Step-1

Consider the equations,

$$-u'' = 2,$$
$$u(0) = 0,$$
$$u(1) = 1$$

By using four intervals and two and an extra half- hat functions, with $h = \frac{1}{3}$, the matrix A(2 by 2) is given by,

$$A = 3\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Let

$$f(x) = x$$

Therefore, we get,

$$\begin{aligned} b &= hf(x) \\ &= \left(\frac{1}{3}\right)2 \\ &= \frac{2}{3} \\ &= \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \end{aligned}$$

Step-2

By using $A_{33} = \int \left(V_3'\right)^2 dx$, we get,

$$A_{33} = 3$$

By using $f_3 = \int 2(V_3)dx$, we get,

$$b_{33} = \frac{1}{3}$$

By substituting A, and b into $Ay = b$, we get,

$$3\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} y = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} y = \frac{1}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$y = \frac{1}{9}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

The inverse matrix A is given by,

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Step-3

On substitution, we get,

$$y = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 2+2+1 \\ 2+4+2 \\ 2+4+3 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 5 \\ 8 \\ 9 \end{bmatrix}$$

Step-4

The linear finite element is given by,

$$U(x) = \frac{5}{9}V_1 + \frac{8}{9}V_2 + V_3$$

Thus, the linear finite element is $\boxed{U(x) = \frac{5}{9}V_1 + \frac{8}{9}V_2 + V_3}$.