Step-1

Given that $A=\begin{bmatrix}0 & 2\\ 2 & 3\end{bmatrix}$ and $A^{-1}=\begin{bmatrix}-3/4 & 1/2\\ 1/2 & 0\end{bmatrix}$

We need to compute the eigenvalues and the eigenvectors of the above two matrices.

Step-2

Now $A=\begin{bmatrix}0 & 2\\ 2 & 3\end{bmatrix}$

$$A-\lambda I=\begin{bmatrix}0 & 2\\ 2 & 3\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 2\\ 2 & 3\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$$

$$=\begin{bmatrix}-\lambda & 2\\ 2 & 3-\lambda\end{bmatrix}$$

Step-3

$$\begin{aligned}|A-\lambda I|&=\begin{vmatrix}-\lambda & 2\\ 2 & 3-\lambda\end{vmatrix}\\ &=(-\lambda)(3-\lambda)-4\\ &=-3\lambda+\lambda^2-4\\ &=\lambda^2-3\lambda-4\end{aligned}$$

Step-4

we know that $|A-\lambda I|=0$

$$\lambda^2-3\lambda-4=0$$

$$\lambda^2-4\lambda+\lambda-4=0$$

$$\lambda(\lambda-4)+1(\lambda-4)=0$$

$$(\lambda-4)(\lambda+1)=0$$

$$\lambda=4,-1$$

$\lambda = 4, -1$

Hence the eigenvalues of A are 4,-1

Step-5

Case(i) Let $\lambda = 4$

Eigenvectors X corresponding to the eigenvalue 4 are given by

$(A - 4I)X = 0$

That is $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Step-6

$By\ 2R_2 + R_1 = R_2$

$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -4x_1 + 2x_2 = 0$

Let $x_1 = k$(say)

Therefore $x_2 = 2k$

Therefore eigenvectors corresponding to eigenvalue 4 are given by $k\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where k is a non-zero parameter.

Step-7

Case(ii) Let $\lambda = -1$

Eigenvectors X corresponding to the eigenvalue -1 are given by

$(A + I)X = 0$

That is $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Step-8

$By\ R_2 - 2R_1 = R_2$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 = 0$

Step-9

Let $x_1 = k(\text{say})$

Therefore $2x_2 = -k$

$x_2 = -k/2$

Therefore eigenvectors corresponding to eigenvalue -1 are given by $k\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ where k is a non-zero parameter

Now $\boxed{A^{-1} = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}}$

$$\left(A^{-1}\right) - \lambda I = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -3/4 - \lambda & 1/2 \\ 1/2 & -\lambda \end{bmatrix}$$

$$= (-3/4 - \lambda)(-\lambda) - 1/4$$

$$= \frac{3}{4}\lambda + \lambda^2 - \frac{1}{4}$$

$$= \lambda^2 + \frac{3}{4}\lambda - \frac{1}{4}$$

Step-10

We know that $\left|A-\lambda I\right|=0$

$$\lambda^2+\frac{3}{4}\lambda-\frac{1}{4}=0$$

$$4\lambda^2+3\lambda-1=0$$

Step-11

Compare this equation with $ax^2+bx+c=0$

$$\lambda=\frac{-3\pm\sqrt{9+16}}{8}$$

$$=\frac{-3\pm\sqrt{25}}{8}$$

$$=\frac{-3\pm 5}{8}$$

$$=\frac{-3+5}{8},\frac{-3-5}{8}$$

$$=\frac{1}{4},-1$$

Step-12

Case(i) Let $\lambda=\frac{1}{4}$

Eigenvectors X corresponding to the eigenvalue $\frac{1}{4}$ are given by

$$\left(\left(A^{-1}\right)-\frac{1}{4}I\right)X=0$$

That is $\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Step-13

$By\ R_2+\frac{R_1}{2}=R_2$

$$\begin{bmatrix} -1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1+\frac{1}{2}x_2=0$$

Step-14

Let $x_1 = k(\text{say})$

$$-k+\frac{1}{2}x_2=0$$

Therefore $x_2 = 2k$

Therefore eigenvectors corresponding to eigenvalue $\frac{1}{4}$ are given by $k\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where k is a non-zero parameter

Step-15

Case(ii) Let $\lambda = -1$

Eigenvectors X corresponding to the eigenvalue -1 are given by

$$\left(\left(A^{-1}\right)+I\right)X=0$$

That is $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Step-16

$By\ R_2/2-R_1=R_2$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{1}{4}x_1+\frac{1}{2}x_2=0$$

$\Rightarrow x_1 + 2x_2 = 0$

Step-17

Let $x_1 = k(\text{say})$

Therefore $x_2 = -k/2$

Therefore eigenvectors corresponding to eigenvalue -1 are given by $k\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ where k is a non-zero parameter.

A^{-1} has the same eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$.