

## Step-1

Given that add the extra column  $b$  and reduce  $A$  to echelon form, A combination of the rows  $A$  has produced the zero row.

$$[A \ b] = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}$$

We need to find the combination, need to find the null space vector and left null space vector.

Let  $R_1, R_2, R_3$  be the three rows of A and apply the row operation  $R_3 \rightarrow R_3 - 2R_2 + R_1$

Then a combination of the rows of  $A$  has produced the zero rows and the rows  $R_1, R_2, R_3$  are linearly dependent.

The combination is Row3-2(Row2)+Row1=zero row

Therefore, the combination is  $\boxed{\text{Row3}-2(\text{Row2})+\text{Row1}=\text{zero row}}$ .

Therefore, the vectors  $\boxed{C\{(1, -2, 1)\}}$  are in null space.

## Step-2

In other way, we write the homogeneous equations using this.

$$Ax = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply  $R_2 \rightarrow R_2 - 4R_1$  and  $R_3 \rightarrow R_3 - 2R_2 + R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 + 3x_3 = 0$$

$$-3x_2 - 6x_3 = 0$$

$$\Rightarrow x_2 = -2x_3$$

$$x_1 = -2x_2 - 3x_3 \quad (\text{Substitute } x_2 \text{ value})$$

$$= x_3$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} \\ = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, the null space of the given matrix is spanned by  $\{(x_3, -2x_3, x_3)\}$

In other words, the basis of the null space of  $A$  is  $N(A) = \{(1, -2, 1)\}$

Therefore, the vectors  $\boxed{C\{(1, -2, 1)\}}$  are in null space.

### Step-3

We need to find the null space vector of  $A^T$ .

The transpose of the matrix  $A$  is  $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

We write the homogeneous equations using this.

$$A^T x = 0$$

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Apply } R_2 \rightarrow R_2 - 2R_1 \text{ and } R_3 \rightarrow \frac{R_3}{3}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Apply } R_2 \rightarrow \frac{R_2}{-3} \text{ and } R_3 \rightarrow R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Apply } R_3 \rightarrow R_3 + 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

## Step-4

$$\Rightarrow x_1 + 4x_2 + 7x_3 = 0$$

$$x_2 + 2x_3 = 0$$

$$\Rightarrow x_2 = -2x_3$$

$$x_1 = -4x_2 - 7x_3 \quad (\text{Substitute } x_2 \text{ value})$$

$$= x_3$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, the null space of  $A^T$  spanned by  $\{(x_3, -2x_3, x_3)\}$

In other words, the basis of the null space of  $A^T$  is  $N(A^T) = \{(1, -2, 1)\}$

Therefore, the vectors  $\boxed{C\{(1, -2, 1)\}}$  are in null space of  $A^T$ .

