Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Step-2

Solve the above differential equation for both initial values:

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$u(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Also find e^{At}.

Step-3

Firstly find the Eigen values and Eigen vectors of matrix A:

To calculate the Eigen values do the following calculations;

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \\ \det(A - \lambda I) &= 0 \\ (3-\lambda)(3-\lambda) - 1 &= 0 \\ \lambda^2 - 6\lambda + 8 &= 0 \end{aligned}$$

After solving following values are obtained:

$$\lambda_1 = 4$$
$$\lambda_2 = 2$$

Step-4

To calculate Eigen vectors do the following calculations:

$$(A-\lambda_1 I)x=0$$
$$\begin{bmatrix}3-\lambda & 1\\ 1 & 3-\lambda\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}3-2 & 1\\ 1 & 3-2\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

On solving values of y and z corresponding to $\lambda=2$ is as follows:

$$x_1=\begin{bmatrix}y\\ z\end{bmatrix}$$
$$=\begin{bmatrix}1\\ -1\end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda=4$ is as follows:

$$(A-\lambda_1 I)x=0$$
$$\begin{bmatrix}3-\lambda & 1\\ 1 & 3-\lambda\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}3-4 & 1\\ 1 & 3-4\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2=\begin{bmatrix}y\\ z\end{bmatrix}$$
$$=\begin{bmatrix}1\\ 1\end{bmatrix}$$

Step-6

To find the value of the exponential matrix e^{At} recall the following:

$$e^{At}=Se^{\Lambda t}S^{-1}$$

Substitute the values in the above equation and solve.

Step-7

Matrix e^{At} can be written as follows:

$$\begin{aligned} e^{At} &= Se^{\Lambda t}S^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{2t}+e^{4t} & -e^{2t}+e^{4t} \\ -e^{2t}+e^{4t} & e^{2t}+e^{4t} \end{bmatrix} \end{aligned}$$

Step-8

Therefore, value of the exponential e^{At} matrix is:

$$\boxed{e^{At} = \frac{1}{2} \begin{bmatrix} e^{2t}+e^{4t} & -e^{2t}+e^{4t} \\ -e^{2t}+e^{4t} & e^{2t}+e^{4t} \end{bmatrix}}$$

Step-9

Recall that $du/dt = Au$ has the following solution:

$$u(t) = e^{At}u(0)$$

Take initial value to be:

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step-10

Solve the following:

$$\begin{aligned} u(t) &= e^{At}u(0) \\ &= \frac{1}{2} \begin{bmatrix} e^{2t}+e^{4t} & -e^{2t}+e^{4t} \\ -e^{2t}+e^{4t} & e^{2t}+e^{4t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{2t}+e^{4t} \\ -e^{2t}+e^{4t} \end{bmatrix} \end{aligned}$$

Therefore, the solution is:

$$\boxed{u(t)=\frac{1}{2}\begin{bmatrix} e^{2t}+e^{4t} \\ -e^{2t}+e^{4t} \end{bmatrix}}$$

Step-11

Take initial value to be:

$$u(0)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solve the following:

$$\begin{aligned} u(t) &= e^{At}u(0) \\ &= \frac{1}{2}\begin{bmatrix} e^{2t}+e^{4t} & -e^{2t}+e^{4t} \\ -e^{2t}+e^{4t} & e^{2t}+e^{4t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -e^{2t}+e^{4t} \\ e^{2t}+e^{4t} \end{bmatrix} \end{aligned}$$

Step-12

Therefore, the solution is:

$$\boxed{u(t)=\frac{1}{2}\begin{bmatrix} -e^{2t}+e^{4t} \\ e^{2t}+e^{4t} \end{bmatrix}}$$