Step-1

(a) We need to show that $\det A = 0$, provided $a+d=0$.

Consider the following:

$$\det A = \begin{vmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{vmatrix}$$

$$= \begin{vmatrix} 2a & c & b & 0 \\ b & 0 & 0 & b \\ c & 0 & 0 & c \\ 0 & c & b & 2d \end{vmatrix}$$

$$= 2a\begin{vmatrix} 0 & 0 & b \\ 0 & 0 & c \\ c & b & 2d \end{vmatrix} - c\begin{vmatrix} b & 0 & b \\ c & 0 & c \\ 0 & b & 2d \end{vmatrix} + b\begin{vmatrix} b & 0 & b \\ c & 0 & c \\ 0 & c & 2d \end{vmatrix}$$

Step-2

Note the following:

$$\begin{vmatrix} 0 & 0 & b \\ 0 & 0 & c \\ c & b & 2d \end{vmatrix} = 0$$

, because the first two columns are dependent

$$\begin{vmatrix} b & 0 & b \\ c & 0 & c \\ 0 & b & 2d \end{vmatrix} = 0$$

, because the first two rows are dependent

$$\begin{vmatrix} b & 0 & b \\ c & 0 & c \\ 0 & c & 2d \end{vmatrix} = 0$$

, because the first two rows are dependent

Therefore, $\boxed{\det A = 0}$.

Step-3

Since, $a+d=0$, we have $d=-a$. Consider the following:

$$\begin{bmatrix} 2a & c & b & 0 \\ b & 0 & 0 & b \\ c & 0 & 0 & c \\ 0 & c & b & -2a \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2au + cv + bw \\ bu + bz \\ cu + cz \\ cv + bw - 2az \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives, $u = -z$. Therefore, the first and the fourth equations are as follows:

$$2au + cv + bw = 0$$
$$cv + bw + 2au = 0$$

Step-4

The two equations that we obtained are the same. So, when we fix u and v, the other two variables w and z get fixed automatically.

Let $u = u$ and $v = v$.

Therefore, $z = -u$ and $w = \frac{-2au - cv}{b}$.

Therefore, the solution of the system $Ax = b$ is given by $\boxed{x = \begin{bmatrix} u \\ v \\ \frac{-2au - cv}{b} \\ -u \end{bmatrix}}$.

(b) Let us show that $\det A = 0$, if $ad = bc$.

We have

$$\det A = \begin{vmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{vmatrix}$$

$$= 2a\begin{vmatrix} a+d & 0 & b \\ 0 & a+d & c \\ c & b & 2d \end{vmatrix} - c\begin{vmatrix} b & 0 & b \\ c & a+d & c \\ 0 & b & 2d \end{vmatrix} + b\begin{vmatrix} b & a+d & b \\ c & 0 & c \\ 0 & c & 2d \end{vmatrix}$$

$$= 2a\left(2d(a+d)^2 - 2(a+d)bc\right) - c\left(2bd(a+d)\right) + b\left(-2cd(a+d)\right)$$

$$
\begin{aligned}
\det A &= 2(a+d)\{a(2d(a+d)-2bc)-c(bd)-b(cd)\} \\
&= 2(a+d)\{2a^2d+2ad^2-2abc-2bcd\} \\
&= 2(a+d)\{2a^2d+2ad^2-2abc-2bcd\}
\end{aligned}
$$

Step-5

We have $ad = bc$. Putting *bc* for *ad*, we get

$$
\begin{aligned}
\det A &= 2(a+d)\{2a^2d+2ad^2-2abc-2bcd\} \\
&= 2(a+d)\{2abc+2bcd-2abc-2bcd\} \\
&= 0
\end{aligned}
$$

Thus, we have shown that $\boxed{\det A = 0}$.