Step-1

To obtain the values for u on a three dimensional grid, the differential equation can be replaced by the finite differences:

$$-u_{i+1,j,k}+2u_{i,j,k}-u_{i-1,j,k}+\left(-u_{i,j+1,k}+2u_{i,j,k}-u_{i,j-1,k}\right)+\left(-u_{i,j,k+1}+2u_{i,j,k}-u_{i,j,k-1}\right)=0$$

Step-2

Note that we are considering the differences along the three directions, x, y, and z. The difference matrix in any one direction is given as follows:

$$A=\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$$

At the same time, the identity matrix in any of the other directions is given by,

$$I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Step-3

The required matrix is given by the following:

$$A_{3\mathrm{D}}=A_{1\mathrm{D}}\otimes I\otimes I+I\otimes A_{1\mathrm{D}}\otimes I+I\otimes I\otimes A_{1\mathrm{D}}$$

Consider the following:

$$\begin{aligned}A_{1\mathrm{D}}\otimes I\otimes I&=A_{1\mathrm{D}}\otimes\left(I\otimes I\right)\\&=A_{1\mathrm{D}}\otimes\begin{bmatrix}I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\\&=\begin{bmatrix}2\left(I\otimes I\right) & -\left(I\otimes I\right) & 0\\ -\left(I\otimes I\right) & 2\left(I\otimes I\right) & -\left(I\otimes I\right)\\ 0 & -\left(I\otimes I\right) & 2\left(I\otimes I\right)\end{bmatrix}\end{aligned}$$

Step-4

Similarly, we get

$$
\begin{aligned}
I \otimes A_{1D} \otimes I &= I \otimes \left(A_{1D} \otimes I \right) \\
&= I \otimes \begin{bmatrix} 2I & -I & 0 \\ -I & 2I & -I \\ 0 & -I & 2I \end{bmatrix} \\
&= \begin{bmatrix} A_{1D} \otimes I & 0 & 0 \\ 0 & A_{1D} \otimes I & 0 \\ 0 & 0 & A_{1D} \otimes I \end{bmatrix}
\end{aligned}
$$

Step-5

Finally, we have

$$
\begin{aligned}
I \otimes I \otimes A_{1D} &= I \otimes \left(I \otimes A_{1D} \right) \\
&= I \otimes \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix} \\
&= \begin{bmatrix} I \otimes A_{1D} & 0 & 0 \\ 0 & I \otimes A_{1D} & 0 \\ 0 & 0 & I \otimes A_{1D} \end{bmatrix}
\end{aligned}
$$

Step-6

Thus, the required matrix is given by,

$$
\begin{aligned}
A_{3D} &= A_{1D} \otimes I \otimes I + I \otimes A_{1D} \otimes I + I \otimes I \otimes A_{1D} \\
&= \begin{bmatrix} 2(I \otimes I) & -(I \otimes I) & 0 \\ -(I \otimes I) & 2(I \otimes I) & -(I \otimes I) \\ 0 & -(I \otimes I) & 2(I \otimes I) \end{bmatrix} + \begin{bmatrix} A_{1D} \otimes I & 0 & 0 \\ 0 & A_{1D} \otimes I & 0 \\ 0 & 0 & A_{1D} \otimes I \end{bmatrix} + \\
&\qquad \begin{bmatrix} I \otimes A_{1D} & 0 & 0 \\ 0 & I \otimes A_{1D} & 0 \\ 0 & 0 & I \otimes A_{1D} \end{bmatrix}
\end{aligned}
$$

That is,

$$
\begin{bmatrix} 2(I \otimes I) + A_{1D} \otimes I + I \otimes A_{1D} & -(I \otimes I) & 0 \\ -(I \otimes I) & 2(I \otimes I) + A_{1D} \otimes I + I \otimes A_{1D} & -(I \otimes I) \\ 0 & -(I \otimes I) & 2(I \otimes I) + A_{1D} \otimes I + I \otimes A_{1D} \end{bmatrix}
$$