Unless otherwise noted, vector spaces are over $\mathbb{F}$ and with finite dimensions, where $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$.

$U_1=\{(x,x)\}$
$U_2=\{(x,0)\}$
$U_3=\{(0,y)\}$

1. (30 points, 6 points each) Label the following statements as **True** or **False**. **Along with your answer, provide an informal proof, counterexample, or other explanation.**

   (a) Suppose U_1,U_2,U_3 are subspaces of V and $U_1\cap U_2=\{0\}$, $U_2\cap U_3=\{0\}$, $U_1\cap U_3=\{0\}$, then $U_1\cap(U_2+U_3)=\{0\}$. F

   (b) Suppose $V=\operatorname{null} T\oplus\operatorname{range} T$, then T is diagonalizable. T

   $(0,y)(x,0)$ (x,x) $(x,0)$

   (c) If U_1,U_2,U_3 are subspaces of V such that $V=U_1\oplus U_3$ and $V=U_2\oplus U_3$, then $U_1=U_2$. F

   (d) Suppose U and W are subsets of V with $U\subset W$. Then $W^\circ\subset U^\circ$. T

   $\boxed{\text{(e)}}$ Suppose the dual basis of $1,x,x^2,x^3$ for $\mathcal{P}_3(\mathbb{R})$ is $\varphi_1,\varphi_2,\varphi_3,\varphi_4$. Then $\varphi_j(p)=\frac{p^{(j)}(0)}{j!}$, $j=$ ~~1, 2, 3, 4.~~ 0, 1, 2, 3 F

2. (10 points) Let $\mathbb{R}^{2\times2}$ be the set of all real 2×2 matrices and

$$V_1=\left\{\begin{bmatrix} a & -a\\ b & c\end{bmatrix}: a,b,c\in\mathbb{R}\right\},\ V_2=\left\{\begin{bmatrix} y & x\\ -y & z\end{bmatrix}: x,y,z\in\mathbb{R}\right\}.$$

   (a) Show that V_1 and V_2 are subspaces of $\mathbb{R}^{2\times2}$. ✓

   (b) Find $\overset{3}{\dim} V_1,\overset{3}{\dim} V_2,\dim(V_1\overset{4}{+}V_2)$, and $\dim(V_1\cap V_2)$. $2\to\begin{bmatrix} a & -a\\ -a & c\end{bmatrix}$

   (c) Is V_1+V_2 a direct sum? Provide an explanation. $V_1\cap V_2\neq\{0\}$ ✗

3. (10 points) Let $V=\mathbb{R}^2$ and

$$v_1=(1,-1), v_2=(2,-1),\ v_3=(-3,1)$$
$$w_1=(1,0), w_2=(0,1),\ w_3=(1,1)$$

$-3=a+2\overset{-4}{b}$
$1=-a-b$
$-4=b$

   Is there a linear map $T\in\mathcal{L}(V)$ such that $T(v_i)=w_i$, $i=1,2,3$? Explain.

v_1, v_2 basis of V $Tv_1=w_1$, $Tv_2=w_2$ ✓

$Tv_3=T(v_1-2v_2)$
$=w_1-2w_2=(1,-2)\neq(1,1)$

$T(v_1,v_2)=(v_1,v_2)\begin{pmatrix}0&0\\0&1\end{pmatrix}=(0,v_2)$

$Tv_1=0$, $Tv_2=v_2$

4. (10 points) Let V be a 2-dimensional complex vector space and v_1,v_2 be a basis of V. Suppose $T\in\mathcal{L}(V)$ and the matrix of T with respect to v_1,v_2 is $\begin{bmatrix}0&0\\0&1\end{bmatrix}$. Find all the invariant subspaces of V under T. V, $\operatorname{span}(v_1)$, $\operatorname{span}(v_2)$ $\boxed{\{0\}}$ 要漏了!

$\alpha=(\alpha_1,\alpha_2)\in\mathbb{F}^2$, $v_1,v_2\in V$

5. (10 points) Show that $V\times V$ and $\mathcal{L}(\mathbb{F}^2,V)$ are isomorphic vector spaces. Where V maybe an infinite-dimensional vector space.

$T(v_1,v_2)(\alpha)=\alpha_1v_1+\alpha_2v_2$

6. (10 points) In a triangle with sides of length a, b, and c, let d be the length of the line segment from the midpoint of the side of length c to the opposite vertex. Prove that $a^2+b^2=\frac{1}{2}c^2+2d^2$. The identity is called the **Apollonius's identity**.