Step-1

Consider the two Jordan matrices *J* and *K* given by:

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \; K = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

Since block sizes for both the matrices are different, so *J* and *K* are not similar matrices.

Objective is to compare the product matrix JM with MK for any matrix *M*. If $JM = MK$ then prove that matrix *M* is not invertible.

Step-2

Consider the condition that $JM = MK$. Use the method of contrary and assume that *M* is some invertible matrix. Then the inverse of *M* will exist, say M^{-1}.

Pre-multiply by M^{-1} in the condition $JM = MK$ gives

$$\begin{aligned} M^{-1}JM &= M^{-1}MK \\ M^{-1}JM &= IK \\ M^{-1}JM &= K \end{aligned}$$

Square both the sides and get,

$$\begin{aligned} \left(M^{-1}JM\right)^2 &= K^2 \\ \left(M^{-1}JM\right)\left(M^{-1}JM\right) &= K^2 \\ M^{-1}J\left(M \cdot M^{-1}\right)JM &= K^2 \\ M^{-1}J^2M &= K^2 \quad \left[M \cdot M^{-1} = I\right] \end{aligned}$$

Step-3

Note that the block matrix *J* is nilpotent matrix with index 2. Therefore, $J^2 = 0$. Then the equation $M^{-1}J^2M = K^2$ implies that

$K^2 = 0$.

But

$$K^2 = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\neq 0$$

This is a contradiction. So, assumed assumption was wrong and M is not invertible. Thus, M^{-1} does not exist and $M^{-1}JM = K$ is not possible.

Step-4

Hence, matrices J and K are not similar.