Step-1

Consider the matrix $A=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$.

The objective is to compute $A^T A$ and AA^T.

Step-2

Compute $A^T A$.

$$A^T=\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}$$

$$\begin{aligned}A^T A &= \begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 1\end{bmatrix}\end{aligned}$$

Compute AA^T.

$$\begin{aligned}AA^T &= \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\end{aligned}$$

Step-3

Find the eigenvalues of AA^T.

$$AA^T-\lambda I=\begin{bmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda)-1=0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-1)(\lambda-3)=0$$

$$\lambda = 1,3$$

The eigenvalues of AA^T are $\lambda_1 = 1, \lambda_2 = 3$.

Step-4

Find an eigenvector of AA^T corresponding to the eigenvalue $\lambda_1 = 1$.

Solve the system $(AA^T - I)\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$$

$$= -x_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

An eigenvector of AA^T corresponding to the eigenvalue $\lambda_1 = 1$ is $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step-5

Find an eigenvector of AA^T corresponding to the eigenvalue $\lambda_2 = 3$.

Solve the system $(AA^T - 3I)\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$

$$x_1 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$$
$$= x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

An eigenvector of AA^T corresponding to the eigenvalue $\lambda_2 = 3$ is $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step-6

Find the eigenvalues of $A^T A$.

$$A^T A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)[(2-\lambda)(1-\lambda)-1] - (1-\lambda) = 0$$
$$\lambda(\lambda-1)(\lambda-3) = 0$$

$$\lambda = 0,1,3$$

Therefore, the eigenvalues of $A^T A$ are $\lambda_1 = 1, \lambda_2 = 3,$ and $\lambda_3 = 0$

Step-7

Find an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_1 = 1$.

Solve the system $[A^T A - I]\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0$$
$$x_1 + x_2 + x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= -x_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Therefore, an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_1 = 1$ is $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Step-8

Find an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_2 = 3$.

Solve the system $[A^T A - 3I]\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix for the system is $\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$.

The reduced row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$
$$x_2 - 2x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix}$$
$$= x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Therefore, an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_2 = 3$ is $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Step-9

Find an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_3 = 0$.

Solve the system $A^T A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix for the system is $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$.

The reduced row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$
$$x_2 + x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$
$$= x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Therefore, an eigenvector of $A^T A$ corresponding to the eigenvalue $\lambda_3 = 0$ is $\mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Step-10

Eigenvectors of AA^T are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

Find the orthonormal eigenvectors of AA^T.

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\mathbf{u}_1, \mathbf{u}_2$ are orthonormal eigenvectors of AA^T.

The matrix U is

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Step-11

Eigenvectors of $A^T A$ are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

The vectors $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$ are orthogonal.

Find the orthonormal eigenvectors of $A^T A$.

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\mathbf{v}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are orthonormal eigenvectors of $A^T A$.

The matrix V is

$$V = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Eigenvalues of AA^T are $\lambda_1 = 1, \lambda_2 = 3$.

Singular values of A are

$$\sigma_1 = \sqrt{\lambda_1}$$
$$= 1$$

$$\sigma_2 = \sqrt{\lambda_2}$$
$$= \sqrt{3}$$

The matrix Σ is

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$$
$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

Step-12

Find $U\Sigma V^T$.

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Step-13

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$V=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

$$V^T=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

$$U\Sigma V^T=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \sqrt{3} & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

$$=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$$

Therefore, $U\Sigma V^T = A$.