Trace

Lecture 29

Dept. of Math., SUSTech

2023.05

$A_{n\times n}$ $\mathrm{tr}(A)=\sum_{i=1}^{n} a_{ii}$

T $\mathrm{trace}\, T=\lambda_1+\lambda_2+\cdots+\lambda_n$

eigenvalues of T

$\lambda_1,\cdots,\lambda_n$ $\mathcal{M}(T,v_1,\cdots,v_n)=A$

$T\in\mathcal{L}(V)$ $\dim V=n$

$\lambda_1,\lambda_2,\cdots,\lambda_n$ eigenvalues

$q(z)=(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$

$=z^n-\underbrace{(\lambda_1+\lambda_2+\cdots+\lambda_n)}_{\mathrm{trace}\, T}z^{n-1}+\cdots+(-1)^n\underbrace{\lambda_1\lambda_2\cdots\lambda_n}_{\det T}$

$\mathrm{tr}(A)=\mathrm{trace}\, T$

$T(u_1,\cdots,u_n)=(u_1,\cdots,u_n)A$

$T(v_1,\cdots,v_n)=(v_1,\cdots,v_n)B$

$B=M^{-1}AM$

$\mathrm{trace}\, A=\mathrm{trace}\, B$

$^*\mathrm{trace}\, AB=\mathrm{trace}\, BA$

$\mathrm{trace}\, B=\mathrm{trace}(M^{-1}AM)$

$=\mathrm{trace}(AMM^{-1})$

$=\mathrm{trace}(A)$

Trace

1. Change of Basis
2. Trace: A Connection Between Operators and Matrices
3. Properties of Trace
4. Homework Assignment 29

Change of Basis

10.2 **Definition** ***identity matrix, I***

Suppose n is a positive integer. The n-by-n diagonal matrix

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

is called the ***identity matrix*** and is denoted I.

Note that we use the symbol I to denote the identity operator and the identity matrix.

Inverse

10.3 Definition ***invertible, inverse,*** A^{-1}

A square matrix A is called ***invertible*** if there is a square matrix B of the same size such that $AB = BA = I$; we call B the ***inverse*** of A and denote it by A^{-1}.

Some mathematicians use the terms **nonsingular**, which means the same as invertible, and **singular**, which means the same as noninvertible.

Matrix

The next result is one of the unusual cases in which we use two different bases even though we have operators from a vector space to itself.

10.4 The matrix of the product of linear maps

Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are all bases of V. Suppose $S, T \in \mathcal{L}(V)$. Then

$$\mathcal{M}\big(ST, (u_1, \ldots, u_n), (w_1, \ldots, w_n)\big) =$$
$$\mathcal{M}\big(S, (v_1, \ldots, v_n), (w_1, \ldots, w_n)\big)\mathcal{M}\big(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n)\big).$$

The next result deals with the matrix of the identity operator I with respect to two bases.

10.5 Matrix of the identity with respect to two bases

Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then the matrices $\mathcal{M}\big(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)\big)$ and $\mathcal{M}\big(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n)\big)$ are invertible, and each is the inverse of the other.

Change of basis formula

10.7 Change of basis formula

Suppose $T \in \mathcal{L}(V)$. Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be bases of V. Let $A = \mathcal{M}\big(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)\big)$. Then

$$\mathcal{M}\big(T, (u_1, \ldots, u_n)\big) = A^{-1}\mathcal{M}\big(T, (v_1, \ldots, v_n)\big)A.$$

Proof. In 10.4, replace w_j with u_j and replace S with I, getting

$$\mathscr{M}(T,(u_1,u_2,\cdots,u_n)) = A^{-1}\mathscr{M}(T,(u_1,u_2,\cdots,u_n),(v_1,v_2,\cdots,v_n)),$$

where we have used 10.5.

Again use 10.4, this time replacing w_j with v_j. Also replace T with I and replace S with T, getting

$$\mathscr{M}(T,(u_1,u_2,\cdots,u_n),(v_1,v_2,\cdots,v_n)) = A^{-1}\mathscr{M}(T,(v_1,v_2,\cdots,v_n))A.$$

Substituting the equation above into 10.8 gives the desired result.

Properties of Trace

In the definition below, the sum of the eigenvalues " with each eigenvalue repeated according to its multiplicity" means that if $\lambda_1, \lambda_2, \cdots, \lambda_m$ are the distinct eigenvalues of T with multiplicities $d_1, d_2, \cdots, d_m$, then the sum is

$$d_1\lambda_1 + d_2\lambda_2 + \cdots + d_m\lambda_m.$$

10.9 **Definition** ***trace of an operator***

Suppose $T \in \mathcal{L}(V)$.

- If $\mathbf{F} = \mathbf{C}$, then the ***trace*** of T is the sum of the eigenvalues of T, with each eigenvalue repeated according to its multiplicity.
- If $\mathbf{F} = \mathbf{R}$, then the ***trace*** of T is the sum of the eigenvalues of $T_{\mathbf{C}}$, with each eigenvalue repeated according to its multiplicity.

The trace of T is denoted by trace T.

Properties of Trace

The trace has a close connection with the characteristic polynomial. The characteristic polynomial of T can be written in the form

$$z^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n) z^{n-1} + \cdots + (-1)^n (\lambda_1 \cdots + \lambda_n).$$

10.12 Trace and characteristic polynomial

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then trace T equals the negative of the coefficient of z^{n-1} in the characteristic polynomial of T.

Most of the rest of this section is devoted to discovering how to compute trace T from the matrix of T (with respect to an arbitrary basis).

Trace of a Matrix

At this point you should suspect that trace T equals the sum of the diagonal entries of the matrix of T with respect to an arbitrary basis. Remarkably, this suspicion turns out to be true.

10.13 Definition ***trace of a matrix***

The ***trace*** of a square matrix A, denoted trace A, is defined to be the sum of the diagonal entries of A.

Trace of AB equals trace of BA:

10.14 Trace of AB equals trace of BA

If A and B are square matrices of the same size, then

$$\operatorname{trace}(AB) = \operatorname{trace}(BA).$$

Properties of Trace

Now we can prove that the sum of the diagonal entries of the matrix of an operator is independent of the basis with respect to which the matrix is computed.

10.15 Trace of matrix of operator does not depend on basis

Let $T \in \mathcal{L}(V)$. Suppose $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are bases of V. Then

$$\operatorname{trace} \mathcal{M}\big(T, (u_1, \dots, u_n)\big) = \operatorname{trace} \mathcal{M}\big(T, (v_1, \dots, v_n)\big).$$

The next result below, which is the most important result in this section, states that the trace of an operator equals the sum of the diagonal entries of the matrix of the operator.

10.16 Trace of an operator equals trace of its matrix

Suppose $T \in \mathcal{L}(V)$. Then $\operatorname{trace} T = \operatorname{trace} \mathcal{M}(T)$.

Properties of Trace

The next result would be difficult to prove without using 10.16.

10.18 Trace is additive

Suppose $S, T \in \mathcal{L}(V)$. Then $\operatorname{trace}(S+T) = \operatorname{trace} S + \operatorname{trace} T$.

The statement of the next result does not involve traces, although the short proof uses traces.

10.19 The identity is not the difference of ST and TS

There do not exist operators $S, T \in \mathcal{L}(V)$ such that $ST - TS = I$.

$\operatorname{trace}(ST-TS) = 0 \neq \operatorname{trace}(I)$

Homework Assignment 29

10.A: 1, 8, 11, 19, 20.