

3

Orthogonality (正交性)

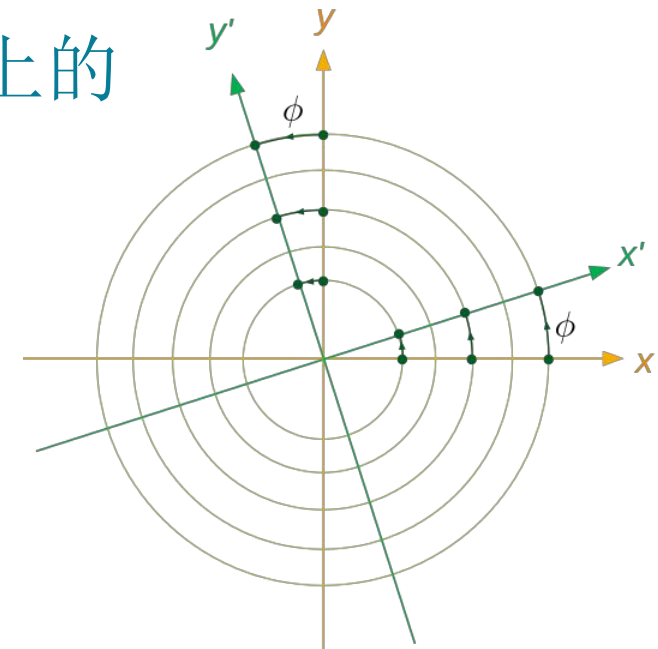
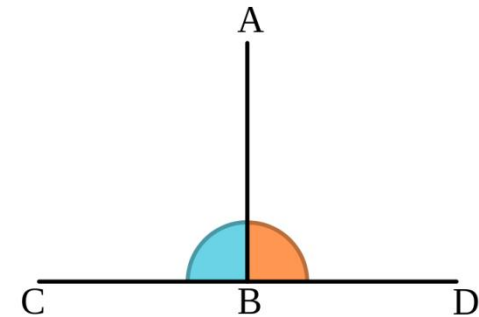
3.2

COSINES AND PROJECTIONS ONTO LINES (余弦; 向量往线上的 投影)

Cosines

Projection onto a line

Projections as linear
transformations

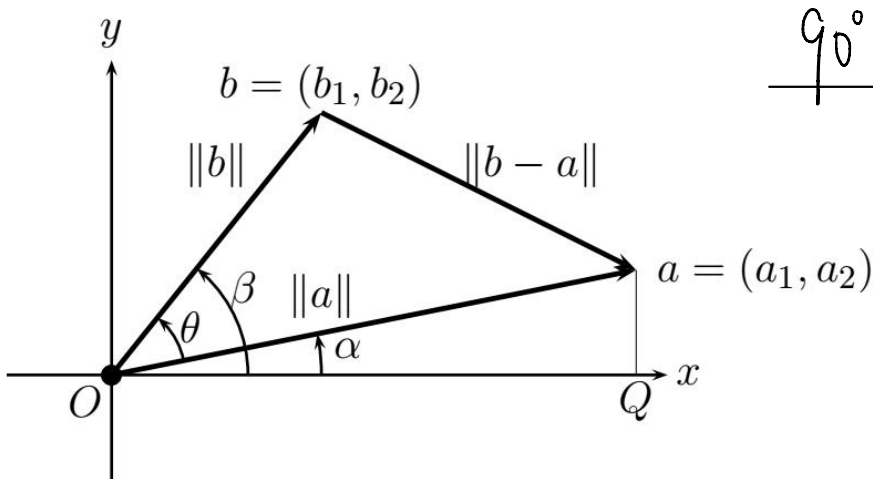
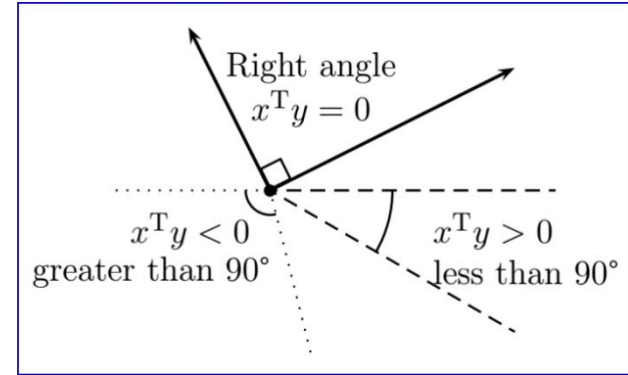


I. Cosines (余弦)

- If $\mathbf{x}^T \mathbf{y} = 0$, then \mathbf{x}, \mathbf{y} are **orthogonal**, also called **perpendicular**.

The orthogonal case is the most important.

Now we allow inner products that are **not zero**, and angles that are **not right angles**.



The cosine of the angle $\theta = \beta - \alpha$ using inner products.

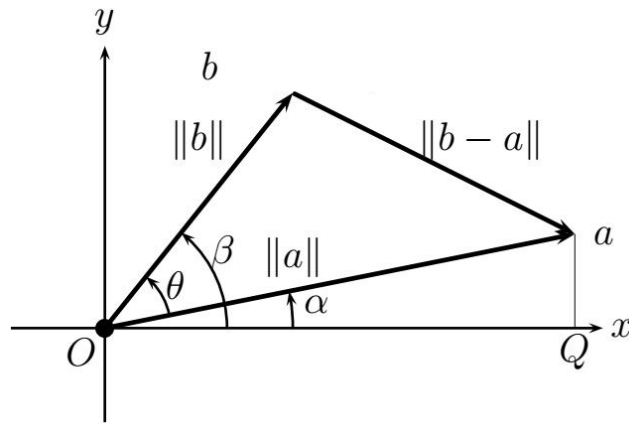
- If $\mathbf{x}^T \mathbf{y} > 0$, their angle is less than 90° ;
- If $\mathbf{x}^T \mathbf{y} < 0$, their angle is greater than 90° .

$$\theta = \beta - \alpha$$

$$\underline{\cos \theta} = \cos \beta \cos \alpha + \sin \beta \sin \alpha$$

$$= \frac{b_1}{\|\mathbf{b}\|} \frac{a_1}{\|\mathbf{a}\|} + \frac{b_2}{\|\mathbf{b}\|} \frac{a_2}{\|\mathbf{a}\|}$$

$$= \frac{a_1 b_1 + a_2 b_2}{\|\mathbf{a}\| \|\mathbf{b}\|} = \underline{\underline{\frac{\mathbf{a}^T \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}}}$$



Law of Cosines

$$\|b - a\|^2 = \|b\|^2 + \|a\|^2 - 2\|b\|\|a\|\cos\theta$$

$$\Rightarrow (b - a)^T (b - a) = b^T b + a^T a - 2\|b\|\|a\|\cos\theta$$

$$\Rightarrow b^T b - 2a^T b + a^T a = b^T b + a^T a - 2\|b\|\|a\|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{a^T b}{\|a\|\|b\|}.$$

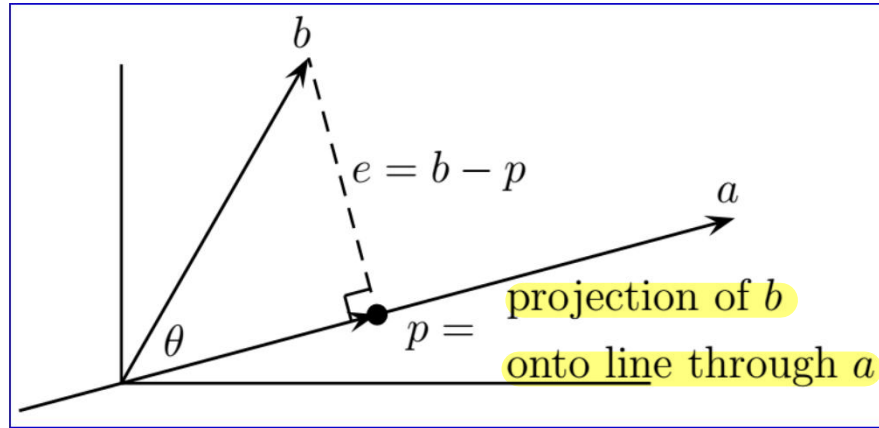
(the cosine of the angle between any *nonzero* vectors a and b)

It holds in n dimensions.

We notice that, since $|\cos\theta| \leq 1$, we have

$$|a^T b| \leq \|a\|\|b\|. \quad (\text{Cauchy-Schwarz inequality})$$

II. Projection onto a Line (往线上的投影)



Goal: find the distance from a point b to the line in the direction of the vector a .

→ *find the projection p*

(The line connecting b to p is perpendicular to a)

Even though a and b are not orthogonal, the distance problem automatically brings in orthogonality.

Let \mathbf{a} be a vector in a vector space V , and let $\text{proj}_{\mathbf{a}}$ be the projection of the vectors of V onto the line in the direction of \mathbf{a} . Then

$$\text{proj}_{\mathbf{a}} : \mathbf{b} \mapsto \mathbf{p} = \hat{x}\mathbf{a}$$

for some *scalar* \hat{x} , and the difference $\mathbf{b} - \hat{x}\mathbf{a}$ is perpendicular to the vector \mathbf{a} . Thus

$$0 = \mathbf{a}^T (\mathbf{b} - \hat{x}\mathbf{a}) = \mathbf{a}^T \mathbf{b} - \hat{x} \mathbf{a}^T \mathbf{a},$$

so that the scalar

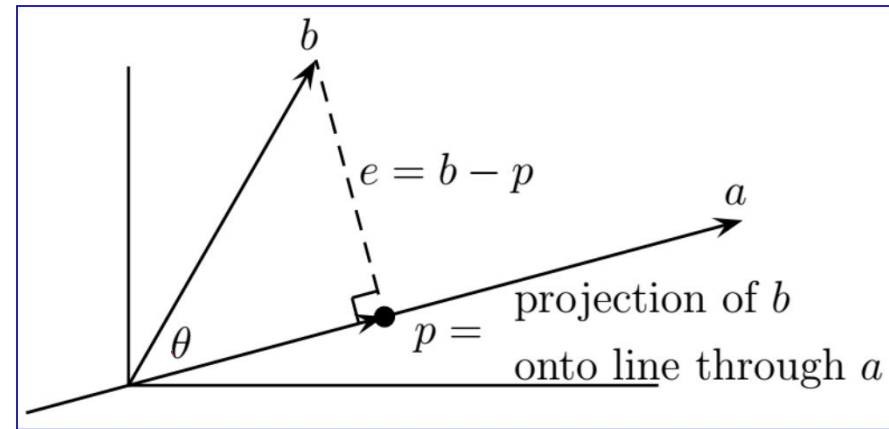
$$\hat{x} = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}}.$$

projection from vector \mathbf{b}
to vector \mathbf{a} .

We therefore have the following result.

Proposition (命题) The projection $\text{proj}_{\mathbf{a}}$ satisfies

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} \mathbf{a}.$$



Example 1 Project $\mathbf{b} = (1, 2, 3)^T$ onto the line through $\mathbf{a} = (1, 1, 1)^T$ to get :

$$\hat{x} = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} = \frac{6}{3} = 2.$$

The projection is

$$\mathbf{p} = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} \mathbf{a} = (2, 2, 2)^T.$$

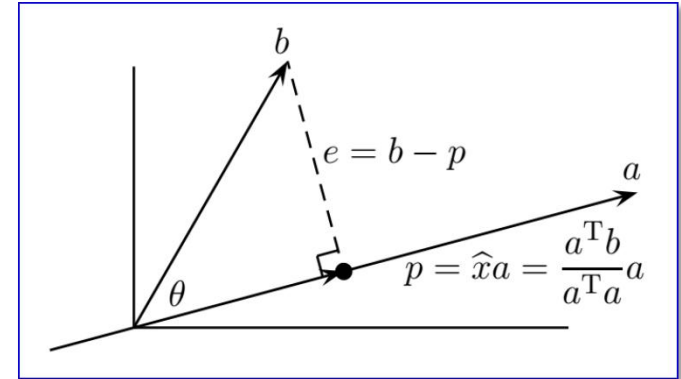
The angle between \mathbf{a} and \mathbf{b} has

$$\cos \theta = \frac{\|\mathbf{p}\|}{\|\mathbf{b}\|} = \frac{\sqrt{12}}{\sqrt{14}}.$$

$$\cos \theta = \frac{\vec{a}^T \vec{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|} = \frac{\|\text{proj}_{\mathbf{a}} \vec{b}\|}{\|\vec{b}\|}$$

(Cauchy- Schwarz inequality)

$$|a^T b| \leq \|a\| \|b\|.$$

**Second proof:**

$$\begin{aligned} \|e\|^2 &= \|b - p\|^2 = \left\| b - \frac{a^T b}{a^T a} a \right\|^2 = \left(b - \frac{a^T b}{a^T a} a \right)^T \left(b - \frac{a^T b}{a^T a} a \right) \\ &= b^T b - 2 \frac{a^T b}{a^T a} a^T b + \left(\frac{a^T b}{a^T a} \right)^2 a^T a \\ &= \frac{(b^T b)(a^T a) - (a^T b)^2}{a^T a} \geq 0 \end{aligned}$$

Cauchy-Schwarz
inequality is equivalent to
 $|\cos \theta| \leq 1.$

Therefore,

$$|a^T b| \leq \|a\| \|b\|.$$

Equality holds if and only if
 b is a multiple of a .
or \vec{a} is a multiple of \vec{b}

Triangle inequality:

$$\|a + b\| \leq \|a\| + \|b\|.$$

III. Projection as a Linear Transformation (Projection Matrix of Rank 1: 秩为1的投影矩阵)

$$\text{proj}_a : \mathbf{b} \mapsto \mathbf{p} = \hat{x}\mathbf{a}.$$

Rewrite the projection $\text{proj}_a(\mathbf{b})$:

$$\text{proj}_a(\mathbf{b}) = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} \mathbf{a} = \mathbf{a} \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} = \boxed{\frac{\mathbf{a} \mathbf{a}^T}{\mathbf{a}^T \mathbf{a}}} \mathbf{b}.$$

Projection onto a line is carried out by a **projection matrix \mathbf{P}** :

$$\mathbf{P} = \frac{\mathbf{a} \mathbf{a}^T}{\mathbf{a}^T \mathbf{a}}. \quad (\text{a column times a row—a square matrix—divided by the number } \mathbf{a}^T \mathbf{a}.)$$

\mathbf{P} is a matrix of rank 1, and as a linear transformation, it transforms a vector \mathbf{b} to its projection $\text{proj}_a(\mathbf{b}) = \mathbf{P}\mathbf{b}$.

Theorem. Let \mathbf{a} be a nonzero vector of a vector space V , and let T be a linear transformation which transforms vector \mathbf{b} to its projection onto the line in the direction of \mathbf{a} . Then the matrix of T is

$$\mathbf{P} = \frac{\mathbf{a} \mathbf{a}^T}{\mathbf{a}^T \mathbf{a}} = \frac{\mathbf{a} \mathbf{a}^T}{\|\mathbf{a}\|^2}.$$

Example 2 Let $\mathbf{a}=(1,1,1)^T$.

Then the matrix that projects onto the line through \mathbf{a} is

$$\mathbf{P} = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

This matrix has two properties (typical of projections):

1. \mathbf{P} is a symmetric matrix: $\mathbf{P}^T = \mathbf{P}$.
2. Its square is itself: $\mathbf{P}^2 = \mathbf{P}$.

The column space consists of the line through $\mathbf{a} = (1,1,1)^T$.

The nullspace consists of the plane perpendicular to \mathbf{a} .

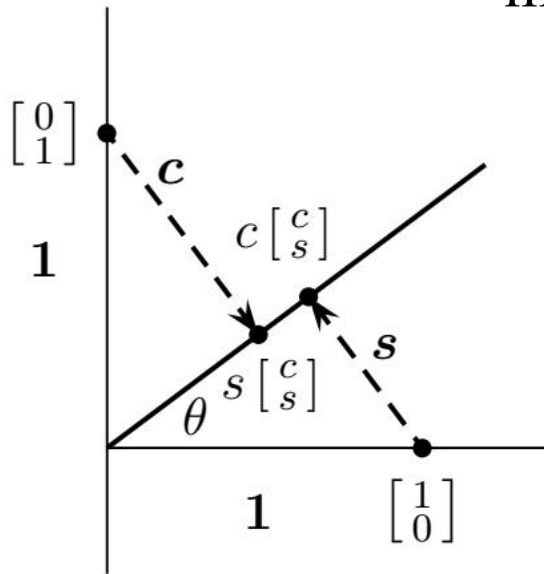
The rank is $r = 1$.

Remark: The nullspace should be orthogonal to the *row space*.

But because \mathbf{P} is symmetric, its row and column spaces are the same.

Example 3 Project onto the “ θ -direction” in the x - y plane. (\mathbf{R}^2)

The line goes through $\mathbf{a} = (\cos\theta, \sin\theta)^T$ and the matrix is symmetric with $\mathbf{P}^2 = \mathbf{P}$.



Projection onto the θ -line

$$\mathbf{P} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

($c = \cos\theta$, $s = \sin\theta$)

$$\mathbf{P} = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{\begin{bmatrix} c \\ s \end{bmatrix} \begin{bmatrix} c & s \end{bmatrix}}{\begin{bmatrix} c & s \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix}} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}.$$

Note:

\mathbf{P} in any number of dimensions: $\mathbf{P} = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}.$

We emphasize that it produces the projection \mathbf{p} :

To project \mathbf{b} onto \mathbf{a} , multiply by the projection matrix \mathbf{P} : $\mathbf{p} = \mathbf{P}\mathbf{b}$.

Key words:

Cosine of the angle

Projection onto a Line

Projection as Linear Transformation: Projection matrix

Homework

See Blackboard

