

Step-1

A graph consists of a set of vertices or nodes, and a set of edges that connect them. The edge goes from node j to node k , then that row has -1 in column j and +1 in column k .

The incidence matrix A is,

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

Step-2

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Let us consider the matrix

To show directly from the columns that every vector b in the column space will satisfy $b_1 + b_2 - b_3 = 0$

So,

Let

$$[A : b] = \begin{bmatrix} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 \\ 1 & 0 & -1 & b_3 \end{bmatrix}$$

Apply $R_3 \rightarrow R_3 - R_1$

$$= \begin{bmatrix} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 \\ 0 & 1 & -1 & b_3 - b_1 \end{bmatrix}$$

Step-3

So,

Apply $R_3 \rightarrow R_3 - R_2$

$$= \begin{bmatrix} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix}$$

Apply $R_3 \rightarrow (-1)R_3$

$$= \begin{bmatrix} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 \\ 0 & 0 & 0 & b_1 + b_2 - b_3 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Therefore, the two columns are independent and every vector b in the column space will satisfy $b_1 + b_2 - b_3 = 0$.

Step-4

To show directly from the rows that every vector b in the row space will satisfy $b_1 + b_2 - b_3 = 0$

Since, the row space of A is equal to the column space of A^T .

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

If

$$\begin{bmatrix} A^T & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & b_1 \\ -1 & 1 & 0 & b_2 \\ 0 & -1 & -1 & b_3 \end{bmatrix}$$

Let

Apply $R_2 \rightarrow R_2 + R_1$

$$= \begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & 1 & b_2 + b_1 \\ 0 & -1 & -1 & b_3 \end{bmatrix}$$

Apply $R_3 \rightarrow R_3 + R_2$

$$= \begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & 1 & b_2 + b_1 \\ 0 & 0 & 0 & b_3 + b_2 + b_1 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Therefore, the first two columns are independent and every vector b in the row space will satisfy $b_1 + b_2 - b_3 = 0$.

Step-5

$$\text{Let } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then,

$$\begin{aligned} AX &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_1 - x_3 \end{bmatrix} \end{aligned}$$

The potential difference at classâ€™s 1, 2, and 3 are $x_1 - x_2, x_2 - x_3$ and $x_1 - x_3$.

Step-6

The space of \mathcal{A} is $x_1 = x_2$ and $x_2 = x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, this means that equal potentials a cross each close.