Step-1

(a)

Consider the eigen values of A are 1, 1, and 2

The objective is to verify that the matrix A is invertible or not.

We have three Eigen values, so the matrix A has order of 3×3 with these eigen values.

The product of the eigen values is the determinant of that matrix.

Product of the eigen values is $1\cdot1\cdot2=2(\neq0)$.

Since determinant of matrix A is 0, A is non-singular matrix.

We know that every non-singular matrix is invertible.

Hence, the matrix A with eigen values 1, 1, and 2 is invertible is **true.**

Step-2

(b)

Consider the eigen values of A are 1, 1, and 2

The statement is â€œ A is diagonalizableâ€

The objective is to verify that the above statement is true or false.

The statement is **false**

Counter example:

Let $A=\begin{bmatrix}1&1&1\\0&1&1\\0&0&2\end{bmatrix}$

Here, A is a triangular matrix, the eigen values are the diagonal entries 1, 1, and 2 which are as given in the question.

Find the eigen vectors corresponding to the Eigen value $\lambda=1$

$$(A-\lambda I)\mathbf{x}=0$$
$$(A-\lambda I)\mathbf{x}=0 \qquad (\text{Since } \lambda=1)$$
$$\left(\begin{bmatrix}1&1&1\\0&1&1\\0&0&2\end{bmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}0&1&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The system of equations for the above matrix form is,

$$x_2+x_3=0$$
$$x_2=0$$
$$x_3=0$$

From the above equations, we have $x_2=x_3=0$ and let $x_1=k$

Therefore, eigen vector is

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}k\\0\\0\end{bmatrix}=k\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Therefore, the eigen space corresponding to the eigen value is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$.

Since the number of elements in eigen space is 1, geometric multiplicity for the eigen value $\lambda=1$ is 1.

The algebraic multiplicity of eigen value $\lambda=1$ is 2 which is not equals to the geometric multiplicity.

Therefore, the matrix A **is not diagonalizable**.

Therefore, the statement â€œ A is diagonalizableâ€ is **false**.

Step-3

c)

Consider the eigen values of A are 1, 1, and 2

The statement is â€œ A is not diagonalizableâ€

The objective is to verify that the above statement is true or false.

The statement is **false.**

Counter example:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

The eigen values are 1, 1, and 2

It is a diagonal matrix already.

So, it can be written as $S^{-1}AS = \Lambda$ where S is the identity matrix and the diagonal matrix Λ is A itself.

This example shows that in every matrix with eigen values 1, 1, and 2 is diagonalizable also.

Therefore, the statement â€œ A is not diagonalizableâ€ is **false**.