# Step-1

We have to find the fourth Legendre polynomial; it is a cubic $x^3+ax^2+bx+c$ that is orthogonal to 1, *x*, and $x^2-\frac{1}{3}$ over the interval $-1 \le x \le 1$

# Step-2

The fourth Legendre polynomial

$$u_4 = x^3 - \frac{(1,x^3)}{(1,1)}1 - \frac{(x,x^3)}{(x,x)}x - \frac{\left(x^2-\frac{1}{3},x^3\right)}{\left(x^2-\frac{1}{3},x^2-\frac{1}{3}\right)}\left(x^2-\frac{1}{3}\right)$$

Â â€¦â€¦ (1)

Now compute the each of the inner products,

$$(1,x^3) = \int_{-1}^{1} x^3.1dx$$
$$= 0\left(\text{Since } x^3 \text{ is odd }\right)$$

# Step-3

$$(1,1) = \int_{-1}^{1} 1.1\,dx$$
$$= [x]_{-1}^{1}$$
$$= 2$$

# Step-4

$$(x,x^3) = \int_{-1}^{1} x^4.1\,dx$$
$$= 2\int_{0}^{1} x^4 dx$$
$$= 2\left[\frac{x^5}{5}\right]_0^1$$
$$= \frac{2}{5}$$

# Step-5

$$(x,x)=\int_{-1}^{1} xxdx$$
$$=2\int_{0}^{1} x^2dx$$
$$=2\left[\frac{x^3}{3}\right]_0^1$$
$$=\frac{2}{3}$$

# Step-6

$$\left(x^2-\frac{1}{3},x^3\right)=\int_{-1}^{1}\left(x^2-\frac{1}{3}\right).x^3dx$$
$$=0\left(\text{since }\left(x^2-\frac{1}{3}\right)x^3\text{ is odd}\right)$$

# Step-7

$$\left(x^2-\frac{1}{3},x^2-\frac{1}{3}\right)=\int_{-1}^{1}\left(x^2-\frac{1}{3}\right).\left(x^2-\frac{1}{3}\right)dx$$
$$=2\int_{0}^{1}\left(x^4-\frac{2}{3}x^2+\frac{1}{9}\right)dx$$
$$=2\left[\frac{x^5}{5}-\frac{2}{3}\left(\frac{x^3}{3}\right)+\frac{1}{9}x\right]_0^1$$
$$=2\left(\frac{1}{5}-\frac{2}{9}+\frac{1}{9}\right)$$
$$=\frac{8}{45}$$

# Step-8

Hence (1) becomes

$$u_4=x^3-\frac{0}{2}.1-\frac{2/5}{2/3}x-\frac{0}{8/45}\left(x^2-\frac{1}{3}\right)$$
$$=x^3-\frac{3}{5}x$$

Therefore the fourth Legendre polynomial is $\boxed{u_4=x^3-\frac{3}{5}x}$