Step-1

Consider the following matrix:

$$A=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

Observe that this matrix has a sub matrix of order 3×3 and also the determinant is zero.

So, the rank of the matrix A is less than 3.

And also, see that any sub matrix of order 2×2 has the determinant zero.

So, the rank of the matrix A is less than 2, that means rank of A is 1.

Step-2

Now, the characteristic equation of A is $|A-\lambda I|=0$.

$$\begin{vmatrix}1-\lambda&1&1&1\\1&1-\lambda&1&1\\1&1&1-\lambda&1\\1&1&1&1-\lambda\end{vmatrix}=0$$

Use elementary row operations on this to reduce into the echelon form as follows.

$R_1\to R_1+R_2+R_3+R_4$

$$\begin{vmatrix}4-\lambda&4-\lambda&4-\lambda&4-\lambda\\1&1-\lambda&1&1\\1&1&1-\lambda&1\\1&1&1&1-\lambda\end{vmatrix}=0$$

$$(4-\lambda)\begin{vmatrix}1&1&1&1\\1&1-\lambda&1&1\\1&1&1-\lambda&1\\1&1&1&1-\lambda\end{vmatrix}=0$$

Step-3

Continue the above steps.

$$R_2 \to R_2 - R_1$$
$$R_3 \to R_3 - R_1$$
$$R_4 \to R_4 - R_1$$

$$(4-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0-\lambda & 0 & 0 \\ 0 & 0 & 0-\lambda & 0 \\ 0 & 0 & 0 & 0-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left[1(-\lambda)(-\lambda)(-\lambda)\right] = 0$$

$$\lambda^3(\lambda-4) = 0$$

So, $\lambda = 0,0,0,4$ are the Eigen values of the given matrix.

Step-4

Now, take the matrix $C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

Here, the elementary operations on the matrix C does not change the rank of C.

So, apply row operations on C to reduce it to echelon form.

Then we confirm the rank of C.

For, $R_1 \leftrightarrow R_4, R_2 \leftrightarrow R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$$R_3 \to R_3 - R_1, R_4 \to R_4 - R_2 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-5

Clearly, the entries in the principal diagonal are either zero or one.

And also all the entries below the main diagonal are 0â€™s.

So, the matrix is in the echelon form.

Hence, the rank of C is equal to the number of non-zero rows in the matrix C.

Here, the number of non-zero rows is 2, and then the rank is 2.

Step-6

Further, to get the Eigen values of C, take the characteristic equation of C.

That is

$$|C-\lambda \mathrm{I}|=0$$

$$\left|\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right|=0$$

$$\left|\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}-\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}\right|=0$$

$$\begin{vmatrix} 0-\lambda & 1 & 0 & 1 \\ 1 & 0-\lambda & 1 & 0 \\ 0 & 1 & 0-\lambda & 1 \\ 1 & 0 & 1 & 0-\lambda \end{vmatrix}=0$$

$$\begin{vmatrix} -\lambda & 1 & 0 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{vmatrix}=0$$

Apply the row operations on this, then

$R_1 \rightarrow R_1+R_2+R_3+R_4 \Rightarrow$

$$\begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda & 2-\lambda \\ 1 & 0-\lambda & 1 & 0 \\ 0 & 1 & 0-\lambda & 1 \\ 1 & 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{vmatrix} = 0$$

Step-7

Apply the following row-operations.

$$R_2 \to R_2 - R_1$$
$$R_3 \to R_3 - R_1$$
$$R_4 \to R_4 - R_1$$

$$(2-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda-1 & 0 & -1 \\ 0 & 1 & -\lambda & 1 \\ 0 & -1 & 0 & -\lambda-1 \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} -\lambda-1 & 0 & -1 \\ 1 & -\lambda & 1 \\ -1 & 0 & -\lambda-1 \end{vmatrix} = 0$$

$$(2-\lambda)(-\lambda)\left\{(-\lambda-1)^2 - 1\right\} = 0$$

$$(2-\lambda)(-\lambda)\left\{\left(\lambda^2 + 2\lambda + 1\right) - 1\right\} = 0$$

$$(2-\lambda)(-\lambda)\left(\lambda^2 + 2\lambda\right) = 0$$

$$(2-\lambda)(-\lambda)(\lambda)(\lambda+2) = 0$$

$$-\lambda^2(2-\lambda)(\lambda+2) = 0$$

Therefore, the eigen values of the matrix C are $0, 0, -2,$ and 2.

Step-8

Now, find the Eigen vectors as follows.

Let $v_1 = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ be the required Eigen vector.

At $\lambda = 2$, the system $(C - \lambda \mathrm{I})v_1 = \mathbf{0}$ becomes as $(C - (2)\mathrm{I})v_1 = \mathbf{0}$.

$$\begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Here, the coefficient matrix is $\begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$.

Apply the row operations to reduce into echelon form.

$R_1 \Leftrightarrow R_2$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_4 \to R_4 - R_1$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & 0 & -2 \end{pmatrix}$$

$R_3 \to 3R_3 + R_2$, $R_4 \to 3R_4 + 2R_2$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & -4 \end{pmatrix}$$

$R_4 \to R_4 + R_3$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now the system becomes as,

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The system of equations is,

$$\begin{aligned} x - 2y + z &= 0 \quad \ldots\ldots (1) \\ -3y + 2z + t &= 0 \quad \ldots\ldots (2) \\ -4z + 4t &= 0 \quad \ldots\ldots (3) \end{aligned}$$

From equation (3), $z = t$.

Substitute $z = t$ in the equation (2),

$$\begin{aligned} -3y + 2t + t &= 0 \\ -3y + 3t &= 0 \\ y &= t \end{aligned}$$

Then, substitute $z = t$, $y = t$ in the equation (1).

$$\begin{aligned} x - 2y + z &= 0 \\ x - 2t + t &= 0 \\ x - t &= 0 \\ x &= t \end{aligned}$$

So, the variable matrix is $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Hence, the Eigen vector for the Eigen value $\lambda = 2$ is $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Step-9

Similarly, the remaining Eigen vectors are $v_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ for $\lambda = -2$.

$v_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda = 0$ and $v_4 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ for $\lambda = 0$.