Step-1

(a)

The square matrix A has independent columns.

The pseudo-inverse of A with independent columns is $A^{+}=\left(A^{T}A\right)^{-1}A^{T}$

It is a left inverse of A as follows:

$$\begin{aligned}A^{+}A&=\left(A^{T}A\right)^{-1}A^{T}A\\&=\left(A^{T}A\right)^{-1}\left(A^{T}A\right)\\&=I\end{aligned}$$

Further,

$$\begin{aligned}A^{+}&=\left(A^{T}A\right)^{-1}A^{T}\\&=A^{-1}A^{-T}A^{T}\\&=A^{-1}I\\&=A^{-1}\end{aligned}$$

Thus, the left inverse $\left(A^{T}A\right)^{-1}A^{T}$ is A^{+}.

Step-2

As the square matrix A has independent columns, the columns of A span the entire vector space $\mathbf{R}^{n}$.

But for a matrix, its row rank is equal to its column rank.

Therefore, the rows of A too span the entire vector space $\mathbf{R}^{n}$.

This indicates that $x^{+}=A^{+}b$ is in the row space.

Now verify the following:

$$\begin{aligned}\left(A^{T}A\right)x^{+}&=\left(A^{T}A\right)\left(A^{+}b\right)\\&=\left(A^{T}A\right)\left[\left(A^{T}A\right)^{-1}A^{T}\right]b\\&=\left(A^{T}A\right)\left[A^{-1}\left(A^{T}\right)^{-1}A^{T}\right]b\end{aligned}$$

$= A^{T}\left(AA^{-1}\right)\left[\left(A^{T}\right)^{-1}A^{T}\right]b$
$= A^{T}\cdot I\cdot I\cdot b$
$= A^{T}b$

Thus, $A^{T}Ax^{+} = A^{T}b$.

Step-3

(b)

The square matrix *A* has independent rows.

Therefore, the rows of *A* span the entire vector space $\mathbf{R}^{n}$.

In this case, the right inverse of *A* is given by,

$A^{+} = A^{T}\left(AA^{T}\right)^{-1}$ This can be verified as follows:

$$
\begin{aligned}
AA^{+} &= AA^{T}\left(AA^{T}\right)^{-1} \\
&= \left(AA^{T}\right)\left(AA^{T}\right)^{-1} \\
&= I
\end{aligned}
$$

Further,

$$
\begin{aligned}
A^{+} &= A^{T}\left(AA^{T}\right)^{-1} \\
&= A^{T}\left(A^{-T}A^{-1}\right) \\
&= \left(A^{T}A^{-T}\right)A^{-1} \\
&= IA^{-1} \\
&= A^{-1}
\end{aligned}
$$

Step-4

Let $x^{+} = A^{+}b$, then $x^{+} = A^{T}\left(AA^{T}\right)^{-1}b$.

Obviously, x^{+} is in the row space, because A^{T} multiplied by any vector has to be in the row space.

Now verify the following:

$$
\begin{aligned}
A^T A x^+ &= \left(A^T A\right)\left[A^T\left(AA^T\right)^{-1} b\right] \\
&= \left(A^T A\right)\left[A^T \cdot \left(A^T\right)^{-1} A^{-1}\right] b \\
&= \left(A^T A\right)\left[A^T \cdot \left(A^T\right)^{-1}\right] A^{-1} b
\end{aligned}
$$

$$
\begin{aligned}
&= \left(A^T A\right) \cdot I \cdot A^{-1} b \\
&= A^T\left(AA^{-1}\right) b \\
&= A^T \cdot I \cdot b \\
&= A^T b
\end{aligned}
$$

Thus, $A^{\mathrm{T}} A x^+ = A^{\mathrm{T}} b$