Step-1

Let us solve the Left Hand Side:

$$\begin{aligned}\int\left(-y_1V_1''-y_2V_2''-\ldots-y_nV_n''\right)V_jdx &= \int\left(-y_1V_1''V_j-y_2V_2''V_j-\ldots-y_nV_n''V_j\right)dx\\ &= \int -y_1V_1''V_jdx+\int -y_2V_2''V_jdx+\ldots+\int -y_nV_n''V_jdx\\ &= -\int y_1V_1''V_jdx-\int y_2V_2''V_jdx-\ldots-\int y_nV_n''V_jdx\end{aligned}$$

Step-2

Consider the following:

$$\begin{aligned}\int_0^1 -V_i''V_jdx &= -V_j\int_0^1 V_i''dx-\int_0^1\left(\int -V_i''dx\right)\left(\frac{d}{dx}V_j\right)dx\\ &= \left[-V_jV_i'\right]_0^1+\int_0^1 V_i'V_j'dx\\ &= \int_0^1 V_i'V_j'dx-\left[V_jV_i'\right]_0^1\end{aligned}$$

$$\begin{aligned}\int_0^1 -V_i''V_jdx &= \int_0^1 V_i'V_j'dx-0\\ &= \int_0^1 V_i'V_j'dx\\ &= A_{ij}\end{aligned}$$

Step-3

This gives the following:

$$\begin{aligned}\int\left(-y_1V_1''-y_2V_2''-\ldots-y_nV_n''\right)V_jdx &= \int\left(A_{ij}y\right)V_jdx\\ &= \int f(x)V_jdx\end{aligned}$$

Thus, the equation $\int\left(-y_1V_1''-y_2V_2''-\ldots-y_nV_n''\right)V_jdx=\int f(x)V_j(x)dx$ is true.