Step-1

We know that Singular Value Decomposition for any m by n matrix A is given by

$$\begin{aligned} A &= U\Sigma V^T \\ &= \begin{pmatrix} \text{orthogonal} \\ U \text{ is } m\times m \end{pmatrix} \begin{pmatrix} m\times n \text{ matrix } \Sigma \\ \sigma_1 \cdots \sigma_r \text{ on diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ V \text{ is } n\times n \end{pmatrix} \end{aligned}$$

Here eigenvectors of AA^T are in U, eigenvectors of $A^T A$ are *in* V and

$$\begin{aligned} \sigma_i &= \sqrt{\lambda_i\left(A^T A\right)} \\ &= \sqrt{\lambda_i\left(AA^T\right)} \end{aligned}$$

We also know that $Av_j = \sigma_j u_j$.

Here σ_j is the length of eigenvector vector Av_j and u_j is unit eigenvector.

Step-2

Since $u = \frac{1}{3}(2,2,1,)$ and $v = \frac{1}{2}(1,1,1,1)$, then matrix A must be 3 by 4.

If the A has rank-1 then $A^T A$ has also rank-1.

So, only one eigenvalue of $A^T A$ is nonzero.

The matrix Σ is given by

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is only one nonzero entry in Σ, we get the following equation

$$Av_1 = \sigma_1 u_1$$

So, u be the first column of U and v be the first column of V.

Step-3

If we find out $A = U\Sigma V^T$, the only nonzero will come from first column of U and first column of V.

This gives the equation $Av = 12u$.

Therefore, the matrix $\boxed{A = 12uv^T}$ with rank 1 that has $Av = 12u$.

Step-4

The length of eigenvector vector Av_j is σ_j and u_j is unit eigenvector.

So, from the equation $A = 12uv^T$, the only one singular value is $\boxed{\sigma_1 = 12}$.