Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

Step-2

Find the Eigen values and Eigen vectors and write the solution in the form of $S\Lambda S^{-1}$. Then find e^{At} from $Se^{\Lambda t}S^{-1}$. Check e^{At} when $t = 0$.

Step-3

First step is to find the Eigen values and Eigen vectors of matrix A. To calculate the Eigen values do the following calculations;

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \\ \det(A - \lambda I) &= 0 \\ (1-\lambda)(3-\lambda) &= 0 \\ \lambda^2 - 4\lambda + 3 &= 0 \end{aligned}$$

After solving following values are obtained:

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

Step-4

To calculate Eigen vectors do the following calculations:

$$\begin{aligned} (A - \lambda I)x &= 0 \\ \begin{bmatrix} 1-3 & 1 \\ 0 & 3-3 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

On solving, values of y and z corresponding to $\lambda = 3$ are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 1$ is as follows:

$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 1-1 & 1 \\ 0 & 3-1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore Eigen values are:

$$x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step-6

Recall that: $A = S\Lambda S^{-1}$.Therefore,

$$A = \boxed{\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix}}$$

Step-7

Recall that $e^{At} = Se^{\Lambda t}S^{-1}$. Therefore,

$$e^{At}=\begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}\begin{bmatrix}e^{3t} & 0\\ 0 & e^{t}\end{bmatrix}\begin{bmatrix}0 & 1/2\\ 1 & -1/2\end{bmatrix}$$
$$=\begin{bmatrix}e^{3t} & e^{t}\\ 2e^{3t} & 0\end{bmatrix}\begin{bmatrix}0 & 1/2\\ 1 & -1/2\end{bmatrix}$$
$$=\begin{bmatrix}e^{t} & \frac{1}{2}e^{3t}-\frac{1}{2}e^{t}\\ 0 & e^{3t}\end{bmatrix}$$

Step-8

Therefore,

$$e^{At}=\boxed{\begin{bmatrix}e^{t} & \frac{1}{2}e^{3t}-\frac{1}{2}e^{t}\\ 0 & e^{3t}\end{bmatrix}}$$

Step-9

At $t=0$:

$$e^{0}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$=I$$

Therefore, at $t=0$, e^{At} is $\boxed{I}$.