

## Step-1

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Given the matrix

In the given matrix, independent columns are 2 and 3. Its rank is 2(non-zero rows).

The basis of column space is  $\{(3,0,1), (3,0,0)\}$  and its dimension is less than or equal to 2.

The basis of row space is  $\{(0 \ 3 \ 3 \ 3), (0 \ 1 \ 0 \ 1)\}$  and its dimension is less than or equal to 2.

## Step-2

In order to find the null space, set it in  $AX = 0$ . So,

$$\begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies;

$$3x_2 + 3x_3 + 3x_4 = 0$$

$$x_2 + x_4 = 0$$

$$x_2 = -x_4$$

$$x_3 = 0$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix}$$
$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Therefore, the basis for null space is and its dimension is 2.

### Step-3

In order to find the left null space basis, the transpose of the matrix,

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Let's write it in  $A^T X = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies;

$$3x_1 + 3x_3 = 0$$

$$3x_1 = 0$$

$$x_3 = 0$$

$$x_1 = 0$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ x_4 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Therefore, the basis for null space is  $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  and its dimension is  $\boxed{2}$ .

## Step-4

$$B = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 5 & 5 \end{bmatrix}$$

Given the matrix

In this matrix, first column is independent.

## Step-5

$$C(B) = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Column space basis is  $\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$  and its dimension is 1.

Null space basis is  $\left( C(B^T) \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and its dimension is 1.

## Step-6

In order to find the null space, set it in  $BX = 0$ . So,

$$\begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies;

$$x_1 + x_2 = 0$$

$$4x_1 + 4x_2 = 0$$

$$5x_1 + 5x_2 = 0$$

$$x_1 = -x_2$$

So,

$$\begin{aligned}
 X &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
 &= \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} \\
 &= x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}
 \end{aligned}$$

Therefore, basis for null space is  $\boxed{B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$  and its dimension is  $\boxed{1}$ .

## Step-7

In order to find the left null space basis, the transpose of the matrix,

$$B^T = \begin{bmatrix} 1 & 4 & 5 \\ 1 & 4 & 5 \end{bmatrix}$$

Let's write it in  $B^T X = 0$

$$\begin{aligned}
 \begin{bmatrix} 1 & 4 & 5 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
 x_1 + 4x_2 + 5x_3 &= 0 \\
 x_1 + 4x_2 + 5x_3 &= 0 \\
 x_1 &= -4x_2 - 5x_3
 \end{aligned}$$

So,

$$\begin{aligned}
 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -4x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} \\
 &= x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}
 \end{aligned}$$

Therefore, the basis for is  $\boxed{N(B^T) = \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}}$  and its dimension is  $\boxed{2}$ .