Step-1

Consider the following first order differential equations:

$$\frac{dv}{dt} = w - v$$
$$\frac{dw}{dt} = v - w$$

Here, $v + w$ is constant. These differential equations can be written in the following form:

$$\frac{du}{dt} = Au$$

Step-2

Now reverse the above system to the following:

$$\frac{du}{dt} = -Au$$
$$\frac{dv}{dt} = -(w - v)$$
$$\frac{dw}{dt} = -(v - w)$$

Here, also $v + w$ is constant. Find the matrix -*A* and its Eigen values.

Step-3

Initial condition:

$$v(0) = 30$$
$$w(0) = 10$$

Find the value of *v* when $t \to \infty$.

Step-4

Changed differential equations can be written as follows:

$$\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}$$

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Bu$$
$$B = -A$$

Matrix $u(t)$ is: $u(t) = (v, w)$

Therefore, matrix *B* is defined as follows:

$$\boxed{B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

Step-5

First step is to find the Eigen values and Eigen vectors of matrix *B*. To calculate the Eigen values do the following calculations;

$$B - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}$$
$$\det(B - \lambda I) = 0$$
$$(1-\lambda)(1-\lambda) - 1 = 0$$
$$\lambda^2 - 2\lambda = 0$$

After solving following values are obtained:

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

Step-6

Therefore, Eigen values are $\boxed{2,0}$

Step-7

To calculate Eigen vectors do the following calculations:

$$(B - \lambda I)x = 0$$
$$\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1-2 & -1 \\ -1 & 1-2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-8

On solving, values of y and z corresponding to $\lambda = 2$ are as follows:

$$
\begin{aligned}
x_1 &= \begin{bmatrix} y \\ z \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

Step-9

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 0$ is as follows:

$$
\begin{aligned}
(B - \lambda I)x &= 0 \\
\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1-0 & -1 \\ -1 & 1-0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

On solving values of y and z are as follows:

$$
\begin{aligned}
x_2 &= \begin{bmatrix} y \\ z \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Step-10

Recall that: $e^{At} = Se^{\Lambda t}S^{-1}$

Here, Eigen value matrix is given as follows:

$$
\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}
$$

Therefore, the general solution of the differential equation is:

$$
\begin{aligned}
u(t) &= c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\end{aligned}
$$

Here, c_1 and c_2 are constants. Their values are determined by the following values:

$$c = S^{-1}u(0)$$

Step-11

So, the solution for differential equation can be written as follows:

$$\begin{aligned} u(t) &= Se^{\Lambda t}S^{-1}u(0) \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 30 \\ 10 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 1 \\ -e^{2t} & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} 10e^{2t} + 20 \\ -10e^{2t} + 20 \end{bmatrix}$$

Step-12

Therefore, specific solution of the differential equation is:

$$v(t) = 10e^{2t} + 20$$
$$w(t) = -10e^{2t} + 20$$

The value of v when $t \to \infty$ is:

$$\boxed{\begin{aligned} v(\infty) &= 10e^{2} + 20 \\ &\to \infty \end{aligned}}$$