Step-1

Given that;

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$$
$$\det A = 2(4-4) - 1(8-8) + 2(4-4)$$
$$= 0$$

Therefore A is singular matrix.

And else all 2×2 sub matrices of A having determinant is 0.

Therefore the rank of A = 1.

Step-2

For finding the Eigen values λ, consider $|A - \lambda I| = 0$

This implies;

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 4 & 2-\lambda & 4 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

This implies;

$$(2-\lambda)\left\{(2-\lambda)^2 - 4\right\} - 1\left\{4(2-\lambda) - 8\right\} + 2\left\{(4 - 2(2-\lambda))\right\} = 0$$
$$(2-\lambda)\left\{4 + \lambda^2 - 4\lambda - 4\right\} - \left\{8 - 4\lambda - 8\right\} + 2\left\{4 - 4 + 2\lambda\right\} = 0$$
$$8 + 2\lambda^2 - 8\lambda - 8 - 4\lambda - \lambda^3 + 4\lambda^2 + 4\lambda + 4\lambda + 4\lambda = 0$$
$$-\lambda^3 + 6\lambda^2 = 0$$

This implies;

$$\lambda^3 - 6\lambda^2 = 0$$
$$\lambda^2(\lambda - 6) = 0$$

Thus, $\lambda = 0, 0, 6$ are the Eigen values of the given matrix A.

Step-3

To find Eigen vector for $\lambda = 0$, consider $(A - \lambda I)x = 0$

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply the row operations on the coefficient matrix,

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$, and get $\begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

This is the reduced matrix and

So, rewrite the homogeneous equations from this.

$$2x_1 + x_2 + 2x_3 = 0$$

Allowing x_1, x_3 being free variables and put;

$$\begin{aligned} x_1 &= k_1, \\ x_3 &= k_2 \\ 2k_1 + x_2 + 2k_2 &= 0 \\ x_2 &= -2k_1 - 2k_2 \end{aligned}$$

So, the solution set is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

Therefore, the Eigen vectors corresponding to $\lambda = 0$ are $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

Step-4

Similarly, to get the Eigen vector corresponding to $\lambda = 6$, we solve $(A - \lambda I)x = 0$

That is;

$$\begin{bmatrix} 2-6 & 1 & 2 \\ 4 & 2-6 & 4 \\ 2 & 1 & 2-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 1 & 2 \\ 4 & -4 & 4 \\ 2 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-5

Apply the row operations on the coefficient matrix, $R_2 \to R_2 + R_1, R_3 \to 2R_3 + R_1$, and get;

$$\begin{bmatrix} -4 & 1 & 2 \\ 0 & -3 & 6 \\ 0 & 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2, \frac{R_2}{-3}\begin{bmatrix} -4 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is the reduced matrix.

So, rewrite the homogeneous equations from this.

Step-6

$$\begin{aligned} x_2 - 2x_3 &= 0 \\ -4x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

Using $x_2 = 2x_3$ in the 2nd equation, and get; $x_1 = x_3$

So, put $x_3 = 1$, the solution set is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 6$