### Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dT} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

### Step-2

Find the general solution of the above differential equation. Also, find the value of T at which the solution u(T) is guaranteed to return to the initial value u(0).

# Step-3

Recall that du/dt = Au has the following solution:

$$u(t) = e^{At}u(0)$$

General solution will be:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + \ldots + c_n e^{\lambda_n t} x_n$$

Here,  $c_1, c_2, \dots, c_n$  are constants.

#### Step-4

Firstly find the Eigen values and Eigen vectors of matrix A.

$$A - \lambda I = \begin{bmatrix} 0 - \lambda & -1 & 0 \\ 1 & 0 - \lambda & -1 \\ 0 & 1 & 0 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(-\lambda)(\lambda^2+1)-\lambda=0$$

$$-\lambda^3 - 2\lambda = 0$$

After solving following values are obtained:

$$\lambda_1 = 0$$

$$\lambda_2 = \sqrt{2}i$$

$$\lambda_3 = -\sqrt{2}i$$

#### Step-5

To calculate Eigen vectors do the following calculations:

$$(A - \lambda_1 I) x = 0$$

$$\begin{bmatrix} 0 - \lambda & -1 & 0 \\ 1 & 0 - \lambda & -1 \\ 0 & 1 & 0 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving values of x, y and z corresponding to  $\lambda = 0$  is as follows:

$$x_{1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

## Step-6

Similarly, Eigen vectors corresponding to Eigen value  $\lambda = \sqrt{2}i$  is as follows:

$$(A - \lambda_1 I) x = 0$$

$$\begin{bmatrix} 0 - \sqrt{2}i & -1 & 0 \\ 1 & 0 - \sqrt{2}i & -1 \\ 0 & 1 & 0 - \sqrt{2}i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\sqrt{2}i & -1 & 0 \\ 1 & -\sqrt{2}i & -1 \\ 0 & 1 & -\sqrt{2}i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

# Step-7

On solving the above matrix equation following values of x, y and z are obtained:

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ \sqrt{2}i \\ 1 \end{bmatrix}$$

Similarly, Eigen vectors corresponding to Eigen value  $\lambda = -\sqrt{2}i$  is as follows:

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{bmatrix}$$

## Step-8

General solution will be:

$$u(T) = c_1 e^{\lambda_1 T} x_1 + \dots + c_n e^{\lambda_n T} x_n$$

$$= c_1 e^{0T} x_1 + c_2 e^{\sqrt{2}iT} x_2 + c_3 e^{\sqrt{2}iT} x_3$$

$$u(T) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{\sqrt{2}iT} \begin{bmatrix} -1 \\ \sqrt{2}i \\ 1 \end{bmatrix} + c_3 e^{-\sqrt{2}iT} \begin{bmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{bmatrix}$$

# Step-9

Therefore at time T general solution is as follows:

$$u(T) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{\sqrt{2}iT} \begin{bmatrix} -1 \\ \sqrt{2}i \\ 1 \end{bmatrix} + c_3 e^{-\sqrt{2}iT} \begin{bmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{bmatrix}$$

#### Step-10

Put T = 0 in the general solution and use  $e^{\lambda \cdot 0} = 1$ :

$$u(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ \sqrt{2}i \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{bmatrix}$$

Therefore, at T = 0 solution u(T) will return to the initial value u(0).