Step-1

Row space:

By the definition, nonzero rows of any matrix form the basis for the row spaces and matrix U have two pivot element. So,

$$\begin{aligned}\dim C(A^T) &= \dim C(U^T)\\ &= 2\end{aligned}$$

And transpose of the nonzero rows $\left\{\begin{bmatrix}1\\2\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$ form the basis for the row spaces.

Step-2

Column space:

In matrix U, only first two columns are pivot columns. So, these will form the basis for column space of U,

$$\dim C(U) = 2, \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\right\}.$$

For the basis of column space of matrix A, one has to take the same pivot column (that is, first two) but the elements from A, that is,

$$\dim C(A) = 2, \left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}2\\1\\2\end{bmatrix}\right\}.$$

Step-3

Null space:

For the null space, find the solution of the system $Ux = 0$:

$$\begin{bmatrix}1&2&0&1\\0&1&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = 0$$

Then there will be two equations

$$x_1+2x_2+x_4=0$$
$$x_2+x_3=0$$

Substitute $x_2=-x_3$ in first equation and get, $x_1=2x_3-x_4$, where x_3, x_4 are arbitrary real numbers. Thus, the solution of the system is:

$$x=\begin{bmatrix}2x_3-x_4\\ -x_3\\ x_3\\ x_4\end{bmatrix}$$
$$=x_3\begin{bmatrix}2\\ -1\\ 1\\ 0\end{bmatrix}+x_4\begin{bmatrix}-1\\ 0\\ 0\\ 1\end{bmatrix}$$

Hence, dimension of null space of matrix *A* and *U* will be 2 with the basis:

$$\left\{\begin{bmatrix}2\\ -1\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-1\\ 0\\ 0\\ 1\end{bmatrix}\right\}.$$

Step-4

Left null space:

The left null space of matrix *U* is the null space of U^T. That is, find the solution of the system $U^T x=0$:

$$\begin{bmatrix}1&0&0\\ 2&1&0\\ 0&1&0\\ 1&0&0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0.$$

Then it gives $x_1=0, x_2=0, x_3\in R$. Thus, the solution of the system is:

$$x=x_3\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}.$$

Hence, $\dim N(U^T)=1$, with basis $\left\{\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\right\}$.

Step-5

The left null space of matrix A is the null space of A^T. That is, find the solution of the system $A^T x = 0$:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$
.

Then it gives $x_1 + x_3 = 0, x_2 = 0, x_3 \in R$. Thus, the solution of the system is:

$$x = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$$
$$= x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Step-6

Hence, $\dim N(A^T) = 1$, with basis $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.