Step-1

Given that $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$

Write the four equations $Ax = b$ for the closest cubic $b = C + Dt + Et^2 + Ft^3$.

Step-2

First, write the equations that would hold if a line could go through all four points.

Then, every $C + Dt + Et^2 + Ft^3$ would agree exactly with *b*.

Now, $Ax = b$ is;

$$C + D(0) + E(0)^2 + F(0)^3 = 0$$
$$C + D(1) + E(1)^2 + F(1)^3 = 8$$
$$C + D(3) + E(3)^2 + F(3)^3 = 8$$
$$C + D(4) + E(4)^2 + F(4)^3 = 20$$

Or,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$
$$x = \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix}$$
$$b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

Step-3

Observe that the least-square solution to a problem is $A^T A\hat{x} = A^T b$

Now,

$A^T A\hat{x} = A^T b$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \\ 0 & 1 & 27 & 64 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}\begin{bmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \\ \widehat{F} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \\ 0 & 1 & 27 & 64 \end{bmatrix}\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 & 26 & 92 \\ 8 & 26 & 92 & 338 \\ 26 & 92 & 338 & 1268 \\ 92 & 338 & 1268 & 4826 \end{bmatrix}\begin{bmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \\ \widehat{F} \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \\ 1504 \end{bmatrix}$$

Step-4

Apply $\frac{R_1}{2}, \frac{R_2}{2}, \frac{R_3}{2}, \frac{R_4}{2}$, and get;

$$\begin{bmatrix} 2 & 4 & 13 & 46 \\ 4 & 13 & 46 & 169 \\ 13 & 46 & 169 & 634 \\ 46 & 169 & 634 & 2413 \end{bmatrix}\begin{bmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \\ \widehat{F} \end{bmatrix} = \begin{bmatrix} 18 \\ 56 \\ 200 \\ 752 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to 2R_3 - 13R_1, R_4 \to 4R_4 - 46R_1$

$$\begin{bmatrix} 2 & 4 & 13 & 46 \\ 0 & 5 & 20 & 77 \\ 0 & 40 & 169 & 670 \\ 0 & 154 & 670 & 2710 \end{bmatrix}\begin{bmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \\ \widehat{F} \end{bmatrix} = \begin{bmatrix} 18 \\ 20 \\ 166 \\ 676 \end{bmatrix}$$

Step-5

Apply $R_3 \to R_3 - 8R_2, R_4 \to 5R_4 - 154R_2$

$$\begin{bmatrix} 2 & 4 & 13 & 46 \\ 0 & 5 & 20 & 77 \\ 0 & 0 & 9 & 54 \\ 0 & 0 & 270 & 1692 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \\ \widehat{F} \end{bmatrix} = \begin{bmatrix} 18 \\ 20 \\ 6 \\ 300 \end{bmatrix}$$

Apply $R_4 \rightarrow R_4 - 30R_3$

$$\begin{bmatrix} 2 & 4 & 13 & 46 \\ 0 & 5 & 20 & 77 \\ 0 & 0 & 9 & 54 \\ 0 & 0 & 0 & 72 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \\ \widehat{F} \end{bmatrix} = \begin{bmatrix} 18 \\ 20 \\ 6 \\ 120 \end{bmatrix}$$

Step-6

From this, get the equations;

Step-7

$2\widehat{C} + 4\widehat{D} + 13\widehat{E} + 46\widehat{F} = 18$ â€¦â€¦ (1)

$5\widehat{D} + 20\widehat{E} + 77\widehat{F} = 20$ â€¦â€¦ (2)

$9\widehat{E} + 54\widehat{F} = 6$ â€¦â€¦ (3)

$72\widehat{F} = 120$ â€¦â€¦ (4)

Step-8

From (4), obtain;

$$\begin{aligned} 72\widehat{F} &= 120 \\ \widehat{F} &= \frac{120}{72} \\ &= \frac{5}{3} \end{aligned}$$

Step-9

Substitute the value of $\widehat{F}$ in (3), and get;

$$\widehat{E} = \frac{6 - 54\left(\frac{5}{3}\right)}{9}$$
$$= \frac{-28}{3}$$

Substitute the value of $\widehat{E}$ and $\widehat{F}$ in (2), and get;

$$\widehat{D} = \frac{20 + \frac{560}{3} - \frac{385}{3}}{5}$$
$$= \frac{-47}{3}$$

Step-10

Substitute, the value of $\widehat{D}, \widehat{E}$ and $\widehat{F}$ in (1), and get;

$$\widehat{C} = \frac{18 + \frac{188}{3} + \frac{364}{3} - \frac{230}{3}}{2}$$
$$= \frac{188}{3}$$

Hence, required cubic equation is $\boxed{b = \frac{188}{3} - \left(\frac{47}{3}\right)t - \left(\frac{28}{3}\right)t^2 + \frac{5}{3}t^3}$.

Step-11

Now, to find the projection matrix p and the error matrix e

Now,

$$p = A\hat{x}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} \frac{188}{3} \\ -\frac{47}{3} \\ -\frac{28}{3} \\ \frac{5}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1\left(\frac{188}{3}\right)+0\left(-\frac{47}{3}\right)+0\left(-\frac{28}{3}\right)+0\left(\frac{5}{3}\right) \\ 1\left(\frac{188}{3}\right)+1\left(-\frac{47}{3}\right)+1\left(-\frac{28}{3}\right)+1\left(\frac{5}{3}\right) \\ 1\left(\frac{188}{3}\right)+3\left(-\frac{47}{3}\right)+9\left(-\frac{28}{3}\right)+27\left(\frac{5}{3}\right) \\ 1\left(\frac{188}{3}\right)+4\left(-\frac{47}{3}\right)+16\left(-\frac{28}{3}\right)+64\left(\frac{5}{3}\right) \end{bmatrix}$$

Step-12

Continuation to the above;

$$= \begin{bmatrix} \frac{188}{3} \\ \frac{118}{3} \\ -\frac{115}{3} \\ -\frac{128}{3} \end{bmatrix}$$

Hence, the projection matrix $\boxed{p = \begin{bmatrix} \frac{188}{3} \\ \frac{118}{3} \\ -\frac{115}{3} \\ -\frac{128}{3} \end{bmatrix}}$

Step-13

Now,

Step-14

$e = b - p$

$$= \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} \frac{188}{3} \\ \frac{118}{3} \\ -\frac{115}{3} \\ -\frac{128}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{188}{3} \\ -\frac{94}{3} \\ \frac{139}{3} \\ \frac{188}{3} \end{bmatrix}$$

Hence $\boxed{e = \begin{bmatrix} -\frac{188}{3} \\ -\frac{94}{3} \\ \frac{139}{3} \\ \frac{188}{3} \end{bmatrix}}$