题 号	1	2	3	4	5	6	7	8
分 值	15 分	25 分	10 分	12 分	10 分	10 分	10 分	8 分

本试卷共 (8) 大题, 满分 (100) 分. **请将所有答案写在答题本上**.

This exam includes **8** questions and the score is 100 in total. **Write all your answers on the examination book**.

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1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

   (共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

   (1) The system

$$\begin{cases} u+2v=b \\ 2u+3v=3b \\ 3u+4v=4 \\ 4u-4v=0 \end{cases}$$

   is consistent

   (A) for any b.

   (B) only for $b=-1$.

   (C) only for $b=1$.

   (D) none of the above.

   下面这个线性方程组

$$\begin{cases} u+2v=b \\ 2u+3v=3b \\ 3u+4v=4 \\ 4u-4v=0 \end{cases}$$

   (A) 对任何的 b 都有解.

   (B) 只有当 $b=-1$ 有解.

   (C) 只有当 $b=1$ 有解.

   (D) 以上都不是.

   (2) Let $A,\ B$ be $n\times n$ square matrices, and $(AB)^2=I$, where I is the $n\times n$ identity matrix, then

   (A) $A^{-1}=B$.

   (B) $AB=-I$.

   (C) $AB=I$.

   (D) $A^{-1}=BAB$.

   设 $A,\ B$ 为 n 阶方阵, 且 $(AB)^2=I$, 其中 I 为 n 阶单位矩阵, 则必有

   (A) $A^{-1}=B$.

   (B) $AB=-I$.

   (C) $AB=I$.

(D) $A^{-1} = BAB$.

(3) Suppose $\eta_1,\ \eta_2$ are two different solutions to the homogeneous system of linear equations $Ax = 0$ in n unknowns, and rank $(A) = n - 1$, then the general solution to $Ax = 0$ can be expressed as

(A) $k\eta_1$, k is an arbitrary constant.

(B) $k\eta_2$, k is an arbitrary constant.

(C) $k(\eta_1 - \eta_2)$, k is an arbitrary constant.

(D) $k(\eta_1 + \eta_2)$, k is an arbitrary constant.

设 $\eta_1,\ \eta_2$ 是 n 元齐次线性方程组 $Ax = 0$ 的两个不同的解. 如果 rank $(A) = n - 1$, 则 $Ax = 0$ 的通解是

(A) $k\eta_1$, k 是任意常数.

(B) $k\eta_2$, k 是任意常数.

(C) $k(\eta_1 - \eta_2)$, k 是任意常数.

(D) $k(\eta_1 + \eta_2)$, k 是任意常数.

(4) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} a_{12}+a_{13} & a_{11} & a_{13} \\ a_{22}+a_{23} & a_{21} & a_{23} \\ a_{32}+a_{33} & a_{31} & a_{33} \end{bmatrix}$, $P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, then $B =$

(A) P_1AP_2.

(B) AP_2P_1.

(C) AP_1P_2.

(D) P_2AP_1.

设 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} a_{12}+a_{13} & a_{11} & a_{13} \\ a_{22}+a_{23} & a_{21} & a_{23} \\ a_{32}+a_{33} & a_{31} & a_{33} \end{bmatrix}$, $P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, 则 $B =$

(A) P_1AP_2.

(B) AP_2P_1.

(C) AP_1P_2.

(D) P_2AP_1.

(5) Let $A,\ B$ be $n \times n$ matrices. Which of the following statements is correct？

(A) If $AB = B$, then B is the identity matrix.

(B) If $A^2 = A$ and A is invertible, then A must be the identity matrix.

(C) If A is invertible, then $ABA^{-1} = B$.

(D) If $AB = BA$, then AB is a symmetric matrix.

设 A, B 都为 n 阶矩阵. 下列哪个论断是正确的？

(A) 如果 $AB = B$, 则 B 是单位方阵.

(B) 如果 $A^2 = A$, 且 A 为可逆矩阵, 则 A 一定为单位矩阵.

(C) 如果 A 是可逆方阵, 则 $ABA^{-1} = B$.

(D) 如果 $AB = BA$, 则 AB 是对称矩阵.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) If $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} X = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, then $X =$ __________.

若 $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} X = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, 则 $X =$ __________.

(2) If the vectors $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 2 \\ 3 \\ t \end{bmatrix}$ are linearly dependent, then $t =$ __________.

已知向量组 $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 2 \\ 3 \\ t \end{bmatrix}$ 线性相关, 则 $t =$ __________.

(3) Let A be a 3×3 matrix with rank $(A) = 1$, $B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & k \\ 5 & 5 & 15 \end{bmatrix}$. If $AB = O$, where O is the zero matrix, then $k =$__________.

设 A 为一个秩为 1 的 3 阶矩阵 , $B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & k \\ 5 & 5 & 15 \end{bmatrix}$. 如果 $AB = O$, 其中 O 为零矩阵, 则 $k =$__________.

(4) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Then $\dim N(A^T A) =$__________.

设 $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. 则 $\dim N(A^T A) =$__________.

(5) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ 1 \\ -5 \end{bmatrix}$.

Then the least squares solution to $Ax = b$ is $\hat{x} =$__________.

设 $A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ 1 \\ -5 \end{bmatrix}$.
则 $Ax = b$ 的最小二乘解是 $\hat{x} =$__________.

3. (10 points) Suppose there are three linearly independent solutions to the system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = -1 \\ 4x_1 + 3x_2 + 5x_3 - x_4 = -1 \\ ax_1 + x_2 + 3x_3 + bx_4 = 1 \end{cases}$$

(a) Prove that the coefficient matrix of the system has the rank: $\operatorname{rank}(A) = 2$;

(b) Find the values of a, b, and solve the system of linear equations.

已知线性方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = -1 \\ 4x_1 + 3x_2 + 5x_3 - x_4 = -1 \\ ax_1 + x_2 + 3x_3 + bx_4 = 1 \end{cases}$$

有三个线性无关的解.

(a) 证明：方程组系数矩阵 A 的秩 $\operatorname{rank}(A) = 2$;

(b) 求 a, b 的值及方程组的通解.

4. (12 points) Let A be the matrix

$$A = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & a^3 & 1 & 0 \\ 0 & a^4 & 0 & 1 \end{bmatrix}, \ a \neq 0.$$

(a) Factor A into LU.

(b) Find A^{-1}.

(c) Find the solution of the equation $Ax = b$, if $b = \begin{bmatrix} 1 \\ a \\ a^2 \\ a^3 \end{bmatrix}$.

设

$$A = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & a^3 & 1 & 0 \\ 0 & a^4 & 0 & 1 \end{bmatrix}, \ a \neq 0.$$

(a) 求 A 的 LU 分解.

(b) 求 A^{-1}.

(c) 如果 $b = \begin{bmatrix} 1 \\ a \\ a^2 \\ a^3 \end{bmatrix}$, 求解 $Ax = b$.

5. (10 points) Let

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}.$$

(a) Find a basis for the nullspace of A.

(b) Find a basis for the row space of A.

(c) Find a basis for the column space of A.

(d) For each column vector which is not in the basis that you obtained in part (c), express it as a linear combination of the basis vectors for the column space of A (as obtained in part (c)) .

设

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}.$$

(a) 求矩阵 A 的零空间的一组基.

(b) 求矩阵 A 的行空间的一组基.

(c) 求矩阵 A 的列空间的一组基.

(d) 把矩阵 A 不在 (c) 中基向量组中的列向量表示成 (c) 中得到的基向量的线性组合.

6. (10 points) Let V and W be the following subspaces of the space $\mathbb{R}^3$:

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x - y + z = 0 \right\}, \ W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 0 \right\}.$$

(a) Find two orthogonal vectors $v_1, v_2 \in \mathbb{R}^3$ such that $V = \text{span}(v_1, v_2)$, i.e., V is spanned by v_1, v_2.

(b) Find a basis for the intersection L of the subspaces V and W (i.e., $L = V \cap W$).

(c) Find the orthogonal projection p of the vector $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto L.

设 V 和 W 为 $\mathbb{R}^3$ 的两个子空间:

$$V=\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix}\in\mathbb{R}^3 : x-y+z=0\right\},\ W=\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix}\in\mathbb{R}^3 : z=0\right\}.$$

(a) 求两个正交的向量 $v_1, v_2 \in \mathbb{R}^3$, 使得 $V=\operatorname{span}(v_1,v_2)$, 也即 V 由 v_1, v_2 生成.

(b) 求子空间 V 和 W 的交 L 的一组基, 这里 $L=V\cap W$.

(c) 求 $v=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 投影到 L 的投影 p.

7. (10 points) Let $A=\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 0 & -4 & -2 \end{bmatrix}$, $\xi_1=\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$.

(a) Find all the vectors ξ_2 and ξ_3 which satisfy the equations $A\xi_2=\xi_1,\ A^2\xi_3=\xi_1$.

(b) For any vectors $\xi_2,\ \xi_3$ as described above, show that $\xi_1,\ \xi_2,\ \xi_3$ are linearly independent.

设 $A=\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 0 & -4 & -2 \end{bmatrix}$, $\xi_1=\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$.

(a) 求满足 $A\xi_2=\xi_1,\ A^2\xi_3=\xi_1$ 的所有向量 $\xi_2,\ \xi_3$.

(b) 对以上的任意向量 $\xi_2,\ \xi_3$, 证明: $\xi_1,\ \xi_2,\ \xi_3$ 线性无关.

8. (8 points) Let $u,\ v\in\mathbb{R}^n$ and $U,\ V$ be $n\times m$ real matrices.

(a) If $v^Tu\neq 1$, show that $A=I_n-uv^T$ is invertible, and find A^{-1}.

(b) If $B=I_n-UV^T$ is invertible, find B^{-1}.

Where I_n is the $n\times n$ identity matrix.

设 $u,\ v\in\mathbb{R}^n$, $U,\ V$ 为 $n\times m$ 实矩阵.

(a) 如果 $v^Tu\neq 1$, 证明 $A=I_n-uv^T$ 是可逆的, 并求 A^{-1}.

(b) 如果 $B=I_n-UV^T$ 可逆, 求 B^{-1}.

其中 I_n 是 n 阶单位阵.