Step-1

Given matrices are $A=\begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix}$ and $A=\begin{bmatrix}1 & 0 & 1\\ 2 & 2 & 0\\ 0 & 2 & 0\end{bmatrix}$.

We have to factor each matrix A into $PA=LU$.

Step-2

We first reduce A into the upper triangular matrix or the row echelon form U by elementary row operations.

In the non singular case, there is a permutation matrix P that reorders the rows of A to avoid zeroes in the pivot positions.

Step-3

We have $A=\begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix}$

$$R_2 \to R_2 - 2R_1 \Rightarrow \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$$

We apply the same operation on the identity matrix while an elementary row operation on A and pre multiplying an elementary row matrix with A are identical procedures.

Let us consider $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$ on this, we get

$$B=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}$$

Now, $BA=U=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$

Step-4

Observe that B is the elementary matrix whose inverse is also an elementary matrix obtained by the operation $R_2 \to R_2 + 2R_1$ on the identity matrix.

i.e. $B^{-1}=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$

So, $BA = U \Rightarrow A = B^{-1}U$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= LU$$

We easily see that the given matrix A has no zeroes in its pivot positions.

So, we are not needed to multiply any matrix P with it to reorder the pivot positions.

In other words, we multiply with the identity matrix to continue A as it is.

i.e., $I = P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that $PA = LU$.

(Observe that the text book answer differs and there can be many times of row transformations which result in different sets of *P, L,* and *U*â€™s.)

Step-5

We have $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

$$\left.\begin{matrix} R_1 \to R_3 \\ R_2 \to R_1 \\ R_3 \to R_2 \end{matrix}\right\} \Rightarrow B = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

This reordering can be done by the multiple P obtained from the identity matrix I when the same operations are performed.

i.e. $\left.\begin{matrix} R_1 \to R_3 \\ R_2 \to R_1 \\ R_3 \to R_2 \end{matrix}\right\} \Rightarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

We now apply row operations on B to change it into the upper triangular matrix U.

Step-6

Now we reduce B to lower triangular matrix.

We have

$$B = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - 0.5R_1 \Rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$ â€¦â€¦ (3)

The respective elementary matrix is obtained by applying $R_3 \to R_3 + 0.5R_1$ on the identity matrix

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}$$

Step-7

Applying $R_3 \to R_3 + 0.5R_2 \Rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This is the upper triangular matrix *U*.

Step-8

The respective elementary matrix is obtained by applying $R_3 \to R_3 - 0.5R_2$ on the identity matrix, we get $L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.5 & 1 \end{bmatrix}$

The above performance can be written as $L_2^{-1} L_1^{-1} B = U$

$\Rightarrow B = L_1 L_2 U$ where $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix}$ and $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}$

So, $L = L_1 L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & 0.5 & 1 \end{bmatrix}$

Step-9

Now $B = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product of lower and upper triangular matrices.

Further, we have $PA = B$

Thus, we obtained $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence the $PA = LU$ decomposition of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ is

$$\boxed{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$