

Step-1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Consider the matrix

Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix}_{3 \times 3}$ and $C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}_{3 \times 4}$

The objective is to find bases for the four fundamental subspaces without computing product A .

Note that the order of the matrix A is 3 by 4 and matrix B is invertible because it is non-singular.

Step-2

Column space of A = column space of C

Reduce the matrix C into reduced echelon form as follows:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$2R_2 - R_1$$

$$\square \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_1 + R_2, \quad 2R_3 - R_2$$

$$\square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$C_4 - C_2, \quad C_4 - 2C_3$$

$$\square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \dots\dots(1)$$

The rank of C is 3.

Thus, the basis for column space of A is $\{(1,0,0), (0,1,0), (0,0,1)\}$

Step-3

And, row space of A = row space of C

A basis for row space of C consists of the nonzero rows in the reduced matrix (1)

Thus, the basis for row space of $A = \{(1, 2, 3, 4), (0, 1, 2, 3), (0, 0, 1, 2)\}$

Step-4

Null space of A = Null space of C

In order to find the null space, set $C\mathbf{x} = 0$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases}$$

$$\begin{cases} x_3 = -2x_4 \\ x_2 = -2x_3 - 3x_4 \\ \quad = 4x_4 - 3x_4 \\ \quad = x_4 \\ x_1 = -2x_2 - 3x_3 - 4x_4 \\ \quad = -2x_4 + 6x_4 - 4x_4 \\ \quad = 0 \end{cases}$$

Thus, the solution is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_4 \\ -2x_4 \\ x_4 \end{bmatrix} \\ = x_4 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad x_4 \text{ is arbitrary}$$

So, basis for null space = $(0, 1, -2, 1)$

Left Null space of A = left Null space of C^T

In order to find the left null space, set $C^T x = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 = 0 \\ 2x_1 + x_2 = 0 \\ 3x_1 + 2x_2 + x_3 = 0 \\ 4x_1 + 3x_2 + 2x_3 = 0 \end{cases}$$

$$\Rightarrow x_1 = x_2 = x_3 = 0$$

So, basis for left null space is empty.

Therefore, the required bases are as follows:

$$\begin{aligned} C(A^T) &= \{(1, 2, 3, 4), (0, 1, 2, 3), (0, 0, 1, 2)\}, \\ N(A) &= (0, 1, -2, 1), \\ C(A) &= \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}, \\ N(A^T) &= \text{empty} \end{aligned}$$