Step-1

Consider the given matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 9 \end{pmatrix}$.

Let $X^T = (x \quad y)$ so that $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Now $X^T A X$

$= (x \quad y) \begin{pmatrix} 1 & 2 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= (x+2y \quad 2x+9y) \begin{pmatrix} x \\ y \end{pmatrix}$

$= (x+2y)x + (2x+9y)y$

$= x^2 + 4xy + 9y^2$

Step-2

Thus the quadratic for the given metro A is $x^2 + 4xy + 9y^2$.

Now,

$f = x^2 + 4xy + 9y^2$

$= x^2 + 2(x)(2y) + 4y^2 - 4y^2 + 9y^2$

$= (x+2y)^2 + 5y^2$

$= d_1 (x+2y)^2 + d_2 y^2$

So that $d_1 = 1$ and $d_2 = 5$.

Therefore, the quadratic $\boxed{f = x^2 + 4xy + 9y^2 = (x+2y)^2 + 5y^2}$.

Step-3

Now consider the second matrix, $A = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$.

$$X^T A X = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x+3y & 3x+9y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= (x+3y)x(3x+9y)y$$

$$= x^2 + 6xy + 9y^2$$

Step-4

Thus the quadratic for the given matrix A is, $x^2 + 6xy + 9y^2$.

Now,

$$f = x^2 + 6xy + 9y^2$$

$$= x^2 + 2(x)(2y) + (3y)^2$$

$$= (x+3y)^2$$

Therefore, the quadratic $\boxed{f = x^2 + 6xy + 9y^2 = (x+3y)^2}$.