Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #10-11

2023/05/07

Name: ____________________

Student Number: ____________________

1. True or false (and give a proof of your answer): there exists $T \in \mathcal{L}(\mathbf{R}^3)$ such that T is not self-adjoint (with respect to the usual inner product) and such that there is a basis of $\mathbf{R}^3$ consisting of eigenvectors of T.

   判断正误（并证明你的结论）：存在 $T \in \mathcal{L}(\mathbf{R}^3)$ 使得 T（关于通常的内积）不是自伴的并且 $\mathbf{R}^3$ 有一个由 T 的本征向量构成的基.

   Proof. Let e_1, e_2, e_3 be standard basis of $\mathbf{R}^3$, we define $T \in \mathcal{L}(\mathbf{R}^3)$, $Te_1 = e_1,\ Te_2 = e_1 + 2e_2, Te_3 = 3e_3,$ then

$$\mathcal{M}(T; e_1, e_2, e_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

□

   T has three linearly independent eigenvectors which form a basis of $\mathbf{R}^3$, but T is not self-adjoint.

2. Find the singular values of the differentiation operator $D \in \mathcal{P}(\mathbf{R}^2)$ defined by $Dp = p'$, where the inner product on $\mathcal{P}(\mathbf{R}^2)$ is $\langle p(x), q(x)\rangle = \int_{-1}^{1} p(x)q(x)dx$, $\forall p(x), q(x) \in \mathcal{P}(\mathbf{R}^2)$, and an orthonormal basis w.r.t this inner product of $\mathcal{P}(\mathbf{R}^2)$ is $\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$.

   求由 $Dp = p'$ 定义的微分算子 $D \in \mathcal{P}(\mathbf{R}^2)$ 的奇异值,这里 $\mathcal{P}(\mathbf{R}^2)$ 上的内积定义为 $\langle p(x), q(x)\rangle = \int_{-1}^{1} p(x)q(x)dx$, $\forall p(x), q(x) \in \mathcal{P}(\mathbf{R}^2)$, 在此内积下的 $\mathcal{P}(\mathbf{R}^2)$ 的一组规范正交基是 $\mathcal{P}(\mathbf{R}^2)$ is $\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$.

   Proof. Let $e_1 = \sqrt{\frac{1}{2}}$, $e_2 = \sqrt{\frac{3}{2}}x$, $e_3 = \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$, then

$$De_1 = 0,\ De_2 = \sqrt{\frac{3}{2}} = \sqrt{3}e_1,\ De_3 = \sqrt{\frac{45}{8}} \cdot 2x = \sqrt{\frac{45}{2}}x = \sqrt{15}e_2.$$

$$M = \mathcal{M}(D; e_1, e_2, e_3) = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow M^* = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{pmatrix}$$

   so $M^*M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$, the eigenvalues of T^*T are $0, 3, 15$, then the singular values of T are $0, \sqrt{3}, \sqrt{15}$. □

3. Give an example of two self-adjoint operators $T_1, T_2 \in \mathcal{L}(\mathbf{F}^4)$ such that the eigenvalues of both operators are $2, 5, 7$ but there does not exist an isometry $S \in \mathcal{L}(\mathbf{F}^4)$ such that $T_1 = S^*T_2S$. Be sure to explain why there is no isometry with the required property.

找出两个自伴算子 $T_1, T_2 \in \mathcal{L}(\mathbf{F}^4)$ 使得它们的本征值为 $2, 5, 7$, 但不存在等距同构 $S \in \mathcal{L}(\mathbf{F}^4)$ 使得 $T_1 = S^*T_2S$. 一定要解释为什么不存在满足条件的等距同构.

Proof. Let e_1, e_2, e_3, e_4 be an orthonormal basis of $\mathbf{F}^4$,

$$\mathcal{M}(T_1; e_1, e_2, e_3, e_4) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}, \quad \mathcal{M}(T_2; e_1, e_2, e_3, e_4) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

we have $T_1e_1 = 2e_1, T_1e_2 = 2e_2$, $T_2e_1 = 2e_1$.

Suppose there exists an isometry $S \in \mathcal{L}(\mathbf{F}^4)$, s.t. $T_1 = S^*T_2S \Rightarrow ST_1 = T_2S$, then

$$ST_1e_1 = T_2Se_1 \Rightarrow T_2Se_1 = 2Se_1, \quad ST_1e_2 = T_2Se_2 \Rightarrow T_2Se_2 = 2Se_2$$

so Se_1, Se_2 are eigenvectors of T_2 corresponding to the eigenvalue 2. And S is an isometry, so Se_1, Se_2 are linearly independent, which contradicts T_2 has only one linearly independent eigenvector of 2.

Thus there does not exist an isometry $S \in \mathcal{L}(\mathbf{F}^4)$, s.t. $T_1 = S^*T_2S$.

根据 eigenvalue 找 contradiction

$\square$

4. Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\text{null } T^k = \text{null } T \quad \text{and} \quad \text{range } T^k = \text{range } T$$

for every positive integer k.

设 $T \in \mathcal{L}(V)$ 是正规的. 证明：对每个正整数 k 均有

$$\text{null } T^k = \text{null } T \quad 且 \quad \text{range } T^k = \text{range } T.$$

Proof. Firstly, WTS: null $T^2 =$ null T, since T is normal, then $V = E(0,T) \oplus E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_l, T)$ (If T has no zero eigenvalue, then denote $E(0,T)$ as $\{0\}$). Let $\lambda_0 = 0$, notice that $E(\lambda_i, T) \subseteq E(\lambda_i, T^2)$, and $V = E(0, T^2) \oplus E(\lambda_1^2, T^2) \oplus \cdots \oplus E(\lambda_l^2, T^2)$, so $E(0,T) = E(0,T^2)$, i.e. null $T^2 =$ null T.

If $K \geqslant 3$, since T is normal, $V = E(0,T) \oplus E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_k, T)$ (If T has no zero eigenvalue, then denote $E(0,T)$ as $\{0\}$). And T^k is also normal, then $V = E(0,T^k) \oplus E(\lambda_1^k, T^k) \oplus \cdots \oplus E(\lambda_l^k, T^k)$, and we notice that $E(\lambda_i, T) \subseteq E(\lambda_i^k, T^k), i = 0, 1, \cdots, l$, so $E(0,T) = E(0,T^k)$, i.e. null $T^k =$ null T.

Next, WTS: range $T^k =$ range T. We have range $T^k \subseteq$ range T, dim $V =$ dim null $T +$ dim range $T =$ dim null $T^k +$ dim range T^k, null $T^k =$ null T, so dim range $T =$ dim range $T^k \Rightarrow$ range $T =$ range T^k.

□