Step-1

Given that V is spanned by $(1,1,0,1)$ and $(0,0,1,0)$

(a) We have to find a basis for the orthogonal complement $\mathbf{V}^{\perp}$

Let $a_1 = (1,1,0,1), a_2 = (0,0,1,0)$ and write a_1, a_2 are rows of A Â Â

Therefore $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Step-2

The row space of A is equal to V

This implies $\mathbf{V}^{\perp}$ = null space of A

By definition of null space $Ax = 0$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 + x_4 = 0$ and $x_3 = 0$

Step-3

Put

$x_2 = a, x_4 = b$

$\Rightarrow x_1 = -a - b$

Therefore $\mathbf{V}^{\perp}$ (null space of A) is given by the vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -a-b \\ a \\ 0 \\ b \end{bmatrix}$$

$$= a\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Step-4

Hence a basis for the orthogonal complement $\mathbf{V}^{\perp}$ is

$$\boxed{\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

Step-5

(b) We have to find the projection matrix *P* onto *V*.

The projection matrix $P = A^T\left(AA^T\right)^{-1}A$

Given $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$\Rightarrow A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Step-6

And

$$AA^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left(AA^T\right)^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Step-7

Therefore

$$A^T\left(AA^T\right)^{-1}A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Therefore the projection matrix P onto **V**

$$= \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix}$$

Step-8

(c) We have to find the vectors in **V** closest to the vector $b = (0,1,0,-1)$ in $\mathbf{V}^{\perp}$

The closest vector to b in **V** = the projection of b onto **V**

$= Pb$

$$= \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$