Step-1

Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$$

A symmetric matrix A is given by

$$A = A^T$$

The entries of a symmetric matrix are symmetric to the main diagonal

Therefore, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$ is a symmetric matrix.

Step-2

Using the Gaussian elimination, we have

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 3 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 3 & 4 & 9 \end{bmatrix} \xrightarrow{-3R_1 + R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{2R_2 + R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -4 \end{bmatrix}$$

Here the pivots are $1, 1, -4$.

Step-3

We know that a symmetric matrix is positive definite, if all pivots (without row exchanges) are positive and it is positive semidefinite, if no pivots are negative.

Since one of the pivots is negative, therefore the symmetric matrix A is $\boxed{\text{indefinite}}$.

Step-4

Consider the following matrix:

$$B=\begin{bmatrix}1&2&0&0\\2&6&-2&0\\0&-2&5&-2\\0&0&-2&3\end{bmatrix}$$

Since entries are symmetric to the main diagonal, so B is symmetric matrix.

Step-5

Using the Gaussian elimination, we have

$$\begin{bmatrix}1&2&0&0\\2&6&-2&0\\0&-2&5&-2\\0&0&-2&3\end{bmatrix}\xrightarrow{-2R_1+R_2}\begin{bmatrix}1&2&0&0\\0&2&-2&0\\0&-2&5&-2\\0&0&-2&3\end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2}\begin{bmatrix}1&2&0&0\\0&1&-1&0\\0&-2&5&-2\\0&0&-2&3\end{bmatrix}$$

$$\xrightarrow{2R_2+R_3}\begin{bmatrix}1&2&0&0\\0&1&-1&0\\0&0&3&-2\\0&0&-2&3\end{bmatrix}$$

Again applying the row operations we have:

$$\begin{bmatrix}1&2&0&0\\0&1&-1&0\\0&0&3&-2\\0&0&-2&3\end{bmatrix}\xrightarrow{\frac{1}{3}R_3}\begin{bmatrix}1&2&0&0\\0&1&-1&0\\0&0&1&\frac{-2}{3}\\0&0&-2&3\end{bmatrix}$$

$$\xrightarrow{2R_3+R_4}\begin{bmatrix}1&2&0&0\\0&1&-1&0\\0&0&1&\frac{-2}{3}\\0&0&0&\frac{5}{3}\end{bmatrix}$$

Here the pivots are $1,1,1,\text{and }\frac{5}{3}$.

We know that a symmetric matrix is positive definite, if all pivots (without row exchanges) are positive and it is positive semidefinite, if no pivots are negative.

Since all pivots is positive, the symmetric matrix B is $\boxed{\text{positive definite}}$.

Step-6

Consider the reduced form of B as

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 0 & \frac{5}{3} \end{bmatrix}$$

Since $C = -B$, so we have

$$C = \begin{bmatrix} -1 & -2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & \frac{2}{3} \\ 0 & 0 & 0 & -\frac{5}{3} \end{bmatrix}$$

We know that a symmetric matrix is negative definite, if all the pivots (without row exchanges) are negative.

Therefore, matrix C is $\boxed{\text{nagative definite}}$.

Step-7

Consider the reduced form of A as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -4 \end{bmatrix}$$

Sinc $D = A^{-1}$, so we have

$$D=\begin{bmatrix}1&2&3\\0&1&-2\\0&0&-4\end{bmatrix}^{-1}$$
$$=\begin{bmatrix}1&-2&\frac{7}{2}\\0&1&-\frac{1}{2}\\0&0&-\frac{1}{4}\end{bmatrix}$$

Here the pivots are $1,1,-\frac{1}{4}$.

Step-8

We know that a symmetric matrix is positive definite, if all pivots (without row exchanges) are positive and it is positive semidefinite, if no pivots are negative.

Since one of the pivots is negative, therefore the symmetric matrix D is $\boxed{\text{indefinite}}$.

Step-9

We have

$$-X^T AX=-\begin{bmatrix}x&y&z\end{bmatrix}\begin{bmatrix}1&2&3\\2&5&4\\3&4&9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$
$$=-x^2-5y^2-9z^2-4xy-6xz-8yz$$

This is the real solution of $-X^T AX=1$.