Step-1

Consider the following differential equation:

$$\frac{du}{dt} = Ju$$

Here, J is a 3 by 3 Jordan block defined as below:

$$J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Whereas,

$$u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Step-2

Above differential equation can be written as follows:

$$\frac{dx}{dt} = 5x + y$$
$$\frac{dy}{dt} = 5y + z$$
$$\frac{dz}{dt} = 5z$$

Add a fourth equation defined as follows:

$$\frac{dw}{dt} = 5w + x$$

Find the value of w.

Step-3

Add fourth equation in differential equation:

$$\frac{du}{dt} = Ju$$

Thus,

$$u = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$J = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Step-4

The system is triangular. So, solve the single variable equation and move upwards in further equation by back substitution method:

$$\frac{dz}{dt} = 5z$$

Recall that solution of the differential equation is given as follows:

$$\frac{du}{dt} = au$$

Solution:

$$u(t) = e^{at}u(0)$$

So, the solution of the differential equation starting with initial values is:

$$z = z(0)e^{5t}$$

Solve the next equation having variable z and substitute the values:

$$\frac{dy}{dt} = 5y + z$$

$$y = \left(y(0) + tz(0)\right)e^{5t}$$

Step-5

Similarly, solutions of other two differential equations are as follows:

$$\frac{dx}{dt} = 5x + y$$

$$x = \left(x(0) + ty(0) + \frac{t^2}{2!}z(0)\right)e^{5t}$$

Step-6

Another added equation solution will be:

$$\frac{dw}{dt} = 5w + x$$

$$w = \left(w(0) + tx(0) + \frac{t^2}{2!} y(0) + \frac{t^3}{3!} z(0) \right) e^{5t}$$

Step-7

Therefore, the solution of the added equation is:

$$\boxed{w = \left(w(0) + tx(0) + \frac{t^2}{2} y(0) + \frac{t^3}{6} z(0) \right) e^{5t}}$$