Step-1

We have to find an orthonormal basis for the plane $x-y+z=0$, and we have to find the matrix P that project onto the plane, and we have to find the nullspace of P.

Given equation is $x-y+z=0$

$$
\begin{aligned}
&\text{Put } y=k,\ z=r \\
&\Rightarrow x=y-z \\
&=k-r
\end{aligned}
$$

Step-2

Therefore

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k-r \\ k \\ r \end{bmatrix}
$$

$$
= k\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
$$

Therefore $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are two vectors in the plane $x-y+z=0$

Step-3

$$
q_1 = \frac{a_1}{\|a_1\|}
$$

$$
= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}
$$

Step-4

$$q_2 = \frac{\beta}{\|\beta\|} \text{ where } \beta = a_2 - \left(q_1^T a_2\right) q_1$$

$$q_1^T a_2 = \frac{1}{\sqrt{2}}(1,1,0)\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(-1+0+0)$$

$$= -\frac{1}{\sqrt{2}}$$

Step-5

$$\left(q_1^T a_2\right) q_1 = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Step-6

$$\beta = \begin{pmatrix}-1\\0\\1\end{pmatrix} + \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}-1/2\\1/2\\1\end{bmatrix}$$

Step-7

$$\|\beta\| = \sqrt{\frac{1}{4}+\frac{1}{4}+1}$$

$$= \sqrt{\frac{6}{4}}$$

Step-8

$$q_2 = \frac{\beta}{\|\beta\|}$$

$$= \frac{2}{\sqrt{6}}\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

Step-9

Therefore the required orthonormal basis is

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$$

Step-10

P = the projection matrix on the plane = $A\left(A^T A\right)^{-1} A^T$

Here $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Step-11

$$\left(A^T A\right)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A\left(A^T A\right)^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Step-12

$$P = A\left(A^T A\right)^{-1} A^T$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \boxed{\frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}}$$

This is the required projection matrix.

Step-13

The nullspace of P is obtained from $Px = 0$

$$\frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-14

Apply $R_2 \to 2R_2 - R_1,\ R_3 \to 2R_3 + R_1$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + y - z = 0$$

$$3x + 3y = 0$$

$$\Rightarrow x = -y$$

Step-15

$$\begin{aligned}
&if\ \ y = k \\
&\Rightarrow x = -k \\
&z = 2x + y \\
&= -k
\end{aligned}$$

Step-16

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ k \\ -k \end{bmatrix}$$

$$= -k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The nullspace of P is represented the vector $= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

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