## Step-1

A graph consists of a set of vertices or nodes, and a set of edges that connect them. The edge goes from node $j$ to node $k$ , then that row has -1 in column $j$ and + 1 in column $k$ .

The incidence matrix $A$ is $A=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{bmatrix}$ and its transpose is $A^T=\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 0\\ 0 & -1 & -1\end{bmatrix}$

Let the diagonal matrix $C=\begin{bmatrix}C_1 & 0 & 0\\ 0 & C_2 & 0\\ 0 & 0 & C_3\end{bmatrix}$

## Step-2

We need to compute $A^TCA$ . So,

$$A^TCA=\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 0\\ 0 & -1 & -1\end{bmatrix}\begin{bmatrix}C_1 & 0 & 0\\ 0 & C_2 & 0\\ 0 & 0 & C_3\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 1 & 0 & -1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 0\\ 0 & -1 & -1\end{bmatrix}\begin{bmatrix}C_1 & -C_1 & 0\\ 0 & C_2 & -C_2\\ C_3 & 0 & -C_3\end{bmatrix}$$

$$=\begin{bmatrix}C_1+C_3 & -C_1 & -C_3\\ -C_1 & C_1+C_2 & -C_2\\ -C_3 & -C_2 & C_2+C_3\end{bmatrix}$$

Therefore, the matrix $\boxed{A^TCA=\begin{bmatrix}C_1+C_3 & -C_1 & -C_3\\ -C_1 & C_1+C_2 & -C_2\\ -C_3 & -C_2 & C_2+C_3\end{bmatrix}}$.

## Step-3

After removing last column of $A$ and last row of $A^T$ , the 2 by 2 matrix is $B=\begin{bmatrix}C_1+C_3 & -C_1\\ -C_1 & C_1+C_2\end{bmatrix}$

We need to show that this matrix is invertible.

So,

$$B = \begin{bmatrix} C_1 + C_3 & -C_1 \\ -C_1 & C_1 + C_2 \end{bmatrix}$$

Apply $R_2 \to (C_1 + C_3) R_2$,

$$= \begin{bmatrix} C_1 + C_3 & -C_1 \\ -C_1^2 - C_3 C_1 & C_1^2 + C_1 C_2 + C_1 C_3 + C_2 C_3 \end{bmatrix}$$

Apply $R_2 \to R_2 + (C_1) R_1$

$$= \begin{bmatrix} C_1 + C_3 & -C_1 \\ 0 & C_1 C_2 + C_1 C_3 + C_2 C_3 \end{bmatrix}$$

Apply $R_2 \to \dfrac{R_2}{(C_1 + C_3)}$

$$= \begin{bmatrix} C_1 + C_3 & -C_1 \\ 0 & \dfrac{C_1 C_2 + C_1 C_3 + C_2 C_3}{C_1 + C_3} \end{bmatrix}$$

So, the determinant of this matrix is not equals to zero. It is non-singular.

Therefore, the matrix $B = \begin{bmatrix} C_1 + C_3 & -C_1 \\ -C_1 & C_1 + C_2 \end{bmatrix}$ $\boxed{\text{is invertible}}$ and it has $\boxed{C_1 + C_3, \dfrac{C_1 C_3 + C_1 C_2 + C_2 C_3}{C_1 + C_3}}$ pivots.