## Step-1

Consider the matrices,

$$A=\begin{bmatrix}1 & 1\\ 1 & -1\\ -2 & 4\end{bmatrix},\quad b=\begin{bmatrix}1\\ 2\\ 7\end{bmatrix}$$

The objective is to find the projection of $b$ onto the column space of $A$.

Split $b$ into $p+q$ with $p$ in the column space and $q$ perpendicular to that space.

Also determine the subspace that contains $q$.

## Step-2

Projection of $b$ onto the column space of $A$ is $p=Pb$, here $P=A\left(A^TA\right)^{-1}A^T$

Compute matrix $P$ as shown:

$$\begin{aligned}P&=A\left(A^TA\right)^{-1}A^T\\
&=\begin{bmatrix}1 & 1\\ 1 & -1\\ -2 & 4\end{bmatrix}\left(\begin{bmatrix}1 & 1 & -2\\ 1 & -1 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\\ -2 & 4\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1 & -2\\ 1 & -1 & 4\end{bmatrix}\\
&=\begin{bmatrix}1 & 1\\ 1 & -1\\ -2 & 4\end{bmatrix}\left(\begin{bmatrix}1 & 1 & -2\\ 1 & -1 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\\ -2 & 4\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1 & -2\\ 1 & -1 & 4\end{bmatrix}\\
&=\begin{bmatrix}1 & 1\\ 1 & -1\\ -2 & 4\end{bmatrix}\begin{bmatrix}6 & -8\\ -8 & 18\end{bmatrix}^{-1}\begin{bmatrix}1 & 1 & -2\\ 1 & -1 & 4\end{bmatrix}\\
&=\begin{bmatrix}1 & 1\\ 1 & -1\\ -2 & 4\end{bmatrix}\left(\frac{1}{22}\begin{bmatrix}9 & 4\\ 4 & 3\end{bmatrix}\right)\begin{bmatrix}1 & 1 & -2\\ 1 & -1 & 4\end{bmatrix}\end{aligned}$$

## Step-3

Continuation to the above steps,

$$P=\frac{1}{22}\begin{bmatrix}1 & 1\\ 1 & -1\\ -2 & 4\end{bmatrix}\begin{bmatrix}9 & 4\\ 4 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & -2\\ 1 & -1 & 4\end{bmatrix}$$

$$=\frac{1}{22}\begin{bmatrix}13 & 7\\ 5 & 1\\ -2 & 4\end{bmatrix}\begin{bmatrix}1 & 1 & -2\\ 1 & -1 & 4\end{bmatrix}$$

$$=\frac{1}{22}\begin{bmatrix}20 & 6 & 2\\ 6 & 4 & -6\\ 2 & -6 & 20\end{bmatrix}$$

## Step-4

Projection of $b$ onto the column space of $A$ is $p=Pb$

$$p=Pb$$

$$=\frac{1}{22}\begin{bmatrix}20 & 6 & 2\\ 6 & 4 & -6\\ 2 & -6 & 20\end{bmatrix}\begin{bmatrix}1\\ 2\\ 7\end{bmatrix}$$

$$=\frac{1}{22}\begin{bmatrix}46\\ -28\\ 130\end{bmatrix}$$

$$=\boxed{\begin{bmatrix}23/11\\ -14/11\\ 65/11\end{bmatrix}}$$

## Step-5

Split $b$ into $p+q$ with $p$ in the column space and $q$ perpendicular to that space.

That is, $b=p+q$ where $p\in\operatorname{Col}(A)$ and $q\in\operatorname{Col}(A)^{\perp}=\operatorname{Nul}(A^{\mathrm{T}})$

As two vectors in columns of matrix $A=\begin{bmatrix}1 & 1\\ 1 & -1\\ -2 & 4\end{bmatrix}$ are linearly independent,

$$\operatorname{Col}(A)=\operatorname{span}\left\{\begin{bmatrix}1\\ 1\\ -2\end{bmatrix},\begin{bmatrix}1\\ -1\\ 4\end{bmatrix}\right\}$$

## Step-6

Find $\mathrm{Nul}\left(A^{\mathrm{T}}\right)$.

Let $\mathbf{x} \in \mathrm{Nul}\left(A^{\mathrm{T}}\right)$

$$A^{\mathrm{T}}\mathbf{x} = 0$$

$$\underbrace{\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

Augmented matrix associated with the above notation is,

$$\begin{aligned}
[\mathbf{M} \mid \mathbf{0}] &= \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -1 & 4 & 0 \end{array}\right] \\
& R_2 \to R_2 - R_1 \\
&\approx \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 6 & 0 \end{array}\right] \\
& R_2 \to -\frac{1}{2} R_2 \\
&\approx \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \\
& R_1 \to R_1 - R_2 \\
&\approx \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]
\end{aligned}$$

## Step-7

From the above one we obtain the system,

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

Suppose that $x_3 = s,\ s \in \mathbb{R}$

Then $x_1 = -s, x_2 = 3s$

Therefore,

$$\operatorname{Nul}\left(A^{\mathrm{T}}\right)=\left\{\begin{bmatrix}-s\\3s\\s\end{bmatrix}:s\in\mathbb{R}\right\}$$

$$=\left\{s\begin{bmatrix}-1\\3\\1\end{bmatrix}:s\in\mathbb{R}\right\}$$

## Step-8

To express $b=p+q$ where $p\in\operatorname{Col}(A)$ and $q\in\operatorname{Col}(A)^{\perp}=\operatorname{Nul}\left(A^{\mathrm{T}}\right)$:

Observe that,

$$b=\begin{bmatrix}1\\2\\7\end{bmatrix}$$

$$=\begin{bmatrix}\frac{11}{11}\\\frac{22}{11}\\\frac{77}{11}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{23-12}{11}\\\frac{-14+36}{11}\\\frac{65+12}{11}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{23}{11}\\-\frac{14}{11}\\\frac{65}{11}\end{bmatrix}+\begin{bmatrix}-\frac{12}{11}\\\frac{36}{11}\\\frac{12}{11}\end{bmatrix}\quad(=p+q)$$

$$=\begin{bmatrix}\frac{9}{22}+\frac{37}{22}\\\frac{9}{22}-\frac{37}{22}\\-\frac{18}{22}+\frac{148}{22}\end{bmatrix}+\begin{bmatrix}-\frac{12}{11}\\\frac{36}{11}\\\frac{12}{11}\end{bmatrix}$$

## Step-9

Continuation to the above step,

$$b=\begin{bmatrix}\frac{9}{22}+\frac{37}{22}\\ \frac{9}{22}-\frac{37}{22}\\ -\frac{18}{22}+\frac{148}{22}\end{bmatrix}+\begin{bmatrix}-\frac{12}{11}\\ \frac{36}{11}\\ \frac{12}{11}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{9}{22}\\ \frac{9}{22}\\ -\frac{18}{22}\end{bmatrix}+\begin{bmatrix}\frac{37}{22}\\ -\frac{37}{22}\\ \frac{148}{22}\end{bmatrix}+\begin{bmatrix}-\frac{12}{11}\\ \frac{36}{11}\\ \frac{12}{11}\end{bmatrix}$$

$$=\frac{9}{22}\begin{bmatrix}1\\1\\-2\end{bmatrix}+\frac{37}{22}\begin{bmatrix}1\\-1\\4\end{bmatrix}+\frac{12}{11}\begin{bmatrix}-1\\3\\1\end{bmatrix}$$

$$=\text{span}\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix},\begin{bmatrix}1\\-1\\4\end{bmatrix}\right\}+\text{span}\left\{\begin{bmatrix}-1\\3\\1\end{bmatrix}\right\}$$

$$=\text{Col}(A)+\text{Nul}\left(A^{T}\right)$$

Thus,

$b=p+q$ Here $p=\begin{bmatrix}\frac{23}{11}\\ -\frac{14}{11}\\ \frac{65}{11}\end{bmatrix}\in\text{Col}(A)$ and $q=\begin{bmatrix}-\frac{12}{11}\\ \frac{36}{11}\\ \frac{12}{11}\end{bmatrix}\in\text{Nul}\left(A^{T}\right)$

Therefore $q\in\text{Nul}\left(A^{T}\right)$