Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #12

2023/05/11

Name: ______________________________

Student Number: ______________________________

1. Prove or give a counterexample: The set of nilpotent operators on V is a subspace of $\mathcal{L}(V)$.

   证明或给出反例：V 上的幂零算子的集合是 $\mathcal{L}(V)$ 的子空间.

   False!

   $\Rightarrow S, T$ nilpotent

   Consider $V = \mathbf{R}^2$, $S, T \in \mathcal{L}(V)$ satisfy

   $S^2e_2 = Se_1 = 0$ $\quad$ $T^2e_1 = Te_2 = 0$

$$Se_1 = 0, Se_2 = e_1, \quad Te_1 = e_2, Te_2 = 0$$

   then S, T are nilpotent, but $(S+T)e_1 = e_2$,$(S+T)e_2 = e_1$, $S+T$ is invertible.

2. Let V be a vector space on the complex field $\mathbf{C}$, prove that $N \in \mathcal{L}(V)$ is nilpotent if and only if 0 is the only eigenvalue of N.

   设 V 是复数域 $\mathbf{C}$ 上的向量空间，$N \in \mathcal{L}(V)$ 是幂零的当且仅的 0 是 N 仅有的本征值.

   Proof. Let λ be an eigenvalue of N, v is the corresponding eigenvector, i.e. $Nv = \lambda v$. Since N is nilpotent, $0 = N^n = \lambda^n v \Rightarrow \lambda^n = 0 \Rightarrow \lambda = 0$.

   Schur 's Theorem

   Conversely, since $\mathbf{F} = \mathbf{C}$, then there exists a basis $v_1, \cdots, v_n$ of V s.t. $\mathcal{M}(N; v_1, \cdots, v_n)$ is an upper triangular matrix. Since N only has zero eigenvalues, the diagonal entries must be 0, so $\mathcal{M}(N; v_1, \cdots, v_n)^n = 0 \Rightarrow N$ is nilpotent. □

   diagonal entries 0 $\Rightarrow$ nilpotent

   □