

Step-1

We have to factor $\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & 0 \end{bmatrix}$ into QR

Let $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$, where $a_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$, $a_2 = \begin{bmatrix} \sin \theta \\ 0 \end{bmatrix}$

Â We know that

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix}$$

Step-2

Now

$$\begin{aligned} \|a_1\| &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

Step-3

$$\begin{aligned} q_1 &= \frac{a_1}{\|a_1\|} \\ &= \frac{1}{1} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \end{aligned}$$

Step-4

Now

$$q_2 = \frac{\beta}{\|\beta\|} \text{ where } \beta = a_2 - (q_1^T a_2) q_1$$

$$\begin{aligned} q_1^T a_2 &= \begin{bmatrix} \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \sin \theta \\ 0 \end{bmatrix} \\ &= \cos \theta \sin \theta \end{aligned}$$

Step-5

$$\begin{aligned}
(q_1^T a_2) q_1 &= \cos \theta \sin \theta \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2 \theta \sin \theta \\ \cos \theta \sin^2 \theta \end{bmatrix} \\
\beta &= \begin{bmatrix} \sin \theta \\ 0 \end{bmatrix} - \begin{bmatrix} \cos^2 \theta \sin \theta \\ \cos \theta \sin^2 \theta \end{bmatrix} \\
&= \begin{bmatrix} \sin^3 \theta \\ -\cos \theta \sin^2 \theta \end{bmatrix}
\end{aligned}$$

Step-6

$$\begin{aligned}
\|\beta\| &= \sqrt{\sin^6 \theta + \cos^2 \theta \sin^4 \theta} \\
&= \sqrt{(\sin^4 \theta)(1)} \\
&= \sin^2 \theta
\end{aligned}$$

Step-7

Therefore

$$\begin{aligned}
q_2 &= \frac{1}{\sin^2 \theta} \begin{bmatrix} \sin^3 \theta \\ -\cos \theta \sin^2 \theta \end{bmatrix} \\
&= \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix}
\end{aligned}$$

Step-8

And

$$\begin{aligned}
q_1^T a_1 &= \begin{bmatrix} \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \\
&= \cos^2 \theta + \sin^2 \theta \\
\hat{A} &= I \quad \hat{A}
\end{aligned}$$

Step-9

$$\begin{aligned}
q_1^T a_2 &= \begin{bmatrix} \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \sin \theta \\ 0 \end{bmatrix} \\
&= \cos \theta \sin \theta
\end{aligned}$$

Step-10

$$q_2^T a_2 = \begin{bmatrix} \sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} \sin \theta \\ 0 \end{bmatrix}$$

$$= \sin^2 \theta + 0$$

$$= \sin^2 \theta$$

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

$$= \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} 1 & \cos \theta \sin \theta \\ 0 & \sin^2 \theta \end{bmatrix}$$

$$= QR$$

Therefore $\boxed{\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & 0 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} 1 & \cos \theta \sin \theta \\ 0 & \sin^2 \theta \end{bmatrix} = QR}$