Step-1

Consider the matrix,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

The objective is to find $PA = LDU$ factorization.

Step-2

In the matrix A, the pivot is zero, 0.

So, to change the pivot entry, exchange the first and second rows so that the entry 1 in the second row moves into the pivot.

For this row exchange, use the permutation matrix P which is the same as an identity matrix.

So, the PA matrix will be as follows:

$$\begin{aligned} PA &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \end{aligned}$$

Here, the permutation matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Step-3

Find the lower and upper (L and U) triangular matrices for the matrix PA.

Find the upper triangular matrix U for the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

Step 1: Subtract two times the first row from the third row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad R_3 \to R_3 - 2R_1 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

Step 2: Subtract three times the second row from the third row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \quad R_3 \to R_3 - 3R_2 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Therefore, the upper triangular matrix U is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

Step-4

Find the lower triangular matrix L for the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

To find the matrix L, do the same row operations reversely on the identity matrix I.

Step 1: Add two times the first row from the third row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 + 2R_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Step 2: Add three times the second row from the third row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 + 3R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Therefore, the lower triangular matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$.

Step-5

Find the matrix D:

Use the fact that D is the diagonal matrix of pivots, and the pivots are the diagonal entries of the upper triangular matrix U.

Here, the pivots are $1, 1, -1$ because, $1, 1, -1$ are the diagonal entries of the upper triangular matrix U.

So, the matrix D is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Step-6

The LU factorization is the following:

$$PA = LU$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Step-7

To make the diagonal entries all 1 in the matrix U, one needs to split LU into LDU such that $DU' = U$ as follows:

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Here, L and U are the lower and upper triangular matrices which contain 1 on diagonal

D is the diagonal matrices which consists pivots 1, 1,-1 on the diagonal.

Therefore, write the $PA = LDU$ factorizations as follows:

$$\boxed{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Step-8

Check:

Find LDU :

$$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$
$$= PA$$

Therefore, by row column multiplication of the above two sides we get the same matrices.

Step-9

Consider the second matrix,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

In the matrix A, the pivot is 4.

So, to change the pivot entry, exchange the second and third rows, so that the entry 1 in the third row moves into the pivot.

So, PA matrix will be the following:

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

Here, the permutation matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Step-10

Find the lower and upper (L and U) triangular matrices for the matrix PA.

Find the upper triangular matrix U for the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$.

Step 1: Subtract one the first row from the second row.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \quad R_2 \to R_2 - R_1 \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 4 & 2 \end{bmatrix}$$

Step 2: Subtract two times the first row from the third row.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 4 & 2 \end{bmatrix} \quad R_3 \to R_3 - 2R_1 \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the upper triangular matrix U is the following:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Step-11

Find the lower triangular matrix L for the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$:

To find the matrix L, do the same row operations reversely on the identity matrix I.

Step-12

Step 1: Add one the first row to the second row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 + R_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2: Add two times the first row to the third row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 + 2R_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Therefore, the lower triangular matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

Step-13

Find the matrix D:

Use the fact that *D* is the diagonal matrix of pivots, and the pivots are the diagonal entries of the upper triangular matrix *U*.

Here, the pivots are $1,-1,0$ because, $1,-1,0$ are the diagonal entries of the upper triangular matrix *U*.

So, the matrix D is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Step-14

The LU factorization is as follows:

$$PA = LU$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To make the diagonal entries all 1 in the matrix *U*, one needs to split *LU* into *LDU* such that $DU' = U$ as follows:

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Here, *L* and *U* are the lower and upper triangular matrices, which contain 1 on diagonal

D is the diagonal matrices which consists pivots 1,-1, 0 on the diagonal.

Therefore, $PA = LDU$ factorization is as follows:

$$\boxed{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&1\\2&4&2\\1&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&2&1\\0&1&0\\0&0&0\end{bmatrix}}.$$

Step-15

Check:

Find LDU:

$$\begin{aligned}LDU&=\begin{bmatrix}1&0&0\\1&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&2&1\\0&1&0\\0&0&0\end{bmatrix}\\&=\begin{bmatrix}1&0&0\\1&-1&0\\2&0&0\end{bmatrix}\begin{bmatrix}1&2&1\\0&1&0\\0&0&0\end{bmatrix}\\&=\begin{bmatrix}1&2&1\\1&1&1\\2&4&2\end{bmatrix}\\&=PA\end{aligned}$$

Therefore, by row column multiplication of the above two sides we get the same matrices.