Step-1

Consider a matrix A with orthogonal columns $w_1, w_2, \ldots, w_n$ of lengths $\sigma_1, \sigma_2, \ldots, \sigma_n$. Objective is to determine the singular value decomposition $U\Sigma V^T$.

Let matrix A is of $m \times n$ order having the columns $w_i \in R^m$ of non-zero lengths σ_i, for $i = 1, 2, \ldots, n$. It is known that $w_1, w_2, \ldots, w_n$ are an orthogonal columns, so the matrix $A^T A$ will be $n \times n$ square diagonal matrix.

Step-2

The diagonal entries of $A^T A$, that will be the eigenvalues of the matrix, are $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. It shows that every nonzero $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the singular values of A.

Assume that there are t number of nonzero singular values. Consider the permutation:

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

such that $\sigma_{f(1)} \geq \sigma_{f(2)} \geq \cdots \geq \sigma_{f(n)}$. Then the matrix Σ will be of order $m \times n$ with

$$\Sigma_{ii} = \begin{cases} \sigma_{f(i)} & \text{for } 1 \leq i \leq t \\ 0 & \text{otherwise} \end{cases}.$$

The matrix V will contain all the eigenvectors of diagonal matrix $A^T A$. These eigenvectors are the standard basis vectors e_i and can be ordered according to the size of corresponding eigenvalue.

Step-3

Then, the matrix V will be equal to some $n \times n$ permutation matrix. Since $A = U\Sigma$, therefore the matrix U will be of order $m \times m$ whose first k columns are $\frac{1}{\sigma_{f(1)}} w_{f(1)}, \frac{1}{\sigma_{f(2)}} w_{f(2)}, \ldots, \frac{1}{\sigma_{f(t)}} w_{f(t)}$ and remaining ones are zero.

One can easily find the simpler singular value decomposition by assuming that the diagonal entries in the matrix Σ are not written in a decreasing order. That is, the matrix Σ is of $m \times n$ order where $\Sigma_{ii} = \sigma_i$, $V = I_{n \times n}$ and U has the nonzero columns as $\frac{1}{\sigma_i} w_1, \frac{1}{\sigma_2} w_2, \ldots, \frac{1}{\sigma_t} w_t$ and the zero columns for the $\sigma_i = 0$. Then the singular value decomposition of $U\Sigma V^T$ will same as matrix A.

Step-4

Hence, $A = U\Sigma V^T$.