Linear Algebra-A

Assignments - Week 5

Supplementary Problem Set

1. If the vector $\boldsymbol{\beta}$ *can* be linearly represented by the set of vectors $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_m\}$, and *cannot* be linearly represented by the set of vectors $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_{m-1}\}$.
   (1) Show that $\boldsymbol{\alpha}_m$ *cannot* be linearly represented by the set of vectors $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_{m-1}\}$.
   Proof by contradiction → β can be l.r. by $\{\alpha_1, \cdots \alpha_{m-1}\}$ ⇒ contradiction
   (2) Show that $\boldsymbol{\alpha}_m$ *can* be linearly represented by the set of vectors $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_{m-1}, \boldsymbol{\beta}\}$.

2. Suppose that a homogeneous system of linear equations (I) is as follows:

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 0 \\ x_1 + 2x_2 + x_3 - x_4 = 0 \end{cases},$$

while the basis of the solution space of another homogeneous system of linear equations (II) is

$$\boldsymbol{\alpha}_1 = (2, -1, a+2, 1)^T, \boldsymbol{\alpha}_2 = (-1, 2, 4, a+8)^T.$$

   (1) Write out the basis of the solution space of the system (I).
   (2) For what value of a for the systems (I) and (II) to have common *nonzero* solutions? Write out all the common *nonzero* solutions.

$$\begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & 2 & 1 & -1 \end{bmatrix} \alpha_1 = 0$$

3. Let $\boldsymbol{A}$ be a 5×4 matrix, and $\mathrm{rank}(\boldsymbol{A}) = 2$. We now know that $\boldsymbol{x}_1 = [1 \quad 2 \quad 0 \quad 1]^T$, $\boldsymbol{x}_2 = [2 \quad 1 \quad 1 \quad 3]^T$ are solutions to the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$, and $\boldsymbol{x}_3 = [1 \quad 0 \quad 1 \quad 0]^T$ is a solution to the corresponding homogeneous system of linear equations $\boldsymbol{Ax} = \boldsymbol{0}$. Please find the general solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

* $(-1, 1, -1, -2)^T$ and $(1, 0, 1, 0)^T$ are linearly independent

4. (1) If $\boldsymbol{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \cdots, \boldsymbol{v}_k = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix}$ are linearly independent, show that

$$\boldsymbol{v}_1^* = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \\ a_{n+1,1} \\ \vdots \\ a_{m1} \end{bmatrix}, \boldsymbol{v}_2^* = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \\ a_{n+1,2} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \boldsymbol{v}_k^* = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \\ a_{n+1,k} \\ \vdots \\ a_{mk} \end{bmatrix}$$ are also linearly independent.

【即证明如下命题：如果一组n维向量线性无关，那么把这些向量任意添加相同个数的若干个分量所得到的新向量组也是线性无关的。反过来，如果一组向量线性相关，那么它们各去掉相同个数对应位置的若干分量所得到的新向量组也是线性相关的。】

4. (2) ① if rank $(A)=n$. $\Rightarrow N(A)=\{0\}$

② if rank$(A)=r<n$. the general solution is

$$\Rightarrow \begin{cases} x_1=-b_{1\,r+1}x_{r+1}-\cdots-b_{1n}x_n \\ x_2=-b_{2\,r+1}x_{r+1}-\cdots-b_{2n}x_n \\ \vdots \\ x_r=-b_{r\,r+1}x_{r+1}-\cdots-b_{rn}x_n \end{cases}$$

Taking $\begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix}$ as $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ respectly

(2) Please explain why the special solutions to a homogeneous system of linear equations $\boldsymbol{Ax}=\mathbf{0}$ form a basis for $N(\boldsymbol{A})$.

[~~*Hint*: Use the conclusion of question (1).~~]

(3) If the rank of an $m\times n$ matrix $\boldsymbol{A}$ is r, then what is the dimension of $N(\boldsymbol{A})$?

$\eta_i=\begin{bmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a solution of $Ax=0$ where the ith component is 1

and $i=1,\cdots,n-r$.

$\because \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$ are linearly independent

$\therefore \eta_1, \eta_2, \cdots \eta_{n-r}$ are linearly independent

$\because \forall \eta\in N(A) \Rightarrow \eta$ can be represented by the linear system

$\Rightarrow \eta$ can be linearly represented by $\eta_1,\eta_2,\cdots,\eta_{n-r}$.

$\therefore \eta_1,\cdots,\eta_{n-r}$ are the special solutions of $Ax=0$

and $\eta_1,\eta_2,\cdots,\eta_{n-r}$ form a basis for $N(A)$

5. (Find the basis and dimension of a subspace)

(1) In $\mathbb{R}^4$, find a basis for the subspace $V_1\cap V_2$ (i.e., the intersection of V_1 and V_2), where $V_1=\mathrm{Span}\{(1,-5,3,2)^T,(4,1,-2,9)^T\}$, and $V_2=\mathrm{Span}\{(2,0,-1,4)^T,(0,3,4,-5)^T\}$.

(2) Generally, if V_1 and V_2 are subspaces of a vector space V.

$l_1\alpha_1+l_2\alpha_2 \in V_1\cap V_2$ ✓ $\Leftarrow$ $l_1\alpha_1+l_2\alpha_2\in V_1 \Leftarrow \alpha_1,\alpha_2\in V_1$

$l_1\alpha_1+l_2\alpha_2\in V_2$ $\quad \alpha_1,\alpha_2\in V_2$

(i) Is $V_1\cap V_2$ a subspace? Why?

(ii) Is $V_1\cup V_2$ a subspace? Why?