Step-1

Given that $S = \{0 = (0,0,0,0)\}$

We know that this is the trivial subspace of $\mathbf{R}^4$

Suppose $S^{\perp} = \{v = (x, y, z, w) : v \in \mathbf{R}^4\}$

Then by definition of orthogonal complement, we get $v^T 0 = 0$

i.e., $$\begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

We know that this condition is satisfied by every $v = (x, y, z, w) : v \in \mathbf{R}^4$

Therefore, $\mathbf{R}^4 \subseteq S^{\perp}$

Since $\mathbf{R}^4$ is the linear space, S is the subspace, we follow that $S^{\perp} \subseteq \mathbf{R}^4$

Putting these observations together, we get $S^{\perp} = \mathbf{R}^4$

Step-2

Suppose $w = (0,0,0,1)$ spans the subspace S.

Then $S = \{x = (0,0,0,k)\}$ where k is any real number.

Suppose $S^{\perp} = \{v = (x, y, z, w) : v \in \mathbf{R}^4\}$

Then by definition, we get $$\begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ k \end{bmatrix} = 0$$

In other words, $x \cdot 0 + y \cdot 0 + z \cdot 0 + w \cdot k = 0$

We easily see that the first three summands obviously become zero with any real numbers x, y, z .

But k is any real number such that $w.\ k = 0$ is possible only when $w = 0$

Therefore, we can write $S^{\perp} = \{v = (x, y, z, 0) : v \in \mathbf{R}^4\}$

Observe that *S* is of dimension 1 and $S^{\perp}$ is of 3 such that their sum is the dimension of $\mathbf{R}^4$.

Step-3

Assuming $S^{\perp} = U$, we follow that *U* is of dimension 3 and so, $U^{\perp}$ is of dimension 1 and thus, $U^{\perp} \subseteq (S^{\perp})^{\perp}$

In other words, $S \subseteq (S^{\perp})^{\perp}$ â€¦â€¦ (1)

On the other hand, *S* spans all the vectors of the form $(0,0,0,k)$ and $(S^{\perp})^{\perp}$ contains vectors of the form $(0,0,0,k)$

Therefore, $(S^{\perp})^{\perp} \subseteq S$ â€¦â€¦ (2)

Putting (1) and (2) together, we get $(S^{\perp})^{\perp} = S$