考试科目: 线 性 代 数 精 讲 **开课单位:** 数 学 系

考试时长: 120 分钟 **命题教师:** 线性代数精讲教学团队

题 号	1	2	3	4	5	6	7
分 值	15分	15 分	15 分	15 分	20 分	10 分	10 分

本试卷共 (7) 大题, 满分 (100) 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

This test includes 7 questions. Write ***all your answers*** on the examination book.

Please put away all books, calculators, cell phones and other devices. Please write carefully and clearly in complete sentences. Your explanations are your only representative when your work is being graded.

Unless otherwise noted, vector spaces are over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

本套试卷为 A 卷 Version A

1. (15 points, 3 points each) Label the following statements as **True** or **False**. **Along with your answer, provide an informal proof, counterexample, or other explanation.**

   (a) The empty set is a subspace of every vector space.

   (b) The dimension of $P_n(\mathbb{F})$ is n.

   (c) Let u, v, w be distinct vectors of a vector space V. If u, v, w is a basis for V, then $u + v + w, v + w, w$ is also a basis for V.

   (d) $P_n(\mathbb{F})$ is isomorphic to $P_m(\mathbb{F})$ if and only if $n = m$.

   (e) If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

2. (15 points) Show that the set of solutions, V, to the system of linear equations

$$\begin{cases} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{cases}$$

   is a subspace of $\mathbb{R}^3$. Find a basis for this subspace, V.

3. (15 points) State and prove the **Fundamental Theorem of Linear Maps**.

4. (15 points) Suppose V is a finite dimensional inner product space and $T \in \mathcal{L}(V)$.

   (a) State the definition of **invariant subspace**.

   (b) Let U be a subspace of V. Show that U and $U^\perp$ are invariant under T if and only if $P_U T = T P_U$.

5. (20 points) Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (-3x_1 + 3x_2 - 2x_3,\ -7x_1 + 6x_2 - 3x_3,\ x_1 - x_2 + 2x_3).$$

   (a) Determine the eigenspace of T corresponding to each eigenvalue.

(b) Find the Jordan form and a Jordan basis of T.

(c) Find the minimal polynomial of T.

(d) Compute trace T and det T.

6. (10 points) Suppose T is a linear operator defined on $\mathbb{R}^4$ with $T^2 = -I$.

   (a) Show that the only eigenvalues of $T_{\mathbb{C}}$ are i and $-i$. Where $T_{\mathbb{C}}$ is the complexification of T.

   (b) Show that v is an eigenvector of $T_{\mathbb{C}}$ with respect to i if and only if $\bar{v}$ is an eigenvector of $T_{\mathbb{C}}$ with respect to $-i$, and hence show that there is a basis consisting of complex eigenvectors of $T_{\mathbb{C}}$ of the form $v_1, v_2, \bar{v}_1, \bar{v}_2$.

   (c) Show that there is a basis of $\mathbb{R}^4$ with respect to which T has the following matrix representation

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

7. (10 points) Let T be a linear operator on a finite-dimensional complex vector space V, and let $\lambda_1,\ \lambda_2,\ \cdots, \lambda_k$ be the distinct eigenvalues of T. Let $S : V \to V$ be the mapping defined by

$$S(x) = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k,$$

where, for each i, v_i is the unique vector in $G(\lambda_i, T)$, such that $x = v_1 + v_2 + \cdots + v_k$.

   (a) Prove that S is a diagonalizable linear operator on V.

   (b) Let $N = T - S$. Prove that N is nilpotent and commutes with S, that is $SN = NS$.