Step-1

Consider the space of 2 by 2 matrices.

Suppose that the transformation T on this space is defined as $T(A) = A^{\mathrm{T}}$.

That is, when the linear transformation T is applied to any matrix, the resultant matrix is the transpose of the original matrix.

Thus, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $T(A) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

The objective is to find the eigenvalues and eigenmatrices for $A^{\mathrm{T}} = \lambda A.$

Step-2

Let us suppose $A^{\mathrm{T}} = \lambda A$.

Then the equation becomes $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

From this, the obtained equation are as follows:

$$\begin{aligned} a &= \lambda a && \ldots\ldots(1) \\ c &= \lambda b && \ldots\ldots(2) \\ b &= \lambda c && \ldots\ldots(3) \\ d &= \lambda d && \ldots\ldots(4) \end{aligned}$$

The equation (1) can be solved as follows:

$$\begin{aligned} a &= \lambda a \\ a - \lambda a &= 0 \\ a(1-\lambda) &= 0 \\ a &= 0 \ \text{ or } 1-\lambda = 0 \\ a &= 0 \ \text{ or } \lambda = 1 \end{aligned}$$

Step-3

Substitute $c = \lambda b$ in $b = \lambda c.$

$$\begin{aligned} b &= \lambda c \\ b &= \lambda(\lambda b) \\ b &= \lambda^2 b \end{aligned}$$

Solve the equation $b = \lambda^2 b$.

$$\begin{aligned} b &= \lambda^2 b \\ b - \lambda^2 b &= 0 \\ b\left(1-\lambda^2\right) &= 0 \\ b &= 0 \text{ or } 1-\lambda^2 = 0 \\ b &= 0 \text{ or } \lambda^2 = 1 \\ b &= 0 \text{ or } \lambda = \pm 1 \end{aligned}$$

Step-4

The equation (4) can be solved as follows:

$$\begin{aligned} d &= \lambda d \\ d - \lambda d &= 0 \\ d\left(1-\lambda\right) &= 0 \\ d &= 0 \text{ or } 1-\lambda = 0 \\ d &= 0 \text{ or } \lambda = 1 \end{aligned}$$

Therefore, the eigenvalues for $A^{\mathrm{T}} = \lambda A$ are 1,1,1, â€“1.

Step-5

When $\lambda = 1$, then from equation (2) obtained that $b = c$.

Then the matrix *A* becomes as follows:

$$\begin{aligned} A &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ b & d \end{bmatrix} \qquad (\text{Use } b = c) \\ &= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

When $\lambda = -1$, then from equation (2) obtained that $b = -c$.

Then the matrix *A* becomes as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ -b & d \end{bmatrix} \qquad (\text{ Use } c = -b)$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, the eigenmatrices of for $A^{\mathrm{T}} = \lambda A$ are $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.