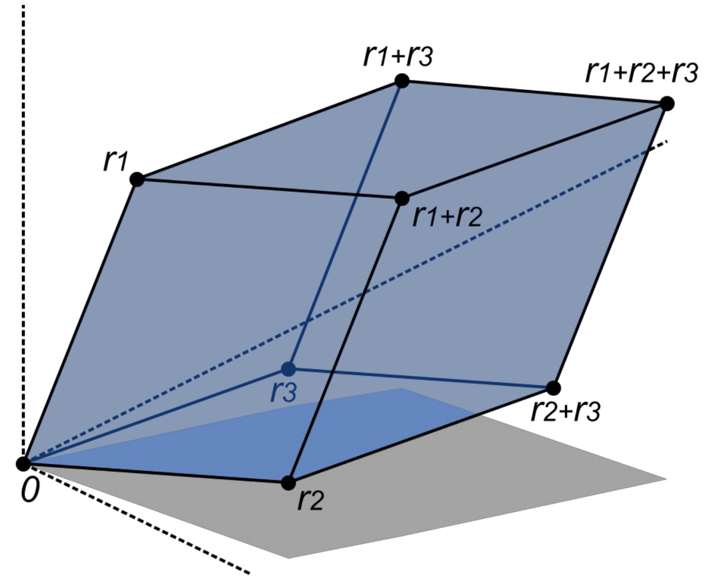
4 Determinants
（行列式）

4.4

APPLICATIONS OF DETERMINANTS

- Computation of A^{-1}
- Cramer's rule
- Volumes of boxes
- Formula for pivots



I. Computation of A^{-1} (用行列式求A的逆矩阵)

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

$$a_{s1} \quad a_{s2} \quad \cdots \quad a_{sn}$$

$$C_{t1} \quad C_{t2} \quad \cdots \quad C_{tn}$$

$$D_1=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

$$a_{t1}C_{t1}+a_{t2}C_{t2}+\ldots+a_{tn}C_{tn}=D$$

$$a_{s1}C_{t1}+a_{s2}C_{t2}+\ldots+a_{sn}C_{tn}=0$$

They have the same cofactor.
(其代数余子式相等!)

Let $\boldsymbol{A} = \left[a_{ij}\right]_{n\times n}$, and let $C_{ij} = (-1)^{i+j}|\boldsymbol{M}_{ij}|$.

Lemma 1 $\quad a_{11}C_{21} + a_{12}C_{22} + \cdots + a_{1n}C_{2n} = 0.$

Proof Let $\boldsymbol{B}$ be the matrix resulted from replacing row 2 of $\boldsymbol{A}$ by $a_{11}, a_{12}, \ldots, a_{1n}$, i.e.,

$$\boldsymbol{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Expanding $|\boldsymbol{B}|$ along the second row gives rise to

$$a_{11}C_{21} + a_{12}C_{22} + \cdots + a_{1n}C_{2n} = 0.$$

as claimed.

For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} (a_{11}a_{22} - a_{12}a_{21} \neq 0)$, we have

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

It can also be written as $A^{-1} = \dfrac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}.$

Define the **cofactor matrix** as

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}, \text{ and } A^* = C^{\mathrm{T}} = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

A^* is called the **adjoint matrix (adjugate matrix, 伴随矩阵)** of A.
Then, by Lemma 1 and its generalization, we have:

Theorem 2 Let A be an $n \times n$ matrix. Then

$$AA^* = A^*A = |A|I.$$

Moreover, A is invertible if and only if $|A| \neq 0$. Then

$$A^{-1} = \frac{1}{|A|}A^*.$$

Proof

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then $A^* = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$,

From the expansion rule,

$$AA^* = \begin{bmatrix} |A| & & & \\ & |A| & & \\ & & \ddots & \\ & & & |A| \end{bmatrix} = |A|I;$$

Similarly,

$$A^*A = |A|I.$$

If A is invertible, then $AA^{-1}=I$, thus $|A|\,|A^{-1}|=|I|=1$,

therefore $|A|\neq 0$.

If $|A|\neq 0$, then from $AA^{*}=A^{*}A=|A|I$ we can get

$$A\left(\frac{A^{*}}{|A|}\right)=\left(\frac{A^{*}}{|A|}\right)A=I,$$ therefore A is invertible, and $A^{-1}=\dfrac{A^{*}}{|A|}$.

Corollary Let A be an �×� matrix. If there exists an �×� matrix M, such that $AM=I$ **or** $MA=I$, then $M=A^{-1}$.

Proof Since $|A|\,|M|=|I|=1$, so $|A|\neq 0$, and A is invertible.

By left multiplying both sides of $AM=I$ by A^{-1}, we have $M=A^{-1}$.

Similarly, $MA=I$ will give us $M=A^{-1}$.

The method to find the inverse of a matrix

By Theorem 2, we can find the inverse of a matrix by the following steps (求逆矩阵的**伴随矩阵法**):

(1) Calculate the determinant of $A = [a_{ij}]$;

(2) If $|A| = 0$, then A is not invertible;

(3) If $|A| \neq 0$, then find the cofactor of each entry a_{ij} and the adjoint matrix of A, denoted by A^*, and finally we get

$$\boxed{A^{-1} = \frac{1}{|A|} A^*.}$$

Example 1 Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution $|A| = 1$, so A is invertible.

Thus $A^{-1} = A^*/|A| = A^* = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 2 Find the inverse of the matrix $\boldsymbol{A}=\begin{bmatrix}2 & 1 & 3\\ 1 & -1 & 1\\ 1 & 4 & -2\end{bmatrix}$.

Solution $|\boldsymbol{A}|=14$, so $\boldsymbol{A}$ is invertible.

The nine cofactors are

$$C_{11}=+\begin{vmatrix}-1 & 1\\ 4 & -2\end{vmatrix}=-2,\quad C_{12}=-\begin{vmatrix}1 & 1\\ 1 & -2\end{vmatrix}=3,\quad C_{13}=+\begin{vmatrix}1 & -1\\ 1 & 4\end{vmatrix}=5$$

$$C_{21}=-\begin{vmatrix}1 & 3\\ 4 & -2\end{vmatrix}=14,\quad C_{22}=+\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix}=-7,\quad C_{23}=-\begin{vmatrix}2 & 1\\ 1 & 4\end{vmatrix}=-7$$

$$C_{31}=+\begin{vmatrix}1 & 3\\ -1 & 1\end{vmatrix}=4,\quad C_{32}=-\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix}=1,\quad C_{33}=+\begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix}=-3$$

The adjugate matrix is the *transpose* of the matrix of cofactors.

Thus

$$\boldsymbol{A}^{*}=\begin{bmatrix}-2 & 14 & 4\\ 3 & -7 & 1\\ 5 & -7 & -3\end{bmatrix},$$

and $$\boldsymbol{A}^{-1}=\frac{1}{14}\begin{bmatrix}-2 & 14 & 4\\ 3 & -7 & 1\\ 5 & -7 & -3\end{bmatrix}=\begin{bmatrix}-1/7 & 1 & 2/7\\ 3/14 & -1/2 & 1/14\\ 5/14 & -1/2 & -3/14\end{bmatrix}.$$

Some Properties of Adjoint Matrices

$$AA^* = A^*A = |A|I.$$

当 $|A| \neq 0$, $|B| \neq 0$时，有

去掉“”

$$|A^*| = |A|^{n-1}.$$

$$\boxed{A^* = |A|A^{-1}};$$

$$(kA)^* = k^{n-1}A^*;$$

$$(A^*)^{-1} = \frac{A}{|A|};$$

$$(A^*)^* = |A|^{n-2}A;$$

$= |A^*|(A^*)^{-1} = |A|^n \cdot |A^{-1}| \frac{A}{|A|} = |A|^{n-2}A$

$$(A^*)^{-1} = (A^{-1})^*;$$

$= \frac{A}{|A|}$ $\quad = |A^{-1}|(A^{-1})^{-1}$

$$(A^*)^{\mathrm{T}} = (A^{\mathrm{T}})^*;$$

$= (|A| \cdot A^{-1})^{T}$ $\quad = |A^{T}|(A^{T})^{-1}$

$$(AB)^* = B^*A^*;$$

When $|A| = 0$, we have $|A^*| = 0$.

Example 3 Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of degree n with $|\boldsymbol{A}|=2$, $|\boldsymbol{B}|=3$. Find $\left|\boldsymbol{A}^{*}\boldsymbol{B}^{*}-\boldsymbol{A}^{*}\boldsymbol{B}^{-1}\right|$.

Solution $\left|\boldsymbol{A}^{*}\boldsymbol{B}^{*}-\boldsymbol{A}^{*}\boldsymbol{B}^{-1}\right|=\left|\boldsymbol{A}^{*}\right|\left|\boldsymbol{B}^{*}-\boldsymbol{B}^{-1}\right|$,

$$\left|\boldsymbol{A}^{*}\right|=\left|\boldsymbol{A}\right|^{n-1}, \qquad \boldsymbol{B}^{*}=\left|\boldsymbol{B}\right|\boldsymbol{B}^{-1},$$

therefore $\left|\boldsymbol{A}^{*}\boldsymbol{B}^{*}-\boldsymbol{A}^{*}\boldsymbol{B}^{-1}\right|=\left|\boldsymbol{A}\right|^{n-1}\left|\left(\left|\boldsymbol{B}\right|-1\right)\boldsymbol{B}^{-1}\right|$

$$=\left|\boldsymbol{A}\right|^{n-1}\left(\left|\boldsymbol{B}\right|-1\right)^{n}\left|\boldsymbol{B}^{-1}\right|=2^{n-1}\cdot 2^{n}\cdot\frac{1}{3}=\frac{2^{2n-1}}{3}.$$

II. The solution of $Ax=b$ --- Cramer's rule (解方程组的Cramer法则)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1,} \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2,} \end{cases}$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \qquad D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

$$x_1 = \frac{\boxed{b_1 a_{22} - a_{12} b_2}}{a_{11}a_{22} - a_{12}a_{21}}, \qquad x_2 = \frac{\boxed{a_{11} b_2 - b_1 a_{21}}}{a_{11}a_{22} - a_{12}a_{21}}.$$

So the solution is

$$x_1 = \frac{D_1}{D},$$

$$x_2 = \frac{D_2}{D}.$$

Consider a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Replacing column j of $\boldsymbol{A}$ with $\boldsymbol{b}$, and then by $\boldsymbol{Ax} = \boldsymbol{b}$, we obtain a matrix whose determinant is D_j .

$$D_j = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & \sum_{k=1}^{n} a_{1k} x_k & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & \sum_{k=1}^{n} a_{nk} x_k & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1j} x_j & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{nj} x_j & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = x_j D,$$

leading to the *Cramer's rule* for solving systems of linear equations.

Theorem 3 ***(Cramer's rule)*** *Let* $\boldsymbol{A}$ *be an invertible* $n\times n$ *matrix. For any* $\boldsymbol{b}$ *in* $\mathbf{R}^n$, *the unique solution* $\boldsymbol{x}$ *of* $A\boldsymbol{x}=\boldsymbol{b}$ *has entries given by*

$$x_j = \frac{D_j}{D}, \quad j = 1,\ldots,n,$$

Each entry of $\boldsymbol{x}$ *is a ratio of two determinants.*

where D *is the determinant of the coefficient matrix, and* D_j *is the determinant of the following matrix:*

$$\begin{bmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}.$$

(其中D_j是把系数行列式D中第j列的元素用方程组右端的常数项代替后所得到的n阶行列式)

Example 4 Using the Cramer's rule, the solution of

$$\begin{cases} x_1 + 3x_2 = 0, \\ 2x_1 + 4x_2 = 6. \end{cases}$$

is

$$x_1 = \frac{\begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{-18}{-2} = 9, \qquad x_2 = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{6}{-2} = -3.$$

Cramer 法则的优点

(1)具有重要的理论价值:

给出了简洁的解的表达公式，揭示了解与系数、常数项的关系;

(2)指出了当 $D \neq 0$ 时，有唯一解.

Cramer 法则的缺点

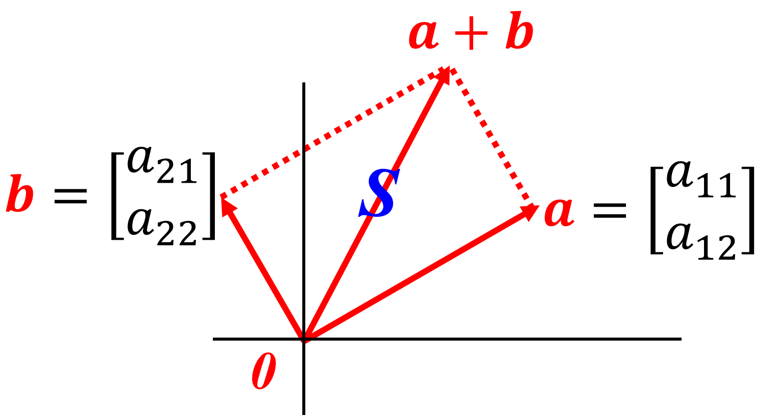
(1) 适用的范围窄:

方程个数与未知量个数必须相等;

系数行列式 $D \neq 0$.

(2) 计算量大.

III. A Geometrical Application – The connection between the determinant and the volume (体积与行列式)



Easy case: when all angles are *right angles*—the edges are perpendicular, and the box is rectangular.

In $\mathbf{R}^2$,The four points $\mathbf{0}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}+\boldsymbol{b}$ form a rectangle S.

If $\boldsymbol{a}$, $\boldsymbol{b}$ are *orthogonal*, then the area of this rectangle equals $||\boldsymbol{a}|| \cdot ||\boldsymbol{b}||$.

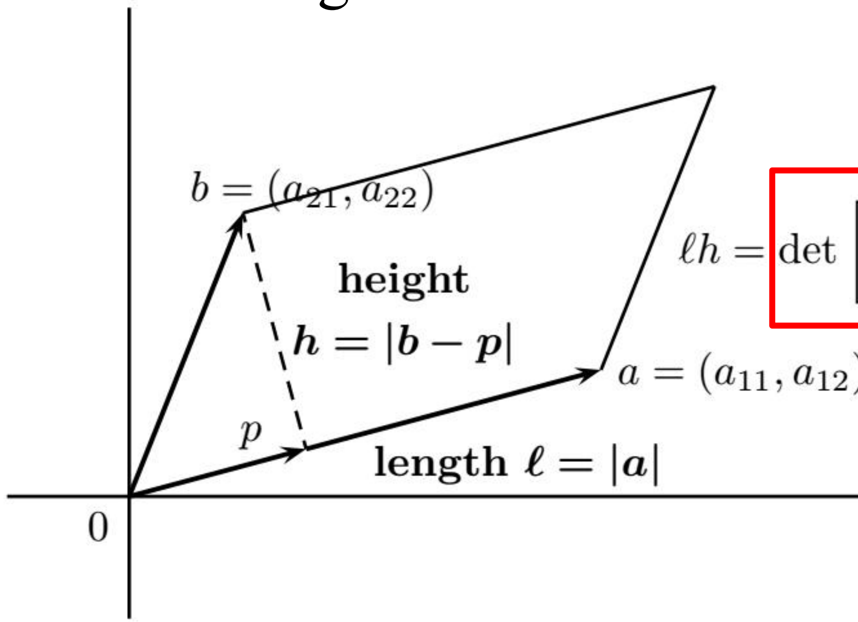
Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$|\boldsymbol{A}|^2 = |\boldsymbol{A}||\boldsymbol{A}^{\mathrm{T}}| = |\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}| = (a_{11}^2 + a_{12}^2)(a_{21}^2 + a_{22}^2)$$
$$= ||\boldsymbol{a}||^2 \cdot ||\boldsymbol{b}||^2.$$

So $|\boldsymbol{A}|$ (actually the absolute value of $|\boldsymbol{A}|$) equals the area of S.

Note: The sign of $|\boldsymbol{A}|$ indicates whether the edges form a “right-handed” set of coordinates, or a “left-handed” system.

General case: If the angles are not *right angles*—the volume is not the product of the lengths.



$$\ell h = \boxed{\det \begin{bmatrix} a \\ b - p \end{bmatrix} = \det \begin{bmatrix} a \\ b \end{bmatrix}}$$

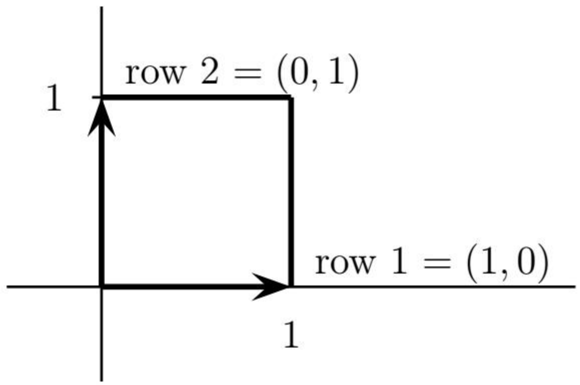
$$\boldsymbol{p} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}$$

The "volume" (area) of a parallelogram equals lh (base times height).

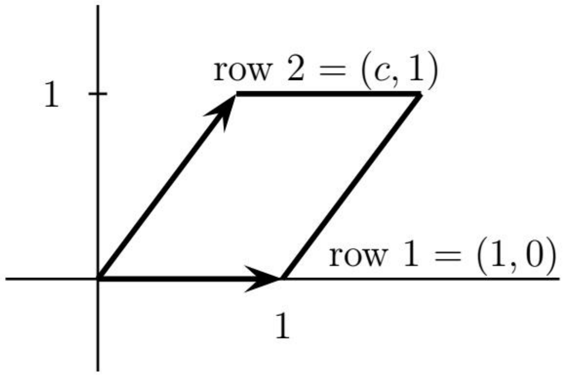
The vector $\boldsymbol{b} - \boldsymbol{p} = \boldsymbol{b} - \dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}$ is the second row $\boldsymbol{b}$ minus its projection $\boldsymbol{p}$ onto the first row.

The key point is: $|\boldsymbol{A}|$ is unchanged when a multiple of row 1 is subtracted from row 2. *We can change the parallelogram to a rectangle*, where it is already proved that volume = determinant.

For instance,



$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$



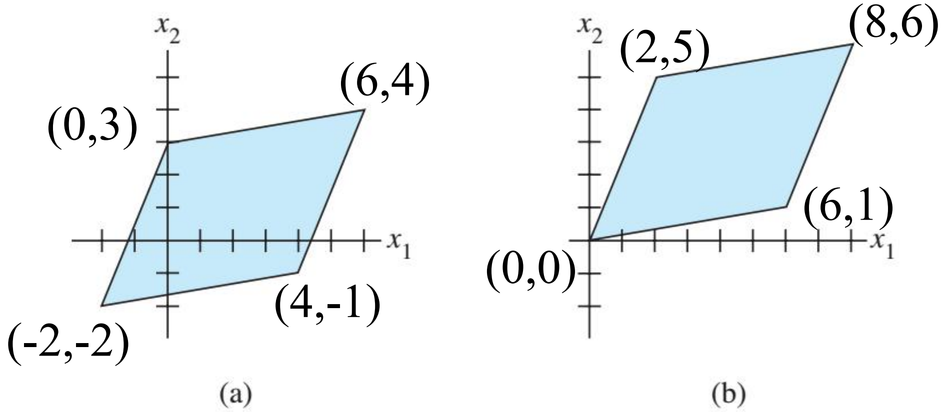
$$\begin{vmatrix} 1 & 0 \\ c & 1 \end{vmatrix} = 1$$

The parallelogram has unit base and unit height; its area is also 1.

These determinants give the volumes (or areas, since we are in two dimensions).

https://www.geogebra.org/m/nemeagcq

Example 5 Calculate the area of the parallelogram determined by the points $(-2,\ -2)$, $(0,\ 3)$, $(4,\ -1)$, and $(6,\ 4)$. See Fig(a).



Solution First translate the parallelogram to one having the origin as a vertex.

For example, subtract the vertex $(-2,\ -2)$ from each of the four vertices.

The new parallelogram has the same area, and its vertices are $(0,\ 0)$, $(2,\ 5)$, $(6,\ 1)$, and $(8,\ 6)$. See Fig(b).

This parallelogram is determined by the rows of

$$\boldsymbol{A} = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} \qquad \begin{bmatrix} \vec{a}^T \\ \vec{b}^T \end{bmatrix}$$

Since $|\boldsymbol{A}| = -28$, the area of the parallelogram is 28.

In n dimensions, it takes longer to make each box rectangular, but the idea is the same.

The volume and determinant are unchanged if we subtract from each row its projection onto the space spanned by the preceding rows—leaving a perpendicular "height vector".

This Gram-Schmidt process produces orthogonal rows, with volume = determinant.

So the same equality must have held for the original rows.

Theorem 4 *The volume of a box formed by n linearly independent vectors* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ *is equal to the determinant* $|\boldsymbol{A}|$, *where* $\boldsymbol{A}$ *is a matrix with n rows being the n vectors* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$.

https://www.geogebra.org/m/hq5fjbtn

IV. A Formula for the Pivots (LU分解与行列式)

Let $\boldsymbol{A}$ be a matrix of degree n, and let $\boldsymbol{A}_k$ be the *leading submatrix* of degree k, consisting of the entries of $\boldsymbol{A}$ in the first k rows and first k columns.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}.$$

$$\boldsymbol{A}_1 = [2],$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix},$$

$$\boldsymbol{A}_3 = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}.$$

Let

$$\boldsymbol{A} = \begin{bmatrix} a & b & e \\ c & d & f \\ g & h & i \end{bmatrix}.$$

Then, the (1,1)-entry a is a pivot if and only if $|\boldsymbol{A}_1| = a \neq 0$.

$$\boldsymbol{A} \to \begin{bmatrix} a & b & e \\ 0 & (ad-bc)/a & (af-ec)/a \\ g & h & i \end{bmatrix}.$$

Thus $(ad-bc)/a$ is a pivot if and only if $|\boldsymbol{A}_2| = ad - bc \neq 0$.
Then in the row echelon form for $\boldsymbol{A}$, the (3,3)-entry is non-zero (pivot) if and only if $|\boldsymbol{A}_3| = |\boldsymbol{A}| \neq 0$.

Furthermore, if $\boldsymbol{A} = \boldsymbol{LDU}$, then

$$\boldsymbol{A} = \boldsymbol{LDU} = \begin{bmatrix} 1 & & \\ c/a & 1 & \\ * & * & 1 \end{bmatrix} \begin{bmatrix} a & & \\ & (ad-bc)/a & \\ & & * \end{bmatrix} \begin{bmatrix} 1 & b/a & * \\ & 1 & * \\ & & 1 \end{bmatrix}$$

Therefore, $\boldsymbol{A}_1 = \boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1$, $\boldsymbol{A}_2 = \boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2$.
This is a general rule if there are no row exchanges.

Theorem 5 *If $\boldsymbol{A}$ is factored into $\boldsymbol{LDU}$, then the upper left corners satisfy $\boldsymbol{A}_k = \boldsymbol{L}_k\boldsymbol{D}_k\boldsymbol{U}_k$, and $|\boldsymbol{A}_k| = d_1 d_2 \ldots d_k$. For every k, the submatrix $\boldsymbol{A}_k$ is going through a Gaussian elimination of its own.*

Note: The product of the first k pivots is the determinant of $\boldsymbol{A}_k$. we can isolate each pivot d_k as a ***ratio of determinants***:

Formula for pivots:

$$\frac{|\boldsymbol{A}_k|}{|\boldsymbol{A}_{k-1}|} = \frac{d_1 d_2 \ldots d_k}{d_1 d_2 \ldots d_{k-1}} = d_k.$$

In our example above, the second pivot was exactly this ratio $(ad - bc)/a = |\boldsymbol{A}_2|/|\boldsymbol{A}_1|$.

By convention $|\boldsymbol{A}_0| = 1$, so that the first pivot is $a/1 = a$.

Multiplying together all the individual pivots, we recover

$$d_1 d_2 \ldots d_n = \frac{|\boldsymbol{A}_1|}{|\boldsymbol{A}_0|}\frac{|\boldsymbol{A}_2|}{|\boldsymbol{A}_1|}\cdots\frac{|\boldsymbol{A}_n|}{|\boldsymbol{A}_{n-1}|} = \frac{|\boldsymbol{A}_n|}{|\boldsymbol{A}_0|} = |\boldsymbol{A}|.$$

$$\frac{|\boldsymbol{A}_k|}{|\boldsymbol{A}_{k-1}|} = \frac{d_1 d_2 \ldots d_k}{d_1 d_2 \ldots d_{k-1}} = d_k.$$

From equation above we can finally get the conclusion:

The pivot entries are all nonzero whenever the numbers $|\boldsymbol{A}_k|$ are all nonzero.

Theorem 6 *A matrix $\boldsymbol{A}$ of degree n can be decomposed as*

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U},$$

*where **L** is lower triangular and **U** is non-singular upper triangular (which means that elimination can be completed without row exchanges) if and only if the leading submatrices $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ are all non-singular.*

Key words:

Applications:

Compute inverse of a matrix;

Cramer's rule;

Volume & Determinant;

LU & Determinant

Homework

See Blackboard