南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线 性 代 数 A　　　　开课单位: 数 学 系
考试时长: 120 分钟　　　　命题教师: 线性代数教学团队

| 题 号 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 分 值 | 15 分 | 25 分 | 10 分 | 10 分 | 10 分 | 10 分 | 10 分 | 10 分 |

本试卷共 ( 8 ) 大题, 满分 ( 100 ) 分. **请将所有答案写在答题本上.**

This exam includes **8** questions and the score is 100 in total. **Write all your answers on the examination book.**

本套试卷为 A 卷 Version A

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

   (共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

   (1) Which one of the following statements must be true? ( ) D

   (A) If $A$ and $B$ are $m \times n$ matrices and $Ax = 0$ has the same solution set as $Bx = 0$, then $A$ and $B$ have the same column space. nullspace

   (B) If $A$ is an $n \times n$ real symmetric positive definite matrix, then all the square submatrices of $A$ have positive determinants.

   (C) If real symmetric matrices $A$ and $B$ are congruent, then they are similar.

   (D) If $AB = 0$ and $A$ and $B$ are not zero matrices, then $A$ has linearly dependent columns and $B$ has linearly dependent rows.

   下列陈述一定正确的是? ( )

   (A) 若 $A$ 和 $B$ 为 $m \times n$ 矩阵, 且 $Ax = 0$ 与 $Bx = 0$ 同解，则 $A$ 和 $B$ 具有相同的列空间.

   (B) 若实对称矩阵 $A$ 正定, 则它的所有正方形子矩阵的行列式都为正.

   (C) 若实对称矩阵 $A$ 和 $B$ 相合 (也称合同), 则它们相似.

   (D) 设 $A, B$ 为满足 $AB = 0$ 的两个非零矩阵, 则 $A$ 的列向量组线性相关, $B$ 的行向量组线性相关.

   (2) The plane curve defined by the equation $2x^2 - 8xy + 2y^2 = 1$ is ( ) B

   (A) an ellipse.

   (B) a hyperbola.

   (C) a parabola.

   (D) a union of two intersecting lines.

$$\begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 0 & -6 \end{bmatrix}$$

$$0 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$2(x-2y)^2 - 6y^2 = 1$$

由方程 $2x^2 - 8xy + 2y^2 = 1$ 定义的平面曲线是 ( )

(A) 椭圆.

(B) 双曲线.

(C) 抛物线.

(D) 一对相交直线.

(3) Let $A$ be an $n \times n$ real matrix. Suppose that for all column vectors $x \in \mathbb{R}^n$ we have $x^T Ax = 0$. Then ( ~~B~~ C )

A is skew-symmetric

choose $x = e_i$

(A) The determinant $|A|$ of $A$ is 0.

(B) $A = 0$.

(C) The trace, trace($A$), of $A$ is 0.

(D) The only eigenvalue of $A$ in $\mathbb{C}$ is 0.

设 $A$ 为 $n \times n$ 实矩阵. 假设对任意列向量 $x \in \mathbb{R}^n$ 都有 $x^T Ax = 0$. 则 ( )

(A) $A$ 的行列式 $|A|$ 为 0.

(B) $A = 0$.

(C) $A$ 的迹, trace($A$), 为 0.

(D) $A$ 在 $\mathbb{C}$ 中唯一的特征值是 0.

$A = uv^T$

(4) Let $n \geq 2$. Let $A$ be an $n \times n$ real matrix of rank 1. Then ( D )

(A) ~~$A$ is necessarily diagonalizable.~~

(B) ~~$A$ has only one nonzero column.~~

(C) ~~The trace, trace($A$), of $A$ is nonzero.~~ $u^T v = 0$ $tr(A) = 0$

(D) $A$ has at least $n-1$ linearly independent eigenvectors.

设 $n \geq 2$, $A$ 是秩为 1 的 $n \times n$ 实矩阵. 则 $v^T u, \underbrace{0, \cdots, 0}_{n-1}$ ( )

(A) $A$ 一定可以对角化.

(B) $A$ 只有一列是非零列.

(C) $A$ 的迹, trace($A$), 不为零.

(D) $A$ 有至少 $n-1$ 个线性无关的特征向量.

(5) Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The matrices $A$ and $B$ are ( D )

A eigenvalue $-1, -1, 5$

(A) congruent and similar.

(B) congruent but not similar.

(C) similar but not congruent.

(D) neither similar nor congruent.

设 $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. 则 $A$ 与 $B$ ( )

(A) 合同且相似. (这里的“合同”也称“相合”.)

(B) 合同但不相似.

(C) 相似但不合同.

(D) 既不相似也不合同.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Evaluate the determinant: $\begin{vmatrix} 0 & 0 & \cdots & 0 & n \\ 0 & 0 & \cdots & n-1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 2 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} =$ ___n!___.

计算行列式: $\begin{vmatrix} 0 & 0 & \cdots & 0 & n \\ 0 & 0 & \cdots & n-1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 2 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} =$ __________.

(2) Let $\mathbb{R}^{2\times 2}$ be the real vector space of $2\times 2$ real matrices. Let $V$ be the subspace of $\mathbb{R}^{2\times 2}$ spanned by the 4 matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix},\ A_3 = \begin{bmatrix} 5 & 1 \\ 1 & -5 \end{bmatrix},\ A_4 = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}.$$

Then $\dim V =$ ___$\begin{bmatrix} b & a \\ a & -b \end{bmatrix}$___. $a, b \in \mathbb{R}$

设 $\mathbb{R}^{2\times 2}$ 为所有 $2\times 2$ 实矩阵构成的实向量空间. 令 $V$ 为以下 4 个矩阵在 $\mathbb{R}^{2\times 2}$ 中张成 (也称"生成") 的子空间:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix},\ A_3 = \begin{bmatrix} 5 & 1 \\ 1 & -5 \end{bmatrix},\ A_4 = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}.$$

则 $\dim V =$ __________.

(3) Let $A$ be a $2\times 2$ matrix and suppose $\alpha_1, \alpha_2$ are 2-dimensional linearly independent column vectors such that $A\alpha_1 = 0$, $A\alpha_2 = 4\alpha_1 + 2\alpha_2$. Then the eigenvalues of $A$ are ___0, 2___.
设 $A$ 是 2 阶方阵, $\alpha_1, \alpha_2$ 是线性无关的二维列向量, 满足 $A\alpha_1 = 0$, $A\alpha_2 = 4\alpha_1 + 2\alpha_2$. 则 $A$ 的所有特征值为 __________.

$A(A\alpha_2) = 2(A\alpha_2)$

(4) Suppose that the quadratic form $f(x_1, x_2, x_3) = x_1^2 + ax_2^2 + x_3^2 + 2bx_1x_2 + 2x_1x_3 + 2x_2x_3$ can be transformed by an orthogonal transformation $(x_1, x_2, x_3)^T = Q(y_1, y_2, y_3)^T$ to $y_2^2 + 4y_3^2$. Then $a =$ ___3___, $b =$ ___1___.
假定二次型 $f(x_1, x_2, x_3) = x_1^2 + ax_2^2 + x_3^2 + 2bx_1x_2 + 2x_1x_3 + 2x_2x_3$ 可由正交变换 $(x_1, x_2, x_3)^T = Q(y_1, y_2, y_3)^T$ 化为 $y_2^2 + 4y_3^2$. 则 $a =$ __________, $b =$ __________.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(5) Let $A$ be a $3\times 3$ matrix with eigenvalues $-1$, $0$, $1$. Suppose that

$$v_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad v_2=\begin{bmatrix}-1\\1\\0\end{bmatrix},\quad v_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

are eigenvectors belonging to the eigenvalues $-1$, $0$, $1$ respectively. Then $A^{2021}=$ $\begin{bmatrix}-1&-1&0\\0&0&0\\0&0&1\end{bmatrix}$.

假设 $A$ 为 $3\times 3$ 矩阵, 其特征值为 $-1$, $0$, $1$. 假设

$$v_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad v_2=\begin{bmatrix}-1\\1\\0\end{bmatrix},\quad v_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

分别为属于特征值 $-1$, $0$, $1$ 的特征向量. 则 $A^{2021}=$__________.

3. (10 points 本题共 10 分) Suppose $A=\begin{bmatrix}a&1&0\\1&a&-1\\0&1&a\end{bmatrix}$ and $A^3=0$.

(a) Find $|A|$. $=0$

(b) Find the value of $a$. $a=0$

$I-A=\begin{bmatrix}1&-1&0\\-1&1&1\\0&-1&1\end{bmatrix}$

(c) Show that $I-A$ is invertible. (Here $I$ denotes the $3\times 3$ identity matrix.)

(d) Find an invertible matrix $X$ of order 3 such that $(I-A)^{-1}X=(X^{-1}-X^{-1}A)^{-1}A^2+I$.

(*Hint:* $X^{-1}-X^{-1}A=X^{-1}(I-A)$.)

$X(I+A)=I$

$X=(I+A)^{-1}=\begin{bmatrix}2&-1&-1\\-1&1&1\\1&-1&0\end{bmatrix}$

设 $A=\begin{bmatrix}a&1&0\\1&a&-1\\0&1&a\end{bmatrix}$, 且 $A^3=0$.

(a) 求 $A$ 的行列式 $|A|$.

(b) 求 $a$ 的值.

(c) 证明 $I-A$ 可逆. (这里 $I$ 表示 3 阶单位矩阵.)

(d) 求一个 3 阶可逆矩阵 $X$ 使得 $(I-A)^{-1}X=(X^{-1}-X^{-1}A)^{-1}A^2+I$.

(提示: $X^{-1}-X^{-1}A=X^{-1}(I-A)$.)

4. (10 points 本题共 10 分) Let

$$H=\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}.$$

$\det(H)=16$　　$\mathrm{trace}(H)=0$

(a) Find the determinant and the trace of $H$.

(b) Find all the singular values of $H$. $2, 2, 2, 2$

(c) Find a real number $\alpha$ such that $\mathrm{rank}(\alpha I_4 - H)$ is the smallest possible. $\alpha=2$ or $-2$.

$\mathrm{rank}(\alpha I_4 - H) = 2$

令

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(a) 求 $H$ 的行列式和迹.

(b) 求 $H$ 的所有奇异值.

(c) 求一个实数 $\alpha$ 使得 $\mathrm{rank}(\alpha I_4 - H)$ 达到最小可能的值.

5. (10 points 本题共 10 分) Suppose that the complex matrix $A = \begin{bmatrix} 1 & 1+i \\ \alpha & 2 \end{bmatrix}$ is a Hermitian matrix.

$1-i$

$\lambda = 0$ or $3$

$x = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$　$x = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

(a) Find the value of $\alpha$. $\alpha = 1-i$

(b) Find all the complex eigenvalues of $A$.

(c) Find a unitary matrix $U$ such that $U^{-1}AU$ is a diagonal matrix. $U = \begin{bmatrix} (1+i)/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{3} & (1-i)/\sqrt{3} \end{bmatrix}$

(d) Let $B = A + A^T$, where $A^T$ denotes the transpose of $A$. Show that $B$ is a real symmetric matrix, and decide whether $B$ is positive definite.

$\times$　　$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

假设复矩阵 $A = \begin{bmatrix} 1 & 1+i \\ \alpha & 2 \end{bmatrix}$ 是个埃尔米特矩阵. semidefinite

(a) 求 $\alpha$ 的值.

(b) 求 $A$ 的所有复特征值.

(c) 求一个酉矩阵 $U$ 使得 $U^{-1}AU$ 为对角阵.

(d) 令 $B = A + A^T$, 其中 $A^T$ 表示 $A$ 的转置. 证明 $B$ 是实对称阵, 并确定 $B$ 是否正定.

$A = [\,p\ q\ r\,]$　　$1/\sqrt{2}$

6. (10 points 本题共 10 分) Let $A$ be a $3 \times 3$ real matrix whose second and third columns are $(1, 0, 0)^T$ and $(2, 1, 0)^T$ respectively. Suppose that the QR factorization of $A$ takes the form $A = QR$ with

$q_1^T\alpha_1$　$q_1^T\alpha_2$　$q_1^T\alpha_3 = \sqrt{2}$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & x \\ 0 & 0 & y \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & z \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{2} & a & b \\ 0 & \frac{1}{\sqrt{2}} & c \\ 0 & 0 & 1 \end{bmatrix}$$

$q_2^T\alpha_2$　$q_2^T\alpha_3 = \sqrt{2}$　$q_3^T\alpha_3$

$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$

(a) Find the values of $x$, $y$, $z$ and $a$, $b$, $c$.

(b) Find the determinant $|A|$. $|A| = 1$

$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \sqrt{2} \\ 0 & 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$

设 $A$ 为 3 阶实方阵, 其第二列和第三列分别为 $(1, 0, 0)^T$ 和 $(2, 1, 0)^T$. 假设 $A$ 的 QR 分解 $A = QR$ 满足

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & x \\ 0 & 0 & y \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & z \end{bmatrix} \quad 及 \quad R = \begin{bmatrix} \sqrt{2} & a & b \\ 0 & \frac{1}{\sqrt{2}} & c \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) 求 $x, y, z$ 和 $a, b, c$ 的值.

(b) 求行列式 $|A|$.

7. (10 points 本题共 10 分) Let $A$ be an $n \times n$ real symmetric positive definite matrix.

(a) Show that there exists an $n \times n$ invertible matrix $R$ such that $A = R^T R$.

(b) Show that for all column vectors $x, y \in \mathbb{R}^n$,

$$(x^T Ay)^2 \le (x^T Ax)(y^T Ay).$$

设 $A$ 为 $n$ 阶正定实对称矩阵.　$x^TAx>0$　$A=Q\Lambda Q^T=(Q\sqrt{\Lambda}Q^T)(Q\sqrt{\Lambda}Q^T)$

(a) 证明: 存在 $n$ 阶可逆阵 $R$ 使得 $A = R^T R$.　choose $R=Q\sqrt{\Lambda}Q^T$

(b) 证明: 对任意列向量 $x, y \in \mathbb{R}^n$ 都有

$$(x^T Ay)^2 \le (x^T Ax)(y^T Ay).$$

$$=\left((Rx)^T(Ry)\right)^2 \le \underbrace{\|Rx\|^2\|Ry\|^2}_{\text{Cauchy's inequality}} = RHS$$

8. (10 points 本题共 10 分) Let $A$, $B$ be two $n \times n$ real symmetric matrices.

(a) Suppose $A$ is positive definite. Show that there exists an invertible $n \times n$ matrix $C$ such that $C^T AC = I_n$ and $C^T BC$ is diagonal. (Here $I_n$ denotes the $n \times n$ identity matrix).

(b) Suppose $B - A$ and $A$ are positive semidefinite matrices. Show that: $\det B \ge \det A$.

设 $A$, $B$ 都为 $n$ 阶实对称矩阵.

(a) 设 $A$ 为正定实对称阵. 证明: 存在 $n$ 阶可逆实矩阵 $C$ 使得 $C^T AC = I_n$ 且 $C^T BC$ 是对角阵. (这里 $I_n$ 为 $n$ 阶单位阵).

(b) 设 $B - A$ 和 $A$ 都是半正定矩阵. 证明: $\det B \ge \det A$.

(a) $A=R^TR$, $R$ is invertible

$C_1^TR^TRC_1=I_n$

Suppose $RC_1=Q \Leftrightarrow Q^TQ=I_n$ ✓

Spectrum Theorem $Q^TBQ=\Lambda$

choose $C=RC_1$ ✓

(b) $B-A=R_1^TR_1$

$A=R_2^TR_2$



$B=R_1^TR_1+R_2^TR_2$　$x^TBx=\|R_1x\|^2+\|R_2x\|^2$

discuss $\underset{=}{\text{semidefiniteness}}$ or $\underset{>}{\text{definiteness}}$ ✓