Span and Linear Independence(生成和线性无关)

Lecture 3

Dept. of Math.

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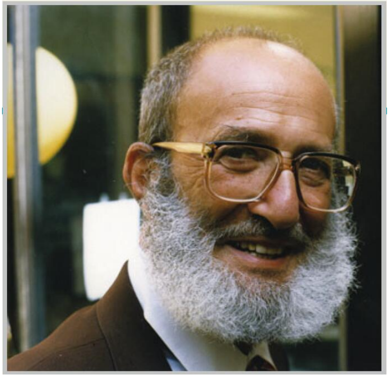
Span and Linear Independence (生成和线性无关)

1. Introduction
2. Linear Combinations and Span
3. Linear Independence
4. Linear Dependence Lemma
5. Homework Assignment 3

Paul Halmos(1916 – 2006)



American mathematician Paul Halmos (1916 – 2006), who in 1942 published the first modern linear algebra book. The title of Halmos's book was the same as the title of this chapter.

Introduction

Let us first consider the following notation:

2.2 Notation ***list of vectors***

$v_1, v_2, \cdots, v_m \Rightarrow$ list

We will usually write lists of vectors without surrounding parentheses.

Adding up scalar multiples of vectors in a list gives what is called a linear combination of the list. Here is the formal definition:

2.3 Definition ***linear combination***

A ***linear combination*** of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

$\operatorname{span}(v_1, v_2, \cdots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_i \in F, v_i \in V\}$

where $a_1, \ldots, a_m \in \mathbf{F}$.

Example

2.4 Example In $\mathbf{F}^3$,

- $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$ because
$$(17, -4, 2) = 6(2, 1, -3) + 5(1, -2, 4).$$
- $(17, -4, 5)$ is not a linear combination of $(2, 1, -3), (1, -2, 4)$ because there do not exist numbers $a_1, a_2 \in \mathbf{F}$ such that
$$(17, -4, 5) = a_1(2, 1, -3) + a_2(1, -2, 4).$$
In other words, the system of equations
$$\begin{aligned} 17 &= 2a_1 + a_2 \\ -4 &= a_1 - 2a_2 \\ 5 &= -3a_1 + 4a_2 \end{aligned}$$
has no solutions (as you should verify).

Span

2.5 Definition ***span***

The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the ***span*** of $v_1, \ldots, v_m$, denoted $\operatorname{span}(v_1, \ldots, v_m)$. In other words,

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbf{F}\}.$$

$v_i \in V$

The span of the empty list ( ) is defined to be $\{0\}$.

Some mathematicians use the term linear span, which means the same as span.

2.6 Example The previous example shows that in $\mathbf{F}^3$,

- $(17, -4, 2) \in \operatorname{span}\big((2, 1, -3), (1, -2, 4)\big)$;
- $(17, -4, 5) \notin \operatorname{span}\big((2, 1, -3), (1, -2, 4)\big)$.

Proposition 2.7

span

2.7 Span is the smallest containing subspace

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Proposition 2.7

span

2.7 Span is the smallest containing subspace

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Proof.

Suppose $v_1, v_2, \cdots, v_m$ is a list of vectors in V. It can be readily checked that $\mathrm{span}(v_1, v_2, \cdots, v_m)$ is a subspace of V and $\mathrm{span}(v_1, v_2, \cdots, v_m)$ contains each v_j. Conversely, because subspaces are closed under scalar multiplication and addition, every subspace of V containing v_j contains $\mathrm{span}(v_1, v_2, \cdots, v_m)$. Thus $\mathrm{span}(v_1, v_2, \cdots, v_m)$ is the smallest subspace of V containing all the vectors $v_1, v_2, \cdots, v_m$. □

spans

$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m \in F^F$

$P_m(F)$ subspace of F^F

$= \operatorname{span}(1, z, z^2, \cdots, z^m)$

finite dimensional

spans

2.8 Definition ***spans***

If $\operatorname{span}(v_1, \ldots, v_m)$ equals V, we say that $v_1, \ldots, v_m$ ***spans*** V.

Finite-Dimensional vector space

2.10 Definition ***finite-dimensional vector space***

A vector space is called ***finite-dimensional*** if some list of vectors in it spans the space.

Polynomials

2.11 **Definition** ***polynomial***, $\mathcal{P}(\mathbf{F})$

- A function $p\colon \mathbf{F} \to \mathbf{F}$ is called a ***polynomial*** with coefficients in $\mathbf{F}$ if there exist $a_0, \ldots, a_m \in \mathbf{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$.

- $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in $\mathbf{F}$.

finite dimension

$\mathscr{P}(\mathbb{F})$ is a subspace of $\mathbb{F}^{\mathbb{F}}$, the vector space of functions from $\mathbb{F}$ to $\mathbb{F}$.

Polynomials

2.12 Definition ***degree of a polynomial***, $\deg p$

- A polynomial $p \in \mathcal{P}(\mathbf{F})$ is said to have ***degree*** m if there exist scalars $a_0, a_1, \ldots, a_m \in \mathbf{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

  for all $z \in \mathbf{F}$. If p has degree m, we write $\deg p = m$.

- The polynomial that is identically 0 is said to have degree $-\infty$.

In the next definition, we use the convention that $-\infty < m$, which means that the polynomial 0 is in $\mathscr{P}_m(\mathbb{F})$.

2.13 Definition $\mathcal{P}_m(\mathbf{F})$

For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.

Linear Combinations and Span

2.15 **Definition** ***infinite-dimensional vector space***

A vector space is called ***infinite-dimensional*** if it is not finite-dimensional.

2.16 **Example** Show that $\mathcal{P}(\mathbf{F})$ is infinite-dimensional.

Solution Consider any list of elements of $\mathcal{P}(\mathbf{F})$. Let m denote the highest degree of the polynomials in this list. Then every polynomial in the span of this list has degree at most m. Thus z^{m+1} is not in the span of our list. Hence no list spans $\mathcal{P}(\mathbf{F})$. Thus $\mathcal{P}(\mathbf{F})$ is infinite-dimensional.

Linear Independence

$\mathbb{C}$ vector space over $\mathbb{R}/\mathbb{C}$

$a_1(1+i)+a_2(1-i)=0$

$\mathbb{R}$: $\begin{cases} a_1=0 \\ a_2=0 \end{cases}$ linearly indep.

2.17 Definition ***linearly independent***

- A list $v_1, \dots, v_m$ of vectors in V is called ***linearly independent*** if the only choice of $a_1, \dots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$.
- The empty list ( ) is also declared to be linearly independent.

The reasoning above shows that $v_1, v_2, \cdots, v_m$ is linearly independent if and only if each vector in span$(v_1, v_2, \cdots, v_m)$ has only one representation as a linear combination of $v_1, v_2, \cdots, v_m$.

$V = \mathrm{span}(v_1, v_2, \cdots, v_n)$

$+\ v_1, \cdots, v_n$ linearly independent

$v \in V \Rightarrow v = c_1 v_1 + \cdots + c_n v_n$

$v = d_1 v_1 + \cdots + d_n v_n$

$c_1 = d_1, \cdots, c_n = d_n \Leftarrow \quad \Rightarrow 0 = (c_1 - d_1)v_1 + \cdots + (c_n - d_n)v_n$

Examples

2.18 Example ***linearly independent lists***

"$\vec{0}$ is linearly dependent"

(a) A list v of one vector $v \in V$ is linearly independent if and only if $v \neq 0$.

(b) A list of two vectors in V is linearly independent if and only if neither vector is a scalar multiple of the other.

(c) $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ is linearly independent in $\mathbf{F}^4$.

(d) The list $1, z, \ldots, z^m$ is linearly independent in $\mathcal{P}(\mathbf{F})$ for each nonnegative integer m.

If some vectors are removed from a linearly independent list, the remaining list is also linearly independent.

Linear Dependence

2.19 **Definition** ***linearly dependent***

- A list of vectors in V is called ***linearly dependent*** if it is not linearly independent.
- In other words, a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

Examples

2.20 **Example** ***linearly dependent lists***

- $(2, 3, 1), (1, -1, 2), (7, 3, 8)$ is linearly dependent in $\mathbf{F}^3$ because
$$2(2, 3, 1) + 3(1, -1, 2) + (-1)(7, 3, 8) = (0, 0, 0).$$
- The list $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$, as you should verify.
- If some vector in a list of vectors in V is a linear combination of the other vectors, then the list is linearly dependent. (Proof: After writing one vector in the list as equal to a linear combination of the other vectors, move that vector to the other side of the equation, where it will be multiplied by -1.)
- Every list of vectors in V containing the 0 vector is linearly dependent. (This is a special case of the previous bullet point.)

Linear Dependence Lemma

The lemma below will often be useful. It states that given a linearly dependent list of vectors, one of the vectors is in the span of the previous ones and furthermore we can throw out that vector without changing the span of the original list.

2.21 Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold:

(a) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$;

(b) if the j^{th} term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

$u \in \operatorname{span}(v_1, \cdots, v_m)$

$\Rightarrow u \in \operatorname{span}(v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_m)$

$u = c_1 v_1 + \cdots + \underbrace{c_j v_j}_{\text{to be replaced}} + \cdots + c_m v_m$

Proof of (a)

Proof.

Because the list $v_1, v_2, \cdots, v_m$ is linearly dependent, there exist numbers $a_1, a_2, \cdots, a_m \in \mathbb{F}$, not all 0, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0.$$

Let j be the largest element of $\{1, 2, \cdots, m\}$ such that $a_j \neq 0$. Then

$$v_j = -\frac{a_1}{a_j} v_1 + - \cdots - \frac{a_{j-1}}{a_j} v_{j-1} \tag{1}$$

proving (a). □

Proof of (b)

To prove (b), suppose $u \in \text{span}(v_1, v_2, \cdots, v_m)$. Then there exist numbers $c_1, c_2, \cdots, c_m \in \mathbb{F}$ such that $u = c_1v_1 + \cdots + c_mv_m$. In this equation, v_j can be replaced with the right side of (1), which shows that u is in the span of the list obtained by removing the jth term from $v_1, v_2, \cdots, v_m$. Thus (b) holds.

2.23

Now we come to a ~~key result~~. It says that no linearly independent list in V is longer than a spanning list in V.

2.23 Length of linearly independent list $\leq$ length of spanning list

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

V finite-dimensional vector space.

$V = \text{span}(\underline{v_1, v_2, \cdots, v_n})$ $\hookrightarrow$ spanning list

$u_1, u_2, \cdots, u_m$ linearly indep. list

Prove: $m \leq n$.

THOUGHT

$\underline{u_1, v_1, v_2, \cdots, v_n}$ spanning list

$\rightarrow$ linearly dependent

$(u_1 \in V, \Rightarrow u_1 = c_1 v_1 + \cdots + c_n v_n)$

$0 = (-1) u_1 + c_1 v_1 + \cdots + c_n v_n$

不全为0 $\rightarrow$ Delete

Add

2.23

Now we come to a key result. It says that no linearly independent list in V is longer than a spanning list in V.

2.23 Length of linearly independent list $\leq$ length of spanning list

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Proof.

Suppose $u_1, \cdots, u_m$ is linearly independent in V. Suppose also that $w_1, w_2, \cdots, w_n$ spans V. We need to prove that $m \leq n$. We do so through the multi-step process described below; note that in each step we add one of the u's and remove one of the w's. (To be continued) □

step 1

- Let B be the list $w_1, \cdots, w_n$, which spans V.
- Thus adjoining any vector in V to this list produces a linearly dependent list (because the newly adjoined vector can be written as a linear combination of the other vectors).
- In particular, the list

$$u_1, w_1, \cdots, w_n$$

  is linearly dependent.
- Thus by the Linear Dependence Lemma (2.21), we can remove one of the w's so that the new list B (of length n) consisting of u_1 and the remaining w's spans V.

step j

- The List B (of length n) from step $j-1$ spans V.
- Thus adjoining any vector to this list produces a linearly dependent list.
- In particular, the list of length $(n+1)$ obtained by adjoining u_j to B, placing it just after $u_1, u_2, \cdots, u_{j-1}$, is linear dependent. By the Linear Dependence Lemma, one of the vectors in this list is in the span of the previous ones, and because $u_1, \cdots, u_j$ is linearly independent, this vector is one of the w's, not one of the u's. We can remove that w from B so that the new list B (of length n) consisting of $u_1, \cdots, u_j$ and the remaining w's spans V.

After step m, we have added all the u's and the process stops. At each step as we add a u to B, the Linear Dependence Lemma implies that there is some w to remove. Thus there are at least as many w's as u's.

Examples

The next two examples show how the result above can be used to show without any computations, that certain lists are not linearly independent and that certain lists do not span a given vector space.

2.24 **Example** Show that the list $(1,2,3), (4,5,8), (9,6,7), (-3,2,8)$ is not linearly independent in $\mathbf{R}^3$.

Solution The list $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbf{R}^3$. Thus no list of length larger than 3 is linearly independent in $\mathbf{R}^3$.

2.25 **Example** Show that the list $(1,2,3,-5), (4,5,8,3), (9,6,7,-1)$ does not span $\mathbf{R}^4$.

Solution The list $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ is linearly independent in $\mathbf{R}^4$. Thus no list of length less than 4 spans $\mathbf{R}^4$.

Finite-dimensional subspaces

Our intuition suggests that every subspace of a finite-dimensional vector space should also be finite-dimensional.

2.26 Finite-dimensional subspaces

Every subspace of a finite-dimensional vector space is finite-dimensional.

U subspace of V. V finite-dimensional $\Rightarrow$ U finite dimensional

$U = \{0\}$ ✓

$U \neq \{0\}$. $u_1 \in U$, $u \neq 0$ $\quad$ $\operatorname{span}(u_1) = U$ ✓

$\operatorname{span}(u_1) \neq U$ $\quad$ $u_2 \in U$, $u_2 \notin \operatorname{span}(u_1)$

Finite-dimensional subspaces

Our intuition suggests that every subspace of a finite-dimensional vector space should also be finite-dimensional.

2.26 Finite-dimensional subspaces

Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof. Suppose V is finite-dimensional and U is a subspace of V. We need to prove that U is finite-dimensional. We do this through the following multi-step construction. Step 1: If $U = \{0\}$, then U is finite-dimensional and we are done. If $U \neq \{0\}$, then choose a nonzero vector $v_1 \in U$. Step 2: If $U = \text{span}(v_1, \cdots, v_{j-1})$, then U is finite-dimensional and we are done. If $U \neq \text{span}(v_1, \cdots, v_{j-1})$, then choose a vector $v_j \in U$ such that $v_j \notin \text{span}(v_1, \cdots, v_{j-1})$.

Proof

After each step, as long as the process continues, we have constructed a list of vectors such that no vector in this list is in the span of the previous vectors.

Thus after each step we have constructed a linearly independent list, by the Linear Dependence Lemma (2.21). This linearly independent list $U \subset V$ can not be longer than any spanning list of V (by 2.23). Thus the process eventually terminates, which means that U is finite-dimensional.

只能做有限步

Homework Assignment 3

2.A: 5, 7, 8, 10, 11, 14, 16, 17.