Step-1

Consider matrices A and B, here A is m by n and B is n by m.

The objective is to show that $\det\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \det(AB)$.

Step-2

It is known that $\det\begin{bmatrix} I & 0 \\ B & I \end{bmatrix} = 1$.

Take $\det\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix}$ and post multiply by $\det\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}$;

$$\det\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} \cdot 1 = \det\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} \det\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}$$

The determinant of AB is the product of $\det A$ times $\det B$, so;

$$\begin{aligned} \det\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} \cdot 1 &= \det\left(\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix}\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}\right) \\ &= \det\begin{bmatrix} AB & A \\ 0 & I \end{bmatrix} \\ &= \det(AB) \end{aligned}$$

Hence showed

Step-3

Consider an example for $m < n$ as $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

So,

$$\begin{aligned} \det\begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} &= \det\begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \\ &= 5 \\ \det(AB) &= 5 \end{aligned}$$

And if take $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \end{bmatrix}$.

Then,

$$\det \begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ -1 & -2 & 1 \end{bmatrix}$$
$$= 0$$

And,

$$\det AB = \det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$
$$= 0$$

In this example, the rank of matrices A and B is 1, and so rank of square matrix AB canâ€™t be greater than 1 therefore determinant of AB is zero.