南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

# Calculus Supplement Homework

# Chapter 2 Limits and Continuity

1. Suppose $\lim\limits_{x\to 0^+} f(x) = a$, $\lim\limits_{x\to 0^-} f(x) = b$, then $\lim\limits_{x\to 0^-}\left(f(x-\sin x) + 2f\left(x^2+x\right)\right) =$
   (A) $a+2b$.  (B) $b+2a$.  (C) $3a$.  (D) $3b$ .
2. Compute the following limits:

   (1) $\lim\limits_{x\to 2} \dfrac{x^2-x-2}{\sqrt[3]{x^2+23}-3}$;  (2) $\lim\limits_{x\to 0} \dfrac{\sin(1-\cos x)}{(\tan x)^2}$;

   (3) $\lim\limits_{x\to -\infty} (x+\sqrt{x^2-x+4})$;  (4) $\lim\limits_{x\to 1} \dfrac{2x^2+5}{x-3}$;

   (5) $\lim\limits_{x\to 1}\left(\dfrac{3}{1-x^3}-\dfrac{4}{1-x^4}\right)$;  (6) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$;

   (7) $\lim\limits_{x\to 1} \dfrac{x^{2020}-1}{x^{2019}-1}$;  (8) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1-x/2}{x^2}$;

   (9) $\lim\limits_{x\to \frac{\pi}{2}-} \tan x$;  (10)★ $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$;

   (11) $\lim\limits_{x\to 1} \dfrac{x^4-1}{x^3-1}$;  (12) $\lim\limits_{x\to 0} \left[\dfrac{1}{x}\right]\sin x$;

   (13) $\lim\limits_{x\to 0} \dfrac{\tan(2x)}{3x}$;  (14) $\lim\limits_{x\to -2} \dfrac{x^3+8}{x^2-x-6}$;

   (15) $\lim\limits_{x\to 0} \dfrac{1-\cos(2x)}{x\sin(2x)}$;  (16) $\lim\limits_{x\to \infty}(\sqrt{x^2+x}-\sqrt{x^2-2x+3})$;

   (17) $\lim\limits_{x\to \infty} \dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{2x+1}}$;  (18) $\lim\limits_{x\to \infty}(\sqrt{x}(\sqrt{x+1}-\sqrt{x}))$;

   (19) $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{\sin(2x)}$;  (20)★ $\lim\limits_{x\to 1} \dfrac{\sqrt{2x-x^4}-\sqrt[3]{x}}{1-\sqrt[4]{x^3}}$.

3. Determine the following statement is correct or not. If so, prove it. If not, find a counter-example.
   (1) ★ $\lim\limits_{x\to c} f(x) = A$ and $\lim\limits_{y\to A} g(y) = B$ imply that $\lim\limits_{x\to c} g(f(x)) = B$.
   (2) $\lim\limits_{x\to c} |f(x)| = |l|$, then $\lim\limits_{x\to c} f(x) = l$.
   (3) $\lim\limits_{x\to c}[f(x)+g(x)]$ exists, then $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ both exist.
   (4) If $f(x) > 0$ and $\lim\limits_{x\to c} f(x) = l$, then $l > 0$.
   (5) If $f^2$ is continuous, then $f$ is continuous.
   (6) If $f^3$ is continuous, then $f$ is continuous.

4. Find the asymptotic lines for the following function:

$$(1) f(x) = \frac{x^3 + x + 1}{(x-1)(x+2)}; \qquad (2) f(x) = \frac{2x^2 - x + 3 + x\sin x}{x^2};$$
$$(3) f(x) = \frac{(|x|+1)^3}{(x-2)(x-3)}; \qquad (4)\bigstar\ f(x) = \frac{x\sqrt{x^2+3x+1}}{3x+10^6}.$$

5. Find $a$ such that the following function is continuous:

$$f(x) = \begin{cases} \dfrac{1-\cos\sqrt{x}}{ax}, & x > 0 \\ 1, & x \le 0 \end{cases}.$$

6. Let $m$ and $n$ be positive integers. Compute

$$\lim_{x\to\pi} \frac{\sin(mx)}{\sin(nx)}.$$

7. Describe that $L$ is not the limit of $f(x)$ as $x \to c$.
8. Find $a$ and $b$ such that

$$\lim_{x\to 0}\left(\frac{\sqrt{x^2+x+1}}{x} - \frac{a}{x} - b\right) = 0.$$

9. Suppose that

$$\lim_{x\to 0}\frac{f(x)}{x^2} = 1,$$

compute

$$\lim_{x\to 0}\frac{f(x)}{1-\cos x}.$$

10. Find $a$ and $b$ such that

$$\lim_{x\to\pi/2}\frac{\sqrt{x}-a}{\cos x} = b$$

11. Prove that

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

Moreover, suppose $\lim_{x\to 0} g(x) = 0$, prove that

$$\lim_{x\to 0} g(x)\sin\frac{1}{x} = 0.$$

12. Given $\lim_{x\to 0^+} f(x) = l$ and $\lim_{x\to 0^-} f(x) = m$, determine the following limits exist or not. If so, find the limit.

$$\lim_{x\to 0} f(-x); \qquad \lim_{x\to 0^+} f(x^2 - x); \qquad \lim_{x\to 0^-}(2f(-x) + f(x^2)).$$

13. Given the function $f(x) = \lceil x \rceil - x$, where $\lceil x \rceil = \min\{n \mid n \geq x\}$ is the smallest integer no less than $x$, find the one-sided limits $\lim_{x\to k^-} f(x)$ and $\lim_{x\to k^+} f(x)$, where $k$ is an integer. What can we say for $\lim_{x\to k} f(x)$ ?
14. Determine if the following limits exist or not. If so, find the limit. If not, explain why.

$$(1)\bigstar \lim_{x\to 0}\frac{(1+x)^{\frac{1}{m}} - 1 - \frac{1}{m}x}{x^2}; \qquad (2)\lim_{x\to 0}\frac{\sqrt{3} - \sqrt{2+\cos x}}{\sin^2 2x};$$

$$(3)\lim_{x\to\infty}\frac{\sin^2 x}{x^2+3x}; \qquad (4)\bigstar \lim_{x\to\infty}(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x})x^{3/2}.$$

15. Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$. Suppose that $f(1) < 0$, $f(2) > 0$, $f(3) < 0$. Prove that the equation $f(x) = 0$ has 4 real roots.
16. Find all discontinuities of $f(x)$ and give their types:

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & 0 < x < 1; \\ \dfrac{1}{x}, & x \geq 1. \end{cases}$$

17. Find $a$ and $b$ such that

$$\lim_{x\to 1}\frac{x^3 + 2x + a}{x - 1} = b.$$

18. Find $k$ such that the following function is continuous:

$$f(x) = \begin{cases} \dfrac{\sin(3x)}{x}, & x \neq 0, \\ x^2 + k, & x = 0. \end{cases}$$

19. Find all removable discontinuities of the function

$$f(x) = \frac{x(x+1)}{\sin(\pi x)}.$$

20. Prove that any polynomial of order 3 has at least one real root. What about polynomial of order $2k+1$?
21. Prove that the equation $2\sin x = 3 - 2x$ has a root in $[0, 1]$.
22. Find $a$ and $b$ such that

$$\lim_{x\to 2}\frac{x-2}{x^2 + ax + 1} = b.$$

23. Find $a$ such that $\lim_{x\to 1} f(x)$ exist:

$$f(x) = \begin{cases} ax^2, & x < 1; \\ 1, & x = 1; \\ \cos(\pi x), & x > 1. \end{cases}$$

24. Suppose $f(x)$ is continuous in $[0, 2a]$ with $f(0) = f(2a)$. Prove that there exist a point $x_0 \in [0, a]$ such that $f(x_0) = f(x_0 + a)$.

# Chapter 3 Derivatives

1. ★ Which of the following functions is not differentiable at $x = 0$?

   (A) $f(x) = |x| \sin |x|$. (B) $f(x) = |x| \sin \sqrt{|x|}$.

   (C) $f(x) = \cos |x|$. (D) $f(x) = \cos \sqrt{|x|}$.

2. Let
$$f(x) = \begin{cases} x^4 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$
   Then the largest positive integer $n$, for which $f^{(n)}(0)$ exists, is

   (A) 1. (B) 2. (C) 3. (D) 4.

3. Compute the derivatives of the following functions:
$$(1)\, y = \frac{3}{(5x^2 + \sin(2x))^{3/2}}; \qquad (2)\, y = \frac{x \sin x + \cos x}{x \cos x - \sin x};$$
$$(3)\, y = \sin(\sin(\sin x)); \qquad (4)\, y = \left(\frac{x+1}{x-1}\right)^2 \sin x;$$
$$(5)\, y = (1 + x^2) \cos(2x); \qquad (6)\, y = x^2 \sin^3(2x);$$
$$(7)\, y = \frac{\sec t}{1 + \tan t}.$$

4. Find the derivatives at $x = 0$ for the following functions, or explain why it does not exist.
$$(1)\, y = x^{2/3}; \qquad (2)\, y = \frac{1 + x^2}{\sin x + \cos x};$$
$$(3)\, y = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

5. Find second order derivatives for the following functions:
$$(1)\, y^4 - 2x = y^2 + x^2; \qquad (2)\, y = f(x^n), n < 0; \qquad (3)\, y = f(f(x)).$$

6. ★ Determine if the following function is continuous at $x = 0$ and if it is differentiable at $x = 0$.
$$f(x) = \begin{cases} x, & x \leq 0; \\ \dfrac{1}{n}, & \dfrac{1}{n+1} < x \leq \dfrac{1}{n}. \end{cases}$$

7. ★ Let $f(x) = x(b^2 - x^2)$ for $x \in [0, 1)$, and $f(x) = af(x + 1)$ when $x \in [-1, 0)$. Find $a$

and $b$ such that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

8. Let $P$ be a point other than the origin on the parabola $x^2 = 2py$. The tangent line passing through $P$ intersects the $x$-axis and $y$-axis at the points $Q$ and $R$. Prove that $PQ = QR$.
9. The point $P(a, b)$ lies on the curve $l: (y - x)^3 = y + x$, and the slope of the tangent line of $l$ at $P(a, b)$ is 3. Find the values of $a$ and $b$.
10. Determine the intervals on which $f(x) = |x^2 + 2x|$ is differentiable.
11. ★ Assume $f(0) = 0$. Determine if the following statement is correct or not. If so, prove it. If not, give a counter-example.
    (1) $\lim\limits_{h\to 0} \frac{1}{h^2} f(1 - \cos h)$ exists, then $f$ is differentiable at $x = 0$.
    (2) $\lim\limits_{h\to 0} \frac{1}{h}[f(2h) - f(h)]$ exists, then $f$ is differentiable at $x = 0$.
12. Assume $f(x)$ is differentiable at $x = 0$ with $f(0) = 0$, compute:

$$\lim_{x\to 0} \frac{x^2 f(x) - 2f(x^3)}{x^3}.$$

13. Determine if the following statement is correct or not, and state your reasons.
    (1) Let $f = g + h$, if $f$ has derivative at $x = x_0$, then $g, h$ have derivatives at $x = x_0$.
    (2) Let $f = g + h$, if $g$ has derivative at $x = x_0$ and $h$ is not differentiable at $x = x_0$, then $f$ is not differentiable at $x = x_0$.
    (3) Let $f = g \cdot h$, if $f$ has derivative at $x = x_0$, then $g, h$ have derivatives at $x = x_0$.
    (4) Let $f = g \cdot h$, if $g$ has derivative at $x = x_0$ and $h$ is not differentiable at $x = x_0$, then $f$ is not differentiable at $x = x_0$.
14. Given $x = \tan y$, justify that

$$(1 + x^2)y'' + 2xy' = 0.$$

15. Suppose that $g(x)$ is differentiable at $x = 0$ and $g(0) = 0, g'(0) = 0$. Compute $f'(0)$ for $f(x)$ is defined below:

$$f(x) = \begin{cases} g(x)\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

16. Prove the Leibniz's Rule for higher-order derivatives of products:

$$\frac{\mathrm{d}^n(uv)}{\mathrm{d}x^n} = \sum_{k=0}^{n} C_n^k \frac{\mathrm{d}^k u}{\mathrm{d}x^k}\frac{\mathrm{d}^{n-k} v}{\mathrm{d}x^{n-k}}.$$

    Here $\frac{\mathrm{d}^0 u}{\mathrm{d}x^0} = u(x)$.
17. Find $y^{(2020)}$ for $y = \sin x$.
18. Let $y = |x|^3$. Find $y'$ and $y''$, and prove that $y^{(3)}(0)$ does not exist.

19. Suppose that $u(x), v(x), w(x)$ are differentiable functions of $x$. Express $\frac{d(uv/w)}{dx}$ by $u$, $v$, $w$, $\frac{du}{dx}$, $\frac{dv}{dx}$, $\frac{dw}{dx}$.
20. Assume that $f'(a)$ exists, compute

$$\lim_{h\to 0} \frac{f(a+h) - f(a-2h)}{h}.$$

21. Assume $y = f(x)$ and $y = \sin x$ have the same tangent at the origin. Compute:

$$\lim_{x\to\infty} \sqrt{xf\left(\frac{2}{x}\right)}.$$

22. Find $a$ and $b$ such that $y = x^3 + ax + b$ and $2y = xy^3 - 1$ are tangent at $(1, -1)$.
23. ★ If $f(x)$ is defined on $(-\infty, +\infty)$, $f(x) \neq 0$, $f'(0) = 1$, and $\forall x, y \in (-\infty, \infty)$, and $f(x+y) = f(x)f(y)$. Prove that, $\forall x \in (-\infty, \infty)$, $f(x)$ is differentiable and $f'(x) = f(x)$.
24. ★ Compute the 100th derivative of the following function:

$$y = x^2 \sin x.$$

25. ★ Find $b$ in $f(x) = x^3 - bx^2 + 7x$ so that there exists some $c \in [0, 1]$ such that $f'(c) = 3$.
26. Using the following information to find the values of $a, b$, and $c$ in the formula $f(x) = \frac{x+a}{bx^2+cx+2}$.
    (1) The values of $a, b$ and $c$ are either 0 or 1.
    (2) The graph of $f$ passes through the point $(-1, 0)$.
    (3) The line $y = 1$ is an asymptote of the graph of $f$.

# Chapter 4

# Applications of Derivatives

1. Suppose $f(x)$ is differentiable in $(0, \infty)$, which one of the following statements is true?
   (A) If $f(x)$ has a unique zero point, then $f'(x)$ has no zero point.
   (B) If $f'(x)$ has at least one zero point, then $f(x)$ has at least two zero points.
   (C) If $f(x)$ has no zero point, then $f'(x)$ has at most one zero point.
   (D) If $f'(x)$ has no zero point, then $f(x)$ has at most one zero point.
2. Let $c > 0$. How many real roots are there for the equation $x^3 - 6x^2 + 9x + c = 0$ ?
   (A) 0. (B) 1. (C) 2. (D) 3.
3. Given $f'(x) = (x-1)^2(x-2)^3$, which of the following statements is wrong?
   (A) $(1, f(1))$ is an inflection point. (B) $(2, f(2))$ is an inflection point.
   (C) $(3, f(3))$ is not the inflection point. (D) There exists other inflection points.
4. Find all of the local extreme of the following functions:

   (1) $f(x) = \sin^3 x + \cos^3 x,\ x \in [0, 2\pi]$; (2) $f(x) = |x(x^2 - 4)|$;

   (3) $f(x) = \dfrac{x(x^2+1)}{x^4 - x^2 + 1}$; (4) $g(t) = |t^2 - 4t + 1|,\ x \in [0, 5]$.

5. Sketch the graph of $f(x) = x^{\frac{2}{3}} - (x^2 - 1)^{\frac{1}{3}}$.
6. Suppose the tangent line at the inflection point $(1, 1)$ for the function $y = ax^3 + bx^2 + cx$ is horizontal. Find $a$, $b$ and $c$.
7. ★ Prove that the equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$ has at least one real root in $(0, 1)$ given the coefficients $a_n, a_{n-1}, ..., a_0$ satisfying $\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + ... + a_0 = 0$.
8. Determine the number of real roots of $x^2 = x \sin x + \cos x$.
9. Determine if the following statements are correct or not. If so, prove it. If not, give a counter-example.
   (1) ★ If $f'(c) > 0$, $f(x)$ is strictly increasing in some neighborhood of $c$.
   (2) If $f'(c) > 0$, $f(x) > f(c)$ for $x \in (c, c+\delta)$ for some $\delta > 0$.
   (3) If $(1, f(1))$is the inflection point, $f''(1) = 0$.
   (4) ★ If $f'(x_0)$ exists and $\lim\limits_{x \to x_0} \frac{f''(x)}{x - x_0} = 1$, then $(x_0, f(x_0))$ is the inflection point of $y = f(x)$.

(5) If $f(x)$ is differentiable on $[a, b]$, and $f'(a)f'(b) < 0$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

(6) If $f(x)$ is differentiable in $[a, b]$ and $f'(a)f'(b) > 0$. Then there exists $c \in (a, b)$ such that $f(c) = 0$.

10. $f(x)$, $g(x)$ are continuous on $[a, b]$, and have second derivative function on $(a, b)$. $f(x)$ and $g(x)$ have the same absolute maximum at interior points, $f(a) = g(a)$, $f(b) = g(b)$. Prove that there exists $c \in (a, b)$ such that $f''(c) = g''(c)$.
11. ★ Find the extreme values of the function $y = f(x)$ defined by $y^3 + xy^2 + x^2y + 6 = 0$.
12. ★ Suppose $f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$ and $f(2) = 3f(0)$. Prove that there exists at least one $c \in (0, 2)$ such that $(1 + c)f'(c) = f(c)$.
13. Suppose that the graph of $f(x)$ is concave up on $I = (a, b)$. Prove that if $f$ has a local minimum value at $x = x_0$ in $I$, then $x_0$ is the unique local minimum point of $f$ on $I$.
14. Find the value of $a$ and $b$ such that $(1, 3)$ is a point of inflection of $y = ax^3 + bx^2$.
15. Suppose that the function $f(x)$ is defined on $\mathbb{R}$, and satisfying that $f(a + b) = f(a)f(b)$. Suppose $f$ is differentiable at $x = 0$, prove that $f'(x) = f'(0)f(x)$ for any $x \in \mathbb{R}$.
16. If $f$ has derivative at $x = 0$ and $|f(x)| \leq |\sin x|$ for all $x$, prove that $f(0) = 0$ and $|f'(0)| \leq 1$.

# Chapter 5 Integrals

1. Let $I_1 = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\tan x}{x}\,\mathrm{d}x$, $I_2 = \int_0^{\frac{\pi}{4}} \frac{x}{\tan x}\,\mathrm{d}x$, then which of the following is correct?

   (A) $I_1 > I_2 > 1$. (B) $1 > I_1 > I_2$.

   (C) $I_2 > I_1 > 1$. (D) $1 > I_2 > I_1$.

2. Let the function $f(x)$ be positive and continuous on $[a,b]$. Then the number of roots of the equation $\displaystyle\int_a^x f(t)\,\mathrm{d}t + \int_b^x f(t)\,\mathrm{d}t = 0$ in $(a,b)$ is

   (A) 0. (B) 1. (C) 2. (D) 3.

3. Let $f(x)$ be a continuous function, and $a$ is a nonzero constant. Which of the following function is an odd function ?

   (A) $\displaystyle\int_a^x \left(\int_0^u tf(t^2)\,\mathrm{d}t\right)\mathrm{d}u$. (B) $\displaystyle\int_0^x \left(\int_a^u f(t^3)\,\mathrm{d}t\right)\mathrm{d}u$.

   (C) $\displaystyle\int_0^x \left(\int_a^u tf(t^2)\,\mathrm{d}t\right)\mathrm{d}u$. (D) $\displaystyle\int_a^x \left(\int_0^u (f(t))^2\,\mathrm{d}t\right)\mathrm{d}u$.

4. Given $f(x) = \begin{cases} 2x, & x \le 0 \\ \sin x, & x > 0 \end{cases}$, and $F(x)$ is the anti-derivative of $f$ with $F(0) = 1$ Then $F(x) = (\quad)$.

5. If $f'(\sin x) = \cos(2x)$, then $f(x) = (\quad)$.

6. ★ Suppose that $f(x) = f(x+\pi)$ and $f(x) = \sin x, 0 \le x \le \pi$, find one anti-derivative $F(x) = (\quad)$.

7. Let $M = \displaystyle\sum_{k=1}^{2019} k^{10}$ and $N = \displaystyle\int_0^{2019} x^{10}\,\mathrm{d}x$. Which one is larger?

8. Compute the following integrals:

   (1) $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}(\sin x + 1)\,\mathrm{d}x$; (2) $\displaystyle\int_0^2 |x-1|\,\mathrm{d}x$;

   (3) $\displaystyle\int (ax^2+b)^m x\,\mathrm{d}x$, $(m \ne -1)$; (4) $\displaystyle\int_0^1 x^2(1-x)^{10}\,\mathrm{d}x$;

   (5) $\displaystyle\int_{\frac{1}{2}}^{1} x^{-\frac{1}{4}}(1-x^{\frac{3}{4}})^{\frac{1}{3}}\,\mathrm{d}x$; (6) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2\left(2x+\frac{\pi}{4}\right)\mathrm{d}x$.

9. If $M = \displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^{100}\sin x\,\mathrm{d}x$, $N = \displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^{100}\sin^{100} x\,\mathrm{d}x$, $P = \displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^{100}\sin^{90} x\,\mathrm{d}x$. Compare them.

10. Express the following limit by a definite integral and compute the limit

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\cos\frac{\pi}{n}}+\sqrt{1+\cos\frac{2\pi}{n}}+\cdots+\sqrt{1+\cos\frac{n\pi}{n}}\right)$$

11. Compute the area between the graphs of $y^2 = 2px$ and $x^2 = 2py$. Here $p$ is a positive constant.
12. Assume that $f$ is a continuous, periodic function ($T$ is the period). Prove

$$\lim_{x\to+\infty}\frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{T}\int_0^T f(t)\,dt.$$

13. The function $f(x)$ is continuous on $[0,1]$ and $|f(x)-f(y)| \le |x-y|$ holds for any $x, y \in [0,1]$. Let $n$ be a positive integer, prove that

$$\left|\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right| \le \frac{1}{2n}$$

14. Compute the derivative for the following functions:

$$(1)\, h(s) = \int_s^{s^2}\sqrt{1+x^2}\,\mathrm{d}x; \qquad (2)\, y = \int_{x^2}^{\sin x}\frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t;$$

$$(3)\, g(x) = \int_0^x \sin t\,\mathrm{d}t.$$

15. Let $F(x) = \displaystyle\int_0^x tf(x^2-t^2)\mathrm{d}t$. Compute $F'(x)$.
16. Assume $f(x)$ is increasing on $(-\infty,\infty)$, and $f(x) > 0$. Show that

$$F(x) = \frac{\int_0^x tf(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t}$$

is increasing on the interval $(0,\infty)$.
17. Assume $f(x)$ is continuous on $[a,b]$, differentiable on $(a,b)$, and $f'(x) < 0$. Let $F(x) = \frac{1}{x-a}\displaystyle\int_a^x f(t)\,\mathrm{d}t$. Show that $F'(x) \le 0$, $\forall x \in (a,b)$.
18. Assume $f(x)$ is a periodic with period $T$. Prove that $\displaystyle\int_x^{x+T} f(t)\,\mathrm{d}t$ is a constant. Furthermore, if $\displaystyle\int_0^T f(t)\,\mathrm{d}t = 0$, then $g(x) = \displaystyle\int_0^x f(t)\,\mathrm{d}t$ is also a periodic function with period $T$.

19. Express

$$\lim_{n\to\infty} \frac{1}{n\sqrt{2n}} \left(\sqrt{1} + \sqrt{3} + \cdots + \sqrt{2n-1}\right)$$

as a definite integral, then evaluate this integral.

20. Assume $f(x)$ is continuous and $\displaystyle\int_0^{x^2-1} f(t)\,\mathrm{d}t = x - 1$, $\forall x \geq 0$. Compute $f(7)$.

21. Compute $\displaystyle\int_{-1}^{x} f(t)\,\mathrm{d}t$ where

$$f(x) = \begin{cases} t, & t \in [0,1]; \\ 0, & t < 0, \text{ or } t > 1. \end{cases}$$

22. Assume $f(x) > 0$ is continuous on $[a, b]$. Prove that the equation has only one real root between $(a, b)$:

$$\int_a^x f(t)\,\mathrm{d}t = 2\int_x^b f(t)\,\mathrm{d}t.$$

23. Assume $f$ is continuous on $[a, b]$. Define

$$F(x) = \int_a^x f(t)(x-t)\,\mathrm{d}t.$$

Prove that $F$ is second order differentiable and $F''(x) = f(x)$ in $(a, b)$.

24. ★ Let $f(n) = \displaystyle\sum_{m=1}^{n} \int_0^m \cos\frac{2\pi n\lfloor x+1\rfloor}{m}\,\mathrm{d}x$, here $\lfloor x+1\rfloor$ is the largest integer which is less than or equal to $x + 1$. Evaluate $f(2021)$.

# Chapter 6

# Applications of Definite Integrals

1. Given a ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, compute the surface area and the volume of the solid generated by revolving this ellipse about $x$-axis and $y$-axis.
2. Compute the arc length of $y=\frac{x^2}{2p}$ from the origin to $\left(x, \frac{x^2}{2p}\right)$, $(p>0)$.
3. Find the volume of the solid generated by revolving the region located in the upper half-plane and bounded by the curves $x=y^2-1$ and $x-y=5$ about the x-axis.
4. ★ Let $S=\{(x,y)|-3\leq x\leq 3, 0\leq y\leq x^3-4x+15\}$, find the volume of the solid generated by revolving $S$ about the y-axis.
5. Find the area of the surface generating by revolving $y=x^{2/3}$, $1\leq x\leq 8$ about the x-axis.
6. Assume $y=kx$ is the tangent line of $y=\sqrt{x-1}$.
   (1) Find $k$ and the point where the line is tangent to the curve.
   (2) Find the area bounded by the tangent line, the curve and $x$-axis.
   (3) Find the volume of the solid generated by revolving the above area about $x$-axis.
7. Find the volume of the "triangular" region bounded on the left by $x+y=2$, on the right by $y=x^2$, and above by $y=2$ revolving by the line $y=\frac{7}{4}$.
8. ★ Find the volume of the solid generated by revolving the region bounded by $y=x$ and $y=x^2$ about the line $y=x$.
9. Let $D_1$ be the region on the plane bounded by $y=2x^2$, $y=0$, $x=2$ and $x=a$; let $D_2$ be the region on the plane bounded by $y=2x^2$, $y=0$ and $x=a$. Here $0<a<2$.
   (1) Find the volume $V_1$ of the solid generated by revolving $D_1$ about the $x$-axis, and the volume $V_2$ of the solid generated by revolving $D_2$ about the $y$-axis.
   (2) Find $a$ such that $V_1+V_2$ reaches its maximum.
10. Let $D$ be the region which includes two parts. The first part is $x^2+y^2\leq 2(y\geq\frac{1}{2})$, and the second part is $x^2+y^2\leq 1(y\leq\frac{1}{2})$. Find the volume and the surface area of the solid by revolving $D$ about $y$-axis.
11. Find the volume of the solids generated by revolving the region bounded by $y=\sqrt{x}, y=2, x=0$ about the line $y=2$.
12. The region R is enclosed by the curve $y=2-\sqrt{x}$, $x=1$ and $y=2$, find the surface area of the solid generated by R revolving about $x=1$. (You only need to write down the formula without calculate to the numbers.)

13. The graph of $y = x^3$ on $[0, 1]$ is revolved about the y-axis to form a tank that is then filled with salt water from the Dead Sea (density is about $11400\text{N}/m^3$). How much work does it take to pump all the water to the top of the tank?
14. Compute the area of the surface generated by revolving the graph of the following curve about the $x$-axis:
$$y = \cos x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

# Chapter 7

# Transcendental Functions

1. Let $f(x) = \frac{1+e^{\frac{1}{x}}}{-1+e^{\frac{1}{x}}}$ for $x \neq 0$ and $f(0) = 1$. Then $x = 0$ is a
   (A) jump discontinuity. (B) removable discontinuity.
   (C) continuous point. (D) infinite discontinuity.
2. If $f(x) = \frac{\ln|x|}{|x-1|}\sin x$, then the function $f(x)$ has
   (A) 1 removable discontinuity and 1 jump discontinuity.
   (B) 2 removable discontinuities.
   (C) 1 removable discontinuity and 1 infinite discontinuity.
   (D) 2 jump discontinuities.
3. Assume that $f'(0) = 3$, $f''(0) = 5$, $f'(1) = -4$, and $f''(1) = -7$. Let $g(x) = f(\ln x)$. Then $g''(1) = (\quad)$.
4. Compute the following limits:

(1) $\displaystyle\lim_{x\to\infty}\left(\frac{x^2}{(x-1)(x+3)}\right)^x$;

(2) $\displaystyle\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}$;

(3) $\displaystyle\lim_{x\to\infty}(\pi-2\arctan x)\ln x$;

(4) $\displaystyle\lim_{x\to\infty}\left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}$;

(5) $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x\sin(x^2)}$;

(6) $\displaystyle\lim_{x\to 0}\frac{x\cot x-1}{x^2\ln(1+x)}$;

(7) $\displaystyle\lim_{x\to 0}\cot x\left(\frac{1}{\sin x}-\frac{1}{x}\right)$;

(8) $\displaystyle\lim_{x\to 1}\frac{x-x^x}{1-x+\ln x}$;

(9) $\displaystyle\lim_{x\to 1}x^{\frac{x}{1-x}}$;

(10) $\displaystyle\lim_{\theta\to\frac{\pi}{4}}\frac{\cos(\pi\tan\theta)+1}{\theta^2-\pi^2/16}$;

(11) $\displaystyle\lim_{x\to 0}\frac{x\arcsin^2 x}{\sin x-x}$;

(12) $\displaystyle\lim_{x\to 0}\left(\frac{1}{\arcsin^2 x}-\frac{1}{x^2}\right)$;

(13) $\displaystyle\lim_{x\to 1-}\ln x\ln(1-x)$;

(14) $\displaystyle\lim_{x\to 0+}(e^x-x-1)^{\frac{1}{\ln x}}$;

(15) $\displaystyle\lim_{x\to 0+}\frac{\arctan^3\sqrt{x}}{\ln(1+\sqrt{x})\sin x}$;

(16) $\displaystyle\lim_{x\to\infty}\frac{x^2\arctan x+x\sin x+1}{3x^2\sec^{-1}(-x)-5}$;

(17) $\displaystyle\lim_{x\to 0+}(\cos x)^{\frac{1}{x^2}}$;

(18) $\displaystyle\lim_{x\to\infty}x^3e^{-x}$;

(19) $\displaystyle\lim_{x\to+\infty}\frac{\int_1^x\left(t^2\left(e^{\frac{1}{t}}-1\right)-t\right)dt}{x^2\ln\left(1+\frac{1}{x}\right)}$;

(20) $\displaystyle\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{\ln(1+x)}\right)$.

5. Prove that for any $a > 1$, $0 < b < 1$ and $k > 0$, the following results hold:
   (1) $x^k = o(a^x)$ as $x \to \infty$. That is, $\lim_{x\to\infty} \frac{x^k}{a^x} = 0$.
   (2) $\log_a x = o(x^k)$ as $x \to \infty$. That is, $\lim_{x\to\infty} \frac{\log_a x}{x^k} = 0$.
   (3) $b^x = o(x^{-k})$ as $x \to \infty$. That is, $\lim_{x\to\infty} \frac{b^x}{x^{-k}} = 0$.
6. Find the order of the following infinitesimals as $x \to 0$ :
   (1) $\sin x - x$;
   (2) $\tan x - \sin x$;
   (3) $x - \ln(1 + x)$;
   (4) $\mathrm{e}^x - x - 1$.
7. Compute the derivatives of the following functions:

$$(1)\, y = \ln(\sin(2x)); \qquad (2)\, y = \frac{(5x+1)(x^3-2x)}{\sqrt{x^2-1}};$$
$$(3)\, y = \mathrm{e}^{x^2+\ln x}; \qquad (4)\, y = x^{\sin x};$$
$$(5)\, y = (\sin x)^x.$$

8. Compute the following limit:

$$\lim_{x\to 0} \frac{\int_0^{x^2} \cos(t^2)\,\mathrm{d}t}{x^2}.$$

9. Find the length of the following curve form $x = \frac{\pi}{6}$ to $x = \frac{\pi}{4}$:

$$y(x) = \int_{\frac{\pi}{6}}^{x} \frac{\sqrt{\cos(2t)}}{\sin t}\,\mathrm{d}t.$$

10. Suppose that $g(x)$ is differentiable at $x = 0$ with $g(0) = 0$ and $g'(0) = g''(0) = 1$. Determine if the following function $f(x)$ is continuous at $x = 0$ or not, and if it is differentiable at $x = 0$. If so, find $f'(0)$. If not, state your reasons.

$$f(x) = \begin{cases} \dfrac{g(x) - \sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

11. Compute the derivatives of the following functions:

$$(1)\, y = \arcsin\left(\frac{1}{x}\right); \quad (2)\, y = \csc^{-1}(x^2); \quad (3)\, y = \arctan(\ln x);$$
$$(4)\, y = \arccos(\mathrm{e}^{-t}); \quad (5)\, y = x^{\arctan x}\ (x > 0); \quad (6)\, y = \sqrt[3]{x + \arcsin x}.$$

12. Find $\frac{dy}{dx}$ if
$$y = \int_{x^2+1}^{2x^2+3} t \tan \sqrt{x+t}\,\mathrm{d}t.$$

13. Find $a$, $b$ and $c$ such that the following function has second order derivatives on $\mathbb{R}$:
$$f(x) = \begin{cases} ax^2 + bx + c, & x \le 0 \\ \sin x + \mathrm{e}^x, & x > 0 \end{cases}.$$

14. For what values of $a$ and $b$ is
$$\lim_{x\to 0}\left(\frac{\tan(2x)}{x^3} + \frac{a}{x^2} + \frac{\sin(bx)}{x}\right) = 0\,?$$

15. Prove the following identities:
$$\begin{aligned} &(1)\, 2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1} = \frac{\pi}{2}, && (x \ge 0) \\ &(2) \int_0^x \int_0^u f(t)\,\mathrm{d}t\,\mathrm{d}u = \int_0^x f(u)(x-u)\,\mathrm{d}u, && f \text{ is continuous.} \end{aligned}$$

16. Assume $f''(a)$ exists, compute the following limit:
$$\lim_{x\to a}\frac{f(x) - f(a) - f'(a)(x-a)}{\sin(x-a)}.$$

17. Determine which of the following is correct when $x \to \infty$:
    (1) $\frac{1}{x+3} = O\left(\frac{1}{x}\right)$;
    (2) $\frac{1}{x} + \frac{1}{x^2} = O\left(\frac{1}{x}\right)$;
    (3) $\frac{1}{x} - \frac{1}{x^2} = o\left(\frac{1}{x}\right)$;
    (4) $x\ln x = o(x^2)$;
    (5) $\mathrm{e}^x + \mathrm{e}^{2x} = O(e^{2x})$;
    (6) $\frac{1}{x} - \sin\frac{1}{x} = O\left(\frac{1}{x^3}\right)$;
    (7) $\ln(\ln x) = O(\ln x)$;
    (8) $\frac{1}{x} - \sin\frac{1}{x} = O\left(\frac{1}{x^2}\right)$.

18. ★ Assume $f(x)$ is continuous and $\int_0^x tf(2x-t)\,\mathrm{d}t = \frac{1}{2}\arctan x^2$ with $f(1) = 1$. Compute $\int_1^2 f(x)\,\mathrm{d}x$.

19. Given the following function

$$f(x) = \begin{cases} \dfrac{\ln(1+ax^3)}{x - \arcsin x}, & x < 0 \\ 6, & x = 0 \\ \dfrac{e^{ax} + x^2 - ax - 1}{x \sin \frac{x}{4}}, & x > 0 \end{cases}$$

(1) If $f(x)$ is continuous at $x = 0$, find the value of $a$.

(2) If $f(x)$ has a removable discontinuity at $x = 0$, find $a$.

20. Show that the inverse of the function $f(x) = \frac{x^3}{1+x^2}$ exists and evaluate the derivative of the inverse function at $x = \frac{1}{2}$.

21. Assume that $y = f(x)$ has second order derivative in $(a, b)$ and $f'(x) \neq 0$ for all $x \in (a, b)$. Let $y = g(x)$ be the inverse function of $y = f(x)$, compute $g''(x)$.

# Chapter 8

# Techniques of Integration

1. Let $I_k = \int_0^{k\pi} \mathrm{e}^x \sin x \,\mathrm{d}x$, $(k = 1, 2, 3)$, then which of the following is correct?

   (A) $I_1 < I_2 < I_3$. (B) $I_3 < I_2 < I_1$.

   (C) $I_2 < I_3 < I_1$. (D) $I_2 < I_1 < I_3$.

2. Among the improper integrals below, which one is convergent?

   (A) $\int_0^{+\infty} \frac{1}{\sqrt{1+x}}\,\mathrm{d}x$. (B) $\int_1^{+\infty} \frac{\ln x}{x+x^2}\,\mathrm{d}x$.

   (C) $\int_0^{1} \frac{1}{\sqrt{x}\sin x}\,\mathrm{d}x$. (D) $\int_1^{2} \frac{1}{x(\ln x)^2}\,\mathrm{d}x$.

3. If the improper integral $\int_0^{+\infty} \frac{\tan^{-1}(x^2)}{x^k}\,\mathrm{d}x$ converges, then the constant $k$ must satisfy

   (A) $k < 1$. (B) $k > 3$.

   (C) $1 < k < 2$. (D) $1 < k < 3$.

4. If $\int f(x)\mathrm{e}^{\frac{1}{x}}\,\mathrm{d}x = \mathrm{e}^{\frac{1}{x}} + C$, then $f(x) =$ (  ).

5. Compute the following integrals:

(1) $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+1}}$; (2) $\int_1^{+\infty} \frac{(x+2)\ln(x^2+1)}{x^3}\,\mathrm{d}x$;

(3) $\int \frac{\sin^5 x}{\cos^4 x}\,\mathrm{d}x$; (4) $\int_{-1}^{1} \frac{4^x}{4^x+4^{-x}}\,\mathrm{d}x$;

(5) $\int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}} \frac{\ln^2 x}{x}\,\mathrm{d}x$; (6) $\int_{-1}^{1} \frac{2x^2 + x\cos x}{1+\sqrt{1-x^2}}\,\mathrm{d}x$;

(7) $\int_0^2 x\sqrt{2x-x^2}\,\mathrm{d}x$; (8) $\int_0^2 \max(x, x^2)\,\mathrm{d}x$;

(9) $\int_2^{2+100\pi} |\sin x|\,\mathrm{d}x$; (10) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,\mathrm{d}x$;

(11) $\int x\tan^2 x\,\mathrm{d}t$; (12) $\int_0^4 \frac{\mathrm{d}x}{1+\sqrt{x}}$;

(13) $\int \frac{\mathrm{e}^{2x}-\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+\mathrm{e}^{-2x}}\,\mathrm{d}x$; (14) $\int_0^{\sqrt{2}} \frac{x^3}{1+x^2}\,\mathrm{d}x$;

(15) $\int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x}\,\mathrm{d}x$; (16) $\int_{\sqrt{\mathrm{e}}}^{\mathrm{e}^{\frac{3}{4}}} \frac{\mathrm{d}x}{x\sqrt{\ln x(1-\ln x)}}$.

6. Find the limit:
$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right).$$
7. ★ If $f(x)$ is continuous on $[0,a]$. Show that $\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a-x)\,\mathrm{d}x$.
   In specific, assume $a=1$, show that
$$\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x; \qquad \int_0^{\pi} xf(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x.$$
   Using this result to compute
$$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x.$$
8. If $f(x)$ is continuous with $f(x) = x\sin x + \int_0^{\frac{\pi}{4}} f(2x)\,\mathrm{d}x$. Find the integral $\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x$.
9. Find the limit
$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{4n^2-k^2}}{n^2}.$$
10. Let $f$ be a strictly decreasing continuous function on $[0,1]$ with $f(0)=2$, $f(1)=0$. Given that $\int_0^1 f(x)\,\mathrm{d}x = 1$, find $\int_0^2 f^{-1}(y)\,\mathrm{d}y$.
11. Assume $f(x)$ is continuous and $f(x) = \sqrt{2x-x^2} + x\int_0^1 f(t)\,\mathrm{d}t$. Find $f(x)$.
12. Find the points on which the function $f(x)$ reaches its minimum and maximum:
$$f(x) = \int_0^x \mathrm{e}^{-t}\cos t\,\mathrm{d}t.$$
13. Find the volume of the solid generated by revolving the region bounded by the curves about the x-axis.
    (1) $y=\tan x \quad (0\le x\le \frac{\pi}{4})$;
    (2) $y=\sec x \quad (0\le x\le \frac{\pi}{4})$;
14. Compute the following integrals:

$$(1)\ \int\frac{\mathrm{d}x}{x\sqrt{x^4-1}}; \qquad (2)\ \int_{\frac{1}{2}}^{\frac{3}{2}}\frac{1}{\sqrt{|x-x^2|}}\,\mathrm{d}x;$$
$$(3)\ \int\frac{\mathrm{d}x}{\mathrm{e}^{-2x}\sqrt{1-\mathrm{e}^{4x}}}; \qquad (4)\ \int\frac{\sqrt{\arctan\sqrt{x}}}{(1+x)\sqrt{x}}\,\mathrm{d}x;$$
$$(5)\ \int\frac{\mathrm{d}x}{(x-1)\sqrt{x^2-2x-48}}; \quad (6)\ \int_0^1\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x;$$

(7) $\displaystyle\int_0^1 \frac{1}{x^4+1}\,\mathrm{d}x$; (8) $\displaystyle\int_0^1 \frac{x^3+x+1}{(x^2+2)^2}\,\mathrm{d}x$;

(9) $\displaystyle\int_{-1}^{-2} \frac{x^3+x+1}{\sqrt{x^2-4x}}\,\mathrm{d}x$.

15. Compute $\displaystyle\int_0^3 f(x-1)\,\mathrm{d}x$ given

$$f(x) = \begin{cases} 1+x^2, & x \le 0 \\ \mathrm{e}^{-x}, & x > 0 \end{cases}.$$

16. Evaluate the following integrals:

(1) $\displaystyle\int \sqrt{a^2-x^2}\,\mathrm{d}x$, where $a > 0$.

(2) $\displaystyle\int \sqrt{ax^2+bx+c}\,\mathrm{d}x$, where $a > 0$ and $b^2-4ac < 0$.

(3) $\displaystyle\int \frac{Mx+N}{x^2+px+q}\,\mathrm{d}x$, where $p^2-4q < 0$.

(4) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{3+\cos 2x+2\sin 2x}\,\mathrm{d}x$.

(5) $\displaystyle\int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}\,\mathrm{d}x$, where $x > 1$.

(6) ★ $\displaystyle\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}\,\mathrm{d}x$, where $x > 2$.

(7) ★ $\displaystyle\int_2^{\infty} \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}\,\mathrm{d}x$.

# Chapter 9

# First-Order Differential Equations

1. Solve the differential equation
$$y' = \tan x \tan y.$$
2. Solve the differential equation
$$y' = \frac{y}{x} \log_y x.$$
3. Show the solution to the differential equation $y' = -\frac{x}{y}$ with $y(1) = 1$ is a circle.
4. Solve the initial value problem:
$$\frac{\mathrm{d}^2 y}{dx^2} = \frac{3}{\sqrt{x}} + 15\sqrt{x}, \quad y'(1) = 8, y(1) = 0.$$
5. ★ Let $g$ be a function that is differentiable throughout an open interval containing the origin. Suppose $g$ has the following properties:
   (1) $g(x+y) = \frac{g(x)+g(y)}{1-g(x)g(y)}$ for all real numbers $x, y$, and $x + y$ in the domain of $g$.
   (2) $\lim\limits_{h\to 0} g(h) = 0$.
   (3) $\lim\limits_{h\to 0} \frac{g(h)}{h} = 1$.
   Prove that:
   (1) Show that $g(0) = 0$.
   (2) Show that $g'(x) = 1 + [g(x)]^2$.
   (3) Find $g(x)$ by solving the differential equation in part (b).
6. Find the particular solution of
$$\frac{dy}{dx} = \frac{y}{x + y^2},$$
satisfying $y(2) = 1$.

# Chapter 10

# Infinite Sequences and Series

~~1.~~ Assume $0 < a_n < \frac{1}{n}$, which of the following series converges for sure? D

(A) $\sum_{n=1}^{\infty} a_n$.   (B) $\sum_{n=1}^{\infty} (-1)^n a_n$.

(C) $\sum_{n=1}^{\infty} \sqrt{a_n}$.   $u_n = n \quad v_n = -n$   (D) $\sum_{n=1}^{\infty} \frac{a_n}{n}$.

~~2.~~ Assume both series $\sum u_n$ and $\sum v_n$ diverge. Then C

(A) $\sum_{n=1}^{\infty}(u_n + v_n)$ diverges.   (B) $\sum_{n=1}^{\infty} u_n v_n$ diverges.

(C) $\sum_{n=1}^{\infty}(|u_n| + |v_n|)$ diverges.   (D) $\sum_{n=1}^{\infty}(u_n^2 + v_n^2)$ diverges.

3. Assume $\sum u_n$ converges, which of the following series converge for sure?

(A) $\sum_{n=1}^{\infty} (-1)^n \frac{u_n}{n}$.   (B) $\sum_{n=1}^{\infty}(u_n)^2$.

(C) $\sum_{n=1}^{\infty}(u_{2n-1} - u_{2n})$.   (D) $\sum_{n=1}^{\infty}(u_n + u_{n+1})$.

4. Suppose $a$ is a real number. Then the series $\sum_{n=1}^{\infty}\left(\frac{\sin a}{n^2} - \frac{1}{\sqrt{n}}\right)$

(A) converges absolutely.   (B) converges conditionally.

(C) diverges.   (D) convergence depends on $a$.

5. Assume that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2$. The radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^{2n+1}$ is

(A) 2.   (B) $\frac{1}{\sqrt{2}}$.   (C) $\sqrt{2}$.   (D) $\infty$.

6. Suppose the series $\sum_{n=0}^{\infty} a_n(x-1)^n$ converges at the point $x = -2$. Then at $x = 0$, the series

(A) diverges.   (B) converges conditionally.

(C) converges absolutely.   (D) is undetermined.

7. The interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ is

(A) $[-1/3, 1/3]$.   (B) $[-1/3, 1/3)$.   (C) $[-3, 3]$.   (D) $[-3, 3)$.

8. Let $a_n = (-1)^n \ln\left(1 + \frac{1}{n}\right)$, then the series $I = \sum a_n$ and $J = \sum a_n^2$

(A) both converge.   (B) both diverge.

(C) $I$ converges but $J$ diverges.   (D) $I$ diverges but $J$ converges.

9. If $\sum_{n=1}^{\infty}(-1)^{n-1}u_n = 2$ and $\sum_{n=1}^{\infty}u_{2n-1} = 5$, then $\sum_{n=1}^{\infty}u_n = (\quad)$.

10. Assume $\sum_{n=2}^{\infty}\frac{1}{n\ln^p n}$ converges, the range of $p$ is ( ).

11. Assume $\sum_{n=1}^{\infty}a_n x^n$ converges conditionally at $x = 2$, the radius of convergence of this series is ( 2 ).

12. Assume the radius of convergence of $\sum_{n=1}^{\infty}a_n x^n$ is 2, then the interval of convergence of $\sum_{n=1}^{\infty}na_n(x+1)^{n+1}$ is ( ).

13. The radius of convergence of $\sum_{n=1}^{\infty}\frac{nx^{2n}}{2^n+(-3)^n}$ is ( ).

14. $\sum_{n=1}^{\infty}\frac{2n+1}{n!} = ( 3e-1 )$. $\sum_{n=0}^{\infty}\frac{1}{n!}x^{2n+1} = xe^{x^2}$ $\sum_{n=0}^{\infty}\frac{2n+1}{n!}x^{2n} = (1+2x^2)e^{x^2}$

15. $f(x) = x\sin x$, then $f^{(50)}(0) = (\quad)$.

16. ★ Determine if the following sequence converges or not. If so, find the limit.

$$x_0 = 1, \quad x_{n+1} = x_n - \frac{\tan(x_n) - 1}{\sec^2(x_n)}.$$

17. Prove that the following sequence converges, and find its limit.

$$a_1 = 3, a_{n+1} = 12 - \sqrt{a_n}.$$

18. Compute

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}.$$

19. ★ Compute

$$\sum_{k=1}^{\infty}\frac{1}{k(k+2)(k+6)}.$$

20. Find $a, b, c$ such that

$$\lim_{n\to\infty} n(an + \sqrt{2+bn+cn^2}) = 2.$$

21. ★ Determine if the sequence converges or not.

$$a_n = \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{\cdots+\sqrt{n}}}}}.$$

22. Find the interval of convergence and identify the sum as a function of the following power

series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(2n-1)} x^{2n+1}.$$

23. ★ Compute

$$\lim_{n\to\infty} \cos\frac{a}{n\sqrt{n}} \cos\frac{2a}{n\sqrt{n}} \cdots \cos\frac{na}{n\sqrt{n}}.$$

24. Find the Taylor series at $x = 2$ for

$$f(x) = \ln x.$$

25. Determine if the following series converges or not

$$\sum_{n=1}^{\infty} \left( n^{\frac{1}{n^2+1}} - 1 \right).$$

26. Determine if the following series converges or not

$$\sum_{n=1}^{\infty} \left( 1 - \cos\frac{1}{n} \right)^p, \quad p > 0.$$

27. Find the radius of convergence and interval of convergence of the following power series.

$$\sum_{n=1}^{\infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) x^n.$$

28. Find the Taylor series at $x = 1$ for the following function

$$f(x) = \frac{x}{2 + x - x^2}.$$

29. Find the interval of convergence and identify the sum as a function for the following power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}.$$

30. Find the interval of convergence and identify the sum as a function for the following power series

$$\sum_{n=1}^{\infty} \frac{2n-1}{2^n} x^{2n-2}.$$

~~31.~~ Let the partial sum of the series $\sum a_n$ be

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}.$$

$\frac{1}{2n-1} - \frac{1}{2n}$

$a_n = S_n - S_{n-1}$

$= \frac{1}{n+1} + \cdots + \frac{1}{n+n} - \left( \frac{1}{n-1+1} + \cdots + \frac{1}{n-1+n-1} \right)$

$= -\frac{1}{n} + \frac{1}{2n-1} + \frac{1}{2n}$

Find $a_n$ and the sum of the series.

32. Compute

$$\lim_{n\to\infty}\left(\frac{1}{n^2+n+1}+\frac{2}{n^2+n+2}+\cdots+\frac{n}{n^2+n+n}\right).$$

33. Compute

$$\lim_{n\to\infty}\sqrt[n]{1+x^n+\left(\frac{x^2}{2}\right)^n},\quad x>0.$$

34. Let $x_1=\sqrt{6}$, $x_{n+1}=\sqrt{6+x_n}$ for $n=1,2,\cdots$. Prove the limit of the sequence exists and find the limit.

35. Determine if the following series converge or not and state your reason.

$$(1)\ \sum_{n=2}^{\infty}\frac{(\ln n)^2}{n};\qquad (2)\ \sum_{n=1}^{\infty}\left(1-\frac{3}{n}\right)^n;$$

$$(3)\ \sum_{n=2}^{\infty}\frac{(\ln n)^2}{n^{\frac{3}{2}}};\qquad (4)\ \sum_{n=1}^{\infty}2^n\sin\frac{\pi}{3^n};$$

$$(5)\ \sum_{n=1}^{\infty}\frac{(n!)^n}{n^{2n}};\qquad (6)\ \sum_{n=1}^{\infty}\frac{1}{n(\ln(1+1/n)+1)};$$

$$(7)\ \sum_{n=1}^{\infty}\left(1-\cos\frac{\lambda\pi}{n}\right),\lambda>0.$$

36. If $\sum a_n$ converges absolutely, prove that $\sum\frac{a_n}{a_n+1}$ converges absolutely.

37. Assume the positive sequence $\{a_n\}$ decreases but $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges. Determine if the following series converges or not and state your reasons.

$$\sum_{n=1}^{\infty}\left(\frac{1}{a_n+1}\right)^n.$$

38. Determine if the following series converges or not and state your reasons.

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n+(-1)^n}}.$$

39. Given the power series $\sum_{n=1}^{\infty}\frac{n}{n!}x^{n+1}$, find the interval of convergence and identify the series as a function. Using this to compute

$$\sum_{n=1}^{\infty}\frac{n-1}{n!}2^n.$$

40. Assume $a_n > 0$ is a increasing sequence with an upper bound. Prove that $\sum_{n=1}^{\infty} b_n$ converges, where $b_n = \ln\left(2 - \frac{a_n}{a_{n+1}}\right)$.
41. Write the function $f(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t$ as a power series of $x$.
42. Write the Taylor series at $x = 4$ for the function $f(x) = \frac{1}{x^2-5x+6}$.
43. Find the sum function for the following series.

$$(1) \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}; \qquad (2) \sum_{n=1}^{\infty} \frac{n(n+1)x^n}{(n-1)!};$$

$$(3) \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n-1}\left(\frac{x-2}{3}\right)^{2n+1}; \qquad (4) \sum_{n=1}^{\infty} \frac{2n-1}{2^n} x^{2n-2};$$

$$(5) \sum_{n=1}^{\infty} n(n+1)x^n; \qquad (6) \sum_{n=1}^{\infty} (-1)^{n-1} n^2 x^n;$$

$$(7) \sum_{n=0}^{\infty} \frac{n^2+1}{2^n n!} x^n; \qquad (8) \sum_{n=1}^{\infty} (-1)^n \frac{n+1}{(2n+1)!} x^{2n+1}.$$

44. Compute the following series.

$$(1) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n(n+1)}{2^n}; \qquad (2) \sum_{n=1}^{\infty} \frac{e^{-n}}{n+1}; \qquad (3) \sum_{n=1}^{\infty} \frac{1}{n2^n}.$$

45. Compute $f^{(n)}(0)$ for $f(x) = \arcsin x$.
46. Use the Taylor series to compute the following limit:

$$(1) \lim_{x\to 0} \frac{\sin x \ln(1+x) - x^2}{x^3}; \qquad (2) \lim_{x\to 0} \frac{xe^{x^2/2} - \sin x}{x^2 \sin x};$$

$$(3) \lim_{n\to+\infty} n^2\left(\arctan\frac{1}{n} - \arctan\frac{1}{n+1}\right).$$

47. Compute the following limit:
    (1) $\lim_{n\to\infty} \frac{1}{n} \sin n \cos(n^2)$;
    (2) $\lim_{n\to\infty} (a^n + b^n)^{\frac{1}{n}}, \quad a > 0, b > 0$;
    (3) $\lim_{n\to\infty} \frac{n^c}{a^n}, \quad a > 1, c > 0$;
    (4) $\lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)^{\frac{1}{n}}$.
48. Let $a > 0$, $x_0 = 1$, and $x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n}$. Determine if the sequence converges or not. If so, find the limit.
49. Let $a_1 = 1$ and $a_2 = 1$. $a_{n+2} = a_{n+1} + a_n$. Prove that $b_n = \frac{a_{n+1}}{a_n}$ converges and find the limit.

50. Find the radius and interval of convergence of the following series.

$$(1)\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}x^n; \qquad (2)\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!};$$

$$\boxed{(3)}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right)(x-2)^n; \qquad (4)\sum_{n=1}^{\infty}\frac{3^n}{2^n n!}(x-1)^n;$$

$$\boxed{(5)}\sum_{n=1}^{\infty}\frac{\ln^2 n}{n^n}x^{n^2}; \qquad (6)\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{n2^n};$$

$$(7)\sum_{n=1}^{\infty}\left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)x^n\ (a>0, b>0).$$

51. Find all $x$ so that the series $\sum_{n=1}^{\infty} n^2(\sin x)^n$ converges and find the sum function.

52. Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ with the radius of convergence $R>0$ and the sum function $f(x)$. Prove that the Taylor series of $f$ at $x=a$ is exactly the power series given above.

53. Find the Taylor series generated by $f(x)=\ln\frac{1+x}{1-x}$ at $x=0$ and find an approximate value of $\ln 2$ to the error no greater than $10^{-4}$.

54. Suppose that $f$ has 3rd derivative on $[-1,1]$ and $f^{(3)}$ is continuous on $[-1,1]$. $f(-1)=f(0)=0$, $f(1)=1$ and $f'(0)=0$. Prove that there is a point $c$ between $(-1,1)$ such that $f^{(3)}(c)=3$.

55. Determine if the sequences converge or not. If so, find the limit.

$$(1)\, a_n=\left(\frac{n}{n+c}\right)^n, \quad c\in\mathbb{R}; \qquad (2)\, a_n=\frac{\ln^3 n}{n}.$$

56. ★ Determine if the following series converges or not:

$$\sum_{n=1}^{\infty}\frac{(2n)!}{n!\,n!\,4^n}.$$

57. Suppose the sequence $\{na_n\}$ and series $\sum n(a_n-a_{n-1})$ both converge. Prove that $\sum a_n$ converges.

58. Determine the first three non-zero terms in the Maclaurin series for $f(x)=\tan x$ and evaluate the limit $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$.

59. Find $f^{(8)}(0)$ for $f(x)=\frac{1}{\sqrt{4-x^2}}$.

60. Determine if the following series converge absolutely, conditionally or diverge.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{(-1)^n n}}{n}$;

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$;

~~(3)~~ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$;

~~(4)~~ $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\ln n}{n^p}$; $(p>0)$

(5) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\int_0^{1/n} \frac{\sqrt{x}}{1+x^2}\,\mathrm{d}x$;

(6) $\displaystyle\sum_{n=1}^{\infty} (-1)^n 2^n \sin\frac{\pi}{3^n}$;

(7) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{\sqrt{n}}\right)$;

(8) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^{n+1/n}}{(n+1/n)^n}$;

(9) $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{\prod_{k=1}^{n}(1+a^k)}$ $(a>0)$.

61. Find the constant $a$ and $b$ in the following limit:

(1) If $\displaystyle\lim_{x\to 0}\frac{\sin ax-\sin x-x}{x^3}=b$;

(2) If $\displaystyle\lim_{x\to 0}\frac{\cos ax-b}{2x^2}=-1$.

62. Compute the following limit:

(1) $\displaystyle\lim_{n\to\infty}\left(\sqrt{n+\sqrt{n+2\sqrt{n}}}-\sqrt{n}\right)$;

(2) $\displaystyle\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}\frac{1}{k}}$;

(3) $\displaystyle\lim_{n\to\infty}(n!)^{1/n^2}$.

63. Compute the limit:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{(n+i-1)(n+i)}}.$$

64. Assume that the sequences $a_n$ and $b_n$ are bounded by 0 and $\pi/2$, satisfy that $\cos a_n - a_n = \cos b_n$. If $\sum b_n$ converges:

(1) Prove $\displaystyle\lim_{n\to\infty} a_n = 0$;

(2) Prove $\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{b_n}$ converges.

65. Assume $x_1>0$ and $x_{n+1}=1+\frac{x_n}{1+x_n}$ for $n\ge 1$. Prove that the sequence converges and find the limit.

66. Compute the following limit:

(1) $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$;  (2) $\lim\limits_{n\to\infty}\left(b_1^n+b_2^n+\cdots+b_m^n\right)^{1/n}$ $(b_i\geq 0)$;

(3) $\lim\limits_{n\to\infty}\left(n\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)\right)$;  (4) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{1}{n+\sqrt{k}}$;

(5) $\lim\limits_{n\to\infty}\sum\limits_{k=n^2}^{n^2+2n+1}\dfrac{1}{\sqrt{k}}$;  (6) $\lim\limits_{n\to\infty}\dfrac{(2n-1)!!}{(2n)!!}$;

(7) $\lim\limits_{n\to\infty}\left(\sqrt{n}\left(\sqrt[4]{n^2+1}-\sqrt{n+1}\right)\right)$;  (8) $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\left(1-\dfrac{1}{k^2}\right)$.

67. Compute the following series:

(1)★ $\sum\limits_{n=1}^{\infty}\arctan\dfrac{1}{2n^2}$;  (2) $\sum\limits_{n=0}^{\infty}\dfrac{n^2+2n+1}{2^n}$;

(3) $\sum\limits_{n=2}^{\infty}(-1)^n\dfrac{1}{2^n(n^2-1)}$;  (4) $\sum\limits_{n=1}^{\infty}\dfrac{n!+1}{2^n(n-1)!}$;

(5) $\sum\limits_{n=1}^{\infty}(2n-1)q^{n-1}$ $(|q|<1)$;  (6) $\sum\limits_{n=1}^{\infty}\dfrac{n(n+2)}{4^{n+1}}$.

68. Assume $f(x)$ is differentiable and $f(0)=1$, $0<f'(x)<1/2$. Define $x_{n+1}=f(x_n)$ for $n\in\mathbb{N}$. Prove that

(1) $\sum\limits_{n=1}^{\infty}(x_{n+1}-x_n)$ converges absolutely.

(2) $\lim\limits_{n\to\infty}x_n$ exists with the limit between $(0,2)$.

69. Assume $a_0=1$, $a_1=0$ and $a_{n+1}=\frac{na_n+a_{n-1}}{n+1}$ for $n\geq 2$. Let $S(x)=\sum\limits_{n=0}^{\infty}a_nx^n$.

(1) Prove that the radius of convergence of $S(x)\geq 1$.

(2) Show that $(1-x)S'(x)-xS(x)=0$ and find the sum function.

# Chapter 11

# Parametric Equations and Polar Coordinates

1. ★ Prove that the cycloid is tautochrone: given a particle on the track of the curve's shape, it will take the same time for this particle to reach the bottom no matter where, high or low, to release the particle.
2. Find the open intervals of $x$ such that the curve $y = f(x)$ is concave down, where the curve is given by:
$$\begin{cases} x = t^3 + 3t + 1 \\ y = t^3 - 3t + 1 \end{cases}.$$
3. Given a curve
$$L : \begin{cases} x = f(t) \\ y = \cos(t) \end{cases}, \quad 0 \leq t \leq \frac{\pi}{2},$$
where $f(t)$ has continuous derivative with $f(0) = 0$, $f'(t) > 0$. Suppose the segment of the tangent line of $L$ between the intersection point on $x$-axis and the tangent point is always 1, find $f(t)$ and the area bounded by axes and $L$.
4. Find the length of the curve
$$x = (1 + 2\sin t)\cos t, \quad y = (1 + 2\sin t)\sin t,\ 0 < t < 2\pi.$$
5. Find the area of the surface generated by revolving the curve about the $y$-axis:
$$x = 2\cos t, \quad y = 3 + \sin t,\ 0 < t < 2\pi.$$
6. Find the length of the curve $r\theta = 1$ and $\frac{3}{4} \leq \theta \leq \frac{4}{3}$.
7. ★ Find the area inside the ellipse given by
$$Ax^2 + 2Bxy + Cy^2 = 1, \quad AC - B^2 > 0, C > 0$$

8. Find the second order derivatives $\frac{d^2y}{dx^2}$ for the function given by

$$x = a\cos^3 t, \quad y = a\sin^2 t.$$

9. Find the surface area generated by the following curve revolving about axis:
   (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$, about $y$-axis;
   (b) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \le t \le 2\pi$, $a > 0$, about $x$-axis.
10. ★ Find the arc length of the cycloid and the area bounded by the cycloid and $x$-axis for $0 < t < 2\pi$.
11. Evaluate the area enclosed by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the line $x = -2a$.
12. Find the area of one leaf of the rose $r = \sin(4\theta)$.
13. Assume that the curve is given by

$$x = \sin t, \quad y = \cos(2t), \quad 0 \le t \le \frac{\pi}{2}.$$

Find the Cartesian equation for the curve and identify the graph. Further, find the area enclosed by the curve and $x$-axis and the volume of the solid generated by revolving the curve about $x$-axis.
14. Assume $f \neq 0$ are differentiable. Let $x = \int_0^{t^2} f(u)\,\mathrm{d}u$ and $y = \int_0^t f(u)f(u^2)\,\mathrm{d}u$. Compute $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.
15. Let $L$ be the curve of astroid given by

$$x = \cos^3 t, \quad y = \sin^3 t.$$

   (a) Given a point $A(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$, compute the slope of the curve at $A$ and find the tangent line to the curve at $A$.
   (b) If $A$ is at $t = t_0$, $0 < t_0 < \frac{\pi}{2}$ repeat the first part.
   (c) Compute the length of the tangent line segment at $A$ within the first quadrant.
   (d) Is the length dependent on the choice of $t_0$?
16. Compute $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ for the function given by

$$x = \ln(1 + t^2), \quad y = t - \arctan t.$$

17. Given the following curve:

$$x = \cos(t^2), \quad y = t\cos(t^2) - \int_1^{t^2} \frac{\cos u}{2\sqrt{u}}\,\mathrm{d}u, \quad 0 < t < \sqrt{2\pi}.$$

(a) Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$;

(b) Find the equation for the tangent line and the normal line at $t = 1$.

18. Find the inflection point (where the convexity changes) for the following curves:

$$(1)\, x = t^2, \quad y = 3t + t^3,\ t > 0;$$
$$(2)\, x = 2a\cot\theta, \quad y = 2a\sin^2\theta,\ 0 < \theta < \pi.$$

19. A curve is defined by the parametric equation as following. Compute the length of the arc of the curve from the origin to the nearest point where there is a vertical tangent line.

$$x = \int_1^t \frac{\cos u}{u}\,\mathrm{d}u, \quad y = \int_1^t \frac{\sin u}{u}\,\mathrm{d}u$$

# Chapter 12

## Vectors and the Geometry of Space

1. Prove all three altitudes of a trangle always intersect at the same point.
2. Prove or disprove $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ and state your reasons.
3. Assume $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq 0$, here $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ stands for the triple product. Solve the system of equations:

   scalar

$$\begin{cases} (\mathbf{x}, \mathbf{a}, \mathbf{b}) = n, \\ (\mathbf{x}, \mathbf{b}, \mathbf{c}) = l, \\ (\mathbf{x}, \mathbf{c}, \mathbf{a}) = m. \end{cases}$$

4. For any vectors $\mathbf{a}$,$\mathbf{b}$, $\mathbf{c}$ and $\mathbf{p}$. Prove that $\mathbf{a} \times \mathbf{p}$, $\mathbf{b} \times \mathbf{p}$, $\mathbf{c} \times \mathbf{p}$ are coplanar.
5. Find the symmetry point of the point $(1, -2, 3)$ with respect to the plane $x+4y+z-14 = 0$.
6. Find the projection of the intersecting line of $x^2+y^2+z^2 = 1$ and $x^2+(y-1)^2+(z-1)^2 = 1$ on the $XOY$-plane.
7. Prove that for any vector $\mathbf{a}$ and $\mathbf{b}$, $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$. First please give a geometric explanation of this equation. Then use this to find $|\mathbf{a}+\mathbf{b}|$ given $|\mathbf{a}| = 1$, $|\mathbf{b}| = 32$ and $|\mathbf{a}-\mathbf{b}| = 30$.
8. Prove the distance between a point $(x_0, y_0, z_0)$ to the plane $Ax + By + Cz + D = 0$ is given by
$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$
9. Suppose the plane $M$ passes through the origin and $(6, -3, 2)$ and is perpendicular to the plane $4x - y + 2z - 8 = 0$. Find $M$.
10. Suppose two line $l_1 : x - 1 = \frac{y-5}{-2} = z + 8$ and $l_2$ is the intersection of planes $x - y = 6$ and $2y + z = 3$. Find the angle between $l_1$ and $l_2$.
11. Suppose $l : z = ky (k > 0)$ is a line the $YOZ$-plane and $S$ is the surface by revolving $l$ about $z$-axis. Find the equation of $S$.
12. Find the angle between the line $x = t$, $y = 2t$ and $z = 3t$ and the plane $ax + by + cz = 0$.
13. Determine if the four points $(1, 3, 2)$, $(3, -1, 6)$, $(5, 2, 0)$ and $(3, 6, -4)$ are coplanar or not.

Determine if the four points $(0,0,0)$, $(2,5,0)$, $(5,2,0)$ and $(1,2,4)$ are coplanar or not.

14. Find the set of equidistant points to $(3,1,2)$, $(4,-2,-2)$ and $(0,5,1)$, and describe the geometric of this set.
15. Given a vector $\mathbf{r} = (a,b,c)$ with the angle to $x$-axis, $y$-axis and $z$-axis being $\alpha$, $\beta$ and $\gamma$. Compute $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$.
16. Suppose $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are perpendicular to each other with the right-handed orientation. Suppose the length of them are 1, 2 and 3, find the angle between $\mathbf{a}+\mathbf{b}+\mathbf{c}$ and $\mathbf{c}$.
17. Let $\overrightarrow{AB} = (-3,0,4)$ and $\overrightarrow{AC} = (5,-2,-15)$. Find a unit vector in the direction of the bisector of $\angle BAC$.
18. Give line $l : x = 1+t, y = -1-t, z = t$ and plane $S : 3x - y + z = 5$:
    (a) Find the plane passing $l$ and perpendicular to $S$;
    (b) Find the equation of the projection of $l$ to $S$.
19. Find the projection of the following line to the $XOY$-plane.

$$\begin{cases} x + 5y + 6z = 3 \\ x - 2y + 3z = 1 \end{cases}.$$

20. ★ Given two lines

$$l_1 = \begin{cases} y = 2x \\ z = x + 1 \end{cases}, \qquad l_2 = \begin{cases} y = x + 3 \\ z = x \end{cases}.$$

Prove that $l_1$ and $l_2$ do not intersect with each other and find the equation for the common perpendicular line.

21. Suppose a line $l$ passing $(1,2,1)$ and perpendicular to the line $\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{1}$. If $l$ intersect with the line $\frac{x}{2} = y = -z$, find the equation for $l$.
22. Determine $\lambda$ such that the following two line intersect and find the position of the intersection.

$$l_1 : \frac{2x-2}{3} = -(y+1) = \frac{z-2}{2}; \qquad l_2 : x = \frac{4y+1}{-4} = \frac{z+2}{\lambda}.$$

23. ★ Find the projection of the line $x - 1 = y = 1 - z$ on the plane $x - y + 2z = 1$.
24. The ellipsoid $S_1$ is generated by revolving $\frac{x^2}{4} + \frac{y^2}{3} = 1$ about the $x$-axis, and the cone $S_2$ is generated by the revolving the tangent line passing $(4,0)$ of $\frac{x^2}{4} + \frac{y^2}{3} = 1$ about the $x$-axis. Find the equation of $S_1$ and $S_2$, and compute the volume of the solid enclosed by $S_1$ and $S_2$.

25. Find the equation for the plane through the origin and parallel to both two lines:

$$l_1 = \begin{cases} x = 1 \\ y = -1 + t \\ z = 2 + t \end{cases}, \qquad l_2 = \begin{cases} x = -1 + t \\ y = -2 + 2t \\ z = 1 + t \end{cases}.$$

26. ★ $L$ is on the plane $x - y + z = 1$, and its projection to the plane $x + y + z = 1$ is give by

$$\begin{cases} x + y + z = 1 \\ 2x - 5y + 3z = 4 \end{cases}.$$

Find the equation for $L$.

# Chapter 13

## Vector-Valued Functions and Motion in Space

1. Find the osculating circle of $y = x^3$ at $x = 1$.
2. ★ Prove that there are at least two lines on the surface:

$$\begin{cases} x = a \sec\phi \cos\theta, \\ y = b \sec\phi \sin\theta, \qquad a, b, c > 0, 0 \le \theta < 2\pi, -\pi/2 < \phi < \pi/2. \\ z = c \tan\phi, \end{cases}$$

3. Let $f(x)$ be twice continuously differentiable on $[a, b]$. Prove that the curvature is

$$\kappa(x) = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}^3}.$$

   And find the maximum of the curvature of function $y = \ln x$.
4. Re-parametrize the curve with respect to the arc length measured starting from $t = 0$ in the direction of $t$ increasing:

$$\boldsymbol{r}(t) = 12t\boldsymbol{i} + (1 - 3t)\boldsymbol{j} + (5 + 4t)\boldsymbol{k}.$$

5. Find the equation of the osculating circle at $t = \frac{\pi}{2}$ for the curve $\boldsymbol{r}(t) = -\sin t\boldsymbol{i} + \cos t\boldsymbol{j} + t^2\boldsymbol{k}$.

# Chapter 14

# Partial Derivatives

1. Define $f(x,y) = \frac{xy}{x^2+y^2}$ when $(x,y) \neq (0,0)$ and $f(x,y) = 0$ when $(x,y) = (0,0)$. Then at $(0,0)$, $f(x,y)$ is
   (A) continuous, and partial derivatives exist.
   (B) continuous, but partial derivatives do not exist.
   (C) not continuous, but partial derivatives exist.
   (D) not continuous, and partial derivatives do not exist.
2. Assume $f(x,y) = y(x-1)^2 + x(y-2)^2$. Which of the following method computing $f_x(1,2)$ is incorrect ?
   (A) Since $f(x,2) = 2(x-1)^2$, $f_x(x,2) = 4(x-1)$, $f_x(1,2) = 0$.
   (B) Since $f(1,2) = 0$, $f_x(1,2) = 0' = 0$.
   (C) Since $f_x(x,y) = 2y(x-1) + (y-2)^2$, $f_x(1,2) = 0$.
   (D) $f_x(1,2) = \lim\limits_{x\to 1} \dfrac{f(x,2) - f(1,2)}{x-1} = 0$.
3. Suppose both partial derivatives for $f(x,y)$ exist at the point $(x_0,y_0)$. Then
   (A) $f(x,y)$ is bounded near $(x_0,y_0)$.
   (B) $f(x,y)$ is continuous near $(x_0,y_0)$.
   (C) $f(x,y_0)$ is continuous at $x_0$, $f(x_0,y)$ is continuous at $y_0$.
   (D) $f(x,y)$ is continuous at $(x_0,y_0)$.
4. Let $x = x(y,z)$, $y = y(z,x)$ and $z = z(x,y)$ are implicit function determined by $F(x,y,z) = 0$. Then which of the following is incorrect?
   (A) $\frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = 1$. (B) $\frac{\partial x}{\partial z}\frac{\partial z}{\partial x} = 1$.
   (C) $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = 1$. (D) $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$.
5. Assume $u = u(x,y)$ and $v = v(x,y)$ are differentiable, and $C$ is a constant. Then which of the following is incorrect?
   (A) $\nabla C = 0$. (B) $\nabla(Cu) = C\nabla u$.
   (C) $\nabla(u+v) = \nabla u + \nabla v$. (D) $\nabla(uv) = v\nabla u + u\nabla v$.
6. ★ The tangent plane of the smooth surface $z = x + f(y-z)$ is
   (A) perpendicular to a given line.
   (B) parallel to a given plane.
   (C) intersect with a coordinate plane with constant angle.

(D) parallel to a given line.

7. Assume $u(x,y)$ is twice continuously differentiable in a bounded closed region $D$ with $\frac{\partial^2 u}{\partial x \partial y} \neq 0$ and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Then for $u(x,y)$,

   (A) both the maximum and minimum points are inside $D$.

   (B) both the maximum and minimum points are on the boundary of $D$.

   (C) the maximum point is inside $D$ while the minimum point is on the boundary.

   (D) the minimum point is inside $D$ while the maximum point on the boundary.

8. Suppose $f$ is continuous at $(0,0)$. Then which of the following statement is correct ?

   (A) $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{|x|+|y|}$ exists, then $f$ is differentiable at $(0,0)$.

   (B) $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2+y^2}$ exists, then $f$ is differentiable at $(0,0)$.

   (C) $f$ is differentiable at $(0,0)$, then $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{|x|+|y|}$ exists.

   (D) $f$ is differentiable at $(0,0)$, then $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2+y^2}$ exists.

9. If $f$ is continuous at the origin with $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2+y^2}$ exists, then $f(x,y)$ is differentiable at the origin. (    )

   (A) True.    (B) False.

10. Let $z = \frac{y}{f(x^2-y^2)}$, where $f(u)$ is differentiable. Then $\frac{\partial z}{\partial x}$ =(    ).

11. Let $u = f(\sqrt{x^2+y^2+z^2})$ is a twice continuously differentiable function. Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ =(    ).

12. The absolute maximum of $u = \sin x \sin y \sin z$ with the restriction $x+y+z = \frac{\pi}{2}$ and $x,y,z > 0$ is (    ).

13. Assume the tangent plane of $xyz = 6$ at the point $P$ is parallel to $6x - 3y + 2z + 1 = 0$. The position of $P$ is (    ).

14. The absolute minimum of $z = x^2 + y^2$ with the restriction $x + y = 1$ is (    ).

15. $u = x^{y^z}$, $x > 0$, $y > 0$, then $\mathrm{d}u$ = (    ).

16. Let $z = z(x,y)$ be the function determined implicitly by $\mathrm{e}^{-xy} - 2z + \mathrm{e}^{xz} = 0$. Then $\left.\frac{\partial^2 z}{\partial x^2}\right|_{(0,1)}$ = (    ).

17. Suppose the tangent plane of the surface $z - \mathrm{e}^z + 2xy = 3$ at $P$ is $x + 2y = C$. Then $P$ = (    ) and $C$ = (    ).

18. ★ The minimum of the distance between a point on $y = x^2$ and a a point on $x - y + 2 = 0$ is (    ).

19. Let $w = g(x,y) = f(x,y,z(x,y)) = \mathrm{e}^x yz^2$, here $z(x,y)$ is determined by $x + y + z + xyz = 2$. Then $\frac{\partial w}{\partial x}(0,1)$ = (    ).

20. Assume $f(u)$ is differentiable and $f'(0) = 0.5$. Then $z = f(4x^2 - y^2)$ at $(1,2)$ increases fastest in the direction of (    ).

21. Assume $f(x,y)$ has continuous partial derivatives. If $f(x,x^2) = 1$ and $f_x(x,x^2) = x$, then $f_y(x,x^2)$ = (    ).

22. Assume $z = f(x, y)$ is differentiable at $(1, 1)$. If $f(1, 1) = 1$, $f_x(1, 1) = 2$ and $f_y(1, 1) = 3$, $\phi(x) = f(x, f(x, x))$, then $\frac{\mathrm{d}}{\mathrm{d}x}\left(\phi^3(1)\right) = (\qquad)$.
23. Determine if the following function continuous on the plane and state your reasons.

$$f(x, y) = \begin{cases} y^2 \ln\left(x^2 + y^2\right), & x^2 + y^2 \neq 0; \\ 0, & x^2 + y^2 = 0. \end{cases}$$

24. Determine if the following limit exists or not. If so, find the limit. If not, state your reasons:

(1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^3 + y^4)}{x^2 + y^2}$; (2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y^{1.5}}{x^4 + y^2}$;

(3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4 + y^2}$; (4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^3 + y^3)}{x^2 + y^2}$;

(5) $\lim\limits_{(x,y)\to(0,2)} \dfrac{\ln(1 + xy^2)}{x}$; (6) $\lim\limits_{(x,y)\to(0,0)} (1 - 2xy)^{1/(x^2+y^2)}$;

(7) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x - y)}{x + y}$; (8) $\lim\limits_{(x,y)\to(0,2)} \dfrac{\arctan \frac{xy}{1+xy}}{x}$;

(9) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 + y^2}{|x| + |y|}$; (10) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{1 + x^2 + y^2} - 1}{x^2 + y^2}$;

(11) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2 \sin x}{x^2 + y^4}$; (12) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\ln(x^2 + e^{y^2})}{x^2 + y^2}$;

(13) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2 - \sqrt{xy + 4}}{xy}$; (14) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y^2}{x^4 + x^2 + y^2}$;

(15)★ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 + y^3}{x^2 + y}$; (16) $\lim\limits_{(x,y)\to(0,0)} (x^2 + y^2)^{x^2 y^2}$.

25. Assume $f(u)$ are twice continuously differentiable and $g(x, y) = f\left(\frac{y}{x}\right) + y f\left(\frac{x}{y}\right)$. Compute $x^2 g_{xx} - y^2 g_{yy}$.
26. If $f(x, y) = \left(\frac{y}{x}\right)^{\frac{x}{y}}$, compute $f_x, f_y$.
27. Find the absolute maximum and minimum for $f(x, y) = x^2 + 2y^2 - x^2 y^2$ on $D = \{(x, y) \mid x^2 + y^2 \leq 4, y \geq 0\}$.
28. Find the absolute maximum and minimum for the function $f(x, y) = x^2 + y^2 + x^2 y + 4$ on $D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$.
29. Assume that $\boldsymbol{n}$ is the normal unit vector pointing inside at $(1, 1, 1)$ for the surface $2x^2 + 3y^2 + z^2 = 6$. Compute the directional derivative for $u = \frac{\sqrt{6x^2+8y^2}}{z}$ at the point in the direction of $\boldsymbol{n}$.
30. Find the tangent plane of $z = x^2 + y^2$ through $(1, 0, 0)$ and $(0, 1, 0)$.
31. Find the maximum of the directional derivative for the function $f(x, y) = x + y + xy$ on

the curve $C: x^2 + y^2 + xy = 3$.

32. Calculating all first order partial derivatives:

$$(1)\ f(x,y,z) = z^{xy}; \qquad (2)\ f(x,y,z) = x^z y^{-z};$$
$$(3)\ f(x,y,z) = \int_{xz}^{yz} e^{t^2}\,dt; \qquad (4)\ f(x,y) = \left(1 + x^2 + y^2\right)^{xy};$$
$$(5)\ f(x,y,z) = \left(\frac{x}{y}\right)^{1/z}.$$

33. Calculating all second order partial derivatives:

$$(1)\ f(x,y) = \arctan\frac{x+y}{1-xy}; \qquad (2)\ f(x,y) = \ln\left(x + y^2\right);$$
$$(3)\ w = f\left(xy, \frac{x}{y}\right); \qquad (4)\ z = f\left(xy, x^2 + y^2\right).$$

34. For the following composite or implicit functions,
   (1) find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = e^u + \ln v$, $u = \sqrt{xy}$ and $v = 2x - 3y$.
   (2) find $\frac{dw}{dt}$ if $w = xyz + \ln(x + y + z)$, $x = e^t$, $y = e^{-t}$, $z = \sqrt{t}$.
   (3) find $\frac{d^2y}{dx^2}$ for $\ln\sqrt{x^2 + y^2} = \arctan\frac{y}{x}$.
   (4) find $\frac{\partial w}{\partial x}$ and $\frac{\partial^2 w}{\partial x^2}$ for $w = f(x,y,z)$ and $z = \ln\sqrt{x^2 + y^2}$. Here $f$ is twice continuously differentiable.
   (5) find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z + e^z = xy$.
   (6) find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ for $z = z(x,y)$ determined by $x^2 + y^2 + z^2 = 4z$.
   (7) ★ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where $F(xz, yz) = 0$.
   (8) ★ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where $F(\frac{x}{z}, \frac{z}{y}) = 0$.

35. Find the local extreme values of the function

$$f(x,y) = (8x^2 - 6xy + 3y^2)e^{2x+3y}.$$

36. Find the extreme values of the function $0 \leq x \leq \frac{\pi}{2}$

$$f(x,y) = \cos^2 x + \cos^2 y$$

with the restriction $x - y = \frac{\pi}{4}$.

37. Use the Taylor's formula to find the quadratic approximation for the function $f(x,y) = \sqrt{1 - x^2 - y^2}$ at the origin.

38. Let $r = \sqrt{x^2 + y^2 + z^2}$. Prove that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$ when $r \neq 0$.

39. Prove that the function $f(x,y) = \sqrt{|xy|}$ at $(0,0)$ is continuous with all partial derivatives

exist, but not differentiable.

40. Find the cuboid with the maximal volume inside the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
41. Use the Lagrange multipliers to find the minimal and maximal value of $f(x, y, z) = x^4 + y^4 + z^4$ on the sphere $g(x, y, z) = x^2 + y^2 + z^2 = 1$.
42. Find all points at which is function is continuous:

$$f(x,y) = \begin{cases} \dfrac{\sin(xy)}{x}, & x \neq 0 \\ y, & x = 0 \end{cases}$$

43. Let

$$f(x,y) = \begin{cases} y \sin \dfrac{1}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

(1) Find $f_x$ and $f_y$ for any point other than the origin;
(2) Determine if $f_x(0,0)$ and $f_y(0,0)$ exist or not;
(3) Find all points at which $f$ is continuous.

44. Let

$$f(x,y) = \begin{cases} (x^2+y^2) \sin \dfrac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

(1) Find $f_x(0,0)$ and $f_y(0,0)$;
(2) Determine if $f_x$ and $f_y$ continuous or not at the origin;
(3) Prove that $f$ is differentiable at the origin.

45. Suppose that $w = f(z)$ is a differentiable function and the equation $f(x^2+y^2)+f(x+y) = y$ defines a differentiable implicit function $y = g(x)$. Given $g(0) = 2$, $f'(2) = 0.5$ and $f'(4) = 1$, find $g'(0)$.
46. Suppose that $w = f(x_1, \cdots, x_n)$, $x_i = g_i(t_1, \cdots, t_m)$ and $t_j = h_j(s_1, \cdots, s_l)$. Compute $\frac{\partial w}{\partial s_k}$.
47. Determine if the following function is continuous at $(0, 0)$ and state your reasons.

$$f(x,y) = \begin{cases} \dfrac{x^2\sqrt{|y|}}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

48. Compute $\frac{\partial M}{\partial x}$ and $\frac{\partial M}{\partial y}$, where $M(x, y) = \ln(u^2 + v^2 + (2xy)^2)$, $u = x + 2y$, $v = 2x - y$ with $x = y = 1$.
49. Use Lagrange multiplier to find the point on the parabola $xy = 8$ closest to the point $(3, 0)$.

50. Let

$$f(x,y)=\begin{cases}\dfrac{\sin(x^2y)}{x}, & xy\neq 0;\\ x, & xy=0.\end{cases}$$

(1) Find $f_x(0,1)$ and $f_y(0,1)$;

(2) Determine if $f_x$ continuous or not at the origin;

(3) Determine if $f$ is differentiable at the origin.

51. Compute $\mathrm{d}z$ given $f(2x+z,3y-z)=0$ where $f$ is differentiable.

52. Assume $f(x,y)$ is differentiable with $f(1,1)=1$, $f_x(1,1)=a$ and $f_y(1,1)=b$. Let $u=f(x,f(x,f(x,x)))$, compute $\frac{\mathrm{d}(u^2)}{\mathrm{d}x}$ at $x=1$.

53. Let $u=f(\ln\sqrt{x^2+y^2})$. Find $f$ if $u$ satisfies that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(x^2+y^2)^{\frac{3}{2}}.$$

54. Find the tangent plane of the surface $x^2+2y^2+z^2=1$ parallel to $x-y+2z=0$.

55. Find the local maxima and minima of $f(x,y)=x^3+8y^3-xy$.

56. Let

$$f(x,y)=\begin{cases}\dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2}, & (x,y)\neq(0,0);\\ x, & (x,y)=(0,0).\end{cases}$$

Determine if $f$ is continuous at the origin or not.

57. Let

$$f(x,y)=\begin{cases}\dfrac{\sqrt{x}}{x^2+y^2}\sin(x^2+y^2), & (x,y)\neq(0,0);\\ x, & (x,y)=(0,0).\end{cases}$$

Compute $f_x(0,0)$ and $f_y(0,0)$.

58. Find the equation for the tangent plane and the normal line for the surface $2^{x/z}+2^{y/z}=8$ at $(2,2,1)$.

59. ★ Find the absolute maximum and minimum for $f(x,y)=ax^2+2bxy+cy^2$ in the unit disk. Here we assume $b^2-ac<0$ and $a,b,c>0$.

60. Determine if $f_x$ and $f_y$ exist or not at the origin for $f(x,y)=\mathrm{e}^{\sqrt{x^2+y^4}}$. If so, find the value. If not, state your reasons.

61. Let

$$f(x,y)=\begin{cases}xy, & |x|\geq|y|;\\ -xy, & |x|<|y|.\end{cases}$$

Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

62. Let $z = z(x, y)$ be the implicit function determined by $xe^x - ye^y = ze^z$, and $f(x, y, z)$ has continuous partial derivatives. Compute the gradient of $u = u(x, y) = f(x, y, z(x, y))$.
63. Let $f(x, y) = |x - y|g(x, y)$ where $g(x, y)$ is continuous at $(0, 0)$ with $g(0, 0) = 0$.
    (1) Determine whether the partial derivative of $f$ exist at the origin. If so, find the values.
    (2) Determine whether $f$ is differentiable at the origin, and state your reasons.

# Chapter 15

# Multiple Integrals

1. Let $D = \{(x, y) \mid x^2 + y^2 \le 4\}$, then $\iint_D \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \mathrm{d}A = (\quad)$.
2. Let $D = \{(x, y) \mid x^2 + y^2 \le x + y\}$, then in polar coordinate $\iint_D f(x, y)\,\mathrm{d}x\mathrm{d}y = (\quad)$.
3. Let $\Omega$ be bounded by $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 2z$. In spherical coordinates, the triple integral $\iiint_\Omega (x^2 + y^2 + z^2)\,\mathrm{d}V = (\quad)$.
4. Write the triple integral in cylindrical coordinates in order of $\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$: $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{x^2+y^2} f(x, y, z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = (\quad)$.
5. Let $f(t) = \int_1^{t^2} \mathrm{e}^{-x^2}\,\mathrm{d}x$, then $\int_0^1 t f(t)\,\mathrm{d}t = (\quad)$.
6. Interchange the order of the integral:
   $\int_{-\pi/4}^{\pi/4}\int_0^{2\cos\theta} f(r, \theta) r\,\mathrm{d}r\mathrm{d}\theta = (\quad)$;
   $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y)\,\mathrm{d}y\mathrm{d}x = (\quad)\ (a > 0)$;
   $\int_0^1 \mathrm{d}y \int_0^{2y} f(x, y)\,\mathrm{d}x + \int_1^3 \mathrm{d}y \int_0^{3-y} f(x, y)\,\mathrm{d}x = (\quad)$.
7. Assume $\iint\limits_{x^2+y^2\le a^2} \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}A = \pi$, then $a = (\quad)$.
8. Let $f(x, y)$ be continuous and positive. Then $\lim\limits_{n\to\infty}\int_0^\pi\int_0^\pi (f(x, y))^{1/n}\,\mathrm{d}x\mathrm{d}y = (\quad)$.
9. Given $\int_0^1 f(x)\,\mathrm{d}x = \sqrt{2}$, $\iint\limits_{\substack{0\le x\le 1\\ 0\le y\le 1}} f(x)f(y)\,\mathrm{d}A = (\quad)$, $\iint\limits_{\substack{0\le x\le 1\\ 0\le y\le x}} f(x)f(y)\,\mathrm{d}A = (\quad)$.
10. Write the triple integral in the order of $\mathrm{d}y\mathrm{d}x\mathrm{d}z$: $\int_0^1\int_x^1\int_x^y f(x, y, z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = (\quad)$.
11. The position of the centroid of the region $1 \le x^2 + y^2 \le 2x$ is $(\quad)$.
12. Suppose the density $\delta = \sqrt{x^2 + y^2 + z^2}$, the center of mass of the upper half sphere $0 \le z \le \sqrt{1 - x^2 - y^2}$ is $(\quad)$.
13. Write the triple integral in order of $\mathrm{d}x\mathrm{d}z\mathrm{d}y$: $\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{x+y} f(x, y, z)\,\mathrm{d}z = (\quad)$.
14. Suppose $f(x, y, z)$ is continuous. Then $\lim\limits_{t\to 0+} \frac{1}{\pi t^3} \iiint\limits_{x^2+y^2+z^2\le t^2} f(x, y, z)\,\mathrm{d}V = (\quad)$.

15. Suppose $f'(t)$ is continuous with $f(0) = 0$ and $f'(0)$ exists. Then $\lim\limits_{t\to 0+} \frac{1}{\pi t^4} \iiint\limits_{x^2+y^2+z^2\le t^2} f(\sqrt{x^2+y^2+z^2})\,\mathrm{d}V = (\qquad)$.

16. Compute the volume of the solid $D$ bounded by surfaces given below:

(1) $x^2+y^2=a^2$, $y^2+z^2=a^2$ and $z^2+x^2=a^2$;

(2) $x^2+y^2=x$ and $x^2+y^2+z^2=1$;

(3) $z=0$, $z=2-x-2y$, $x=2y$ and $x=0$;

(4) $x^2+y^2=az$ and $z=2a-\sqrt{x^2+y^2}$ $(a>0)$;

(5) $z=6-x^2-y^2$ and $z=\sqrt{x^2+y^2}$;

(6) $x^2+y^2+z^2=2az(a>0)$ and $x^2+y^2=z^2$, choose the part containing $z$-axis;

(7) $z=\sqrt{5-x^2-y^2}$ and $x^2+y^2=4z$;

(8) $z^2=\frac{x^2}{4}+\frac{y^2}{9}$ and $2z=\frac{x^2}{4}+\frac{y^2}{9}$.

17. Compute the following integral:

(1) $\iint\limits_{|x|+|y|\le 1} (|x|+|y|)\,\mathrm{d}x\mathrm{d}y$;

(2) $\int_0^2\int_0^2 \max(xy,1)\,\mathrm{d}x\mathrm{d}y$;

(3) $\int_0^2\int_x^2 \mathrm{e}^{-y^2}\,\mathrm{d}y\mathrm{d}x$;

(4) $\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,\mathrm{d}y\mathrm{d}x + \int_1^4\int_{x-2}^{\sqrt{x}} f(x,y)\,\mathrm{d}y\mathrm{d}x$;

(5) $\iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} \mathrm{e}^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}\,\mathrm{d}V$;

(6) $\iint\limits_{x^2+y^2\le 1, x\ge 0} \frac{1+xy}{1+x^2+y^2}\,\mathrm{d}A$;

(7) $\iint\limits_{\substack{x+y\le 1\\ x,y\ge 0}} \mathrm{e}^{\frac{x}{x+y}}\,\mathrm{d}A$;

(8) $\iiint\limits_{x^2+y^2+z^2\le R^2} \sqrt{x^2+y^2+z^2}\,\mathrm{d}V$;

(9)★ $\iint\limits_{\substack{x+y\le 1\\ x,y\ge 0}} \mathrm{e}^{\frac{y}{x+y}}\,\mathrm{d}x\mathrm{d}y$;

(10) $\iint\limits_{\substack{0\le x\le 1\\ 0\le y\le 1}} |x^2+y^2-1|\,\mathrm{d}A$;

(11) $\int_0^1\int_{3y}^3 \mathrm{e}^{x^2}\,\mathrm{d}x\mathrm{d}y$;

(12) $\iint\limits_{\substack{x^2+y^2\le 9\\ 0\le y\le \sqrt{3}x}} (x^2+y^2)^{3/2}\,\mathrm{d}A$;

(13) $\iint\limits_{|x|+|y|\le 1} (|x|+y\mathrm{e}^{-x^2})\,\mathrm{d}A$;

(14) $\iiint\limits_{x^2+y^2\le 4} |x^2+y^2-2y|\,\mathrm{d}A$;

(15) $\iint\limits_{x^2+y^2\le x+y} (x+y)\,\mathrm{d}A$;

(16) $\iint\limits_{|x|+|y|\le 1} \mathrm{e}^{x+y}\,\mathrm{d}A$;

(17) $\iint\limits_{x^2+y^2\le R^2} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}A$;

(18) $\iint\limits_{\substack{x^2+y^2\le 1\\ x,y\ge 0}} \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,\mathrm{d}A$;

(19) $\displaystyle\iiint\limits_{\substack{x^2+y^2+z^2\le 1\\ z\ge 0}} z\mathrm{e}^{-x^2-y^2-z^2}\,\mathrm{d}V$; (20) $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\,\mathrm{d}x\mathrm{d}y$;

(21) $\displaystyle\iiint\limits_{\substack{x^2+y^2+z^2\ge z\\ x^2+y^2+z^2\le 2z}} z\,\mathrm{d}V$; (22) $\displaystyle\iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} z^2\,\mathrm{d}V$;

(23) $\displaystyle\iint\limits_{x^2+y^2\le 1} (x^2+xy\mathrm{e}^{x^2+y^2})\,\mathrm{d}A$; (24) $\displaystyle\iint\limits_{\substack{0\le x\le 1\\ 0\le y\le 1}} \mathrm{e}^{\max\{x^2,y^2\}}\,\mathrm{d}A$.

18. ★ Compute

$$\iint\limits_D r^2\sin\theta\sqrt{1-r^2\cos(2\theta)}\,\mathrm{d}r\mathrm{d}\theta,\qquad D=\left\{(r,\theta)\mid 0\le r\le\sec\theta, 0\le\theta\le\frac{\pi}{4}\right\}$$

19. Write the other 5 equivalent form of the following integral by changing the order of the variables:

$$\int_0^2\int_0^{4-y^2}\int_0^{2-y} f(x,y,z)\,\mathrm{d}z\mathrm{d}x\mathrm{d}y$$

20. Compute

$$\iint\limits_R xy^2\,\mathrm{d}x\mathrm{d}y;$$

Where $R$ is bounded by $x=p/2$ and $y^2=2px$ $(p<0)$.

21. Compute

$$\iint\limits_D |\cos(x+y)|\,\mathrm{d}A$$

Where $D$ is bounded by $y=x$, $y=0$ and $x=\frac{\pi}{2}$.

22. Use the transformation $x=u^2-v^2$, $y=2uv$ $(u\ge 0, v\ge 0)$ to evaluate the integral

$$\int_0^1\int_0^{2\sqrt{1-x}}\sqrt{x^2+y^2}\,\mathrm{d}y\mathrm{d}x.$$

23. Evaluate the integrals:

$$\int_0^1\int_{\sqrt[3]{y}}^1 \frac{2\pi\sin(\pi x^2)}{x^2}\,\mathrm{d}x\mathrm{d}y.$$

24. Compute

$$\iiint\limits_D (x^2+y^2)\,\mathrm{d}V,$$

Where $D$ is determined by $0<a\le\sqrt{x^2+y^2+z^2}\le A$ and $z>0$.

25. Compute $\iiint_V (\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2})\,\mathrm{d}V$. Here the region is enclosed by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$V=\frac{4}{3}\pi(\frac{\sqrt{3}}{2})^3=\frac{\sqrt{3}}{2}\pi$

$\boxed{26.}$ Compute $\iiint_V (x^2+y^2+z^2)\,\mathrm{d}V$ where the region is inside the sphere $x^2+y^2+z^2 \le x+y+z$.

$x=r\cos\varphi\cos\phi+\frac{1}{2},\ y=r\sin\varphi\cos\phi+\frac{1}{2}$ $(x-\frac{1}{2})^2+(y-\frac{1}{2})^2+(z-\frac{1}{2})^2\le\frac{3}{4}$

$z=\frac{1}{2}+r\sin\phi$

27. Compute

$$\iint_D (x^3+y^3)\,\mathrm{d}A,$$

Where $D$ is bounded by $x^2=2y$, $x^2=3y$, $x=y^2$ and $x=2y^2$.

28. Compute

$$\iiint_D z^2\,\mathrm{d}V,$$

where $D$ is bounded by $x^2+y^2+z^2=a^2$ and $x^2+y^2=ax$.

29. Compute

$$\iiint_D (x+y+z)\cos(x+y+z)^2\,\mathrm{d}V,$$

where $D$ is the solid bounded by $0\le x-y\le 1$, $0\le x-z\le 1$ and $0\le x+y+z\le 1$.

30. Find the centroid of of the region bounded by $r=2\sin\theta$ and $r=4\sin\theta$.

31. Compute

$$\iint_D \frac{x\sin y}{y}\,\mathrm{d}A,$$

Where $D$ is bounded by $y=x$ and $y=x^2$.

32. Compute

$$\iiint_\Omega (x+z)\,\mathrm{d}V,\qquad \iiint_\Omega xyz\,\mathrm{d}V,$$

Where $\Omega$ is bounded by $z=\sqrt{x^2+y^2}$ and $z=\sqrt{1-x^2-y^2}$.

33. Compute that

$$\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\int_0^y \frac{\sin z}{(1-z)^2}\,\mathrm{d}z.$$

34. ★ Suppose $f(x)>0$ is continuous on $[a,b]$. Prove that

$$\int_a^b f(x)\,\mathrm{d}x\int_a^b \frac{\mathrm{d}x}{f(x)} \ge (b-a)^2.$$

35. Let $R=\{(x,y)\mid x^2+y^2\le 4, x\ge 0, y\ge 0\}$. Suppose $f(x)$ is a positive continuous function

on $R$, $a$ and $b$ are two real numbers. Compute

$$\iint\limits_R \frac{a\sqrt{f(x)} + b\sqrt{f(y)}}{\sqrt{f(x)} + \sqrt{f(y)}}\, \mathrm{d}A.$$

36. Use symmetry to compute the integral:

$$\iint\limits_{(x-1)^2+(y-1)^2\le 2} (\cos^2(x^2+y) + \sin^2(x+y^2))\, \mathrm{d}A.$$

37. Compute

$$\iiint\limits_D xz\, \mathrm{d}V,$$

Where $D$ is bounded by $z = 0$, $z = y$, $y = 1$ and $y = x^2$.

38. Which of the following definite integral is larger?

$$I_1 = \iiint\limits_D \ln^2(x+y+z+3)\, \mathrm{d}V, \qquad I_2 = \iiint\limits_D (x+y+z)^2\, \mathrm{d}V,$$

Where $D$ is bounded by $x + y + z + 1 = 0$, $x + y + z + 2 = 0$, $x = 0$, $y = 0$ and $z = 0$.

39. Let $R = \{(x, y) \mid x^2 + y^2 \le 1\}$. Order the three following integral:

$$I_1 = \iint\limits_R \cos\sqrt{x^2+y^2}\, \mathrm{d}A, \quad I_2 \iint\limits_R \cos(x^2+y^2)\, \mathrm{d}A, \quad I_3 = \iint\limits_R \cos(x^2+y^2)^2\, \mathrm{d}A.$$

40. ★ Given $F(t) = \displaystyle\int_1^t \int_y^t f(x)\, \mathrm{d}x\mathrm{d}y$ where $f$ is a single variable continuous function. Compute $F'(2)$.

41. Suppose $f(x, y)$ and its first and second order partial derivatives are continuous on $R = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. If $f(1, y) = 0$ and $f(x, 1) = 0$, compute

$$\iint\limits_R xyf_{xy}(x, y)\, \mathrm{d}x\mathrm{d}y.$$

42. Compute

$$\iiint\limits_D x^2 \mathrm{e}^y\, \mathrm{d}V,$$

Where $D$ is bounded by parabolic cylinder $z = 1 - y^2$ and planes $z = 0$, $x = \pm 1$.

43. Find the centroid of a right circular cone of height $h$ and base radius $a$.

44. Suppose $f(x, y)$ is continuous function on the region $0 \le |x| \le 1$ and $0 \le |y| \le 1$ with

$f(0,0) = -1$. Compute the limit:

$$\lim_{x\to 0+} \frac{\int_0^{x^2} \mathrm{d}t \int_x^{\sqrt{t}} f(t,u)\,\mathrm{d}u}{1-\mathrm{e}^{-x^3}}.$$

45. Find the mass of a thin plate of the region bounded by curves $xy = 1$, $y = x+1$ and $y = x$, whose density is $\delta = |x+y|$.

46. Compute

$$\iint_R \cos\frac{x-y}{x+y}\,\mathrm{d}A,$$

Where $R$ is bounded by $x + y = 1$, $x = 0$ and $y = 0$.

47. Let the region $D = \{(x,y,z) \mid \sqrt{3(x^2+y^2)} \le z \le \sqrt{16-x^2-y^2}\}$. Write the triple integral $\iiint_D f(x,y,z)\,\mathrm{d}V$

(1) in Cartesian coordinates.

(2) in cylindrical coordinates.

Evaluate the integral given $f(x,y,z) = x^2 + y^2 + z$.

48. Suppose $f(x)$ is continuous. Compute $F'(t)$ given

$$F(t) = \iiint_{\substack{x^2+y^2\le t^2\\ 0\le z\le 2}} (f(x^2+y^2)+z^2)\,\mathrm{d}V.$$

49. ★ Assume $f(x,y)$ is continuous, prove

$$\iint_{\substack{|x|\le 1\\ |y|\le 1}} f(x-y)\,\mathrm{d}A = \int_{-2}^{2} f(t)(2-|t|)\,\mathrm{d}t.$$

50. Compute

$$\iiint_D (x^2+y^2)\,\mathrm{d}V,$$

Where $D$ is bounded by $z = 2$, $z = 8$ and the surface generated by revolving the following curve about $z$-axis:

$$\begin{cases} y^2 = 2z \\ x = 0 \end{cases}$$

51. Compute

$$\iiint_D z\sqrt{x^2+y^2+z^2}\,\mathrm{d}V,$$

Where $D$ is defined by $x^2 + y^2 + z^2 \leq 1$ and $z \geq \sqrt{3(x^2 + y^2)}$.

52. Compute
$$\iint\limits_{\substack{1 \leq x^2+y^2 \leq 4 \\ x,y \geq 0}} \frac{\pi \sin(\pi \sqrt{x^2 + y^2})}{x + y} \, \mathrm{d}A.$$

53. Compute
$$\iiint\limits_D y \cos(x + z) \, \mathrm{d}V,$$
Where $D$ is bounded by $z = 0$, $y = 0$, $y = \sqrt{x}$ and $x + z = \pi/2$.

54. Compute
$$\iiint\limits_D |z - x^2 - y^2| \, \mathrm{d}V,$$
Where $D$ is bounded by $x^2 + y^2 = 1$, $z = 0$ and $z = 1$.

55. Compute
$$\iiint\limits_D xz \, \mathrm{d}V,$$
Where $D$ is bounded by $z = 0$, $z = y$, $y = 1$ and $y = x^2$.

56. Compute
$$\iiint\limits_D z \, \mathrm{d}V,$$
Where $D$ is bounded by $z = \frac{h}{R}\sqrt{x^2 + y^2}$ and $z = h$. Here $R$ and $h$ are positive.

# Chapter 16

# Integrals and Vector Fields

1. Assume $L$ is the clockwise circle $x^2 + y^2 = a^2$, then $\oint_L (x^3 - x^2 y)\,\mathrm{d}x + (xy^2 - y^3)\,\mathrm{d}y = ($ $)$.
2. Assume $L$ is the ellipse $4x^2 + y^2 = 8x$ in the counterclockwise direction, then $\oint_L \mathrm{e}^{y^2}\,\mathrm{d}x + x\,\mathrm{d}y = (\quad)$.
3. Assume $\Sigma$ is the sphere $x^2 + y^2 + z^2 = a^2$ with the direction point outward. Then $\oiint_\Sigma \dfrac{x\,\mathrm{d}y\mathrm{d}z + y\,\mathrm{d}z\mathrm{d}x + z\,\mathrm{d}x\mathrm{d}y}{\sqrt{x^2 + y^2 + z^2}^3} = (\quad)$.
4. Assume $\frac{(x+ay)\,\mathrm{d}y - y\,\mathrm{d}x}{(x+y)^2}$ is exact, then $a = (\quad)$.
5. Let $L$ be the counterclockwise circle $x^2 + (y-1)^2 = 4$, then $\oint_L \dfrac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + (y-1)^2} = (\quad)$.
6. Let $L$ be the line segment $|x| + |y| = 1$ in the counterclockwise direction, then $\oint_L x^2 y^2\,\mathrm{d}x - \cos(x+y)\,\mathrm{d}y = (\quad)$.
7. Let $u = x^2 + 2y + yz$, then $\nabla \cdot (\nabla u) = (\quad)$.
8. The centroid of the curve $y = \sqrt{2x - x^2}$ $(0 \le x \le 2)$ is $(\quad)$.
9. Let $C : x^2 + y^2 = 2$, $\int_C (x+y)^2\,\mathrm{d}s = (\quad)$.
10. Let $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ and $C$ be a simple smooth closed curve in the clockwise direction. Then the flux outward through $C$ is $(\quad)$.
11. Let $L$ be the polygonal line connecting $(0,0,0)$, $(0,0,2)$, $(1,0,2)$ and $(1,3,2)$ successively. Then $\int_L x^2 yz\,\mathrm{d}s = (\quad)$.
12. Let $L$ be the circle $x^2 + y^2 = 4x$, then $\oint_L \sqrt{x^2 + y^2}\,\mathrm{d}s = (\quad)$.
13. Let $L$ be the line segment from $(1,1,1)$ to $(2,3,4)$, then $\int_L x\,\mathrm{d}x + y\,\mathrm{d}y + (x+y-1)\,\mathrm{d}z = ($ $)$.
14. Assume that the integral $\int_C (f(x) - \mathrm{e}^x)\sin y\,\mathrm{d}x - f(x)\cos y\mathrm{d}y$ is independent with respect to the path. If $f$ has continuous derivatives and $f(0) = 0$, find $f(x)$.
15. Compute $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$, where $\boldsymbol{F}(x,y) = (3x^2 y, x^3 + x - 2y)$, and the curve $C$ starts at $(0,0)$, moves to $(2,0)$ along $x^2 + y^2 = 2x$ in the first quadrant, then travels to $(0,2)$ along $x^2 + y^2 = 4$ in the first quadrant.
16. Among all rectangular region $0 \le x \le a$, $0 \le y \le b$, find the one for which the total

outward flux of $\boldsymbol{F} = (x^2+4xy)\boldsymbol{i} - 6y\boldsymbol{j}$ across the four sides is least. What is the least flux?

17. Compute $\displaystyle\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$, where $\boldsymbol{F}(x, y) = (y^2 - z^2, 2z^2 - x^2, 3x^2 - y^2)$ and $C$ is the intersection of plane $x + y + z = 2$ and cylinder $|x| + |y| = 1$. In the counter-clockwise direction if seen from the positive direction of $z$.

18. Compute $\displaystyle\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$, where $\boldsymbol{F}(x, y) = (\mathrm{e}^x \sin y - my, \mathrm{e}^x \cos y - m)$, and $C$ is the curve starting from $(a, 0)$ to $(0, 0)$ along the upper half circle $x^2 + y^2 = ax$.

19. ★ Assume $L$ is the intersection line of sphere $x^2 + y^2 + z^2 = 1$ and plane $x + y + z = 0$. Evaluate $\displaystyle\oint_L xy\,\mathrm{d}s$.

20. Let $\boldsymbol{F} = (x + y, x - y)$ and curve $C$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the counterclockwise direction. Compute $\displaystyle\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$.

21. Given the curve $C : x = t, y = t^2$ and $z = t^3$, $(0 \le t \le 1)$, evaluate the integral

$$\int_C \left(y^2 - z^2\right) \mathrm{d}x + 2yz\,\mathrm{d}y - x^2\,\mathrm{d}z.$$

22. Consider the following line integral:

$$\int_{(1,1,1)}^{(1,3,\pi)} \mathrm{e}^x \ln y\,\mathrm{d}x + \left(\frac{\mathrm{e}^x}{y} + \sin z\right) \mathrm{d}y + y \cos z\,\mathrm{d}z.$$

Show that the differential form in the integral is exact and evaluate the integral.

23. Find the outward flux of $\boldsymbol{F} = (6x + y, -x - z, 4yz)$ across the boundary of $D$, where $D$ is the region in the first octant bounded by $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 1$ and the coordinate planes.

24. Use the Stokes' Theorem to calculate the circulation of the field $\boldsymbol{F} = (y, -xz, x^2)$ around the curve $C$, which is the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when views form above.

25. Let $L$ be the segment on $y = x^2$ from $(1, 1)$ to $(-1, 1)$, then the line segment to $(0, 2)$. Compute

$$\int_L y^2\,\mathrm{d}x - x\,\mathrm{d}y.$$

26. Let $L$ be the segment on the circle $y = \sqrt{2x - x^2}$ from $(2, 0)$ to $(0, 0)$. Compute

$$\int_L \left(x^2 - y\right) \mathrm{d}x - (x + \sin y)\,\mathrm{d}y.$$

27. Let $\Sigma$ be the surface cut from the cone $z = \sqrt{x^2+y^2}$ by $x^2+y^2 = 2x$. Compute

$$\iint\limits_{\Sigma} \frac{1}{z}\,\mathrm{d}\sigma$$

28. Let $L$ be the intersection of $y+z=2$ and $x^2+y^2=1$ in the counterclockwise direction if view from the positive part of $z$-axis. Compute

$$\oint\limits_{L} -y^2\,\mathrm{d}x + x\,\mathrm{d}y + z^2\,\mathrm{d}z.$$

29. Let $\boldsymbol{F} = (-2xz, 0, y^2)$.
    (1) Compute the curl of $\boldsymbol{F}$;
    (2) Show that $\displaystyle\iint\limits_{R} \nabla\times\boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = 0$ for $R$ is any portion of the unit sphere.
    (3) Show that $\displaystyle\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = 0$ for any simple closed curve $C$ on the unit sphere.
30. Let $\boldsymbol{F} = (y^2, x^2yz, x^2)$. Use the divergence theorem to compute the outward flux of the vector field $\boldsymbol{F}$ across the upper hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$.
31. Find the mass of a thin wire $C$ which is the intersection of the cylinder $x^2+y^2 = 1$ and $x+z=0$ with the density function $\delta = |xy|$.
32. Let $C$ be the segment on the curve $x = 1-y^2$ from $(0,-1)$ to $(0,1)$, compute

$$\int_C y^2\,\mathrm{d}x + x^2\,\mathrm{d}y.$$

33. Show the vector field $\boldsymbol{F} = (\sin y, x\cos y, -\sin z)$ is conservative, and find its potential function.
34. Use Green's Theorem to evaluate the circulation by the velocity field $\boldsymbol{F} = x^2y\boldsymbol{i} - xy^2\boldsymbol{j}$ along the counterclockwise circle $C : x^2+y^2 = 4$.
35. Parametrize the following surfaces and derive the factor $|\boldsymbol{r}_u\times\boldsymbol{r}_v|$.
    (1) the sphere centered at the origin of radius $R$;
    (2) a circular cylinder $y^2+z^2 = a^2$;
    (3) a cone $x = 2\sqrt{y^2+z^2}$;
    (4) a paraboloid $z = 16 - x^2 - y^2$.
36. Let $\boldsymbol{r} = (x,y,z)$ and $\boldsymbol{F} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|^p}$. Find the value of $p$ such that $\mathrm{div}\boldsymbol{F} = 0$.
37. Let $S$ be the part of the plane $x+y+z=1$ in the first octant, compute

$$\iint_S yz\,\mathrm{d}\sigma.$$

38. Assume $C$ is a simple smooth closed curve in the counterclockwise direction. Prove that the area enclosed by $C$ is $A = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x$.
Using this to compute the area enclosed by the following curve:

$$\begin{cases} x = 2\cos t + \cos(2t) \\ y = 2\sin t - \sin(2t) \end{cases} \quad 0 \le t \le 2\pi.$$

39. Assume the vector field $\boldsymbol{F} = \left(2xy(x^4+y^2)^k, -x^2(x^4+y^2)^k\right)$. Find $k$ such that there exists the potential function $f(x,y)$; i.e., $\boldsymbol{F} = \nabla(f)$ for $x > 0$. Using this value of $k$ to compute

$$\int_{(1,0)}^{(\sqrt{3},1)} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}.$$

40. Let $C : (x-1)^2 + y^2 = 2$ in the counterclockwise direction. Compute

$$\oint_C \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}.$$

41. Let $C : y = \sqrt{2x-x^2}$ from $(2,0)$ to $(0,0)$. Compute

$$\int_C (\mathrm{e}^x \sin y - b(x+y))\,\mathrm{d}x + (\mathrm{e}^x\cos y - ax)\,\mathrm{d}y.$$

42. Let $f$ be continuously differentiable and $L$ be the line segment from $(1,2)$ to $(2,8)$. Compute

$$\int_L (2xy - \frac{2y}{x^2}f(\frac{y}{x^2}))\,\mathrm{d}x + (x^2 + \frac{1}{x^2}f(\frac{y}{x^2}))\,\mathrm{d}y.$$

43. Let $C$ be the circle $x^2+y^2 = 9$ in the counterclockwise direction. Compute

$$\oint_C (3y - \mathrm{e}^{\sin x})\,\mathrm{d}x + (7x + \sqrt{y^4+1})\,\mathrm{d}y.$$

44. Let $\boldsymbol{F} = (z^2, -3xy, x^3y^3)$ and $S$ is part of the paraboloid $z = 5 - x^2 - y^2$ with $z \ge 1$.
    (1) Compute $\nabla\times\boldsymbol{F}$.
    (2) Parametrize $S$.
    (3) Compute $\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,\mathrm{d}\sigma$ where $\boldsymbol{n}$ is in the direction of $z$ increases.
45. Let $L$ be the boundary of the region bounded by $x^2+y^2 = a$, $y = x$ and $x$-axis in the first quadrant. Compute

$$\oint_L \mathrm{e}^{\sqrt{x^2+y^2}}\,\mathrm{d}s.$$

46. Let $L$ be the counterclockwise circle $x^2 + y^2 = a^2$. Compute

$$\oint_L \frac{(x+y)\,\mathrm{d}x - (x-y)\,\mathrm{d}y}{x^2+y^2}.$$

47. Let $L$ be the segment on the curve $y = \sqrt[3]{x}$ from $(0,0)$ to $(1,1)$. Compute

$$\int_L y\mathrm{e}^{y^2}\,\mathrm{d}x + (x\mathrm{e}^{y^2} + 2xy^2\mathrm{e}^{y^2})\,\mathrm{d}y.$$

48. Let $L$ be the boundary of the square with end vertices being $(1,0)$, $(2,0)$, $(2,1)$ and $(1,1)$ in the counterclockwise direction. Compute

$$\oint_L (x^2+y^2)\,\mathrm{d}x + (x^2-y^2)\,\mathrm{d}y.$$

49. Let $L$ be the circle cut from $x^2+y^2+z^2 = 1$ by the plane $y = x\tan a$ $(0 < a < \pi)$ in the counterclockwise direction if view from the positive part of $x$-axis. Compute

$$\int_L (y-z)\,\mathrm{d}x + (z-x)\,\mathrm{d}y + (x-y)\,\mathrm{d}z.$$

50. Let $L$ be the boundary of the sphere $x^2+y^2+z^2 = 1$ cut by the first octant in the counterclockwise direction if view from the outside of the sphere in the first octant. Compute

$$\int_L (y^2-z^2)\,\mathrm{d}x + (z^2-x^2)\,\mathrm{d}y + (x^2-y^2)\,\mathrm{d}z.$$

51. Compute $\displaystyle\int_L y^2\,\mathrm{d}x + x^2\,\mathrm{d}y$ given $L$ to be
   (1) The upper half counterclockwise circle $x^2+y^2 = R^2$, $y \geq 0$.
   (2) The line segment from $(R,0)$ to $(-R,0)$.
52. Prove that the following integrals are independent to the path, and evaluate these integrals.
   (1) $\displaystyle\int_{(0,0)}^{(1,1)} (x-y)\,(\mathrm{d}x - \mathrm{d}y)$.
   (2) $\displaystyle\int_{(2,1)}^{(1,2)} \phi(x)\,\mathrm{d}x + \varphi(y)\,\mathrm{d}y$ (both functions are continuous).
   (3) $\displaystyle\int_{(1,0)}^{(6,8)} \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{x^2+y^2}}$ (the origin is not on the path).
53. Prove that the differential form $\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2+y^2}$ is exact on the plane without the origin and the negative half of the $y$-axis, and find the potential function.

# 第二章　极限和连续性

1. 已知 $\lim\limits_{x\to0^+} f(x)=a$, $\lim\limits_{x\to0^-} f(x)=b$, 则 $\lim\limits_{x\to0^-}\left(f(x-\sin x)+2f\left(x^2+x\right)\right)=$

   (A) $a+2b$.　　(B) $b+2a$.　　(C) $3a$.　　(D) $3b$ .

2. 求下列极限:

(1) $\lim\limits_{x\to2}\dfrac{x^2-x-2}{\sqrt[3]{x^2+23}-3}$;

(2) $\lim\limits_{x\to0}\dfrac{\sin(1-\cos x)}{(\tan x)^2}$;

(3) $\lim\limits_{x\to-\infty}(x+\sqrt{x^2-x+4})$;

(4) $\lim\limits_{x\to1}\dfrac{2x^2+5}{x-3}$;

(5) $\lim\limits_{x\to1}\left(\dfrac{3}{1-x^3}-\dfrac{4}{1-x^4}\right)$;

(6) $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$;

(7) $\lim\limits_{x\to1}\dfrac{x^{2020}-1}{x^{2019}-1}$;

(8) $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-1-x/2}{x^2}$;

(9) $\lim\limits_{x\to\frac{\pi}{2}-}\tan x$;

(10)★ $\lim\limits_{x\to1}(1-x)\tan\dfrac{\pi x}{2}$;

(11) $\lim\limits_{x\to1}\dfrac{x^4-1}{x^3-1}$;

(12) $\lim\limits_{x\to0}\left[\dfrac{1}{x}\right]\sin x$;

(13) $\lim\limits_{x\to0}\dfrac{\tan(2x)}{3x}$;

(14) $\lim\limits_{x\to-2}\dfrac{x^3+8}{x^2-x-6}$;

(15) $\lim\limits_{x\to0}\dfrac{1-\cos(2x)}{x\sin(2x)}$;

(16) $\lim\limits_{x\to\infty}(\sqrt{x^2+x}-\sqrt{x^2-2x+3})$;

(17) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{2x+1}}$;

(18) $\lim\limits_{x\to\infty}(\sqrt{x}(\sqrt{x+1}-\sqrt{x}))$;

(19) $\lim\limits_{x\to0}\dfrac{\sin(3x)}{\sin(2x)}$;

(20)★ $\lim\limits_{x\to1}\dfrac{\sqrt{2x-x^4}-\sqrt[3]{x}}{1-\sqrt[4]{x^3}}$.

3. 判断题. 若正确, 请给出证明过程. 若错误, 请给出反例.

   (1) ★ 若 $\lim\limits_{x\to c} f(x)=A$, 且 $\lim\limits_{y\to A} g(y)=B$, 则 $\lim\limits_{x\to c} g(f(x))=B$.

   (2) 若 $\lim\limits_{x\to c}|f(x)|=|l|$, 则 $\lim\limits_{x\to c} f(x)=l$.

   (3) 若 $\lim\limits_{x\to c}[f(x)+g(x)]$ 存在, 则 $\lim\limits_{x\to c} f(x)$ 和 $\lim\limits_{x\to c} g(x)$ 都存在.

   (4) 若 $f(x)>0$, 且 $\lim\limits_{x\to c} f(x)=l$, 则 $l>0$.

   (5) 若 $f^2$ 连续, 则 $f$ 也连续.

   (6) 若 $f^3$ 连续, 则 $f$ 也连续.

4. 求下列函数的渐近线:

$$(1) f(x) = \frac{x^3 + x + 1}{(x-1)(x+2)}; \qquad (2) f(x) = \frac{2x^2 - x + 3 + x\sin x}{x^2};$$

$$(3) f(x) = \frac{(|x|+1)^3}{(x-2)(x-3)}; \qquad (4)\bigstar\ f(x) = \frac{x\sqrt{x^2+3x+1}}{3x+10^6}.$$

5. 求$a$ 的值, 使下列函数连续:

$$f(x) = \begin{cases} \dfrac{1-\cos\sqrt{x}}{ax}, & x > 0 \\ 1, & x \le 0 \end{cases}.$$

6. 若$m$ 和$n$ 是正整数，求极限

$$\lim_{x\to\pi} \frac{\sin(mx)}{\sin(nx)}.$$

7. 用极限的定义陈述: $L$ 不是函数$f(x)$ 在$x \to c$ 的极限.

8. 确定常数$a$ 和$b$ 的值, 使

$$\lim_{x\to 0}\left(\frac{\sqrt{x^2+x+1}}{x} - \frac{a}{x} - b\right) = 0.$$

9. 已知

$$\lim_{x\to 0}\frac{f(x)}{x^2} = 1,$$

求

$$\lim_{x\to 0}\frac{f(x)}{1-\cos x}.$$

10. 确定常数$a$ 和$b$ 的值, 使

$$\lim_{x\to\pi/2}\frac{\sqrt{x}-a}{\cos x} = b$$

11. 证明

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

进一步, 若$\lim\limits_{x\to 0} g(x) = 0$, 证明

$$\lim_{x\to 0} g(x)\sin\frac{1}{x} = 0.$$

12. 设$\lim\limits_{x\to 0^+} f(x) = l$, 且$\lim\limits_{x\to 0^-} f(x) = m$, 确定下列极限存在与否. 若存在, 求出极限值.

$$\lim_{x\to 0} f(-x); \qquad \lim_{x\to 0^+} f(x^2 - x); \qquad \lim_{x\to 0^-}(2f(-x) + f(x^2)).$$

13. 设函数$f(x) = \lceil x\rceil - x$, 其中$\lceil x\rceil = \min\{n \mid n \ge x\}$ 不小于$x$ 的最小整数, 求单边极

限 $\lim\limits_{x\to k^-} f(x)$ 和 $\lim\limits_{x\to k^+} f(x)$, 其中$k$ 是一个整数. 进一步, 极限$\lim\limits_{x\to k} f(x)$ 是否存在，若存在，极限是多少?

14. 确定下列极限存在与否. 若极限存在, 求出极限值. 若极限不存在, 给出理由.

$$(1)\bigstar\ \lim_{x\to 0}\frac{(1+x)^{\frac{1}{m}}-1-\frac{1}{m}x}{x^2};\qquad (2)\lim_{x\to 0}\frac{\sqrt{3}-\sqrt{2+\cos x}}{\sin^2 2x};$$

$$(3)\lim_{x\to\infty}\frac{\sin^2 x}{x^2+3x};\qquad (4)\bigstar\ \lim_{x\to\infty}(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x})x^{3/2}.$$

15. 设$f(x)=x^4+ax^3+bx^2+cx+d$, 其中$a,b,c,d\in\mathbb{R}$. 若$f(1)<0$, $f(2)>0$, $f(3)<0$. 证明方程$f(x)=0$ 有4 个实根.

16. 求函数$f(x)$ 的所有间断点, 且说明它们的类型:

$$f(x)=\begin{cases}\dfrac{\sin x}{x}, & 0<x<1;\\ \dfrac{1}{x}, & x\geq 1.\end{cases}$$

17. 确定常数$a$ 和$b$ 的值, 使

$$\lim_{x\to 1}\frac{x^3+2x+a}{x-1}=b.$$

18. 求$k$ 的值, 使下列函数连续:

$$f(x)=\begin{cases}\dfrac{\sin(3x)}{x}, & x\neq 0;\\ x^2+k, & x=0.\end{cases}$$

19. 求函数$f(x)$ 的所有可去间断点:

$$f(x)=\frac{x(x+1)}{\sin(\pi x)}.$$

20. 证明：任意3 阶多项式至少有1个根. 进一步, 任意$2k+1$ 阶多项式至少几个根?

21. 证明：方程$2\sin x=3-2x$ 在区间$[0,1]$ 至少有一个根.

22. 确定常数$a$ 和$b$ 的值, 使

$$\lim_{x\to 2}\frac{x-2}{x^2+ax+1}=b.$$

23. 求$a$ 的值, 使$\lim\limits_{x\to 1} f(x)$ 存在:

$$f(x)=\begin{cases}ax^2, & x<1;\\ 1, & x=1;\\ \cos(\pi x), & x>1.\end{cases}$$

24. 若$f(x)$ 在$[0,2a]$ 上连续, 且$f(0)=f(2a)$. 证明：存在一个点$x_0\in[0,a]$ 使$f(x_0)=f(x_0+a)$.

# 第三章　导数

1. ★ 下列函数在$x = 0$ 处不可导的是?

   (A) $f(x) = |x| \sin |x|$.　　(B) $f(x) = |x| \sin \sqrt{|x|}$.

   (C) $f(x) = \cos |x|$.　　(D) $f(x) = \cos \sqrt{|x|}$.

2. 设

$$f(x) = \begin{cases} x^4 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

   则使$f^{(n)}(0)$ 存在的最大正整数$n$ 是

   (A) 1.　　(B) 2.　　(C) 3.　　(D) 4.

3. 求下列函数的导函数:

$$(1)\, y = \frac{3}{(5x^2 + \sin(2x))^{3/2}}; \qquad (2)\, y = \frac{x \sin x + \cos x}{x \cos x - \sin x};$$

$$(3)\, y = \sin(\sin(\sin x)); \qquad (4)\, y = \left(\frac{x+1}{x-1}\right)^2 \sin x;$$

$$(5)\, y = (1 + x^2) \cos(2x); \qquad (6)\, y = x^2 \sin^3(2x);$$

$$(7)\, y = \frac{\sec t}{1 + \tan t}.$$

4. 求下列函数在$x = 0$ 处的导数. 若导数不存在, 说明理由.

$$(1)\, y = x^{2/3}; \qquad (2)\, y = \frac{1 + x^2}{\sin x + \cos x}; \qquad (3)\, y = x^2 \sin \frac{1}{x}, \text{ with } f(0) = 0.$$

5. 求下列函数的二阶导函数:

$$(1)\, y^4 - 2x = y^2 + x^2; \qquad (2)\, y = f(x^n), n < 0;$$

$$(3)\, y = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

6. ★ 判断下列函数在$x = 0$ 是否连续和可导, 并给出理由.

$$f(x) = \begin{cases} x, & x \leq 0; \\ \dfrac{1}{n}, & \dfrac{1}{n+1} < x \leq \dfrac{1}{n}. \end{cases}$$

7. ★ 设$f(x) = x(b^2 - x^2)$, $x \in [0, 1)$, 且满足$f(x) = af(x + 1)$, $x \in [-1, 0)$. 试确定$a$ 和$b$

的值, 使$f(x)$ 在$x=0$ 处可导, 并求出$f'(0)$ 的值.

8. 设点$P$ 是抛物线$x^2=2py$ 上除了原点的任意一点. 经过$P$ 点的切线与$x$-轴和$y$-轴分别相交于点$Q$ 和$R$. 证明$PQ=QR$.
9. 若点$P(a,b)$ 在曲线$l:\ (y-x)^3=y+x$ 上, 且在点$P(a,b)$ 处的切线$l$ 的斜率等于3. 求$a$ 和$b$ 的值.
10. 求$x$ 的取值范围，使得函数$f(x)=|x^2+2x|$ 的导数存在.
11. ★ 若$f(0)=0$. 判断下列命题正确与否. 若正确, 给出证明. 若错误, 给出反例.
    (1) 若$\lim\limits_{h\to 0}\frac{1}{h^2}f(1-\cos h)$ 存在, 则$f$ 在$x=0$ 处可导.
    (2) 若$\lim\limits_{h\to 0}\frac{1}{h}[f(2h)-f(h)]$ 存在, 则$f$ 在$x=0$ 处可导.
12. 若$f(x)$ 在$x=0$ 处可导, 且$f(0)=0$, 求:

$$\lim_{x\to 0}\frac{x^2f(x)-2f(x^3)}{x^3}.$$

13. 判断下列命题的正确与否, 并说明理由.
    (1) 已知函数$f=g+h$, 若$f$ 在$x=x_0$ 可导, 则函数$g,h$ 在$x=x_0$ 也可导.
    (2) 已知函数$f=g+h$, 若$g$ 在$x=x_0$ 可导, 且函数$h$ 在$x=x_0$ 不可导, 则$f$ 在$x=x_0$ 不可导.
    (3) 已知函数$f=g\cdot h$, 若$f$ 在$x=x_0$ 可导, 则函数$g,h$ 在$x=x_0$ 也可导.
    (4) 已知函数$f=g\cdot h$, 若$g$ 在$x=x_0$ 可导, 且函数$h$ 在$x=x_0$ 不可导, 则$f$ 在$x=x_0$ 不可导.
14. 令$x=\tan y$, 证明:

$$(1+x^2)y''+2xy'=0.$$

15. 若$g(x)$ 在$x=0$ 可导, 且$g(0)=0, g'(0)=0$. 求下列函数$f(x)$ 在$x=0$ 的导数$f'(0)$:

$$f(x)=\begin{cases} g(x)\sin\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}.$$

16. 证明：乘积的高阶导数公式（即莱布尼兹公式）为:

$$\frac{\mathrm{d}^n(uv)}{\mathrm{d}x^n}=\sum_{k=0}^{n}C_n^k\frac{\mathrm{d}^k u}{\mathrm{d}x^k}\frac{\mathrm{d}^{n-k}v}{\mathrm{d}x^{n-k}}.$$

    其中$\frac{\mathrm{d}^0u}{\mathrm{d}x^0}=u(x)$.
17. 求$y^{(2020)}$, 其中$y=\sin x$.
18. 已知$y=|x|^3$. 求$y'$ 和$y''$, 并证明$y^{(3)}(0)$ 不存在.
19. 若$u(x),v(x),w(x)$ 是关于$x$ 的可导函数. 用函数$u$, $v$, $w$, $\frac{du}{dx}$, $\frac{dv}{dx}$, $\frac{dw}{dx}$ 表示$\frac{d(uv/w)}{dx}$.

20. 若$f'(a)$ 存在, 求
$$\lim_{h\to 0}\frac{f(a+h)-f(a-2h)}{h}.$$
21. 若$y=f(x)$, 且与$y=\sin x$ 在原点有相同的切线. 求:
$$\lim_{x\to\infty}\sqrt{xf\left(\frac{2}{x}\right)}.$$
22. 确定$a$ 和$b$ 的值, 使$y=x^3+ax+b$ 和$2y=xy^3-1$ 在点$(1,-1)$ 处相切.
23. ★ 若函数$f(x)$ 定义在$(-\infty,+\infty)$上, $f(x)\neq 0$, $f'(0)=1$ , 且对于$\forall x,y\in(-\infty,\infty)$, 满足$f(x+y)=f(x)f(y)$. 证明, $\forall x\in(-\infty,\infty)$, $f(x)$ 可导, 且$f'(x)=f(x)$.
24. ★ 求下列函数的100 阶导函数:
$$y=x^2\sin x.$$
25. ★ 设$f(x)=x^3-bx^2+7x$，且存在某个$c\in[0,1]$ 使$f'(c)=3$，求$b$ 的取值范围.
26. 设$f(x)=\frac{x+a}{bx^2+cx+2}$. 确定$a,b$, 和$c$ 的值, 满足:
    (1) $a,b$ 和$c$ 是0 或者1.
    (2) 函数$f$ 经过点$(-1,0)$.
    (3) 直线$y=1$ 是函数$f$ 的一条渐近线.

# 第四章　导数的应用

1. 若$f(x)$ 在$(0,\infty)$ 上可导, 下面哪个命题正确?
   (A) 若$f(x)$ 有唯一零点, 则$f'(x)$ 没有零点.
   (B) 若$f'(x)$ 至少有一个零点, 则$f(x)$ 至少有两个零点.
   (C) 若$f(x)$ 没有零点, 则$f'(x)$ 至多有一个零点.
   (D) 若$f'(x)$ 没有零点, 则$f(x)$ 至多有一个零点.
2. 若$c>0$，则方程$x^3-6x^2+9x+c=0$ 的实根的个数为
   (A) 0.　(B) 1.　(C) 2.　(D) 3.
3. 设$f'(x)=(x-1)^2(x-2)^3$, 下列命题错误的是?
   (A) $(1,f(1))$ 是拐点.　(B) $(2,f(2))$ 是拐点.
   (C) $(3,f(3))$ 不是拐点.　(D) 存在其他拐点.
4. 求下列函数的所有局部极值:

   (1) $f(x)=\sin^3 x+\cos^3 x,\ x\in[0,2\pi]$;　(2) $f(x)=|x(x^2-4)|$;

   (3) $f(x)=\dfrac{x(x^2+1)}{x^4-x^2+1}$;　(4) $g(t)=|t^2-4t+1|,\ x\in[0,5]$.

5. 设函数$f(x)=x^{\frac{2}{3}}-(x^2-1)^{\frac{1}{3}}$, 请画出该函数的简图.
6. 设函数$y=ax^3+bx^2+cx$ 在拐点$(1,1)$ 处的切线是水平的. 求$a$, $b$ 和$c$ 的值.
7. ★　证明方程$a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0=0$ 在$(0,1)$ 上至少有一个实根. 其中系数$a_n,a_{n-1},...,a_0$ 满足$\frac{a_n}{n+1}+\frac{a_{n-1}}{n}+...+a_0=0$.
8. 确定方程$x^2=x\sin x+\cos x$ 的实根数目.
9. 判断下列命题正确与否. 若正确, 给出证明. 若错误, 给出反例.
   (1) ★　若$f'(c)>0$, 则$f(x)$ 在包含$c$ 的某个邻域严格单调递增.
   (2) 若$f'(c)>0$, 则存在某个$\delta>0$, 使得任意的$x\in(c,c+\delta)$, 满足$f(x)>f(c)$.
   (3) 若$(1,f(1))$ 是拐点, 则$f''(1)=0$.
   (4) ★　若$f'(x_0)$ 存在, 且$\lim\limits_{x\to x_0}\frac{f''(x)}{x-x_0}=1$, 则$(x_0,f(x_0))$ 是$y=f(x)$ 的一个拐点.
   (5) 若函数$f(x)$ 在$[a,b]$ 上可导, 且$f'(a)f'(b)<0$. 则存在某个$c\in(a,b)$ 使得$f'(c)=0$.
   (6) 若函数$f(x)$ 在$[a,b]$ 上可导, 且$f'(a)f'(b)>0$. 存在某个$c\in(a,b)$ 使得$f(c)=0$.
10. 若函数$f(x)$, $g(x)$ 在$[a,b]$ 上连续, 且在$(a,b)$ 上二阶可导. $f(x)$ 和$g(x)$ 在区间$(a,b)$ 内有相同的最大值, $f(a)=g(a)$, $f(b)=g(b)$. 证明：存在$c\in(a,b)$ 使得$f''(c)=g''(c)$.
11. ★　若$y=f(x)$ 是由方程$y^3+xy^2+x^2y+6=0$ 确定的函数, 求$y=f(x)$ 的局部极值.
12. ★　若函数$f(x)$ 在$[0,2]$ 上连续, 在$(0,2)$ 上可导, 且$f(2)=3f(0)$. 证明：至少存在某

个$c \in (0,2)$ 使得$(1+c)f'(c) = f(c)$.

13. 设函数$f(x)$ 在$I = (a,b)$ 上是凹函数. 证明：若存在$x_0 \in I$, 使得函数$f$ 在$x = x_0$ 处取局部极小值, 则在区间$I$ 上, 函数$f$ 在$x_0$ 处取唯一局部极小值.
14. 确定$a$ 和$b$ 的值, 使得$(1,3)$ 是函数$y = ax^3 + bx^2$ 的拐点.
15. 设函数$f(x)$ 的定义域是$\mathbb{R}$ , 且满足$f(a+b) = f(a)f(b)$. 若$f$ 在$x = 0$ 处可导, 证明：对任意的$x \in \mathbb{R}$, 使得$f'(x) = f'(0)f(x)$.
16. 若$f$ 在$x = 0$ 处可导, 且对所有的$x$ 满足$|f(x)| \leq |\sin x|$, 证明：$f(0) = 0$ 且$|f'(0)| \leq 1$.

# 第五章　积分

1. 设$I_1 = \int_0^{\frac{\pi}{4}} \frac{\tan x}{x}\,\mathrm{d}x$, $I_2 = \int_0^{\frac{\pi}{4}} \frac{x}{\tan x}\,\mathrm{d}x$, 则下列命题正确的是?

   (A) $I_1 > I_2 > 1$.　　(B) $1 > I_1 > I_2$.

   (C) $I_2 > I_1 > 1$.　　(D) $1 > I_2 > I_1$.

2. 函数在$[a, b]$ 上连续, 且满足$f(x) > 0$. 则方程$\int_a^x f(t)\,\mathrm{d}t + \int_b^x f(t)\,\mathrm{d}t = 0$ 在区间$(a, b)$ 上的实根个数是

   (A) 0.　(B) 1.　(C) 2.　(D) 3.

3. 函数$f(x)$ 是连续函数, 且$a$ 是一个非零常数. 则下列函数是奇函数的是?

   (A) $\int_a^x \left(\int_0^u t f(t^2)\,\mathrm{d}t\right)\mathrm{d}u$.　　(B) $\int_0^x \left(\int_a^u f(t^3)\,\mathrm{d}t\right)\mathrm{d}u$.

   (C) $\int_0^x \left(\int_a^u t f(t^2)\,\mathrm{d}t\right)\mathrm{d}u$.　　(D) $\int_a^x \left(\int_0^u (f(t))^2\,\mathrm{d}t\right)\mathrm{d}u$.

4. 设函数$f(x) = \begin{cases} 2x, & x \le 0 \\ \sin x, & x > 0 \end{cases}$. 若函数$F(x)$ 是函数$f(x)$ 的一个原函数, 且$F(0) = 1$, 则函数$F(x) = ($　　$)$.

5. 若$f'(\sin x) = \cos(2x)$, 则函数$f(x) = ($　　$)$.

6. ★　设$f(x) = f(x + \pi)$, 且$f(x) = \sin x, 0 \le x \le \pi$, 则它的一个原函数是$F(x) = ($　　$)$.

7. 设$M = \sum_{k=1}^{2019} k^{10}$, $N = \int_0^{2019} x^{10}\,\mathrm{d}x$. 比较$M, N$ 的大小.

8. 求积分:

   (1) $\int_{-1}^{1} \sqrt{1 - x^2}(\sin x + 1)\,\mathrm{d}x$;　　(2) $\int_0^2 |x - 1|\,\mathrm{d}x$;

   (3) $\int (ax^2 + b)^m x\,\mathrm{d}x$, $(m \neq -1)$;　　(4) $\int_0^1 x^2(1 - x)^{10}\,\mathrm{d}x$;

   (5) $\int_{\frac{1}{2}}^{1} x^{-\frac{1}{4}}(1 - x^{\frac{3}{4}})^{\frac{1}{3}}\,\mathrm{d}x$;　　(6) $\int_0^{\frac{\pi}{4}} \sin^2\left(2x + \frac{\pi}{4}\right)\mathrm{d}x$.

9. 设$M = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^{100}\sin x\,\mathrm{d}x$, $N = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^{100}\sin^{100} x\,\mathrm{d}x$, $P = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^{100}\sin^{90} x\,\mathrm{d}x$. 比较它们的大小.

10. 用定积分来表示下列极限, 并求出极限值.

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\cos\frac{\pi}{n}}+\sqrt{1+\cos\frac{2\pi}{n}}+\cdots+\sqrt{1+\cos\frac{n\pi}{n}}\right)$$

11. 求由图像$y^2=2px$ 和$x^2=2py$ 所围成的面积, 这里$p$ 是一个正实数.

12. 若函数$f$ 连续, 且是以$T$ 为周期的周期函数. 证明

$$\lim_{x\to+\infty}\frac{1}{x}\int_0^x f(t)\,dt=\frac{1}{T}\int_0^T f(t)\,dt.$$

13. 设函数$f(x)$ 在$[0,1]$ 上连续, 且满足对任意的$x,y\in[0,1]$ 满足$|f(x)-f(y)|\le|x-y|$. 若$n$ 是一个正整数, 证明

$$\left|\int_0^1 f(x)\,\mathrm{d}x-\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\right|\le\frac{1}{2n}$$

14. 求下列函数的导函数:

$$(1)\,h(s)=\int_s^{s^2}\sqrt{1+x^2}\,\mathrm{d}x;\qquad(2)\,y=\int_{x^2}^{\sin x}\frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t;$$
$$(3)\,g(x)=\int_0^x\sin t\,\mathrm{d}t.$$

15. 设$F(x)=\displaystyle\int_0^x tf(x^2-t^2)\mathrm{d}t$，求$F'(x)$.

16. 设函数$f(x)$ 在$(-\infty,\infty)$ 上单调递增, 且$f(x)>0$. 证明

$$F(x)=\frac{\int_0^x tf(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t}$$

在$(0,\infty)$ 上也单调递增.

17. 设函数$f(x)$ 在$[a,b]$ 上连续, 在$(a,b)$ 上可导, 且$f'(x)<0$. 令$F(x)=\frac{1}{x-a}\displaystyle\int_a^x f(t)\,\mathrm{d}t$. 证明: $F'(x)\le 0,\ \forall x\in(a,b)$.

18. 若函数$f(x)$ 是一个以$T$ 为周期的周期函数，证明：函数$\displaystyle\int_x^{x+T}f(t)\,\mathrm{d}t$ 是一个常值函数.

进一步, 若$\displaystyle\int_0^T f(t)\,\mathrm{d}t=0$, 证明：$g(x)=\displaystyle\int_0^x f(t)\,\mathrm{d}t$ 也是一个以$T$ 为周期的周期函数.

19. 把极限

$$\lim_{n\to\infty}\frac{1}{n\sqrt{2n}}\left(\sqrt{1}+\sqrt{3}+\cdots+\sqrt{2n-1}\right)$$

表示成定积分, 并计算该定积分.

20. 设函数$f(x)$ 连续, 且$\int_0^{x^2-1} f(t)\,\mathrm{d}t = x-1, \forall x \geq 0$. 求$f(7)$.

21. 求$\int_{-1}^{x} f(t)\,\mathrm{d}t$, 其中

$$f(x)=\begin{cases} t, & t\in[0,1]; \\ 0, & t<0,\ 或\ t>1. \end{cases}$$

22. 设函数$f(x)>0$, 且在$[a,b]$ 上连续. 证明：方程

$$\int_a^x f(t)\,\mathrm{d}t = 2\int_x^b f(t)\,\mathrm{d}t$$

在$(a,b)$ 上有唯一根.

23. 设函数$f$ 在$[a,b]$ 上连续. 定义

$$F(x)=\int_a^x f(t)(x-t)\,\mathrm{d}t.$$

证明：函数$F$ 二阶可导, 且在$(a,b)$ 上满足$F''(x)=f(x)$ .

24. ★ 设$f(n) = \sum_{m=1}^{n}\int_0^m \cos\frac{2\pi n\lfloor x+1\rfloor}{m}\,\mathrm{d}x$, 其中$\lfloor x+1\rfloor$ 是小于等于$x+1$ 的最大整数. 求$f(2021)$.

# 第六章　定积分的应用

1. 求把椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 分别绕$x$-轴和绕$y$-轴旋转形成的旋转体的表面积和体积.
2. 求在曲线$y=\frac{x^2}{2p}$ 上从坐标原点到点$\left(x,\frac{x^2}{2p}\right)$, $(p>0)$ 的弧长.
3. 区域$R$ 位于上半平面, 且由曲线$x=y^2-1$ 和直线$x-y=5$ 所围成，把区域$R$ 绕着$x$-轴旋转, 求此旋转体的体积.
4. ★　区域$S=\{(x,y)|-3\le x\le 3, 0\le y\le x^3-4x+15\}$，把区域$S$ 绕着$y$-轴旋转, 求此旋转体的体积.
5. 设曲线$R$ 为$y=x^{2/3}$, $1\le x\le 8$. 把曲线$R$ 绕着$x$-轴旋转, 求此旋转面的表面积.
6. 设直线$y=kx$ 是曲线$y=\sqrt{x-1}$ 的一条切线.
   (1) 求$k$ 的值, 并求出对应的切点.
   (2) 求由此切线, 该曲线和$x$-轴所围成的区域的面积.
   (3) 把由此切线, 该曲线和$x$-轴所围成的区域绕着$x$-轴旋转, 求此旋转体的体积.
7. 设区域$R$ 是以直线$x+y=2$ 为左边, 以曲线$y=x^2$ 为右边, 以直线$y=2$ 为上边, 所围成的区域, 把此区域绕直线$y=\frac{7}{4}$ 旋转, 求此旋转体的体积.
8. ★　已知区域$R$ 由直线$y=x$ 和曲线$y=x^2$ 所围成, 把此区域绕直线$y=x$ 旋转, 求此旋转体的体积.
9. 设区域$D_1$ 是由曲线$y=2x^2$, 和直线$y=0$, $x=2$, $x=a$ 所围成; 区域$D_2$ 是由曲线$y=2x^2$, 和直线$y=0$, $x=a$ 所围成，这里$0<a<2$.
   (1) 把区域$D_1$ 绕着$x$-轴旋转形成旋转体$V_1$ , 把区域$D_2$ 绕着$y$-轴旋转形成旋转体$V_2$. 求旋转体$V_1$ 和$V_2$ 的体积.
   (2) 当$V_1+V_2$ 取最大值时, 求$a$ 的值.
10. 设区域$D$ 由两部分组成，第一部分是$x^2+y^2\le 2\,(y\ge\frac{1}{2})$，第二部分是$x^2+y^2\le 1\,(y\le\frac{1}{2})$. 求把$D$ 绕着$y$-轴旋转形成的旋转体的体积和表面积.
11. 若区域$R$ 由$y=\sqrt{x}, y=2, x=0$ 所围成, 把区域$R$ 绕着直线$y=2$ 旋转, 求此旋转体的体积.
12. 若区域$R$ 由曲线$y=2-\sqrt{x}$, 和直线$x=1$, $y=2$ 所围成, 把区域$R$ 绕着直线$x=1$ 旋转, 求此旋转体的表面积. (只列出积分表达式, 不需要计算)
13. 把曲线$y=x^3\,(0\le x\le 1)$ 绕着$y$-轴旋转形成一个容器, 将此容器注满来自死海的盐水(盐水的密度大约是11400N/$m^3$). 求将此容器的所有盐水抽到容器顶部所需做的功.
14. 把曲线$y=\cos x\left(-\frac{\pi}{2}\le x\le\frac{\pi}{2}\right)$ 绕着$x$-轴旋转可得一个旋转体, 求此旋转体的表面积.

# 第七章　超越函数

1. 设函数$f(x) = \frac{1+e^{\frac{1}{x}}}{-1+e^{\frac{1}{x}}}$, 其中$x \neq 0$, 且$f(0) = 1$. 则$f(x)$ 在$x = 0$ 处是一个
   (A) 跳跃间断点.　　(B) 可去间断点.
   (C) 连续点.　　(D) 无穷间断点.
2. 设函数$f(x) = \frac{\ln|x|}{|x-1|}\sin x$, 则函数$f(x)$ 有
   (A) 1 个可去间断点和1 个跳跃间断点.
   (B) 2 个可去间断点.
   (C) 1 个可去间断点and 1 个无穷间断点.
   (D) 2 个跳跃间断点.
3. 设$f'(0) = 3$, $f''(0) = 5$, $f'(1) = -4$, 且$f''(1) = -7$. 令$g(x) = f(\ln x)$. 则$g''(1) = (\qquad)$.
4. 求下列极限:

(1) $\lim\limits_{x\to\infty}\left(\frac{x^2}{(x-1)(x+3)}\right)^x$;　　(2) $\lim\limits_{x\to0}\frac{(1+x)^{\frac{1}{x}}-e}{x}$;

(3) $\lim\limits_{x\to\infty}(\pi-2\arctan x)\ln x$;　　(4) $\lim\limits_{x\to\infty}\left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}$;

(5) $\lim\limits_{x\to0}\frac{e^x-e^{-x}-2x}{x\sin(x^2)}$;　　(6) $\lim\limits_{x\to0}\frac{x\cot x-1}{x^2\ln(1+x)}$;

(7) $\lim\limits_{x\to0}\cot x\left(\frac{1}{\sin x}-\frac{1}{x}\right)$;　　(8) $\lim\limits_{x\to1}\frac{x-x^x}{1-x+\ln x}$;

(9) $\lim\limits_{x\to1}x^{\frac{x}{1-x}}$;　　(10) $\lim\limits_{\theta\to\frac{\pi}{4}}\frac{\cos(\pi\tan\theta)+1}{\theta^2-\pi^2/16}$;

(11) $\lim\limits_{x\to0}\frac{x\arcsin^2 x}{\sin x-x}$;　　(12) $\lim\limits_{x\to0}\left(\frac{1}{\arcsin^2 x}-\frac{1}{x^2}\right)$;

(13) $\lim\limits_{x\to1-}\ln x\ln(1-x)$;　　(14) $\lim\limits_{x\to0+}(e^x-x-1)^{\frac{1}{\ln x}}$;

(15) $\lim\limits_{x\to0+}\frac{\arctan^3\sqrt{x}}{\ln(1+\sqrt{x})\sin x}$;　　(16) $\lim\limits_{x\to\infty}\frac{x^2\arctan x+x\sin x+1}{3x^2\sec^{-1}(-x)-5}$;

(17) $\lim\limits_{x\to0+}(\cos x)^{\frac{1}{x^2}}$;　　(18) $\lim\limits_{x\to\infty}x^3e^{-x}$;

(19) $\lim\limits_{x\to+\infty}\frac{\int_1^x\left(t^2\left(e^{\frac{1}{t}}-1\right)-t\right)dt}{x^2\ln\left(1+\frac{1}{x}\right)}$;　　(20) $\lim\limits_{x\to0}\left(\frac{1}{e^x-1}-\frac{1}{\ln(1+x)}\right)$.

5. 证明: 对任意的$a > 1$, $0 < b < 1$ 和$k > 0$, 以下结论成立:
   (1) $x^k = o(a^x)$ as $x\to\infty$. 也就是, $\lim\limits_{x\to\infty}\frac{x^k}{a^x} = 0$.

(2) $\log_a x = o(x^k)$ as $x \to \infty$. 也就是, $\lim\limits_{x\to\infty} \frac{\log_a x}{x^k} = 0$.

(3) $b^x = o(x^{-k})$ as $x \to \infty$. 也就是, $\lim\limits_{x\to\infty} \frac{b^x}{x^{-k}} = 0$.

6. 当$x \to 0$, 求下列无穷小量的阶数:

(1) $\sin x - x$;

(2) $\tan x - \sin x$;

(3) $x - \ln(1+x)$;

(4) $\mathrm{e}^x - x - 1$.

7. 求下列函数的导函数:

$$(1)\, y = \ln(\sin(2x)); \qquad (2)\, y = \frac{(5x+1)(x^3-2x)}{\sqrt{x^2-1}};$$

$$(3)\, y = \mathrm{e}^{x^2+\ln x}; \qquad (4)\, y = x^{\sin x};$$

$$(5)\, y = (\sin x)^x.$$

8. 计算极限

$$\lim_{x\to 0} \frac{\int_0^{x^2} \cos(t^2)\,\mathrm{d}t}{x^2}.$$

9. 求曲线

$$y(x) = \int_{\frac{\pi}{6}}^{x} \frac{\sqrt{\cos(2t)}}{\sin t}\,\mathrm{d}t.$$

从$x = \frac{\pi}{6}$ 到$x = \frac{\pi}{4}$ 的弧长.

10. 设函数$g(x)$ 在$x = 0$ 处可导, 且$g(0) = 0$, $g'(0) = g''(0) = 1$. 判断下列函数$f(x)$ 是否在$x = 0$ 处连续和可导. 若可导, 求$f'(0)$; 若不可导, 说明理由.

$$f(x) = \begin{cases} \dfrac{g(x) - \sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

11. 求下列函数的导函数:

$$(1)\, y = \arcsin\left(\frac{1}{x}\right); \quad (2)\, y = \csc^{-1}(x^2); \quad (3)\, y = \arctan(\ln x);$$

$$(4)\, y = \arccos\left(\mathrm{e}^{-t}\right); \quad (5)\, y = x^{\arctan x}\, (x > 0); \quad (6)\, y = \sqrt[3]{x + \arcsin x}.$$

12. 设

$$y = \int_{x^2+1}^{2x^2+3} t \tan \sqrt{x+t}\,\mathrm{d}t.$$

求$\frac{dy}{dx}$.

13. 确定$a, b$ 和$c$ 的值, 使得下列函数在$x \in \mathbb{R}$ 上存在二阶导数:

$$f(x) = \begin{cases} ax^2 + bx + c, & x \le 0 \\ \sin x + \mathrm{e}^x, & x > 0 \end{cases}.$$

14. 求$a$ 和$b$ 的值, 使得

$$\lim_{x\to 0}\left(\frac{\tan(2x)}{x^3} + \frac{a}{x^2} + \frac{\sin(bx)}{x}\right) = 0.$$

15. 证明下列恒等式成立:

$$(1)\ 2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1} = \frac{\pi}{2}, \qquad (x \ge 0)$$

$$(2)\ \int_0^x \int_0^u f(t)\,\mathrm{d}t\,\mathrm{d}u = \int_0^x f(u)(x-u)\,\mathrm{d}u, \qquad f \text{ 连续}.$$

16. 若$f''(a)$ 存在, 求下列极限:

$$\lim_{x\to a}\frac{f(x) - f(a) - f'(a)(x-a)}{\sin(x-a)}.$$

17. 判断当$x \to \infty$, 下列命题哪些是正确的:

$$(1)\ \frac{1}{x+3} = O\left(\frac{1}{x}\right); \qquad (2)\ \frac{1}{x} + \frac{1}{x^2} = O\left(\frac{1}{x}\right);$$

$$(3)\ \frac{1}{x} - \frac{1}{x^2} = o\left(\frac{1}{x}\right); \qquad (4)\ x\ln x = o(x^2);$$

$$(5)\ \mathrm{e}^x + \mathrm{e}^{2x} = O(e^{2x}); \qquad (6)\ \frac{1}{x} - \sin\frac{1}{x} = O\left(\frac{1}{x^3}\right);$$

$$(7)\ \ln(\ln x) = O(\ln x); \qquad (8)\ \frac{1}{x} - \sin\frac{1}{x} = O\left(\frac{1}{x^2}\right).$$

18. ★ 若$f(x)$ 连续, 且满足$\int_0^x tf(2x-t)\,\mathrm{d}t = \frac{1}{2}\arctan x^2$ , $f(1) = 1$. 求$\int_1^2 f(x)\,\mathrm{d}x$.

19. 设函数

$$f(x) = \begin{cases} \dfrac{\ln(1+ax^3)}{x - \arcsin x}, & x < 0; \\ 6, & x = 0; \\ \dfrac{\mathrm{e}^{ax} + x^2 - ax - 1}{x\sin\frac{x}{4}}, & x > 0. \end{cases}$$

(1) 若$f(x)$ 在$x = 0$ 处连续, 求$a$ 的值.

(2) 若$x = 0$ 为$f(x)$ 的可去间断点, 求$a$ 的值.

20. 证明函数$f(x) = \frac{x^3}{1+x^2}$ 存在反函数, 并求此反函数在$x = \frac{1}{2}$ 处的导数.

21. 函数$y = f(x)$ 在$(a, b)$ 上存在二阶导函数, 且对任何$x \in (a, b)$, 都满足$f'(x) \neq 0$. 设函数$y = g(x)$ 是函数$y = f(x)$ 在$(a, b)$ 上的反函数, 求$g''(x)$.

# 第八章 积分的技巧

1. 设$I_k = \int_0^{k\pi} e^x \sin x\,dx \quad (k = 1, 2, 3)$, 则下列哪个命题正确?

   (A) $I_1 < I_2 < I_3$. (B) $I_3 < I_2 < I_1$.

   (C) $I_2 < I_3 < I_1$. (D) $I_2 < I_1 < I_3$.

2. 下列哪一个反常积分是收敛的?

   (A) $\int_0^{+\infty} \frac{1}{\sqrt{1+x}}\,dx$. (B) $\int_1^{+\infty} \frac{\ln x}{x+x^2}\,dx$.

   (C) $\int_0^{1} \frac{1}{\sqrt{x}\sin x}\,dx$. (D) $\int_1^{2} \frac{1}{x(\ln x)^2}\,dx$.

3. 若反常积分$\int_0^{+\infty} \frac{\tan^{-1}(x^2)}{x^k}\,dx$ 收敛, 则常数$k$ 必须满足

   (A) $k < 1$. (B) $k > 3$.

   (C) $1 < k < 2$. (D) $1 < k < 3$.

4. 若$\int f(x)e^{\frac{1}{x}}\,dx = e^{\frac{1}{x}} + C$, 则$f(x) = (\qquad)$.

5. 求积分:

(1) $\int \frac{dx}{x^2\sqrt{x^2+1}}$; (2) $\int_1^{+\infty} \frac{(x+2)\ln(x^2+1)}{x^3}\,dx$;

(3) $\int \frac{\sin^5 x}{\cos^4 x}\,dx$; (4) $\int_{-1}^{1} \frac{4^x}{4^x+4^{-x}}\,dx$;

(5) $\int_{\frac{1}{e}}^{e} \frac{\ln^2 x}{x}\,dx$; (6) $\int_{-1}^{1} \frac{2x^2+x\cos x}{1+\sqrt{1-x^2}}\,dx$;

(7) $\int_0^2 x\sqrt{2x-x^2}\,dx$; (8) $\int_0^2 \max(x, x^2)\,dx$;

(9) $\int_2^{2+100\pi} |\sin x|\,dx$; (10) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx$;

(11) $\int x\tan^2 x\,dt$; (12) $\int_0^4 \frac{dx}{1+\sqrt{x}}$;

(13) $\int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\,dx$; (14) $\int_0^{\sqrt{2}} \frac{x^3}{1+x^2}\,dx$;

(15) $\int_0^{\pi} \sqrt{\sin^3 x-\sin^5 x}\,dx$; (16) $\int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{dx}{x\sqrt{\ln x(1-\ln x)}}$.

6. 求极限:

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right).$$

7. ★ 若函数$f(x)$ 在$[0,a]$ 上连续，证明: $\displaystyle\int_0^a f(x)\,\mathrm{d}x=\int_0^a f(a-x)\,\mathrm{d}x$.
特别的, 若$a=1$, 证明

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x=\int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x;\quad \int_0^{\pi} xf(\sin x)\,\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x.$$

用上面的结论求积分

$$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x.$$

8. 设函数$f(x)$ 连续, 且$f(x)=x\sin x+\displaystyle\int_0^{\frac{\pi}{4}} f(2x)\,\mathrm{d}x$. 求积分$\displaystyle\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x$.
9. 求极限

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{4n^2-k^2}}{n^2}.$$

10. 设函数$f$ 在$[0,1]$上连续且严格单调递减, 且满足$f(0)=2$, $f(1)=0$. 若$\displaystyle\int_0^1 f(x)\,\mathrm{d}x=1$, 求$\displaystyle\int_0^2 f^{-1}(y)\,\mathrm{d}y$.
11. 设函数$f(x)$ 连续, 且满足$f(x)=\sqrt{2x-x^2}+x\displaystyle\int_0^1 f(t)\,\mathrm{d}t$. 求$f(x)$.
12. 下列函数$f(x)$ 在何处取得最大, 最小值:

$$f(x)=\int_0^x \mathrm{e}^{-t}\cos t\,\mathrm{d}t.$$

13. 求由下列曲线所围成的区域, 绕x-轴旋转, 形成的旋转体的体积:
    (1) $y=\tan x\quad (0\le x\le \frac{\pi}{4})$;
    (2) $y=\sec x\quad (0\le x\le \frac{\pi}{4})$;
14. 求积分:
    (1) $\displaystyle\int\frac{\mathrm{d}x}{x\sqrt{x^4-1}}$;
    (2) $\displaystyle\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{1}{\sqrt{|x-x^2|}}\,\mathrm{d}x$;
    (3) $\displaystyle\int\frac{\mathrm{d}x}{\mathrm{e}^{-2x}\sqrt{1-\mathrm{e}^{4x}}}$;
    (4) $\displaystyle\int\frac{\sqrt{\arctan\sqrt{x}}}{(1+x)\sqrt{x}}\,\mathrm{d}x$;
    (5) $\displaystyle\int\frac{\mathrm{d}x}{(x-1)\sqrt{x^2-2x-48}}$;
    (6) $\displaystyle\int_0^1\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x$;
    (7) $\displaystyle\int_0^1\frac{1}{x^4+1}\,\mathrm{d}x$;
    (8) $\displaystyle\int_0^1\frac{x^3+x+1}{(x^2+2)^2}\,\mathrm{d}x$;
    (9) $\displaystyle\int_{-1}^{-2}\frac{x^3+x+1}{\sqrt{x^2-4x}}\,\mathrm{d}x$.

15. 求 $\int_0^3 f(x-1)\,\mathrm{d}x$, 这里

$$f(x)=\begin{cases}1+x^2, & x\le 0\\ \mathrm{e}^{-x}, & x>0\end{cases}.$$

16. 求下列积分:

(1) $\int \sqrt{a^2-x^2}\,\mathrm{d}x$, 其中$a>0$.

(2) $\int \sqrt{ax^2+bx+c}\,\mathrm{d}x$, 其中$a>0$ 且$b^2-4ac<0$.

(3) $\int \frac{Mx+N}{x^2+px+q}\,\mathrm{d}x$, 其中$p^2-4q<0$.

(4) $\int_0^{\frac{\pi}{2}} \frac{1}{3+\cos 2x+2\sin 2x}\,\mathrm{d}x$.

(5) $\int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}\,\mathrm{d}x$, 其中$x>1$.

(6) ★ $\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}\,\mathrm{d}x$, 其中$x>2$.

(7) ★ $\int_2^{\infty} \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}\,\mathrm{d}x$.

# 第九章　一阶常微分方程

1. 求微分方程
$$y' = \tan x \tan y$$
的通解.
2. 求微分方程
$$y' = \frac{y}{x}\log_y x$$
的通解.
3. 证明微分方程$y' = -\frac{x}{y}, y(1) = 1$ 的解是一个圆.
4. 求解微分方程
$$\frac{\mathrm{d}^2 y}{dx^2} = \frac{3}{\sqrt{x}} + 15\sqrt{x}, \quad y'(1) = 8, y(1) = 0.$$
5. ★ 设函数$g$ 在包含原点的某个开区间可导，且满足以下性质:
   (1) $g(x+y) = \frac{g(x)+g(y)}{1-g(x)g(y)}$ 对函数$g$ 定义域内的所有实数$x, y$, 和$x+y$ 成立.
   (2) $\lim\limits_{h\to 0} g(h) = 0$.
   (3) $\lim\limits_{h\to 0} \frac{g(h)}{h} = 1$.
   证明:
   (1) 证明: $g(0) = 0$.
   (2) 证明: $g'(x) = 1 + [g(x)]^2$.
   (3) 解上面的微分方程(b), 求出$g(x)$ 的表达式.
6. 求解微分方程
$$\frac{dy}{dx} = \frac{y}{x+y^2}, \qquad y(2) = 1.$$

# 第十章　数列和级数

1. 若$0 < a_n < \frac{1}{n}$, 则下列哪个级数一定收敛?

   (A) $\sum\limits_{n=1}^{\infty} a_n$.　　(B) $\sum\limits_{n=1}^{\infty} (-1)^n a_n$.

   (C) $\sum\limits_{n=1}^{\infty} \sqrt{a_n}$.　　(D) $\sum\limits_{n=1}^{\infty} \dfrac{a_n}{n}$.

2. 若级数$\sum u_n$ 和$\sum v_n$ 都发散. 则

   (A) $\sum\limits_{n=1}^{\infty}(u_n + v_n)$ 发散.　　(B) $\sum\limits_{n=1}^{\infty} u_n v_n$ 发散.

   (C) $\sum\limits_{n=1}^{\infty}(|u_n| + |v_n|)$ 发散.　　(D) $\sum\limits_{n=1}^{\infty}(u_n^2 + v_n^2)$ 发散.

3. 若级数$\sum u_n$ 收敛, 下列哪个级数一定收敛?

   (A) $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{u_n}{n}$.　　(B) $\sum\limits_{n=1}^{\infty}(u_n)^2$.

   (C) $\sum\limits_{n=1}^{\infty}(u_{2n-1} - u_{2n})$.　　(D) $\sum\limits_{n=1}^{\infty}(u_n + u_{n+1})$.

4. 设$a$ 是一个实数，则级数$\sum\limits_{n=1}^{\infty}\left(\dfrac{\sin a}{n^2} - \dfrac{1}{\sqrt{n}}\right)$

   (A) 绝对收敛.　　(B) 条件收敛.

   (C) 发散.　　(D) 收敛性取决于$a$ 的值.

5. 若$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 2$. 则幂级数$\sum_{n=1}^{\infty} a_n x^{2n+1}$ 的收敛半径是

   (A) 2.　　(B) $\frac{1}{\sqrt{2}}$.　　(C) $\sqrt{2}$.　　(D) $\infty$.

6. 设级数$\sum\limits_{n=0}^{\infty} a_n(x-1)^n$ 在$x = -2$ 处收敛. 则在$x = 0$ 处, 级数

   (A) 发散.　　(B) 条件收敛.

   (C) 绝对收敛.　　(D) 收敛性不确定.

7. 幂级数$\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n3^n}$ 的收敛域是

   (A) $[-1/3, 1/3]$.　　(B) $[-1/3, 1/3)$.　　(C) $[-3, 3]$.　　(D) $[-3, 3)$.

8. 设$a_n = (-1)^n \ln\left(1 + \frac{1}{n}\right)$, 则级数$I = \sum a_n$ 和$J = \sum a_n^2$

   (A) 同时收敛.　　(B) 同时发散.

   (C) $I$ 收敛但$J$ 发散.　　(D) $I$ 发散但$J$ 收敛.

9. 若$\sum\limits_{n=1}^{\infty} (-1)^{n-1} u_n = 2$ 和$\sum\limits_{n=1}^{\infty} u_{2n-1} = 5$, 则$\sum\limits_{n=1}^{\infty} u_n = (\qquad)$.

10. 若$\sum_{n=2}^{\infty}\frac{1}{n\ln^p n}$ 收敛, 则$p$ 的取值范围是(　　).

11. 若$\sum_{n=1}^{\infty}a_nx^n$ 在$x=2$ 处条件收敛, 则这个级数的收敛半径是(　　).

12. 若级数 $\sum_{n=1}^{\infty}a_nx^n$ 的收敛半径是2, 则级数$\sum_{n=1}^{\infty}na_n(x+1)^{n+1}$ 的收敛区间是(　　).

13. 级数 $\sum_{n=1}^{\infty}\frac{nx^{2n}}{2^n+(-3)^n}$ 的收敛半径是(　　).

14. $\sum_{n=1}^{\infty}\frac{2n+1}{n!}=($　　$)$.

15. $f(x)=x\sin x$, 则$f^{(50)}(0)=($　　$)$.

16. ★ 判断下列数列是否收敛. 若收敛, 则计算此极限.

$$x_0=1,\quad x_{n+1}=x_n-\frac{\tan(x_n)-1}{\sec^2(x_n)}.$$

17. 证明下列数列收敛, 且求出极限的值.

$$a_1=3, a_{n+1}=12-\sqrt{a_n}.$$

18. 计算级数

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}.$$

19. ★ 计算级数

$$\sum_{k=1}^{\infty}\frac{1}{k(k+2)(k+6)}.$$

20. 求$a, b, c$ 的值, 使得

$$\lim_{n\to\infty}n(an+\sqrt{2+bn+cn^2})=2.$$

21. ★ 判断数列

$$a_n=\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{\cdots+\sqrt{n}}}}}$$

的收敛性.

22. 求级数

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(2n-1)}x^{2n+1}$$

的收敛域和和函数.

23. ★ 求

$$\lim_{n\to\infty}\cos\frac{a}{n\sqrt{n}}\cos\frac{2a}{n\sqrt{n}}\cdots\cos\frac{na}{n\sqrt{n}}.$$

24. 求函数

$$f(x)=\ln x.$$

在$x=2$ 处的泰勒展开式.

25. 判断级数

$$\sum_{n=1}^{\infty}\left(n^{\frac{1}{n^2+1}}-1\right)$$

的收敛性.

26. 判断级数

$$\sum_{n=1}^{\infty}\left(1-\cos\frac{1}{n}\right)^{p}\quad(p>0)$$

的收敛性.

27. 求幂级数

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)x^n$$

的收敛半径和收敛域.

28. 求函数

$$f(x)=\frac{x}{2+x-x^2}$$

在$x=1$ 处的泰勒展开式.

29. 求幂级数

$$\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$$

的收敛域和和函数.

30. 求幂级数

$$\sum_{n=1}^{\infty}\frac{2n-1}{2^n}x^{2n-2}$$

的收敛域和和函数.

31. 设级数$\sum a_n$ 的前$n$ 项之和是

$$S_n=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}.$$

求$a_n$ 和级数$\sum a_n$ 的值.

32. 求极限
$$\lim_{n\to\infty}\left(\frac{1}{n^2+n+1}+\frac{2}{n^2+n+2}+\cdots+\frac{n}{n^2+n+n}\right).$$

33. 求极限
$$\lim_{n\to\infty}\sqrt[n]{1+x^n+\left(\frac{x^2}{2}\right)^n},\quad x>0.$$

34. 设$x_1=\sqrt{6}$, $x_{n+1}=\sqrt{6+x_n}$ 对任意的$n=1,2,\cdots$ 成立. 证明此数列的极限存在, 且求出极限值.

35. 判断下列级数的收敛性, 并说明理由.

$$(1)\ \sum_{n=2}^{\infty}\frac{(\ln n)^2}{n};\qquad (2)\ \sum_{n=1}^{\infty}\left(1-\frac{3}{n}\right)^n;$$
$$(3)\ \sum_{n=2}^{\infty}\frac{(\ln n)^2}{n^{\frac{3}{2}}};\qquad (4)\ \sum_{n=1}^{\infty}2^n\sin\frac{\pi}{3^n};$$
$$(5)\ \sum_{n=1}^{\infty}\frac{(n!)^n}{n^{2n}};\qquad (6)\ \sum_{n=1}^{\infty}\frac{1}{n(\ln(1+1/n)+1)};$$
$$(7)\ \sum_{n=1}^{\infty}\left(1-\cos\frac{\lambda\pi}{n}\right),\lambda>0.$$

36. 若级数$\sum a_n$ 绝对收敛, 证明级数$\sum\frac{a_n}{a_n+1}$ 也绝对收敛.

37. 设正项数列$\{a_n\}$ 单调递减, 且级数$\sum\limits_{n=1}^{\infty}(-1)^n a_n$ 发散. 判断级数
$$\sum_{n=1}^{\infty}\left(\frac{1}{a_n+1}\right)^n.$$
的收敛性, 并说明理由.

38. 判断级数
$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n+(-1)^n}}$$
的收敛性, 并说明理由.

39. 求幂级数$\sum\limits_{n=1}^{\infty}\frac{n}{n!}x^{n+1}$ 的收敛域和和函数，并利用和函数来计算级数
$$\sum_{n=1}^{\infty}\frac{n-1}{n!}2^n.$$

40. 设数列$a_n > 0$, 且单调递增有上界. 证明级数$\sum_{n=1}^{\infty} b_n$ 收敛, 其中$b_n = \ln\left(2 - \frac{a_n}{a_{n+1}}\right)$.

41. 将函数$f(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t$ 写成关于$x$ 的幂级数.

42. 写出函数$f(x) = \frac{1}{x^2-5x+6}$ 在$x = 4$ 处的泰勒展开式.

43. 求下列级数的和函数.

$$(1)\ \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1};\qquad (2)\ \sum_{n=1}^{\infty}\frac{n(n+1)x^n}{(n-1)!};$$

$$(3)\ \sum_{n=1}^{\infty}(-1)^n\frac{1}{2n-1}\left(\frac{x-2}{3}\right)^{2n+1};\qquad (4)\ \sum_{n=1}^{\infty}\frac{2n-1}{2^n}x^{2n-2};$$

$$(5)\ \sum_{n=1}^{\infty}n(n+1)x^n;\qquad (6)\ \sum_{n=1}^{\infty}(-1)^{n-1}n^2x^n;$$

$$(7)\ \sum_{n=0}^{\infty}\frac{n^2+1}{2^n n!}x^n;\qquad (8)\ \sum_{n=1}^{\infty}(-1)^n\frac{n+1}{(2n+1)!}x^{2n+1}.$$

44. 求下列级数的值.

$$(1)\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n(n+1)}{2^n};\qquad (2)\ \sum_{n=1}^{\infty}\frac{e^{-n}}{n+1};\qquad (3)\ \sum_{n=1}^{\infty}\frac{1}{n2^n}.$$

45. 设函数$f(x) = \arcsin x$, 求$f^{(n)}(0)$.

46. 使用泰勒级数求极限:

$$(1)\ \lim_{x\to 0}\frac{\sin x\ln(1+x)-x^2}{x^3};\qquad (2)\ \lim_{x\to 0}\frac{x\mathrm{e}^{x^2/2}-\sin x}{x^2\sin x};$$

$$(3)\ \lim_{n\to+\infty} n^2\left(\arctan\frac{1}{n}-\arctan\frac{1}{n+1}\right).$$

47. 求极限:

(1) $\lim\limits_{n\to\infty}\frac{1}{n}\sin n\cos(n^2)$;

(2) $\lim\limits_{n\to\infty}(a^n+b^n)^{\frac{1}{n}},\quad a>0, b>0$;

(3) $\lim\limits_{n\to\infty}\frac{n^c}{a^n},\quad a>1, c>0$;

(4) $\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)^{\frac{1}{n}}$.

48. 已知$a > 0$, $x_0 = 1$, 且$x_{n+1} = x_n - \frac{x_n^2-a}{2x_n}$. 判断这个数列是否收敛. 若收敛, 求出极限值.

49. 已知$a_1 = 1$ 且$a_2 = 1$. $a_{n+2} = a_{n+1} + a_n$. 证明数列$b_n = \frac{a_{n+1}}{a_n}$ 收敛, 并求出极限值.

50. 求下列级数的收敛半径和收敛域.

$$(1)\ \sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}x^n; \qquad (2)\ \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!};$$

$$(3)\ \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right)(x-2)^n; \qquad (4)\ \sum_{n=1}^{\infty}\frac{3^n}{2^n n!}(x-1)^n;$$

$$(5)\ \sum_{n=1}^{\infty}\frac{\ln^2 n}{n^n}x^{n^2}; \qquad (6)\ \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{n2^n};$$

$$(7)\ \sum_{n=1}^{\infty}\left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)x^n\ (a>0, b>0).$$

51. 求满足条件的所有$x$, 使级数$\sum\limits_{n=1}^{\infty} n^2(\sin x)^n$ 收敛, 并求和函数.
52. 设幂级数$\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ 的收敛半径$R>0$, 且和函数是$f(x)$. 证明函数$f$ 在$x=a$ 处的泰勒展开式就是此幂级数.
53. 求函数$f(x)=\ln\frac{1+x}{1-x}$ 在$x=0$ 处的泰勒展开式, 且求$\ln 2$ 的一个估计值, 使误差不大于$10^{-4}$.
54. 设函数$f$ 在$[-1,1]$ 上三阶可导, 且三阶导函数$f^{(3)}$ 在$[-1,1]$ 上连续. $f(-1)=f(0)=0$, $f(1)=1$ 且$f'(0)=0$. 证明: 存在$(-1,1)$ 上的一点$c$, 使得$f^{(3)}(c)=3$.
55. 判断下列数列是否收敛. 若收敛, 求出极限值.

$$(1)\ a_n=\left(\frac{n}{n+c}\right)^n, \quad c\in\mathbb{R}; \qquad (2)\ a_n=\frac{\ln^3 n}{n}.$$

56. ★ 判断级数

$$\sum_{n=1}^{\infty}\frac{(2n)!}{n!\,n!\,4^n}$$

是否收敛.
57. 若数列$\{na_n\}$ 和级数$\sum n(a_n-a_{n-1})$ 都收敛，证明数列$\sum a_n$ 收敛.
58. 求函数$f(x)=\tan x$ 的麦克劳林展开式的前三项非零项, 并求极限$\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$.
59. 设函数$f(x)=\frac{1}{\sqrt{4-x^2}}$, 求$f^{(8)}(0)$.
60. 判断下列级数是绝对收敛, 条件收敛, 还是发散.

(1) $\sum_{n=1}^{\infty} \frac{e^{(-1)^n n}}{n}$; (2) $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$;

(3) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$; (4) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\ln n}{n^p}$; $(p>0)$

(5) $\sum_{n=1}^{\infty} (-1)^{n+1}\int_0^{1/n} \frac{\sqrt{x}}{1+x^2}\,dx$; (6) $\sum_{n=1}^{\infty} (-1)^n 2^n \sin\frac{\pi}{3^n}$;

(7) $\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{\sqrt{n}}\right)$; (8) $\sum_{n=1}^{\infty} (-1)^n \frac{n^{n+1/n}}{(n+1/n)^n}$;

(9) $\sum_{n=1}^{\infty} \frac{a^n}{\prod_{k=1}^{n}(1+a^k)}$ $(a>0)$.

61. 求满足下列极限的$a$ and $b$ 的值:

(1) 若$\lim\limits_{x\to 0}\frac{\sin ax-\sin x-x}{x^3}=b$;

(2) 若$\lim\limits_{x\to 0}\frac{\cos ax-b}{2x^2}=-1$.

62. 求极限:

(1) $\lim\limits_{n\to\infty}\left(\sqrt{n+\sqrt{n+2\sqrt{n}}}-\sqrt{n}\right)$; (2) $\lim\limits_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}\frac{1}{k}}$;

(3) $\lim\limits_{n\to\infty}(n!)^{1/n^2}$.

63. 求极限:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{(n+i-1)(n+i)}}.$$

64. 设数列$a_n$ 和$b_n$ 以0 和$\pi/2$ 为上下界, 且满足$\cos a_n - a_n = \cos b_n$. 若$\sum b_n$ 收敛:

(1) 证明: $\lim\limits_{n\to\infty} a_n = 0$;

(2) 证明: $\sum_{n=1}^{\infty}\frac{a_n}{b_n}$ 收敛.

65. 设$x_1>0$, 且$x_{n+1}=1+\frac{x_n}{1+x_n}$ 对任意的$n\geq 1$ 成立. 证明该数列收敛且求出极限值.

66. 求极限:

(1) $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$; (2) $\lim\limits_{n\to\infty}\left(b_1^n+b_2^n+\cdots+b_m^n\right)^{1/n}$ $(b_i\geq 0)$;

(3) $\lim\limits_{n\to\infty}\left(n\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)\right)$; (4) $\lim\limits_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+\sqrt{k}}$;

(5) $\lim\limits_{n\to\infty}\sum_{k=n^2}^{n^2+2n+1}\frac{1}{\sqrt{k}}$; (6) $\lim\limits_{n\to\infty}\frac{(2n-1)!!}{(2n)!!}$;

(7) $\lim\limits_{n\to\infty}\left(\sqrt{n}\left(\sqrt[4]{n^2+1}-\sqrt{n+1}\right)\right)$; (8) $\lim\limits_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{1}{k^2}\right)$.

67. 求下列级数的值:

(1)★ $\sum_{n=1}^{\infty} \arctan \frac{1}{2n^2}$; (2) $\sum_{n=0}^{\infty} \frac{n^2+2n+1}{2^n}$;

(3) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{2^n(n^2-1)}$; (4) $\sum_{n=1}^{\infty} \frac{n!+1}{2^n(n-1)!}$;

(5) $\sum_{n=1}^{\infty} (2n-1)q^{n-1}$ $(|q|<1)$; (6) $\sum_{n=1}^{\infty} \frac{n(n+2)}{4^{n+1}}$.

68. 设函数$f(x)$ 可导, 且$f(0)=1$, $0<f'(x)<1/2$. 定义$x_{n+1}=f(x_n)$ 对所有的$n\in\mathbb{N}$ 成立. 证明:

(1) $\sum_{n=1}^{\infty}(x_{n+1}-x_n)$ 绝对收敛.

(2) $\lim_{n\to\infty} x_n$ 存在, 且极限值的范围是$(0,2)$ .

69. 设$a_0=1$, $a_1=0$, 且$a_{n+1}=\frac{na_n+a_{n-1}}{n+1}$ 对所有的$n\geq 2$ 成立. 令$S(x)=\sum_{n=0}^{\infty} a_n x^n$.

(1) 证明级数$S(x)$ 的收敛半径大于或等于 1.

(2) 证明$(1-x)S'(x)-xS(x)=0$, 并求该级数的和函数.

# 第十一章 参数方程和极坐标

1. ★ 证明摆线的等时性:若一个粒子从摆线的任意点出发，在重力作用下沿摆线向下滑，则此粒子到达最低点所需的时间与出发点的位置无关。
2. 求曲线
$$\begin{cases} x = t^3 + 3t + 1 \\ y = t^3 - 3t + 1 \end{cases}$$
的所有下凹的开区间.
3. 定义曲线为
$$L : \begin{cases} x = f(t) \\ y = \cos(t) \end{cases}, \quad 0 \le t \le \frac{\pi}{2},$$
这里函数$f(t)$ 的导函数连续且$f(0) = 0$, $f'(t) > 0$. 若在$L$上任取一点$A$，设过点$A$的切线与$x$-轴的交点为$B$. 若线段$AB$ 的长度恒为1，试求$f(t)$，然后求坐标轴与曲线$L$所围的面积.
4. 求曲线
$$x = (1 + 2\sin t)\cos t, \quad y = (1 + 2\sin t)\sin t, \quad 0 < t < 2\pi,$$
的弧长.
5. 求曲线:
$$x = 2\cos t, \quad y = 3 + \sin t,\ 0 < t < 2\pi,$$
绕$y$ 轴旋转一周所形成的旋转体的表面积.
6. 求曲线 $r\theta = 1$ $\left(\frac{3}{4} \le \theta \le \frac{4}{3}\right)$ 的弧长.
7. ★ 求椭圆
$$Ax^2 + 2Bxy + Cy^2 = 1, \quad AC - B^2 > 0, C > 0,$$
的面积.
8. 求函数
$$x = a\cos^3 t, \quad y = a\sin^2 t.$$
的二阶导数$\frac{d^2y}{dx^2}$.
9. 求由下列曲线绕坐标轴旋转形成的旋转体的表面积:
   (1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$, 绕$y$-轴;
   (2) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \le t \le 2\pi$, $a > 0$, 绕$x$-轴.
10. ★ 求摆线 $(0 \le t \le 2\pi)$ 的弧长，并求夹在这一段摆线和$x$-轴之间的面积.

11. 求夹在双曲线 $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1(a,b>0)$ 和直线 $x=-2a$ 之间区域的面积.
12. 求曲线 $r=\sin(4\theta)$ 中的一片叶子的面积.
13. 已知曲线
$$x=\sin t,\quad y=\cos(2t),\quad 0\le t\le\frac{\pi}{2}.$$
求此曲线在直角坐标系下的方程，并求夹在曲线与$x$轴之间区域的面积，以及曲线绕$x$轴旋转一周所形成的旋转体的体积.
14. 设$f(x)$可微，且 $f\neq 0$. 若 $x=\int_0^{t^2}f(u)\,\mathrm{d}u, y=\int_0^t f(u)f(u^2)\,\mathrm{d}u$. 求$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.
15. 星形线$L$ 的方程如下
$$x=\cos^3 t,\quad y=\sin^3 t.$$
   (1) 已知点 $A\left(\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4}\right)$, 求曲线在点 $A$ 处的斜率以及点 $A$ 处的切线方程.
   (2) 若在点 $A$ 处 $t=t_0, 0<t_0<\frac{\pi}{2}$, 再求曲线在点 $A$ 处的斜率以及点 $A$ 处的切线方程.
   (3) 求曲线在点 $A$ 处第一象限内的切线段的长度.
   (4) 请问在 (3) 中所计算的切线段的长度与 $t_0$有关吗?
16. 求函数
$$x=\ln(1+t^2),\quad y=t-\arctan t$$
的三阶导数 $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$.
17. 已知曲线:
$$x=\cos(t^2),\quad y=t\cos(t^2)-\int_1^{t^2}\frac{\cos u}{2\sqrt{u}}\,\mathrm{d}u,\quad 0<t<\sqrt{2\pi}.$$
   (1) 求 $\frac{\mathrm{d}y}{\mathrm{d}x}$ 和$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.
   (2) 求 $t=1$ 处的切线和法线方程.
18. 求下列曲线的拐点(凹凸性改变的点):
$$(1)\ x=t^2,\quad y=3t+t^3,\quad t>0;$$
$$(2)\ x=2a\cot\theta,\quad y=2a\sin^2\theta,\quad 0<\theta<\pi.$$
19. 求曲线
$$x=\int_1^t\frac{\cos u}{u}\,\mathrm{d}u,\quad y=\int_1^t\frac{\sin u}{u}\,\mathrm{d}u$$
上从原点 $O$ 到离原点最近且为垂直切线的点 $A$ 之间的弧长.

# 第十二章　向量和几何空间

1. 证明三角形的三条高相交于同一点.
2. 判断正误: $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$，并说明理由.
3. 设 $(\mathbf{a},\mathbf{b},\mathbf{c})\neq 0$, 这里 $(\mathbf{a},\mathbf{b},\mathbf{c})$ 表示混合积. 求解

$$\begin{cases}(\mathbf{x},\mathbf{a},\mathbf{b})=n,\\(\mathbf{x},\mathbf{b},\mathbf{c})=l,\\(\mathbf{x},\mathbf{c},\mathbf{a})=m.\end{cases}$$

4. 对任意向量 $\mathbf{a}$,$\mathbf{b}$, $\mathbf{c}$ 和 $\mathbf{p}$, 证明: $\mathbf{a}\times\mathbf{p}$, $\mathbf{b}\times\mathbf{p}$, $\mathbf{c}\times\mathbf{p}$ 是共面的.
5. 求点 $(1,-2,3)$ 关于平面 $x+4y+z-14=0$的对称点.
6. 求球面 $x^2+y^2+z^2=1$ 和球面 $x^2+(y-1)^2+(z-1)^2=1$ 的交线在 $XY$ 平面上的投影.
7. 证明对任意向量 $\mathbf{a}$ , $\mathbf{b}$, 必有

$$|\mathbf{a}+\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2=2(|\mathbf{a}|^2+|\mathbf{b}|^2).$$

   请写出这个方程的几何意义. 若 $|\mathbf{a}|=1$, $|\mathbf{b}|=32$, $|\mathbf{a}-\mathbf{b}|=30$, 用这个方程求 $|\mathbf{a}+\mathbf{b}|$.
8. 证明: 点 $(x_0,y_0,z_0)$ 到平面 $Ax+By+Cz+D=0$ 的距离公式为

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}.$$

9. 设平面 $M$ 经过原点和点 $(6,-3,2)$ 且与平面 $4x-y+2z-8=0$ 垂直. 求平面$M$的方程.
10. 直线 $l_1: x-1=\frac{y-5}{-2}=z+8$, 直线 $l_2$ 是平面 $x-y=6$ 与平面 $2y+z=3$的交线, 求 $l_1$ 和 $l_2$的夹角.
11. 设 $l: z=ky(k>0)$ 是 $YZ$ 平面上的一条直线, $l$ 绕 $z$ 轴旋转一周形成曲面 $S$. 求曲面 $S$ 的方程.
12. 求直线 $x=t, y=2t, z=3t$ 与平面 $ax+by+cz=0$的夹角.
13. 判断四个点 $(1,3,2)$, $(3,-1,6)$, $(5,2,0)$ 及 $(3,6,-4)$ 是否共面.
    判断四个点 $(0,0,0)$, $(2,5,0)$, $(5,2,0)$ 及 $(1,2,4)$ 是否共面.
14. 求到点 $(3,1,2)$, $(4,-2,-2)$ 和 $(0,5,1)$ 距离相等的点的集合, 并从几何角度描述这个集合.

15. 已知向量 $\mathbf{r} = (a,b,c)$ 与 $x$ 轴, $y$ 轴以及 $z$ 轴之间的夹角分别为 $\alpha, \beta$ 以及 $\gamma$. 求 $\cos\alpha, \cos\beta, \cos\gamma$ 以及 $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$.
16. 设向量 $\mathbf{a}$, $\mathbf{b}$ ,$\mathbf{c}$ 相互垂直且满足右手定律, 若它们的长度分别为 1, 2 , 3, 求向量 $\mathbf{a}+\mathbf{b}+\mathbf{c}$ 和向量 $\mathbf{c}$之间的夹角.
17. 若 $\overrightarrow{AB} = (-3,0,4), \overrightarrow{AC} = (5,-2,-15)$，求$\angle BAC$ 的角平分线所在方向的单位向量.
18. 设直线 $l: x = 1+t, y = -1-t, z = t$ 和平面 $S: 3x - y + z = 5$:
    (a) 求经过直线 $l$ 并与平面 $S$ 垂直的平面方程;
    (b) 求直线 $l$ 在平面 $S$ 上的投影.
19. 求下面直线在 $XY$ 平面的投影.

$$\begin{cases} x + 5y + 6z = 3 \\ x - 2y + 3z = 1 \end{cases}.$$

20. 设两条直线

$$l_1 = \begin{cases} y = 2x \\ z = x + 1 \end{cases}, \qquad l_2 = \begin{cases} y = x + 3 \\ z = x \end{cases}.$$

证明 $l_1$ 与 $l_2$ 不相交，并求出它们的公垂线方程.

21. 若直线 $l$ 经过点 $(1,2,1)$ 并与直线 $\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{1}$ 垂直. 且 $l$ 还与直线 $\frac{x}{2} = y = -z$相交, 求 $l$ 的方程.
22. 求使两条直线

$$l_1 : \frac{2x-2}{3} = -(y+1) = \frac{z-2}{2}; \qquad l_2 : x = \frac{4y+1}{-4} = \frac{z+2}{\lambda},$$

相交的 $\lambda$ 的值并求出交点坐标.

23. ★ 求直线 $x - 1 = y = 1 - z$ 在平面 $x - y + 2z = 1$上的投影.
24. 设椭圆 $\frac{x^2}{4} + \frac{y^2}{3} = 1$ 绕 $x$ 轴旋转一周形成的椭球面 $S_1$, 该椭圆在点 $(4,0)$ 的切线绕 $x$ 轴旋转一周形成锥面 $S_2$. 求 $S_1$ 和 $S_2$的方程, 并计算 $S_1$ 和 $S_2$所围成的体积.
25. 求经过原点且平行于下面两条直线的平面方程:

$$l_1 = \begin{cases} x = 1 \\ y = -1 + t \\ z = 2 + t \end{cases}, \qquad l_2 = \begin{cases} x = -1 + t \\ y = -2 + 2t \\ z = 1 + t \end{cases}.$$

26. ★ 设直线 $L$ 在平面 $x - y + z = 1$ 上, 且有 $L$ 到平面 $x + y + z = 1$ 的投影为

$$\begin{cases} x + y + z = 1 \\ 2x - 5y + 3z = 4 \end{cases}.$$

求$L$的方程.

# 第十三章

# 向量值函数和空间运动

1. 求 $y = x^3$ 在 $x = 1$ 处的曲率圆方程.
2. ★ 证明: 在曲面

$$\begin{cases} x = a\sec\phi\cos\theta \\ y = b\sec\phi\sin\theta \qquad a, b, c > 0, 0 \leq \theta < 2\pi, -\pi/2 < \phi < \pi/2, \\ z = c\tan\phi \end{cases}$$

上至少存在两条直线.

3. 函数 $y = f(x)$ 在 $[a, b]$ 上二阶连续可微. 证明: 曲线的曲率计算公式为

$$\kappa(x) = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}^3}.$$

并用上述公式计算函数 $y = \ln x$ 的曲率.

4. 从 $t = 0$ 处沿 $t$ 增加的方向使用弧长作为参数写出曲线

$$\boldsymbol{r}(t) = 12t\boldsymbol{i} + (1 - 3t)\boldsymbol{j} + (5 + 4t)\boldsymbol{k}$$

的参数方程.

5. 求曲线 $\boldsymbol{r}(t) = -\sin t\boldsymbol{i} + \cos t\boldsymbol{j} + t^2\boldsymbol{k}$ 在 $t = \frac{\pi}{2}$ 处的曲率圆方程.

# 第十四章　偏导数

1. 设函数$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$. 在 $(0,0)$ 处, $f(x,y)$
   (A) 连续且偏导数存在.
   (B) 连续，但偏导数不存在.
   (C) 不连续, 但偏导数存在.
   (D) 不连续, 且偏导数不存在.
2. 若 $f(x,y)=y(x-1)^2+x(y-2)^2$. 则下面 $f_x(1,2)$ 的 4 种计算方法中哪一种是不正确的?
   (A) 由于 $f(x,2)=2(x-1)^2$, 则 $f_x(x,2)=4(x-1)$, $f_x(1,2)=0$.
   (B) 由于 $f(1,2)=0$, 则 $f_x(1,2)=0'=0$.
   (C) 由于 $f_x(x,y)=2y(x-1)+(y-2)^2$, 则 $f_x(1,2)=0$.
   (D) $f_x(1,2)=\lim\limits_{x\to 1}\dfrac{f(x,2)-f(1,2)}{x-1}=0$.
3. 若函数 $f(x,y)$ 在点 $(x_0,y_0)$ 处的偏导数都存在. 则
   (A) $f(x,y)$ 在 $(x_0,y_0)$ 附近有界.
   (B) $f(x,y)$ 在 $(x_0,y_0)$ 附近连续.
   (C) $f(x,y_0)$ 在 $x_0$ 处连续, $f(x_0,y)$ 在 $y_0$ 处连续.
   (D) $f(x,y)$ 在 $(x_0,y_0)$ 处连续.
4. 若 $x=x(y,z)$, $y=y(z,x)$ 以及 $z=z(x,y)$ 是由 $F(x,y,z)=0$ 所确定的隐函数. 则下面哪一个选项是错误的?
   (A) $\frac{\partial x}{\partial y}\frac{\partial y}{\partial x}=1$.　　(B) $\frac{\partial x}{\partial z}\frac{\partial z}{\partial x}=1$.
   (C) $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x}=1$.　　(D) $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x}=-1$.
5. 若 $u=u(x,y)$ 和 $v=v(x,y)$ 可微, $C$ 是常数. 则下面哪一个选项是错误的?
   (A) $\nabla C=0$.　　(B) $\nabla(Cu)=C\nabla u$.
   (C) $\nabla(u+v)=\nabla u+\nabla v$.　　(D) $\nabla(uv)=v\nabla u+u\nabla v$.
6. ★ 光滑曲面 $z=x+f(y-z)$ 的所有切平面
   (A) 垂直于一条特定直线.
   (B) 平行于一个特定平面.
   (C) 与某一个坐标平面的夹角是固定的.
   (D) 平行于一条特定直线.
7. 设 $u(x,y)$ 是有界闭区域 $D$ 上的二阶可微函数，$\frac{\partial^2 u}{\partial x\partial y}\neq 0$，$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. 则对于 $u(x,y)$，

(A) 最大值点和最小值点都在区域 $D$ 的内部.

(B) 最大值点和最小值点都在区域 $D$ 的边界上.

(C) 最大值点在 $D$ 的内部而最小值点在 $D$ 的边界上.

(D) 最小值点在 $D$ 的内部而最大值点在 $D$的边界上.

8. 设函数 $f$ 在 $(0,0)$处连续, 则下面哪一个选项是正确的?

   (A) 若 $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{|x|+|y|}$ 存在, 则函数 $f$ 在 $(0,0)$处可微.

   (B) 若 $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2+y^2}$ 存在, 则函数 $f$ 在 $(0,0)$处可微.

   (C) 若函数 $f$ 在 $(0,0)$ 处可微, 则 $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{|x|+|y|}$ 存在.

   (D) 若函数 $f$ 在 $(0,0)$处可微, 则 $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2+y^2}$ 存在.

9. 若函数 $f$ 在原点处连续，且 $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2+y^2}$ 存在, 则 $f(x,y)$ 在原点处可微. (　　)

   (A) 正确.　　　(B) 错误.

10. 若 $z = \frac{y}{f(x^2-y^2)}$,$f(u)$ 是可微函数, 则 $\frac{\partial z}{\partial x}$ =(　　).

11. 若 $u = f(\sqrt{x^2+y^2+z^2})$ 是二阶连续可微函数, 则 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = ($　　$)$.

12. 函数 $u = \sin x \sin y \sin z$ 在约束条件 $x+y+z = \frac{\pi}{2}, x, y, z > 0$ 下的最大值是(　　).

13. 若 $xyz = 6$ 在点 $P$ 处的切平面平行于平面 $6x - 3y + 2z + 1 = 0$，则点$P$ 的坐标是(　　).

14. 函数 $z = x^2 + y^2$ 在约束条件 $x + y = 1$ 下的最小值是(　　).

15. 若 $u = x^{y^z}, x > 0, y > 0$, 则 $\mathrm{d}u = ($　　$)$.

16. 设 $z = z(x,y)$ 是由方程 $\mathrm{e}^{-xy} - 2z + \mathrm{e}^{xz} = 0$ 所确定的隐函数. 则 $\left.\frac{\partial^2 z}{\partial x^2}\right|_{(0,1)} = ($　　$)$.

17. 若曲面 $z - \mathrm{e}^z + 2xy = 3$ 在点 $P$ 处的切平面方程为 $x + 2y = C$，则$P = ($　　$), C = ($　　$)$.

18. ★ 从曲线 $y = x^2$ 上的点到平面 $x - y + 2 = 0$ 上的点所得线段的最短距离为(　　).

19. 若 $w = g(x,y) = f(x,y,z(x,y)) = \mathrm{e}^x yz^2$, 这里 $z(x,y)$ 满足方程$x + y + z + xyz = 2$. 则$\frac{\partial w}{\partial x}(0,1) = ($　　$)$.

20. 若函数 $f(u)$ 可微, 且 $f'(0) = 0.5$, 则 $z = f(4x^2 - y^2)$ 在点 $(1,2)$ 处沿单位向量(　　)的方向增加最快.

21. 设函数 $f(x,y)$ 有连续的偏导数, 且 $f(x,x^2) = 1, f_x(x,x^2) = x$, 则 $f_y(x,x^2) = ($　　$)$.

22. 设 $z = f(x,y)$ 在点 $(1,1)$处可微, 且满足 $f(1,1) = 1, f_x(1,1) = 2, f_y(1,1) = 3$, $\phi(x) = f(x, f(x,x))$, 则 $\frac{\mathrm{d}}{\mathrm{d}x}\left(\phi^3(1)\right) = ($　　$)$.

23. 判断函数

$$f(x,y) = \begin{cases} y^2 \ln\left(x^2+y^2\right), & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

   在全平面上是否连续，并说明原因.

24. 判断以下极限是否存在. 若存在, 则求出其极限; 若不存在，请说明理由.

(1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^3+y^4)}{x^2+y^2}$;　　(2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^{1.5}}{x^4+y^2}$;

(3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$;　　(4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^3+y^3)}{x^2+y^2}$;

(5) $\lim\limits_{(x,y)\to(0,2)} \dfrac{\ln(1+xy^2)}{x}$;　　(6) $\lim\limits_{(x,y)\to(0,0)} (1-2xy)^{1/(x^2+y^2)}$;

(7) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x-y)}{x+y}$;　　(8) $\lim\limits_{(x,y)\to(0,2)} \dfrac{\arctan\frac{xy}{1+xy}}{x}$;

(9) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{|x|+|y|}$;　　(10) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{1+x^2+y^2}-1}{x^2+y^2}$;

(11) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2\sin x}{x^2+y^4}$;　　(12) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\ln(x^2+e^{y^2})}{x^2+y^2}$;

(13) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2-\sqrt{xy+4}}{xy}$;　　(14) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+x^2+y^2}$;

(15)★ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y}$;　　(16) $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2}$.

25. 设函数 $f(u)$ 二阶连续可微, 且满足 $g(x,y)=f\left(\frac{y}{x}\right)+yf\left(\frac{x}{y}\right)$. 求 $x^2g_{xx}-y^2g_{yy}$.

26. 若 $f(x,y)=\left(\frac{y}{x}\right)^{\frac{x}{y}}$, 求 $f_x$, $f_y$.

27. 求函数 $f(x,y)=x^2+2y^2-x^2y^2$ 在区域 $D=\{(x,y)\mid x^2+y^2\le 4, y\ge 0\}$ 上的最大值和最小值.

28. 求函数 $f(x,y)=x^2+y^2+x^2y+4$ 在区域 $D=\{(x,y)\mid |x|\le 1, |y|\le 1\}$ 上的最大值和最小值.

29. 设 $\boldsymbol{n}$ 是曲面 $2x^2+3y^2+z^2=6$ 在点 $(1,1,1)$ 处向内的法向量, 求 $u=\frac{\sqrt{6x^2+8y^2}}{z}$ 在该点处沿方向 $\boldsymbol{n}$ 的方向导数.

30. 若曲面 $z=x^2+y^2$ 的一个切平面包含点 $(1,0,0)$ 和 $(0,1,0)$，求此切平面的方程.

31. 设曲线 $C: x^2+y^2+xy=3$ 上的任一点 $A$，求函数 $f(x,y)=x+y+xy$ 在点 $A$ 的方向导数的最大值(这里是求所有的 $A\in C$ 中的最大值).

32. 计算下列函数的所有的一阶偏导数:

(1) $f(x,y,z)=z^{xy}$;　　(2) $f(x,y,z)=x^zy^{-z}$;

(3) $f(x,y,z)=\displaystyle\int_{xz}^{yz} e^{t^2}\,dt$;　　(4) $f(x,y)=\left(1+x^2+y^2\right)^{xy}$;

(5) $f(x,y,z)=\left(\dfrac{x}{y}\right)^{1/z}$.

33. 计算下列函数的所有的二阶偏导数:

$$(1)\ f(x,y)=\arctan\frac{x+y}{1-xy};\qquad (2)\ f(x,y)=\ln\left(x+y^2\right);$$
$$(3)\ w=f\left(xy,\frac{x}{y}\right);\qquad (4)\ z=f\left(xy,x^2+y^2\right).$$

34. 求下列复合函数或隐函数的偏导数.
   (1) 若 $z=\mathrm{e}^u+\ln v, u=\sqrt{xy}, v=2x-3y$, 求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.
   (2) 若 $w=xyz+\ln(x+y+z), x=\mathrm{e}^t, y=\mathrm{e}^{-t}, z=\sqrt{t}$, 求 $\frac{\mathrm{d}w}{\mathrm{d}t}$.
   (3) 若 $\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$，求 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$
   (4) 若函数 $f$ 二阶连续可微, 且满足 $w=f(x,y,z), z=\ln\sqrt{x^2+y^2}$, 求 $\frac{\partial w}{\partial x}$ 和 $\frac{\partial^2 w}{\partial x^2}$.
   (5) 若 $z+\mathrm{e}^z=xy$, 求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.
   (6) 若 $z=z(x,y)$ 是由方程 $x^2+y^2+z^2=4z$ 所确定的隐函数，求$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$, 及$\frac{\partial^2 z}{\partial x\partial y}$.
   (7) ★ 若 $F(xz,yz)=0$, 求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.
   (8) ★ 若 $F(\frac{x}{z},\frac{z}{y})=0$, 求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.

35. 求函数

$$f(x,y)=(8x^2-6xy+3y^2)\mathrm{e}^{2x+3y}$$

   的局部极值.

36. 求函数

$$f(x,y)=\cos^2 x+\cos^2 y$$

   在约束条件 $x-y=\frac{\pi}{4}$ 下的极值.

37. 用泰勒公式求函数 $f(x,y)=\sqrt{1-x^2-y^2}$ 在原点处的二次逼近.

38. 若 $r=\sqrt{x^2+y^2+z^2}$, 证明: 当 $r\neq 0$ 时, 必有

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}=\frac{2}{r}.$$

39. 证明函数 $f(x,y)=\sqrt{|xy|}$ 在点 $(0,0)$ 处连续且所有的偏导数都存在，但在点 $(0,0)$ 处不可微.

40. 求在椭球 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 内体积最大的长方体.

41. 用拉格朗日乘数法求函数 $f(x,y,z)=x^4+y^4+z^4$ 在球面 $g(x,y,z)=x^2+y^2+z^2=1$ 上的最大值和最小值.

42. 求函数

$$f(x,y)=\begin{cases}\dfrac{\sin(xy)}{x}, & x\neq 0\\ y, & x=0\end{cases}$$

   的所有连续点.

43. 设二元函数

$$f(x,y)=\begin{cases} y\sin\dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$$

(1) 求 $f_x$ 和 $f_y$ 在除原点之外的点处的表达式.

(2) 判断 $f_x(0,0)$ 和 $f_y(0,0)$ 是否存在.

(3) 求 $f$ 的所有连续点.

44. 设二元函数

$$f(x,y)=\begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$$

(1) 求 $f_x(0,0)$ 和 $f_y(0,0)$;

(2) 判断 $f_x$ 和 $f_y$ 在原点处是否连续;

(3) 证明 $f$ 在原点处可微.

45. 设函数 $f(z)$ 可微, 且由方程 $f(x^2+y^2)+f(x+y)=y$ 可定义隐函数 $y=g(x)$. 若 $g(0)=2$, $f'(2)=0.5$, $f'(4)=1$, 求 $g'(0)$.

46. 设 $w=f(x_1,\cdots,x_n)$, $x_i=g_i(t_1,\cdots,t_m)$),$t_j=h_j(s_1,\cdots,s_l)$. 求 $\frac{\partial w}{\partial s_k}$.

47. 函数

$$f(x,y)=\begin{cases} \dfrac{x^2\sqrt{|y|}}{x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$$

在点 $(0,0)$ 处是否连续, 并说明理由.

48. 若 $M(x,y)=\ln(u^2+v^2+(2xy)^2)$, 这里 $u$、$v$ 满足 $u=x+2y$, $v=2x-y$, 求 $\frac{\partial M}{\partial x}(1,1)$ 和 $\frac{\partial M}{\partial y}(1,1)$.

49. 用拉格朗日乘数法求在抛物线 $xy=8$ 上与点 $(3,0)$ 的距离最近的点的坐标.

50. 设二元函数

$$f(x,y)=\begin{cases} \dfrac{\sin(x^2y)}{x}, & xy\neq 0;\\ x, & xy=0.\end{cases}$$

(1) 求 $f_x(0,1)$ 和 $f_y(0,1)$;

(2) 判断 $f_x$ 在原点处是否连续;

(3) 判断 $f$ 在原点处是否可微.

51. 若 $f(2x+z,3y-z)=0$ 且 $f$ 可微, 求 $\mathrm{d}z$.

52. 设 $f(x,y)$ 可微, 且 $f(1,1)=1$, $f_x(1,1)=a$, $f_y(1,1)=b$. 若 $u=f(x,f(x,f(x,x)))$, 求 $\frac{\mathrm{d}(u^2)}{\mathrm{d}x}$ 在点 $x=1$处的值.

53. 设 $u = f(\ln\sqrt{x^2+y^2})$. 若 $u$ 满足方程

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^2+y^2)^{\frac{3}{2}}.$$

求$f$.

54. 求曲面 $x^2+2y^2+z^2=1$ 平行于平面 $x-y+2z=0$ 的切平面方程.

55. 求函数 $f(x,y)=x^3+8y^3-xy$的局部极值.

56. 设二元函数

$$f(x,y)=\begin{cases}\dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2}, & (x,y)\neq(0,0);\\ x, & (x,y)=(0,0).\end{cases}$$

判断函数 $f$在原点处是否连续.

57. 设二元函数

$$f(x,y)=\begin{cases}\dfrac{\sqrt{x}}{x^2+y^2}\sin(x^2+y^2), & (x,y)\neq(0,0);\\ x, & (x,y)=(0,0).\end{cases}$$

求 $f_x(0,0)$ 和 $f_y(0,0)$.

58. 求曲面 $2^{x/z}+2^{y/z}=8$ 在点 $(2,2,1)$处的切平面方程和法线方程.

59. ★ 求函数 $f(x,y)=ax^2+2bxy+cy^2$ 在单位圆上的最大值和最小值, 这里 $a,b,c>0$, 且满足 $b^2-ac<0$.

60. 对于函数 $f(x,y)=\mathrm{e}^{\sqrt{x^2+y^4}}$, 判断 $f_x$ 和 $f_y$ 在原点处是否存在. 若存在, 求出它们的值; 若不存在, 请说明理由.

61. 设二元函数

$$f(x,y)=\begin{cases}xy, & |x|\geq|y|;\\ -xy, & |x|<|y|.\end{cases}$$

求 $f_{xy}(0,0)$ 和 $f_{yx}(0,0)$.

62. 若 $z=z(x,y)$ 是由 $x\mathrm{e}^x-y\mathrm{e}^y=z\mathrm{e}^z$ 所确定的隐函数, 且 $f(x,y,z)$ 有连续的偏导数. 求二元函数 $u=u(x,y)=f(x,y,z(x,y))$ 的梯度.

63. 设 $f(x,y)=|x-y|g(x,y)$, 这里 $g(x,y)$ 在点 $(0,0)$ 处连续且满足 $g(0,0)=0$.
   (1) 判断函数 $f$ 在原点处的偏导数是否存在. 若存在, 求出它们的值.
   (2) 判断函数 $f$ 在原点处是否可微, 并说明理由.

# 第十五章　重积分

1. 若 $D=\{(x,y)\mid x^2+y^2\le 4\}$, 则 $\iint_D\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\mathrm{d}A=(\qquad)$.
2. 若 $D=\{(x,y)\mid x^2+y^2\le x+y\}$, 则在极坐标系下 $\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y=(\qquad)$.
3. 设 $\Omega$ 是由 $z=\sqrt{x^2+y^2}$ 和 $x^2+y^2+z^2=2z$ 所围成的区域. 在球坐标系下, 三重积分 $\iiint_\Omega (x^2+y^2+z^2)\,\mathrm{d}V=(\qquad)$.
4. 在柱坐标系下按照 $\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$ 的顺序, 三重积分
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{x^2+y^2} f(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x=(\qquad).$$
5. 若 $f(t)=\int_1^{t^2}\mathrm{e}^{-x^2}\,\mathrm{d}x$, 则 $\int_0^1 tf(t)\,\mathrm{d}t=(\qquad)$.
6. 交换下列积分的积分次序.
$$\int_{-\pi/4}^{\pi/4}\int_0^{2\cos\theta} f(r,\theta)r\,\mathrm{d}r\mathrm{d}\theta=(\qquad);$$
$$\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,\mathrm{d}y\mathrm{d}x=(\qquad)\ (a>0);$$
$$\int_0^1\mathrm{d}y\int_0^{2y} f(x,y)\,\mathrm{d}x+\int_1^3\mathrm{d}y\int_0^{3-y} f(x,y)\,\mathrm{d}x=(\qquad).$$
7. 若 $\displaystyle\iint_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,\mathrm{d}A=\pi$, 则 $a=(\qquad)$.
8. 若 $f(x,y)$ 连续且恒正, 则 $\lim\limits_{n\to\infty}\int_0^\pi\int_0^\pi (f(x,y))^{1/n}\,\mathrm{d}x\mathrm{d}y=(\qquad)$.
9. 若 $\int_0^1 f(x)\,\mathrm{d}x=\sqrt{2}$, 则 $\displaystyle\iint_{\substack{0\le x\le 1\\0\le y\le 1}} f(x)f(y)\,\mathrm{d}A=(\qquad)$, $\displaystyle\iint_{\substack{0\le x\le 1\\0\le y\le x}} f(x)f(y)\,\mathrm{d}A=(\qquad)$.
10. 以 $\mathrm{d}y\mathrm{d}x\mathrm{d}z$ 为积分次序, 则 $\int_0^1\int_x^1\int_x^y f(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x=(\qquad)$.
11. 区域 $1\le x^2+y^2\le 2x$ 的形心坐标是$(\qquad)$.
12. 若密度函数 $\delta=\sqrt{x^2+y^2+z^2}$, 则上半球 $0\le z\le\sqrt{1-x^2-y^2}$ 的质心坐标为$(\qquad)$.
13. 以 $\mathrm{d}x\mathrm{d}z\mathrm{d}y$ 为积分顺序，则 $\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int_0^{x+y} f(x,y,z)\,\mathrm{d}z=(\qquad)$.
14. 若函数 $f(x,y,z)$ 连续，则 $\lim\limits_{t\to0+}\frac{1}{\pi t^3}\displaystyle\iiint_{x^2+y^2+z^2\le t^2} f(x,y,z)\,\mathrm{d}V=(\qquad)$.
15. 若 $f'(t)$ 连续, $f(0)=0$ 且 $f'(0)$ 存在, 则 $\lim\limits_{t\to0+}\frac{1}{\pi t^4}\displaystyle\iiint_{x^2+y^2+z^2\le t^2} f(\sqrt{x^2+y^2+z^2})\,\mathrm{d}V=(\qquad)$.
16. 计算由下列曲面所围成的区域 $D$ 的体积:

(1) $x^2+y^2=a^2, y^2+z^2=a^2$ 和 $z^2+x^2=a^2$;

(2) $x^2+y^2=x$ 和 $x^2+y^2+z^2=1$;

(3) $z=0, z=2-x-2y, x=2y$ 和 $x=0$;

(4) $x^2+y^2=az$ 和 $z=2a-\sqrt{x^2+y^2}\ (a>0)$;

(5) $z=6-x^2-y^2$ 和 $z=\sqrt{x^2+y^2}$;

(6) $x^2+y^2+z^2=2az(a>0)$ 和 $x^2+y^2=z^2$ 所围成的包含 $z$轴的部分;

(7) $z=\sqrt{5-x^2-y^2}$ 和 $x^2+y^2=4z$;

(8) $z^2=\frac{x^2}{4}+\frac{y^2}{9}$ 和 $2z=\frac{x^2}{4}+\frac{y^2}{9}$.

17. 计算下列积分:

(1) $\displaystyle\iint\limits_{|x|+|y|\le 1}(|x|+|y|)\,\mathrm{d}x\mathrm{d}y$;

(2) $\displaystyle\int_0^2\int_0^2\max(xy,1)\,\mathrm{d}x\mathrm{d}y$;

(3) $\displaystyle\int_0^2\int_x^2 \mathrm{e}^{-y^2}\,\mathrm{d}y\mathrm{d}x$;

(4) $\displaystyle\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,\mathrm{d}y\mathrm{d}x+\int_1^4\int_{x-2}^{\sqrt{x}} f(x,y)\,\mathrm{d}y\mathrm{d}x$;

(5) $\displaystyle\iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} \mathrm{e}^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}\,\mathrm{d}V$;

(6) $\displaystyle\iint\limits_{x^2+y^2\le 1, x\ge 0}\frac{1+xy}{1+x^2+y^2}\,\mathrm{d}A$;

(7) $\displaystyle\iint\limits_{\substack{x+y\le 1\\ x,y\ge 0}} \mathrm{e}^{\frac{x}{x+y}}\,\mathrm{d}A$;

(8) $\displaystyle\iiint\limits_{x^2+y^2+z^2\le R^2}\sqrt{x^2+y^2+z^2}\,\mathrm{d}V$;

(9) ★ $\displaystyle\iint\limits_{\substack{x+y\le 1\\ x,y\ge 0}} \mathrm{e}^{\frac{y}{x+y}}\,\mathrm{d}x\mathrm{d}y$;

(10) $\displaystyle\iint\limits_{\substack{0\le x\le 1\\ 0\le y\le 1}} |x^2+y^2-1|\,\mathrm{d}A$;

(11) $\displaystyle\int_0^1\int_{3y}^3 \mathrm{e}^{x^2}\,\mathrm{d}x\mathrm{d}y$;

(12) $\displaystyle\iint\limits_{\substack{x^2+y^2\le 9\\ 0\le y\le\sqrt{3}x}} (x^2+y^2)^{3/2}\,\mathrm{d}A$;

(13) $\displaystyle\iint\limits_{|x|+|y|\le 1}(|x|+y\mathrm{e}^{-x^2})\,\mathrm{d}A$;

(14) $\displaystyle\iiint\limits_{x^2+y^2\le 4}|x^2+y^2-2y|\,\mathrm{d}A$;

(15) $\displaystyle\iint\limits_{x^2+y^2\le x+y}(x+y)\,\mathrm{d}A$;

(16) ★ $\displaystyle\iint\limits_{|x|+|y|\le 1}\mathrm{e}^{x+y}\,\mathrm{d}A$;

(17) $\displaystyle\iint\limits_{x^2+y^2\le R^2}\mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}A$;

(18) $\displaystyle\iint\limits_{\substack{x^2+y^2\le 1\\ x,y\ge 0}}\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,\mathrm{d}A$;

(19) $\displaystyle\iiint\limits_{\substack{x^2+y^2+z^2\le 1\\ z\ge 0}} z\mathrm{e}^{-x^2-y^2-z^2}\,\mathrm{d}V$;

(20) $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}}\frac{1}{\sqrt{1+x^2+y^2}}\,\mathrm{d}x\mathrm{d}y$;

(21) $\displaystyle\iiint\limits_{\substack{x^2+y^2+z^2\geq z\\ x^2+y^2+z^2\leq 2z}} z\,\mathrm{d}V$;　　(22) $\displaystyle\iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1} z^2\,\mathrm{d}V$;

(23) $\displaystyle\iint\limits_{x^2+y^2\leq 1} (x^2+xy\mathrm{e}^{x^2+y^2})\,\mathrm{d}A$;　　(24) $\displaystyle\iint\limits_{\substack{0\leq x\leq 1\\ 0\leq y\leq 1}} \mathrm{e}^{\max\{x^2,y^2\}}\,\mathrm{d}A$.

18. ★ 计算

$$\iint\limits_{D} r^2\sin\theta\sqrt{1-r^2\cos(2\theta)}\,\mathrm{d}r\mathrm{d}\theta,\qquad D=\left\{(r,\theta)\mid 0\leq r\leq \sec\theta, 0\leq\theta\leq\frac{\pi}{4}\right\}.$$

19. 通过改变积分次序，写出与累次积分

$$\int_0^2\int_0^{4-y^2}\int_0^{2-y} f(x,y,z)\,\mathrm{d}z\mathrm{d}x\mathrm{d}y$$

等价的其它五种不同积分次序的累次积分.

20. 计算

$$\iint\limits_{R} xy^2\,\mathrm{d}x\mathrm{d}y,$$

这里 $R$ 是由 $x=p/2$ 和 $y^2=2px\ (p<0)$所围成的区域.

21. 计算

$$\iint\limits_{D} |\cos(x+y)|\,\mathrm{d}A,$$

这里 $D$ 是由 $y=x, y=0$ 和 $x=\frac{\pi}{2}$所围成的区域.

22. 用变换 $x=u^2-v^2, y=2uv\ (u\geq 0, v\geq 0)$ 来计算

$$\int_0^1\int_0^{2\sqrt{1-x}}\sqrt{x^2+y^2}\,\mathrm{d}y\mathrm{d}x.$$

23. 计算

$$\int_0^1\int_{\sqrt[3]{y}}^1 \frac{2\pi\sin(\pi x^2)}{x^2}\,\mathrm{d}x\mathrm{d}y.$$

24. 计算

$$\iiint\limits_{D}(x^2+y^2)\,\mathrm{d}V,$$

这里 $D$ 是由 $0<a\leq\sqrt{x^2+y^2+z^2}\leq A$ 和 $z>0$ 所围成的区域.

25. 计算 $\displaystyle\iiint_V(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2})\,\mathrm{d}V$, 这里积分区域是由 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 所围的区域.

26. 计算 $\iiint_V (x^2+y^2+z^2)\,\mathrm{d}V$, 其中积分区域是球 $x^2+y^2+z^2 \le x+y+z$ 的内部.

27. 计算

$$\iint\limits_D (x^3+y^3)\,\mathrm{d}A,$$

这里 $D$ 是由 $x^2=2y, x^2=3y, x=y^2$ 以及 $x=2y^2$ 所围的区域.

28. 计算

$$\iiint\limits_D z^2\,\mathrm{d}V,$$

这里 $D$ 是由 $x^2+y^2+z^2=a^2$ 和$x^2+y^2=ax$所围的区域.

29. 计算

$$\iiint\limits_D (x+y+z)\cos(x+y+z)^2\,\mathrm{d}V,$$

这里 $D$ 是由 $0 \le x-y \le 1, 0 \le x-z \le 1$ 以及 $0 \le x+y+z \le 1$所围的立体区域.

30. 求由 $r=2\sin\theta$ 和 $r=4\sin\theta$ 所围图形的形心.

31. 计算

$$\iint\limits_D \frac{x\sin y}{y}\,\mathrm{d}A,$$

这里 $D$ 是由 $y=x$ 和 $y=x^2$ 所围的区域.

32. 计算

$$\iiint\limits_\Omega (x+z)\,\mathrm{d}V, \qquad \iiint\limits_\Omega xyz\,\mathrm{d}V,$$

这里 $\Omega$ 是由 $z=\sqrt{x^2+y^2}$ 和 $z=\sqrt{1-x^2-y^2}$ 所围的区域.

33. 计算

$$\int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \frac{\sin z}{(1-z)^2}\,\mathrm{d}z.$$

34. ★ 若 $f(x)>0$ 且在 $[a,b]$上连续,证明:

$$\int_a^b f(x)\,\mathrm{d}x \int_a^b \frac{\mathrm{d}x}{f(x)} \ge (b-a)^2.$$

35. 若函数 $f(x)$ 在 $R$ 上连续且恒正, $a$ 和 $b$ 是实数. 计算

$$\iint\limits_R \frac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}\,\mathrm{d}A,$$

这里 $R=\{(x,y)\mid x^2+y^2\le 4, x\ge 0, y\ge 0\}$.

36. 利用对称性计算

$$\iint\limits_{(x-1)^2+(y-1)^2\le 2} \left(\cos^2(x^2+y)+\sin^2(x+y^2)\right) \mathrm{d}A.$$

37. 计算

$$\iiint\limits_D xz \,\mathrm{d}V,$$

这里 $D$ 是由 $z=0, z=y, y=1$ 以及 $y=x^2$所围成的区域.

38. 下面哪一个定积分的值更大?

$$I_1=\iiint\limits_D \ln^2(x+y+z+3)\,\mathrm{d}V, \qquad I_2=\iiint\limits_D (x+y+z)^2\,\mathrm{d}V,$$

这里 $D$ 是由 $x+y+z+1=0, x+y+z+2=0, x=0, y=0$ 以及 $z=0$ 所围成的区域.

39. 把下列三个积分按大小顺序排序.

$$I_1=\iint\limits_R \cos\sqrt{x^2+y^2}\,\mathrm{d}A, \quad I_2\iint\limits_R \cos(x^2+y^2)\,\mathrm{d}A, \quad I_3=\iint\limits_R \cos(x^2+y^2)^2\,\mathrm{d}A,$$

这里 $R=\{(x,y)\mid x^2+y^2\le 1\}$.

40. ★ 若 $F(t)=\int_1^t\int_y^t f(x)\,\mathrm{d}x\mathrm{d}y$, 这里 $f$ 是单变量连续函数, 计算$F'(2)$.

41. 设函数 $f(x,y)$ 以及它的一阶和二阶偏导数都在 $R=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$ 上连续. 若 $f(1,y)=0$ 且 $f(x,1)=0$, 计算

$$\iint\limits_R xyf_{xy}(x,y)\,\mathrm{d}x\mathrm{d}y.$$

42. 计算

$$\iiint\limits_D x^2 e^y\,\mathrm{d}V,$$

这里 $D$ 是由抛物柱面 $z=1-y^2$ 和平面 $z=0, x=\pm 1$ 所围成的区域.

43. 求一个高为 $h$, 底面半径为 $a$ 的圆锥的形心.

44. 若函数 $f(x,y)$ 在区域 $0\le |x|\le 1, 0\le |y|\le 1$ 上连续且 $f(0,0)=-1$, 计算

$$\lim_{x\to 0+}\frac{\int_0^{x^2}\mathrm{d}t\int_x^{\sqrt{t}} f(t,u)\,\mathrm{d}u}{1-e^{-x^3}}.$$

45. 求由曲线 $xy=1, y=x+1$ 以及 $y=x$ 所围成的薄板的质量, 这里薄板的密度函数

为$\delta = |x+y|$.

46. 计算
$$\iint\limits_R \cos\frac{x-y}{x+y}\,\mathrm{d}A,$$
这里 $R$ 是由 $x+y=1$, $x=0$ 以及 $y=0$所围的区域.

47. ,写出三重积分 $\iiint\limits_D f(x,y,z)\,\mathrm{d}V$ 在下面不同坐标系下对应的累次积分，这里区域 $D=\{(x,y,z)\mid \sqrt{3(x^2+y^2)}\le z\le \sqrt{16-x^2-y^2}\}$.
(1) 直角坐标系.
(2) 柱坐标系.
若 $f(x,y,z)=x^2+y^2+z$, 计算积分的值.

48. 设
$$F(t)=\iiint\limits_{\substack{x^2+y^2\le t^2\\ 0\le z\le 2}}(f(x^2+y^2)+z^2)\,\mathrm{d}V.$$
若函数 $f(x)$ 连续, 计算 $F'(t)$.

49. ★ 若 $f(x,y)$ 连续, 证明:
$$\iint\limits_{\substack{|x|\le 1\\ |y|\le 1}} f(x-y)\,\mathrm{d}A=\int_{-2}^{2} f(t)(2-|t|)\,\mathrm{d}t.$$

50. 计算
$$\iiint\limits_D (x^2+y^2)\,\mathrm{d}V,$$
这里 $D$ 是由 $z=2, z=8$ 以及曲线
$$\begin{cases} y^2=2z \\ x=0 \end{cases}$$
绕 $z$ 轴旋转一周所形成的曲面所围成的区域.

51. 计算
$$\iiint\limits_D z\sqrt{x^2+y^2+z^2}\,\mathrm{d}V,$$
这里 $D$ 是满足$x^2+y^2+z^2\le 1$ 和 $z\ge\sqrt{3(x^2+y^2)}$ 的所有点形成的区域.

52. 计算

$$\iint\limits_{\substack{1\le x^2+y^2\le 4\\ x,y\ge 0}} \frac{\pi\sin(\pi\sqrt{x^2+y^2})}{x+y}\,\mathrm{d}A.$$

53. 计算

$$\iiint\limits_{D} y\cos(x+z)\,\mathrm{d}V,$$

这里 $D$ 是由 $z=0, y=0, y=\sqrt{x}$ 以及 $x+z=\pi/2$ 所围成的区域.

54. 计算

$$\iiint\limits_{D} |z-x^2-y^2|\,\mathrm{d}V,$$

这里 $D$ 是由 $x^2+y^2=1, z=0$ 以及 $z=1$所围成的区域.

55. 计算

$$\iiint\limits_{D} xz\,\mathrm{d}V,$$

这里 $D$ 是由 $z=0, z=y, y=1$ 以及$y=x^2$所围成的区域.

56. 计算

$$\iiint\limits_{D} z\,\mathrm{d}V,$$

这里 $D$ 是由 $z=\frac{h}{R}\sqrt{x^2+y^2}\ (R>0, h>0)$ 和 $z=h$ 所围成的区域.

# 第十六章　积分和向量场

1. 若 $L$ 为圆周 $x^2+y^2=a^2$，沿顺时针方向, 则 $\oint_L (x^3-x^2y)\,\mathrm{d}x+(xy^2-y^3)\,\mathrm{d}y=(\qquad)$.
2. 若 $L$ 为椭圆 $4x^2+y^2=8x$, 沿逆时针方向, 则 $\oint_L \mathrm{e}^{y^2}\,\mathrm{d}x+x\,\mathrm{d}y=(\qquad)$.
3. 若 $\Sigma$ 为球面 $x^2+y^2+z^2=a^2$ ,法向量向外, 则 $\oiint_\Sigma \dfrac{x\,\mathrm{d}y\mathrm{d}z+y\,\mathrm{d}z\mathrm{d}x+z\,\mathrm{d}x\mathrm{d}y}{\sqrt{x^2+y^2+z^2}^3}=(\qquad)$.
4. 若 $\frac{(x+ay)\,\mathrm{d}y-y\,\mathrm{d}x}{(x+y)^2}$ 是恰当的, 则 $a=(\qquad)$.
5. 若 $L$ 为圆周$x^2+(y-1)^2=4$, 沿逆时针方向, 则 $\oint_L \dfrac{x\,\mathrm{d}y-y\,\mathrm{d}x}{x^2+(y-1)^2}=(\qquad)$.
6. 若 $L$ 是封闭曲线 $|x|+|y|=1$, 沿逆时针方向, 则 $\oint_L x^2y^2\,\mathrm{d}x-\cos(x+y)\,\mathrm{d}y=(\qquad)$.
7. 若 $u=x^2+2y+yz$, 则 $\nabla\cdot(\nabla u)=(\qquad)$.
8. 曲线 $y=\sqrt{2x-x^2}\ (0\le x\le 2)$ 的形心为$(\qquad)$.
9. 若 $C: x^2+y^2=2$, 则 $\int_C (x+y)^2\,\mathrm{d}s=(\qquad)$.
10. 若 $\boldsymbol{F}=M\boldsymbol{i}+N\boldsymbol{j}$, $C$ 是沿逆时针方向的光滑简单闭曲线, 则 $\boldsymbol{F}$ 通过曲线 $C$ 向外的通量为$(\qquad)$.
11. 若 $L$ 是依次连接 $(0,0,0)$, $(0,0,2)$, $(1,0,2)$ 以及 $(1,3,2)$ 的折线段, 则 $\int_L x^2yz\,\mathrm{d}s=(\qquad)$.
12. 若 $L$ 是圆周$x^2+y^2=4x$, 则 $\oint_L \sqrt{x^2+y^2}\,\mathrm{d}s=(\qquad)$.
13. 若 $L$ 是从 $(1,1,1)$ 到 $(2,3,4)$ 的线段, 则 $\int_L x\,\mathrm{d}x+y\,\mathrm{d}y+(x+y-1)\,\mathrm{d}z=(\qquad)$.
14. 设曲线积分 $\int_C (f(x)-\mathrm{e}^x)\sin y\ \mathrm{d}x\ -f(x)\cos y\mathrm{d}y$ 与路径无关，若 $f$ 的导数连续且 $f(0)=0$, 求 $f(x)$.
15. 求 $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$, 这里 $\boldsymbol{F}(x,y)=(3x^2y, x^3+x-2y)$, 曲线 $C$ 有两部分，第一部分是从点 $(0,0)$ 在第一象限中沿 $x^2+y^2=2x$ 到 $(2,0)$ 的圆弧, 第二部分是从 $(2,0)$ 在第一象限中沿 $x^2+y^2=4$ 到 $(0,2)$ 的圆弧.
16. 在所有形如 $0\le x\le a, 0\le y\le b$ 的矩形区域中, 求使向量场 $\boldsymbol{F}=(x^2+4xy)\boldsymbol{i}-6y\boldsymbol{j}$ 通过矩形区域边界向外的通量最小的矩形区域. 最小的通量是多少?
17. 求 $\oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$, 这里 $\boldsymbol{F}(x,y)=(y^2-z^2, 2z^2-x^2, 3x^2-y^2)$, $C$ 是平面 $x+y+z=2$ 和柱面 $|x|+|y|=1$ 的交线, 从 $z$ 轴正向看沿逆时针方向.
18. 求 $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$, 这里 $\boldsymbol{F}(x,y)=(\mathrm{e}^x\sin y-my, \mathrm{e}^x\cos y-m)$, $C$ 从点 $(a,0)$ 沿上半圆周 $x^2+y^2=ax$ 到 $(0,0)$.
19. ★ 若 $L$ 是球面 $x^2+y^2+z^2=1$ 和平面 $x+y+z=0$ 的交线. 求 $\oint_L xy\,\mathrm{d}s$.

20. 若 $\boldsymbol{F}=(x+y,x-y)$, 曲线 $C$ 是椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, 沿逆时针方向. 求 $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$.

21. 设 $C: x=t, y=t^2$ and $z=t^3$, $(0\le t\le 1)$, 计算

$$\int_C \left(y^2-z^2\right)\mathrm{d}x+2yz\,\mathrm{d}y-x^2\,\mathrm{d}z.$$

22. 证明曲线积分

$$\int_{(1,1,1)}^{(1,3,\pi)} \mathrm{e}^x\ln y\,\mathrm{d}x+\left(\frac{\mathrm{e}^x}{y}+\sin z\right)\mathrm{d}y+y\cos z\,\mathrm{d}z$$

中的微分是恰当微分形式, 并计算积分值.

23. 求向量场 $\boldsymbol{F}=(6x+y,-x-z,4yz)$ 通过区域 $D$ 的边界向外的通量, 这里 $D$ 是在第一卦限中由锥面 $z=\sqrt{x^2+y^2}$, 柱面 $x^2+y^2=1$ 以及坐标平面所围成的区域.

24. 用斯托克斯定理求向量场 $\boldsymbol{F}=(y,-xz,x^2)$ 沿曲线 $C$的环流量, 这里 $C$ 是平面 $x+y+z=1$ 被第一卦限所截的三角形，从上往下看沿逆时针方向.

25. 计算

$$\int_L y^2\,\mathrm{d}x-x\,\mathrm{d}y,$$

这里曲线 $L$ 包含两部分，第一部分是从点 $(1,1)$ 沿抛物线 $y=x^2$ 到点 $(-1,1)$ 的曲线, 第二部分是从点 $(1,-1)$ 到点 $(0,2)$ 的线段.

26. 计算

$$\int_L \left(x^2-y\right)\mathrm{d}x-(x+\sin y)\,\mathrm{d}y.$$

这里 $L$ 是上半圆周 $y=\sqrt{2x-x^2}$ 中从 $(2,0)$ 到 $(0,0)$的一段圆弧.

27. 设 $\Sigma$ 是锥面 $z=\sqrt{x^2+y^2}$ 被柱面 $x^2+y^2=2x$ 所截的曲面, 求

$$\iint_\Sigma \frac{1}{z}\,\mathrm{d}\sigma.$$

28. 设 $L$ 是平面 $y+z=2$ 和柱面 $x^2+y^2=1$ 的交线, 曲线从 $z$ 轴的正向看是沿逆时针方向. 求

$$\oint_L -y^2\,\mathrm{d}x+x\,\mathrm{d}y+z^2\,\mathrm{d}z.$$

29. 设向量场 $\boldsymbol{F}=(-2xz,0,y^2)$,
    (1) 求 $\boldsymbol{F}$的旋度.
    (2) 证明: 若 $R$ 是单位球面上的任一闭区域, 则 $\iint_R \nabla\times\boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}\sigma=0$.
    (3) 证明: 若 $C$ 是单位球面上的任一简单闭曲线, 则 $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}=0$.

30. 用散度定理求向量场 $\boldsymbol{F}=(y^2,x^2yz,x^2)$ 通过上半球面$x^2+y^2+z^2=1\ \ (z\ge 0)$ 向外的

通量.

31. 求细金属丝 $C$ 的质量，其中 $C$ 是柱面 $x^2+y^2=1$ 和平面 $x+z=0$的交线，其密度函数为 $\delta=|xy|$.

32. $C$ 是在曲线 $x=1-y^2$ 上从 $(0,-1)$ 到 $(0,1)$ 的一段, 求

$$\int_C y^2\,\mathrm{d}x+x^2\,\mathrm{d}y.$$

33. 证明向量场 $\boldsymbol{F}=(\sin y, x\cos y, -\sin z)$ 是保守场, 并求出其势函数.

34. 用格林定理来计算向量场 $\boldsymbol{F}=x^2y\boldsymbol{i}-xy^2\boldsymbol{j}$ 沿逆时针方向绕圆周 $C: x^2+y^2=4$ 的环流量.

35. 参数化下列曲面并计算 $|\boldsymbol{r}_u\times\boldsymbol{r}_v|$.
   (1) 以原点为中心以 $R$为半径的球面.
   (2) 圆柱面 $y^2+z^2=a^2$.
   (3) 锥面 $x=2\sqrt{y^2+z^2}$.
   (4) 椭圆抛物面 $z=16-x^2-y^2$.

36. 设 $\boldsymbol{r}=(x,y,z)$,$\boldsymbol{F}=\frac{\boldsymbol{r}}{|\boldsymbol{r}|^p}$. 若 $\mathrm{div}\boldsymbol{F}=0$, 求 $p$ 的值.

37. 计算

$$\iint_S yz\,\mathrm{d}\sigma,$$

这里 $S$ 是平面 $x+y+z=1$ 在第一卦限中的部分.

38. 若 $C$ 是沿逆时针方向的简单光滑闭曲线, 证明: 曲线 $C$ 所围区域的面积为

$$A=\frac{1}{2}\oint_C x\,\mathrm{d}y-y\,\mathrm{d}x.$$

并用上述公式来计算曲线

$$\begin{cases} x=2\cos t+\cos(2t) \\ y=2\sin t-\sin(2t) \end{cases} \quad (0\le t\le 2\pi)$$

围成的区域的面积.

39. 设向量场 $\boldsymbol{F}=\left(2xy(x^4+y^2)^k, -x^2(x^4+y^2)^k\right)$, 则当 $k$ 取何值时，向量场存在势函数 $f(x,y)$ (即当 $x>0$, $\boldsymbol{F}=\nabla(f)$)? 并用此 $k$的值来计算

$$\int_{(1,0)}^{(\sqrt{3},1)}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}.$$

40. 计算

$$\oint_C\frac{x\,\mathrm{d}y-y\,\mathrm{d}x}{x^2+y^2}.$$

这里曲线 $C:(x-1)^2+y^2=2$, 沿逆时针方向.

41. 计算
$$\int_C (\mathrm{e}^x \sin y - b(x+y))\,\mathrm{d}x + (\mathrm{e}^x \cos y - ax)\,\mathrm{d}y.$$
这里曲线 $C$ 是从点 $(2,0)$ 沿 $y=\sqrt{2x-x^2}$ 到点 $(0,0)$的上半圆周.

42. 设函数 $f$ 连续可微, $L$ 是从点 $(1,2)$ 到点 $(2,8)$的线段. 计算
$$\int_L (2xy - \frac{2y}{x^2} f(\frac{y}{x^2}))\,\mathrm{d}x + (x^2 + \frac{1}{x^2} f(\frac{y}{x^2}))\,\mathrm{d}y.$$

43. 计算
$$\oint_C (3y - \mathrm{e}^{\sin x})\,\mathrm{d}x + (7x + \sqrt{y^4+1})\,\mathrm{d}y,$$
这里曲线 $C$ 为圆周 $x^2+y^2=9$, 沿逆时针方向.

44. 设向量场 $\boldsymbol{F}=(z^2,-3xy,x^3y^3)$, 曲面 $S$ 是椭圆抛物面 $z=5-x^2-y^2$ 中 $z\geq 1$的部分.
   (1) 计算 $\nabla\times\boldsymbol{F}$.
   (2) 用参数方程表示曲面 $S$.
   (3) 计算 $\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,\mathrm{d}\sigma$, 这里法向量 $\boldsymbol{n}$ 指向 $z$ 增加的方向.

45. 计算
$$\oint_L \mathrm{e}^{\sqrt{x^2+y^2}}\,\mathrm{d}s,$$
这里 $L$ 是由$x^2+y^2=a$, $y=x$ 以及 $x$轴所围区域在第一象限中的边界.

46. 计算
$$\oint_L \frac{(x+y)\,\mathrm{d}x-(x-y)\,\mathrm{d}y}{x^2+y^2},$$
这里 $L$ 是圆周 $x^2+y^2=a^2$，沿逆时针方向.

47. 计算
$$\int_L y\mathrm{e}^{y^2}\,\mathrm{d}x + (x\mathrm{e}^{y^2} + 2xy^2\mathrm{e}^{y^2})\,\mathrm{d}y,$$
这里 $L$ 是 $y=\sqrt[3]{x}$ 上从 $(0,0)$ 到 $(1,1)$的一段曲线.

48. 设 $L$ 是以 $(1,0),(2,0),(2,1)$ 以及 $(1,1)$ 为顶点的正方形的边界, 且沿逆时针方向. 计算
$$\oint_L (x^2+y^2)\,\mathrm{d}x + (x^2-y^2)\,\mathrm{d}y.$$

49. 设 $L$ 是单位球面 $x^2+y^2+z^2=1$ 被平面 $y=x\tan a\ (0<a<\pi)$ 所截的圆周，从 $x$ 轴正向看是沿逆时针方向. 计算
$$\int_L (y-z)\,\mathrm{d}x + (z-x)\,\mathrm{d}y + (x-y)\,\mathrm{d}z.$$

50. 设 $L$ 是球面 $x^2+y^2+z^2=1$ 与第一卦限的交线, 在第一卦限从球面外看是沿逆时针方向. 计算

$$\int_L (y^2-z^2)\,\mathrm{d}x+(z^2-x^2)\,\mathrm{d}y+(x^2-y^2)\,\mathrm{d}z.$$

51. 计算 $\displaystyle\int_L y^2\,\mathrm{d}x+x^2\,\mathrm{d}y$, 这里 $L$ 为

(1) 上半圆周 $x^2+y^2=R^2\ (y\geq 0)$, 沿逆时针方向.

(2) 从 $(R,0)$ 到 $(-R,0)$的线段.

52. 证明下列曲线积分与路径无关, 并求积分值.

(1) $\displaystyle\int_{(0,0)}^{(1,1)}(x-y)\,(\mathrm{d}x-\mathrm{d}y)$.

(2) $\displaystyle\int_{(2,1)}^{(1,2)}\phi(x)\,\mathrm{d}x+\varphi(y)\,\mathrm{d}y$ (这里 $\phi(x)$ 和 $\varphi(y)$ 是连续函数).

(3) $\displaystyle\int_{(1,0)}^{(6,8)}\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{\sqrt{x^2+y^2}}$ (原点不在曲线积分的积分路径上).

53. 证明微分形式 $\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{x^2+y^2}$ 在除了原点以及 $y$ 轴负半轴之外的区域是恰当的, 并求出其势函数.

# Exam # 1

1. (15 pts) Determine whether the following statements are **true** or **false**? No justification is necessary.
   (1) If $k > 0$, then $\ln^{100} x < x^{0.0001} < 2^{kx}$ for sufficiently large $x$.
   (2) If $f$ is continuous on $\mathbf{R}$, then $\displaystyle\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$.
   (3) If the graph of a differentiable function $f(x)$ is concave up on an open interval $(a,b)$, then $f(x)$ has a local minimum value at a point $c \in (a,b)$ if and only if $f'(c) = 0$.
   (4) If $|f(x)|$ is continuous at $x = a$, then so is $(f(x))^2$.
   (5) Suppose that $f(a) = g(a) = 0$, that $f$ and $g$ are differentiable on an open interval $I$ containing $a$, and that $g'(x) \neq 0$ on $I$ if $x \neq a$. If $\displaystyle\lim_{x\to a} \frac{f'(x)}{g'(x)}$ does not exist, then neither does $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$.
2. (15pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)
   (1) If $g(x)$ is one-to-one, and $g(1) = 3$, $g(3) = 1$, $g'(1) = 4$, $g'(3) = 28$, then $(g^{-1})'(3) =$
   (A) $\frac{1}{4}$. (B) $\frac{1}{28}$. (C) $\frac{1}{3}$. (D) 4 .
   (2) Let $c > 0$. How many real roots are there for the equation $x^3 - 6x^2 + 9x + c = 0$ ?
   (A) 0. (B) 1. (C) 2. (D) 3.
   (3) Suppose $\displaystyle\lim_{x\to 0^+} f(x) = a$, $\displaystyle\lim_{x\to 0^-} f(x) = b$, then $\displaystyle\lim_{x\to 0^-} \left(f(x-\sin x) + 2f\left(x^2+x\right)\right) =$
   (A) $a + 2b$. (B) $b + 2a$. (C) $3a$. (D) $3b$ .
   (4) If $f(x) = \frac{\ln|x|}{|x-1|}\sin x$, then the function $f(x)$ has
   (A) 1 removable discontinuity and 1 jump discontinuity.
   (B) 2 removable discontinuities.
   (C) 1 removable discontinuity and 1 infinite discontinuity.
   (D) 2 jump discontinuities.
   (5) Let $f(x)$ be a continuous function, and $a$ is a nonzero constant. Which of the following function is an odd function ?
   (A) $\displaystyle\int_a^x \left(\int_0^u tf(t^2)\,dt\right) du$. (B) $\displaystyle\int_0^x \left(\int_a^u f(t^3)\,dt\right) du$.
   (C) $\displaystyle\int_0^x \left(\int_a^u tf(t^2)\,dt\right) du$. (D) $\displaystyle\int_a^x \left(\int_0^u (f(t))^2\,dt\right) du$.
3. (6 pts) If the function

$$f(x) = \begin{cases} a \cdot \sin x, & x \le \frac{\pi}{4} \\ 1 + b \cdot \tan x, & \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$$

is differentiable at $x = \frac{\pi}{4}$, find the values of $a$ and $b$.

4. (8 pts) Evaluate the following limits.
   (1) $\lim\limits_{x\to 0} \dfrac{\tan^{-1} x - x}{x\tan^2 x}$.
   (2) $\lim\limits_{x\to\infty} \dfrac{(x+100)^{100x}}{x^{100x}}$.
5. (6 pts) Find the area of the region enclosed by the curve $y = |x^2 - 4|$ and $y = \frac{x^2}{2} + 4$.
6. (6 pts) The graph of the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ is an astroid. Find the area of the surface generated by revolving the curve about the $x$-axis.
7. (6 pts) The point $P(a, b)$ lies on the curve $l:\ (y-x)^3 = y + x$, and the slope of the tangent line of $l$ at $P(a, b)$ is 3. Find the values of $a$ and $b$.
8. (6 pts) Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \displaystyle\int_2^{\frac{x^2}{2}} \frac{t}{1+t^4}\, dt$.
9. (16 pts) Evaluate the integrals.
   (1) $\displaystyle\int \frac{dx}{\sqrt{1+e^x}}$.
   (2) $\displaystyle\int \frac{3x+6}{(x-1)^2(x^2+x+1)}\, dx$.
   (3) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2 x \sec x\, dx$.
   (4) $\displaystyle\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{|x-x^2|}}\, dx$.
10. (6 pts) An 1600-L tank is half full of fresh water; i.e., contains 800-L of fresh water. At the time $t = 0$, a solution containing 0.0625 kg/L of salt runs into the tank at the rate of 16 L/min, and the mixture is pumped out of the tank at the rate of 8 L/min. At the time the tank is full, how many kilograms of salt will it contain ?
11. (5 pts) $f(x)$ is differentiable, and $f'(x) > 0$ on $(0, +\infty)$. Let $F(x) = \displaystyle\int_{\frac{1}{x}}^{1} xf(u)\, du + \int_1^{\frac{1}{x}} \frac{f(u)}{u^2}\, du$.
    (1) Identify the open intervals on which $F(x)$ is decreasing and the open intervals on which $F(x)$ is increasing.
    (2) Find the open intervals on which the graph of $y = F(x)$ is concave up and the open intervals on which it is concave down.
12. (5 pts) Let $g$ be a function that is differentiable throughout an open interval containing the origin. Suppose $g$ has the following properties:
    (i) $g(x+y) = \frac{g(x)+g(y)}{1-g(x)g(y)}$ for all real numbers $x$, $y$, and $x + y$ in the domain of $g$.
    (ii) $\lim\limits_{h\to 0} g(h) = 0$.
    (iii) $\lim\limits_{h\to 0} \dfrac{g(h)}{h} = 1$.
    Find $g(x)$.

# Exam # 2

1. (15pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)
   (1) The number of the real roots for the equation $x^3 - 3x + 3 = 0$ is
   (A) 0. (B) 1. (C) 2. (D) 3.
   (2) If $f(x)$ is continuous on $(-\infty, +\infty)$, which of the following statements is **wrong** ?
   (A) $\int_0^1 f(x)\,dx = \int_0^1 f(t)\,dt$. (B) $\int_0^1 f(x)\,dx = \int_0^1 f(\sin x)\,d(\sin x)$.
   (C) $d\left(\int_0^x f(t)\,dt\right) = f(x)\,dx$. (D) $d\left(\int_0^{x^2} f(t)\,dt\right) = f(x^2)\,d(x^2)$.
   (3) Let
   $$f(x) = \begin{cases} x^4 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$
   Then the largest positive integer $n$, for which $f^{(n)}(0)$ exists, is
   (A) 1. (B) 2. (C) 3. (D) 4.
   (4) If $f(x)$ is twice-differentiable on $(-\infty, +\infty)$, and $g(x) = (1-x)f(0) + xf(1)$, then which of the following statements is **correct** on $(0, 1)$ ?
   (A) $f(x) > g(x)$ if $f'(x) > 0$ . (B) $f(x) > g(x)$ if $f'(x) < 0$ .
   (C) $f(x) > g(x)$ if $f''(x) > 0$ . (D) $f(x) > g(x)$ if $f''(x) < 0$ .
   (5) If the improper integral $\displaystyle\int_0^{+\infty} \frac{\tan^{-1}(x^2)}{x^k}\,dx$ converges, then the constant $k$ must satisfy
   (A) $k < 1$. (B) $k > 3$.
   (C) $1 < k < 2$. (D) $1 < k < 3$.
2. (15 pts) Fill in the blanks.
   (1) Function $f(x) = x^2$ has a tangent line $y = Kx - 1$ if $K =$ ______, or ______.
   (2) Assume that $f'(0) = 3$, $f''(0) = 5$, $f'(1) = -4$, and $f''(1) = -7$. Let $g(x) = f(\ln x)$. Then $g''(1) =$ ______.
   (3) The average value for $f(x) = \sin^3 x$ on $[0, \pi]$ is ______.
   (4) Let $y = (\cos x)^x$ for $0 < x < \frac{\pi}{2}$, then $y'(x) =$ ______.
   (5) If $f''(a)$ exists, and $f'(a) \neq 0$, then $\displaystyle\lim_{x\to a}\left(\frac{1}{f'(a)(x-a)} - \frac{1}{f(x)-f(a)}\right) =$ ______.
3. (10 pts) The region $D$ is enclosed by the curve $y = \ln\sqrt{x-1}$, the straight line $x = 5$, and the $x$-axis.
   (1) Find the area of the region $D$.
   (2) Find the volumes generated by revolving the region $D$ about the line $x = 5$.

4. (10 pts) Find the particular solution of

$$xy' + (x-2)y = 3x^3e^{-x}, \quad x > 0,$$

satisfying $y(1) = 0$.

5. (10 pts) Evaluate the following limits.

(1) $\displaystyle\lim_{n\to+\infty}\left(\frac{n}{2n^2+3n+1^2} + \frac{n}{2n^2+6n+2^2} + \cdots \frac{n}{2n^2+3nk+k^2} + \cdots + \frac{n}{2n^2+3n^2+n^2}\right)$.

(2) $\displaystyle\lim_{x\to 0}\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{e^x-1}}$.

6. (10 pts)

(1) For $y = \frac{x^2+1}{x+1}$, identify the coordinates of any local and absolute extreme points and inflection points that may exist.

(2) Sketch the graph of the above function. (Please identify all the asymptotes and some specific points, such as local maximum and minimum points, inflection points, and intercepts.)

7. (10 pts) Find $\frac{dy}{dx}$ if

$$y = \int_{x^2+1}^{2x^2+3} t \tan\sqrt{x+t}\, dt.$$

8. (16 pts) Evaluate the integrals.

(1) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc^3 x\, dx$.

(2) $\displaystyle\int \sqrt{\frac{x}{x-2}}\, dx$, where $x > 2$.

(3) $\displaystyle\int_1^e \ln^3 x\, dx$.

(4) $\displaystyle\int_1^{+\infty} \frac{(x+2)\ln(x^2+1)}{x^3}\, dx$.

9. (4 pts) Let $f(n) = \displaystyle\sum_{m=1}^{n}\int_0^m \cos\frac{2\pi n\lfloor x+1\rfloor}{m}\, dx$, here $\lfloor x+1\rfloor$ is the largest integer which is less than or equal to $x+1$. Evaluate $f(2021)$.

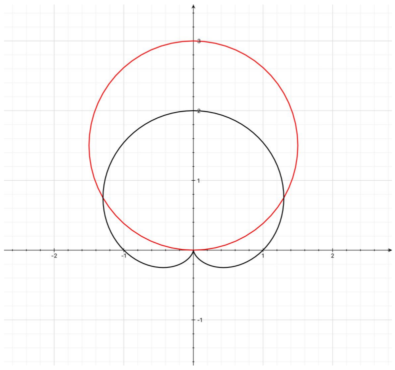
# Exam # 3

1. (9 pts) Determine whether the following statements are **true** or **false**? No justification is necessary.
   (1) Equation $r = 2\sin(\theta)$ $(0 \le \theta \le \pi)$ in polar form is a circle of radius 1 centered at (0,1).
   (2) If $f(x, y) = \sin x + \sin y$, then for any direction $\mathbf{u}$, the directional derivative of $f(x, y)$ satisfies $-\sqrt{2} \le D_{\mathbf{u}} f(x, y) \le \sqrt{2}$.
   (3) If $\mathbf{u} \neq 0$, and if $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
2. (9 pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)
   (1) Let $\mathbf{R} : (x-1)^2 + y^2 \le 1$, then the integral $\iint_{\mathbf{R}} f(x, y)\, dA$ is **not equal to**
   (A) $\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} f(x, y)\, dydx$. (B) $\int_{-1}^{1} \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x, y)\, dxdy$.
   (C) $\int_0^{2\pi} \int_0^1 f(1 + r\cos\theta, r\sin\theta) \cdot rdrd\theta$. (D) $\int_0^{2\pi} \int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta) \cdot rdrd\theta$.
   (2) Which formula satisfies the conditions that function $f(x, y)$ has both partial derivatives at $(0, 0)$ when $f(0, 0) = 0$ ?
   (A) $\frac{xy}{x^2+y^2}$. (B) $\frac{x^2-y^2}{x^2+y^2}$.
   (C) $\sqrt{x^2 + y^2}\ \sin\frac{1}{x^2+y^2}$. (D) $\frac{x^4+y^2}{x^2+y^2}$.
   (3) If $f(x, y) = 3x + 4y - ax^2 - 2ay^2 - 2bxy$ has only local maxima, then
   (A) $2a^2 > b^2$, and $a < 0$. (B) $2a^2 > b^2$, and $a > 0$.
   (C) $2a^2 < b^2$, and $a < 0$. (D) $2a^2 < b^2$, and $a > 0$.
3. (12 pts) Please fill in the blank for the questions below.
   (1) If a plane is tangent to the surface $x^2 - 2y^2 + z^2 = 2$, and parallel to $x - y + 2z = 0$, then the equation of the plane is ______________.
   (2) Let $f(x, y, z) = \left(\frac{x}{y}\right)^{\frac{1}{z}}$, then $df(1, 1, 1) =$ ______________.
   (3) The equation of the plane through the line $x = -1 + 2t, y = 3 + t, z = -t$ and parallel to the line $x = -2t$, $y = t$, $z = 1 - t$ is ______________.
   (4) The circulation of the field $\mathbf{F} = \nabla\,(xy^2z^3)$ around the ellipse

$$C: \ \mathbf{r}(t) = (\cos t)\,\mathbf{i} + (4\sin t)\,\mathbf{j}\,, \quad 0 \le t \le 2\pi,$$

   is ______________.

4. (7 pts) Find the area of region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



5. (7 pts) Find the points on the curve

$$\mathbf{r}(t) = (12\sin t)\,\mathbf{i} - (12\cos t)\,\mathbf{j} + 5t\,\mathbf{k}$$

at a distance $26\pi$ units **along the curve** from the point $(0, -12, 0)$.

6. (8 pts) Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{\ln(n+2)} x^n$.

7. (8 pts) Find the real numbers $a$, $b$ $(b \neq 0)$, which satisfy

$$\lim_{x\to 0} \frac{\cos(\sin x) - \sqrt{1-x^2}}{x^a} = b.$$

8. (8 pts) Find the absolute maximum and minimum values of $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$ on the close disk $x^2 + y^2 \leq 4$.

9. (8 pts) Evaluate the integral $\displaystyle\iiint_D z\sqrt{x^2 + y^2 + z^2}\,dV$, where $D$ is the solid bounded above by $z = 1$ and below by $z = \sqrt{x^2 + y^2}$.

10. (8 pts) Calculate the line integral $\displaystyle\int_L \sin 2x\,dx + 2(x^2 - 1)y\,dy$, here $L$ is the curve $y = \sin x$, from $(0, 0)$ to $(\pi, 0)$.

11. (8 pts) Use the Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve $C$ in the indicated direction, here $\mathbf{F} = y\,\mathbf{i} + xz\,\mathbf{j} + x^2\,\mathbf{k}$ , and $C$ is the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above.

12. (8 pts) Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region $D$, here $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ ; and $D$ is the region cut from the solid cylinder $x^2 + y^2 \leq 4$ by the planes $z = 0$, and $z = 1$ .

# Exam # 4

1. (15 pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)

   (1) Let $a$ be a constant, the series $\displaystyle\sum_{n=2}^{\infty}\left(\frac{\sin(n+a)}{n^{1.01}}-\frac{1}{n\ln n}\right)$

   (A) converges absolutely.

   (B) converges conditionally.

   (C) diverges.

   (D) converges or not depending on the value of $a$.

   (2) The function $f(x,y)=2x^2+5xy+3y^2-7x+10y$ has

   (A) an absolute minimum point. (B) an absolute maximum point.

   (C) a saddle point. (D) none of the above.

   (3) Let $f(x,y)$ be a function which is defined on $D=\{(x,y):x^2+y^2\leq 1\}$. Assume $f(0,0)=0$, $f_x(0,0)=-2$, and $f_y(0,0)=5$, then which of the following statements must be **correct** ?

   (A) $f(x,y)$ is continuous at $(0,0)$.

   (B) The directional derivative of $f$ at $(0,0)$ in the direction of $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ is $\frac{7}{2}\sqrt{2}$.

   (C) $\displaystyle\lim_{y\to 0}f(0,y)=0$.

   (D) $f(x,y)$ is differentiable at $(0,0)$.

   (4) The direction of the gradient for the function $z=\sqrt{1-x^2-y^2}$ at the point $\left(\frac{1}{2},\frac{1}{2}\right)$ is the same with the direction of

   (A) the outward normal vector on the plane curve $x^2+y^2=\frac{1}{2}$ at the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

   (B) the inward normal vector on the plane curve $x^2+y^2=\frac{1}{2}$ at the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

   (C) the outward normal vector on the surface $x^2+y^2+z^2=1$ at the point $\left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)$.

   (D) the inward normal vector on the surface $x^2+y^2+z^2=1$ at the point $\left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)$.

   (5) The region is given by $R:\ x^2+2y^2\leq 4$. Then $\displaystyle\iint_R(4-x^2-2y^2)\,dxdy\ =$

   (A) $4\sqrt{2}\pi$. (B) $8\pi$.

   (C) $8\sqrt{2}\pi$. (D) none of the above.

2. (15 pts) Please fill in the blank for the questions below.

   (1) If a plane $\Pi$ is parallel to $3y+z=2021$ and tangent to the ellipsoid $3x^2+y^2+z^2=10$, then the equation of the plane $\Pi$ is ______________.

   (2) $\displaystyle\lim_{x\to 0}\frac{\sin x-x}{(\cos x-1)(e^{2x}-\cos x)}=$ ______________.

(3) The sum of the series $\frac{1}{2}+\frac{1}{4\cdot 2!}+\frac{1}{8\cdot 3!}+\frac{1}{16\cdot 4!}+\cdots+\frac{1}{2^n\cdot n!}+\cdots$ is ______________.

(4) The area of the region enclosed by $r^2 = \cos 2\theta$ is ______________.

(5) Let $C$ be the curve $x^2 + y^2 = a^2$ $(a > 0)$, then $\int_C x^2\, ds =$ ______________.

3. (10 pts) Find the equation of the plane through point $(1, 0, 1)$, and perpendicular to the plane $x - 2y + 3z + 2 = 0$ and the plane $x + 2y - 3z - 2 = 0$.

4. (10 pts) Find the Maclaurin series for $f(x) = \int_0^{x^2} \frac{1}{1-t}\, dt$, $-1 < x < 1$.

5. (10 pts) If $f(x, y) = \int_0^{xy} e^{-t^2}\, dt$, then $\frac{x}{y} f_{xx} - 2 f_{xy} + \frac{y}{x} f_{yy} = ?$

6. (10 pts) Find

$$J = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_{|x|}^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dydx.$$

7. (10 pts) Find the absolute maximum and minimum values of the function $u = xy + 2yz$ on the surface $x^2 + y^2 + z^2 = 10$.

8. (10 pts) Evaluate the flux of the velocity vector field $\mathbf{F} = xz\,\mathbf{i} + (y^2 + e^{xz})\,\mathbf{j} + \sin(x + y)\,\mathbf{k}$ outward the region bounded above by $z = \sqrt{1 - x^2 - y^2}$, below by $z = \sqrt{x^2 + y^2}$.

9. (10 pts) Calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (y^2 - y)\,\mathbf{i} + (z^2 - z)\,\mathbf{j} + (x^2 - x)\,\mathbf{k}$, and $C$ is the curve of intersection of the sphere $x^2 + y^2 + z^2 = 1$ and $x + y + z = 0$, counterclockwise when viewed from above.

# 试卷 #1

一、 （15分）**判断题：**

(1) 若 $k>0$，那么对充分大的 $x$，必有 $\ln^{100}x<x^{0.0001}<2^{kx}$.

(2) 若函数 $f(x)$ 在 $\mathbf{R}$ 上连续，那么 $\int_0^a f(a-x)\,dx=\int_0^a f(x)\,dx$.

(3) 若可微函数 $f(x)$ 的图形在开区间 $(a,b)$ 上是上凹的，那么 $f(x)$ 在一点 $c\in(a,b)$ 处取得局部极小值当且仅当 $f'(c)=0$.

(4) 若函数 $|f(x)|$ 在 $x=a$ 处连续，则 $(f(x))^2$ 在 $x=a$ 处也连续.

(5) 设 $f(a)=g(a)=0$，函数 $f$ 和 $g$ 在包含 $a$ 的一个开区间 $I$ 上可微，且对任意 $x\in I$，只要 $x\neq a$，必有 $g'(x)\neq 0$. 如果极限 $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ 不存在，则$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ 也不存在.

二、 （15分）**单项选择题：**

(1) 若 $g(x)$ 是一对一的函数，且 $g(1)=3, g(3)=1, g'(1)=4, g'(3)=28$，则 $(g^{-1})'(3)=$

(A) $\frac{1}{4}$. (B) $\frac{1}{28}$. (C) $\frac{1}{3}$. (D) 4 .

(2) 设 $c>0$，方程 $x^3-6x^2+9x+c=0$ 有多少个实根?

(A) 0. (B) 1. (C) 2. (D) 3.

(3) 若 $\lim\limits_{x\to 0^+}f(x)=a$，$\lim\limits_{x\to 0^-}f(x)=b$，则 $\lim\limits_{x\to 0^-}\left(f(x-\sin x)+2f\left(x^2+x\right)\right)=$

(A) $a+2b$. (B) $b+2a$. (C) $3a$. (D) $3b$ .

(4) 设函数 $f(x)=\frac{\ln|x|}{|x-1|}\sin x$，则 $f(x)$ 有

(A) 1 个可去间断点，1个跳跃间断点. (B) 2个可去间断点.

(C) 1个可去间断点，一个无穷间断点. (D) 2个跳跃间断点.

(5) 设 $f(u)$ 为连续函数，$a$ 是非零常数，则为奇函数的是

(A) $\displaystyle\int_a^x\left(\int_0^u tf(t^2)\,dt\right)du$. (B) $\displaystyle\int_0^x\left(\int_a^u f(t^3)\,dt\right)du$.

(C) $\displaystyle\int_0^x\left(\int_a^u tf(t^2)\,dt\right)du$. (D) $\displaystyle\int_a^x\left(\int_0^u (f(t))^2\,dt\right)du$.

三、 （6分）已知函数

$$f(x)=\begin{cases} a\cdot\sin x, & x\le\frac{\pi}{4}\\ 1+b\cdot\tan x, & \frac{\pi}{4}<x<\frac{\pi}{2}\end{cases}$$

在 $x=\frac{\pi}{4}$ 处可导，求常数 $a$，$b$ 的值.

四、 （8分）求下列极限.

(1) $\lim\limits_{x\to 0}\dfrac{\tan^{-1}x-x}{x\tan^2 x}$.

(2) $\lim\limits_{x\to\infty}\dfrac{(x+100)^{100x}}{x^{100x}}$.

五、（6分）求夹在两条曲线 $y=|x^2-4|$ 和 $y=\frac{x^2}{2}+4$ 之间的区域面积.

六、（6分）方程 $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$ 所对应的曲线为一个星形线. 求把此星形线绕 $x$ 轴旋转所形成的旋转面的面积.

七、（6分）点 $P$ 在曲线 $l:(y-x)^3=y+x$ 上，且 $l$ 在 $P$ 处的切线斜率为 3，求点 $P$ 的坐标.

八、（6分）设 $f(x)=e^{g(x)}$，这里 $g(x)=\displaystyle\int_2^{\frac{x^2}{2}}\frac{t}{1+t^4}\,dt$. 求 $f'(2)$.

九、（16分）计算积分.

(1) $\displaystyle\int\frac{dx}{\sqrt{1+e^x}}$.

(2) $\displaystyle\int\frac{3x+6}{(x-1)^2(x^2+x+1)}\,dx$.

(3) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\tan^2 x\sec x\,dx$.

(4) $\displaystyle\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{1}{\sqrt{|x-x^2|}}\,dx$.

十、（6分）一个容积为1600升的蓄水池装有800升的水. 在时间 $t=0$，浓度为每升 0.0625 公斤的盐水以每分钟 16 升的速度流入蓄水池，同时混合液以每分钟 8 升的速度被抽出蓄水池. 请问：当蓄水池正好装满混合液的那一刻，蓄水池内含有多少公斤的盐?

十一、（5分）设函数 $f(x)$ 在区间 $(0,+\infty)$ 上可导，且对任意 $x\in(0,+\infty)$，都有 $f'(x)>0$. 定义函数$F(x)=\displaystyle\int_{\frac{1}{x}}^{1}xf(u)\,du+\int_1^{\frac{1}{x}}\frac{f(u)}{u^2}\,du$.

(1) 求函数 $F(x)$ 的单调区间.

(2) 求函数 $y=F(x)$ 的图形的凹凸区间（即上凹、下凹的开区间）.

十二、（5分）设函数 $g$ 在一个包含原点的开区间上有定义且可微，并且 $g$ 有下列性质：

(i) 对任意在 $g$ 的定义域内的实数 $x,y$ 和$x+y$，满足 $g(x+y)=\frac{g(x)+g(y)}{1-g(x)g(y)}$.

(ii) $\lim\limits_{h\to 0}g(h)=0$.

(iii) $\lim\limits_{h\to 0}\dfrac{g(h)}{h}=1$.

求 $g(x)$.

# 试卷 # 2

一、 （15分）**单项选择题:**

(1) 方程 $x^3 - 3x + 3 = 0$ 的实根个数为

(A) 0. (B) 1. (C) 2. (D) 3.

(2) 设函数 $f(x)$ 在 $(-\infty, +\infty)$ 上连续, 则下列等式中**错误**的是

(A) $\int_0^1 f(x)\,dx = \int_0^1 f(t)\,dt$. (B) $\int_0^1 f(x)\,dx = \int_0^1 f(\sin x)\,d(\sin x)$.

(C) $d\left(\int_0^x f(t)\,dt\right) = f(x)\,dx$. (D) $d\left(\int_0^{x^2} f(t)\,dt\right) = f(x^2)\,d(x^2)$.

(3) 设

$$f(x) = \begin{cases} x^4 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

那么使得 $f^{(n)}(0)$ 存在的最大的正整数 $n$ 是

(A) 1. (B) 2. (C) 3. (D) 4.

(4) 若 $f(x)$ 在 $(-\infty, +\infty)$ 上二阶可导，且 $g(x) = (1-x)f(0) + xf(1)$，则在开区间 $(0, 1)$ 里，下列哪一个叙述是**正确**的?

(A) 当 $f'(x) > 0$ 时，必有 $f(x) > g(x)$.(B) 当 $f'(x) < 0$ 时，必有 $f(x) > g(x)$.

(C) 当 $f''(x) > 0$ 时，必有 $f(x) > g(x)$.(D) 当 $f''(x) < 0$ 时，必有 $f(x) > g(x)$.

(5) 设 $k$ 是常数，若反常积分 $\displaystyle\int_0^{+\infty} \frac{\tan^{-1}(x^2)}{x^k}\,dx$ 收敛，则必有

(A) $k < 1$. (B) $k > 3$.

(C) $1 < k < 2$. (D) $1 < k < 3$.

二、 （15分）**填空题:**

(1) 若 $y = Kx - 1$ 是曲线 $f(x) = x^2$ 的一条切线，则$K =$ ______ 或者______.

(2) 已知 $f'(0) = 3$, $f''(0) = 5$, $f'(1) = -4$, $f''(1) = -7$. 令 $g(x) = f(\ln x)$. 则 $g''(1) =$ ______.

(3) 函数 $f(x) = \sin^3 x$ 在 $[0, \pi]$ 上的平均值为 ______.

(4) 若 $y = (\cos x)^x$，其中 $0 < x < \frac{\pi}{2}$, 则 $y'(x) =$ ______.

(5) 若 $f''(a)$ 存在，且 $f'(a) \neq 0$，则 $\displaystyle\lim_{x \to a}\left(\frac{1}{f'(a)(x-a)} - \frac{1}{f(x) - f(a)}\right) =$ ______.

三、 （10分）设区域 $D$ 是由曲线 $y = \ln\sqrt{x-1}$ 和直线 $x = 5$ 以及 $x$-轴围成.

(1) 求 $D$ 的面积.

(2) 求 $D$ 绕直线 $x = 5$ 旋转一周所成的旋转体的体积.

四、 （10分）求解初值问题

$$xy' + (x-2)y = 3x^3 e^{-x}, \quad x > 0,$$

初始条件为 $y(1) = 0$.

五、 （10分）求下列极限.

(1) $\lim\limits_{n\to+\infty}\left(\dfrac{n}{2n^2+3n+1^2}+\dfrac{n}{2n^2+6n+2^2}+\cdots\dfrac{n}{2n^2+3nk+k^2}+\cdots+\dfrac{n}{2n^2+3n^2+n^2}\right)$.

(2) $\lim\limits_{x\to0}\left(\dfrac{\ln(1+x)}{x}\right)^{\frac{1}{e^x-1}}$.

六、 （10分）

(1) 设 $f(x)=\frac{x^2+1}{x+1}$，求函数的所有（局部）极值、最值以及拐点.

(2) 给 $f(x)$ 画个草图. (请注明所有的极值、最值、拐点、渐近线以及与 $x$ 轴和 $y$ 轴的交点)

七、 （10分）求 $\frac{dy}{dx}$，这里

$$y=\int_{x^2+1}^{2x^2+3} t\tan\sqrt{x+t}\,dt.$$

八、 （16分）计算积分.

(1) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\csc^3 x\,dx$.

(2) $\displaystyle\int\sqrt{\frac{x}{x-2}}\,dx$, where $x>2$.

(3) $\displaystyle\int_1^e\ln^3 x\,dx$.

(4) $\displaystyle\int_1^{+\infty}\frac{(x+2)\ln(x^2+1)}{x^3}\,dx$.

九、 （4分）设 $f(n)=\displaystyle\sum_{m=1}^{n}\int_0^m\cos\frac{2\pi n\lfloor x+1\rfloor}{m}\,dx$，这里 $\lfloor x+1\rfloor$ 表示不超过 $x+1$ 的最大整数. 计算 $f(2021)$.

# 试卷 # 3

一、 （9分）**判断题：**

(1) 极坐标方程 $r = 2\sin(\theta)$ $(0 \le \theta \le \pi)$ 在 $xy$-平面所对应的图形是以 $(0,1)$ 为圆心、半径为 1 的圆.

(2) 设$f(x,y) = \sin x + \sin y$, 则对任意方向 $\mathbf{u}$, 函数 $f(x,y)$ 的方向导数满足$-\sqrt{2} \le D_{\mathbf{u}} f(x,y) \le \sqrt{2}$.

(3) 若 $\mathbf{u} \ne 0$, 且满足 $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ 以及 $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, 则必有 $\mathbf{v} = \mathbf{w}$.

二、 （9分）**单项选择题：**

(1) 设 $\mathbf{R} : (x-1)^2 + y^2 \le 1$, 则积分 $\iint_{\mathbf{R}} f(x,y)\, dA$ 不等于

(A) $\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} f(x,y)\, dydx$.　　(B) $\int_{-1}^1 \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\, dxdy$.

(C) $\int_0^{2\pi} \int_0^1 f(1 + r\cos\theta, r\sin\theta) \cdot r dr d\theta$.　　(D) $\int_0^{2\pi} \int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta) \cdot r dr d\theta$.

(2) 设 $f(0,0) = 0$, 当 $(x,y) \ne (0,0)$ 时，$f(x,y)$ 为如下四式之一，则 $f(x,y)$ 在点 $(0,0)$ 处两个偏导数都存在的是

(A) $\frac{xy}{x^2+y^2}$.　　(B) $\frac{x^2-y^2}{x^2+y^2}$.

(C) $\sqrt{x^2+y^2}\, \sin \frac{1}{x^2+y^2}$.　　(D) $\frac{x^4+y^2}{x^2+y^2}$.

(3) 若 $f(x,y) = 3x + 4y - ax^2 - 2ay^2 - 2bxy$ 只有局部极大值, 则

(A) $2a^2 > b^2$, 且 $a < 0$.　　(B) $2a^2 > b^2$, 且 $a > 0$.

(C) $2a^2 < b^2$, 且 $a < 0$.　　(D) $2a^2 < b^2$, 且 $a > 0$.

三、 （12分）**填空题：**

(1) 与曲面 $x^2 - 2y^2 + z^2 = 2$ 相切, 且与平面 $x - y + 2z = 0$ 平行的平面方程为______________.

(2) 设 $f(x,y,z) = \left(\frac{x}{y}\right)^{\frac{1}{z}}$, 则 $df(1,1,1) =$ ______________.

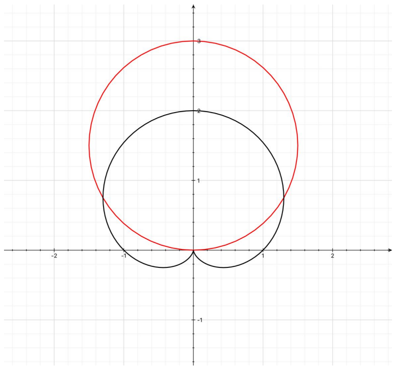
(3) 过直线 $x = -1 + 2t, y = 3 + t, z = -t$ 且平行于直线 $x = -2t, y = t, z = 1 - t$ 的平面方程为______________.

(4) 向量场 $\mathbf{F} = \nabla(xy^2z^3)$ 绕椭圆

$$C: \ \mathbf{r}(t) = (\cos t)\,\mathbf{i} + (4\sin t)\,\mathbf{j}, \quad 0 \le t \le 2\pi,$$

的环量为 ______________.

四、 （7分）设 $D$ 是（如下图所示）在圆 $r = 3\sin\theta$ 的内部，而不在心形线 $r = 1 + \sin\theta$ 的内部，的区域. 求区域 $D$ 的面积.



五、（7分）求在曲线

$$\mathbf{r}(t) = (12\sin t)\,\mathbf{i} - (12\cos t)\,\mathbf{j} + 5t\,\mathbf{k}$$

上且距离点 $(0,-12,0)$ 的弧长为 $26\pi$ 的点的坐标.

六、（8分）求幂级数 $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{\ln(n+2)}x^n$ 的收敛域.

七、（8分）若

$$\lim_{x\to 0}\frac{\cos(\sin x)-\sqrt{1-x^2}}{x^a} = b,$$

这里 $a$、$b$ 为实常数，且 $b \neq 0$, 求 $a$ 和 $b$ 的值.

八、（8分）求函数 $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ 在闭圆盘 $x^2+y^2 \le 4$ 上的最大值和最小值（即全局极大值和全局极小值）.

九、（8分）计算积分 $\displaystyle\iiint_D z\sqrt{x^2+y^2+z^2}\,dV$, 这里 $D$ 是夹在平面 $z = 1$ 和曲面 $z = \sqrt{x^2+y^2}$ 之间的区域.

十、（8分）计算曲线积分 $\displaystyle\int_L \sin 2x\,dx + 2(x^2-1)y\,dy$，其中 $L$ 是曲线 $y = \sin x$ 上从点 $(0,0)$ 到点 $(\pi,0)$ 的一段.

十一、（8分）用Stokes' 定理计算向量场 $\mathbf{F}$ 绕有向闭曲线 $C$ 的环量，这里 $\mathbf{F} = y\,\mathbf{i} + xz\,\mathbf{j} + x^2\,\mathbf{k}$ , 而闭曲线 $C$ 是平面 $x+y+z=1$ 在第一卦限的区域边界，当从上方往下看时，$C$ 是逆时针方向.

十二、（8分）用散度定理计算向量场 $\mathbf{F}$ 通过区域 $D$ 的边界从内向外的通量, 这里 $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ ；区域 $D$ 是圆柱体 $x^2+y^2 \le 4$ 夹在平面 $z=0$ 和 $z=1$ 之间的部分.

# 试卷 # 4

一、 （15分）**单项选择题：**

(1) 设 $a$ 为常数，则级数 $\displaystyle\sum_{n=2}^{\infty}\left(\frac{\sin(n+a)}{n^{1.01}}-\frac{1}{n\ln n}\right)$

(A) 绝对收敛.

(B) 条件收敛.

(C) 发散.

(D) 收敛性与$a$的取值有关.

(2) 函数$f(x,y)=2x^2+5xy+3y^2-7x+10y$ 有

(A) 一个全局极小值点. (B) 一个全局极大值点.

(C) 一个鞍点. (D) 以上都不对.

(3) 设 $f(x,y)$ 是一个定义在 $D=\{(x,y): x^2+y^2\le 1\}$ 上的函数. 若 $f(0,0)=0$, $f_x(0,0)=-2$, 且 $f_y(0,0)=5$, 则下列哪一个叙述是**正确**的?

(A) $f(x,y)$ 在点 $(0,0)$ 处连续.

(B) $f$ 在点 $(0,0)$ 处沿方向 $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ 的方向导数是 $\frac{7}{2}\sqrt{2}$.

(C) $\lim\limits_{y\to 0} f(0,y)=0$.

(D) $f(x,y)$ 在点 $(0,0)$ 处可微.

(4) 函数 $z=\sqrt{1-x^2-y^2}$ 在点 $\left(\frac{1}{2},\frac{1}{2}\right)$ 的梯度方向与下面哪一个向量的方向相同?

(A) 平面曲线 $x^2+y^2=\frac{1}{2}$ 在点 $\left(\frac{1}{2},\frac{1}{2}\right)$ 处的外法向方向.

(B) 平面曲线 $x^2+y^2=\frac{1}{2}$ 在点 $\left(\frac{1}{2},\frac{1}{2}\right)$ 处的内法向方向.

(C) 曲面 $x^2+y^2+z^2=1$ 在点 $\left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)$ 处的外法向方向.

(D) 曲面 $x^2+y^2+z^2=1$ 在点 $\left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)$ 处的内法向方向.

(5) 区域 $R: x^2+2y^2\le 4$，则 $\displaystyle\iint_R (4-x^2-2y^2)\,dxdy=$

(A) $4\sqrt{2}\pi$. (B) $8\pi$.

(C) $8\sqrt{2}\pi$. (D) 以上都不对.

二、 （15分）**填空题：**

(1) 与平面 $3y+z=2021$ 平行, 且与椭球面 $3x^2+y^2+z^2=10$ 相切的平面的方程为____________.

(2) $\displaystyle\lim_{x\to 0}\frac{\sin x-x}{(\cos x-1)(e^{2x}-\cos x)}=$ ____________.

(3) $\displaystyle\frac{1}{2}+\frac{1}{4\cdot 2!}+\frac{1}{8\cdot 3!}+\frac{1}{16\cdot 4!}+\cdots+\frac{1}{2^n\cdot n!}+\cdots$ 的和为 ____________.

(4) 由曲线 $r^2=\cos 2\theta$ 所围成的平面区域的面积为 ____________.

(5) 设 $C$ 为 $x^2+y^2=a^2\ (a>0)$, 那么 $\int_C x^2\,ds=$ ______________.

三、（10分）求通过点 $(1,0,1)$ 且同时垂直于平面 $x-2y+3z+2=0$ 和平面 $x+2y-3z-2=0$ 的平面的方程.

四、（10分）求函数 $f(x)=\int_0^{x^2}\frac{1}{1-t}\,dt,\ -1<x<1,$ 的 Maclaurin 级数.

五、（10分）设 $f(x,y)=\int_0^{xy}e^{-t^2}\,dt$, 则 $\frac{x}{y}f_{xx}-2f_{xy}+\frac{y}{x}f_{yy}=?$

六、（10分）计算

$$J=\int_{\frac{-\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\int_{|x|}^{\sqrt{1-x^2}}\sqrt{1-y^2}\,dydx.$$

七、（10分）求函数 $u=xy+2yz$ 在球面 $x^2+y^2+z^2=10$ 的最大值和最小值.

八、（10分）设速度场为 $\mathbf{F}=xz\,\mathbf{i}+(y^2+e^{xz})\,\mathbf{j}+\sin(x+y)\,\mathbf{k}$，且 $D$ 是夹在曲面 $z=\sqrt{1-x^2-y^2}$（顶部）和曲面 $z=\sqrt{x^2+y^2}$（底部）之间的区域. 求 $\mathbf{F}$ 向外穿过 $D$ 的边界的通量.

九、（10分）计算曲线积分 $\oint_C \mathbf{F}\cdot d\mathbf{r}$, 这里$\mathbf{F}=(y^2-y)\,\mathbf{i}+(z^2-z)\,\mathbf{j}+(x^2-x)\,\mathbf{k}$, 曲线$C$ 为球面 $x^2+y^2+z^2=1$ 与平面 $x+y+z=0$ 的交线，从上往下看， $C$ 是逆时针方向.

# 2020年第十二届全国大学生数学竞赛初赛(非数学类)试题

一、 填空题 (每小题6分, 共30分)

(1) 极限 $\lim\limits_{x\to 0}\frac{(x-\sin x)e^{-x^2}}{\sqrt{1-x^3}-1}=$ ______.

(2) 设函数 $f(x)=(x+1)^n e^{-x^2}$, 则 $f^{(n)}(-1)=$ ______.

(3) 设$y=f(x)$ 是由方程 $\arctan\frac{x}{y}=\ln\sqrt{x^2+y^2}-\dfrac{1}{2}\ln 2+\dfrac{\pi}{4}$ 确定的隐函数, 且满足 $f(1)=1$, 则曲线 $y=f(x)$ 在点 $(1,1)$ 处的切线方程为 ______.

(4) 已知 $\displaystyle\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$, 则 $\displaystyle\int_0^{+\infty}\int_0^{+\infty}\frac{\sin x\sin(x+y)}{x(x+y)}dxdy=$ ______.

(5) 设 $f(x), g(x)$ 在 $x=0$ 的某一邻域 $U$ 内有定义, 对任意 $x\in U, f(x)\neq g(x)$, 且 $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}g(x)=a>0$, 则 $\lim\limits_{x\to 0}\frac{(f(x))^{g(x)}-(g(x))^{g(x)}}{f(x)-g(x)}=$ ______.

二、 (10分) 设数列 $\{a_n\}$ 满足: $a_1=1$, 且 $a_{n+1}=\dfrac{a_n}{(n+1)(a_n+1)}$, $n\geq 1$. 求极限 $\lim\limits_{n\to\infty}n!a_n$.

三、 (12分) 设 $f(x)$ 在 $[0,1]$ 上连续, $f(x)$ 在 $(0,1)$ 内可导, 且$f(0)=0$, $f(1)=1$. 证明:

(1) 存在 $x_0\in(0,1)$ 使得 $f(x_0)=2-3x_0$.

(2) 存在 $\xi,\eta\in(0,1)$, 且$\xi\neq\eta$, 使得 $(1+f'(\xi))(1+f'(\eta))=4$.

四、 (12分) 已知 $z=xf\left(\frac{y}{x}\right)+2y\varphi\left(\frac{x}{y}\right)$, 其中 $f,\varphi$ 均为二阶可微函数.

(1) 求 $\frac{\partial z}{\partial x},\frac{\partial^2 z}{\partial x\partial y}$.

(2) 当 $f=\varphi$, 且 $\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{x=a}=-by^2$ 时, 求 $f(y)$.

五、 (12分) 计算 $I=\displaystyle\oint_\Gamma\left|\sqrt{3}y-x\right|dx-5z\,dz$, 其中$\Gamma:\begin{cases}x^2+y^2+z^2=8\\x^2+y^2=2z\end{cases}$, 从 $z$ 轴正向往坐标原点看去取逆时钟方向.

六、 (12分) 证明 $f(n)=\displaystyle\sum_{m=1}^{n}\int_0^m\cos\frac{2\pi n[x+1]}{m}dx$ 等于 $n$ 的所有因子(包括 1 和 $n$ 本身)之和, 其中 $[x+1]$ 表示不超过 $x+1$ 的最大整数, 并计算 $f(2021)$.

七、 (14分) 设 $u_n=\displaystyle\int_0^1\frac{dt}{(1+t^4)^n}$ $(n\geq 1)$.

(1) 证明数列 $\{u_n\}$ 收敛, 并求极限 $\lim\limits_{n\to\infty}u_n$.

(2) 证明级数 $\sum\limits_{n=1}^{\infty}(-1)^n u_n$ 条件收敛.

(3) 证明当 $p\geq 1$ 时级数 $\displaystyle\sum_{n=1}^{\infty}\frac{u_n}{n^p}$ 收敛, 并求级数 $\displaystyle\sum_{n=1}^{\infty}\frac{u_n}{n}$ 的和.

# 2021年第十三届全国大学生数学竞赛初赛(非数学类)试题

一、 填空题 (本题满分30分,每小题6分).

(1) 极限 $\lim_{x\to+\infty}\sqrt{x^2+x+1}\dfrac{x-\ln(e^x+x)}{x}=$ ______.

(2) 设 $z=z(x,y)$ 是由方程 $2\sin(x+2y-3z)=x+2y-3z$ 所确定的二元隐函数, 则 $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=$ ______.

(3) 设函数 $f(x)$ 连续, 且 $f(0)\neq 0$, 则 $\lim_{x\to 0}\dfrac{2\int_0^x(x-t)f(t)\,dt}{x\int_0^x f(x-t)\,dt}=$ ______.

(4) 过三条直线

$$L_1:\begin{cases}x=0\\ y-z=2\end{cases},\qquad L_2:\begin{cases}x=0\\ x+y-z+2=0\end{cases},\qquad \text{与 } L_3:\begin{cases}x=\sqrt{2}\\ y-z=0\end{cases}$$

的圆柱面方程为 ______.

(5) 记 $D=\left\{(x,y)|\,x^2+y^2\leq\pi\right\}$, 则 $\iint\limits_D\left(\sin x^2\cos y^2+x\sqrt{x^2+y^2}\right)dxdy=$ ______.

二、 (14分) 设 $x_1=2021, x_n^2-2(x_n+1)x_{n+1}+2021=0\,(n\geq 1)$, 证明数列 $\{x_n\}$ 收敛, 并求极限 $\lim_{n\to\infty}x_n$.

三、 (14分) 设 $f(x)$ 在 $[0,+\infty)$ 上是有界连续函数, 证明: 方程 $y''+14y'+13y=f(x)$ 的每一个解在 $[0,+\infty)$ 上都是有界函数.

四、 (14分) 对于4次齐次函数

$$f(x,y,z)=a_1x^4+a_2y^4+a_3z^4+3a_4x^2y^2+3a_5y^2z^2+3a_6x^2z^2$$

计算曲面面积 $\oiint\limits_\Sigma f(x,y,z)\,dS$, 其中 $\Sigma: x^2+y^2+z^2=1$.

五、 (14分) 设函数 $f(x)$ 在闭区间 $[a,b]$ 上有连续的二阶导数, 证明:

$$\lim_{n\to\infty}n^2\left(\int_a^b f(x)dx-\frac{b-a}{n}\sum_{k=1}^n f\left(a+\frac{2k-1}{2n}(b-a)\right)\right)=\frac{(b-a)^2}{24}\left(f'(b)-f'(a)\right).$$

六、 (本题满分14分) 设 $\{a_n\}$ 与 $\{b_n\}$ 均为正实数列, 满足: $a_1=b_1=1$, 且 $b_n=a_nb_{n-1}-2$, $n=2,3,\cdots$. 又设 $b_n$ 为有界数列, 证明级数 $\sum_{n=1}^{+\infty}\dfrac{1}{a_1a_2\cdots a_n}$ 收敛, 并求该级数的和.

# 中英文词汇对照

**A**

abscissa axis 横坐标轴

absolute convergence 绝对收敛

absolute extrema 最值

absolute (global) maximum 最大值

absolute (global) minimum 最小值

absolute value 绝对值

acute angle 锐角

acute triangle 锐角三角形

addition 加法

alternative series 交错级数

amplitude 振幅

angle 角

antiderivative 原函数

applied optimizations 应用优化

arc length 弧长

arccosine function 反余弦函数

arcsine function 反正弦函数

arctangent function 反正切函数

arithmetic mean 算术平均数

associative law 结合律

astroid 星形线

asymptote 渐近线

axiom 公理

**B**

base of logarithmic function

对数函数的底

binary system 二进制

binomial 二项式

bisection method 二分法

boundary point 边界点

bounded function 有界函数

bounded interval 有界区间

bounded region 有界区域

brace 大括号

brachistochrone 最速降线

**C**

calculus 微积分

Cantor set 康托尔集

cardioid 心形线

Cartesian coordinates 笛卡尔坐标

Cauchy’s mean value theorem

柯西中值定理

Cauchy-Schwarz inequality

柯西-施瓦茨不等式

center of mass 质心

centroid 矩心

chain rule 链式法则

circumcircle 外接圆

circle 圆

circulation density 环流当量密度

circumscribed 外切的

closed interval 闭区间

closed region 闭区域

coefficient 系数

common factor 公因子

complex number 复数

compliment 补集

composite function 复合函数

concave 凹的

concave down 下凹

concave up 上凹

concavity 凹凸性

conditional convergence 条件收敛

cone 锥面

congruent 全等

conjugate 共轭

connected 连通的

conservative field 保守场

conservation law 守恒律

constant 常数，常量

continuous 连续的

contour curve 等高线

contradiction 矛盾

convergent 收敛的

converse proposition 逆命题

convex 凸的

coordinate planes 坐标平面

coordinate system 坐标系

coplanar 共面的

corner 拐角

corollary 推论

cosecant function 余割函数

cosine function 余项函数

cotangent function 余切函数

critical point 临界点

cross product 叉乘，向量积，外积

cross-sections 截面

cube 立方体，三次方

cube root 三次方根，立方根

cubic function 三次函数

curl 旋度

curvature 曲率

cusp 尖点

cycloid 摆线，旋轮线

cylinder 柱面

cylindrical coordinate 柱坐标

**D**

Decay 衰减

decimal 小数

decimal point 小数点

decimal system 十进制

definite integral 定积分

degree 角度

denominator 分子

dependent variable 因变量

derivative 导数

determinant 行列式

diameter 直径

difference 差

differential 微分

dimensionality 维数

directional derivative 方向导数

discontinuity 间断

discriminant 判别式

disjoint 不相交的

distributive law 分配律

divergent 发散的

divergence theorem

散度定理，高斯公式

dividend 被除数

division 除法

divisor 除数

domain 定义域

dot product 数量级，点乘，内积

double integral 二重积分

double-angle formula 二倍角公式

**E**

element 元素

ellipse 椭圆

ellipsoid 椭球面

elliptical cone 椭圆锥

elliptical cylinder 椭圆柱面

elliptical paraboloid 椭圆抛物面

endpoint 终点，端点

equation 方程式

equilateral polygon 正多边形

equilateral triangle 等边三角形

even number 偶数

even function 偶函数

evolution 开方

exact differential form 恰当微分形式

expected value 期望值

exponent 指数

exponential function 指数函数

extreme point 极值点

**F**

factorial 阶乘

Fibonacci number 斐波那契数

first derivative 一阶导数

first order differential equation 一阶微分方程

fraction 分数

frequency 频率

frustum of a cone 圆台

function 函数

fundamental theorem of Calculus 微积分基本定理，牛顿莱布尼兹公式

**G**

game theory 博弈论

general solution 通解

geometric series 几何（等比）级数

global maximum 最大值

global minimum 最小值

gradient 梯度

gravitational field 重力场

Green’s theorem 格林定理

**H**

half angle formula 半角公式

harmonic function 调和函数

harmonic series 调和级数

heat equation 热传导方程

height 高

hemisphere 半球

hexadecimal system 十六进制

hexagon 六边形

horizontal asymptote 水平渐近线

hydrodynamics 流体力学，水动力学

hyperbola 双曲线

hyperbolic cylinder 双曲柱面

hyperbolic paraboloid 双曲抛物面

hyperboloid of one sheet 单叶双曲面

hyperboloid of two sheet 双叶双曲面

hypotenuse 斜边

hypothesis 假设

**I**

identity function

单位函数，恒等函数

imaginary number 虚数

implicit differentiation 隐微分法

implicit function 隐函数

implicit function theorem 隐函数定理

improper integral 反常积分

independent variable 自变量

indeterminate form 不定式

induction 归纳法

inertia 惯性

infinite discontinuity 无穷间断点

infinite sequence 无穷序列

infinite series 无穷级数

infinitesimal 无穷小

infinity 无穷大

inflection point 拐点

integer 整数

integral 积分

integration by parts 分部积分

interior point 内点

intermediate value theorem 介值定理

intersection 交点，交集

interval of convergence 收敛域

inverse function 反函数

irrational number 无理数

irreducible polynomial 不可约多项式

inscribed circle 内切圆

isosceles trapezoid 等腰梯形

iterated integral 累次积分

**J**

Jacobian determinant 雅克比行列式

joint 连接

jump discontinuity 跳跃间断点

**K**

kinetic energy 动能

**L**

Lagrange multiplier 拉格朗日乘子法

Laplace equation 拉普拉斯方程

law of association 结合律

law of commutation 交换律

law of distribution 分配律

left-handed derivative 左导数

leg （直角三角形的）直角边

lemma 引理

length 长

level curve 等高线

level surface 等值面

L'Hôpital's rule 洛必达法则

limit 极限

line 线

linear approximation 线性近似

linear equation 线性方程

linear transformation 线性变换

linearization 线性化

line integral 曲线积分

line segment 线段

local extreme 局部极值，极值

local maximum 极大值

local minimum 极小值

locus 轨迹

logarithmic function 对数函数

lower bound 下界

lower sum 下和

**M**

Maclaurin series 麦克劳林级数

mapping 映射

mathematical induction 数学归纳法

matrix 矩阵

mean value theorem 中值定理

midpoint 中点

minus 减

Möbius band 莫比乌斯带

monotonic function 单调函数

monotonic sequence 单调序列

monotonicity 单调性

multiple integral 重积分

multiplication 乘法

**N**

natural exponential function 自然指数函数

natural logarithm function 自然对数函数

natural number 自然数

negate proposition 否命题

negative number 负数

Newton’s method 牛顿法

nondecreasing sequence 非减序列

non-increasing sequence 非增序列

nonlinear 非线性

normal distribution 正态分布

normal line 法线

normal plane 法平面

normal vector 法向量

number 数

numerator 分母

numerical integration 数值积分

**O**

oblique asymptote 斜渐近线

obtuse angle 钝角

obtuse triangle 钝角三角形

octagon 八边形

octant 卦限

odd number 奇数

odd function 奇函数

one-sided derivative 单边导数

one-sided limit 单边极限

one-to-one function 一对一函数

operator 运算符

optimize 最优化

ordinary differential equation 常微分方程

oriented 有向的

origin 原点

orthogonal 正交的

oscillating discontinuity 振荡间断点

osculating circle 曲率圆

**P**

parabola 抛物线

paraboloid 抛物面

parallel 平行

parallel lines 平行线

parallel planes 平行面

parallelepiped 平行六面体

parallelogram 平行四边形

parametric curve 参数曲线

parametric equation 参数方程

parametrization 参数化

parity 奇偶性

partial derivative 偏导数

partial fraction 部份分式

partial sum 部分和

particular solution 特解

path independence 与路径无关

pentagon 五边形

period 周期

periodic function 周期函数

periodicity 周期性

perpendicular 垂直的

phase 相位

piecewise-continuous 分段连续

piecewise-smooth 分段光滑

plane 面

plane curve 平面曲线

point 点

polar coordinate 极坐标

polygon 多边形

polyhedron 多面体

polynomial 多项式

positive number 正数

potential function 势函数

power 乘方

power series 幂级数

prime number 质数

probability 概率

product 乘积

projectile 投射物

projection 投影

proportional 成比例的

proposition 命题

pythagorean theorem

毕达哥拉斯定理，勾股定理

**Q**

quadrant 象限

quadratic polynomial 二次多项式

quadric surface 二次曲面

quadrilateral 四边形

quotient 商

**R**

radial 射线

radian 弧度

radical sign 根号

radius 半径

radius of convergence 收敛半径

random variable 随机变量

range 值域

ratio 比例，比率

rational function 有理函数

rational number 有理数

real number 实数

reciprocal 倒数

rectangle 长方形

rectangular coordinates 直角坐标

recursion formula 递推公式

recursive definition 递归定义

reduction of a fraction 约分

reduction to a common denominator

通分

regression analysis 回归分析

relative extrema 极值

removable discontinuity 可去间断点

revolution 旋转

Riemann sum 黎曼和

right angle 直角

right trapezoid 直角梯形

right triangle 直角三角形

right-handed derivative 右导数

Rolle's theorem 罗尔定理

rotation 旋转

**S**

saddle point 鞍点

sandwich theorem 夹逼定理

scalar function 标量函数

scalar multiplication 数乘，点乘

secant function 正割函数

secant line 割线

semicircle 半圆

sequence 序列

series 级数

set 集合

shearing flow 剪流

side 边

significant digit 有效数字

similar 相似

simply connected 单连通的

Simpsons rule 辛普森公式

sine function 正弦函数

slant line asymptote 斜渐近线

slope 斜率

solid 体

sphere 球

spherical coordinate 球坐标

square 平方，正方形

square root 平方根

squeeze theorem 夹逼定理

standard deviation 标准差

Stokes' theorem 斯托克斯定理

subset 子集

substitution 换元

subtraction 减法

surface area 表面积

surface integral 曲面积分

symmetric 对称的

**T**

tangent function 正切函数

tangent line 切线

tangent plane 切平面

tangent vector 切向量

Taylor polynomial 泰勒多项式

Taylor series 泰勒级数

Taylor's formula 泰勒公式

term-by-term 逐项

terminal point 终点，端点

tetrahedron 四面体

theorem 定理

torque 转矩，扭矩

torsion 扭转

total differential 全微分

transcendental function 超越函数

transcendental number 超越数

trapezoid 梯形

trapezoidal rule 梯形公式

triangle 三角形

trigonometric function 三角函数

**U**

unbounded 无界的

uniform distribution 均匀分布

uniformly continuous 一致连续

union of set 并集

unique solution 唯一解

unit 单元，单位

upper bound 上界

upper sum 上和

**V**

variable 变量

variance 方差，偏差

vector 向量

velocity 速度

vertex 顶点

vertical 垂直的

vertical asymptote 垂直渐近线

void 空集

volume 体积

**W**

wave equation 波动方程

weighted 加权的

width 宽

# 中英文相关概念的细微区别

本节所有的比较都是针对 Thomas’Calculus (13th edition) [2] 和同济版的高等数学（第七版）[1] 在相关概念的细微差别.

## (1) 关于（局部）极值的区别

在[2] 中第201页，local (relative) maximum 的定义为：

“ A function $f$ has a local maximum value at a point $c$ within its domain $D$ if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c. ”

而在[1] 上册中第153页，极大值的定义为：

“ 设函数 $f(x)$ 在点 $x_0$ 的某邻域$\mathrm{U}(x_0)$ 内有定义，如果对于去心邻域 $\mathring{\mathrm{U}}(x_0)$ 内的任一 $x$，有

$$f(x) < f(x_0),$$

那么就称 $f(x_0)$ 是函数 $f(x)$ 的一个极大值. ”

**注意：** 在[2] 中，条件是$f(x) \leq f(c)$；而在[1] 中，条件则变为$f(x) < f(x_0)$，缺少了等号。所以对一些判断题就会造成答案不一样的情况，比方说对于常值函数，用英文教材[2]的定义，每个点都是极大值（极小值），而在中文教材[1] 就没有局部极值.

除此之外，关于闭区间的边界是否为极大值的问题，在[2] 中第201页，有如下补充说明：

“ If the domain of $f$ is the closed interval $[a, b]$, then $f$ has a local maximum at the endpoint $x = a$, if $f(x) \leq f(a)$ for all $x$ in some half-open interval $[a, a + \delta)$, $\delta > 0$, a local maximum at the endpoint $x = b$, if $f(x) \leq f(b)$ for all $x$ in some half-open interval $(b - \delta, b]$, $\delta > 0$.”

**注意：** 用英文教材[2]的定义，边界点可以为极值点；而在中文教材[1]中，则没有专门提及闭区间端点的情况，由于在定义里要求是在开邻域内考虑，所以闭区间端点

不是极值点。更进一步，用英文教材[2]，最大值一定是极大值（因为边界点可以为极值点），所以如果一个函数在闭区间上存在最大值，就一定存在极大值；而中文教材则可能出现存在最大值却不存在极大值的情况（最大值点是区间端点）.

**注意：** 上述关于极大值的区别同样出现在极小值的情况.

### (2) 关于临界点（驻点）的区别

在[2] 中第203页，critical point（翻译过来应为临界点）的定义为：

“ An interior point of the domain of a function $f$ where $f'$ is zero or undefined is a critical point of $f$.”

而在[1] 上册中第126页，驻点（临界点）的定义为：

“通常称导数等于零的点为函数的驻点（或稳定点，临界点）.”

**注意：** 用英文教材[2]的定义，临界点是指 $f' = 0$ 的点或者导数不存在的点；而在中文教材[1]中，则专指 $f' = 0$ 的点（不包括导数不存在的情况）.

### (3) 关于拐点的区别

在[2] 中第219页，a point of inflection（拐点）的定义为：

“ A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a point of inflection. ”

而在[1] 上册中第149页，拐点的定义为：

“一般地，设 $y = f(x)$ 在区间 $I$ 上连续，$x_0$ 是 $I$ 内的点. 如果曲线 $y = f(x)$ 在经过点 $(x_0, f(x_0))$ 时，曲线的凹凸性改变了，那么就称点 $(x_0, f(x_0))$ 为这曲线的拐点.”

**注意：** 用英文教材[2]的定义，除了要求在这一点的左右，函数的凹凸性发生了变化，还要求在这一点的切线存在（包括垂直切线（vertical tangent））；而在中文教材[1]中，只要求在这一点的左右，函数的凹凸性发生了变化，换句话说，函数在这一

点的左导数和右导数可能不相等.

### (4) 关于收敛区间的区别

在[2] 中第219页，interval of convergence 的定义为:

“ $R$ is called the radius of convergence of the power series, and the interval of radius $R$ centered at $x = a$ is called the interval of convergence. The interval of convergence may be open, closed, or half-open, depending on the particular series.”

而在[1] 下册中第275页，收敛区间的定义为:

“ 正数 $R$ 通常叫做幂级数的收敛半径. 开区间 $(-R, R)$ 叫做幂级数的收敛区间. ”

在[1] 下册中第273页，收敛域的定义为:

“函数项级数的收敛点的全体称为它的收敛域.”

**注意:** 在中文教材[1]中，收敛区间是开区间，而收敛域则要考虑在区间端点的收敛性；所以虽然interval of convergence 按照直译是收敛区间，但是根据在英文教材[2] 里的定义，应该翻译为中文教材中的收敛域.

### (5) 关于方向导数的区别

在[2] 中第833页，directional derivative (方向导数) 的定义为:

“ The derivative of $f$ at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is the number

$$\left(\frac{df}{ds}\right)_{\mathrm{u},P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s},$$

provided the limit exists. ”

而在[1] 下册中第104页，在方向导数的定义则为单边极限，即

$$\left(\frac{df}{ds}\right)_{\mathrm{u},P_0} = \lim_{s\to 0^+} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}.$$

**注意：** 用英文教材[2]的定义，圆锥的顶点处对于任何方向都不存在方向导数；而用中文教材[1]的定义，任何方向的方向导数在此处都存在.

## References

[1] 同济大学数学系. 高等数学. 高等教育出版社, 7th edition, 2014.

[2] G. B. Thomas, M. D. Weir, and J. Hass. Thomas' Calculus. Pearson Education Limited, 13th edition, 2016.