Step-1

Let a and b be any two vectors such that the angle between them is θ, then cosine of the angle θ is given by as follows:

$$\cos\theta = \frac{a^T b}{\|a\|\|b\|}.$$

By using this, to get

$$a^T b = \|a\|\|b\|\cos\theta.$$

Step-2

Since $|\cos\theta| \le 1$ for any θ, to get the Schwartzâ€™s Inequality as follows:

$$\begin{aligned} \left|a^T b\right| &= \left|\|a\|\|b\|\cos\theta\right| \\ &= \|a\|\|b\||\cos\theta| \\ &\le \|a\|\|b\| \qquad \text{since } |\cos\theta| \le 1. \end{aligned}$$

So, $\left|a^T b\right| \le \|a\|\|b\|.$

Step-3

Note that the Schwartzâ€™s inequality can be proved, if a and b are unit vectors, as follows:

$$\begin{aligned} \left|a^T b\right| &= \left|\sum a_i b_i\right| \\ &\le \sum |a_i||b_i| \\ &\le \sum \frac{|a_i|^2 + |b_i|^2}{2} \end{aligned}$$

$$= \frac{1}{2} + \frac{1}{2}$$
$$= \|a\|\|b\|$$

Step-4

In this case, notice the step

$$\left|\sum a_i b_i\right| \le \sum |a_i||b_i|.$$

Since $|a_i - b_i|^2 = |a_i|^2 + |b_i|^2 - 2|a_i||b_i|$, to get

$$\frac{|a_i - b_i|^2}{2} + |a_i||b_i| = \frac{|a_i|^2 + |b_i|^2}{2}.$$

So, $\sum |a_i||b_i| \le \sum \frac{|a_i|^2 + |b_i|^2}{2}.$

Step-5

Since $\sum |a_i|^2 = 1, \sum |b_i|^2 = 1,$ to get

$$\begin{aligned}\sum \frac{|a_i|^2 + |b_i|^2}{2} &= \frac{1}{2} + \frac{1}{2} \\ &= 1 \\ &= 1 \cdot 1 \\ &= \|a\|\|b\|\end{aligned}$$

Hence, $|a^T b| \le \|a\|\|b\|.$