Step-1

Consider the matrix A_1 as shown below:

$$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$$

In this matrix,

$a = 5$
$b = 6$
$c = 7$

Since $a > 0$ but $ac < b^2$, the matrix A_1 has one positive and one negative eigenvalue.

Step-2

Consider the matrix A_2 as shown below:

$$A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$$

In this matrix,

$a = -1$
$b = -2$
$c = -5$

Since $a < 0$ and $ac > b^2$, the matrix A_2 has negative eigenvalues.

Step-3

Consider the matrix A_3 as shown below:

$$A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$$

In this matrix,

$a = 1$
$b = 10$
$c = 100$

Since $a > 0$ and $ac = b^2$, the matrix A_3 has one positive and one zero eigenvalue.

Step-4

Consider the matrix A_4 as shown below:

$$A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$$

In this matrix,

$$a = 1$$
$$b = 10$$
$$c = 101$$

Since $a > 0$ and $ac > b^2$, the matrix A_4 has positive eigenvalues.

Step-5

We want a vector *x*, such that $x^{\mathrm{T}} A_1 x < 0$. Let $x = (x_1, x_2)^{\mathrm{T}}$.

Therefore, we get

$$\begin{aligned} x^{\mathrm{T}} A_1 x &< 0 \\ (x_1, x_2)\begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &< 0 \\ (x_1, x_2)\begin{bmatrix} 5x_1 + 6x_2 \\ 6x_1 + 7x_2 \end{bmatrix} &< 0 \\ 5x_1^2 + 6x_1x_2 + 6x_1x_2 + 7x_2^2 &< 0 \end{aligned}$$

This gives $5x_1^2 + 12x_1x_2 + 7x_2^2 < 0$. Note that the quantities $5x_1^2$ and $7x_2^2$ cannot be negative. Therefore, in order to satisfy this inequality, the necessary condition is that exactly one of the quantities x_1 and x_2 should be negative and the other should be positive.

Fix $x_1 = a$ and let $x_2 = x$. We will vary *x* so that $5a^2 + 12ax + 7x^2 < 0$.

Observe the following:

$$\begin{aligned} 5a^2 + 12ax + 7x^2 &< 0 \\ 7x^2 + 12ax + 5a^2 &< 0 \\ \left(\sqrt{7}x\right)^2 + 2\left(\sqrt{7}\right)\left(\frac{6}{\sqrt{7}}\right)ax + \frac{36a^2}{7} - \frac{a^2}{7} &< 0 \end{aligned}$$

$$\left(\sqrt{7}x\right)^2+2\left(\sqrt{7}\right)\left(\frac{6}{\sqrt{7}}\right)ax+\frac{36a^2}{7}<\frac{a^2}{7}$$

$$\left(x\sqrt{7}+\frac{6a}{\sqrt{7}}\right)^2<\left(\frac{a}{\sqrt{7}}\right)^2$$

Therefore, whenever $x\sqrt{7}+\frac{6a}{\sqrt{7}}<\frac{a}{\sqrt{7}}$, we are through. That is, $x\sqrt{7}<-\frac{5a}{\sqrt{7}}$. This is same as $x<-\frac{5a}{7}$.

Step-6

Therefore, if $x_1=a$, x_2 should be less than $-\frac{5a}{7}$. Let $x_1=7$ and $x_2=-6$. Now observe the following:

$$\begin{aligned} x^{\mathrm{T}}A_1x &= (7,-6)\begin{bmatrix}5 & 6\\ 6 & 7\end{bmatrix}\begin{bmatrix}7\\ -6\end{bmatrix} \\ &= (7,-6)\begin{bmatrix}-1\\ 0\end{bmatrix} \\ &= -7 \end{aligned}$$

Thus, we have obtained $\boxed{x^{\mathrm{T}}A_1x<0}$.