Step-1

That is true because Transpose $A-\lambda I$: det $(A-\lambda I) =$ det $(A-\lambda I)^T =$ det$(A^T - \lambda I)$

Example:

Suppose $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

Step-2

Now $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$$\begin{aligned} A-\lambda I &= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} \end{aligned}$$

Step-3

$$\begin{aligned} |A-\lambda I| &= \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} \\ &= (1-\lambda)(4-\lambda)+2 \\ &= 4-\lambda-4\lambda+\lambda^2+2 \\ &= \lambda^2-5\lambda+6 \end{aligned}$$

Step-4

we know that $|A-\lambda I| = 0$

$\lambda^2 - 5\lambda + 6 = \lambda^2 - 3\lambda - 2\lambda + 6$

$= \lambda(\lambda-3) - 2(\lambda-3)$

$=(\lambda-3)(\lambda-2)$

Now $(\lambda-3)(\lambda-2)=0$

$\lambda=3,2$

Hence the eigenvalues of A are 3,2

Step-5

Case(i) Let $\lambda = 3$

Eigenvectors X corresponding to the eigenvalue 3 are given by

$(A\text{-}3I)X=0$

That is $\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Step-6

$By\ R_2+R_1=R_2$

$\begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1-x_2=0$

Let $x_1=k$(say)

$-2k-x_2=0$

$x_2=-2k$

Step-7

Therefore eigenvectors corresponding to eigenvalue 3 are given by $k\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ where k is a non-zero parameter.

Case(ii) Let $\lambda = 2$

Eigenvectors X corresponding to the eigenvalue 2 are given by

$(A\text{-}2I)X=0$

That is $\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Step-8

By $R_2 + 2R_1 = R_2$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 - x_2 = 0$$

Step-9

Let $x_1 = k(\text{say})$

Therefore $x_2 = -k$

Therefore eigenvectors corresponding to eigenvalue 2 are given by $k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ where k is a non-zero parameter

Step-10

Now $A^T = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} \end{aligned}$$

Step-11

$$|A-\lambda I|=\begin{vmatrix}1-\lambda & 2\\ -1 & 4-\lambda\end{vmatrix}$$
$$=(1-\lambda)(4-\lambda)+2$$
$$=4-\lambda-4\lambda+\lambda^2+2$$
$$=\lambda^2-5\lambda+6$$

Step-12

we know that $|A-\lambda I|=0$

$\lambda^2-5\lambda+6=0$
$\lambda^2-3\lambda-2\lambda+6=0$

$\lambda(\lambda-3)-2(\lambda-3)=0$

$(\lambda-3)(\lambda-2)=0$

Now $(\lambda-3)(\lambda-2)=0$

$\lambda=3,2$
Hence the eigenvalues of A^T are 3,2

Step-13

Case(i) Let $\lambda=3$

Eigenvectors X corresponding to the eigenvalue 3 are given by

$(A\text{-}3I)X=0$

That is $\begin{bmatrix}-2 & 2\\ -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

Step-14

By $2R_2-R_1=R_2$

$\begin{bmatrix}-2 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$\Rightarrow -2x_1+2x_2=0$

Let $x_1 = k$(say)

Therefore $x_2 = k$

Therefore eigenvectors corresponding to eigenvalue 3 are given by $k\begin{bmatrix}1\\1\end{bmatrix}$ where k is a non-zero parameter.

Step-15

Case(ii) Let $\lambda = 2$

Eigenvectors X corresponding to the eigenvalue 2 are given by

$(A - 2I)X = 0$

That is $\begin{bmatrix}-1 & 2\\-1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Step-16

By $R_2 - R_1 = R_2$

$\begin{bmatrix}-1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\Rightarrow -x_1 + 2x_2 = 0$

Step-17

Let $x_1 = k$(say)

Therefore $x_2 = k/2$

Therefore eigenvectors corresponding to eigenvalue 2 are given by $k\begin{bmatrix}1\\1/2\end{bmatrix}$ where k is a non-zero parameter

The above example shows that the eigenvectors of A and A^T are not the same.