Step-1

Given the complete solution to $Ax = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Any vector x_n in the null space can be added to a particular solution x_p. The solutions to all linear equations have this form, $x = x_n + x_p$

Complete solution $Ax_p = b$ and $Ax_n = 0$ produce $A\left(x_p + x_n\right) = b$

Step-2

So, the particular solution x_p is $(1,0)$ and the solution x_n is $(0,c)$

Since, the linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $C\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Therefore, the order of matrix A is 2 by 2.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Given $Ax = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow ax_1 + bx_2 &= 1 \\ cx_1 + dx_2 &= 3 \end{aligned}$$ â€¦â€¦ (1)

Step-3

From the complete solution $Ax_p = b$, we need to plug the particular solution in equation â€“ (1).

So,

$$\begin{aligned} ax_1 + bx_2 &= 1 \\ cx_1 + dx_2 &= 3 \end{aligned}$$

$$a(1)+b(0)=1$$
$$\Rightarrow a=1$$
$$c(1)+d(0)=3$$
$$\Rightarrow c=3$$

Step-4

Form the complete solution $Ax_n = 0$,

So,

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow ax_1 + bx_2 = 0$$
$$cx_1 + dx_2 = 0$$

Plug the solution of x_n ,

$$a(0)+b(c)=0$$
$$\Rightarrow b=0 \qquad (\text{Since}, c \neq 0)$$
$$c(0)+d(c)=0$$
$$\Rightarrow d=0 \qquad (\text{Since}, c \neq 0)$$

Therefore, the values are $a=1,\ b=0,\ c=3$ and $d=0$.

Therefore, the matrix is $\boxed{A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}}$.