

Step-1

Given that the only solution to $Ax = 0$ (m equations in n unknowns) is $x = 0$. We have to find the rank of A and we have to verify that whether the columns of A are linearly independent or not.

Step-2

Given that the only solution of $Ax = 0$ is $x = 0$

So the null space of $A = \{0\}$

Therefore $\dim \mathbf{N}(A) = 0$

From a known theorem,

$$\dim(\mathbf{N}(A)) + \dim(\mathbf{C}(A)) = \text{Number of columns of } A$$

Step-3

$$\Rightarrow \dim(\mathbf{C}(A)) = \text{Number of columns of } A \quad (\text{since } \dim \mathbf{N}(A) = 0)$$

Since A is m by n matrix, numbers of columns of A is n

$$\text{Therefore } \dim(\mathbf{C}(A)) = n$$

Hence the rank of $A = n$, and hence the columns of A are linearly independent.