

Step-1

Suppose we diagonalize the symmetric matrix A using an orthogonal matrix Q . That is, if $\hat{\Lambda}$ denotes the diagonal matrix, which has all the eigenvalues of A along its diagonal, then we get the following:

$$Q^T A Q = \Lambda$$

Let $x = Qy$. Thus, we get

$$\begin{aligned} R(x) &= \frac{x^T A x}{x^T x} \\ &= \frac{(Qy)^T A (Qy)}{(Qy)^T (Qy)} \\ &= \frac{y^T \Lambda y}{y^T y} \\ &= \frac{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2}{y_1^2 + y_2^2 + \dots + y_n^2} \end{aligned}$$

Step-2

Since $\hat{\lambda}_n$ is the largest eigenvalue of the matrix A , we have $\lambda_k \leq \lambda_n$, where $\hat{\lambda}_k$ is any eigenvalue of A .

Thus, we have

$$\begin{aligned} R(x) &= \frac{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2}{y_1^2 + y_2^2 + \dots + y_n^2} \\ &\leq \frac{\lambda_n y_1^2 + \lambda_n y_2^2 + \dots + \lambda_n y_n^2}{y_1^2 + y_2^2 + \dots + y_n^2} \\ &= \frac{\lambda_n (y_1^2 + y_2^2 + \dots + y_n^2)}{y_1^2 + y_2^2 + \dots + y_n^2} \\ &= \lambda_n \end{aligned}$$

Step-3

Thus, we have shown that $\boxed{R(x) \leq \lambda_n}$.