Southern University of Science and Technology
Linear Algebra I Final Examination

Department:____Math_____Class:________

Student ID:______________________Name:_____________

Answer all parts of Questions (1)-(8). Total is 100 points.

试卷包含八道大题. 总分100.

(1) (15 points)Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

(i) Describe the row space of A. $\{(1,0,2)^T, (1,1,4)^T\}$

(ii) Find a basis for the orthogonal complement of the row space of A.(Hint: Given a subspace V of $\mathbb{R}^n$, the space of all the vectors in $\mathbb{R}^n$ orthogonal to V is called the orthogonal complement of V.) $\{(-2,-2,1)^T\}$

(~~iii~~) Split $\mathbf{x} = (\mathbf{3}, \mathbf{3}, \mathbf{3})^{\mathbf{T}}$ into a row space component $\mathbf{x_r}$ and a nullspace component $\mathbf{x_n}$, i.e., $\mathbf{x} = \mathbf{x_r} + \mathbf{x_n}$.

$A = \begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix}$ $P = A(A^TA)^{-1}A^T$

$\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

(1) (15 分)假定

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

(i) 写出 A 的行空间.

(ii) 找出 A 的行空间的正交补的一组基.(提示: 给定 $\mathbb{R}^n$的子空间 V, 则 V 的正交补是指所有与 V 中每个向量正交的向量构成的空间)

(iii) 将向量 $\mathbf{x} = (\mathbf{3}, \mathbf{3}, \mathbf{3})^{\mathbf{T}}$ 分解为 $\mathbf{x} = \mathbf{x_r} + \mathbf{x_n}$, 其中 $\mathbf{x_r}$ 和 $\mathbf{x_n}$ 分别属于 A 的行空间和零空间.

(2) (10 points)Let

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}, \quad i = \sqrt{-1}.$$

(i) Is A Hermitian? Yes

(ii) Find all the eigenvalues and eigenvectors of A. $\lambda_1 = 0 \longrightarrow \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$ $\lambda_2 = 3 \longrightarrow \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

(iii)Find a unitary matrix U (namely, $U^{-1} = U^H$) that diagonalizes A, in other words, $U^{-1}AU = \Lambda$, Λ is a diagonal matrix with the eigenvalues on the main diagonal.

$U = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 & 1-i \\ 1+i & 1 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

(2) (10 分)考虑

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}, \quad i = \sqrt{-1}.$$

(i) A 是否为厄米特型矩阵?

(ii) 找出 A 的所有的特征值及其对应的特征向量.

(iii)找到酉矩阵 U (即 $U^{-1} = U^H$) 使 A 对角化, 换言之, $U^{-1}AU = \Lambda$, Λ 是一个对角阵，对角元为矩阵 A 的特征值.

(3) (10 points) (i) Describe the positive definiteness of a matrix.

(ii)Decide whether the following matrices are positive definite, positive semidefinite or indefinite. (Hint: Use the determinant test)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 3 \end{bmatrix}.$$ indefinite

positive definite   positive semidefinite

(3) (10 分) (i) 给出正定矩阵的定义.

(ii)判断下列矩阵是否正定，半正定，或者不定.（提示：利用行列式判别法）

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 3 \end{bmatrix}.$$

(4) (10 points) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(i) Check the solvability of the system of linear equations $A\mathbf{x} = \mathbf{b}$. inconsistent

(ii) If the above system is not solvable, find the best estimate $\hat{\mathbf{x}}$ by least squares. (Hint: $\hat{\mathbf{x}}$ minimizes $||A\mathbf{x} - \mathbf{b}||^2$). $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\hat{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

(iii) Suppose $\mathbf{p}$ is the projection of $\mathbf{b}$ onto the column space of A, that is, $\mathbf{p} = A\hat{\mathbf{x}}$. Verify that the error $\mathbf{b} - \mathbf{p}$ is perpendicular to the columns of A. ✓ $p = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

$= \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$

(4) (10 分) 考虑

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(i) 判断$A\mathbf{x} = \mathbf{b}$是否可解.

(ii) 如果上述方程不可解, 用最小二乘法求出一个最佳估计$\hat{\mathbf{x}}$.(提示: $\hat{x}$ 使得$||A\mathbf{x} - \mathbf{b}||^2$达到最小).

(iii) 设 $\mathbf{p}$ 为 $\mathbf{b}$ 在 A 的列空间中的投影, 即: $\mathbf{p} = A\hat{\mathbf{x}}$, 证明误差向量 $\mathbf{b} - \mathbf{p}$ 正交于 A 的列向量.

(5) (15 points)Let

$$A = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

(i) Find the cofactor of x. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1$

(ii) If $x = 0$, find $\det A$. $= 1$

(iii) Find $\det A$ for $x \neq 0$. $\det(A) = 1 - x$

(5) (15 分)假定

$$A = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

(i) 求 x 对应的代数余子式.

(ii) 如果 $x = 0$, 求 $\det A$.

(iii) 如果 $x \neq 0$, 求 $\det A$.

(6) (10 points)Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}.$$

(i) Find the determinant of A. $\det(A) = (b-a)(c-a)(c-b)$

(ii) Find the condition under which A is invertible, and then find the inverse of A.

$a \neq b \neq c$

(6) (10 分)假设

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}.$$

(i) 求出A 的行列式.

(ii) 找出A 可逆的条件，并在该条件下求出它的逆.

(7) (15 points)Let

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

(i) Find AA^T and A^TA.

(ii) Find all the singular values of A.

(iii) Find all the eigenvectors of both AA^T and A^TA.

(iv) Find the singular value decomposition of A, in other words, find orthogonal matrices U and V, such that $A = U\Sigma V^T$.

(7) (15 分)考虑

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

(i) 求出 AA^T 和 A^TA.

(ii) 求出 A 的所有奇异值.

(iii) 分别找出 AA^T 和 A^TA 的所有特征向量.

(iv) 将 A 进行奇异值分解, 换言之, 找出正交矩阵 U 和 V, 使得 $A = U\Sigma V^T$.

(8) (15 points) Let

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}.$$

$\lambda_1 = 5,\ \lambda_2 = -1,\ \lambda_3 = -1$

$$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(i) Find the eigenvalues and eigenvectors of A.

(ii) Find an orthogonal matrix Q (namely, $Q^{-1} = Q^T$) that diagonalizes A (i.e. $Q^{-1}AQ = \Lambda$, Λ is a diagonal matrix).

$$\begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

(iii) Compute A^k, where k is a positive integer.

$$A = Q^T \begin{bmatrix} 5 & & \\ & -1 & \\ & & -1 \end{bmatrix} Q \quad \checkmark$$

(8) (15 分) 考虑

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}.$$

(i) 找出 A 的所有的特征值及其对应的特征向量.

(ii) 找到正交矩阵 Q (即 $Q^{-1} = Q^T$) 把 A 对角化(i.e., $Q^{-1}AQ = \Lambda$, Λ 是一个对角矩阵).

(iii) 计算A^k, 这里 k 是一个正整数.