Linear Algebra-A

Assignments - Week 13

Supplementary Problem Set

1. Suppose there exist a 3×3 matrix $\boldsymbol{A}$ and a 3-dimensional column vector $\boldsymbol{x}$ such that the set of vectors $\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x}, \boldsymbol{A}^2\boldsymbol{x}$ are linearly independent, and

$$\boldsymbol{A}^3\boldsymbol{x} = 3\boldsymbol{A}\boldsymbol{x} - 2\boldsymbol{A}^2\boldsymbol{x}$$

   (1) Let $\boldsymbol{P} = [\boldsymbol{x},\ \boldsymbol{A}\boldsymbol{x},\ \boldsymbol{A}^2\boldsymbol{x}]$. Find a matrix $\boldsymbol{B}$, such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}$.

   (2) Compute the determinant $|\boldsymbol{A}^2 + \boldsymbol{A} + \boldsymbol{I}|$.

   【**Hint**: You may use the following fact:

   Please show that if $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{B}$, then $\boldsymbol{P}^{-1}f(\boldsymbol{A})\boldsymbol{P} = f(\boldsymbol{B})$, i.e., if $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then $f(\boldsymbol{A})$ is similar to $f(\boldsymbol{B})$, where $f(x)$ is a polynomial of degree n: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, and $a_n, a_{n-1}, \cdots a_1, a_0$ are constants.

   Please prove it before applying it.】

2. Suppose $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$, and $\boldsymbol{A}$ is similar to $\boldsymbol{B}$. Find a, b and an invertible matrix $\boldsymbol{S}$, such that $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{B}$.

3. Suppose $A = \begin{bmatrix} 1 & -2 & -1 \\ -a & a & -a \\ -1 & 2 & 1 \end{bmatrix}$ cannot be diagonalized (不能相似对角化), please find the value of a.

4. If $\boldsymbol{A} = \begin{bmatrix} a & -1 & c \\ 5 & b & 3 \\ 1-c & 0 & -a \end{bmatrix}$, and the determinant of $|\boldsymbol{A}| = -1$. The matrix $\boldsymbol{A}^*$ ($\boldsymbol{A}$的伴随矩阵) has an eigenvector $\boldsymbol{x} = (-1, -1, 1)^{\mathrm{T}}$ corresponding to it eigenvalue λ_0. Find a, b, c and λ_0.

5. (1) Find an orthogonal matrix $\boldsymbol{Q}$ (and a unitary matrix $\boldsymbol{U}$) to diagonalize the following matrix $\boldsymbol{A}$ (and $\boldsymbol{B}$):

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix}.$$

(2) Find all the eigenvalues of the matrix $\boldsymbol{C} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{bmatrix}$, and a unitary matrix to diagonalize $\boldsymbol{C}$.

【**Hint**: You may use the following fact:

For a block matrix

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_{m1} & \boldsymbol{C}_{m2} & \cdots & \boldsymbol{C}_{mm} \end{bmatrix}, \quad \text{(the block } \boldsymbol{C}_{ii} \text{ is a matrix of order } r_i\text{)}$$

the eigenvalues of $\boldsymbol{C}$ come from the union set of the eigenvalues of $\boldsymbol{C}_{ii}$ $(i = 1,2,\cdots,m)$.

Please prove it before applying it.】

$$|\lambda I - C| = \begin{vmatrix} \lambda I - A & 0 \\ 0 & \lambda I - B \end{vmatrix} = |\lambda I - A|\,|\lambda I - B|$$
$$= (\lambda-1)^3(\lambda+3)(\lambda+1)(\lambda-2)\lambda$$

$\sigma(C) = \{0, 1, -1, 2, -3\}$.