Step-1

Let $A = QR$

Given that the orthogonal matrices have norm $\|Q\| = 1$.

We have to show that $\|A\| \le \|R\|$ and $\|R\| \le \|A\|$.

Step-2

If Q is an orthogonal matrix, then by definition, we have $Q^{-1} = Q^T$ â€¦â€¦ (1)

Also, we have by the definition of norm that $\|Q\|^2 = \max \frac{\|Qx\|^2}{\|x\|^2}$ â€¦â€¦ (2)

$= \max \frac{x^T Q^T Qx}{x^T x}$

$= \max \frac{x^T Ix}{x^T x}$ (Since by (1))

= 1

While norm is non negative, we get $\|Q\| = 1$

Step-3

By definition of norm, we have $\|A\| = \max_{x \ne 0} \frac{\|Ax\|}{\|x\|}$

In other words, $\|A\| \le \|A\| \|x\|$ for every matrix x

Now, A is decomposed into QR where Q is an orthogonal matrix.

Then $\|A\| \le \|(QR)x\|$

$\le \|Q\| \|Rx\|$

$\le \|Q\| \|R\| \|x\|$

Step-4

Continuation to the above

$$\Rightarrow \frac{\|(QR)x\|}{\|x\|} \le \|Q\|\|R\|$$

$$\Rightarrow \max \frac{\|(QR)x\|}{\|x\|} \le \|Q\|\|R\|$$

$\Rightarrow \|QR\| \le \|Q\|\|R\|$ (By (2))

$\Rightarrow \|A\| \le \|R\|$ â€¦â€¦ (3)

Step-5

On the other hand, we have $\|R\| \le \|XR\|$ for every non singular matrix *X*.

Since *Q* is the orthogonal matrix non singular, we can write $\|R\| \le \|QR\|$

Consequently $\|R\| \le \|A\|$ â€¦â€¦ (4)

Putting (3) and (4) together, we get $\|A\| = \|R\|$

Step-6

We have to find an example of $A = LU$ with $\|A\| < \|L\|\|U\|$

Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

Suppose $L = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}, U = \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$ such that $LU = A$

Then it follows that $a_{11} = 2,\ a_{11}u_{12} = 1,\ a_{21} = 3,\ a_{21}u_{12} + a_{22} = 2$

From this, we get $u_{12} = \frac{1}{2}$ and $a_{22} = \frac{1}{2}$

So, $L = \begin{bmatrix} 2 & 0 \\ 3 & \frac{1}{2} \end{bmatrix}, U = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$

Step-7

Now

$$A^T A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$$

The characteristic equation of $A^T A$ is

$$\left| A^T A - \lambda I \right| = 0$$
$$\Rightarrow \begin{vmatrix} 5-\lambda & 8 \\ 8 & 13-\lambda \end{vmatrix} = 0$$
$$\Rightarrow \lambda^2 - 18\lambda + 1 = 0$$
$$\Rightarrow \lambda = \frac{18+\sqrt{324-4}}{2}, \frac{18-\sqrt{324-4}}{2}$$
$$= 17.94, 0.0557$$

So the eigenvalues of $A^T A$ are 0.0557 , 17.94

Step-8

Now

$$\sqrt{\lambda_{\max}\left(A^T A\right)} = \sqrt{17.94}$$
$$= 4.24$$
$$= \|A\| \quad \text{â€¦â€¦ (5)}$$

Step-9

On the other hand, $L^T L = \begin{bmatrix} 2 & 3 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & \frac{1}{2} \end{bmatrix}$

$$= \begin{bmatrix} 13 & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{4} \end{bmatrix}$$

The characteristic equation of $L^T L$ is

$$\left|L^T L-\lambda I\right|=0$$

$$\Rightarrow\begin{vmatrix}13-\lambda & \frac{3}{2}\\ \frac{3}{2} & \frac{1}{4}-\lambda\end{vmatrix}=0$$

$$\Rightarrow(13-\lambda)\left(\frac{1}{4}-\lambda\right)-\frac{9}{4}=0$$

Step-10

Continuation to the above

$$\Rightarrow\lambda^2-\frac{53}{4}\lambda+1=0$$

$$\lambda=\frac{(53/4)+\sqrt{(53/4)^2-4}}{2},\frac{(53/4)+\sqrt{(53/4)^2-4}}{2}$$

$$=13.174,0.076$$

So

$$\begin{aligned}\|L\|&=\sqrt{\lambda_{\max}\left(L^T L\right)}\\&=\sqrt{13.174}\\&\approx 3.627\end{aligned}\qquad \text{â€¦â€¦ (6)}$$

Step-11

Now

$$U^T U=\begin{bmatrix}1 & 0\\ \frac{1}{2} & 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2}\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0.5\\ 0.5 & 1\end{bmatrix}$$

The characteristic equation of $U^T U$ is

$$\left|U^T U-\lambda I\right|=0$$

$$\Rightarrow\begin{vmatrix}1-\lambda & 0.5\\ 0.5 & 1-\lambda\end{vmatrix}=0$$

$$\Rightarrow(1-\lambda)(1-\lambda)-0.25=0$$

Step-12

Continuation to the above

$$\Rightarrow \lambda^2 - 2\lambda + 0.75 = 0$$

$$\Rightarrow \lambda = \frac{2+\sqrt{4-1}}{2}, \frac{2-\sqrt{4-1}}{2}$$

$$= 1.866, 0.134$$

Therefore,

$$\begin{aligned} \|U\| &= \sqrt{\lambda_{\max}\left(U^T U\right)} \\ &= \sqrt{1.866} \\ &\approx 1.366 \qquad \text{â€¦â€¦ (7)} \end{aligned}$$

Step-13

Therefore,

$$\begin{aligned} \|A\| = 4.24 &< 4.95 \\ &< 3.627 \times 1.366 \\ &< \|L\|\|U\| \end{aligned}$$

Hence the required matrices are $\boxed{A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, L = \begin{bmatrix} 2 & 0 \\ 3 & \frac{1}{2} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}$.