Step-1

Orthogonal Matrix: If matrix A is skew- symmetric then e^{At} is an orthogonal matrix.

Skew symmetric: If transpose of matrix A $\left(A^T\right)$ is equal to negative of matrix A, then matrix A is skew-symmetric.

In a conservative system following are observed:

$$\begin{aligned} A^T &= -A \\ \left(e^{At}\right)^T &= e^{-At} \\ \left\|e^{At}u(0)\right\| &= \left\|u(0)\right\| \end{aligned}$$

Step-2

Consider the skew-symmetric equation:

$$\begin{aligned} \frac{du}{dt} &= Au \\ &= \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \end{aligned}$$

Step-3

(a) Compute u_1', u_2', u_3' and confirm the following:

$$u_1u_1' + u_2u_2' + u_3u_3' = 0$$

Calculate the following from the skew-symmetric equation

$$\begin{aligned} u_1' &= cu_2 - bu_3 \\ u_2' &= -cu_1 + au_3 \\ u_3' &= bu_1 - au_2 \end{aligned}$$

Step-4

Now,

$$u_1u_1' = cu_2u_1 - bu_3u_1$$
$$u_2u_2' = -cu_1u_2 + au_3u_2$$
$$u_3u_3' = bu_1u_3 - au_2u_3$$
$$u_1u_1' + u_2u_2' + u_3u_3' = 0$$

Therefore, $\boxed{u_1u_1' + u_2u_2' + u_3u_3' = 0}$.

Step-5

(b) Evaluate the length of $u_1^2 + u_2^2 + u_3^2$.

Length of $u_1^2 + u_2^2 + u_3^2$ can be written as $\|u(t)\|^2$. Now,

$$\|u(t)\|^2 = u_1^2 + u_2^2 + u_3^2$$

Differentiate the following:

$$\frac{d\|u(t)\|^2}{dt} = 2u_1u_1' + 2u_2u_2' + 2u_3u_3'$$

Substitute the above result, to get derivate of $\|u(t)\|^2$ equal to zero. This shows that $\|u(t)\|^2$ is constant.

Step-6

Recall that $u(t) = e^{At}u(0)$, then

$$\begin{aligned}\|u(t)\|^2 &= \|e^{At}u(0)\|^2 \\ &= \|u(0)\|^2 \\ &= \text{constant}\end{aligned}$$

Therefore, $\boxed{\|u(t)\|^2 = \|u(0)\|^2}$.

Step-7

(c) Find the Eigen values of matrix *A*.

To calculate the Eigen values do the following calculations;

$$A-\lambda I=\begin{bmatrix}0-\lambda & c & -b\\ -c & 0-\lambda & a\\ b & -a & 0-\lambda\end{bmatrix}$$

$$\det\left(A-\lambda I\right)=0$$

$$\left(-\lambda\right)^3-\lambda\left(a^2+b^2+c^2\right)=0$$

$$-\lambda\left(\lambda^2+\left(a^2+b^2+c^2\right)\right)=0$$

After solving following values are obtained:

$$\lambda_1=0$$

$$\lambda_2=\sqrt{\left(a^2+b^2+c^2\right)}$$

$$\lambda_3=-\sqrt{\left(a^2+b^2+c^2\right)}$$

Step-8

Therefore, Eigen values are $\boxed{0,\pm\sqrt{\left(a^2+b^2+c^2\right)}}$.