Step-1

a) Given set is the plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

We have to verify that the given set is a subspace of $\mathbf{R}^3$ or not.

Step-2

Let $A = \{(b_1, b_2, b_3) \mid b_1, b_2, b_3 \in \mathbf{R},\, b_1 = 0\}$

Let $(b_1, b_2, b_3) \in A, (c_1, c_2, c_3) \in A$

then $b_1 = 0, c_1 = 0$

Step-3

Now,

$$
\begin{aligned}
(b_1, b_2, b_3) + (c_1, c_2, c_3) &= (b_1 + c_1, b_2 + c_2, b_3 + c_3) \\
&= (0, b_2 + c_2, b_3 + c_3) \in A \qquad \begin{pmatrix} \text{Since } b_1 = 0, c_1 = 0 \\ \Rightarrow b_1 + c_1 = 0 \end{pmatrix}
\end{aligned}
$$

Therefore, A is closed under vector addition.

Let $c \in \mathbf{R}$ and $(b_1, b_2, b_3) \in A$

$$
\begin{aligned}
c(b_1, b_2, b_3) &= (cb_1, cb_2, cb_3) \\
&= (0, cb_2, cb_3) \in A \qquad \begin{pmatrix} \text{Since } b_1 = 0 \\ \Rightarrow cb_1 = 0 \end{pmatrix}
\end{aligned}
$$

Therefore A is closed under scalar multiplication

Thus A is a subspace of $\mathbf{R}^3$

Step-4

b) Given set is the plane of vectors b with $b_1 = 1$.

We have to verify that the given set is a subspace of $\mathbf{R}^3$ or not.

Step-5

Let $B = \{(b_1, b_2, b_3) \mid b_1, b_2, b_3 \in \mathbf{R}, b_1 = 1\}$

B is not closed under vector addition.

Since $(1,2,3), (1,5,6) \in B$

But $(1,2,3) + (1,5,6) = (2,7,9) \notin B$

The first component is not equal to 1.

Hence B is not a subspace of $\mathbf{R}^3$.

Step-6

c) Given set is the set of vectors b with $b_2 b_3 = 0$.

We have to verify that the given set is a subspace of $\mathbf{R}^3$ or not.

Step-7

Let $C = \{(b_1, b_2, b_3) \mid b_1, b_2, b_3 \in \mathbf{R}, b_2 b_3 = 0\}$

[This union of two subspaces $A = \{(b_1, b_2, b_3) \mid b_2 = 0\}, B = \{(b_1, b_2, b_3) \mid b_3 = 0\}$]

But C is not a subspace of $\mathbf{R}^3$

Since $(1,0,2) \in C$ and $(1,5,0) \in C$

Now $(1,0,2) + (1,5,0) = (2,5,2) \notin C$

since $5 \cdot 2 = 10 \neq 0$

Therefore C is not a subspace of $\mathbf{R}^3$

Step-8

d) Given set is the set all combiantions of two vectors $(1,1,0)$ and $(2,0,1)$.

We have to verify that the given set is a subspace of $\mathbf{R}^3$ or not.

Step-9

Let D =the linear combinations of the vectors $(1,1,0)$ and $(2,0,1)$

That is $D=\{a(1,1,0)+b(2,0,1)\mid a,b\in\mathbf{R}\}$

D is closed and vector addition

Since

$$\begin{aligned}&\left(a_1(1,1,0)+b_1(2,0,1)\right)+\left(a_2(1,1,0)+b_2(2,0,1)\right)\\&\qquad=(a_1+a_2)(1,1,0)+(b_1+b_2)(2,0,1)\in D\end{aligned}$$

Let c be a scalar and $a_1(1,1,0)+b_1(2,0,1)\in D$

Now $c\left(a_1(1,1,0)+b_1(2,0,1)\right)=\left(ca_1(1,1,0)+cb_1(2,0,1)\right)\in D$

Hence D is a subspace of $\mathbf{R}^3$

Step-10

e) Given set is the plane of vectors (b_1,b_2,b_3) that satisfy $b_3-b_2+3b_1=0$.

We have to verify that the given set is a subspace of $\mathbf{R}^3$ or not.

Step-11

Let E be the plane of vectors b_1,b_2,b_3 satisfy $b_3-b_2+3b_1=0$

$E=\{(b_1,b_2,b_3)\mid b_3-b_2+3b_1=0\}$

Let $(b_1,b_2,b_3)\in E,(c_1,c_2,c_3)\in E$

Now, $(b_1,b_2,b_3)+(c_1,c_2,c_3)=(b_1+c_1,b_2+c_2,b_3+c_3)\in E$

Since,

Step-12

$$\begin{aligned}(b_3+c_3)-(b_2+c_2)+3(b_1+c_1)&=(b_3-b_2+3b_1)+(c_3-c_2+3c_1)\\&=0+0\\&=0\end{aligned}$$

Therefore, E is closed under vector addition

Step-13

Let $a \in \mathbf{R}$, $(b_1, b_2, b_3) \in E$

$\Rightarrow a(b_1, b_2, b_3) = (ab_1, ab_2, ab_3) \in E$

Since

$$\begin{aligned} ab_3 - ab_2 + 3ab_1 &= a(b_3 - b_2 + 3b_1) \\ &= a \cdot 0 \\ &= 0 \end{aligned}$$

Therefore, E is closed under scalar multiplication

Therefore E is a subspace of $\mathbf{R}^3$.