Step-1

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

Consider the vectors and

The objective is to find the values of c and d such that the above matrices have rank 2.

Step-2

Apply brow operations on A.

Add -1 times the second row to the third row:

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \xrightarrow{R_3:R_3+(-1)R_2} \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & -c & d-2 & 0 \end{bmatrix}$$

Notice that rank of any matrix is the number of non-zero rows after applying row operations.

The above matrix has rank 2 if the last row contains 0's.

All entries in third row are zero's if $-c = 0$ and $d - 2 = 0$

This implies that $c = 0$ and $d = 2$.

Therefore, the rank of the matrix A is 2 for $\boxed{c = 0 \text{ and } d = 2}$.

Step-3

Consider the matrix $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$

Method 1:

Case (1):

Let $c = d = 0$, then $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix} \Rightarrow B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So here rank of B is 0.

Case (2):

Let $c = d$ then $B = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$

Add -1 times the second row to the first row:

$$B = \begin{bmatrix} c & c \\ c & c \end{bmatrix} \xrightarrow{R_2 : R_2 + (-1)R_1} B = \begin{bmatrix} c & c \\ 0 & 0 \end{bmatrix}$$

The number of non-zero rows is 1, so rank of matrix B is 1.

Case (3):

Let $c = -d$ then $B = \begin{bmatrix} c & c \\ -c & -c \end{bmatrix}$

Add 1 times the second row to the first row:

$$B = \begin{bmatrix} c & c \\ c & c \end{bmatrix} \xrightarrow{R_2 : R_2 + R_1} B = \begin{bmatrix} c & c \\ 0 & 0 \end{bmatrix}$$

The number of non-zero rows is 1, so rank of matrix B is 1.

Remaining all cases, the matrix B has rank 2.

Therefore, conclude that the matrix B has rank 2 for except $c = d$ or $c = -d$

Step-4

Method 2:

Consider the matrix $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$

Notice that rank of any matrix is the number of non-zero rows after applying row operations.

Multiply the first row by $\dfrac{1}{c}$

$$B = \begin{bmatrix} c & d \\ d & c \end{bmatrix} \xrightarrow{R_1 : \left(\frac{1}{c}\right)R_1} \begin{bmatrix} 1 & \dfrac{d}{c} \\ d & c \end{bmatrix}$$

Add -d times the first row to the second row.

2

$$\begin{bmatrix} 1 & \dfrac{d}{c} \\ d & c \end{bmatrix} \xrightarrow{R_2:R_2+(-d)R_1} \begin{bmatrix} 1 & \dfrac{d}{c} \\ 0 & c-\dfrac{d^2}{c} \end{bmatrix}$$

The above matrix has rank 2 if $c-\dfrac{d^2}{c} \neq 0$

$$c-\dfrac{d^2}{c} \neq 0$$
$$c^2-d^2 \neq 0$$
$$(c+d)(c-d) \neq 0$$
$$c \neq d \text{ or } c \neq -d$$

Therefore, the matrix has rank 2 if $\boxed{c \neq d \text{ or } c \neq -d}$