Step-1

We need to show that $v_1v_1^{\mathrm{T}}+v_2v_2^{\mathrm{T}}+\ldots+v_nv_n^{\mathrm{T}}=I$

We have

$$v_1v_1^{\mathrm{T}}=\begin{bmatrix}a_{11}\\a_{21}\\a_{31}\\\vdots\\a_{n1}\end{bmatrix}(a_{11},a_{21},a_{31},\ldots,a_{n1})$$
$$=\begin{bmatrix}a_{11}^2 & a_{11}a_{21} & a_{11}a_{31} & \ldots & a_{11}a_{n1}\\ a_{21}a_{11} & a_{21}^2 & a_{21}a_{31} & \ldots & a_{21}a_{n1}\\ a_{31}a_{11} & a_{31}a_{21} & a_{31}^2 & \ldots & a_{31}a_{n1}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ a_{n1}a_{11} & a_{n1}a_{21} & a_{n1}a_{31} & \ldots & a_{n1}^2\end{bmatrix}$$

Step-2

Similarly, we get

$$v_2v_2^{\mathrm{T}}=\begin{bmatrix}a_{12}\\a_{22}\\a_{32}\\\vdots\\a_{n2}\end{bmatrix}(a_{12},a_{22},a_{32},\ldots,a_{n2})$$
$$=\begin{bmatrix}a_{12}^2 & a_{12}a_{22} & a_{12}a_{32} & \ldots & a_{12}a_{n2}\\ a_{22}a_{12} & a_{22}^2 & a_{22}a_{32} & \ldots & a_{22}a_{n2}\\ a_{32}a_{12} & a_{32}a_{22} & a_{32}^2 & \ldots & a_{32}a_{n2}\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ a_{n2}a_{12} & a_{n2}a_{22} & a_{n2}a_{32} & \ldots & a_{n2}^2\end{bmatrix}$$

Step-3

From $v_1v_1^{\mathrm{T}}$ and $v_2v_2^{\mathrm{T}}$, we can write down $v_iv_i^{\mathrm{T}}$.

$$v_i v_i^{\mathrm{T}} = \begin{bmatrix} a_{1i} \\ a_{2i} \\ a_{3i} \\ \vdots \\ a_{ni} \end{bmatrix} \left(a_{1i}, a_{2i}, a_{3i}, \ldots, a_{ni}\right)$$

$$= \begin{bmatrix} a_{1i}^2 & a_{1i}a_{2i} & a_{1i}a_{3i} & \ldots & a_{1i}a_{ni} \\ a_{2i}a_{1i} & a_{2i}^2 & a_{2i}a_{3i} & \ldots & a_{2i}a_{ni} \\ a_{3i}a_{1i} & a_{3i}a_{2i} & a_{3i}^2 & \ldots & a_{3i}a_{ni} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{ni}a_{1i} & a_{ni}a_{2i} & a_{ni}a_{3i} & \ldots & a_{ni}^2 \end{bmatrix}$$

Step-4

Therefore, $v_i v_i^{\mathrm{T}}$ is always an *n* by *n* matrix. When we add all these *n* matrices, the resultant matrix is also an *n* by *n* matrix.

The first diagonal entry of this matrix is $a_{11}^2 + a_{12}^2 + \ldots + a_{1n}^2$, which is equal to 1. Similarly, all the diagonal entries can be shown to be equal to 1.

Consider the entry in the first row and second column: It is $a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{33} + \ldots + a_{1n}a_{nn}$, which has to be equal to be zero, because this is the inner product of v_1 and v_2.

On similar lines, it can be shown that the entry in the i^{th} row and j^{th} column is the inner product of v_i and v_j.

Step-5

Therefore, we observe that the diagonal entries are 1 each and the non diagonal entries are 0 each. Thus, the resultant matrix is the identity matrix.

Therefore, we get $v_1 v_1^{\mathrm{T}} + v_2 v_2^{\mathrm{T}} + \ldots + v_n v_n^{\mathrm{T}} = I$.