南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线 性 代 数 A **开课单位:** 数 学 系
考试时长: 150 分钟 **命题教师:** 线性代数教师团队

题 号	1	2	3	4	5	6	7	8	9	10
分 值	10 分	15 分	12 分	8 分	15 分	10 分	8 分	12 分	10 分	10 分

本试卷共 (10) 大题, 满分 (110) 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)
This exam paper contains 10 questions and the score is 110 in total. (Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends.)

本套试卷为 A 卷 Version A

1. (10 points, 2 points each) True or false？No justification is necessary.
   (10 分, 每小题 2 分) 判断对错. 不需要给出解释.

   (1) Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are column vectors of length n, and A is an $n \times n$ matrix. If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent, then $A\alpha_1, A\alpha_2, \cdots, A\alpha_n$ are linearly independent.
   False
   设 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 均为 n 维列向量, A 是 $n \times n$ 矩阵, 若 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性无关, 则 $A\alpha_1, A\alpha_2, \cdots, A\alpha_n$ 线性无关.

   (2) For the matrices $A = \begin{bmatrix} 2018 & & & \\ & 2019 & & \\ & & 2019 & \\ & & & 2020 \end{bmatrix}$ and $A' = \begin{bmatrix} 2020 & & & \\ & 2019 & & \\ & & 2018 & \\ & & & 2019 \end{bmatrix}$, there exists an invertible real matrix P of order 4 such that $P^{-1}AP = A'$.
   True
   对于矩阵 $A = \begin{bmatrix} 2018 & & & \\ & 2019 & & \\ & & 2019 & \\ & & & 2020 \end{bmatrix}$ 和 $A' = \begin{bmatrix} 2020 & & & \\ & 2019 & & \\ & & 2018 & \\ & & & 2019 \end{bmatrix}$, 存在可逆的 4 阶实矩阵 P 使得 $P^{-1}AP = A'$.

   (3) Let A be a real square matrix. Then a real number λ is an eigenvalue of A if and only if it is an eigenvalue of the transpose A^T.
   True
   设 A 为实方阵. 则一个实数 λ 是 A 的特征值当且仅当它是转置矩阵 A^T 的特征值.

   (4) If H is a Hermitian matrix, then $I + iH$ is an invertible matrix. True
   若 H 为 Hermite 矩阵, 则 $I + iH$ 是可逆矩阵.

$|a|>0$, $\begin{vmatrix} a & b \\ 0 & c \end{vmatrix} > 0$　　不是对称阵没有"正定"之说

(5) Suppose $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, where a, b, c are positive real numbers. Then for all nonzero column vectors x in $\mathbb{R}^2$, $x^T A x \geq 0$.

~~True~~ False

设 $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, 其中 a, b, c 均为正实数. 则对于 $\mathbb{R}^2$ 中所有的非零列向量 x, 均有 $x^T A x \geq 0$.

2. (15 points, 3 points each) Write down your answers to the following questions. No further explanation is needed.

(15 分, 每小题 3 分) 请直接写出以下问题的答案. 不需要做进一步解释.

(1) Let U be the subspace of $\mathbb{R}^4$ spanned by the two column vectors $u = (1, 0, 1, -1)^T$ and $v = (0, 1, 0, -1)^T$. Let $W = U^\perp$ be the orthogonal complement of U in $\mathbb{R}^4$, that is, W is the subspace of $\mathbb{R}^4$ consisting of vectors orthogonal to all vectors in U.

Find a basis of W. $(-1, 0, 1, 0)^T$, $(1, 1, 0, 1)^T$

设 U 为 $\mathbb{R}^4$ 中的两个列向量 $u = (1, 0, 1, -1)^T$ 和 $v = (0, 1, 0, -1)^T$ 张成 (生成) 的子空间. 设 $W = U^\perp$ 为 U 在 $\mathbb{R}^4$ 中的正交补, 即, W 由 $\mathbb{R}^4$ 中与 U 中向量全都正交的向量组成.

找出 W 的一组基.

(2) Consider the following points in the plane $\mathbb{R}^2$:

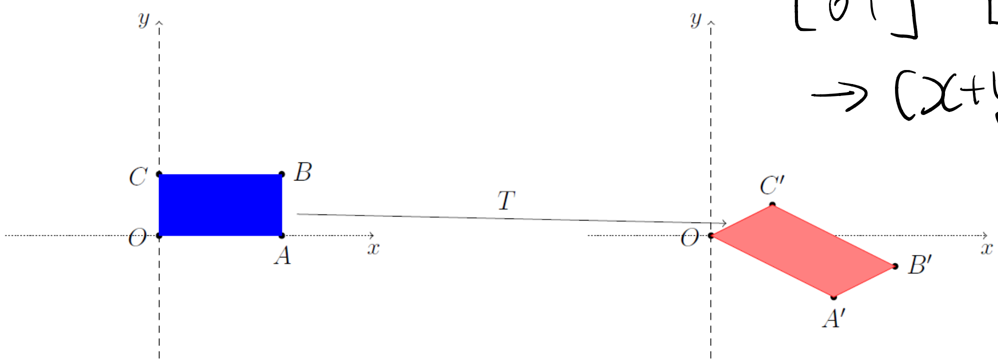
$$O(0, 0)\,; \quad A(2, 0)\,, \; B(2, 1)\,, \; C(0, 1)\,; \quad A'(2, -1)\,, \; B'(3, -0.5)\,, \; C'(1, 0.5)\,.$$

(Here we write $M(a, b)$ to mean that the point M has coordinates (a, b) in the plane.)

Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that transforms the rectangular $OABC$ (with sides and interior) onto the parallelogram $OA'B'C'$. (*Please write down the expression $T(x, y)$ in terms of the coordinate (x, y) of a general point in $\mathbb{R}^2$.*)

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\to (x+y, \frac{y-x}{2})$



考虑平面 $\mathbb{R}^2$ 内的下列各点:

$$O(0, 0)\,; \quad A(2, 0)\,, \; B(2, 1)\,, \; C(0, 1)\,; \quad A'(2, -1)\,, \; B'(3, -0.5)\,, \; C'(1, 0.5)\,.$$

(这里我们用 $M(a, b)$ 表示点 M 的坐标为 (a, b).)

找出一个线性变换 $T:\mathbb{R}^2\to\mathbb{R}^2$, 它将长方形 $OABC$ (包含四条边和内部) 变换成 (映射为) 平行四边形 $OA'B'C'$. (请通过 $\mathbb{R}^2$ 中一般点的坐标 (x, y) 写出 $T(x, y)$ 的表达式.)

(3) Let $A=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&-1\end{bmatrix}$. Compute A^{2020}.

设 $A=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&-1\end{bmatrix}$. 求 A^{2020}. $=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

(4) Find a real number t such that the matrix $\begin{bmatrix}1&t&0\\t&4&-4\\0&t&1\end{bmatrix}$ is positive semidefinite but not positive definite. $t=2$

找出一个实数 t 使得矩阵 $\begin{bmatrix}1&t&0\\t&4&-4\\0&t&1\end{bmatrix}$ 半正定但不正定.

(5) Let $A=\begin{bmatrix}2&0&2\\0&1&0\\0&0&0\end{bmatrix}$. Write down all the singular values of A.

设 $A=\begin{bmatrix}2&0&2\\0&1&0\\0&0&0\end{bmatrix}$. 写出 A 的所有奇异值.

$1, 2\sqrt{2}$

3. (12 points) For each natural number $n\geq 3$, consider the $n\times n$ matrix $A_n=\begin{bmatrix}2&-1&&&\\-1&2&-1&&\\&-1&2&\ddots&\\&&\ddots&\ddots&-1\\&&&-1&2\end{bmatrix}$ and define $a_n=\det(A_n)$. $a_n=n+1$

(a) Compute a_3 and a_4. 4 5

(b) For each $n\geq 5$, find a recursive formula relating a_n to a_{n-1} and a_{n-2}. $a_n=2a_{n-1}-a_{n-2}$

(c) For general $n\geq 3$, find an explicit expression of a_n (in terms of n). $a_n=n+1$

(12 分) 对每个自然数 $n\geq 3$, 考虑 $n\times n$ 矩阵 $A_n=\begin{bmatrix}2&-1&&&\\-1&2&-1&&\\&-1&2&\ddots&\\&&\ddots&\ddots&-1\\&&&-1&2\end{bmatrix}$ 并设 $a_n=\det(A_n)$.

(a) 计算 a_3 和 a_4.

(b) 对每个 $n \geq 5$, 找出一个递推公式将 a_n 和 a_{n-1}, a_{n-2} 联系起来.

(c) 对一般的 $n \geq 3$, 找出 a_n (关于 n) 的一个显式表达式.

4. (8 points) Suppose that a data set consists of points $(-6,-1)$, $(-2,2)$, $(1,1)$ and $(7,6)$ on the xy-plane. Find an equation for the line that best models the relation between the x and y coordinates of these sample values in the sense of least-squares. $y = \frac{1}{2}x + 2$

(8 分) 假设一组数据由 xy-平面内的点 $(-6,-1)$, $(-2,2)$, $(1,1)$ 和 $(7,6)$ 给出. 求能够在最小二乘法意义下最好地拟合这些样本点 x 和 y 坐标关系的直线方程.

5. (15 points) Consider the quadratic form $f(x_1,x_2,x_3) = -x_1^2 - 5x_2^2 - 9x_3^2 - 4x_1x_2 - 6x_1x_3 - 8x_2x_3$.

(a) Find the matrix A of the quadratic form $f(x_1,x_2,x_3)$. $\begin{bmatrix} -1 & -2 & -3 \\ -2 & -5 & -4 \\ -3 & -4 & -9 \end{bmatrix}$

(b) Decide for or against the positive definiteness of A. X

(c) Find an orthogonal matrix Q (i.e., $Q^TQ = QQ^T = I$) to diagonalize A, namely,

$$Q^TAQ = \Lambda.$$

Here Λ is a diagonal matrix.

(d) Is there a real solution to the quadratic equation $f(x_1,x_2,x_3) = 1$ (in the unknowns x_1, x_2, x_3)? Explain why. YES

(15 分) 考虑二次型 $f(x_1,x_2,x_3) = -x_1^2 - 5x_2^2 - 9x_3^2 - 4x_1x_2 - 6x_1x_3 - 8x_2x_3$.

(a) 求二次型 $f(x_1,x_2,x_3)$ 的矩阵 A.

(b) 判定矩阵 A 的正定性.

(c) 求一个正交矩阵 Q (Q 满足 $Q^TQ = QQ^T = I$) 把 A 对角化, 换言之,

$$Q^TAQ = \Lambda.$$

这里 Λ 是一个对角矩阵.

(d) (以 x_1, x_2, x_3 为未知数的) 二次方程 $f(x_1,x_2,x_3) = 1$ 是否有实数解? 请阐述理由.

6. (10 points) For any real matrix M, let $C(M)$ be its column space, $N(M)$ be its null space and $\mathrm{rank}(M)$ be its rank.

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(a) Write down a 2×3 real matrix A such that $C(A)$ has dimension 2.

(b) For the matrix A you give in the previous question, find $\dim N(A)$ $=1$ and $\mathrm{rank}(A^TA)$. $=2$

(c) Is there a real matrix M such that $\mathrm{rank}(M^TM) < \mathrm{rank}(M)$? If yes, provide such an example; otherwise, explain why such a matrix cannot exist. No $\mathrm{rank}(M^TM) = \mathrm{rank}(M)$

(10 分) 对任意实矩阵 M, 以 $C(M)$ 表示它的列空间, $N(M)$ 表示它的零空间 (也称零化空间), $\mathrm{rank}(M)$ 表示它的秩.

(a) 写出一个 2×3 实矩阵 A 使 $C(A)$ 的维数是 2.

(b) 对于你在上一个问题中写出的矩阵 A, 求 $\dim N(A)$ 和 $\operatorname{rank}(A^T A)$.

(c) 是否存在实矩阵 M 满足 $\operatorname{rank}(M^T M) < \operatorname{rank}(M)$? 若是, 请给出一个这样的例子; 若否, 请解释为什么这样的矩阵不存在.

$\boxed{7.}$ (8 points) Let A be a square matrix of order n and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be column vectors in $\mathbb{R}^n$. Suppose that

$$\alpha_i^T A \alpha_j = 0 \quad \text{whenever } i \neq j\,, \quad \text{and} \;\; \alpha_i^T A \alpha_i = 1 \text{ for all } i = 1, 2, \cdots, n\,.$$

Show that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent.

(8 分) 设 A 是一个 n 阶方阵, $\alpha_1, \alpha_2, \cdots, \alpha_n$ 是 $\mathbb{R}^n$ 中的列向量. 假设

$$\text{当 } i \neq j \text{ 时, 总有} \;\; \alpha_i^T A \alpha_j = 0 \quad \text{且} \;\; \alpha_i^T A \alpha_i = 1 \text{ 对所有 } i = 1, 2, \cdots, n \text{ 成立}.$$

证明 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性无关.

$0 = (C_1\alpha_1 + \cdots + C_n\alpha_n)^T A (C_1\alpha_1 + \cdots + C_n\alpha_n)$
$= C_1^2 + C_2^2 + \cdots + C_n^2 \Leftrightarrow C_1 = C_2 = \cdots = C_n = 0$

8. (12 points) Let A be a square matrix of order n such that $A^2 = A$.

(a) Show that if λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$. ✓

(b) Suppose $A \neq I$ (where I is the identity matrix of order n). Show that $\det(A) = 0$. ✓

(c) Suppose B is a square matrix of order n, and the only eigenvalues of B are 0 and 1. Is it necessarily true that $B^2 = B$? If yes, provide a proof. Otherwise give a counterexample.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(12 分) 设 A 为 n 阶方阵, $A^2 = A$.

(a) 证明: 若 λ 是 A 的特征值, 则 $\lambda = 0$ 或 $\lambda = 1$.

(b) 假设 $A \neq I$ (这里 I 表示 n 阶单位矩阵). 证明 $\det(A) = 0$.

(c) 假设 B 是 n 阶方阵, 且 B 的特征值只有 0 和 1. 等式 $B^2 = B$ 是否一定成立? 若是, 请给出证明. 若否, 请举出反例.

9. (10 points) Let

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \quad f(x) = x^3 - 2x + 5\,, \quad B = f(A)\,.$$

(a) Prove that every eigenvector of A is an eigenvector of B. ✓

(b) Show that B is diagonalizable. ✓

(10 分) 设

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \quad f(x) = x^3 - 2x + 5\,, \quad B = f(A)\,.$$

(a) 证明 A 的特征向量都是 B 的特征向量.

(b) 证明 B 可以对角化.

10. (10 points) Let v be a nonzero column vector in $\mathbb{R}^n$ with $n \geq 2$.

(a) Find all the eigenvalues of the $n \times n$ matrix $v\,v^T$.

(b) Let I_n be the identity matrix of order n and

$$H = I_n - 2\frac{vv^T}{v^Tv} .$$

Find the rank of the matrix $I_n + H$.

(10 分) 设 v 为 $\mathbb{R}^n$ 中的非零列向量, 其中 $n \geq 2$.

$v^Tv, 0$

(a) 求出 $n \times n$ 矩阵 $v\,v^T$ 的所有特征值.

(b) 令 I_n 为 n 阶单位矩阵,

$$H = I_n - 2\frac{vv^T}{v^Tv} .$$

求矩阵 $I_n + H$ 的秩. $I_n + H = 2I_n - 2\frac{vv^T}{v^Tv}$ $\det(I_n + H) = 0$

$n-1$