Step-1

Given 4 by 4 Hadamard Matrix is $H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

We have to find the inverse of the given matrix.

Step-2

The matrix H has diagonal columns of length 2.

So the inverse of $H = \frac{H^T}{4} = \frac{H}{4}$

Therefore, $\boxed{H^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}}$

Step-3

Now we verify that $HH^{-1} = H^{-1}H = I$

Therefore,

$$HH^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Step-4

Now we have to write $v=(7,5,3,1)$ as a combination of the columns of H.

Consider

$$a\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+b\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}+c\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}+d\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}=\begin{bmatrix}7\\5\\3\\1\end{bmatrix}$$

$$\begin{aligned}\Rightarrow a+b+c+d&=7\\ a-b+c-d&=5\\ a+b-c-d&=3\\ a-b-c+d&=1\end{aligned}$$

Step-5

The augmented matrix of the above system is

$$\begin{bmatrix}1&1&1&1&7\\1&-1&1&-1&5\\1&1&-1&-1&3\\1&-1&-1&1&1\end{bmatrix}$$

$$\left.\begin{matrix}R_2-R_1\\R_3-R_1\\R_4-R_1\end{matrix}\right\}\rightarrow\begin{bmatrix}1&1&1&1&7\\0&-2&0&-2&-4\\0&0&-2&-2&-4\\0&-2&-2&0&-6\end{bmatrix}$$

$$\left.\begin{matrix}\frac{-1}{2}R_2\\R_1+\frac{1}{2}R_3\\-\frac{1}{2}R_4\end{matrix}\right\}\rightarrow\begin{bmatrix}1&0&1&0&6\\0&1&0&1&1\\0&0&1&1&2\\0&-1&1&0&3\end{bmatrix}$$

Step-6

Continuation to the above

$$R_4 - R_2 \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{bmatrix}$$

$$\left.\begin{matrix} R_1 - R_3 \\ R_4 - R_3 \end{matrix}\right\} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\left.\frac{-1}{2} R_4 \right\} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Continuation to the above

$$\left.\begin{matrix} R_1 + R_4 \\ R_2 - R_4 \\ R_3 - R_4 \end{matrix}\right\} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

From this, we get $a = 4, b = 1, c = 2, d = 0$

Hence $\boxed{\begin{bmatrix} 7 \\ 5 \\ 3 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} - \\ -1 \\ 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}$