Step-1

Consider the matrix:

$$A = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix}$$

Thus,

$$A^T = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 0 \\ 6 & 8 \end{bmatrix}$$

Therefore, $A^T A$ is given by,

$$\begin{aligned} A^T A &= \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 0 \\ 6 & 8 \end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} 100 & 60 \\ 60 & 100 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \end{aligned}$$

Step-2

The eigenvalues of $A^T A$ are given by,

$$\begin{vmatrix} 10-\lambda & 6 \\ 6 & 10-\lambda \end{vmatrix} = 0$$

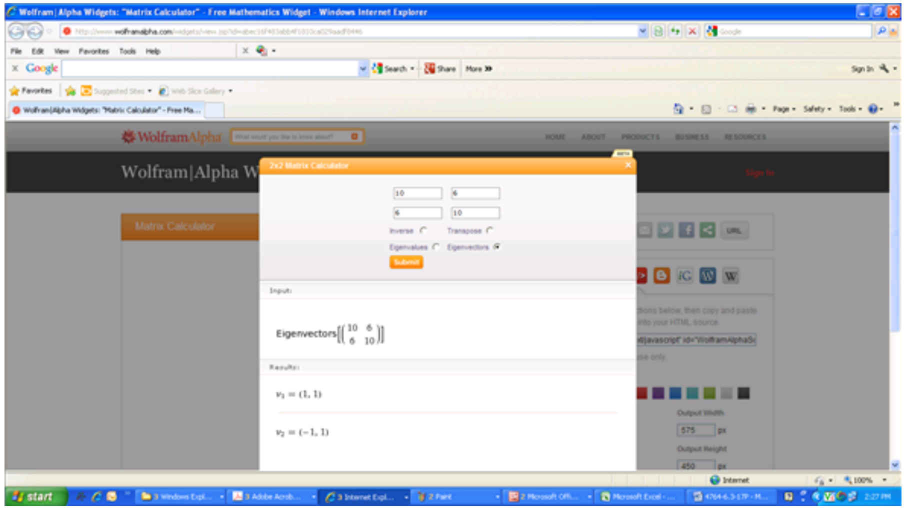
$$(10-\lambda)(10-\lambda) = 36$$

$$\begin{aligned} 100 - 20\lambda + \lambda^2 &= 36 \\ \lambda^2 - 20\lambda + 64 &= 0 \\ (\lambda-16)(\lambda-4) &= 0 \\ \lambda &= 16, 4 \end{aligned}$$

Therefore, the eigenvalues are 16 and 4.

Step-3

By using matrix calculator (the screenshot is given below), the eigenvectors of AA^T are given by,



The eigenvectors of AA^T are given by,

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Step-4

Therefore,

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 \\ 0 & 4 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Step-5

The positive definite square root $S = V\Sigma^{\frac{1}{2}}V^T$ is given by,

Step-6

$$S = V\Sigma^{\frac{1}{2}}V^T$$
$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix}^{\frac{1}{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T$$
$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

The inverse of S is given by,

$$S^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

Therefore, we get,

$$Q = AS^{-1}$$
$$= \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{bmatrix}$$
$$= \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

Step-7

Thus, $\boxed{S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}}$ and $\boxed{Q = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}}$.