Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Initial condition:

$$u(0) = (3,1)$$

Find the specific solution that matches the initial condition. Also, show that it blows up instead of decaying as $t \to \infty$.

Step-2

First step is to find the Eigen values and Eigen vectors of matrix A. To calculate the Eigen values do the following calculations;

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} \\ \det(A - \lambda I) &= 0 \\ (1-\lambda)(1-\lambda) - 1 &= 0 \\ \lambda^2 - 2\lambda &= 0 \end{aligned}$$

After solving following values are obtained:

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

Step-3

Therefore, Eigen values are $0, 2$

Step-4

To calculate Eigen vectors do the following calculations:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix}1-2 & -1\\ -1 & 1-2\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

On solving, values of *y* and *z* corresponding to $\lambda=2$ are as follows:

$$x_1=\begin{bmatrix}y\\ z\end{bmatrix}$$

$$=\begin{bmatrix}1\\ -1\end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda=0$ is as follows:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix}1-0 & -1\\ -1 & 1-0\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

On solving values of *y* and *z* corresponding to $\lambda=0$ is as follows:

$$x_2=\begin{bmatrix}y\\ z\end{bmatrix}$$

$$=\begin{bmatrix}1\\ 1\end{bmatrix}$$

Step-6

Therefore Eigen values are:

$$x_1=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$x_2=\begin{bmatrix}1\\ 1\end{bmatrix}$$

Step-7

Recall that: $e^{At} = Se^{\Lambda t}S^{-1}$

Here, Eigen value matrix is given as follows:

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Step-8

Therefore, the general solution of the differential equation is:

$$\begin{aligned} u(t) &= c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

Here, c_1 and c_2 are constants. Their values are determined by the following values:

$$c = S^{-1}u(0)$$

Step-9

So, the solution for differential equation can be written as follows:

$$\begin{aligned} u(t) &= Se^{\Lambda t}S^{-1}u(0) \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{0t} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{2t} & 1 \\ -e^{2t} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{2t}+1 & -e^{2t}+1 \\ -e^{2t}+1 & e^{2t}+1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2e^{2t}+4 \\ -2e^{2t}+4 \end{bmatrix} \end{aligned}$$

$$u(t) = \begin{bmatrix} e^{2t}+2 \\ -e^{2t}+2 \end{bmatrix}$$

Step-10

Therefore, specific solution of the differential equation is:

$$\boxed{u(t) = \begin{bmatrix} e^{2t} + 2 \\ -e^{2t} + 2 \end{bmatrix}}$$

When $t \to \infty$, e^{2t} becomes ∞. This blows up the solution of u_∞.