Step-1

Given $A=\begin{bmatrix}3 & 1 & -1\end{bmatrix}$ and V is the nullspace of A.

(a) We have to find a basis for V and a basis for V^T

V is the nullspace of A

Implies by definition, $Ax=0$

Step-2

$$\begin{bmatrix}3 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$3x_1+x_2-x_3=0$$

$$\text{Put } x_2=k_1,\ x_3=k_2$$

$$\Rightarrow 3x_1=-k_1+k_2$$

$$\Rightarrow x_1=-\frac{1}{3}k_1+\frac{1}{3}k_2$$

Step-3

Therefore

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-\frac{1}{3}k_1+\frac{1}{3}k_2\\k_1\\k_2\end{bmatrix}$$

$$=k_1\begin{bmatrix}-1/3\\1\\0\end{bmatrix}+k_2\begin{bmatrix}1/3\\0\\1\end{bmatrix}$$

Basis of $\boxed{V=\left\{\begin{bmatrix}-1/3\\1\\0\end{bmatrix},\begin{bmatrix}1/3\\0\\1\end{bmatrix}\right\}}$

Step-4

Basis of $V^{\perp}$

=Row space of V

=span $\left(\begin{bmatrix}3 & 1 & -1\end{bmatrix}^T\right)$

Step-5

(b) We have to write an orthonormal basis for $V^{\perp}$ and find the projection matrix P_1 that projects vectors in R^3 onto $V^{\perp}$.

$$a=\begin{bmatrix}3\\1\\-1\end{bmatrix}$$
$$\Rightarrow \|a\|=\sqrt{9+1+1}$$
$$=\sqrt{11}$$

Step-6

$$\alpha=\frac{a}{\|a\|}$$
$$=\frac{1}{\sqrt{11}}\begin{bmatrix}3\\1\\-1\end{bmatrix}$$

Orthonormal basis for $V^{\perp}=\left\{\begin{bmatrix}3/\sqrt{11}\\1/\sqrt{11}\\-1/\sqrt{11}\end{bmatrix}\right\}$

Step-7

Projection of $\bar{i}$ onto α

$$= \frac{a.\bar{i}}{\|a\|}\alpha$$

$$= \frac{3}{\sqrt{11}}\frac{1}{\sqrt{11}}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 9/11 \\ 3/11 \\ -3/11 \end{bmatrix}$$

Step-8

Projection of $\bar{j}$ onto

$$= \frac{a.\bar{j}}{\|a\|}\alpha$$

$$= \frac{1}{\sqrt{11}}\frac{1}{\sqrt{11}}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/11 \\ 1/11 \\ -1/11 \end{bmatrix}$$

Step-9

Projection of $\bar{k}$ onto

$$= \frac{a.\bar{k}}{\|a\|}\alpha$$

$$= \frac{-1}{\sqrt{11}}\frac{1}{\sqrt{11}}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/11 \\ -1/11 \\ 1/11 \end{bmatrix}$$

Projection matrix $P_1 = \begin{bmatrix} 9/11 & 3/11 & -3/11 \\ 3/11 & 1/11 & -1/11 \\ -3/11 & -1/11 & 1/11 \end{bmatrix}$

Step-10

(c) We have to find the projection matrix P_2 that projects vectors in R^3 onto *V*.

$$A = \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10/9 & -1/9 \\ -1/9 & 10/9 \end{bmatrix}$$

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Step-11

$$\left(A^T A\right)^{-1} = \frac{81}{99} \begin{bmatrix} 10/9 & 1/9 \\ 1/9 & 10/9 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$$

Step-12

$$A\left(A^T A\right)^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{11} \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} -3 & 3 \\ 10 & 1 \\ 1 & 10 \end{bmatrix}$$

Step-13

$$A\left(A^T A\right)^{-1} A^T = \frac{1}{11} \begin{bmatrix} -3 & 3 \\ 10 & 1 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 2 & -3 & 3 \\ -3 & 10 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$

Hence Projection matrix $P_2 = \begin{bmatrix} 2/11 & -3/11 & 3/11 \\ -3/11 & 10/11 & 1/11 \\ 3/11 & 1/11 & 10/11 \end{bmatrix}$