Linear Algebra I Final Examination
Fall 2018 A

Department: Math Class:________

Student ID:____________________Name:____________

Answer all parts of Questions (1)-(11). Total is 100 points.

(1) (12 points, 2 points each) True or false. No need to justify.

(a) The diagonal entries of an $n \times n$ $(n > 1)$ real symmetric positive definite matrix are positive. True

(b) If A is similar to B, then A^2 is similar to B^2. True

(c) If A and B are diagonalizable, so is AB. False

$|A| = |A^T| = -|A|$

(d) If A is a 3×3 skew-symmetric $(A^T = -A)$, then $|A| = 0$. True

(e) If A is negative definite, then all the upper left submatrices A_k of A have negative determinants. False

(f) Let A be an $n \times n$ matrix, then the number of $\boxed{\text{nonzero eigenvalues}}$ of A (counting the multiplicities) is equal to the rank of A. False

$X\ n-r \quad Ax=0$

(2) (9 points, 3 points each) Fill in the blanks.

(a) Let A be a 3×3 real matrix whose column vectors $\alpha_1, \alpha_2, \alpha_3$ are linearly independent. If $A\alpha_1 = \alpha_1 + \alpha_2, A\alpha_2 = \alpha_2 + \alpha_3, A\alpha_3 = \alpha_3 + \alpha_1$, then $|A| =$ <u>2</u>.

$A[\alpha_1\ \alpha_2\ \alpha_3] = [\alpha_1+\alpha_2,\ \alpha_2+\alpha_3\ \ \alpha_3+\alpha_1] = [\alpha_1\ \alpha_2\ \alpha_3]\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}$

(b) If $A \in \mathbb{R}^{3\times 3}$ has eigenvalues $0, 1, 2$, then the eigenvalues of $A(A-I)(A-2I)$ are <u>0, 0, 0</u>

(c) A box has edges from $(0,0,0)$ to $(3,1,1), (1,3,1), (1,1,3)$, then its volume is <u>20</u>.

$$\begin{vmatrix}3&1&1\\1&3&1\\1&1&3\end{vmatrix}$$

(3) (10 points) Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = \lambda_4 = i$

(i) Find all the eigenvalues of A and their associated eigenvectors.

(ii) Is A diagonalizable? Explain why.

(4) (9 points) Let

$$A = \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix}.$$

(i) Verify that A is Hermitian. $\checkmark$ $A^H = \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix} = A$ $\checkmark$

(ii) Find a unitary matrix U that diagonalizes A.

$\lambda_1 = -1,\ \lambda_2 = 6$

$x_1 = \begin{bmatrix} 3+i \\ -2 \end{bmatrix}$ $x_2 = \begin{bmatrix} 3+i \\ 5 \end{bmatrix}$

$U = \begin{bmatrix} \frac{1}{\sqrt{14}}(3+i) & \frac{1}{\sqrt{35}}(3+i) \\ \frac{1}{\sqrt{14}}(-2) & \frac{1}{\sqrt{35}} \times 5 \end{bmatrix}$

(5) (12 points) Let

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(i) Find all the singular values of A.

(ii) Find the singular value decomposition of A, in other words, find orthogonal matrices U and V, such that $A = U\Sigma V^T$.

(6) (8 points) Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 2$

(i) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

(ii) Find A^k, where k is a positive integer.

$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$A^k = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 2^k \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 2^{k-1} & 0 & -2^{k-1} \\ 0 & 2^k & 0 \\ -2^{k-1} & 0 & 2^{k-1} \end{bmatrix}$

$t>0 \quad t^2-1>0$

(7) (8 points) Consider the following quadratic form

$$f(x_1, x_2, x_3, x_4) = t(x_1^2 + x_2^2 + x_3^2) + x_4^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3.$$

(i) Find A, such that $f(x_1, x_2, x_3, x_4) = x^T A x$.

$$A = \begin{bmatrix} t & 1 & 1 & 0 \\ 1 & t & -1 & 0 \\ 1 & -1 & t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) For which t is $f(x_1, x_2, x_3, x_4)$ positive definite?

$t>2$

(8) (10 points) Let N be a normal matrix ($N^H N = N N^H$).

(i) Show that $||Nx|| = ||N^H x||$ for every vector x. ✓

(ii) Deduce that the ith row of N has the same length as the ith column.

Let ith $= e_i$

(iii) If N is upper triangular, then N must be diagonal.

(9) (8 points) Prove the following two statements:

$A = Q \Lambda Q^T \quad = |Q\Lambda Q^T + Q I_n Q^T|$

(i) Suppose A is an $n \times n$ real symmetric positive definite matrix, then $|A + I_n| > 1$.

$= |Q| \cdot \underbrace{|\Lambda + I|}_{>1} \cdot |Q^T| > 1$

SVD !

(ii) Let A be an $n \times n$ matrix, then $A^T A$ is similar to AA^T.

$A = U \Sigma V^T \quad A^T = V \Sigma U^T$ ✓

(10) (6 points) Let A be an $n \times n$ real matrix. If $A^k = O$ for some $\boxed{\text{positive}}$ integer k, then A is called a "nilpotent" matrix. O is the $n \times n$ zero matrix.

$Ax = \lambda x$

(i) Show that all the eigenvalues of a nilpotent matrix must be zero. $\Rightarrow A^k x = \lambda^k x$

$(A^k = 0,\ x \neq 0) \Rightarrow \lambda = 0$

$\Leftrightarrow A = Q \Lambda Q^T$

(ii) Prove that a nonzero nilpotent matrix can not be symmetric.

if $A = A^T$ and $A^k = 0$ $\quad A = Q \underset{0}{\Lambda} Q^T = 0$ contradiction

(11) (8 points) Let A be an $n \times n$ real symmetric positive definite matrix, and $\alpha \in \mathbb{R}^n$ be a nonzero vector. Consider

$$M = \begin{bmatrix} A & \alpha \\ \alpha^T & b \end{bmatrix} = \begin{bmatrix} I & 0 \\ \alpha^T A^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & \alpha \\ 0 & b - \alpha^T A^{-1} \alpha \end{bmatrix}$$

Here b is a real number.

$\det(M) = |A| \cdot |b - \alpha^T A^{-1} \alpha|$

(i) Under what condition on b is M positive definite? $b > \alpha^T A^{-1} \alpha$

(ii) In the case that M is positive semidefinite (not positive definite), find a basis for the nullspace of M, $N(M)$.

$b = \alpha^T A^{-1} \alpha$

$$\left\{ \begin{bmatrix} -A^{-1}\alpha \\ 1 \end{bmatrix} \right\}$$ ✓