Step-1

Given that $A = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}$

We have to diagonalize the given matrix by constructing its eigenvalue matrix Λ and its eigenvector matrix *S*.

Step-2

First we find the eigenvalues of *A*.

The characteristic equation of *A* is

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1-i \\ 1+i & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda) - (1-i^2) = 0$$

Step-3

Continuation to the above

$$\Rightarrow \lambda^2 - 5\lambda + 6 - 2 = 0$$
$$\Rightarrow \lambda^2 - 9\lambda + 4 = 0$$
$$\Rightarrow (\lambda-1)(\lambda-4) = 0$$
$$\Rightarrow \lambda = 1, 4$$

Hence the eigenvalues of *A* are 1, 4

Therefore, the eigenvalue matrix is $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

Step-4

Now we find the eigenvectors of the given matrix.

We know that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of *A* if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the nonzero solution of the system $(A - \lambda I)x = 0$

That is $\begin{bmatrix} 2-\lambda & 1-i \\ 1+i & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$ â€¦â€¦ (1)

Step-5

For $\lambda=1$, (1) becomes

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

The Augmented matrix

$$\begin{bmatrix} 1 & 1-i & 0 \\ 1+i & 2 & 0 \end{bmatrix}$$

Add $(1-i)$ times of row 1 to row 2.

$$\begin{bmatrix} 1 & 1-i & 0 \\ 2 & 2-2i & 0 \end{bmatrix}$$

Step-6

Add (-2) times of row 1 to row 2.

$$\begin{bmatrix} 1 & 1-i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this, we get

$$x_1+(1-i)x_2=0$$

Here x_2 is free variable.

Let $x_2=k$, where *k* is a free variable.

Then $x_1=-(1-i)k$

Step-7

Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} -(1-i)k \\ k \end{bmatrix}$$
$$= -k\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue $\lambda = 1$ is $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.

Step-8

For $\lambda = 4$, (1) becomes

$$\begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

The Augmented matrix is

$$\begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Add $(1(1-i))$ times of row 1 to 2 times of row 2

$$\begin{bmatrix} -2 & 1-i & 0 \\ 4 & -2+2i & 0 \end{bmatrix}$$

Step-9

Add 2 times of row 1 to row 2

$$\begin{bmatrix} -2 & 1-i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this, we get

$$-2x_1 + (1-i)x_2 = 0$$

Here x_2 is free variable.

Let $x_2 = k$, where k is a free variable.

Then $x_1 = \left(\frac{1-i}{2}\right)k$

Step-10

Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{(1-i)}{2}k \\ k \end{bmatrix}$$
$$= \frac{k}{2}\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue $\lambda = 4$ is $\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$.

Let $S = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}$

Then

$$S^{-1} = \frac{1}{2(1-i)+1(1-i)}\begin{bmatrix} 2 & -1+i \\ 1 & 1-i \end{bmatrix}$$

$$= \frac{1}{2-2i+1-i}\begin{bmatrix} 2 & -1+i \\ 1 & 1-i \end{bmatrix}$$
$$= \frac{1}{2-2i+1-i}\begin{bmatrix} 2 & -1+i \\ 1 & 1-i \end{bmatrix}$$

Step-11

Continuation to the above

$$= \frac{1}{3-3i}\begin{bmatrix} 2 & -1+i \\ 1 & 1-i \end{bmatrix}$$

$$= \frac{3+3i}{18}\begin{bmatrix} 2 & -1+i \\ 1 & 1-i \end{bmatrix}$$

$$= \frac{1+i}{6}\begin{bmatrix} 2 & -1+i \\ 1 & 1-i \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix}$$

Therefore, $S^{-1} = \frac{1}{6}\begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix}$

Step-12

Now

$$S\Lambda S^{-1} = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{6}(2+2i)+0(1+i) & \frac{1}{6}(-2)+0(2) \\ 0(2+2i)+\frac{4}{6}(1+i) & 0(-2)+\frac{4}{6}(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1+i}{3} & -\frac{1}{3} \\ \frac{2}{3}(1+i) & \frac{4}{3} \end{bmatrix}$$

Step-13

Continuation to the above

$$= \begin{bmatrix} (1-i)\left(\frac{1+i}{3}\right)+(1-i)\left(\frac{2}{3}(1+i)\right) & (1-i)\left(-\frac{1}{3}\right)+(1-i)\frac{4}{3} \\ -1\left(\frac{1+i}{3}\right)+2\left(\frac{2}{3}(1+i)\right) & (-1)\left(-\frac{1}{3}\right)+(2)\frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}+\frac{4}{3} & -(1-i)\frac{1}{3}+(1-i)\frac{4}{3} \\ -\frac{1}{2}(1+i)+\frac{4}{3}(1+i) & \frac{1}{3}+\frac{8}{3} \end{bmatrix}$$

Step-14

Continuation to the above

$$= \begin{bmatrix} \frac{6}{3} & (1-i)\frac{3}{3} \\ \frac{3}{3}(1+i) & \frac{9}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}$$

Therefore, $A = S\Lambda S^{-1}$

Hence the matrix that diagonalizes the given matrix A is $\boxed{S = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}}$.