Step-1

Given that $Q=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

The characteristic equation of this matrix is $\det(Q-\lambda I)=0$

$$\Rightarrow\begin{vmatrix}\cos\theta-\lambda & -\sin\theta\\ \sin\theta & \cos\theta-\lambda\end{vmatrix}=0$$

$$\Rightarrow(\cos-\lambda)^2+\sin^2\theta=0$$
$$\Rightarrow\cos^2\theta+\lambda^2-2\lambda\cos\theta+\sin^2\theta=0$$
$$\Rightarrow\lambda^2-2\lambda\cos\theta+1=0$$

$$\Rightarrow\lambda=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}$$
$$=\frac{2\left(\cos\theta\pm\sqrt{1-\cos^2\theta}\right)}{2}$$

$$\Rightarrow\cos\theta\pm\sqrt{\sin^2\theta}$$

$\Rightarrow\cos\theta+i\sin\theta,\cos\theta-i\sin\theta$ are the eigen values of the given matrix that rotates the *xy* â€“ plane through an angle θ.

Step-2

We now find the eigen vectors of Q for the eigen value $\lambda=\cos\theta+i\sin\theta$

For, we solve $(Q-\lambda I)x=0$ where x is the required eigen vector.

$$\Rightarrow\begin{bmatrix}-i\sin\theta & -\sin\theta\\ \sin\theta & -i\sin\theta\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

Using the row operation $R_2\to iR_2+R_1$, we get

$$\begin{bmatrix}-i\sin\theta & -\sin\theta\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

This is the row reduced matrix and so, writing the homogeneous equations, we get

$$ix_1+x_2=0$$

Putting $x_1 = 1,$ we get the solution set $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ which is the eigen vector corresponding to $\lambda = \cos\theta + i\sin\theta$

Step-3

Similarly, the eigen vector corresponding to the eigen value $\lambda = \cos\theta - i\sin\theta$ is obtained by $\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

Applying $R_2 \to iR_2 - R_1$, we get $\begin{bmatrix} i\sin\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow ix_1 - x_2 = 0$

Putting $x_1 = 1,$ we get the solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ the eigen vector corresponding to $\lambda = \cos\theta - i\sin\theta$

Step-4

Thus, the eigen values of the rotation matrix are $\lambda = \cos\theta + i\sin\theta$, $\lambda = \cos\theta - i\sin\theta$ and the respective eigen vectors are $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$