2022 *MIT Integration Bee Qualifier Test* 2022年麻省理工积分大赛资格考试题

$$\int\frac{1+\cos x}{x+\sin x}\mathrm{d}x$$

$$\int\frac{1+\cos x}{x+\sin x}\mathrm{d}x=\int\frac{1}{x+\sin x}\mathrm{d}(x+\sin x)=\ln|x+\sin x|+\mathrm{C}$$

$$\int_1^{\sqrt{3}}\frac{\arctan x+\operatorname{arccot}x}{x}\mathrm{d}x$$

令$\arctan x=\theta$, $\operatorname{arccot}x=\phi$, $\tan\theta=\frac{x}{1}$, $\cot\phi=\frac{x}{1}$

画个直角三角形，直角边分别为1和x，那么易得$\theta+\phi=\frac{\pi}{2}$

$$\int_1^{\sqrt{3}}\frac{\arctan x+\operatorname{arccot}x}{x}\mathrm{d}x=\int_1^{\sqrt{3}}\frac{\pi}{2}\cdot\frac{1}{x}\mathrm{d}x=\frac{\pi}{2}\ln\sqrt{3}$$

$$\int_0^{2022}(x^2-\lfloor x\rfloor\lceil x\rceil)\mathrm{d}x$$

$$\int_0^{2022}(x^2-\lfloor x\rfloor\lceil x\rceil)\mathrm{d}x=\int_0^1(x^2-0\cdot1)\mathrm{d}x+\int_1^2(x^2-1\cdot2)\mathrm{d}x+\int_2^3(x^2-2\cdot3)\mathrm{d}x+\cdots+\int_{2021}^{2022}(x^2-2021\cdot2022)\mathrm{d}x$$

$$=\int_0^{2022}x^2\mathrm{d}x-0\cdot1-1\cdot2-2\cdot3-\cdots-2021\cdot2022$$

$$=\frac{1}{3}x^3\Big|_0^{2022}-\sum_{n=1}^{2021}n(n+1)$$

$$=\frac{2022^3}{3}-\sum_{n=1}^{2021}n^2-\sum_{n=1}^{2021}n$$

$$=\frac{2022^3}{3}-\frac{2021(2021+1)(2\cdot2021+1)}{6}-\frac{1+2021}{2}\cdot2021$$

$$=674$$

$$\int\frac{\sinh x}{\cosh x-\sinh x}\mathrm{d}x$$

$$\cosh x=\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2},\ \sinh x=\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2}$$

$$\int\frac{\sinh x}{\cosh x-\sinh x}\mathrm{d}x=\int\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2\mathrm{e}^{-x}}\mathrm{d}x=\int\left(\frac{\mathrm{e}^{2x}}{2}-\frac{1}{2}\right)\mathrm{d}x=\frac{\mathrm{e}^{2x}}{4}-\frac{1}{2}x+\mathrm{C}$$

$$\int\frac{x}{\sqrt{x-1}+\sqrt{x+1}}\mathrm{d}x$$

$$\int\frac{x}{\sqrt{x-1}+\sqrt{x+1}}\mathrm{d}x=\int\frac{x\left(\sqrt{x-1}-\sqrt{x+1}\right)}{-2}\mathrm{d}x=\frac{1}{2}\int x\sqrt{x+1}\,\mathrm{d}x-\frac{1}{2}\int x\sqrt{x-1}\,\mathrm{d}x$$

$$x+1=u,\ x-1=v$$

$$L=\frac{1}{2}\int(u-1)\sqrt{u}\,\mathrm{d}u-\frac{1}{2}\int(v+1)\sqrt{v}\,\mathrm{d}v$$

$$=\frac{1}{2}\int u^{\frac{3}{2}}\mathrm{d}u-\frac{1}{2}\int u^{\frac{1}{2}}\mathrm{d}u-\frac{1}{2}\int v^{\frac{3}{2}}\mathrm{d}v-\frac{1}{2}\int v^{\frac{1}{2}}\mathrm{d}v$$

$$=\frac{(x+1)^{\frac{5}{2}}}{5}-\frac{(x+1)^{\frac{3}{2}}}{3}-\frac{(x-1)^{\frac{5}{2}}}{5}-\frac{(x-1)^{\frac{3}{2}}}{3}+\mathrm{C}$$

$$\int_0^{\pi}\cos(x+\cos x)\mathrm{d}x$$

$$\int_0^{\pi}\cos(x+\cos x)\mathrm{d}x\overset{u=\pi-x}{=\!=\!=}\int_{\pi}^{0}\cos(\pi-u+\cos(\pi-u))\mathrm{d}(-u)=\int_0^{\pi}\cos(\pi-u-\cos u)\mathrm{d}u$$

$$=\int_0^{\pi}\cos(\pi-(u+\cos u))\mathrm{d}u=-\int_0^{\pi}\cos(u+\cos u)\mathrm{d}u=-\int_0^{\pi}\cos(x+\cos x)\mathrm{d}x=0$$

$$\int x^3\sin(x^2)\mathrm{d}x$$

$$\int x^3\sin(x^2)\mathrm{d}x\overset{u=x^2}{=\!=\!=}\frac{1}{2}\int u\sin u\,\mathrm{d}u=\frac{1}{2}\sin u-\frac{1}{2}u\cos u+C=\frac{1}{2}\sin x^2-\frac{1}{2}x^2\cos x^2+\mathrm{C}$$

$$\int\frac{x}{1-x^4}\mathrm{d}x$$

$$\int\frac{x}{1-x^4}\mathrm{d}x\overset{\text{裂项}}{=\!=\!=}\frac{1}{4}\ln(x^2+1)-\frac{1}{4}\ln|x+1|-\frac{1}{4}\ln|x-1|+\mathrm{C}$$

$$\int\frac{1}{\cosh^2x}\mathrm{d}x$$

$$\int\frac{1}{\cosh^2x}\mathrm{d}x=\int\frac{1}{\left(\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2}\right)^2}\mathrm{d}x=\int\frac{4\mathrm{e}^{2x}}{\mathrm{e}^{4x}+2\mathrm{e}^{2x}+1}\mathrm{d}x=\int\frac{4\mathrm{e}^{2x}}{(\mathrm{e}^{2x}+1)^2}\mathrm{d}x$$

$$\overset{\mathrm{e}^{2x}+1=u}{=\!=\!=}\int\frac{2}{u^2}\mathrm{d}u=-\frac{2}{\mathrm{e}^{2x}+1}+\mathrm{C}$$

$$\int_0^1\mathrm{e}^{\mathrm{e}^x}-\mathrm{e}^{\mathrm{e}^x-x}\mathrm{d}x$$

$$\int_0^1\mathrm{e}^{\mathrm{e}^x}-\mathrm{e}^{\mathrm{e}^x-x}\mathrm{d}x=\mathrm{e}^{\mathrm{e}^x-x}\Big|_0^1=\mathrm{e}^{\mathrm{e}-1}-\mathrm{e}$$

$$\lim_{n\to\infty}\int_0^3\sin\left(\frac{\pi}{3}\sin\left(\frac{\pi}{3}\sin\left(\cdots\sin\left(\frac{\pi}{3}x\right)\cdots\right)\right)\right)\mathrm{d}x$$

$$u=\lim_{n\to\infty}\sin\left(\frac{\pi}{3}\sin\left(\frac{\pi}{3}\sin\left(\cdots\sin\left(\frac{\pi}{3}x\right)\cdots\right)\right)\right)$$

因此 $u=\sin\left(\frac{\pi}{3}u\right)$, 即 $\frac{\pi}{3}=\frac{\arcsin u}{u}$, $u=\frac{1}{2}$

$$\lim_{n\to\infty}\int_0^3\sin\left(\frac{\pi}{3}\sin\left(\frac{\pi}{3}\sin\left(\cdots\sin\left(\frac{\pi}{3}x\right)\cdots\right)\right)\right)\mathrm{d}x=\int_0^3\lim_{n\to\infty}\sin\left(\frac{\pi}{3}\sin\left(\frac{\pi}{3}\sin\left(\cdots\sin\left(\frac{\pi}{3}x\right)\cdots\right)\right)\right)\mathrm{d}x=\int_0^3\frac{1}{2}\mathrm{d}x=\frac{3}{2}$$

$$\int_0^1\sqrt{1-\sqrt{x}}\,\mathrm{d}x$$

$$\int_0^1\sqrt{1-\sqrt{x}}\,\mathrm{d}x\overset{u=1-\sqrt{x}}{=\!=\!=}\int_1^0\sqrt{u}\cdot(-2)(1-u)\mathrm{d}u=2\int_0^1u^{\frac{1}{2}}-u^{\frac{3}{2}}\mathrm{d}u=2\cdot\left(\frac{2}{3}u^{\frac{3}{2}}-\frac{2}{5}u^{\frac{5}{2}}\right)\Big|_0^1=\frac{8}{15}$$

$$\int\frac{x^3}{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3}\mathrm{d}x$$

$$\int\frac{x^3}{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3}\mathrm{d}x=6\int\frac{\frac{1}{6}x^3+\left(1+x+\frac{1}{2}x^2\right)-\left(1+x+\frac{1}{2}x^2\right)}{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3}\mathrm{d}x$$

$$=6\int1-\frac{1+x+\frac{1}{2}x^2}{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3}\mathrm{d}x=6\left(x-\ln\left|1+x+\frac{1}{2}x^2+\frac{1}{6}x^3\right|\right)+\mathrm{C}$$

$$\int\sin(x+\sin x)-\sin(x-\sin x)\mathrm{d}x$$

$$\int\sin(x+\sin x)-\sin(x-\sin x)\mathrm{d}x=2\int\cos x\sin(\sin x)\mathrm{d}x\overset{\sin x=u}{=\!=\!=}2\int\frac{\sqrt{1-u^2}\sin u}{\sqrt{1-u^2}}\mathrm{d}u=-2\cos(\sin x)+\mathrm{C}$$

$$\int\tan^4x\sec^3x+\tan^2x\sec^5x\,\mathrm{d}x$$

$$\int\tan^4x\sec^3x+\tan^2x\sec^5x\,\mathrm{d}x=\boxed{\frac{1}{3}\tan^3x\sec^3x}+\mathrm{C}$$

$$\int(1+\ln x)\ln(\ln x)\mathrm{d}x$$

$$\int\overset{(x\ln x)'}{(1+\ln x)}\ln(\ln x)\mathrm{d}x=\ln(\ln x)\cdot x\ln x-\int x\ln x\frac{1}{\ln x}\frac{1}{x}\mathrm{d}x=\ln(\ln x)\cdot x\ln x-x+\mathrm{C}$$

$$\int\frac{1}{1+\sin x}+\frac{1}{1+\cos x}+\frac{1}{1+\tan x}+\frac{1}{1+\cot x}+\frac{1}{1+\sec x}+\frac{1}{1+\csc x}\mathrm{d}x$$

$$L=\int\underbrace{\left(\frac{1}{1+\sin x}+\frac{1}{1+\csc x}\right)}_{=1}+\underbrace{\left(\frac{1}{1+\cos x}+\frac{1}{1+\sec x}\right)}_{=1}+\underbrace{\left(\frac{1}{1+\tan x}+\frac{1}{1+\cot x}\right)}_{=1}\mathrm{d}x=3x+\mathrm{C}$$

$$\int\frac{1}{\sqrt{x-x^2}}\mathrm{d}x$$

$$\int\frac{1}{\sqrt{x-x^2}}\mathrm{d}x=\int\frac{1}{\sqrt{x}\sqrt{1-x}}\mathrm{d}x\overset{\sqrt{x}=u}{=\!=\!=}\int\frac{2}{\sqrt{1-u^2}}\mathrm{d}u=2\arcsin\sqrt{x}+\mathrm{C}$$

$$\int_0^{\frac{1}{2}}\sum_{n=0}^{\infty}\binom{n+3}{n}x^n\mathrm{d}x$$

$$\sum_{n=0}^{\infty}\binom{n+3}{n}x^n=\sum_{n=0}^{\infty}\frac{(n+3)!}{n!\,((n+3)-n)!}x^n=\sum_{n=0}^{\infty}\frac{(n+1)(n+2)(n+3)}{6}x^n$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n,\ \frac{x^3}{1-x}=\sum_{n=0}^{\infty}x^{n+3}$$

$$\sum_{n=0}^{\infty}\frac{(n+1)(n+2)(n+3)}{6}x^n=\frac{1}{6}\left(\sum_{n=0}^{\infty}x^{n+3}\right)'''=\frac{1}{6}\left(\frac{x^3}{1-x}\right)'''$$

$$\int_0^{\frac{1}{2}}\sum_{n=0}^{\infty}\binom{n+3}{n}x^n\mathrm{d}x=\int_0^{\frac{1}{2}}\frac{1}{6}\left(\frac{x^3}{1-x}\right)'''\mathrm{d}x=\frac{1}{6}\left(\frac{x^3}{1-x}\right)''\Big|_0^{\frac{1}{2}}=82$$

$$\int\frac{1}{1+\cos^2x}\mathrm{d}x$$

$$\int\frac{1}{1+\cos^2x}\boxed{\frac{\sec^2x}{\sec^2x}}\mathrm{d}x=\int\frac{\sec^2x}{\tan^2x+2}\mathrm{d}x\overset{u=\tan x}{=\!=\!=}\int\frac{1}{u^2+2}\mathrm{d}u=\frac{\sqrt{2}}{2}\arctan\frac{\tan x}{\sqrt{2}}+\mathrm{C}$$