Step-1

We have to decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution or infinitely many solutions:

$$\begin{aligned} v-w&=2 \\ u-v&=2 \\ u-w&=2 \end{aligned}, \quad \begin{aligned} v-w&=0 \\ u-v&=0 \\ u-w&=0 \end{aligned}, \text{ and } \begin{aligned} v+w&=1 \\ u+v&=1 \\ u+w&=1 \end{aligned}$$

Step-2

Consider the system

$$\begin{aligned} v-w&=2 \\ u-v&=2 \\ u-w&=2 \end{aligned}$$

Converting the given equations into $Ax=b$ form gives

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Step-3

First coefficient is zero, so it needs row exchange hence exchange row 1 and row 2, then

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Subtracting row 1 from row 3, we have

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Step-4

Subtracting row 2 from row 3 gives

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$$

From the above the last equation is 0 = -2,

$\boxed{\text{So the system has no solution, and hence it is singular}}$.

Step-5

Consider the second system

$$\begin{aligned}v-w&=0\\u-v&=0\\u-w&=0\end{aligned}$$

Converting the given equations into $Ax=b$ form gives

$$\begin{pmatrix}0&1&-1\\1&-1&0\\1&0&-1\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Step-6

First coefficient is zero, so it needs row exchange hence exchange row 1 and row 2, then

$$\begin{pmatrix}1&-1&0\\0&1&-1\\1&0&-1\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Subtracting row 1 from row 3 gives

$$\begin{pmatrix}1&-1&0\\0&1&-1\\0&1&-1\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Step-7

Subtracting row 2 from row 3 gives

$$\begin{pmatrix}1&-1&0\\0&1&-1\\0&0&0\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Now from the system, we obtained the last equation as 0 = 0 and no further elimination is possible, so we have two equations in three variables.

$\boxed{\text{Hence the system has infinitely many solutions, and hence it is nonsingular}}$.

Step-8

Consider the third system

$v + w = 1$
$u + v = 1$
$u + w = 1$

Converting the given equations into $Ax = b$ form gives

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Step-9

First coefficient is zero, so it needs row exchange hence exchange row 1 and row 2

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Subtracting row 1 from row 3, we have

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Adding row 2 to row 3 gives

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Step-10

Now the pivot is 2, so subtracting $\frac{1}{2}$ times row 3 from row 2 gives

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}$$

Subtracting row 2 from row 1 and divide row 3 by 2 gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

Hence the system has unique solution which is $\boxed{u=v=w=\frac{1}{2}}$

$\boxed{\text{Hence it is a nonsingular system}}$.