Step-1

Kernel of a linear transformation $T:\mathbf{R}^2 \to \mathbf{R}^2$ is $\ker T = \{(v_1, v_2) \in \mathbf{R}^2 / T(v_1, v_2) = 0\}$

And the **range of** T is $\text{Range } T = \{T(v_1, v_2) / (v_1, v_2) \in \mathbf{R}^2\}$

Step-2

(a)

Given that $T:\mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(v_1, v_2) = (v_2, v_1)$.

Now the kernel is;

$$\ker T = \{(v_1, v_2) \in \mathbf{R}^2 / T(v_1, v_2) = 0\}$$

This implies;

$$\begin{aligned}
&= \{(v_1, v_2) \in \mathbf{R}^2 / (v_2, v_1) = 0\} \\
&= \left\{ \begin{matrix} (v_1, v_2) \in \mathbf{R}^2 / v_2 = 0 \\ v_1 = 0 \end{matrix} \right\} \\
&= \{(v_1, v_2) \in \mathbf{R}^2 / (v_1, v_2) = (0,0)\} \\
&= \{(0,0)\}
\end{aligned}$$

Step-3

And the range of T is

$$\begin{aligned}
\text{Range } T &= \{T(v_1, v_2) / (v_1, v_2) \in \mathbf{R}^2\} \\
&= \{(v_2, v_1) / v_1, v_2 \in \mathbf{R}\} \\
&= \mathbf{R}^2
\end{aligned}$$

Hence the kernel of the given transformation is $\boxed{\ker T = \{(0,0)\}}$ and the range is $\boxed{\text{Range} T = \mathbf{R}^2}$.

Step-4

(b)

Given that $T:\mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(v_1, v_2, v_3) = (v_1, v_2)$.

Now the kernel is;

$$\ker T = \{(v_1, v_2, v_3) \in \mathbf{R}^3 / T(v_1, v_2, v_3) = 0\}$$

This implies;

$$\begin{aligned}
&= \{(v_1, v_2, v_3) \in \mathbf{R}^3 / (v_1, v_2) = 0\} \\
&= \{(v_1, v_2, v_3) \in \mathbf{R}^3 / v_1 = 0, v_2 = 0\} \\
&= \{(v_1, v_2, v_3) \in \mathbf{R}^3 / (v_1, v_2, v_3) = (0, 0, v_3)\} \\
&= \{(0, 0, v_3) / v_3 \in \mathbf{R}\}
\end{aligned}$$

And the range of T is;

$$\begin{aligned}
\text{Range } T &= \{T(v_1, v_2, v_3) / (v_1, v_2, v_3) \in \mathbf{R}^3\} \\
&= \{(v_1, v_2) / v_1, v_2 \in \mathbf{R}\} \\
&= \mathbf{R}^2
\end{aligned}$$

Hence the kernel of the given transformation is $\boxed{\ker T = \{(0, 0, v_3) / v_3 \in \mathbf{R}\}}$ and the range is $\boxed{\text{Range} T = \mathbf{R}^2}$.

Step-5

(c)

Given that $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(v_1, v_2) = (0, 0)$.

Now the kernel is;

$$\begin{aligned}
\ker T &= \{(v_1, v_2) \in \mathbf{R}^2 / T(v_1, v_2) = 0\} \\
&= \{(v_1, v_2) \in \mathbf{R}^2 / (0, 0) = 0\} \\
&= \{(v_1, v_2) \in \mathbf{R}^2 / v_1, v_2 \in \mathbf{R}\} \\
&= \mathbf{R}^2
\end{aligned}$$

Step-6

And the range of T is;

$$\begin{aligned}
\text{Range } T &= \{T(v_1, v_2) / (v_1, v_2) \in \mathbf{R}^3\} \\
&= \{(0, 0) / v_1, v_2 \in \mathbf{R}\} \\
&= \{(0, 0)\}
\end{aligned}$$

Hence the kernel of the given transformation is $\boxed{\ker T = \mathbf{R}^2}$ and the range is $\boxed{\text{Range}\,T = \{(0,0)\}}$.

Step-7

(d)

Given that $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(v_1, v_2) = (v_1, v_1)$.

Now the kernel is;

$$\begin{aligned}
\ker T &= \{(v_1, v_2) \in \mathbf{R}^2 / T(v_1, v_2) = 0\} \\
&= \{(v_1, v_2) \in \mathbf{R}^2 / (v_1, v_1) = 0\} \\
&= \{(v_1, v_2) \in \mathbf{R}^2 / v_1 = 0\} \\
&= \{(0, v_2) / v_2 \in \mathbf{R}\}
\end{aligned}$$

Step-8

And the range of T is;

$$\begin{aligned}
\text{Range } T &= \{T(v_1, v_2) / (v_1, v_2) \in \mathbf{R}^3\} \\
&= \{(v_1, v_1) / v_1, v_2 \in \mathbf{R}\} \\
&= \{(v_1, v_2) / v_1 = v_2 \text{ and } v_1, v_2 \in \mathbf{R}\}
\end{aligned}$$

Hence the kernel of the given transformation is $\boxed{\ker T = \{(0, v_2) / v_2 \in \mathbf{R}\}}$ and the range is $\boxed{\text{Range}\,T = \{(v_1, v_2) / v_1 = v_2 \text{ and } v_1, v_2 \in \mathbf{R}\}}$.