Step-1

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The 3 ×3 Hilbert matrix is $H=\begin{pmatrix}1 & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5}\end{pmatrix}$

To compute the inverse of this matrix, we consider the matrix $\left[H|I\right]$ where I is the identity matrix of order 3 and apply the same row operations on either matrices H and I and reduce it to $\left[I|G\right]$

We observe that H is reduced to I and on the other side of the bar, I is changed to G.

Therefore, G is nothing but the inverse matrix of H.

$$\left[H|I\right]=\left[\begin{array}{ccc|ccc}1 & 1/2 & 1/3 & 1 & 0 & 0\\ 1/2 & 1/3 & 1/4 & 0 & 1 & 0\\ 1/3 & 1/4 & 1/5 & 0 & 0 & 1\end{array}\right]$$

Step-2

We apply the elementary row operations on this matrix as

$$R_2\to R_2-\frac{1}{2}R_1;R_3\to R_3-\frac{1}{3}R_1\Rightarrow\left[\begin{array}{ccc|ccc}1 & 1/2 & 1/3 & 1 & 0 & 0\\ 0 & 1/12 & 1/12 & -1/2 & 1 & 0\\ 0 & 1/12 & 4/45 & -1/3 & 0 & 1\end{array}\right]$$

$$R_3\to R_3-R_2\Rightarrow\left[\begin{array}{ccc|ccc}1 & 1/2 & 1/3 & 1 & 0 & 0\\ 0 & 1/12 & 1/12 & -1/2 & 1 & 0\\ 0 & 0 & 1/180 & 1/6 & -1 & 1\end{array}\right]$$

$$R_3\to R_3(180),R_2\to R_2(12)\Rightarrow\left[\begin{array}{ccc|ccc}1 & 1/2 & 1/3 & 1 & 0 & 0\\ 0 & 1 & 1 & -6 & 12 & 0\\ 0 & 0 & 1 & 30 & -180 & 180\end{array}\right]$$

$$R_2\to R_2-R_3;R_1\to R_1-\frac{1}{3}R_3\Rightarrow\left[\begin{array}{ccc|ccc}1 & 1/2 & 0 & -9 & 60 & -60\\ 0 & 1 & 0 & -36 & 192 & -180\\ 0 & 0 & 1 & 30 & -180 & 180\end{array}\right]$$

$$R_1 \to R_1 - \frac{1}{2}R_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right]$$

$$\boxed{H^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}}$$ â€¦â€¦ (1)

Step-3

On the other hand, we adjust the entries of the Hilbert matrix to three decimals and proceed as above again.

$$H = \begin{bmatrix} 1 & 0.5 & 0.333 \\ 0.5 & 0.333 & 0.25 \\ 0.333 & 0.25 & 0.2 \end{bmatrix}$$

We write
$$\left[H|I\right] = \left[\begin{array}{ccc|ccc} 1 & 0.5 & 0.333 & 1 & 0 & 0 \\ 0.5 & 0.333 & 0.25 & 0 & 1 & 0 \\ 0.333 & 0.25 & 0.2 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - 0.5R_1; R_3 \to R_3 - 0.333R_1 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0.5 & 0.333 & 1 & 0 & 0 \\ 0 & 0.083 & 0.139 & -0.5 & 1 & 0 \\ 0 & 0.084 & 0.089 & -0.333 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3(0.083) - R_2(0.084) \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0.5 & 0.333 & 1 & 0 & 0 \\ 0 & 0.083 & 0.139 & -0.5 & 1 & 0 \\ 0 & 0 & -0.004 & 0.014 & -0.084 & 0.083 \end{array}\right]$$

Step-4

$$R_3(-250), R_2(12.048) \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0.5 & 0.333 & 1 & 0 & 0 \\ 0 & 1 & 1.675 & -6.024 & 12.048 & 0 \\ 0 & 0 & 1 & -3.5 & 21 & 20.75 \end{array}\right]$$

$$R_2 \to R_2 - 1.675R_3; R_1 \to R_1 - 0.333R_3 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0.5 & 0 & -1.166 & -6.993 & 6.91 \\ 0 & 1 & 0 & -39.508 & 210.185 & -196.951 \\ 0 & 0 & 1 & 33.283 & -196.951 & 195.771 \end{array}\right]$$

$$R_1 \to R_1 - \frac{1}{2}R_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9.671 & -39.508 & 33.283 \\ 0 & 1 & 0 & -39.508 & 210.185 & -196.951 \\ 0 & 0 & 1 & 33.283 & -196.951 & 195.771 \end{array}\right]$$

Thus, we have reduced the Hilbert matrix and identity matrix into the identity and inverse matrices respectively.

Therefore, $\boxed{H^{-1} = \begin{bmatrix} 9.671 & -39.508 & 33.283 \\ -39.508 & 210.185 & -196.951 \\ 33.283 & -196.951 & 195.771 \end{bmatrix}}$ â€¦â€¦ (2)

We see that the respective entries of (1) and (2) are largely different and thus, the Hilbert matrix is ill conditioned.