Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #6

2023/03/30

Name: ______________________________

Student Number: ______________________________

1. Show that there exist $\alpha_1, \alpha_2, \alpha_3$ are three roots of $f(x) = x^3 - 1$ and $\alpha_2 = \alpha_1^2, \alpha_3 = \alpha_1^3$.

   证明 $f(x) = x^3 - 1$ 存在三个根 $\alpha_1, \alpha_2, \alpha_3$，并且满足 $\alpha_2 = \alpha_1^2, \alpha_3 = \alpha_1^3$.

   Proof.
$$\alpha_1 = \frac{-1+\sqrt{3}i}{2}, \quad \alpha_2 = \frac{-1-\sqrt{3}i}{2} = \alpha_1^2, \quad \alpha_3 = 1 = \alpha_1^3$$

□

* Matrix representation of linear map

$Tu_j = a_{1j}v_1 + \cdots + a_{nj}v_n$

2. Suppose $T \in \mathcal{L}(V)$, and $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ are bases of V. Prove that T is invertible if and only if the columns of $\mathcal{M}(T)$ spans $\mathbf{F}^{n,1}$. Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \cdots, u_n), (v_1, \cdots, v_n))$. $\Leftrightarrow$ Tu_i basis of V

basis + right length $\Leftrightarrow$ spans

设 $T \in \mathcal{L}(V)$，$u_1, \cdots, u_n$ 和 $v_n, \cdots, v_n$ 是 V 的两组基. 证明 T 是可逆线性映射当且仅当 $\mathcal{M}(T)$ 的列向量组可以张成 $\mathbf{F}^{n,1}$. 此处 $\mathcal{M}(T)$ 是 $\mathcal{M}(T, (u_1, \cdots, u_n), (v_1, \cdots, v_n))$.

$\mathbb{F}^{n\times 1}$

Proof. "$\Rightarrow$" : Since T is invertible, then $Tu_1, \cdots, Tu_n$ is a basis of V.

Let $Tu_i = a_{1i}v_1 + \cdots + a_{ni}v_n$, if

column vectors linearly independent

$Tu_1 = a_{11}v_1 + \cdots + a_{n1}v_n$
$Tu_2 = a_{12}v_1 + \cdots + a_{n2}v_n$
$\vdots$
$Tu_n = a_{1n}v_1 + \cdots + a_{nn}v_n$

$$k_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + k_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + k_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix} = 0$$

then $k_1Tu_1 + k_2Tu_2 + \cdots + k_nTu_n = 0 \Rightarrow$

$$k_1Tu_1 + k_2Tu_2 + \cdots + k_nTu_n = (k_1a_{11} + \cdots + k_na_{1n})v_1 + \cdots + (k_1a_{n1} + \cdots + k_na_{nn})v_n = 0$$

so $k_1 = k_2 = \cdots = k_n = 0$, then the columns of $\mathcal{M}(T, (u_1, \cdots, u_n), (v_1, \cdots, v_n))$ is linearly independent and also is a basis of $\mathbf{F}^{n,1}$, thus they can span $\mathbf{F}^{n,1}$.

"$\Leftarrow$" : Since the columns of $\mathcal{M}(T)$ spans $\mathbf{F}^{n,1}$, then they can be a basis of $\mathbf{F}^{n,1}$.

If $k_1Tu_1 + \cdots + k_nTu_n = 0$, then

$$(k_1a_{11} + \cdots + k_na_{1n})v_1 + \cdots + (k_1a_{n1} + \cdots + k_na_{nn})v_n = 0$$

Since $v_1, \cdots, v_n$ is linearly independent, then $k_1a_{11} + \cdots + k_na_{1n} = 0, \cdots, k_1a_{n1} + \cdots + k_na_{nn} = 0$, that is,

$$k_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots + k_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix} = 0$$

Since the columns of $\mathcal{M}(T)$ is linearly independent, then $k_1 = \cdots = k_n = 0$, so $Tu_1, \cdots, Tu_n$ is also linearly independent, which is also a basis of V, thus T is invertible.

□