Step-1

The objective is to find that the trace of the matrix is equals the sum of the eigenvalues.

Step-2

Consider that a $n \times n$ matrix A.

Let $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ be the eigenvalues of the matrix A

Recall that characteristic equation is kept to zero to get the eigenvalues. This means that determinant value of $A - \lambda I$ must be equal to zero.

That is,

The characteristic equation of the matrix is given by,

$$\begin{aligned} p(\lambda) &= |\lambda I - A| \\ &= \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0 \qquad \ldots\ldots(1) \end{aligned}$$

Since the set $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are all eigenvalues of the matrix A

Hence, the characteristic equation factories as $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$

Step-3

From the equation (1) the coefficient of λ^{n-1} is c_{n-1}

The coefficient of the λ^{n-1} calculated in two different ways as follows:

First By expanding the characteristic equation $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$

Hence, by expansion the λ^{n-1} term will be

$$-\lambda_1\lambda^{n-1} - \lambda_2\lambda^{n-1} \ldots - \lambda_n\lambda^{n-1} = -(\lambda_1 + \lambda_2 + \ldots + \lambda_n)\lambda^{n-1}$$

By comparing the λ^{n-1} term with equation (1) obtain $c_{n-1} = -(\lambda_1 + \lambda_2 + \ldots + \lambda_n)$ â€¦â€¦ (2)

Step-4

Second the coefficients can be calculated by expanding $\det(A - \lambda I)$ as,

$$\det(A-\lambda I)=\det\begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix}$$

From the determinant of the matrix $\det(A-\lambda I)$ one of the product is $(\lambda-a_{11})(\lambda-a_{22})...(\lambda-a_{nn})$ and every other possible products can contain at most $n-2\ \lambda's$.

Hence, the product of all other possible products will be a polynomial of degree at most $n-2$.

Let that polynomial is denoted by $q(\lambda)$

Thus, by expanding $\det(A-\lambda I)$ obtain,

The characteristic equation $p(\lambda)=(\lambda-a_{11})(\lambda-a_{22})...(\lambda-a_{nn})+q(\lambda)$

Since the degree of the polynomial $q(\lambda)$ is $n-2$ it has no λ^{n-1} term.

Hence, the λ^{n-1} term of the characteristic equation $p(\lambda)$ is in the form $(\lambda-a_{11})(\lambda-a_{22})...(\lambda-a_{nn})$

By expanding the product $(\lambda-a_{11})(\lambda-a_{22})...(\lambda-a_{nn})$ obtain,

$-(a_{11}+a_{22}...+a_{nn})\lambda^{n-1}$ â€¦â€¦ (3)

Compare the equation (1) and (3) obtain,

$c_{n-1}=-(a_{11}+a_{22}+...+a_{nn})$ â€¦â€¦ (4)

From equation (2) and (4) obtain,

$$-(\lambda_1+\lambda_2+...+\lambda_n)=-(a_{11}+a_{22}+...+a_{nn})$$
$$(\lambda_1+\lambda_2+...+\lambda_n)=(a_{11}+a_{22}+...+a_{nn})$$

Hence, trace of the matrix is equal to the sum of the eigenvalues.