

## Step-1

As per the definition, we can write  $A \otimes B$  as follows:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix} \end{aligned}$$

## Step-2

Let  $A$  and  $B$  be invertible and let  $A^{-1}$  be as follows:

$$A^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{bmatrix}$$

Therefore, we have

$$\begin{aligned} A^{-1} \otimes B^{-1} &= \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{bmatrix} \otimes B^{-1} \\ &= \begin{bmatrix} a'_{11}B^{-1} & a'_{12}B^{-1} & \cdots & a'_{1n}B^{-1} \\ a'_{21}B^{-1} & a'_{22}B^{-1} & \cdots & a'_{2n}B^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ a'_{n1}B^{-1} & a'_{n2}B^{-1} & \cdots & a'_{nn}B^{-1} \end{bmatrix} \end{aligned}$$

## Step-3

Now consider the following:

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix} \begin{bmatrix} a_{11}'B^{-1} & a_{12}'B^{-1} & \cdots & a_{1n}'B^{-1} \\ a_{21}'B^{-1} & a_{22}'B^{-1} & \cdots & a_{2n}'B^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}'B^{-1} & a_{n2}'B^{-1} & \cdots & a_{nn}'B^{-1} \end{bmatrix}$$

Consider the  $ij^{\text{th}}$  entry of this matrix. The entry will be the sum of the product of the consecutive terms of the  $i^{\text{th}}$  row of  $A \otimes B$  and the  $j^{\text{th}}$  column of  $A^{-1} \otimes B^{-1}$ .

## Step-4

Thus, we have the  $ij^{\text{th}}$  entry as follows:

$$\begin{aligned} (A \otimes B)(A^{-1} \otimes B^{-1})_{ij} &= a_{i1}Ba_{1j}'B^{-1} + a_{i2}Ba_{2j}'B^{-1} + \dots + a_{in}Ba_{nj}'B^{-1} \\ &= a_{i1}a_{1j}'BB^{-1} + a_{i2}a_{2j}'BB^{-1} + \dots + a_{in}a_{nj}'BB^{-1} \\ &= a_{i1}a_{1j}'I + a_{i2}a_{2j}'I + \dots + a_{in}a_{nj}'I \\ &= a_{i1}a_{1j}' + a_{i2}a_{2j}' + \dots + a_{in}a_{nj}' \end{aligned}$$

From the property of the inverse of matrices, the following should be clear:

$$\text{When } i = j, \quad a_{i1}a_{1j}' + a_{i2}a_{2j}' + \dots + a_{in}a_{nj}' = 1$$

$$\text{When } i \neq j, \quad a_{i1}a_{1j}' + a_{i2}a_{2j}' + \dots + a_{in}a_{nj}' = 0.$$

This gives us the following:

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}$$

In the above matrix, each identity matrix is of the order  $n$ . Thus, we can say that  $(A \otimes B)(A^{-1} \otimes B^{-1})$  is itself an identity matrix of the order  $n^2$ .

Therefore,  $\boxed{(A \otimes B)(A^{-1} \otimes B^{-1}) = I_{2D}}$ .