Step-1

(a)

The objective is to determine whether the following statement is true or false.

â€œIf A is Hermitian, then $A+iI$ is invertibleâ€.

If A is Hermitian, then $A^H = A$.

Consider the following matrix:

$$A = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

Here,

$$A^H = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \text{ Since } A^H = \left(\overline{A}\right)^T$$

$$= A$$

Therefore, the matrix A is Hermitian.

Step-2

Matrix $A+iI$ is as follows:

$$\begin{aligned} A+iI &= \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} + i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \\ &= \begin{bmatrix} 1+i & -i \\ i & 1+i \end{bmatrix} \end{aligned}$$

This matrix will be invertible if determinant is non-zero. So, calculate the determinant:

$$\begin{aligned} \det(A+iI) &= (1+i)^2 + i^2 \\ &= 1+i^2+2i+i^2 \qquad \text{Since } i^2 = -1 \\ &= -1+2i \\ &\neq 0 \end{aligned}$$

The eigenvalues of Hermitian matrix are real, and the eigenvalues of skew-Hermitian matrix is purely imaginary or 0.

Since, iI is a skew-Hermitian matrix, so the matrix $A+iI$ is sum of Hermitian and skew-Hermitian matrices, which is a square matrix.

Inverse exist if, eigenvalues are non-zero.

Therefore, if A is Hermitian, then $A+iI$ is invertible is true.

Step-3

(b)

The objective is to determine whether the following statement is true or false.

â€œIf Q is Orthogonal, then $Q+\frac{1}{2}I$ is invertibleâ€.

Consider the following matrix:

$$Q=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Orthogonal: If $QQ^T=I$, then Q is said to be orthogonal matrix.

$$\begin{aligned} QQ^T &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

Therefore, matrix Q is orthogonal matrix.

Step-4

Matrix $Q+\frac{1}{2}I$ is as follows:

$$Q+\frac{1}{2}I=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$
$$=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}+\begin{bmatrix}1/2&0&0\\0&1/2&0\\0&0&1/2\end{bmatrix}$$
$$=\begin{bmatrix}1/2&1&0\\0&1/2&1\\1&0&1/2\end{bmatrix}$$

This matrix will be invertible if determinant is non-zero.

So, calculate the determinant:

$$\det\left(Q+\frac{1}{2}I\right)=\frac{1}{2}\left(\frac{1}{4}\right)-1(-1)+0$$
$$=\frac{1}{8}+1$$
$$=\frac{9}{8}$$
$$\neq 0$$

Since, every eigenvalue of unitary matrix has absolute value 1.

Inverse not exist if, at least one of the eigenvalue is 0.

Therefore, if Q is Orthogonal, then $Q+1/2\,I$ is invertible is true.

(c)

The objective is to determine whether the following statement is true or false.

â€œIf A is real then $A+iI$ is invertibleâ€.

Consider the matrix,

$$A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

As all the elements of matrix A is real elements, so A is real.

Step-5

Matrix $A+iI$ is as follows:

$$
\begin{aligned}
A+iI &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \\
&= \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}
\end{aligned}
$$

This matrix will be invertible if determinant is non-zero.

So, calculate the determinant:

$$
\begin{aligned}
\det(A+iI) &= (i)^2 + 1 \\
&= -1+1 \\
&= 0
\end{aligned}
$$

Therefore, if A is real then $A+iI$ will be invertible is, false.