Step-1

Given that $A=\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}$

To find eigen values, we consider the characteristic equation $|A-\lambda I|=0$.

$$\Rightarrow\begin{vmatrix}5-\lambda & 4\\ 4 & 5-\lambda\end{vmatrix}=0$$

$$\Rightarrow(5-\lambda)^2-16=0$$
$$\Rightarrow\lambda^2+25-10\lambda-16=0$$
$$\Rightarrow\lambda^2-10\lambda+9=0$$
$$\Rightarrow(\lambda-1)(\lambda-9)=0$$
$$\Rightarrow\lambda=1,9$$

The eigen values are $\lambda=1,9$

Step-2

To find the eigen vector for $\lambda=1$, we solve $(A-I)(x)=0$

$$\Rightarrow\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

Applying row operations $R_2\to R_2-R_1, R_1/4$R, we get $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

The coefficient matrix is the reduced matrix and so, we rewrite the homogeneous equations from this.

$x_1+x_2=0$

Putting $x_1=1,$ we get $x_2=-1$ and thus, the solution set is $\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}$ is the eigen vector corresponding to $\lambda=1$

Step-3

Similarly, for the eigen value $\lambda=9$, we solve $(A-9I)(x)=0$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying the row operations and reducing it to the echelon form, the homogeneous equation is $x_1 - x_2 = 0$

Putting $x_1 = 1$, the eigen vector is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda = 9$

Step-4

Using the eigen vectors as the columns of the matrix, we get $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and

$S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ such that

$$\begin{aligned} S^{-1}AS &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 9 \\ -1 & 9 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 18 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \\ &= \Lambda \end{aligned}$$

Now, we can write this equation as $A = S\Lambda S^{-1}$

Applying the n powers on both sides, it becomes $A^n = S\Lambda^n S^{-1}$ and observe Λ^n is the n^{th} powers of the diagonal entries.

Putting $n = \frac{1}{2}$, we get $\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^{\frac{1}{2}} = S\Lambda^{\frac{1}{2}}S^{-1}$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1^{1/2} & 0 \\ 0 & 9^{1/2} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
&= R
\end{aligned}
$$

While $1^{1/2} = \pm 1, 9^{1/2} = \pm 3$, we follow that there are three more square roots to the given matrix.