Step-1

Consider the following equation

$$\|b_1 \sin x - \cos x\|^2 = \int_0^{2\pi} (b_1 \sin x - \cos x)^2 \, dx$$

â€¦â€¦ (1)

By taking derivative of both sides, we get

$$\begin{aligned}
\frac{\partial}{\partial b_1}\left(\|b_1 \sin x - \cos x\|^2\right) &= \frac{\partial}{\partial b_1}\left(\int_0^{2\pi} (b_1 \sin x - \cos x)^2 \, dx\right) \\
&= \frac{\partial}{\partial b_1}\left(\int_0^{2\pi} \left(b_1^2 \sin^2 x + \cos^2 x - 2b_1 \sin x \cos x\right) dx\right) \\
&= \frac{\partial}{\partial b_1}\left(b_1^2 \int_0^{2\pi} \sin^2 x dx + \int_0^{2\pi} \cos^2 x dx - 2b_1 \int_0^{2\pi} \sin x \cos x dx\right)
\end{aligned}$$

Step-2

By integrating we get:

$$\begin{aligned}
\frac{\partial}{\partial b_1}\left(\|b_1 \sin x - \cos x\|^2\right) &= \frac{\partial}{\partial b_1}\left(\begin{array}{l} b_1^2\left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{2\pi} + \left[\frac{x}{2} + \frac{1}{4}\sin 2x\right]_0^{2\pi} \\ -2b_1\left[-\frac{1}{2}\cos^2 x\right]_0^{2\pi} \end{array}\right) \\
&= \frac{\partial}{\partial b_1}\left(\begin{array}{l} b_1^2\left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{2\pi} + \left[\frac{x}{2} + \frac{1}{4}\sin 2x\right]_0^{2\pi} \\ +b_1\left[\cos^2 x\right]_0^{2\pi} \end{array}\right) \\
&= \frac{\partial}{\partial b_1}\left(\begin{array}{l} b_1^2\left[\frac{2\pi}{2} - \frac{1}{4}\sin(4\pi)\right] + \left[\frac{2\pi}{2} + \frac{1}{4}\sin(4\pi)\right] \\ +b_1\left[\cos^2(2\pi) - \cos^2(0)\right] \end{array}\right) \\
&= \frac{\partial}{\partial b_1}\left(b_1^2 \pi + \pi\right)
\end{aligned}$$

Step-3

By integrating we get:

$$\begin{aligned}
\frac{\partial}{\partial b_1}\left(\|b_1 \sin x - \cos x\|^2\right) &= \frac{\partial}{\partial b_1}\left(b_1^2 \pi + \pi\right) \\
&= 2b_1 \pi
\end{aligned}$$

To find the value of b_1 that minimizes the equation (1), setting the above derivative equal to 0.

$$\begin{aligned}\frac{\partial}{\partial b_1}\left(\|b_1 \sin x - \cos x\|^2\right) &= 0 \\ 2b_1\pi &= 0 \\ b_1 &= 0\end{aligned}$$

Therefore, $\boxed{b_1 = 0}$ minimizes the equation (1).

Step-4

We know that the fourier coefficient b_1 is as follows:

$$\begin{aligned}b_1 &= \frac{\int_0^{2\pi} f(x)\sin x dx}{\int_0^{2\pi} (\sin x)^2 dx} \\ &= \frac{(f, \sin x)}{(\sin x, \sin x)}\end{aligned}$$

Since $b_1 \sin x - \cos x$ is the vector between the function and the basis ($\sin x$), so we get

$$f(x) = \cos x$$

Step-5

Substitute $f(x) = \cos x$ in the following equation:

$$\begin{aligned}b_1 &= \frac{\int_0^{2\pi} f(x)\sin x dx}{\int_0^{2\pi} (\sin x)^2 dx} \\ &= \frac{\int_0^{2\pi} (\cos x)\sin x dx}{\int_0^{2\pi} (\sin x)^2 dx} \\ &= \frac{\left[\frac{1}{2}(\cos^2 x)\right]_0^{2\pi}}{\left[\frac{x}{2} - \frac{1}{4}\sin(2x)\right]_0^{2\pi}}\end{aligned}$$

By simplifying we get:

$$b_1 = \frac{\frac{1}{2}\left[\cos^2(2\pi) - \cos^2(0)\right]}{\left[\frac{2\pi}{2} - \frac{1}{4}\sin(4\pi) - 0\right]}$$
$$= \frac{\frac{1}{2}(1-1)}{\pi}$$
$$= \frac{0}{\pi}$$
$$= 0$$

By using the Fourier coefficient, we are getting $b_1 = 0$.

Therefore, $\boxed{b_1 = 0}$ minimizes the equation (1).