Step-1

Given

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} \det(A_2) &= 0 - 1 \\ &= -1 \end{aligned}$$

Step-2

And

$$A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} \det(A_3) &= (-1)\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ &= (-1)(-1) + 1(1) \\ &= 2 \end{aligned}$$

Step-3

Then

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$ Then

$$\det(A_4)=\begin{vmatrix}0&1&0&0\\1&0&1&1\\1&1&-1&0\\1&1&0&-1\end{vmatrix}=-\begin{vmatrix}1&1&1\\1&-1&0\\1&0&-1\end{vmatrix}$$

$$=-\begin{vmatrix}1&1&0\\0&-2&-1\\0&-1&-2\end{vmatrix}$$

$$=-(4-1)$$
$$=-3$$

Step-4

Now

$$-3\begin{vmatrix}1&1&1\\1&0&1\\1&1&0\end{vmatrix}=-3\left[\begin{vmatrix}0&1\\1&0\end{vmatrix}-\begin{vmatrix}1&1\\1&0\end{vmatrix}+\begin{vmatrix}1&0\\1&1\end{vmatrix}\right]$$

$$=-3[-1+1+1]$$
$$=3$$
$$=-3\left[\det(A_2)+\det(A_3)\right]$$

Step-5

And

$$A_5=\begin{bmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&1&0\end{bmatrix}$$

Expanding by 1st row

$$\det(A_5)=-\begin{vmatrix}1&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{vmatrix}+\begin{vmatrix}1&0&1&1\\1&1&1&1\\1&1&0&1\\1&1&1&0\end{vmatrix}-\begin{vmatrix}1&0&1&1\\1&1&0&1\\1&1&1&1\\1&1&1&0\end{vmatrix}+\begin{vmatrix}1&0&1&1\\1&1&0&1\\1&1&1&0\\1&1&1&1\end{vmatrix}$$

Step-6

Observe that the determinants on R.HS. Are each formed by one interchange of rows from the previous one so we can see that

$$\det(A_5) = -4\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$\begin{pmatrix} \text{if we call the determinants on R.H.S as } d_1, d_2, d_3, d_4 \text{ then } d_2 = -d_1, \\ d_3 = d_1, d_4 = -d_1 \end{pmatrix}$

Step-7

On solving

$$= -4\left[\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}\right] \qquad (\text{working on similar lines})$$

$$= -4\left[\det(A_3) + \det(A_4)\right]$$

$$= -4(2-3)$$
$$= 4$$

So, in general we can predict $\boxed{\det(A_n) = (-1)^{n-1}(n-1)}$

$\left(\text{so we get } \det A_2 = -(2-1) = -1\right)$

$$\det(A_3) = (3-1)$$
$$= 2$$
$$\det(A_4) = -(4-1)$$
$$= -3$$

$$\det(A_5) = 5-1$$
$$= 4 \text{ etc}$$