Step-1

Consider $A = Q\Lambda Q^T$ is symmetric positive definite and then $R = Q\sqrt{\Lambda}Q^T$ is its symmetric positive definite square root.

The objective is to identify whether R have positive eigenvalues or not.

Step-2

Consider the matrix $A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$.

The objective is to compute R and verify $R^2 = A$.

Since $R = Q\sqrt{\Lambda}Q^T$ is positive definite, then for all $x \neq 0$,

$x^T Rx > 0$

To show that R have positive eigenvalues, so consider:

$$\begin{aligned} Rx &= \lambda x \\ x^T Rx &= x^T \lambda x \\ x^T Rx &= \lambda \|x\|^2 \end{aligned}$$

We know that $x^T Rx > 0$, so $\lambda \|x\|^2 > 0$, thus $\lambda > 0$.

Therefore, R have positive eigenvalues, $\boxed{\lambda > 0}$.

Step-3

Consider the matrix $A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$.

Now, to find R, first find Q.

Using $l_{21} = \dfrac{6}{10}$

$$A=\begin{bmatrix}10 & 6\\ 6 & 10\end{bmatrix}\xrightarrow{R_2\to R_2-\frac{6}{10}R_1}\begin{bmatrix}10 & 6\\ 6-\left(\frac{6}{10}\cdot 10\right) & 10-\left(\frac{6}{10}\cdot 6\right)\end{bmatrix}$$

$$\to\begin{bmatrix}10 & 6\\ 0 & \frac{32}{5}\end{bmatrix}$$

$$=U$$

Step-4

Here, $U=\begin{bmatrix}10 & 6\\ 0 & \frac{32}{5}\end{bmatrix}$ and $L=\begin{bmatrix}1 & 0\\ \frac{6}{10} & 1\end{bmatrix}$,

$$\begin{aligned}LDU&=\begin{bmatrix}1 & 0\\ \frac{6}{10} & 1\end{bmatrix}\begin{bmatrix}10 & 0\\ 0 & \frac{32}{5}\end{bmatrix}\begin{bmatrix}10 & 6\\ 0 & \frac{32}{5}\end{bmatrix}\\ &=\begin{bmatrix}1 & 0\\ \frac{6}{10} & 1\end{bmatrix}\begin{bmatrix}10 & 0\\ 0 & \frac{32}{5}\end{bmatrix}\begin{bmatrix}1 & \frac{6}{10}\\ 0 & 1\end{bmatrix}\\ &=\begin{bmatrix}1 & 0\\ \frac{6}{10} & 1\end{bmatrix}\begin{bmatrix}10 & 0\\ 0 & \frac{32}{5}\end{bmatrix}\begin{bmatrix}1 & 0\\ \frac{6}{10} & 1\end{bmatrix}^T\end{aligned}$$

Step-5

Compare LDU with $Q\sqrt{\Lambda}Q^T$,

$$Q=\begin{bmatrix}1 & 0\\ \frac{6}{10} & 1\end{bmatrix}=L$$

$$\Lambda=\begin{bmatrix}10 & 0\\ 0 & \frac{32}{5}\end{bmatrix}=D$$

And

$$Q^T=\begin{bmatrix}1 & \frac{6}{10}\\ 0 & 1\end{bmatrix}=L^T$$

Here $A = Q\Lambda Q^T$ is symmetric positive definite and $R = Q\sqrt{\Lambda}Q^T$ symmetric positive definite square root, so take $R = \sqrt{\Lambda}Q^T$.

Step-6

To compute R, use $Q^T = \begin{bmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{bmatrix}$ and $\sqrt{\Lambda} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{\frac{32}{5}} \end{bmatrix}$.

$$\begin{aligned} R &= \sqrt{\Lambda}Q^T \\ &= \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{\frac{32}{5}} \end{bmatrix}\begin{bmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{10} & \frac{3}{5}\left(\sqrt{10}\right) \\ 0 & \frac{4}{5}\left(\sqrt{10}\right) \end{bmatrix} \end{aligned}$$

Therefore, $\boxed{R = \begin{bmatrix} \sqrt{10} & \frac{3}{5}\left(\sqrt{10}\right) \\ 0 & \frac{4}{5}\left(\sqrt{10}\right) \end{bmatrix}}$.

Step-7

Verify that $R^2 = A$.

$$\begin{aligned} R^2 &= R^T R \\ &= \begin{bmatrix} \sqrt{10} & 0 \\ \frac{3}{5}\left(\sqrt{10}\right) & \frac{4}{5}\left(\sqrt{10}\right) \end{bmatrix}\begin{bmatrix} \sqrt{10} & \frac{3}{5}\left(\sqrt{10}\right) \\ 0 & \frac{4}{5}\left(\sqrt{10}\right) \end{bmatrix} \\ &= \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \\ &= A \end{aligned}$$

Therefore, $\boxed{R^2 = A}$.

Step-8

Consider the matrix $A=\begin{bmatrix}10 & -6\\ -6 & 10\end{bmatrix}$.

The objective is to compute R and verify $R^2=A$.

Now, to find R, again find Q.

Using $l_{21}=\frac{-6}{10}$

$$A=\begin{bmatrix}10 & -6\\ -6 & 10\end{bmatrix}\xrightarrow{R_2\to R_2-\frac{6}{10}R_1}\begin{bmatrix}10 & -6\\ -6-\left(-\frac{6}{10}\cdot 10\right) & 10-\left(-\frac{6}{10}\cdot(-6)\right)\end{bmatrix}$$
$$\to\begin{bmatrix}10 & -6\\ 0 & \frac{32}{5}\end{bmatrix}$$
$$=U$$

Step-9

Here, $U=\begin{bmatrix}10 & -6\\ 0 & \frac{32}{5}\end{bmatrix}$ and $L=\begin{bmatrix}1 & 0\\ -\frac{6}{10} & 1\end{bmatrix}$,

$$LDU=\begin{bmatrix}1 & 0\\ -\frac{6}{10} & 1\end{bmatrix}\begin{bmatrix}10 & 0\\ 0 & \frac{32}{5}\end{bmatrix}\begin{bmatrix}1 & -\frac{6}{10}\\ 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\ -\frac{6}{10} & 1\end{bmatrix}\begin{bmatrix}10 & 0\\ 0 & \frac{32}{5}\end{bmatrix}\begin{bmatrix}1 & 0\\ -\frac{6}{10} & 1\end{bmatrix}^T$$

Step-10

Now, compare LDU with $Q\sqrt{\Lambda}Q^T$,

$$Q=\begin{bmatrix}1 & 0\\ -\frac{6}{10} & 1\end{bmatrix}=L$$

$$\Lambda=\begin{bmatrix}10 & 0\\ 0 & \frac{32}{5}\end{bmatrix}=D$$

$$Q^T = \begin{bmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{bmatrix} = L^T$$

Here $A = Q\Lambda Q^T$ is symmetric positive definite and $R = Q\sqrt{\Lambda}Q^T$ symmetric positive definite square root, so take $R = \sqrt{\Lambda}Q^T$.

Step-11

To compute R, use $Q^T = \begin{bmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{bmatrix}$ and $\sqrt{\Lambda} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{\frac{32}{5}} \end{bmatrix}$.

$$\begin{aligned} R &= \sqrt{\Lambda}Q^T \\ &= \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{\frac{32}{5}} \end{bmatrix}\begin{bmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{10} & -\frac{3}{5}\left(\sqrt{10}\right) \\ 0 & \frac{4}{5}\left(\sqrt{10}\right) \end{bmatrix} \end{aligned}$$

Therefore, $\boxed{R = \begin{bmatrix} \sqrt{10} & -\frac{3}{5}\left(\sqrt{10}\right) \\ 0 & \frac{4}{5}\left(\sqrt{10}\right) \end{bmatrix}}$.

Step-12

Verify that $R^2 = A$.

$$\begin{aligned} R^2 &= R^T R \\ &= \begin{bmatrix} \sqrt{10} & 0 \\ -\frac{3}{5}\left(\sqrt{10}\right) & \frac{4}{5}\left(\sqrt{10}\right) \end{bmatrix}\begin{bmatrix} \sqrt{10} & -\frac{3}{5}\left(\sqrt{10}\right) \\ 0 & \frac{4}{5}\left(\sqrt{10}\right) \end{bmatrix} \\ &= \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix} \\ &= A \end{aligned}$$

Therefore, $\boxed{R^2 = A}$.