# Southern University of Science and Technology
# Advanced Linear Algebra Spring 2023

# MA109– Quiz #3

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**Name**: ____________________________________________

**Student Number**: __________________________________

1. Suppose $V$ is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\text{range } S \subset \text{null } T.$$

Prove that $(ST)^2 = 0$.

设 $V$ 是一个向量空间，$S, T \in \mathcal{L}(V, V)$，使得 range $S \subset$ null $T$, 证明 $(ST)^2 = 0$.

*Proof.* Let $v \in V$, then $(ST)^2 v = ST(S(Tv))$

Since $S(Tv) \in \text{range}S \subset \text{null}T$, then $T(S(Tv)) = 0$, so $(ST)^2 v = 0$ for all $v \in V$, then $(ST)^2 = 0$.

$\square$

2. Suppose $V$ and $W$ are finite-dimensional and that $U$ is a subspace of $V$. Prove that there exists $T \in \mathcal{L}(V, W)$ such that null $T = U$ if and only if $\dim U \geqslant \dim V - \dim W$.

   设 $V$ 和 $W$ 是有限维的，且 $U$ 是 $V$ 的子空间. 证明存在 $T \in \mathcal{L}(V, W)$，使得 null $T = U$ 当且仅当 $\dim U \geqslant \dim V - \dim W$.

   *Proof.* "$\Rightarrow$": Assume there exists $T \in \mathcal{L}(V, W)$ such that null $T = U$, then by the Fundamental Theorem of Linear Maps, $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. Since range $T$ is a subspace of $W$, we have $\dim \operatorname{range} T \leqslant \dim W$, then

$$\dim U = \dim \operatorname{null} T = \dim V - \dim \operatorname{range} T \geqslant \dim V - \dim W.$$

   "$\Leftarrow$": Assume $\dim U \geqslant \dim V - \dim W$, then $\dim W \geqslant \dim V - \dim U$.

   Let $u_1, \cdots, u_r$ be a basis for $U$ and extend it to a basis $u_1, \cdots, u_r, v_1, \cdots, v_n$ for $V$. Let $w_1, \cdots, w_m$ be a basis for $W$.

   For any $v \in V$, we can write $v = a_1u_1 + \cdots a_ru_r + b_1v_1 + \cdots b_nv_n$, we define a map $T : V \to W$

$$T(a_1u_1 + \cdots a_ru_r + b_1v_1 + \cdots b_nv_n) = b_1w_1 + \cdots b_nw_n$$

   Since $\dim W \geqslant \dim V - \dim U$, we have $m \geqslant n$ and so $T$ is well defined. Clearly $T \in \mathcal{L}(V, W)$ and null $T = U$.

   $\square$