Step-1

Consider the following skew-Hermitian matrix:

$$K = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$$

Diagonalize matrix K and compute $e^{Kt} = Se^{\Lambda t}S^{-1}$. Verify that e^{Kt} is unitary. Also find the derivative of e^{Kt} at $t = 0$.

Step-2

First step is to find the Eigen values of matrix K. To calculate the Eigen values do the following calculations:

$$K - \lambda I = \begin{bmatrix} i-\lambda & i \\ i & i-\lambda \end{bmatrix}$$

$$\det(K - \lambda I) = 0$$

$$(i-\lambda)^2 + 1 = 0$$

$$(\lambda^2 - 2\lambda i) = 0$$

After solving following values are obtained:

$$\lambda_1 = 2i$$

$$\lambda_2 = 0$$

Step-3

To calculate Eigen vectors do the following calculations:

$$(K - \lambda I)x = 0$$

$$\begin{bmatrix} i-\lambda & i \\ i & i-\lambda \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & i \\ i & -i \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda = 2i$ are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Step-4

Similarly, Eigen vector corresponding to Eigen value $\lambda = 0$ is as follows:

$$(K - \lambda I)x = 0$$

$$\begin{bmatrix} i-\lambda & i \\ i & i-\lambda \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & i \\ i & i \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-5

Therefore, diagonalisation of a matrix can be done as below:

$$K = S\Lambda S^{-1}$$
$$= \boxed{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2i & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$$

Step-6

Now, compute the following:

$$e^{Kt} = Se^{\Lambda t}S^{-1}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{2it} & 0 \\ 0 & e^{0t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{2it} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} e^{2it}+1 & e^{2it}-1 \\ e^{2it}-1 & e^{2it}+1 \end{bmatrix}$$

Step-7

Therefore,

$$\boxed{e^{Kt}=\frac{1}{2}\begin{bmatrix} e^{2it}+1 & e^{2it}-1 \\ e^{2it}-1 & e^{2it}+1 \end{bmatrix}}$$

Step-8

Recall that a matrix U is unitary if $U^H U = I$. To verify that e^{Kt} is unitary, do the following calculations:

$$\begin{aligned}
\left(e^{Kt}\right)^H e^{Kt} &= \frac{1}{2}\begin{bmatrix} e^{-2it}+1 & e^{-2it}-1 \\ e^{-2it}-1 & e^{-2it}+1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} e^{2it}+1 & e^{2it}-1 \\ e^{2it}-1 & e^{2it}+1 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} e^{-2it}+1 & e^{-2it}-1 \\ e^{-2it}-1 & e^{-2it}+1 \end{bmatrix}\begin{bmatrix} e^{2it}+1 & e^{2it}-1 \\ e^{2it}-1 & e^{2it}+1 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\
&= I
\end{aligned}$$

Therefore, e^{Kt} is unitary.

Step-9

Derivative of e^{Kt} at $t=0$ will be:

$$\begin{aligned}
e^{Kt} &= \frac{1}{2}\begin{bmatrix} e^{2it}+1 & e^{2it}-1 \\ e^{2it}-1 & e^{2it}+1 \end{bmatrix} \\
\frac{de^{Kt}}{dt} &= \frac{1}{2}\begin{bmatrix} 2ie^{2it} & 2ie^{2it} \\ 2ie^{2it} & 2ie^{2it} \end{bmatrix} \\
\left(\frac{de^{Kt}}{dt}\right)_{t=0} &= \begin{bmatrix} i & i \\ i & i \end{bmatrix} \\
&= K
\end{aligned}$$

Therefore, derivative of e^{Kt} at $t=0$ is K matrix.