Step-1

Given circulant matrix $C = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}$

If the eigen vector of C is $x = (1, w_j, w_j^2, w_j^3)$ then its corresponding eigen value is

$e_j = c_0 + c_1 w_j + c_2 w_j^2 + c_3 w_j^3$ for $j = 0,1$ and $n = 4$

If $x = (1,1,1,1)$ then its corresponding eigen value is

$$\begin{aligned} e_0 &= c_0 + c_1 w_0 + c_2 w_0^2 + c_3 w_0^3 \\ &= c_0 + c_1 + c_2 + c_3 \end{aligned}$$

If $x = (1, i, i^2, i^3)$ then its corresponding eigen value is

$$\begin{aligned} e_1 &= c_0 + c_1 w_1 + c_2 w_1^2 + c_3 w_1^3 \\ &= c_0 + ic_1 + i^2 c_2 + i^3 c_3 \\ &= c_0 + ic_1 - c_2 - ic_3 \\ &= c_0 - c_2 + i(c_1 - c_3) \end{aligned}$$

Therefore, the required eigen values of C are $e_0 = c_0 + c_1 + c_2 + c_3$ and $e_1 = c_0 - c_2 + i(c_1 - c_3)$