题 号	1	2	3	4	5	6	7	8
分 值	15 分	25 分	10 分	16 分	10 分	6 分	16 分	12 分

本试卷共 (8) 大题, 满分 (110) 分. 请将所有答案写在答题本上.

This exam includes 8 questions and the score is 110 in total. Write all your answers on the examination book.

---

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

   (共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

   (1) Let A be an $m \times n$ real matrix and b be a column vector in $\mathbb{R}^m$. Which of the following statements is correct? (   )

   (A) If $Ax = b$ has infinite many solutions, then $Ax = 0$ has a nonzero solution.

   (B) If the system $Ax = 0$ has only zero solution, then $Ax = b$ has one and only one solution.

   (C) If the rank of A is n, then the system $Ax = b$ must have a solution.

   (D) If A is a square matrix (i.e., $m = n$), then the system $Ax = b$ is consistent if and only if A is invertible.

   设 A 为 $m \times n$ 实矩阵, b 是 $\mathbb{R}^m$ 中的列向量. 下列陈述中哪个是正确的? (   )

   (A) 如果 $Ax = b$ 有无穷多个解, 则 $Ax = 0$ 有非零解.

   (B) 如果方程组 $Ax = 0$ 只有零解, 则 $Ax = b$ 有且仅有一个解.

   (C) 如果 A 的秩为 n, 则方程组 $Ax = b$ 必有解.

   (D) 如果 A 是方阵 (即 $m = n$), 则方程组 $Ax = b$ 是相容的当且仅当 A 可逆.

   (2) Suppose A is an $m \times n$ matrix, B is an $n \times m$ matrix, and I is the $m \times m$ identity matrix. If $AB = I$, then (   )

   (A) the column vectors of A are linearly independent, and the row vectors of B are linearly independent.

   (B) the column vectors of A are linearly independent, and the column vectors of B are linearly independent.

   (C) the row vectors of A are linearly independent, and the column vectors of B are linearly independent.

   (D) the row vectors of A are linearly independent, and the row vectors of B are linearly independent.

   设 A 为 $m \times n$ 型矩阵, B 为 $n \times m$ 型矩阵, I 为 m 阶单位矩阵. 若 $AB = I$, 则 (   )

   (A) A 的列向量组线性无关, B 的行向量组线性无关.

   (B) A 的列向量组线性无关, B 的列向量组线性无关.

   (C) A 的行向量组线性无关, B 的列向量组线性无关.

(D) A 的行向量组线性无关, B 的行向量组线性无关.

(3) Let A be a 3×3 matrix, and let B be the matrix formed by adding the second column of A to its first column. Suppose that after exchanging the second and third rows of B, the resulting matrix is the 3×3 identity matrix. Let $P_1=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}, P_2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$. Then $A=($ $)$

(A) P_1P_2.

(B) $P_1^{-1}P_2$.

(C) P_2P_1.

(D) $P_2P_1^{-1}$.

设 A 为 3 阶方阵, 将 A 的第二列加到第一列得矩阵 B. 假设交换 B 的第二行与第三行可以得到 3 阶单位矩阵. 记 $P_1=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}, P_2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$. 则 $A=($ $)$

(A) P_1P_2.

(B) $P_1^{-1}P_2$.

(C) P_2P_1.

(D) $P_2P_1^{-1}$.

(4) Let $A=\begin{bmatrix}-1&2&3\\-3&6&8\\2&-4&t\end{bmatrix}$, where $t\in\mathbb{R}$. Suppose $\operatorname{rank}(A)=2$. Then ( )

(A) $t=-6$.

(B) $t=6$.

(C) $t\neq 0$.

(D) t can be any real number.

设 $A=\begin{bmatrix}-1&2&3\\-3&6&8\\2&-4&t\end{bmatrix}$, 其中 $t\in\mathbb{R}$. 假设 $\operatorname{rank}(A)=2$. 则 ( )

(A) $t=-6$.

(B) $t=6$.

(C) $t\neq 0$.

(D) t 可以是任意实数.

(5) Which of the following statements is incorrect? ( )

(A) For any matrix A, $\operatorname{rank}(A)=\dim C(A)$.

(B) If $v_1,\cdots,v_m$ are pairwise orthogonal nonzero vectors, then the vectors $v_1,\cdots,v_m$ are linear independent.

(C) If A is an upper triangular $n \times n$ matrix such that $A^2 = 0$, then $A = 0$.

(D) Let A, B be $n \times n$ matrices such that AB is invertible. Then both A and B are invertible.

下列哪个论断是错误的？(　　)

(A) 对于任意矩阵 A, $\mathrm{rank}(A) = \dim C(A)$.

(B) 如果 $v_1, \cdots, v_m$ 是一组两两正交的非零向量, 则向量组 $v_1, \cdots, v_m$ 线性无关.

(C) 如果 A 是 $n \times n$ 上三角矩阵且 $A^2 = 0$, 则 $A = 0$.

(D) 设 A, B 为 $n \times n$ 矩阵且 AB 可逆. 则 A 和 B 都可逆.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let A, B be invertible $n \times n$ matrices. Then the inverse of the block matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ is ___________

设 A, B 均为 $n \times n$ 可逆矩阵. 则分块矩阵 $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ 的逆为 ___________

(2) Suppose A is a 3×4 matrix and $\dim N(A) = 2$. Then $\dim N(A^T) =$ ___________

设 A 为 3×4 矩阵且 $\dim N(A) = 2$. 则 $\dim N(A^T) =$ ___________

(3) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. Then $A^{-1} =$___________

设 $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. 则 $A^{-1} =$___________

(4) Let u, v be vectors in $\mathbb{R}^n$ such that $\|u\| = 3$, $\|v\| = 4$ and $u^T v = -3$.

Then $\|2u + 3v\| =$___________

设 u, v 为 $\mathbb{R}^n$ 中的向量, 满足 $\|u\| = 3$, $\|v\| = 4$ 以及 $u^T v = -3$. 则 $\|2u+3v\| =$___________

(5) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

Then the least squares solution to $Ax = b$ is $\hat{x} =$___________

设 $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

则 $Ax = b$ 的最小二乘解是 $\hat{x} =$___________

3. (10 points) Find the LU factorization of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

求矩阵 $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ 的 LU 分解.

4. (16 points) Let $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

Please give a basis for each of the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$ and $N(A^T)$, respectively.

(16 分) 设 $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

请对四个基本子空间 $C(A)$, $N(A)$, $C(A^T)$ 和 $N(A^T)$ 分别给出各自的一组基.

5. (10 points) Let $E = \{u_1, u_2, u_3\}$ and $F = \{v_1, v_2\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \; u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \; u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}.$$

Find the matrix A representing T with respect to the ordered bases E and F.

(10 分) 设 $E = \{u_1, u_2, u_3\}$, $F = \{v_1, v_2\}$, 其中

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \; u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \; u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

定义线性变换 $T : \mathbb{R}^3 \to \mathbb{R}^2$ 如下

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}.$$

求 T 在 E 和 F 这两组有序基下的矩阵表示 A.

6. (6 points) Let A, B be $n \times n$ matrices. Suppose A and B are both symmetric. Is AB necessarily symmetric? If yes, please give a proof. Otherwise please give a counterexample.

(6 分) 设 A, B 均为 $n\times n$ 矩阵. 假设 A 和 B 都是对称矩阵. AB 是否一定是对称矩阵？若是, 请给出证明. 否则请给出一个反例.

7. (16 points) The following two questions are independent:

   (a) Let A be the 2×2 matrix such that the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto Av$ rotates every vector in $\mathbb{R}^2$ through $60°$ counter-clockwise (about the origin).
   Find A and A^{2020}.

   (b) Three planes Π_1, Π_2, Π_3 in the space $\mathbb{R}^3$ are given by the equations

$$\begin{aligned}\Pi_1 &: \quad x+y+z=0\,,\\ \Pi_2 &: \quad 2x-y+4z=0\,,\\ \Pi_3 &: \quad -x+2y-z=0\,.\end{aligned}$$

   Determine a matrix representative (in the standard basis of $\mathbb{R}^3$) of a linear transformation taking the xy plane to Π_1, the yz plane to Π_2 and the zx plane to Π_3.

(16 分) 以下两个小题是相互独立的:

   (a) 设 A 是 2×2 矩阵使得线性变换 $\mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto Av$ 把 $\mathbb{R}^2$ 中每个向量 (绕原点) 逆时针转动 $60°$.
   求 A 和 A^{2020}.

   (b) 在空间 $\mathbb{R}^3$ 中由以下方程给出三个平面 Π_1, Π_2, Π_3:

$$\begin{aligned}\Pi_1 &: \quad x+y+z=0\,,\\ \Pi_2 &: \quad 2x-y+4z=0\,,\\ \Pi_3 &: \quad -x+2y-z=0\,.\end{aligned}$$

   求一个矩阵, 使它 (在 $\mathbb{R}^3$ 的标准基下) 表示的线性变换将 xy 平面映射成 Π_1, 将 yz 平面映射成 Π_2 并将 zx 平面映射成 Π_3.

8. (12 points) Let A be a 3×3 matrix such that $\operatorname{rank}(A)=2$ and $A^3=0$.

   (a) Prove that $\operatorname{rank}(A^2)=1$.

   (b) Let $\alpha_1 \in \mathbb{R}^3$ be a nonzero vector such that $A\alpha_1=0$. Prove that there exist vectors α_2, α_3 such that $A\alpha_2=\alpha_1$, $A^2\alpha_3=\alpha_1$.

   (c) For any vectors α_2, α_3 described as above, show that $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.

(*In this problem, you are allowed to assume the statements of some questions to answer subsequent questions.*)

(12 分) 设 A 是 3×3 矩阵, 它满足 $\operatorname{rank}(A)=2$ 及 $A^3=0$.

(a) 证明 $\mathrm{rank}(A^2) = 1$.

(b) 设 $\alpha_1 \in \mathbb{R}^3$ 是满足 $A\alpha_1 = 0$ 的非零向量. 证明: 存在向量 α_2, α_3 使得 $A\alpha_2 = \alpha_1$, $A^2\alpha_3 = \alpha_1$.

(c) 证明: 对于任意满足上述条件的向量 α_2, α_3, 向量组 $\alpha_1, \alpha_2, \alpha_3$ 线性无关.

(本题中, 允许承认前面小题的结果来用于后续问题的解答.)