

Step-1

Properties of the determinant

1. The determinant of the identity matrix is 1
2. The determinant changes sign when two rows are exchanged
3. The determinant depends linearly on the first row.
4. If two rows of A are equal, then $\det A = 0$
5. Subtracting a multiple of one row from another row leaves the same determinant.
6. If A has a row of zeros, then $\det A = 0$
7. If A is triangular, then $\det A$ is the product $a_{11}a_{22}a_{33}\dots a_{nn}$ of the diagonal entries.
8. If A is singular, then $\det A = 0$. If A is invertible, then $\det A \neq 0$.
9. The determinant of AB is product of $\det A$ times $\det B$
10. The transpose of A has the same determinant as A itself; $\det A^T = \det A$

Step-2

a) We have

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} 0 & 0 & b \\ 0 & a & 0 \\ c & 0 & 0 \end{vmatrix} \text{ interchanging first and second rows}$$

$$= - \begin{vmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{vmatrix} \text{ interchanging first and third rows}$$

Step-3

Therefore

$$|A| = \begin{vmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{vmatrix}$$
$$= \boxed{abc}$$

Step-4

$$B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix}$$

b) Given

Now

$$\det B = \begin{vmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{vmatrix}$$
$$= - \begin{vmatrix} 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{vmatrix} \text{interchanging first and second rows}$$

Step-5

On solving

$$= \begin{vmatrix} 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{vmatrix} \text{interchanging fourth and third rows}$$
$$= - \begin{vmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{vmatrix} \text{interchanging first and third rows}$$

$$\det B = -(d)(a)(b)(c)$$

$$= \boxed{-abcd}$$

Step-6

iii) We have

$$c = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}$$

$$\det c = \begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} a & a & a \\ a & b & b \\ 0 & 0 & c-b \end{vmatrix} \text{ Adding } -1 \text{ time second row to the third row}$$

Step-7

$$= \begin{vmatrix} a & a & a \\ a & b-a & b-a \\ 0 & 0 & c-b \end{vmatrix} \text{ Adding } -1 \text{ time the first row to the second row}$$

$$= U$$

$$\det C = \det U$$

$$= a(b-a)(c-b)$$

$$= \boxed{a(a-b)(b-c)}$$