Step-1

$$A=\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

To find the eigen values of A, we consider $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} \frac{8}{10}-\lambda & \frac{3}{10} \\ \frac{2}{10} & \frac{7}{10}-\lambda \end{vmatrix}=0$$

$$\Rightarrow (.8-\lambda)(.7-\lambda)-(.3)(.2)=0$$
$$\Rightarrow 0.56-1.5\lambda+\lambda^2-0.6=0$$
$$\Rightarrow \lambda^2-1.5\lambda+0.5=0$$

$$\Rightarrow \lambda^2-\lambda-0.5\lambda+0.5=0$$
$$\Rightarrow \lambda(\lambda-1)-0.5(\lambda-1)=0$$
$$\Rightarrow (\lambda-1)(\lambda-0.5)=0$$

$\Rightarrow \lambda=1, \lambda=0.5$ are the eigen values of the given matrix

Step-2

To find the eigen vector corresponding to $\lambda=1$, we consider $|A-\lambda I|x=0$

$$\begin{bmatrix} .8-1 & .3 \\ .2 & .7-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying row operation $R_2 \to R_2+R_1$, we get $\begin{bmatrix} -0.2 & 0.3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This is the reduced matrix.

So, back substituting the equations, we get $x_1=1.5x_2$

Putting $x_2=2$, we get $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=1$

Step-3

Similarly, to find the eigen vector corresponding to $\lambda = 0.5$, we consider $|A-\lambda I|x = 0$

$$\begin{bmatrix} .8-0.5 & .3 \\ .2 & .7-0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} .3 & .3 \\ .2 & .2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applying row operation $R_2 \to R_2 - \frac{2}{3}R_1$, we get $\begin{bmatrix} 0.3 & 0.3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This is the reduced matrix.

So, back substituting the equations, we get $x_1 = -x_2$

Putting $x_1 = 1$, we get $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 0.5$

Step-4

The matrix whose columns are the eigen vectors of *A* is $P = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$

$$P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$$

So, by the diagonalizability, we have $P^{-1}AP = D$ where *D* is the diagonal matrix.

$$\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & .5 \\ 2 & -.5 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 2.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

This is the diagonal matrix whose diagonal entries are nothing but the eigen values of *A*.

Step-5

The above equation can otherwise be written as $A = PDP^{-1}$

Raising this equation to the power *n*, we easily see

$$A^n = \left(PDP^{-1}\right)\left(PDP^{-1}\right)\ldots\left(PDP^{-1}\right)$$
$$= PD^nP^{-1}$$

So, $A^2 = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1^2 & 0 \\ 0 & .5^2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$ â€¦â€¦ (1)

$$= \frac{1}{5}\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0.5 & -0.75 \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} 3.5 & 2.25 \\ 1.5 & 2.75 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}$$

Step-6

$$\lim_{n\to\infty} A^n = P\left\{\lim_{n\to\infty} D^n\right\}P^{-1}$$
$$= \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\left\{\lim_{n\to\infty}\begin{bmatrix} 1^n & 0 \\ 0 & 0.5^n \end{bmatrix}\right\}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$$

$= \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$ â€¦â€¦ (2)

Step-7

We have the equation $A^n = PD^nP^{-1}$

So, the eigen values of A^n are nothing but the n^{th} powers of the diagonal matrix of *A*.

That is $1^n, 0.5^n$

We know that $1^2 = 1, \lim_{n\to\infty} 1^n = 1$

The average of $1, \lim_{n\to\infty} 1^n$ is 1^2 â€¦â€¦ (3)

But $0.5^2 = \frac{1}{2}(0.5), \lim_{n\to\infty} 0.5^n = 0$

The average of 0.5 and 0 is 0.5^2 â€¦â€¦ (4)

Putting (3) and (4) together, we can say that the eigen values of A^2 are nothing but the averages of the eigen values of A, and $\lim_{n\to\infty} A^n$.