南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线 性 代 数 A　　　　**开课单位:** 数 学 系

考试时长: 120 分钟　　　　**命题教师:** 线性代数教学团队

题 号	1	2	3	4	5	6	7	8
分 值	15 分	25 分	10 分	12 分	16 分	6 分	8 分	8 分

本试卷共 (8) 大题, 满分 (100) 分. **请将所有答案写在答题本上.**

This exam includes **8** questions and the score is 100 in total. **Write all your answers on the examination book.**

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1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Let A, B, C, D be $n \times n$ matrices. Which of the following statements is true? ( ) A

$B=C$

(A) Suppose A is invertible. If $AB = AC$, then $BA = CA$.

(B) If A is invertible, then $AB = BA$.

(C) If $A \neq O$ and $AB = AC$, then $B = C$.

(D) If $B \neq C$, then $AB \neq AC$. $A=0$

设 A, B, C, D 为 $n \times n$ 矩阵. 下列陈述正确的是? ( )

(A) 假设 A 可逆. 如果 $AB = AC$, 则 $BA = CA$.

(B) 假设 A 可逆, 则 $AB = BA$.

(C) 假设 $A \neq O$ 并且 $AB = AC$, 则 $B = C$.

(D) 假设 $B \neq C$, 则 $AB \neq AC$.

(2) Let A be an $m \times n$ matrix and b be an m-dimensional column vector. Suppose that $Ax = b$ has infinitely many solutions. Which of the following statements is true? ( ) D

(A) $\text{rank}(A) < m$. $A_{m\times n} x_{n\times 1} = b_{x\times 1}$

(B) $m < n$. $r=m<n$

(C) A is the zero matrix.

(D) $Ax = 0$ has infinitely many solutions.

设 A 为 $m \times n$ 矩阵, b 为 m 维列向量. 假设 $Ax = b$ 有无穷多解. 下列陈述正确的是? ( )

(A) $\text{rank}(A) < m$.

(B) $m < n$.

(C) A 为零矩阵.

(D) $Ax = 0$ 有无穷多解.

(3) Let P and Q be $n \times n$ matrices satisfying $P - Q = PQ$. Let I be the identity matrix of order n. Which of the following statements is true? (A)

$PQ - P + Q - I = -I$

$(P+I)(Q-I) = -I$

$(P+I)(I-Q) = I$

(A) $I + P$ is invertible. (B) $I + Q$ is invertible.

(C) $I - P$ is invertible. (D) $I - Q$ is not invertible.

设 P 和 Q 为 $n \times n$ 矩阵且 $P - Q = PQ$. 设 I 为 n 阶单位阵. 下列说法正确的是? ( )

(A) $I + P$ 可逆. (B) $I + Q$ 可逆.

(C) $I - P$ 可逆 . (D) $I - Q$ 不可逆.

(4) Assume $AB = 0$, where A and B are nonzero matrices. Then we must have (D) A

(A) A has linearly dependent columns and B has linearly dependent rows.

(B) A has linearly dependent columns and B has linearly dependent columns.

(C) A has linearly dependent rows and B has linearly dependent rows.

(D) A has linearly dependent rows and B has linearly dependent columns.

设 A, B 是满足 $AB = 0$ 的两个非零矩阵. 则必有 ( )

$\sum$(Rows of A)·(Cols of B) = 0

(A) A 的列向量线性相关, B 的行向量线性相关.

(B) A 的列向量线性相关, B 的列向量线性相关.

(C) A 的行向量线性相关, B 的行向量线性相关.

(D) A 的行向量线性相关, B 的列向量线性相关.

(5) Let A be a symmetric matrix of order n, and B be a skew-symmetric matrix of order n. Which of the following matrices is skew-symmetric? $A^T = A \quad B^T = -B$ (C)

(A) $AB - BA$

(B) $(AB)^2$

(C) $AB + BA$

(D) BAB

$(AB+BA)^T = B^TA^T + A^TB^T$

$= -BA - AB$

$(BAB)^T = B^TA^TB^T = BAB$

设 A 是 n 阶对称矩阵, B 是 n 阶反对称矩阵. 则下列矩阵必为反对称矩阵的是 ( )

(A) $AB - BA$

(B) $(AB)^2$

(C) $AB + BA$

(D) BAB

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let $A = \begin{bmatrix} 1 & x & y \\ 0 & 2 & x \\ 0 & 0 & 4 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2}x & \frac{x^2}{8}-\frac{y}{4} \\ 0 & \frac{1}{2} & -\frac{x}{8} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$.

设 $A = \begin{bmatrix} 1 & x & y \\ 0 & 2 & x \\ 0 & 0 & 4 \end{bmatrix}$. 则 $A^{-1} =$____________.

(2) Let A be a 4×5 matrix of rank 2. Then $\dim N(A^T) =$ 2.

假定 A 是秩为 2 的 4×5 矩阵. 则 $\dim N(A^T) =$__________.

(3) The least squares solution of the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 + 3x_2 &= 2 \\ 2x_1 + 4x_2 &= 3 \end{aligned}$$

is $\begin{bmatrix} 2/15 \\ 7/15 \end{bmatrix}$

方程组

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 + 3x_2 &= 2 \\ 2x_1 + 4x_2 &= 3 \end{aligned}$$

$$\begin{bmatrix} \alpha \\ 2\alpha \\ 3\alpha \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix}$$

的最小二乘解是 __________.

(4) Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 3 & 0 & 4 \end{bmatrix}$, $\alpha = \begin{bmatrix} a \\ 1 \\ 1 \end{bmatrix}$. Suppose that $A\alpha$ and α are linearly dependent. Then $a =$ -1.

$$\begin{bmatrix} \alpha \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} \alpha \\ 2\alpha+3 \\ 3\alpha+4 \end{bmatrix}$$

设 $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 3 & 0 & 4 \end{bmatrix}$, $\alpha = \begin{bmatrix} a \\ 1 \\ 1 \end{bmatrix}$. 假设 $A\alpha$ 与 α 线性相关. 则 $a =$ -1.

(5) Let A be an $n \times n$ matrix such that $A^3 = I_n$. (Here I_n denotes the identity matrix of order n.) Let $M = \begin{bmatrix} 0 & -I_n \\ A & 0 \end{bmatrix}$. Then $M^{2021} =$ $\cancel{\begin{bmatrix} 0 & -I_n \\ A^2 & 0 \end{bmatrix}}$ $\begin{bmatrix} 0 & -A^2 \\ I_n & 0 \end{bmatrix}$

设 A 为 $n \times n$ 矩阵, 满足 $A^3 = I_n$. (这里 I_n 表示 n 阶单位阵.) 令 $M = \begin{bmatrix} 0 & -I_n \\ A & 0 \end{bmatrix}$. 则 $M^{2021} =$__________.

3. (10 points) Find the LU factorization of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$.

(10 分) 求矩阵 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ 的 LU 分解.

4. (12 points) Let $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 2 & 8 & 0 \end{bmatrix}$.

(a) Find a basis for the column space $C(A)$.

(b) Does there exist a vector u in $C(A)$ and a vector v in $N(A^T)$ such that $v^Tu + u^Tv = 2$? Why?

orthogonal

(c) Write down a matrix B such that $C(A) = N(B)$. (You should explain why your matrix has the required property.) $\Leftrightarrow BA=0 \Leftarrow$ B is the basis for the left-nullspace of A

(12 分) 设 $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 2 & 8 & 0 \end{bmatrix}$.

(a) 找出列空间 $C(A)$ 的一组基.

(b) 是否存在 $C(A)$ 中的向量 u 和 $N(A^T)$ 中的向量 v 使得 $v^Tu + u^Tv = 2$? 为什么?

(c) 写出一个矩阵 B 使得 $C(A) = N(B)$. (请解释为何你给出的矩阵满足要求.)

5. (16 points) Let $V = \mathbb{R}^{2\times 2}$ be the vector space of 2×2 real matrices. Consider the map

$$f : V \to V, \quad f(A) = A - A^T.$$

(a) Show that f is a linear transformation, and that the subset $\operatorname{Ker}(f) = \{A \in V \mid f(A) = 0\}$ is a subspace of V.

(b) Find a basis for $\operatorname{Ker}(f)$.

(c) Find the matrix representation of f with respect to the following ordered basis of V:

$$v_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; v_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; v_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

变换的是坐标

(d) Is there a nonzero matrix A such that $f(A)$ is a nonzero scalar multiple of A (i.e., there exists a nonzero real number λ such that $f(A) = \lambda A$)? If yes, please find a matrix with this property; otherwise please explain why such a matrix cannot exist.

(16 分) 设 $V = \mathbb{R}^{2\times 2}$ 为 2 阶实方阵构成的向量空间. 考虑如下映射

$$f : V \to V, \quad f(A) = A - A^T.$$

(a) 证明 f 是线性变换, 而且 $\operatorname{Ker}(f) = \{A \in V \mid f(A) = 0\}$ 是 V 的子空间.

(b) 找出 $\operatorname{Ker}(f)$ 的一组基.

(c) 求 f 在 V 的以下有序基下的矩阵:

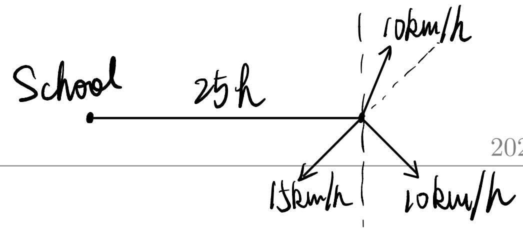
$$v_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; v_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; v_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

(d) 是否存在一个非零矩阵 A 使得 $f(A)$ 是 A 的某个非零常数倍 (即, 存在非零实数 λ 使得 $f(A) = \lambda A$)? 若是, 请找出一个这样的矩阵; 若否, 请解释为什么这样的矩阵不存在.

$\lambda = 2$

$(1-\lambda)A = A^T$

$(1-2)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



6. (6 points) Xiaomeng wants to take a hot air balloon ride from her home on the top of Wutong mountain to her Linear Algebra exam at SUSTech; a journey of 25 km West. She can adjust the height above ground of the hot air balloon by changing the temperature of the air in the balloon. Assume that the balloon travels at a velocity relative to the ground equal to that of the air current, and that the air currents stay constant throughout her journey. Her phone app tells her that at low altitudes, the air current is 15 km per hour southwest. At medium altitudes, it is 10 kilometers per hour north north east (this direction is half way between north east and north), and high up, it is 10 kilometers per hour south east.

   Show that she can get to SUSTech by hot air balloon.

   If the air current at medium altitudes shifts to 10 km per hour north east, can she still get to SUSTech by hot air balloon? Why? * positive solution

   (If needed, you may use the numerical values: $\sin(45°) \approx 0.707$, $\sin(67.5°) \approx 0.924$, $\cos(67.5°) \approx 0.383$.)

   (6 分) 小萌的家在梧桐山山上. 她希望乘坐热气球从家里来南科大参加线性代数考试, 而学校位于她家正西 25 公里. 通过调整热气球内的空气温度, 小萌可以调节热气球距离地面的高度. 假设热气球相对地面的飞行速度等于风速, 而同样高度的风速在她的旅途中保持不变. 根据她的手机 app 显示, 在低空高度的风速是朝向西南方向 15 公里每小时, 在中空高度的风速是朝向东北偏北方向 (该方向位于正北方向和东北方向正中间) 10 公里每小时, 在高空高度的风速是朝向东南方向 10 公里每小时.

   证明: 小萌乘坐热气球是可以从家到达南科大的.

   如果中空高度的风速改变为朝向东北方向 10 公里每小时, 小萌的同一行程是否仍然能够到达? 为什么?

   (如有需要, 可参考这些数值: $\sin(45°) \approx 0.707$, $\sin(67.5°) \approx 0.924$, $\cos(67.5°) \approx 0.383$.)

7. (8 points) Let R be the 2 by 2 matrix representing a counter-clockwise rotation of $60°$ (about the origin) in the plane. Let T be the 2 by 2 matrix representing the reflection (also called mirror symmetry) about the line ℓ defined by the equation $x + y = 0$.

   (a) Find the matrix $S = RT$.

   (b) Is it possible to factorize S into a product $S = T'R'$, where T' represents a reflection and R' represents a rotation (about the origin)? If yes, please find such a factorization. Otherwise please explain why such a factorization cannot exist.

   (8 分) 设 2×2 矩阵 R 表示平面上 (以原点为中心) 逆时针方向 $60°$ 的旋转, 而 2×2 矩阵 T 表示平面上关于直线 ℓ 的反射 (也称镜像对称, 对称轴为 ℓ), 这里 ℓ 的方程是 $x + y = 0$.

   (a) 求矩阵 $S = RT$.

   (b) 是否可以将 S 分解为一个乘积 $S = T'R'$ 的形式, 使得其中 T' 表示某个反射, R' 表示某个 (以原点为中心的) 旋转? 若是, 请找出一个这样的分解. 若否, 请解释为什么这种分解不存在.

8. (8 points) Let A and B be two invertible n by n matrices. Assume that A and B commute, i.e., $AB = BA$. Let $M = \begin{bmatrix} A & B \\ B^{-1} & A^{-1} \end{bmatrix}$.

   (a) Show that M is not invertible.

   (b) Find the rank of M.

   (8 分) 设 A 和 B 是可逆的 n 阶方阵, 且 A 和 B 可交换, 即 $AB = BA$. 令 $M = \begin{bmatrix} A & B \\ B^{-1} & A^{-1} \end{bmatrix}$.

   (a) 证明 M 不可逆. √

   (b) 求 M 的秩. n.