Step-1

Consider the matrices as shown below:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}, \text{ and } B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$$

The objective is to choose the number q so that the ranks of A and B are,

(a) 1

(b) 2

(c) 3

Step-2

Consider the matrix $A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$.

Convert the above matrix into echelon form.

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \xrightarrow{R_1 : \left(\frac{1}{6}\right) R_1} \begin{bmatrix} 1 & \frac{4}{6} & \frac{2}{6} \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$$

$$\xrightarrow[R_3 : R_3 - 9R_1]{R_2 : R_2 + 3R_1} \begin{bmatrix} 1 & \frac{4}{6} & \frac{2}{6} \\ 0 & 0 & 0 \\ 0 & 0 & q-3 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & \frac{4}{6} & \frac{2}{6} \\ 0 & 0 & q-3 \\ 0 & 0 & 0 \end{bmatrix} = E$$

Here, A and E are similar matrices and they have same rank.

(a)

The matrix E has only one non-zero row if $q-3=0$.

So for $q=3$, the matrix E has only one linearly independent row.

So, the rank of the matrix E is 1.

Hence, the rank of the matrix A is 1 when $q=3$.

Step-3

(b)

The matrix E has two non-zero rows if $q-3\neq 0$.

So for $q\neq 3$, the matrix E has two linearly independent rows.

So, the rank of the matrix E is 2.

Hence, the rank of the matrix A is 2 when $q\neq 3$.

Step-4

(c)

In the matrix E, the maximum number of independent rows is 2.

So rank of E never exceeds 2.

Therefore, for any value of q, rank of E never equals to 3.

Step-5

Consider the matrix $B=\begin{bmatrix}3 & 1 & 3\\ q & 2 & q\end{bmatrix}$.

Convert the above matrix into echelon form.

$$B=\begin{bmatrix}3 & 1 & 3\\ q & 2 & q\end{bmatrix}\xrightarrow{\frac{1}{3}R_1}\begin{bmatrix}1 & \frac{1}{3} & 1\\ q & 2 & q\end{bmatrix}$$

$$\xrightarrow{R_2:R_2-qR_1}\begin{bmatrix}1 & \frac{1}{3} & 1\\ 0 & \frac{6-q}{3} & 0\end{bmatrix}=F$$

(a)

The matrix F has only one non-zero row if $\frac{6-q}{3}=0$.

That is $6-q=0 \Rightarrow q=6$

So for $q=6$, the matrix E has only one linearly independent row.

So, the rank of the matrix E is 1.

Hence, the rank of the matrix B is 1 when $q=6$.

Step-6

(b)

The matrix F has two non-zero rows if $\frac{6-q}{3}\neq 0$.

So for $q\neq 6$, the matrix F has two linearly independent rows.

So, the rank of the matrix F is 2.

Hence, the rank of the matrix A is 2 when $q\neq 6$.

Step-7

(c)

Here rank of B less than or equals to number of rows.

That is $rank(B)\leq 2$.

Therefore, for any value of q the rank of B never be equals to 3.