Step-1

Consider the matrix:

$$A=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Thus, matrix D is given by,

$$D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We know that,

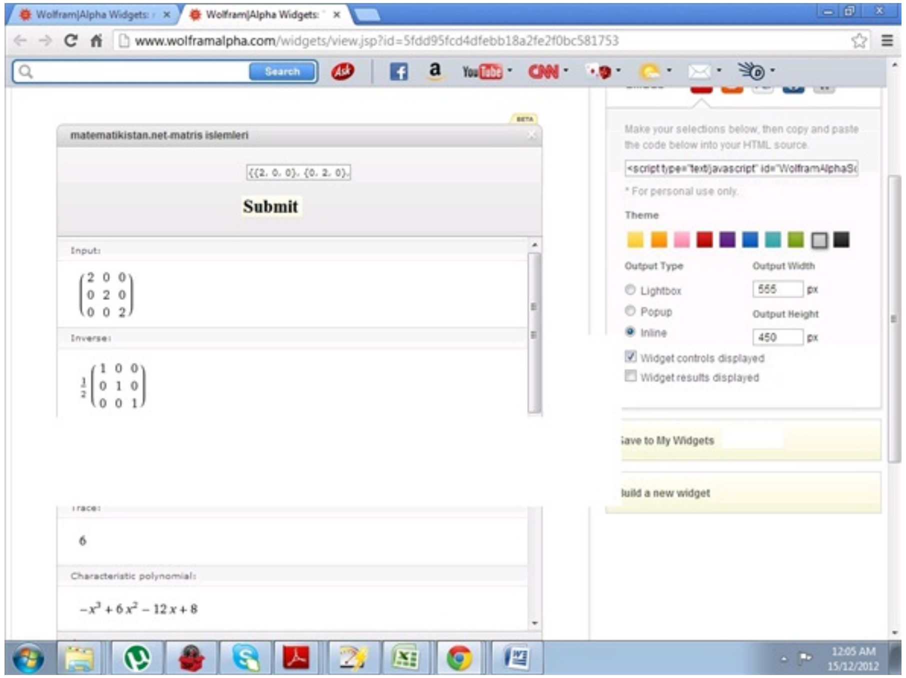
$$A=D+L+U$$

Therefore, we get,

$$\begin{aligned} -L-U &= D-A \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}-\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Step-2

By using matrix calculator (the screenshot is given below), the inverse of D is given by,



Therefore,

$$D^{-1}=\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}$$

Step-3

By multiplying D^{-1} and â€“ *L* â€“ *U*, we get,

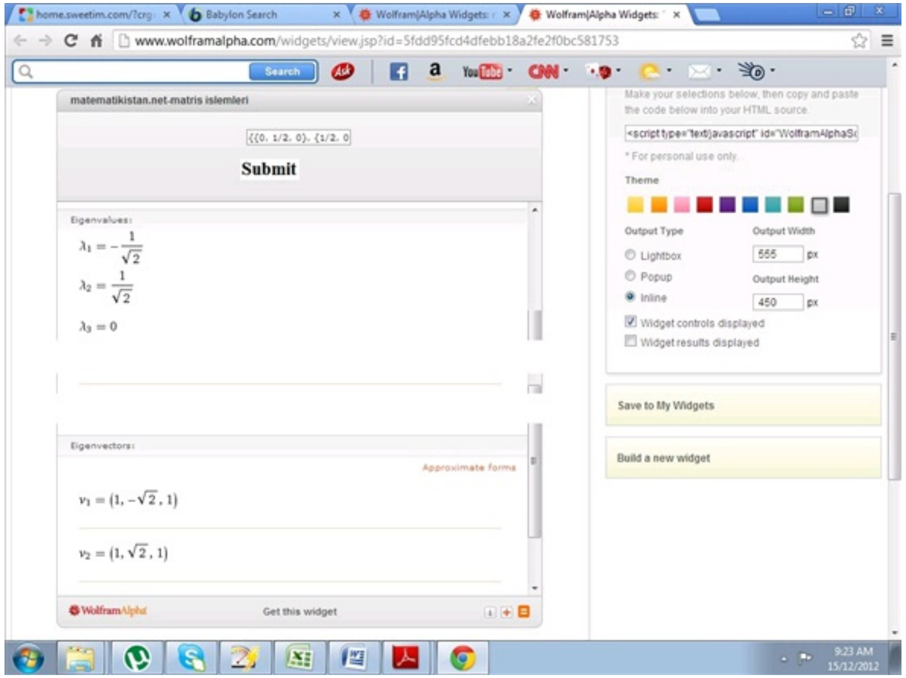
$$D^{-1}(-L-U)=\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Step-4

By using matrix calculator (the screenshot is given below), the eigenvalues of $D^{-1}(-L-U)D$ are given by,

Step-5



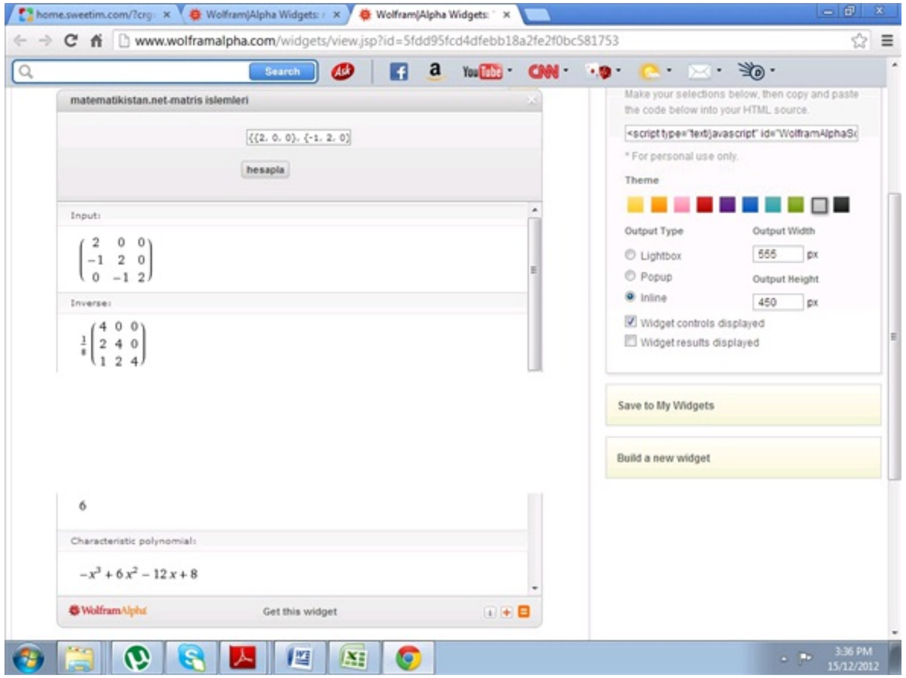
Similarly,

$D+L=A-U$

$$=\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}-\begin{bmatrix}0 & -1 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 0 & 0\\ -1 & 2 & 0\\ 0 & -1 & 2\end{bmatrix}$$

By using matrix calculator (the screenshot is given below), the inverse of $D+L$ is given by,



Step-6

Thus,

$$(D+L)^{-1}=\frac{1}{8}\begin{bmatrix}4 & 0 & 0\\ 2 & 4 & 0\\ 1 & 2 & 4\end{bmatrix}$$

By multiplying $(D+U)^{-1}$ and â€“ U, we get,

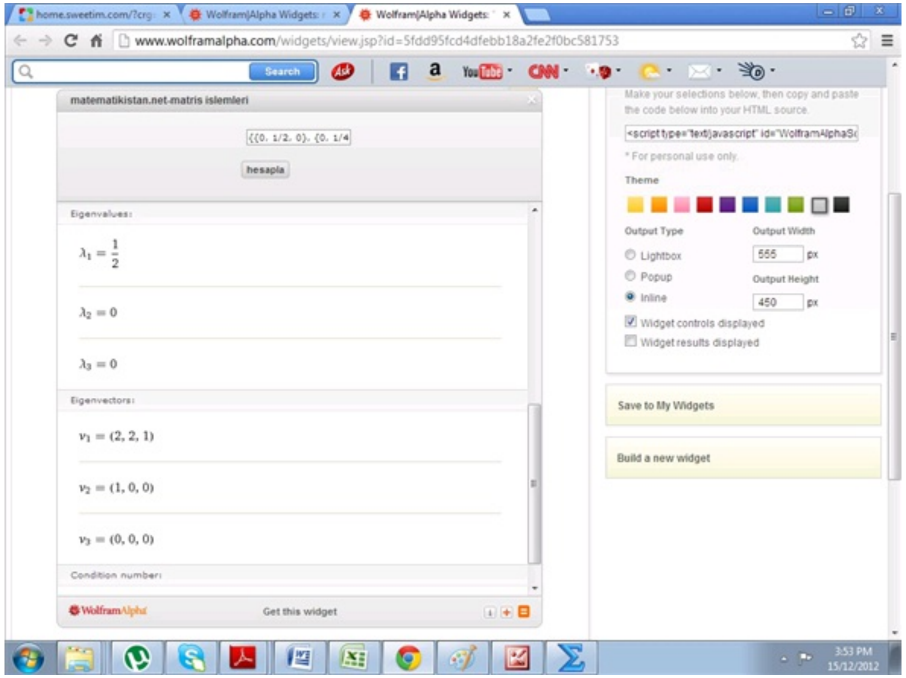
$$(D+L)^{-1}(-U)=\frac{1}{8}\begin{bmatrix}4 & 0 & 0\\ 2 & 4 & 0\\ 1 & 2 & 4\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix}0 & 4 & 0\\ 0 & 2 & 4\\ 0 & 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}0 & \frac{1}{2} & 0\\ 0 & \frac{1}{4} & \frac{1}{2}\\ 0 & \frac{1}{8} & \frac{1}{4}\end{bmatrix}$$

Step-7

By using matrix calculator (the screenshot is given below), the eigenvalues of $(D+U)^{-1}(-U)$ are given by,



The maximum eigenvalue is given by,

$$\mu_{\max}=\frac{1}{\sqrt{2}}$$

By substituting $\mu_{\max} = \frac{1}{\sqrt{2}}$ into $\omega_{opt} = \frac{2\left(1-\sqrt{1-\mu_{\max}^2}\right)}{\mu_{\max}^2}$, we get,

$$\begin{aligned}\omega_{opt} &= \frac{2\left(1-\sqrt{1-\mu_{\max}^2}\right)}{\mu_{\max}^2} \\ &= \frac{2\left(1-\sqrt{1-\left(\frac{1}{2}\right)}\right)}{\left(\frac{1}{\sqrt{2}}\right)^2}\end{aligned}$$

On simplification, we get,

$$\begin{aligned}\omega_{opt} &= 4\left(1-\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\right) \\ &= 4\left(1-\frac{\sqrt{2}}{2}\right) \\ &= 2\left(2-\sqrt{2}\right) \\ &= 4-2\sqrt{2}\end{aligned}$$

Step-8

The $\lambda_{\max}$ is given by,

$$\begin{aligned}\lambda_{\max} &= \omega_{opt} - 1 \\ &= 4-2\sqrt{2}-1 \\ &= 3-2\sqrt{2} \\ &\approx 0.2\end{aligned}$$

Step-9

Thus, $\boxed{\omega_{opt} = 4-2\sqrt{2}}$ and $\boxed{\lambda_{\max} = 3-2\sqrt{2} \approx 0.2}$.