Step-1

(a)

Find a 2 by 3 system $Ax = b$ whose complete solution is;

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Suppose A be a 2×3 matrix such that the complete solution of $Ax = b$ is;

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Step-2

If $x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ then $Ax = b$ reduces to $Rx = C$ where R is row reduced echelon form.

$$u = 1 + w$$
$$v = 2 + 3w$$
$$u - w = 1$$
$$v - 3w = 2$$

Step-3

And w is an independent vowel.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If

$$A = \begin{bmatrix} 1 & 0 - 1 \\ 0 & 1 - 3 \end{bmatrix}$$
$$b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

,

Then the system whose solution $x=\begin{bmatrix}1\\2\\0\end{bmatrix}+w\begin{bmatrix}1\\3\\1\end{bmatrix}$ is;

$$\begin{bmatrix}1 & 0 & -1\\0 & 1 & -3\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

Step-4

(b)

Find a 3 by 3 system with the solutions $x_1=\begin{bmatrix}1\\2\\0\end{bmatrix}+w\begin{bmatrix}1\\3\\1\end{bmatrix}$ exactly when $b_1+b_2=b_3$.

Let *A* be a 3×3 matrix such that ;

$x_1=\begin{bmatrix}1\\2\\0\end{bmatrix}+w\begin{bmatrix}1\\3\\1\end{bmatrix}$ Is the complete solution of $Ax=B$

Step-5

Then,

$$\begin{bmatrix}1 & 0 & -1\\0 & 1 & -3\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1+R_2}\begin{bmatrix}1 & 1 & -4\\0 & 1 & -3\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\xrightarrow{R_{13}}\begin{bmatrix}1 & 0 & -1\\0 & 1 & -3\\1 & 1 & -4\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Step-6

Let,

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & -4 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$b_1 = 1$$
$$b_2 = 2$$
$$b_3 = 3$$

So it is clear that $b_1 + b_2 = b_3$.

Therefore, the system whose solution $x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$