Step-1

a) The intersection of two planes through $(0,0,0)$ is probably a **line** but it could be a **subspace**. It canâ€™t be the zero vector **Z**.

Step-2

b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably

$\{(0,0,0)\}$ but it could be a line itself if the line is in the plane and if could be a **subspace**.

Step-3

c) Let **S** and **T** be the subspaces of $\mathbf{R}^5$

Now we have to show that $\mathbf{S} \cap \mathbf{T}$ is also a subspace of $\mathbf{R}^5$.

Step-4

First we verify that $\mathbf{R}^5$ is a vector space.

Let

$$x = (x_1, x_2, x_3, x_4, x_5)$$
$$y = (y_1, y_2, y_3, y_4, y_5) \in \mathbf{R}^5$$

And $r \in \mathbf{R}$

Now,

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + y_5) \in \mathbf{R}^5$$
$$y + x = (y_1 + x_1, y_2 + x_2, y_3 + x_3, y_4 + x_4, y_5 + x_5) \in \mathbf{R}^5$$

1. Therefore, $x + y = y + x$ (since $x_1 + y_1 = y_1 + x_1, x_2 + y_2 = y_2 + x_2$

The addition is commutative in $\mathbf{R}$)

2. since addition in $\mathbf{R}$ is associative

Therefore if $x, y, z \in \mathbf{R}^5$ then $x + (y + z) = (x + y) + z$

3. $0 = (0,0,0,0,0)$

$$
\begin{aligned}
x+0 &= \left(x_1+0, x_2+0, x_3+0, x_4+0, x_5+0\right) \\
&= \left(x_1, x_2, x_3, x_4, x_5\right) \\
&= x
\end{aligned}
$$

Therefore, $x+0=x$ for all x

Step-5

4.

$$
\begin{aligned}
x+(-x) &= \left(x_1+(-x_1), x_2+(-x_2), x_3+(-x_3), x_4+(-x_4), x_5+(-x_5)\right) \\
&= (0,0,0,0,0) \\
&= 0
\end{aligned}
$$

Therefore, $x+(-x)=0$

5.

$$
\begin{aligned}
1x &= 1\left(x_1, x_2, x_3, x_4, x_5\right) \\
&= \left(1x_1, 1x_2, 1x_3, 1x_4, 1x_5\right) \\
&= \left(x_1, x_2, x_3, x_4, x_5\right) \\
&= x
\end{aligned}
$$

Step-6

6. Let c_1, c_2 be any scalars.

Then

$$
\begin{aligned}
(c_1c_2)x &= \left(c_1c_2x_1, c_1c_2x_2, c_1c_2x_3, c_1c_2x_4, c_1c_2x_5\right) \\
&= c_1\left(c_2x_1, c_2x_2, c_2x_3, c_2x_4, c_2x_5\right) \\
&= c_1\left(c_2\left(x_1, x_2, x_3, x_4, x_5\right)\right) \\
&= c_1\left(c_2x\right)
\end{aligned}
$$

Step-7

7. Let c be any scalar

$$
\begin{aligned}
c(x+y) &= c(x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5) \\
&= (cx_1+cy_1, cx_2+cy_2, cx_3+cy_3, cx_4+cy_4, cx_5+cy_5) \\
&= (cx_1, cx_2, cx_3, cx_4, cx_5) + (cy_1, cy_2, cy_3, cy_4, cy_5) \\
&= cx + cy
\end{aligned}
$$

8. Let c_1, c_2 be any scalars.

$$
\begin{aligned}
(c_1+c_2)x &= (c_1+c_2)(x_1, x_2, x_3, x_4, x_5) \\
&= ((c_1+c_2)x_1, (c_1+c_2)x_2, (c_1+c_2)x_3, (c_1+c_2)x_4, (c_1+c_2)x_5) \\
&= (c_1x_1+c_2x_1, c_1x_2+c_2x_2, c_1x_3+c_2x_3, c_1x_4+c_2x_4, c_1x_5+c_2x_5) \\
&= (c_1x_1, c_1x_2, c_1x_3, c_1x_4, c_1x_5) + (c_2x_1, c_2x_2, c_2x_3, c_2x_4, c_2x_5) \\
&= c_1x + c_2x
\end{aligned}
$$

Step-8

Therefore all the four properties are satisfied

Therefore $\mathbf{R}^5$ is a vector space

Now let S, T are subspace of $\mathbf{R}^5$

$$
\begin{aligned}
& x, y, \in S \cap T \\
\Rightarrow\ & x, y \in S, \quad x, y \in T \\
\Rightarrow\ & x+y \in S, \quad x+y \in T \qquad \text{(since } S, T \text{ are subspace of } \mathbf{R}^5\text{)} \\
\Rightarrow\ & x+y \in S \cap T
\end{aligned}
$$

Step-9

Let

$$
\begin{aligned}
& x \in S \cap T \text{ and } r \in \mathbf{R} \\
& x \in S \text{ and } x \in T \text{ and } r \in \mathbf{R} \\
\Rightarrow\ & rx \in S,\ rx \in T \qquad \text{(since } S, T \text{ are subspace of } \mathbf{R}^5\text{)} \\
\Rightarrow\ & rx \in S \cap T
\end{aligned}
$$

Therefore, $S \cap T$ is a subspace of $\mathbf{R}^5$