Step-1

Let *W* be the space of all polynomials p(x) of degree ≤ 3 .

The objective is to find the basis for W.

Step-2

Since W consists the polynomials of degree ≤ 3 , so all the polynomials in W are the linear combination of the vectors $1, x, x^2, x^3$.

And the vectors $1, x, x^2, x^3$ are linearly independent.

Hence, the basis for W is $[1, x, x^2, x^3]$.

Step-3

Now find the basis for the subspace of space of all polynomials of degree ≤ 3 with p(1) = 0.

Let S be the subspace of W such that p(1) = 0.

Let
$$p(x) = ax^3 + bx^2 + cx + d \in S$$
.

Then

$$p(1) = 0$$

 $a(1)^3 + b(1)^2 + c(1) + d = 0$
 $a+b+c+d=0$

Step-4

Here, b, c, and d are free variables.

Therefore, a = -b - c - d.

Thus, polynomial $p(x) = ax^3 + bx^2 + cx + d$ can be written as,

$$p(x) = ax^{3} + bx^{2} + cx + d$$

$$= (-b - c - d)x^{3} + bx^{2} + cx + d$$

$$= (-1 + x^{3})d + (-1 + x^{2})b + (-1 + x)c$$

$$= (x^{3} - 1)d + (x^{2} - 1)b + (x - 1)c$$

Thus, $p(x) \in S$ can be written as a linear combination of the vectors $(x^3-1),(x^2-1),(x-1)$ and note that the vectors $(x^3-1),(x^2-1),(x-1)$ are linearly independent.

Hence, the basis for the subspace with p(1) = 0 is $[(x^3-1),(x^2-1),(x-1)]$.