot x = 0$ and $(2,2,2) \cdot x = 0$ can be written matrix form as,

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Step-3

Note that the nullspace of a matrix A is the solution to the system $Ax = \mathbf{0}$.

Here, $\mathbf{S}^{\perp} = \{x / (1,5,1) \cdot x = 0 \text{ and } (2,2,2) \cdot x = 0\}$ contains all the solutions of the system $Ax = \mathbf{0}$.

Therefore, the $\mathbf{S}^{\perp}$ is the nullspace of the matrix $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$.

Hence, the correct matrix that fills the blank is $\boxed{A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}}$.

Since $\mathbf{S}^{\perp}$ is the nullspace of the matrix $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$, so $\mathbf{S}^{\perp}$ is a subspace even **S** is not.