Linear Algebra-A

Assignments - Week 9

Supplementary Problem Set

1. Let $\mathbf{0} \neq \boldsymbol{v} \in \mathbf{R}^n$. Please give a matrix $\boldsymbol{P}$ such that
$$\begin{cases} \boldsymbol{P}\boldsymbol{v} = \mathbf{0} \\ \boldsymbol{P}\boldsymbol{x} = \boldsymbol{x}, \forall \boldsymbol{x} \in N(\boldsymbol{v}^T) \end{cases},$$
where $N(\boldsymbol{v}^T)$ is the nullspace of $\boldsymbol{v}^T$.
$P = I - \frac{vv^T}{v^Tv}$
In addition, please show that
   a) $\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{P}$ and $\boldsymbol{P}^2 = \boldsymbol{P}$.
   b) Please show that $\boldsymbol{P}\boldsymbol{b}$ is the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{P}$. The error vector $\boldsymbol{b} - \boldsymbol{P}\boldsymbol{b}$ is orthogonal to the space. In other words, please show that the inner product $(\boldsymbol{b} - \boldsymbol{P}\boldsymbol{b})^{\mathrm{T}}\boldsymbol{P}\boldsymbol{c} = 0$ for any $\boldsymbol{c} \in \mathbf{R}^n$.

2. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 3 & 5 & 4 & 6 \end{bmatrix}$.
Please give a 4 by 4 orthogonal matrix $\boldsymbol{Q} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4]$, such that $\boldsymbol{q}_1, \boldsymbol{q}_2 \in C(\boldsymbol{A}^T)$ and $\boldsymbol{q}_3, \boldsymbol{q}_4 \in N(\boldsymbol{A})$.

3. Calculate the following determinant:
$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & 0 & \cdots & 0 \end{vmatrix}.$$ ("reverse-triangular" matrix)
$D_n = (-1)^{\frac{n(n-1)}{2}} a_{n1} a_{(n-1)2} \cdots a_{1n}$

4. If $\begin{vmatrix} 1+x & 2 & 3 \\ 2 & 1+x & 2 \\ 3 & 3 & 1+x \end{vmatrix} = 0$, find x.

5. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two invertible n by n matrices. Assume that $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, i.e., $\boldsymbol{AB} = \boldsymbol{BA}$. Let $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^{-1} & \boldsymbol{A}^{-1} \end{bmatrix}$.
(a) Show that $\boldsymbol{M}$ is not invertible.
(b) Find the rank of $\boldsymbol{M}$.