

Step-1

Let T be a linear transformation then $T(0) = 0$.

We have to prove that $T(0) = 0$ from $T(v+w) = T(v) + T(w)$ by choosing $w = 0$.

Step-2

Now choose $w = 0$

Therefore,

$$\begin{aligned}T(v+0) &= T(v) + T(0) \\ \Rightarrow T(v) &= T(v) + T(0) \\ \Rightarrow 0 + T(v) &= T(v) + T(0) \\ \Rightarrow 0 &= T(0)\end{aligned}$$

Hence by choosing $w = 0$, we can get $T(0) = 0$

Step-3

We have to prove that $T(0) = 0$ from $T(cv) = cT(v)$.

Now $T(cv) = cT(v)$

Choose $c = 0$

$$\begin{aligned}T(0v) &= 0T(v) \\ \Rightarrow T(0) &= 0\end{aligned}$$

By choosing $c = 0$, we have $T(0) = 0$ by using the condition $T(cv) = cT(v)$.

Hence $\boxed{T(0) = 0}$, for every linear transformation T