

## Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix  $A$  is defined as follows:

$$A = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$$

## Step-2

Initial condition:

$$u(0) = (5, -2)$$

Find the Eigen values and Eigen vectors, so that  $u = e^{\lambda t} x$  solves the differential equation. Also, compute the general solution that starts from the initial value.

## Step-3

First step is to find the Eigen values and Eigen vectors of matrix  $A$ . To calculate the Eigen values do the following calculations;

$$A - \lambda I = \begin{bmatrix} 4 - \lambda & 3 \\ 0 & 1 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(4 - \lambda)(1 - \lambda) = 0$$

After solving following values are obtained:

$$\lambda_1 = 4$$

$$\lambda_2 = 1$$

## Step-4

Therefore, Eigen values are  $\boxed{4, 1}$

## Step-5

To calculate Eigen vectors do the following calculations:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4-4 & 3 \\ 0 & 1-4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of  $y$  and  $z$  corresponding to  $\lambda = 4$  are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

## Step-6

Similarly, Eigen vectors corresponding to Eigen value  $\lambda = 1$  is as follows:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4-1 & 3 \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of  $y$  and  $z$  are as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

## Step-7

Therefore Eigen values are:

$$\begin{aligned} x_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ x_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

## Step-8

Therefore, values of  $u = e^{\lambda t} x$  corresponding to both Eigen values are as follows:

$$\begin{aligned} u_1 &= e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ u_2 &= e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

## Step-9

Recall that:  $e^{At} = Se^{At}S^{-1}$

Here, Eigen value matrix is given as follows:

$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, the general solution of the differential equation is:

$$\begin{aligned} u(t) &= c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 \\ &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

Here,  $c_1$  and  $c_2$  are constants. Their values are determined by the following values:

$$c = S^{-1}u(0)$$

## Step-10

So, the solution for differential equation can be written as follows:

$$\begin{aligned}
 u(t) &= Se^{At} S^{-1} u(0) \\
 &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}
 \end{aligned}$$

## Step-11

Therefore, general solution of the differential equation is:

$$\boxed{u(t) = 3e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$