Step-1

Def: suppose A is n by n matrix has n linearly independent eigen vectors. If these eigen vectors are the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ.

The eigen values of A are on the diagonal of Λ.

Step-2

Given that $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

To find the eigen values and the respective eigen vectors of this matrix, we consider the characteristic equation of A to be $\det(A-\lambda I)=0$

i.e., $\begin{vmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix}=0$

$\Rightarrow \lambda^2-2\lambda=0$

$\Rightarrow \lambda_1=0, \lambda_2=2$ are the eigen values of A.

Step-3

To get the respective eigen vectors, suppose x_1 is a vector such that it satisfies $(A-\lambda_1 I)x_1=0$

$\Rightarrow \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}t_1\\ t_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

Using the row operation $R_2 \to R_2-R_1$, we get $\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}t_1\\ t_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

Writing the homogeneous equation from this, we get $t_1+t_2=0$

Putting $t_1=1$, we get $t_2=-1$ and thus, $x_1=\begin{pmatrix}t_1\\ t_2\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}$ is the eigen vector corresponding to $\lambda_1=0$

Step-4

Similarly, suppose x_2 is the eigen vector satisfying $(A-\lambda_2 I)x_2=0$

i.e., $\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Consequently, $x_2 = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigen vector corresponding to $\lambda_2 = 2$

Since the eigen vectors of the distinct eigen vectors are linearly independent, the matrix whose columns are these eigen vectors will be non singular.

i.e., $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is the non â€“ singular matrix and $S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Step-5

Now, $S^{-1}AS = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$
$$= \Lambda$$

This is the diagonal matrix whose diagonal entries are the eigen values 0 , 2 of the given matrix.

Step-6

Therefore, $A = S\Lambda S^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Step-7

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

We consider the characteristic equation of A to be $\det(A - \lambda I) = 0$

i.e., $\begin{vmatrix} 2-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 2\lambda = 0$

$\Rightarrow \lambda_1 = 0, \lambda_2 = 2$ are the eigen values of A.

Step-8

To get the respective eigen vectors, suppose x_1 is a vector such that it satisfies $(A - \lambda_1 I)x_1 = 0$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Writing the homogeneous equation from this, we get $2t_1 + t_2 = 0$

Putting $t_1 = 1$, we get $t_2 = -2$ and thus, $x_1 = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is the eigen vector corresponding to $\lambda_1 = 0$

Step-9

Similarly, suppose x_2 is the eigen vector satisfying $(A - \lambda_2 I)x_2 = 0$

Step-10

i.e., $\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Using the row operation $R_2 \to R_2 + 2R_1$, we get $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Writing the homogeneous equations, we get $t_2 = 0$

But t_1 satisfy any scalar value and so, the solution is $t_2 = 0$ and $t_1 = k$

When k = 1, we get $x_2 = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the eigen vector corresponding to $\lambda_2 = 2$

Step-11

Since the eigen vectors of the distinct eigen vectors are linearly independent, the matrix whose columns are these eigen vectors will be non singular.

i.e., $S=\begin{pmatrix}1 & 1\\ -2 & 0\end{pmatrix}$ is the non â€“ singular matrix and $S^{-1}=\frac{1}{2}\begin{pmatrix}0 & -1\\ 2 & 1\end{pmatrix}$

Step-12

Now, $S^{-1}AS=\frac{1}{2}\begin{pmatrix}0 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ -2 & 0\end{pmatrix}$

$$=\frac{1}{2}\begin{pmatrix}0 & 0\\ 4 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ -2 & 0\end{pmatrix}$$
$$=\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}$$
$$=\Lambda$$

This is the diagonal matrix whose diagonal entries are the eigen values 0, 2 of the given matrix.

Step-13

Therefore, $A=S\Lambda S^{-1}$

$$=\begin{pmatrix}1 & 1\\ -2 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}\frac{1}{2}\begin{pmatrix}0 & -1\\ 2 & 1\end{pmatrix}$$