

## Step-1

$$U = \begin{bmatrix} 0 & 5 & 4 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is

The objective is to find a basis for the column space of  $U$  by locating the pivots.

Recollect that the subspace that is spanned by the column vectors of  $U$  is called the column space of  $U$ .

The basis for the column space will be the columns that contain leading 1's.

Consider the matrix,

$$U = \begin{bmatrix} 0 & 5 & 4 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply column operations to get pivots.

Subtract third column from second matrix and fourth column from third matrix.  $(C_2 - C_3, C_3 - C_4)$

$$U \sim \begin{bmatrix} 0 & 1 & 1 & 3 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Locate the pivots in this matrix.

$$U \sim \begin{bmatrix} 0 & \boxed{1} & 1 & 3 \\ 0 & -2 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the matrix, the second and third columns of  $U$  contain leading 1 and so will form a basis for the column space of  $U$ .

$$\left\{ \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Therefore, the basis for the column space will be

## Step-2

Express each column that is not in the basis as a combination of basic columns.

Columns 1, 4 are not in the basis.

First column can be written as,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

Fourth column can be written as,

$$\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

## Step-3

The matrix in equation (1) represents the echelon form.

$$U \sim \begin{bmatrix} 0 & 1 & 1 & 3 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in the echelon form but with different column space.

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

The basis for this column space is

$$A = \begin{bmatrix} 0 & 1 & 1 & 3 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, required matrix is