## Step-1

The objective is to decide the stability of $u' = Au$ for the given matrices.

We know that the differential equation $\frac{du}{dt} = Au$ is

stable, when all $\operatorname{Re}\lambda_i < 0$

neutrally stable, when all $\operatorname{Re}\lambda_i \le 0$ and $\operatorname{Re}\lambda_1 = 0$

unstable and $e^{At}$ is unbounded if any Eigen values has $\operatorname{Re}\lambda_i > 0$.

## Step-2

(a)

Consider $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, we have to find Eigen values of matrix *A*.

To find the Eigen values of matrix *A*, we have to solve for $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} 2-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$
$$(2-\lambda)(5-\lambda) - 12 = 0$$
$$\lambda^2 - 7\lambda - 2 = 0$$

## Step-3

Solve the equation $\lambda^2 - 7\lambda - 2 = 0$, using the following formula.

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\lambda = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-2)}}{2(1)}$$
$$= \frac{7 \pm \sqrt{49+8}}{2}$$
$$= \frac{7 \pm \sqrt{57}}{2}$$

Solving the above equation, we get $\lambda_1$ and $\lambda_2$ as

$$\lambda_1 = \frac{7+\sqrt{57}}{2}$$
$$\lambda_2 = \frac{7-\sqrt{57}}{2}$$

## Step-4

Hence, the Eigen values of matrix $A$ are $\lambda_1 = \frac{7+\sqrt{57}}{2}$ and $\lambda_2 = \frac{7-\sqrt{57}}{2}$.

Since the Eigen value $\lambda_1 = \frac{7+\sqrt{57}}{2}$ is greater than 0, that is $\operatorname{Re}\lambda_i > 0$, so the differential equation $\frac{du}{dt} = Au$ is $\boxed{\text{unstable}}$.

## Step-5

(b)

Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, we have to find Eigen values of matrix $A$.

To find the Eigen values of matrix $A$, we have to solve for $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)(-1-\lambda) - 6 = 0$$
$$-1 - \lambda + \lambda + \lambda^2 - 6 = 0$$
$$\lambda^2 = 7$$

## Step-6

Taking square root of both side of equation $\lambda^2 = 7$, we get.

$$\lambda = \pm\sqrt{7}$$

Hence, the Eigen values of matrix $A$ are $\lambda_1 = \sqrt{7}$ and $\lambda_2 = -\sqrt{7}$.

Since the Eigen value $\lambda_1 = \sqrt{7}$ is greater than 0, that is $\operatorname{Re}\lambda_i > 0$, so the differential equation $\frac{du}{dt} = Au$ is $\boxed{\text{unstable}}$.

## Step-7

(c)

Consider $A=\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}$, we have to find Eigen values of matrix *A*.

To find the Eigen values of matrix *A*, we have to solve for $\det(A-\lambda I)=0$.

$$\begin{aligned}\det(A-\lambda I)&=0\\ \begin{bmatrix}1-\lambda & 1\\ 1 & -2-\lambda\end{bmatrix}&=0\\ (1-\lambda)(-2-\lambda)-1&=0\\ \lambda^2+\lambda-3&=0\end{aligned}$$

Solve the equation $\lambda^2+\lambda-3=0$, using the following formula.

$$\begin{aligned}\lambda&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\ \lambda&=\frac{-1\pm\sqrt{(1)^2-4(1)(-3)}}{2(1)}\\ &=\frac{-1\pm\sqrt{13}}{2}\end{aligned}$$

Solving the above equation, we get $\lambda_1$ and $\lambda_2$ as

$$\lambda_1=\frac{-1+\sqrt{13}}{2}$$
$$\lambda_2=\frac{-1-\sqrt{13}}{2}$$

## Step-8

Hence, the Eigen values of matrix *A* are $\lambda_1=\frac{-1+\sqrt{13}}{2}$ and $\lambda_2=\frac{-1-\sqrt{13}}{2}$.

Since the Eigen value $\lambda_1=\frac{-1+\sqrt{13}}{2}$ is greater than 0, that is $\operatorname{Re}\lambda_i>0$, so the differential equation $\frac{du}{dt}=Au$ is $\boxed{\text{unstable}}$.

## Step-9

(d)

Consider $A=\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}$, we have to find Eigen values of matrix *A*.

To find the Eigen values of matrix *A*, we have to solve for $\det(A-\lambda I)=0$.

$$\begin{aligned}\begin{bmatrix}-1-\lambda & -1\\ -1 & -1-\lambda\end{bmatrix}&=0\\ (-1-\lambda)(-1-\lambda)-1&=0\\ (-1)(1+\lambda)(-1)(1+\lambda)-1&=0\\ (1+\lambda)^2-1&=0\end{aligned}$$

## Step-10

Solve the equation $(1+\lambda)^2-1=0$, to find the value of $\lambda$.

$$\begin{aligned}(1+\lambda)^2-1&=0\\ (1+\lambda)^2&=1\\ 1+\lambda&=\pm\sqrt{1}\end{aligned}$$

Solving the above equation, we get $\lambda_1$ and $\lambda_2$ as

$$\lambda_1=0$$
$$\lambda_2=-2$$

## Step-11

Hence, the Eigen values of matrix *A* are $\lambda_1=0$ and $\lambda_2=-2$.

Since the Eigen values $\lambda_1=0$ and $\lambda_2=-2$, that is $\operatorname{Re}\lambda_i\le 0$ and $\operatorname{Re}\lambda_1=0$, so the differential equation $\frac{du}{dt}=Au$ is $\boxed{\text{neutrally stable}}$.