Problem Set 1

Problem 1. *Let*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 11 \end{bmatrix}$$

(a) Find the symmetric factorization of $A = LDL^T$.
(b) Use the Gauss-Jordan method to find A^{-1}.

Problem 2. *Computing high power of matrices is definitely a complicated thing. Suppose we have*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 0 & 4 & -2 \\ -1 & 0 & -2 & 1 \\ -3 & 0 & -6 & 3 \end{bmatrix}$$

Try to compute A^5, B^5, C^5.

Problem Set 2

Problem 1. *Let*

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}$$

(a) Find the complete solution to $Ax = 0$.

(b) Explain why $Ax = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ *is inconsistent.*

(c) Find the complete solution to $Ax = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Problem 2. ***True*** *or* ***False****. Give explanations or find counter examples.*

1. *If* x_p *is a particular solution to* $Ax = b$*, then* x_p *is in the nullspace of* A.
2. *Linear equation systems* $Ax = 0$ *always have a solution.*
3. *The column space and the nullspace of the* 5×3 *rectangular matrix* A *have the same dimension.*

Problem Set 2

Problem 1. *Let*

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}$$

(*a*) *Find the complete solution to* $Ax = 0$.

(*b*) *Explain why* $Ax = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ *is inconsistent.*

(*c*) *Find the complete solution to* $Ax = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Solution 1.

(*a*)

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_3 + 2x_4 + 3x_5 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_2 + 2x_4 + 3x_5 \\ x_3 = -2x_4 - 3x_5 \end{cases}$$

The nullspace solution $x_n = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_4 + 3x_5 \\ x_2 \\ -2x_4 - 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

(*b*)

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 2 \\ 1 & 2 & 2 & 2 & 3 & 2 \\ -1 & -2 & 0 & 2 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

The third row gives $0 = 5$, *so the equation system is inconsistent.*

(*c*)

One column 3 gives b, so a particular solution is $x_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

The complete solution is

$$x = x_p + x_n = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Problem Set 3

Problem 1. *(2020 Fall Midterm - 16 points)Let*

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Please give a basis for each of the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$.

Problem 2. *(2019 Fall Midterm - 14 points)Let*

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}$$

Determine the dimension and also give a basis for each of the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$.

Problem 2. ***True*** *or* ***False****. Give explanations or find counter examples.*

1. *If x_p is a particular solution to $Ax = b$, then x_p is in the nullspace of A.*
2. *Linear equation systems $Ax = 0$ always have a solution.*
3. *The column space and the nullspace of the 5×3 rectangular matrix A have the same dimension.*

Solution 2.

1. ***False****.*

If the right-hand side $b = 0$, that is the case. The nullspace contains all the solutions to linear equation system $Ax = 0$. But if $b \neq 0$, we can not guarantee x_p is in the nullspace of A.

2. ***True****.*

Zero solution! Zero vector is in the nullspace of any matrix A.

3. ***False****.*

The 5×3 zero matrix have the column space of only the origin in $\mathbb{R}^5$ space (0 dimension), while the nullspace contains all the $\mathbb{R}^3$ vectors (3 dimensions).

Problem Set 3

Problem 1. *(2020 Fall Midterm - 16 points)Let*

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Please give a basis for each of the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$.

Solution 1. *Simplify to RREF form firstly.*

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for $C(A)$ (for reference):

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

A basis for $C(A^T)$ (for reference):

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

A basis for $N(A)$ (for reference):

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4 & 6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

A basis for $N(A^T)$ (for reference):

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for $C(A^T)$ (for reference):

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix}$$

A basis for $N(A^T)$ (for reference):

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Problem 2. *(2019 Fall Midterm - 14 points)Let*

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}$$

Determine the dimension and also give a basis for each of the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$.

Solution 2. *Simplify to RREF form firstly.*

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 & -3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $r = 2$, $dim(C(A)) = r = 2$, $dim(C(A^T)) = r = 2$, $dim(N(A)) = n - r = 3$, $dim(N(A^T)) = m - r = 1$.
A basis for $C(A)$ (for reference):

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

A basis for $C(A^T)$ (for reference):

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

A basis for $N(A)$ (for reference):

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 & 0 & 1 & 0 \\ -1 & -2 & 0 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 1 \end{bmatrix}$$

A basis for $N(A^T)$ (for reference):

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for $C(A^T)$ (for reference):

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

Problem Set 4

Problem 1. *(2020 Fall Midterm - 10 points)Let* $E = \{u_1, u_2, u_3\}$ *and* $F = \{v_1, v_2\}$*, where*

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ *by*

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}$$

Find the matrix A *representing* T *with respect to the ordered bases* E *and* F*.*

Problem 2. *(2020 Fall Midterm - 8 points)Three planes Π_1, Π_2, Π_3 are given by the equations*

$$\Pi_1 : x + y + z = 0$$
$$\Pi_2 : 2x - y + 4z = 0$$
$$\Pi_3 : -x + 2y - z = 0$$

geometry

Determine a matrix representative (in the standard basis of $\mathbb{R}^3$) of a linear transformation taking the xy plane to Π_1, the yz plane to Π_2 and the zx plane to Π_3.

$(1,0,0) \rightarrow$ intersection of Π_1 and Π_3

$\times 3$

$$A = \begin{bmatrix} -1 & -5 & -7 \\ 0 & 2 & -2 \\ 1 & 3 & 3 \end{bmatrix}$$

Problem Set 7

Problem 1. *Find the QR decomposition for the matrix*

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$

Problem 2. *Calculate the area of triangle on the plane* $\mathbb{R}^2$ *with vertices* $(2,1),(3,4),(0,5)$ *using determinants. Also calculate the volume of parallelepiped on* $\mathbb{R}^3$ *created by vectors* $(2,1,1),(3,4,1),(0,5,1)$.

Problem 3. *Consider the following matrix* A*:*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

If there exist λ *that makes* $\mathbf{det}(A - \lambda I) = 0$*? Find all of them. (Those are the eigenvalues of matrix* A*.)*

Problem Set 7

Problem 1. *Find the QR decomposition for the matrix*

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$

Solution 1.

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$a_2' = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6/5 \\ -3/5 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} \frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{bmatrix}, q_2 = \begin{bmatrix} \frac{2\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} \end{bmatrix}$$

$${q_1}^T a_1 = \sqrt{5}, {q_1}^T a_2 = -\frac{\sqrt{5}}{5}, {q_2}^T a_2 = \frac{3\sqrt{5}}{5}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix}\begin{bmatrix} \sqrt{5} & -\frac{\sqrt{5}}{5} \\ 0 & \frac{3\sqrt{5}}{5} \end{bmatrix} = QR$$

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{\sqrt{2}}{2} \\ \frac{2}{3} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix} = QR$$

Problem 2. *Calculate the area of triangle on the plane* $\mathbb{R}^2$ *with vertices* $(2, 1), (3, 4), (0, 5)$ *using determinants. Also calculate the volume of parallelepiped on* $\mathbb{R}^3$ *created by vectors* $(2, 1, 1), (3, 4, 1), (0, 5, 1)$.

Solution 2.
Firstly, calculate the area by adding an one column:

$$A = \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5$$

Then, calculate the volume of parallelepiped:

$$V = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 10$$

Problem 3. *Consider the following matrix* A*:*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

If there exist λ *that makes* $\boldsymbol{det}(A - \lambda I) = 0$*? Find all of them. (Those are the eigenvalues of matrix* A*.)*

Solution 3.

$$\begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = \lambda^3(\lambda - 4) = 0$$

$$\lambda_1 = 0, \lambda_2 = 4$$

Problem Set 8

Problem 1. *(Final Exam, Fall 2020, Version A, 16 marks) Compute the nth order determinant:*

$$\det A = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, n \geqslant 2$$

Problem 2. *(Final Exam, Fall 2020, Version B, 10 marks) Compute the nth order determinant:*

$$D_n(x, y) = \begin{vmatrix} x+y & xy & & & & \\ 1 & x+y & xy & & & \\ & 1 & x+y & xy & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & x+y & xy \\ & & & & 1 & x+y \end{vmatrix}, n \geqslant 2$$

Problem 3. *Compute the nth order determinant:*

$$\begin{vmatrix} 1+{x_1}^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & 1+{x_2}^2 & \cdots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \cdots & 1+{x_n}^2 \end{vmatrix}$$

Problem 4. *Compute the nth order determinant:*

$$\det A = \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 \end{vmatrix}_{n\times n}$$

Problem 5. *Compute the determinant:*

$$\begin{vmatrix} a & a^2 & a^3 & a^5 \\ b & b^2 & b^3 & b^5 \\ c & c^2 & c^3 & c^5 \\ d & d^2 & d^3 & d^5 \end{vmatrix}$$

Problem 6. *Compute the determinant:*

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$$

Problem Set 8

Problem 1. *(Final Exam, Fall 2020, Version A, 16 marks) Compute the nth order determinant:*

$$\det A = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, n \geqslant 2$$

Solution 1.
Cofactor expansion on row 2, row 3, ..., row n-1:

$$\det A = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix} = a^{n-2} \begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} = a^n - a^{n-2}$$

Problem 2. *(Final Exam, Fall 2020, Version B, 10 marks) Compute the nth order determinant:*

$$D_n(x,y) = \begin{vmatrix} x+y & xy & & & & \\ 1 & x+y & xy & & & \\ & 1 & x+y & xy & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & x+y & xy \\ & & & & 1 & x+y \end{vmatrix}, n \geqslant 2$$

Solution 2.
Cofactor expansion on row 1, following by cofactor expansion on column 1:

$$D_n(x,y) = (x+y) D_{n-1} - xy \begin{vmatrix} 1 & xy & & & \\ 0 & x+y & xy & & \\ 0 & 1 & x+y & \ddots & \\ 0 & & \ddots & \ddots & xy \\ 0 & & & 1 & x+y \end{vmatrix} = (x+y) D_{n-1} - xy D_{n-2}$$

Check $D_1(x,y) = x+y$ *and* $D_2(x,y) = x^2 + xy + y^2$.
By mathematical induction, $D_n(x,y) = x^n + x^{n-1}y + \ldots + xy^{n-1} + y^n$. *(Process omitted here.)*

Problem 3. *Compute the nth order determinant:*

$$\begin{vmatrix} 1+{x_1}^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & 1+{x_2}^2 & \cdots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \cdots & 1+{x_n}^2 \end{vmatrix}$$

Solution 3.
Add column 1:

$$\det A = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ x_1 & 1+{x_1}^2 & x_1x_2 & \cdots & x_1x_n \\ x_2 & x_2x_1 & 1+{x_2}^2 & \cdots & x_2x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_nx_1 & x_nx_2 & \cdots & 1+{x_n}^2 \end{vmatrix} = \begin{vmatrix} 1 & -x_1 & -x_2 & \cdots & -x_n \\ x_1 & 1 & & & \\ x_2 & & 1 & & \\ \vdots & & & \ddots & \\ x_n & & & & 1 \end{vmatrix} = 1 + \sum_{i=1}^{n} {x_i}^2$$

Problem 4. *Compute the nth order determinant:*

$$\det A = \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 \end{vmatrix}_{n\times n}$$

Solution 4.
Add column 1:

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & \cdots & -1 \\ 1 & -1 & & & & \\ 1 & & -1 & & & \\ 1 & & & -1 & & \\ \vdots & & & & \ddots & \\ 1 & & & & & -1 \end{vmatrix} = \begin{vmatrix} 1-n & & & & & \\ 1 & -1 & & & & \\ 1 & & -1 & & & \\ 1 & & & -1 & & \\ \vdots & & & & \ddots & \\ 1 & & & & & -1 \end{vmatrix} = (-1)^n (1-n)$$

Problem 5. *Compute the determinant:*

$$\begin{vmatrix} a & a^2 & a^3 & a^5 \\ b & b^2 & b^3 & b^5 \\ c & c^2 & c^3 & c^5 \\ d & d^2 & d^3 & d^5 \end{vmatrix}$$

Solution 5.

$$\begin{vmatrix} a & a^2 & a^3 & a^5 \\ b & b^2 & b^3 & b^5 \\ c & c^2 & c^3 & c^5 \\ d & d^2 & d^3 & d^5 \end{vmatrix} = abcd \begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix}$$

Let $S = (d-c)(d-b)(d-a)(c-b)(c-a)(b-a)$*:*

$$\det A = \begin{vmatrix} 1 & a & a^2 & a^3 & a^4 \\ 1 & b & b^2 & b^3 & b^4 \\ 1 & c & c^2 & c^3 & c^4 \\ 1 & d & d^2 & d^3 & d^4 \\ 1 & x & x^2 & x^3 & x^4 \end{vmatrix} = S(x-a)(x-b)(x-c)(x-d) = C_{51} + C_{52}x + C_{53}x^2 + C_{54}x^3 + C_{55}x^4$$

Compare the coefficient of x^3*:*

$$C_{54} = (-a-b-c-d)\,S$$

$$M_{54} = (-1)^{5+4}(-a-b-c-d)\,S = (a+b+c+d)S$$

Therefore the final result is:

$$\begin{vmatrix} a & a^2 & a^3 & a^5 \\ b & b^2 & b^3 & b^5 \\ c & c^2 & c^3 & c^5 \\ d & d^2 & d^3 & d^5 \end{vmatrix} = abcd(a+b+c+d)(d-c)(d-b)(d-a)(c-b)(c-a)(b-a)$$

Problem 6. *Compute the determinant:*

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$$

Solution 6.
Add column 1:

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1+x & 1 & 1 & 1 \\ 1 & 1 & 1-x & 1 & 1 \\ 1 & 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1 & 1-y \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 \\ 1 & x & & & \\ 1 & & -x & & \\ 1 & & & y & \\ 1 & & & & -y \end{vmatrix} = \begin{vmatrix} 1 & & & & \\ 1 & x & & & \\ 1 & & -x & & \\ 1 & & & y & \\ 1 & & & & -y \end{vmatrix} = x^2y^2$$

MA113 Linear Algebra
Peer Supporting Class
Provided by Zhang Ce

Problem Set 10

December 6, 2021

Problem 1. *Suppose that a* 3×3 *real symmetric matrix* A *has the eigenvalues* $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$. *The eigenvectors corresponding to* λ_1, λ_2 *are* $p_1 = (1, 2, 2)^T, p_2 = (2, 1, -2)^T$. *Find the matrix* A.

✓

$A = Q \Lambda Q^T \quad Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$

Problem 2. *Find an orthogonal diagonalizing matrix for the following matrix:*

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & 5-\lambda & -4 \\ -2 & -4 & 5-\lambda \end{vmatrix} = 0$$

Problem 3. *Find a unitary diagonalizing matrix for the following matrix:*

$$A = \begin{bmatrix} 0 & 1-i \\ 1+i & 1 \end{bmatrix}$$

Problem Set 10

Problem 1. *Suppose that a 3×3 real symmetric matrix A has the eigenvalues $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$. The eigenvectors corresponding to λ_1, λ_2 are $p_1 = (1, 2, 2)^T, p_2 = (2, 1, -2)^T$. Find the matrix A.*

Solution 1. *Real symmetric matrix can be written as $A = Q\Lambda Q^T$.*
We can set the diagonalizing matrix to:

$$Q = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$$

As Q is an orthogonal matrix, we can get:

$$\begin{cases} a + 2b + 2c = 0 \\ 2a + b - 2c = 0 \\ a^2 + b^2 + c^2 = 1 \end{cases}, \begin{cases} a = 2/3 \\ b = -2/3 \\ c = 1/3 \end{cases}$$

Q and λ are all known, so

$$A = Q\Lambda Q^T = \begin{bmatrix} -1/3 & 0 & 2/3 \\ 0 & 1/3 & 2/3 \\ 2/3 & 2/3 & 0 \end{bmatrix}$$

Problem 2. *Find an orthogonal diagonalizing matrix for the following matrix:*

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$$

Solution 2.

$$\det(A - \lambda I) = (1 - \lambda)(1 - \lambda)(10 - \lambda) = 0, \lambda_1 = \lambda_2 = 1, \lambda_3 = 10$$

For $\lambda = 1$:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, a_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 10$:

$$\begin{bmatrix} -8 & 2 & -2 \\ 2 & -5 & -4 \\ -2 & -4 & -5 \end{bmatrix} \to \begin{bmatrix} -8 & 2 & -2 \\ 0 & -9/2 & -9.2 \\ 0 & 0 & 0 \end{bmatrix}, a_3 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

Do Gram-Schmidt:

$$a_2' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 1 \\ 4/5 \end{bmatrix}$$

So, the diagonalizing matrix is:

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} & -\frac{1}{3} \\ 0 & \frac{5}{\sqrt{45}} & -\frac{2}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} & \frac{2}{3} \end{bmatrix}$$

(Note that the first 2 columns can be exchanged and the vector in every column can be reversed.)

Problem 3. *Find a unitary diagonalizing matrix for the following matrix:*

$$A = \begin{bmatrix} 0 & 1 - i \\ 1 + i & 1 \end{bmatrix}$$

Solution 3.

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda & 1-i\\ 1+i & 1-\lambda\end{vmatrix}=\lambda^2-\lambda+2=0, \lambda_1=-1, \lambda_2=2$$

For eigenvalue $\lambda=-1$:

$$\begin{bmatrix}1 & 1-i\\ 1+i & 2\end{bmatrix}\to\begin{bmatrix}1 & 1-i\\ 0 & 0\end{bmatrix}, x_1=\begin{bmatrix}-1+i\\ 1\end{bmatrix}, q_1=\begin{bmatrix}\frac{-1+i}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}$$

For eigenvalue $\lambda=2$:

$$\begin{bmatrix}-2 & 1-i\\ 1+i & -1\end{bmatrix}\to\begin{bmatrix}-2 & 1-i\\ 0 & 0\end{bmatrix}, x_2=\begin{bmatrix}\frac{1-i}{2}\\ 1\end{bmatrix}, q_2=\begin{bmatrix}\frac{1-i}{2\sqrt{3/2}}\\ \frac{1}{\sqrt{3/2}}\end{bmatrix}=\begin{bmatrix}\frac{1-i}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\end{bmatrix}$$

$$A=\begin{bmatrix}0 & 1-i\\ 1+i & -1\end{bmatrix}=\begin{bmatrix}\frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}\frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\end{bmatrix}^H=U\Lambda U^H$$

Problem Set 12

Problem 1. *(2020 Fall Final Exam, 12 marks)Consider the quadratic form*

$$f(x_1, x_2, x_3) = (x_1 - x_2 + x_3)^2 + (x_2 + x_3)^2 + (x_1 + ax_3)^2$$

where $a \in \mathbb{R}$ is a parameter.
(a) What are the possible values of a if the quadratic form f is positive definite?
(b) What are the possible values of a if the equation $f(x_1, x_2, x_3) = 0$ has infinitely many solutions.
(c) Let y be a new system of variables and equation $f(x_1, x_2, x_3) = 0$ has infinitely many solutions. Find an invertible linear transformation $y = Px$, such that the quadratic form f has a diagonal form.

Problem 2. *(2019 Fall Final Exam, 15 marks) Consider the quadratic form*

$$f(x_1, x_2, x_3) = -2{x_1}^2 - 4{x_2}^2 - 5{x_3}^2 + 4x_1x_3$$

(a) Find the matrix A for the quadratic form $f(x_1, x_2, x_3)$.
(b) Decide for or against the positive definiteness of A.
(c) Find an orthogonal matrix Q to diagonalize A.
(d) Is there a real solution to the quadratic form $f(x_1, x_2, x_3) = 1$? Explain why.

Problem 3. *(2020 Fall Final Exam, 4 marks) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$. Find all the singular values of A.*

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Solution 1.
(a) If the quadratic form f is positive definite, then $f(x_1, x_2, x_3) = 0$ if and only if x_1, x_2, x_3 all equal to 0. So the following equation system can only have zero solution.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So the matrix must be a full-rank matrix, leading $a \neq 2$.
(b) The above equation system should have infinitely many solutions, which means $a = 2$.
(c) Let $y_1 = x_1 - x_2 + x_3$, $y_2 = x_2 + x_3$, $y_3 = x_1 + 2x_3$, the quadratic form will be transformed to a standard form. The required matrix P:

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

Problem 2. *(2019 Fall Final Exam, 15 marks) Consider the quadratic form*

$$f(x_1, x_2, x_3) = -2x_1{}^2 - 4x_2{}^2 - 5x_3{}^2 + 4x_1x_3$$

(a) Find the matrix A for the quadratic form $f(x_1, x_2, x_3)$.
(b) Decide for or against the positive definiteness of A.
(c) Find an orthogonal matrix Q to diagonalize A.
(d) Is there a real solution to the quadratic form $f(x_1, x_2, x_3) = 1$? Explain why.

Solution 2.
(a)

$$A = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -5 \end{bmatrix}$$

(b) Adding Row 1 to Row 3, giving 3 negative pivots, the matrix is negative definite.
(c)

$$\det(A - \lambda I) = \begin{bmatrix} -2-\lambda & 0 & 2 \\ 0 & -4-\lambda & 0 \\ 2 & 0 & -5-\lambda \end{bmatrix} = -\lambda^3 - 11\lambda^2 - 34\lambda - 24 = -(\lambda+6)(\lambda+4)(\lambda+1)$$

For eigenvalue $\lambda = -1$:

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \Rightarrow a_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow q_1 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$$

For eigenvalue $\lambda = -4$:

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} \Rightarrow a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For eigenvalue $\lambda = -6$:

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow a_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Rightarrow q_3 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ -2/\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

(d) No. Negative definite, $f(x_1, x_2, x_3) \leqslant 0$.

Problem 3. *(2020 Fall Final Exam, 4 marks) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$. Find all the singular values of A.*

Solution 3. *The singular values are $\sqrt{2}$ and 2.*