Step-1

Given that $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$

Write an equation for the closest parabola $b = C + Dt + Et^2$

Step-2

First write the equations that would hold if a line could go through all four points.

Then, every $C + Dt + Et^2$ agree exactly with *b*.

Now $Ax = b$ is;

$$C + D(0) + E(0)^2 = 0$$
$$C + D(1) + E(1)^2 = 8$$
$$C + D(3) + E(3)^2 = 8$$
$$C + D(4) + E(4)^2 = 20$$

Step-3

The matrix form of the above system is;

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

$$x = \begin{bmatrix} C \\ D \\ E \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

Therefore, the closest parabola is $\boxed{\begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 3 & 9\\1 & 4 & 16\end{bmatrix}\begin{bmatrix}C\\D\\E\end{bmatrix}=\begin{bmatrix}0\\8\\8\\20\end{bmatrix}}$.

Step-4

Now, to find the normal equations $A^T A\hat{x} = A^T b$

Now,

$A^T A\hat{x} = A^T b$

$$\begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 3 & 4\\0 & 1 & 9 & 16\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 3 & 9\\1 & 4 & 16\end{bmatrix}\begin{bmatrix}\widehat{C}\\\widehat{D}\\\widehat{E}\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 3 & 4\\0 & 1 & 9 & 16\end{bmatrix}\begin{bmatrix}0\\8\\8\\20\end{bmatrix}$$

Step-5

Continuation to the above

$$\begin{bmatrix}\begin{pmatrix}1(1)+1(1)\\+1(1)+1(1)\end{pmatrix} & \begin{pmatrix}1(0)+1(1)\\+1(3)+1(4)\end{pmatrix} & \begin{pmatrix}1(0)+1(1)\\+1(9)+1(16)\end{pmatrix}\\\begin{pmatrix}0(1)+1(1)\\+3(1)+4(1)\end{pmatrix} & \begin{pmatrix}0(0)+1(1)\\+3(3)+4(4)\end{pmatrix} & \begin{pmatrix}0(0)+1(1)\\+3(9)+4(16)\end{pmatrix}\\\begin{pmatrix}0(1)+1(1)\\+9(1)+16(1)\end{pmatrix} & \begin{pmatrix}0(0)+1(1)\\+9(3)+16(4)\end{pmatrix} & \begin{pmatrix}0(0)+1(1)\\+9(9)+16(16)\end{pmatrix}\end{bmatrix}\begin{bmatrix}\widehat{C}\\\widehat{D}\\\widehat{E}\end{bmatrix}$$

$$=\begin{bmatrix}\begin{pmatrix}1(0)+1(8)\\+1(8)+1(20)\end{pmatrix}\\\begin{pmatrix}0(0)+1(8)\\+3(8)+4(20)\end{pmatrix}\\\begin{pmatrix}0(0)+1(8)\\+9(8)+16(20)\end{pmatrix}\end{bmatrix}$$

$$\begin{bmatrix}4 & 8 & 26\\8 & 26 & 92\\26 & 92 & 338\end{bmatrix}\begin{bmatrix}\widehat{C}\\\widehat{D}\\\widehat{E}\end{bmatrix}=\begin{bmatrix}36\\112\\400\end{bmatrix}$$

Hence the three normal equations are;

$$\boxed{\begin{aligned}&4\widehat{C}+8\widehat{D}+26\widehat{E}=36\\&8\widehat{C}+26\widehat{D}+92\widehat{E}=112\\&26\widehat{C}+92\widehat{D}+338\widehat{E}=400\end{aligned}}$$