Step-1

Let $A=\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}2-\lambda & 1\\1 & 2-\lambda\end{vmatrix}=0$$

$$(2-\lambda)^2-1=0$$

$$\lambda^2-4\lambda+3=0$$
$$(\lambda-1)(\lambda-3)=0$$
$$\lambda=1,3$$

The eigen values are $\lambda_1=1,\lambda_2=3$

Step-2

The eigen vector corresponding to $\lambda_1=1$ is obtained by solving $(\mathrm{A}-\lambda_1 I)x=0$

$$\Rightarrow\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

After reducing the coefficient matrix, we rewrite the homogeneous equation $x_1+x_2=0$

So, the solution set $X_1=\begin{bmatrix}1\\-1\end{bmatrix}$ is the eigen vector corresponding to $\lambda_1=1$.

Similarly, solving $(\mathrm{A}-\lambda_2 I)x=0$, we get $X_2=\begin{bmatrix}1\\1\end{bmatrix}$, the eigen vector corresponding to $\lambda_2=3$

Step-3

The eigen vectors corresponding to the distinct eigen values are linearly independent and so, the matrix S whose columns are these eigen vectors is non singular.

Therefore, $S=\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix},\Lambda=\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix},S^{-1}=\frac{1}{2}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$ such that $A=S\Lambda S^{-1}$

Further, $A^k=S\Lambda^k S^{-1}$

$$=\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} 1 & 3^k \\ -1 & -3^k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} 1+3^k & 3^k-1 \\ -1-3^k & 1-3^k \end{bmatrix}$$

By interchanging X_1 and X_2 in *S*, we get the stipulated form given.