Operators on Real Vector Spaces

$$\underset{\text{real}}{V} \underset{\longleftarrow}{\overset{\text{eigenvalue}}{\longrightarrow}} \underset{\text{complex}}{V_{\mathbb{C}}}$$

$$T \in \mathcal{L}(V) \underset{\longleftarrow}{\longrightarrow} T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$$

$T \in \mathcal{L}(V)$, $V$ real vector space

$$V \to V_{\mathbb{C}}$$

$$\underset{\in \mathcal{L}(V)}{T} \to T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$$

# Complexification

Lecture 27

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# Complexification

# Introduction

As we will soon see:

1. a real vector space $V$ can be embedded, in a natural way, in a complex vector space called the complexification of $V$.
2. Each operator on $V$ can be extended to an operator on the complexification of $V$.
3. Our results about operators on complex vector spaces can then be translated to information about operators on real vector spaces.

We begin by defining the complexification of a real vector space.

# Complexification of a Vector Space

$V$ real vector space $\mathbb{F}=\mathbb{R}$
$\underline{\underline{V_{\mathbb{C}}}}$ $u, v \in V$
$u+iv$.

**9.2 Definition** ***complexification of*** $V$, $V_{\mathbf{C}}$

Suppose $V$ is a real vector space.

- The ***complexification*** of $V$, denoted $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair $(u, v)$, where $u, v \in V$, but we will write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

  for $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

  for $a, b \in \mathbf{R}$ and $u, v \in V$.

$T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}})$ $T_{\mathbb{C}}(u+iv)=\lambda(u+iv)$
$\lambda \in \mathbb{C}$

# Basis

We think of $V$ as a subset of $V_{\mathbb{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbb{C}}$ from $V$ can be thought of as generalizing the construction of $\mathbb{C}^n$ from $\mathbb{R}^n$.

**9.3 $V_{\mathbf{C}}$ is a complex vector space.**

Suppose $V$ is a real vector space. Then with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

The proof of the result above is left as an exercise for the reader.

# Basis

Probably everything that you think should work concerning compexification does work, usually with a straightforward verification, as illustrated by the next result.

**9.4 Basis of $V$ is basis of $V_{\mathbf{C}}$**

Suppose $V$ is a real vector space.

$u+iv \in V_{\mathbb{C}}$
$u, v \in \operatorname{span}(v_1, \cdots, v_n)$

(a) If $v_1, \ldots, v_n$ is a basis of $V$ (as a real vector space), then $v_1, \ldots, v_n$ is a basis of $V_{\mathbf{C}}$ (as a complex vector space).

$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$
$\lambda_1, \cdots, \lambda_n \in \mathbb{C}$

(b) The dimension of $V_{\mathbf{C}}$ (as a complex vector space) equals the dimension of $V$ (as a real vector space).

# Proof.

To prove (a), suppose $v_1, v_2, \cdots, v_n$ is a basis of the real vector space $V$. Then span$(v_1, v_2, \cdots, v_n)$ in the complex vector space $V_{\mathbb{C}}$ contains all the vectors $v_1, v_2, \cdots, v_n, iv_1, iv_2, \cdots, iv_n$. Thus $v_1, v_2, \cdots, v_n$ spans the complex vector space $V_{\mathbb{C}}$.

# Proof.

To show that $v_1, v_2, \cdots, v_n$ is linearly independent in the complex vector space $V_{\mathbb{C}}$, suppose $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{C}$ and

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0.$$

Then the equation above and our definitions imply that

$$(Re\ \lambda_1)v_1 + \cdots + (Re\ \lambda_n)v_n = 0 \text{ and } (Im\ \lambda_1)v_1 + \cdots + (Im\ \lambda_n)v_n = 0.$$

Because $v_1, v_2, \cdots, v_n$ is linearly independent in $V$, the equations above imply $Re\ \lambda_1 = \cdots = Re\ \lambda_n = 0$ and $Im\ \lambda_1 = \cdots = Im\ \lambda_n = 0$. Thus we have $\lambda_1 = \cdots = \lambda_n = 0$. Hence $v_1, v_2, \cdots, v_n$ is linearly independent in $V_{\mathbb{C}}$, completing the proof of (a).

Clearly (b) follows immediately from (a).

# Complexification of an Operator

Now we can define the complexification of an operator.

9.5 **Definition** ***complexification of $T$, $T_{\mathbf{C}}$***

Suppose $V$ is a real vector space and $T \in \mathcal{L}(V)$. The ***complexification*** of $T$, denoted $T_{\mathbf{C}}$, is the operator $T_{\mathbf{C}} \in \mathcal{L}(V_{\mathbf{C}})$ defined by

$$\boxed{T_{\mathbf{C}}(u + iv) = Tu + iTv}$$

for $u, v \in V$.

You should verify that if $V$ is a real vector space and $T \in \mathscr{L}(V)$, then $T_{\mathbb{C}}$ is indeed in $\mathscr{L}(V_{\mathbb{C}})$. The key point here is that our definition of complex scalar multiplication can be used to show that that

$$T_{\mathbb{C}}(\lambda(u+iv)) = \lambda T_{\mathbb{C}}(u+iv)$$

for all $u, v \in V$ and all complex numbers $\lambda$.

# Complexification of an Operator

The next result makes sense because 9.4 tells us that a basis of a real vector space is also a basis of its complexification. The proof of the next result follows immediately from the definitions.

**9.7 Matrix of $T_{\mathbf{C}}$ equals matrix of $T$**

Suppose $V$ is a real vector space with basis $v_1, \ldots, v_n$ and $T \in \mathcal{L}(V)$. Then $\mathcal{M}(T) = \mathcal{M}(T_{\mathbf{C}})$, where both matrices are with respect to the basis $v_1, \ldots, v_n$.

# Every operator has an invariant subspace of dimension 1 or 2

We now show that every operator has an invariant subspace of dimension 1 or 2.

dim $U$ = 1 or 2

Existence of invariant subspace

**9.8** Every operator has an invariant subspace of dimension 1 or 2

Every operator on a nonzero finite-dimensional vector space has an invariant subspace of dimension 1 or 2.

$V = U \oplus U^{\perp}$ (induction)

Proof. Every operator on a nonzero finite-dimensional complex vector space has an eigenvalue (5.21) and thus has a 1-dimensional invariant subspace.

# Proof.

Hence assume $V$ is a real vector space and $T \in \mathscr{L}(V)$. The complexification $T_{\mathbb{C}}$ has an eigenvalue $a+bi$(by 5.21), where $a,b \in \mathbb{R}$. Thus there exist $u,v \in V$, not both 0, such that $T_{\mathbb{C}}(u+iv) = (a+bi)(u+iv)$. Using definition of $T_{\mathbb{C}}$, the last equation can be rewritten as

$$Tu + iTv = (au - bv) + (av + bu)i.$$

Thus

U=span (u, v)

$$Tu = au - bv \text{ and } Tv = av + bu.$$ invariant under T

Let $U$ equal the span in $V$ of the list $u,v$. Then $U$ is a subspace of $V$ with dimension 1 or 2. The equations above show that $U$ is invariant under $T$, completing the proof.

# The Minimal Polynomial of the Complexification

Suppose $V$ is a real vector space and $T \in \mathscr{L}(V)$. Repeated application of the definition of $T_{\mathbb{C}}$ shows that

$$(T_{\mathbb{C}})^n(u+iv) = T^n u + iT^n v$$

for every positive integer $n$ and all $u, v \in V$.

Notice that the next result implies that the minimal polynomial of $T_{\mathbb{C}}$ has real coefficients.

**9.10 Minimal polynomial of $T_{\mathbf{C}}$ equals minimal polynomial of $T$**

Suppose $V$ is a real vector space and $T \in \mathcal{L}(V)$. Then the minimal polynomial of $T_{\mathbf{C}}$ equals the minimal polynomial of $T$.

# Eigenvalues of the Complexification

Now we turn to questions about the eigenvalues of the complexification of an operator. Again, everything that we expect to work indeed works easily.

**9.11 Real eigenvalues of $T_{\mathbf{C}}$**

Suppose $V$ is a real vector space, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{R}$. Then $\lambda$ is an eigenvalue of $T_{\mathbf{C}}$ if and only if $\lambda$ is an eigenvalue of $T$.

Our next result shows that $T_{\mathbb{C}}$ behaves symmetrically with respect to an eigenvalue $\lambda$ and its complex conjugate $\bar{\lambda}$.

# Proof.

9.12 $T_{\mathbf{C}} - \lambda I$ and $T_{\mathbf{C}} - \bar{\lambda} I$

Suppose $V$ is a real vector space, $T \in \mathcal{L}(V)$, $\lambda \in \mathbf{C}$, $j$ is a nonnegative integer, and $u, v \in V$. Then

$$(T_{\mathbf{C}} - \lambda I)^j (u + iv) = 0 \quad \text{if and only if} \quad (T_{\mathbf{C}} - \bar{\lambda} I)^j (u - iv) = 0.$$

We will prove this result by induction on $j$. To get started, note that if $j = 0$ then (because an operator raised to the power $0$ equals the identity operator) the result claims that $u + iv = 0$ if and only if $u - iv = 0$, which is clearly true.

Thus assume by induction that $j \geq 1$ and the desired result holds for $j - 1$. Suppose $(T_{\mathbb{C}} - \lambda I)^j (u + iv) = 0$. Then

$$(T_{\mathbb{C}} - \lambda I)^{j-1} (T_{\mathbb{C}} - \lambda I)(u + iv) = 0.$$

# Proof.

Writing $\lambda = a + ib$, where $a, b \in \mathbb{R}$, we have

$$(T_{\mathbb{C}} - \lambda I)(u + iv) = (Tu - au + bv) + i(Tv - av - bu)$$

and

$$(T_{\mathbb{C}} - \overline{\lambda} I)(u - iv) = (Tu - au + bv) - i(Tv - av - bu)$$

Our induction hypothesis, 9.13 , and 9.14 imply that $(T_{\mathbb{C}} - \overline{\lambda} I)^{j-1}((Tu - au + bv) - i(Tv - av - bu)) = 0$. Now this equation and the equation above imply that $(T_{\mathbb{C}} - \bar{\lambda} I)^{j}(u - iv) = 0$, completing the proof in one direction. The other direction is proved by replacing $\lambda$ with $\bar{\lambda}$, replacing $v$ with $-v$, and then using the first direction.

# Nonreal eigenvalues come in pairs

An important consequence of the result above is the next result, which states that if a number is an eigenvalue of $T_{\mathbb{C}}$, then its complex conjugate is also an eigenvalue of $T_{\mathbb{C}}$.

**9.16 Nonreal eigenvalues of $T_{\mathbf{C}}$ come in pairs**

Suppose $V$ is a real vector space, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{C}$. Then $\lambda$ is an eigenvalue of $T_{\mathbf{C}}$ if and only if $\bar{\lambda}$ is an eigenvalue of $T_{\mathbf{C}}$.

# Characteristic Polynomial of the Complexification

The next result states that the multiplicity of an eigenvalue of a complexification equals the multiplicity of its complex conjugate.

**9.17 Multiplicity of $\lambda$ equals multiplicity of $\bar{\lambda}$**

Suppose $V$ is a real vector space, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{C}$ is an eigenvalue of $T_{\mathbf{C}}$. Then the multiplicity of $\lambda$ as an eigenvalue of $T_{\mathbf{C}}$ equals the multiplicity of $\bar{\lambda}$ as an eigenvalue of $T_{\mathbf{C}}$.

# Existence of Eigenvalues

We have seen an example [5.8(a)] of an operator on $\mathbb{R}^2$ with no eigenvalues. The next result shows that no such example exists on $\mathbb{R}^3$.

**9.19 Operator on odd-dimensional vector space has eigenvalue**

Every operator on an odd-dimensional real vector space has an eigenvalue.

# Characteristic polynomial of $T_{\mathbb{C}}$

In the previous chapter we defined the characteristic polynomial of an operator on a finite-dimensional complex vector space (see 8.34). The next result is a key step toward defining the characteristic polynomial for operators on finite-dimensional real vector spaces.

**9.20 Characteristic polynomial of $T_{\mathbf{C}}$**

Suppose $V$ is a real vector space and $T \in \mathcal{L}(V)$. Then the coefficients of the characteristic polynomial of $T_{\mathbf{C}}$ are all real.

# Characteristic polynomial

Now we can define the characteristic polynomial of an operator on a finite-dimensional of its compelxification.

**9.21 Definition** ***Characteristic polynomial***

Suppose $V$ is a real vector space and $T \in \mathcal{L}(V)$. Then the ***characteristic polynomial*** of $T$ is defined to be the characteristic polynomial of $T_{\mathbf{C}}$.

$\lambda$ and $\bar{\lambda}$ in pairs

+ same multiplicity

# Eigenvalues

In the next result, the eigenvalues of $T$ are all real (because $T$ is an operator on a real vector space).

**9.23 Degree and zeros of characteristic polynomial**

Suppose $V$ is a real vector space and $T \in \mathcal{L}(V)$. Then

(a) the coefficients of the characteristic polynomial of $T$ are all real;

(b) the characteristic polynomial of $T$ has degree $\dim V$;

(c) the eigenvalues of $T$ are precisely the real zeros of the characteristic polynomial of $T$.

# Cayley-Hamilton Theorem

In the previous chapter, we proved the Cayley-Hamilton Theorem (8.37) from complex vector spaces. Now we can also prove it for real vector spaces.

**9.24 Cayley–Hamilton Theorem**

Suppose $T \in \mathcal{L}(V)$. Let $q$ denote the characteristic polynomial of $T$. Then $q(T) = 0$.

# Characteristic polynomial is a multiple of minimal polynomial

We can now prove another result that we previously knew only in the complex case.

**9.26 Characteristic polynomial is a multiple of minimal polynomial**

Suppose $T \in \mathcal{L}(V)$. Then

(a) the degree of the minimal polynomial of $T$ is at most $\dim V$;

(b) the characteristic polynomial of $T$ is a polynomial multiple of the minimal polynomial of $T$.

# Homework Assignment 27

9.A: 6, 8, 9, 10, 12, 15, 16, 18.