Step-1

(a)

The objective is to comment on the nature of the sum of a complex number and itâ€™s conjugate.

Let $a+ib$ be a complex number.

Then the conjugate of $a+ib=\overline{(a+ib)}$

$=a-ib$

Now the sum of $a+ib$ and $a-ib$ is;

$$=(a+ib)+(a-ib)$$
$$=(a+a)+i(b-b)$$
$$=2a$$

Therefore the sum of the complex number and its conjugate is **twice the real part** of the given complex number.

Step-2

(b)

The objective is to comment on the nature of the conjugate of a complex number on the unit circle

Let $a+ib$ be a complex number on the unit circle.

Then $r=\sqrt{a^2+b^2}=1$

The conjugate of $a+ib=a-ib$

And the radius of the complex number $r'=\sqrt{a^2+b^2}=1$

Therefore, the conjugate of a complex number is also **lies on the unit circle**.

Step-3

(c)

The objective is to comment on the nature of the product of two complex numbers on the unit circle

Let $(a+ib),(c+id)$ are the complex numbers on unit circle that is,

$a^2+b^2=1$ and $c^2+d^2=1$

Then the product

$$\begin{aligned}(a+ib)(c+id) &= ac+ibc+iad+i^2bd \\ &= ac+ibc+iad-bd \\ &= (ac-bd)+i(ad+bc)\end{aligned}$$

The radius $r=\sqrt{(ac-bd)^2+(ad+bc)^2}$

Without loss of generality consider $a=\cos\theta_1, b=\sin\theta_1; c=\cos\theta_2, d=\sin\theta_2$ where $\theta_1,\theta_2\in[0,2\pi)$.

$$\begin{aligned}r &= \sqrt{(ac-bd)^2+(ad+bc)^2} \\ &= \sqrt{(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)^2+(\cos\theta_1\sin\theta_2+\cos\theta_2\sin\theta_1)^2} \\ &= \sqrt{\cos(\theta_1+\theta_2)^2+\sin(\theta_1+\theta_2)^2} \\ &= 1\end{aligned}$$

Radius $\boxed{r=1}$ thus the product lies **on the same unit circle**.

Step-4

(d)

The objective is to comment on the nature of the sum of two complex numbers on the unit circle

Let $(a+ib),(c+id)$ are the complex numbers on unit circle that is,

$a^2+b^2=1$ and $c^2+d^2=1$

Then the sum of two complex numbers

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

The radius $r=\sqrt{(a+c)^2+(b+d)^2}$

Without loss of generality consider $a=\cos\theta_1, b=\sin\theta_1; c=\cos\theta_2, d=\sin\theta_2$ where $\theta_1,\theta_2\in[0,2\pi)$.

$$\begin{aligned}r &= \sqrt{(a+c)^2+(b+d)^2} \\ &= \sqrt{a^2+c^2+2ac+b^2+d^2+2bd} \\ &= \sqrt{(a^2+b^2)+(c^2+d^2)+2(ac+bd)} \\ &= \sqrt{2(1+ac+bd)}\end{aligned}$$

Substituting *a, b, c* and *d* in the above equation gives:

$$r = \sqrt{2\left(1+\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\right)}$$
$$= \sqrt{2\left(1+\cos\left(\theta_1-\theta_2\right)\right)}$$

As,

$$-1 \le \cos\left(\theta_1-\theta_2\right) \le 1$$
$$1-1 \le 1+\cos\left(\theta_1-\theta_2\right) \le 1+1$$
$$0 \le 2\left(1+\cos\left(\theta_1-\theta_2\right)\right) \le 4$$
$$0 \le \sqrt{2\left(1+\cos\left(\theta_1-\theta_2\right)\right)} \le 2$$

Thus,

$$0 \le r \le 2$$

Step-5

It implies the sum of two complex number on a unit circle lies in a region of $\boxed{r \le 2}$.