Inner Products and Norms

Lecture 16

Dept. of Math., SUSTech

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Inner Product Space

- vector space $(+, \cdot)$
- Inner Product

$$(u, v) \longmapsto \langle u, v\rangle \in \mathbb{F}$$

Example

- $u = (x_1, y_1, z_1) \in \mathbb{F}^3$

  $v = (x_2, y_2, z_2) \in \mathbb{F}^3$

$$\langle u, v\rangle = x_1\overline{x_2} + y_1\overline{y_2} + z_1\overline{z_2}$$

- $V = \mathbb{F}^2$. A, B

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\langle A, B\rangle = \operatorname{tr}(B^H A)$$

- $V = \mathcal{P}_m(\mathbb{F})$ inner product

Inner Products Spaces

Introduction

In making the definition of a vector space, we generalized the linear structure (addition and scalar multiplication) of $\mathbb{R}^2$ and $\mathbb{R}^3$. We ignored other important features, such as the notions of length and angle. These ideas are embedded in the concept we now investigate, inner products.

Learning Objectives for This Chapter:

1. Cauchy-Schwarz Inequality.
2. Gram-Schmidt Procedure.
3. linear functionals on inner product spaces.
4. calculating minimum distance to a subspace.

Inner Products

The length of a vector x in $\mathbb{R}^2$ or $\mathbb{R}^3$ is called the norm of x, denoted by $||x||$. The generalization to $\mathbb{R}^n$ is obvious: we define the norm of $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ by

$$||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$$

The norm is not linear on $\mathbb{R}^n$. To inject linearity into the discussion, we introduce the dot product.

6.2 Definition ***dot product***

For $x, y \in \mathbf{R}^n$, the ***dot product*** of x and y, denoted $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Inner Product

V vector space over $\mathbb{F} <^{\mathbb{R}}_{\mathbb{C}}$

$(u, v) \mapsto \underline{\langle u, v\rangle} \in \mathbb{F}$

6.3 Definition ***inner product***

An ***inner product*** on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbf{F}$ and has the following properties:

positivity

$\langle v, v\rangle \geq 0$ for all $v \in V$;

definiteness

$\langle v, v\rangle = 0$ if and only if $v = 0$;

↳ zero vector in V

additivity in first slot

$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$;

$\langle v, u+w\rangle = \overline{\langle u+w, v\rangle}$

$= \overline{\langle u, v\rangle + \langle w, v\rangle}$

$= \overline{\langle u, v\rangle} + \overline{\langle w, v\rangle}$

$= \langle v, u\rangle + \langle v, w\rangle$

homogeneity in first slot

$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$;

$\langle u, \lambda v\rangle = \overline{\langle \lambda v, u\rangle}$

$= \overline{\lambda\langle v, u\rangle}$

$= \boxed{\overline{\lambda}}\langle u, v\rangle$

conjugate symmetry

$\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

Example

$V = \mathcal{P}_m(\mathbb{R})$ $\langle f, g \rangle = \int_{-1}^{1} f(x) \cdot g(x)\, dx$

$f, g \in \mathcal{P}_m(\mathbb{R})$

6.4 Example ***inner products***

(a) The ***Euclidean inner product*** on $\mathbf{F}^n$ is defined by

$$\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}.$$

different inner product

(b) If $c_1, \ldots, c_n$ are positive numbers, then an inner product can be defined on $\mathbf{F}^n$ by

$$\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle = c_1 w_1 \overline{z_1} + \cdots + c_n w_n \overline{z_n}.$$

(c) An inner product can be defined on the vector space of continuous real-valued functions on the interval $[-1, 1]$ by

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx.$$

$V = \mathcal{P}(\mathbb{R})$

$f, g \in \mathcal{P}(\mathbb{R})$

$\langle f, g \rangle = \int_0^{\infty} f(x) g(x) e^{-x}\, dx$

(d) An inner product can be defined on $\mathcal{P}(\mathbf{R})$ by

$$\langle p, q \rangle = \int_0^{\infty} p(x) q(x) \boxed{e^{-x}}\, dx.$$

convergence

Inner Product Space

6.5 Definition ***inner product space***

An ***inner product space*** is a vector space V along with an inner product on V.

The most important example of an inner product space is $\mathbb{F}^n$ with the Euclidean inner product given by part (a) of the last example. When $\mathbb{F}^n$ is referred to as an inner product space, you should assume that the inner product is the Euclidean inner product unless explicitly told otherwise.

6.6 Notation V

For the rest of this chapter, V denotes an inner product space over $\mathbf{F}$.

Basic Properties of an inner product

6.7 Basic properties of an inner product

$f(v_1) \quad f(v_2)$

$f(v_1+v_2) = \langle v_1+v_2, u\rangle = \langle v_1, u\rangle + \langle v_2, u\rangle$

(a) For each fixed $u \in V$, the function that takes v to $\langle v, u\rangle$ is a linear map from V to $\mathbf{F}$.

$f(\lambda v) = \langle \lambda v, u\rangle = \lambda\langle v, u\rangle = \lambda f(v)$

(b) $\langle 0, u\rangle = 0$ for every $u \in V$.

(c) $\langle u, 0\rangle = 0$ for every $u \in V$.

(d) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$.

(e) $\langle u, \lambda v\rangle = \bar{\lambda}\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and $u, v \in V$.

Norms

Our motivation for defining inner products came initially from the norms of vectors on $\mathbb{R}^2$ and $\mathbb{R}^3$. Now we see that each inner product determines a norm.

V inner product space

$\langle u, v \rangle$

6.8 Definition ***norm***, $\|v\|$

norm $\|v\| = \sqrt{\langle v, v \rangle}$

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norm $\to$ inner product space

For $v \in V$, the ***norm*** of v, denoted $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

Example

6.9 **Example** ***norms***

(a) If $(z_1, \ldots, z_n) \in \mathbf{F}^n$ (with the Euclidean inner product), then

$$\|(z_1, \ldots, z_n)\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

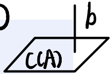
(b) In the vector space of continuous real-valued functions on $[-1, 1]$ [with inner product given as in part (c) of 6.4], we have

$$\|f\| = \sqrt{\int_{-1}^{1} \big(f(x)\big)^2 \, dx}.$$

Basic Properties of The Norm

$Ax=b$

$A^TA\hat{x}=A^Tb$



6.10 Basic properties of the norm

Suppose $v \in V$.

(a) $\|v\| = 0$ if and only if $v = 0$.

(b) $\|\lambda v\| = |\lambda| \, \|v\|$ for all $\lambda \in \mathbf{F}$.

Now we come to a crucial definition.

6.11 Definition ***orthogonal*** $u, v \in V$ $\langle u, v\rangle = 0$ $\Rightarrow$ orthogonal

Two vectors $u, v \in V$ are called ***orthogonal*** if $\langle u, v\rangle = 0$.

x, x^2

$P_2(\mathbb{R})$ $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$

$\langle x, x^2\rangle = \int_{-1}^{1} x \cdot x^2\,dx = 0$

Orthogonality

We begin our study of orthogonality with an easy result.

6.12 Orthogonality and 0

(a) 0 is orthogonal to every vector in V.

(b) 0 is the only vector in V that is orthogonal to itself.

Pythagorean Theorem

6.13 Pythagorean Theorem

Suppose u and v are orthogonal vectors in V. Then

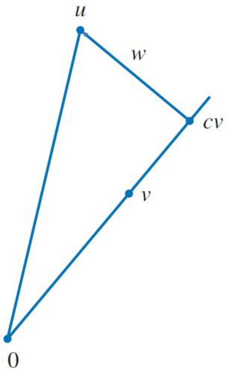
$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof. We have

$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v\rangle \\ &= \langle u,v\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle \\ &= ||u||^2 + ||v||^2, \end{aligned}$$

as desired.

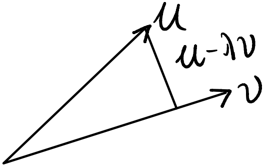
An orthogonal decomposition



An orthogonal decomposition.

An orthogonal decomposition

Suppose $u, v \in V$, with $v \neq 0$. We would like to write u as a scalar multiple of v plus a vector w orthogonal to v, as suggested in the above picture.



$$0 = \langle u - \lambda v, v \rangle$$
$$= \langle u, v \rangle - \lambda \langle v, v \rangle$$
$$\lambda = \frac{\langle u, v \rangle}{\langle v, v \rangle}$$

An orthogonal decomposition

To discover how to write u as a scalar multiple of v plus a vector orthogonal to v, let $c \in \mathbb{F}$ denote a scalar. Then

$$u = cv + (u - cv).$$

Thus we need to choose c so that v is orthogonal to $(u - cv)$. In other words, we want

$$0 = \langle u - cv, v \rangle = \langle u, v \rangle - c\|v\|^2.$$

6.14 An orthogonal decomposition

Suppose $u, v \in V$, with $v \neq 0$. Set $c = \dfrac{\langle u, v \rangle}{\|v\|^2}$ and $w = u - \dfrac{\langle u, v \rangle}{\|v\|^2} v$. Then

$$\langle w, v \rangle = 0 \quad \text{and} \quad u = cv + w.$$

$\langle w, v \rangle = 0$

Cauchy-Schwarz Inequality

6.15 Cauchy–Schwarz Inequality

Suppose $u, v \in V$. Then

$u, v \in V$ inner product space

$|<u, v>| \leq \|u\| \cdot \|v\|$

$$\boxed{|\langle u, v\rangle| \leq \|u\| \, \|v\|}.$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

$$u = \frac{<u, v>}{\|v\|^2} \cdot v + w$$

$$\Rightarrow \|u\|^2 = \left\| \frac{<u, v>}{\|v\|^2} v \right\|^2 + \|w\|^2$$

$$\geq \frac{<u, v>^2 \|v\|^2}{\|v\|^4} = \frac{|<u, v>|^2}{\|v\|^2}.$$

Examples

6.17 **Example** ***examples of the Cauchy–Schwarz Inequality***

(a) If $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{R}$, then

$$|x_1 y_1 + \cdots + x_n y_n|^2 \leq ({x_1}^2 + \cdots + {x_n}^2)({y_1}^2 + \cdots + {y_n}^2).$$

(b) If f, g are continuous real-valued functions on $[-1, 1]$, then

$$\left|\int_{-1}^{1} f(x)g(x)\,dx\right|^2 \leq \left(\int_{-1}^{1} \big(f(x)\big)^2\,dx\right)\left(\int_{-1}^{1} \big(g(x)\big)^2\,dx\right).$$

Triangle Inequality

6.18 Triangle Inequality

Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

This inequality is an equality if and only if one of u, v is a nonnegative multiple of the other.

Proof We have

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle = \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle \\
&= \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \overline{\langle u,v\rangle} \\
&= \langle u,u\rangle + \langle v,v\rangle + 2\mathrm{Re}\langle u,v\rangle \leq \langle u,u\rangle + \langle v,v\rangle + 2|\langle u,v\rangle| \\
&\underset{\text{Cauchy-Schwarz}}{\leq} \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| = (\|u\| + \|v\|)^2.
\end{aligned}$$

Proof.

Taking square roots of both sides of the inequality above gives the desired inequality.

We have equality in the Triangle Inequality if and only if

$$\langle u, v \rangle = ||u|| ||v||.$$

If one of u, v is a nonnegative multiple of the other, then the above equality holds. Then the condition for equality in the Cauchy-Schwarz inequality implies that one of u, v is a scalar multiple of the other. Clearly, the above equality forces the scalar in question to be nonnegative, as desired.

The triangle inequality is an equality if and only if one of u, v is a nonnegative multiple of the other.

Parallelogram Equality

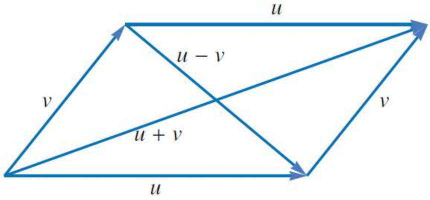
6.22 Parallelogram Equality

Suppose $u, v \in V$. Then

$=\langle u-v, u-v\rangle$

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

$=\langle u+v, u+v\rangle$



The parallelogram equality.

US Supreme Court, 2010

Law professor Richard Friedman presenting a case before the U.S. Supreme Court in 2010:

Mr. Friedman: I think that issue is entirely orthogonal to the issue here because the Commonwealth is acknowledging—

Chief Justice Roberts: I'm sorry. Entirely what?

Mr. Friedman: Orthogonal. Right angle. Unrelated. Irrelevant.

Chief Justice Roberts: Oh.

Justice Scalia: What was that adjective? I liked that.

Mr. Friedman: Orthogonal.

Chief Justice Roberts: Orthogonal.

Mr. Friedman: Right, right.

Justice Scalia: Orthogonal, ooh. (Laughter.)

Justice Kennedy: I knew this case presented us a problem. (Laughter.)

Homework Assignment 16

6.A: 6, 14, 16, 19, 20, 24, 25.