

Step-1

(a)

Consider the subspace of all vectors whose components are equal.

The objective is to find a basis for this subspaces of \mathbf{R}^4 .

Step-2

The space is of all vectors of the form $(t, t, t, t) = t(1, 1, 1, 1)$ that is all multiples of $(1, 1, 1, 1)$.

This expresses that $\{(1, 1, 1, 1)\}$ spans the space, and meanwhile it's a singleton, it is also linearly independent, and henceforth a basis. \hat{A}

Step-3

(b)

Consider the subspace of all vectors whose components add to equal.

The objective is to find a basis for this subspaces of \mathbf{R}^4 .

Step-4

Now look for $(a, b, c, d) \in \mathbf{R}^4$ such that: \hat{A}

$$a + b + c + d = 0$$

This implies that,

$$d = -a - b - c \quad \text{That is,}$$

$$\begin{aligned}(a, b, c, -a - b - c) &= (a, 0, 0, -a) + (0, b, 0, -b) + (0, 0, c, -c) \\ &= a(1, 0, 0, -1) + b(0, 1, 0, -1) + c(0, 0, 1, -1)\end{aligned}$$

Thus, $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1)\}$ spans the space, and is provably linearly independent, consequently is a basis.

Step-5

(c)

Consider the subspace of all vectors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.

The objective is to find a basis for this subspaces of \mathbf{R}^4 .

Step-6

Two vectors perpendicular if their dot product is zero.

Now observe for $(a, b, c, d) \in \mathbf{R}^4$ such that,

$$(a, b, c, d) \cdot (1, 1, 0, 0) = 0$$

This implies that,

$$a + b = 0$$

And,

$$(a, b, c, d) \cdot (1, 0, 1, 1) = 0$$

This implies that,

$$a + c + d = 0$$

From this, $a = -c - d$ and from $a + b = 0$;

$$-c - d + b = 0$$

$$b = c + d$$

Therefore, vectors of the form obtained are:

$$\begin{aligned}(a, b, c, d) &= (-c - d, c + d, c, d) \\ &= (-c, c, c, 0) + (-d, d, 0, d) \\ &= c(-1, 1, 1, 0) + d(-1, 1, 0, 1)\end{aligned}$$

The vectors $(-1, 1, 1, 0)$ and $(-1, 1, 0, 1)$ are linearly independent because first vector has fourth component zero and second vector has third component zero.

Hence, the set $\boxed{\{(-1, 1, 1, 0), (-1, 1, 0, 1)\}}$ is a basis for this space.

Step-7

(d)

Consider the below matrix;

$$U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The objective is to find a basis for the column space (in \mathbf{R}^2) and null space (in \mathbf{R}^5) of U .

Step-8

The matrix U is already in Echelon form.

Therefore, the columns contain pivots are basis for the column space.

Thus, the basis for column space is $\{(1,0), (0,1)\}$.

Step-9

The null space of a matrix A is the solution x of the system $Ax = 0$.

Consider $Ux = 0$ where $x = (a, b, c, d, e)^T$.

So,

$$Ux = 0$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives;

$$a + c + e = 0$$

$$b + d = 0$$

This implies;

$$a = -c - e$$

$$b = -d$$

So,

$$\begin{aligned}
 x &= \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \\
 &= \begin{bmatrix} -c-e \\ -d \\ c \\ d \\ e \end{bmatrix} \\
 &= c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + e \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
 \end{aligned}$$

$$\left[\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right].$$

Hence, the basis of null space is