Step-1

Consider the matrix $A=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$.

The objective is to find the Eigen values and Eigen vectors, and also find the diagonal zing matrix S.

To find the Eigen values of A as follows:

The Eigen equation of A is $\det(A-\lambda I)=0$. Where λ is an Eigen value.

Consider,

$$\det(A-\lambda I)=0$$
$$\left|\begin{pmatrix}1 & 0\\ 2 & 3\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right|=0$$
$$\begin{vmatrix}1-\lambda & 0\\ 2 & 3-\lambda\end{vmatrix}=0$$

$$(1-\lambda)(3-\lambda)-0=0$$
$$3-3\lambda-\lambda+\lambda^2=0$$
$$\lambda^2-4\lambda+3=0$$

$$\lambda^2-3\lambda-\lambda+3=0$$
$$\lambda(\lambda-3)-1(\lambda-3)=0$$
$$(\lambda-1)(\lambda-3)=0$$

$\lambda=3,1.$

Thus the eigenvalues of A are $\lambda_1=3,\lambda_2=1.$

Step-2

To find the Eigen vectors corresponding to the Eigen values as follows:

The Eigen vector corresponding to Eigen value $\lambda=3$ as shown below:

By definition, $X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if X is a nontrivial solution of $(A-\lambda I)X=0$

$\begin{bmatrix}1-\lambda & 0\\ 2 & 3-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ â€¦â€¦. (1)

If $\lambda = 3$, then (1) becomes,

$$\begin{bmatrix} 1-3 & 0 \\ 2 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-2x_1 = 0$$
$$x_1 = 0$$

Let $x_2 = t$ be any scalar.

Thus, the eigenvector of *A*, corresponding to $\lambda = 3$ are the nonzero vectors of the form

$$X = \begin{bmatrix} 0 \\ t \end{bmatrix}$$
$$= t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, the eigenvector corresponding to $\lambda = 3$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Step-3

If $\lambda = 1$, then (1) becomes,

$$\begin{bmatrix} 1-1 & 0 \\ 2 & 3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$2x_1 + 2x_2 = 0$$
$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

Let $x_2 = t$ for some scalar *t*, implies $x_1 = -t$.

Thus, the eigenvector of *A*, corresponding to $\lambda = 1$ are the nonzero vectors of the form

$$X = \begin{bmatrix} -t \\ t \end{bmatrix}$$
$$= t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Thus, the eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Step-4

Use the above Eigen vectors to write the Eigen vector matrix as follows:

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Then

$$S^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{0(1)-(-1)(1)}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

Step-5

Consider,

$$S^{-1}AS = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= D$$

Thus, $A = SDS^{-1}$, where $S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

Step-6

Consider the matrix $B = \begin{bmatrix} 7 & 2 \\ -15 & -4 \end{bmatrix}$.

The objective is to find the Eigen values and Eigen vectors, and also find the diagonal zing matrix S.

To find the Eigen values of B as follows:

The Eigen equation of B is $\det(B-\lambda I)=0$. Where λ is an Eigen value.

Consider,

$$\det(B-\lambda I)=0$$
$$\left|\begin{pmatrix} 7 & 2 \\ -15 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$$
$$\begin{vmatrix} 7-\lambda & 2 \\ -15 & -4-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)(-4-\lambda)+30=0$$
$$-28+4\lambda-7\lambda+\lambda^2+30=0$$
$$\lambda^2-3\lambda+2=0$$

$$(\lambda-1)(\lambda-2)=0$$

$\lambda = 1, 2.$

Thus the eigenvalues of B are $\lambda_1 = 2, \lambda_2 = 1.$

Step-7

To find the Eigen vectors corresponding to the Eigen values as follows:

The Eigen vector corresponding to Eigen value $\lambda = 2$ as shown below:

By definition, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of B corresponding to λ if and only if X is a nontrivial solution of $(B-\lambda I)X=0$

$$\begin{bmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ â€¦â€¦. (2)

If $\lambda = 2$, then (2) becomes,

$$\begin{bmatrix} 7-2 & 2 \\ -15 & -4-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 5 & 0 \\ -15 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$5x_1+2x_2=0$$
$$-15x_1-6x_2=0$$

Clearly, $5x_1=-2x_2$.

Let $x_2=t$ be any scalar.

Then $x_1=-\frac{2}{5}x_2$.

Thus, $x_1=-\frac{2}{5}t$

Thus, the eigenvector of B , corresponding to $\lambda=3$ are the nonzero vectors of the form

$$X=\begin{bmatrix} -\frac{2}{5}t \\ t \end{bmatrix}$$
$$=t\begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$$

Thus, the eigenvector corresponding to $\lambda=2$ is $\begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$.

Step-8

If $\lambda=1$, then (2) becomes,

$$\begin{bmatrix} 7-1 & 2 \\ -15 & -4-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 2 \\ -15 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$6x_1+2x_2=0$$
$$-15x_1-5x_2=0$$

Clearly, $6x_1=-2x_2$.

Let $x_2 = t$ be any scalar.

Then, $x_1 = -\frac{2}{6}x_2$.

Thus, $x_1 = -\frac{1}{3}t$

Thus, the eigenvector of B , corresponding to $\lambda = 1$ are the nonzero vectors of the form

$$X = \begin{bmatrix} -\frac{1}{3}t \\ t \end{bmatrix}$$
$$= t\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Thus, the eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$.

Step-9

Use the above Eigen vectors to write the Eigen vector matrix as follows:

$$S = \begin{bmatrix} \frac{-2}{5} & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

Then

$$S^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{\left(-\frac{2}{5}\right)(1) - \left(-\frac{1}{3}\right)(1)}\begin{bmatrix} 1 & \frac{1}{3} \\ -1 & -\frac{2}{5} \end{bmatrix}$$
$$= \frac{-1}{15}\begin{bmatrix} 1 & \frac{1}{3} \\ -1 & -\frac{2}{5} \end{bmatrix}$$

Consider,

$$S^{-1}BS = \frac{-1}{15}\begin{bmatrix} 1 & \frac{1}{3} \\ -1 & -\frac{2}{5} \end{bmatrix}\begin{bmatrix} 7 & 2 \\ -15 & -4 \end{bmatrix}\begin{bmatrix} \frac{-2}{5} & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

$$= \frac{-1}{15}\begin{bmatrix} 2 & \frac{2}{3} \\ -1 & -\frac{2}{5} \end{bmatrix}\begin{bmatrix} \frac{-2}{5} & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= D$$

Thus, $B = SDS^{-1}$, where $S = \begin{bmatrix} \frac{-2}{5} & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.