7.24 complex spectral theorem $\mathbb{F}=\mathbb{C}$ normal

7.29. real spectral theorem $\mathbb{F}=\mathbb{R}$. self-adjoint

The Spectral Theorem (谱定理)

Lecture 20

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Operators on Inner Product Spaces

1. The Complex Spectral Theorem
2. The Real Spectral Theorem
3. Homework Assignment 20

Introduction

Recall:

1. A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.
2. An operator on V has a diagonal matrix with respect to a basis if and only if the basis consists of $\boxed{\text{eigenvectors}}$ of the operator (see 5.41).
3. The nicest operators on V are those for which there is an orthonormal basis of V with respect to which the operator has a diagonal matrix. These are precisely the operators $T \in \mathscr{L}(V)$ such that there is an orthonormal basis of V consisting of eigenvectors of T.

Our goal in this section is to prove the spectral theorem, which characterizes these operators as the normal operators when $\mathbb{F} = \mathbb{C}$ and as the self-adjoint operators when $\mathbb{F} = \mathbb{R}$. The spectral theorem is probably the most useful tool in the study of operators on inner product spaces.

Several remarks

- Because the conclusion of the Spectral Theorem depends on $\mathbb{F}$, we will break the Spectral Theorem into two pieces, called the Complex Spectral Theorem and the Real Spectral Theorem.
- As is often the case in linear algebra, complex vector spaces are easier to deal with than real vector spaces. Thus we present the Complex Spectral Theorem first.

The Complex Spectral Theorem

The key part of the Complex Spectral Theorem states that if $\mathbb{F} = \mathbb{C}$ and $T \in \mathscr{L}(V)$ is normal, then T has a diagonal matrix with respect to some orthonormal basis of V. The next example illustrates this conclusion.

7.23 **Example** Consider the normal operator $T \in \mathcal{L}(\mathbf{C}^2)$ from Example 7.19, whose matrix (with respect to the standard basis) is

$Te_1 = 2e_1 + 3e_2$

$Te_2 = -3e_1 + 2e_2$

$T(x,y) = \lambda(x,y)$

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

$|A - \lambda I| = \begin{pmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{pmatrix} = 0$

$\Rightarrow \lambda = 2 \pm 3i$

As you can verify, $\frac{(i,1)}{\sqrt{2}}, \frac{(-i,1)}{\sqrt{2}}$ is an orthonormal basis of $\mathbf{C}^2$ consisting of eigenvectors of T, and with respect to this basis the matrix of T is the diagonal matrix

$$\begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix}.$$

The Complex Spectral Theorem

In the next result, the equivalence of (b) and (c) is easy (see 5.41).
Thus we prove only that (c) implies (a) and that (a) implies (c).

7.24 Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

(b) $\Leftrightarrow$ (c) ✓

(c) $\Rightarrow$ (a) $T(e_1, e_2, \cdots, e_n) = (e_1, \cdots, e_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{pmatrix} = m(T)$

$m(TT^*) = m(T^*T)$

$m(T)m(T^*) = m(T^*)\cdot m(T)$

$\xrightarrow{7.10} T^*(e_1, \cdots, e_n) = (e_1, \cdots, e_n)\begin{pmatrix} \overline{\lambda_1} & 0 \\ 0 & \ddots\ \overline{\lambda_n} \end{pmatrix} = m(T^*)$

$\underbrace{TT^* = T^*T}_{\checkmark}$

Proof.

First suppose (c) holds, so T has a diagonal matrix with respect to some orthonormal basis of V. The matrix of T^* (with respect to the same basis) is obtained by taking the conjugate transpose of the matrix of T; hence T^* also has a diagonal matrix. Any two diagonal matrices commute; thus T commutes with T^*, which means that T is normal. In other words, (a) holds.

Now suppose (a) holds, so T is normal. By Schur's Theorem (6.38), there is an orthonormal basis $e_1, e_2, \cdots, e_n$ of V with respect to which T has an upper triangular matrix.

(a) $\Rightarrow$ (c) (6.38) $T(e_1, \cdots, e_n) = (e_1, \cdots, e_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & 0 & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$

(7.10) $T^*(e_1, \cdots, e_n) = (e_1, \cdots, e_n)\begin{pmatrix} \overline{a_{11}} & & 0 & \\ \overline{a_{12}} & \overline{a_{22}} & & \\ \vdots & & \ddots & \\ \overline{a_{1n}} & & & \overline{a_{nn}} \end{pmatrix}$

$\|Te_1\|^2 = |a_{11}|^2$

$\|T^*e_1\|^2 = |\overline{a_{11}}|^2 \underbrace{+ \cdots + |\overline{a_{1n}}|^2}_{=0}$ ✓

$\Downarrow$ normal

diagonal matrix

Proof

Thus we can write

$$\mathscr{M}(T,(e_1,e_2,\cdots,e_n)) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{pmatrix} \cdots\cdots(*)$$

We will show that this matrix is actually a diagonal matrix.

We see from the matrix above that

$$||Te_1||^2 = |a_{1,1}|^2$$

Proof.

and

$$\|T^*e_1\|^2 = |a_{1,1}|^2 + |a_{1,2}|^2 + \cdots + |a_{1,n}|^2$$

Because T is normal, $\|Te_1\| = \|T^*e_1\|$ (see 7.20). Thus the two equations above imply that all entries in the first row of the matrix in 7.25, except possibly the first entry $a_{1,1}$ equal 0.

Now from (*) we see that

$$\|Te_2\|^2 = |a_{2,2}|^2$$

(because $a_{1,2} = 0$, as we showed in the paragraph above) and

$$\|T^*e_2\|^2 = |a_{2,2}|^2 + |a_{2,3}|^2 + \cdots + |a_{2,n}|^2$$

Proof.

Because T is normal, $||Te_2|| = ||T^*e_2||$. Thus the two equations above imply that all entries in the second row of the matrix in (*), except possibly the diagonal entry $a_{2,2}$ equal 0. Continuing in this fashion, we see that all the nondiagonal entries in the matrix (*) equal 0. Thus (c) holds.

Invertible quadratic expressions

- $x^2 + bx + c$ is an invertible real number (a convoluted way of saying that it is not 0).
- Replacing the real number x with a self-adjoint operator (recall the analogy between real numbers and self-adjoint operators), we are led to the result below.

7.26 Invertible quadratic expressions

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbf{R}$ are such that $b^2 < 4c$. Then

$$T^2 + bT + cI$$

is invertible. 3.69

Proof

Proof.

Let v be a nonzero vector in V. Then

$$\langle (T^2+bT+cI)v, v\rangle = \langle Tv, Tv\rangle + b\langle Tv, v\rangle + c\langle v, v\rangle \geq ||Tv||^2 - |b|||Tv||||v|| + c||v||^2$$
$$= \left(||Tv|| - \frac{|b|||v||}{2}\right)^2 + (c - \frac{b^2}{4})||v||^2 > 0$$

where the inequality holds by Cauchy-Schwarz Inequality. The last inequality implies that $(T^2+bT+cI)v \neq 0$. Thus $T^2+bT+cI$ is injective, which implies that it is invertible (see 3.69). □

Self-adjoint operators have eigenvalues

- We know that every operator, self-adjoint or not, on a finite dimensional nonzero complex vector space has an eigenvalue (see 5.21).
- Thus the next result tells us something new only for real inner product spaces.

7.27 Self-adjoint operators have eigenvalues

Suppose $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ is a self-adjoint operator. Then T has an eigenvalue.

Check the proof given in the textbook on Page 220.

Self-adjoint operators and invariant subspaces

The next result shows that if U is a subspace of V that is invariant under a $\boxed{\text{self-adjoint}}$ operator T, then $U^{\perp}$ is also invariant under T. Later we will show that the hypothesis that T is self-adjoint can be replaced with the weaker hypothesis that T is normal (see 9.30).

7.28 Self-adjoint operators and invariant subspaces

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T. Then

$\Rightarrow T|_U \in \mathcal{L}(U)$ self-adjoint

$T|_{U^\perp} \in \mathcal{L}(U^\perp)$ self-adjoint

(a) $U^{\perp}$ is invariant under T;

(b) $T|_U \in \mathcal{L}(U)$ is self-adjoint;

(c) $T|_{U^{\perp}} \in \mathcal{L}(U^{\perp})$ is self-adjoint.

The Real Spectral Theorem

We can now prove the next result, which is one of the major theorems in linear algebra.

(c) $\Rightarrow$ (a)
(a) $\Rightarrow$ (b)
(b) $\Rightarrow$ (c) ✓

7.29 Real Spectral Theorem

Suppose $\mathbf{F} = \mathbf{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is self-adjoint.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

(a) $\Rightarrow$ (b) $\dim V = 1$ ✓

$n = \dim V > 1$. Induction.

$T \in \mathcal{L}(V)$. self-adjoint

$\xRightarrow{7.27}$ λ eigenvalue of T.

$Tu = \lambda u$. $\|u\| = 1$.

$U = \operatorname{span}(u)$ invariant under T

$V = U \oplus U^{\perp}$ (6.47)

$\xRightarrow{(7.28)}$ $T|_{U^{\perp}} \in \mathcal{L}(U^{\perp})$ self-adjoint

$\dim U^{\perp} < \dim V$

Proof

First suppose (c) holds, so T has a diagonal matrix with respect to some orthonormal basis of V. A diagonal matrix equals its transpose. Hence $T = T^*$, and thus T is self-adjoint. In other words, (a) holds.

We will prove that (a) implies (b) by induction on $\dim V$. To get started note that if $\dim V = 1$, then (a) implies (b). Now assume that $\dim V > 1$ and that (a) implies (b) for all real inner product spaces of smaller dimension.

Suppose (a) holds, so $T \in \mathscr{L}(V)$ is self-adjoint. Let u be an eigenvector of T with $\|u\| = 1$. Let $U = \text{span}(u)$. Then U is a 1-dimensional subspace of V that is invariant under T. By 7.28(c), the operator $T|_{U^\perp} \in \mathscr{L}(U^\perp)$ is self-adjoint.

Proof

By our induction hypothesis, there is an orthonormal basis of $U^\perp$ consisting of eigenvectors of $T|_{U^\perp}$. Adjoining u to this orthonormal basis of $U^\perp$ gives an orthonormal basis of V consisting of eigenvectors of T, completing the proof that (a) implies (b).

We have proved that (c) implies (a) and that (a) implies (b). Clearly (b) implies (c), completing the proof.

Example

7.30 Example Consider the self-adjoint operator T on $\mathbf{R}^3$ whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 14 & -13 & 8 \\ -13 & 14 & 8 \\ 8 & 8 & -7 \end{pmatrix}.$$

As you can verify,

$$\frac{(1,-1,0)}{\sqrt{2}}, \frac{(1,1,1)}{\sqrt{3}}, \frac{(1,1,-2)}{\sqrt{6}}$$

is an orthonormal basis of $\mathbf{R}^3$ consisting of eigenvectors of T, and with respect to this basis, the matrix of T is the diagonal matrix

$$\begin{pmatrix} 27 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -15 \end{pmatrix}.$$

Remarks

If $\mathbf{F} = \mathbf{C}$, then the Complex Spectral Theorem gives a complete description of the normal operators on V. A complete description of the self-adjoint operators on V then easily follows (they are the normal operators on V whose eigenvalues all are real; see Exercise 6).

If $\mathbf{F} = \mathbf{R}$, then the Real Spectral Theorem gives a complete description of the self-adjoint operators on V. In Chapter 9, we will give a complete description of the normal operators on V (see 9.34).

Homework Assignment 20

7.B: 2, 5, 6, 9, 10, 12, 13.