南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 高等数学(上) A　　**开课单位:** 数 学 系

考试时长: 120 分钟　　**命题教师:** 王融 等

题 号	1	2	3	4	5	6	7	8	9	10
分 值	15 分	15 分	10 分	10 分	10 分	9 分	9 分	9 分	8 分	5 分

1. (15 pts) Determine whether the following statements are **true** or **false**? No justification is necessary.

   (1) If $f'(x)$ is bounded on $(0,1)$，so is $f(x)$. T

   (2) Let $f(x)$ is defined on $(-\infty,+\infty)$. There must be a local maximum point of $f(x)$ between two local minimum points of $f(x)$. F

   (3) If $f(x)$ is differentiable on $(-1,1)$, and $f(-1)=f(1)$, then $f'(c)=0$ for some number $|c|<1$. ~~T~~ F continuous on $[-1,1]$

   (4) If $f(x)$ is a continuous, even function on $[-1,1]$, then $g(x)=\int_0^x f(t)\,dt$ is odd and differentiable on $[-1,1]$. 偶 T ✓

   (5) If $f(x)$ is a continuous, periodic function on $\mathbf{R}$ (T is the period), then $g(x)=\int_0^x f(t)\,dt$ is also a periodic function with the period T. F

Solution: (1) T; (2) F; (3) F; (4) T; (5) F.

2. (15pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)

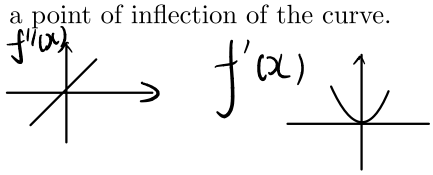
   (1) Which of the following functions is not differentiable at $x=0$? D
   (A) $|x|\sin|x|$. (B) $|x|\sin(\sqrt{|x|})$. (C) $\cos|x|$. (D) $\cos\sqrt{|x|}$.

   (2) Suppose that $f(x)$ is differentiable at $x=0$ and $f(0)=0$. Then $\lim\limits_{x\to0}\frac{x^2f(x)-2f(x^3)}{x^3}=$ B
   (A) $-2f'(0)$. (B) $-f'(0)$. (C) $f'(0)$. (D) 0.

   $\lim\limits_{x\to0}\frac{f(x)}{x}-\frac{2f(x^3)}{x^3}=-f'(0)$

   (3) Suppose that $f(x)$ has a second derivative and $f'(0)=0$, $\lim\limits_{x\to0}\frac{f''(x)}{x}=1$. Then
   (A) $f(0)$ is a local minimum value. ✗
   (B) $f(0)$ is a local maximum value. ✗
   (C) $(0,f(0))$ is a point of inflection of the curve. ✓
   (D) $(0,f(0))$ is neither a local extrema nor a point of inflection of the curve.

   $f''(0)=0$
   $f'''(0)=1$ C



(4) Suppose that $f(x)$ is defined on $(-\infty, +\infty)$. Which of the following statements is equivalent to the statement that "$f(x)$ is differentiable at $x = a$" ? D

(A) $\lim\limits_{h\to 0}(f(a+h) + f(a-h) - 2f(a)) = 0$.

(B) $\lim\limits_{h\to 0}\dfrac{f(a+h) - f(a-h)}{2h}$ exists.

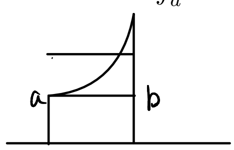
(C) $\lim\limits_{h\to 0}\dfrac{f(a+h^2) - f(a)}{h^2}$ exists. 不能保证左右极限一致，

(D) $\lim\limits_{h\to 0}\dfrac{f(a+h^3) - f(a)}{h^3}$ exists.

(5) Suppose that $f(x) > 0$, $f'(x) > 0$, and $f''(x) > 0$ for all $x \in [a, b]$. Let $M = \displaystyle\int_a^b f(x)\,dx$, $N = f(a)(b-a)$, and $P = \frac{f(a)+f(b)}{2}(b-a)$. Then B

(A) $N < P < M$.　　$P > M > N$　　(B) $N < M < P$.

(C) $M < N < P$.　　(D) $M < P < N$.



Solution: (1) D; (2) B; (3) C; (4) D; (5) B.

3. (10 pts) Let $f(x) = \frac{x^3}{x^2+1}$.

(1) Identify the inflection points and local maxima and minima of the function that may exist.

(2) Identify the horizontal, vertical, or oblique asymptotes that may exist.

(3) Graph the function.

Solution:

(1)

$$f'(x) = \frac{x^2(x^2+3)}{(x^2+1)^2}, \qquad f''(x) = \frac{2x(3-x^2)}{(x^2+1)^3}.$$

There is no local extrema. There are three inflection points on $(-\sqrt{3}, -\frac{3}{4}\sqrt{3})$, $(0,0)$, $(\sqrt{3}, \frac{3}{4}\sqrt{3})$.

(2) There exists an oblique asymptote on $y = x$.

4. (10 pts) Find the limits.

(1) $\lim\limits_{x\to 1}\left(\dfrac{\sin 5x}{x} + \dfrac{x^3+x^2-2}{x^2+2x-3}\right)$. $= \sin 5 + \dfrac{\cancel{(x-1)}(x^2+2x+2)}{(x+3)\cancel{(x-1)}} = \sin 5 + \dfrac{5}{4}$

(2) $\lim\limits_{n\to\infty}\dfrac{1}{n}\left(\sqrt{1-\left(\dfrac{1}{n}\right)^2} + \sqrt{1-\left(\dfrac{2}{n}\right)^2} + \cdots + \sqrt{1-\left(\dfrac{n}{n}\right)^2}\right)$.

$= \displaystyle\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$

Solution:

(1) $\sin 5 + \frac{5}{4}$.

(2) $\frac{\pi}{4}$.

5. (10 pts) Evaluate the definite integral.

$= \int_{-1}^{a}(a-t)t^2\,dt + \int_{a}^{1}(t-a)t^2\,dt$

(1) $\displaystyle\int_{-1}^{1} |a-t|t^2\,dt$, where $a \in (-1,1)$. $= \frac{1}{6}a^4 + \frac{1}{2}$

$u = 1-\cos x$ $\quad du = \sin x\,dx$

(2) $\displaystyle\int_{0}^{\pi} \frac{\overset{=2\sin x\cos x}{\sin 2x}}{\sqrt{1-\cos x}}\,dx. = \int_0^2 \frac{2(1-u)}{\sqrt{u}}\,du = 2\int_0^2 \left((u)^{-\frac{1}{2}} - u^{\frac{1}{2}}\right)du = 2\left[\left[2u^{\frac{1}{2}}\right]_0^2 - \frac{2}{3}u^{\frac{3}{2}}\right]_0^2$

$= 2\left(2\sqrt{2} - \frac{2}{3}\times 2\sqrt{2}\right) = \frac{4\sqrt{2}}{3}$

Solution:

(1) $\displaystyle = \int_{-1}^{a}(a-t)t^2\,dt + \int_{a}^{1}(t-a)t^2\,dt = \frac{1}{6}a^4 + \frac{1}{2}.$

(2) Let $u = 1-\cos x$.

$$\int_0^2 \frac{2(1-u)}{\sqrt{u}}\,du = \left(4\sqrt{u} - \frac{4}{3}u^{\frac{3}{2}}\right)_0^2 = \frac{4}{3}\sqrt{2}.$$

6. (9 pts) Find the volume of the solid generated by revolving the region bounded by $x = 12(y^2-y^3)$ $(0 \le y \le 1)$ and y-axis about the line $y = 2$.

Solution:

$$\int_0^1 2\pi 12(y^2-y^3)(2-y)\,dy = 24\pi\int_0^1 (y^4-3y^3+2y^2)\,dy = \frac{14}{5}\pi.$$

7. (9 pts) Use the linear approximation of $f(x) = \tan x$ at $a = \frac{\pi}{6}$ to estimate the value of $\tan\frac{11\pi}{60}$. Comparing the estimation with the true value, which one is larger ?

$f'(\frac{\pi}{6}) = \sec^2\frac{\pi}{6} = \frac{4}{3}$

$L(x) = \frac{1}{\sqrt{3}} + \frac{4}{3}(x-\frac{\pi}{6})$

$f(\frac{11\pi}{60}) \approx \frac{1}{\sqrt{3}} + \frac{4}{3}\frac{\pi}{60} = \frac{1}{\sqrt{3}} + \frac{\pi}{45}$

concave up $\Rightarrow$ true > estimation

Solution: The linear approximation is

$$L(x) = \frac{1}{\sqrt{3}} + \frac{4}{3}(x-\frac{\pi}{6}).$$

Thus $\tan\frac{11\pi}{60} \approx \frac{1}{\sqrt{3}} + \frac{\pi}{45}$. Because $f(x)$ is concave up on $\left(0, \frac{\pi}{2}\right)$, the true value is larger than the estimation.

8. (9 pts) Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y-1)^2$, and above right by the line $x = 3-y$.

Solution:

$$\int_0^1 \left((1+\sqrt{x}) - \frac{x^2}{4}\right)dx + \int_1^2 \left((3-x) - \frac{x^2}{4}\right)dx = \frac{19}{12} + \frac{11}{12} = \frac{5}{2}.$$

9. (8 pts) Let $F(x) = \displaystyle\int_{2019}^{x^2} \cos(2t^2)\,dt$. Find all the critical points for $F(x)$ on $[-1,1]$.

Solution: $F'(x) = \cos(2x^4)\cdot 2x = 0$. Thus $x = 0, \pm\frac{\sqrt[4]{4\pi}}{2}$ $\quad F'(x) = \cos(2x^4)\cdot 2x$

10. (5 pts) (**Use Rolle's theorem to prove the mean value theorem.**) If the function $f(x)$ is continuous on $[a,b]$, and differentiable on (a,b), prove that there exists a number c in (a,b), such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$