

Step-1

The matrix corresponding to the quadratic $ax_1^2 + bx_2^2 + cx_3^2 + 2hx_1x_2 + 2gx_1x_3 + 2fx_2x_3$ is

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}.$$

This is the conversion of quadratic form into matrix.

Step-2

(a)

Given that $f_1 = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

The symmetric matrix corresponding to it is;

$$A_1 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

Step-3

Now the matrix A_2 corresponding to $f_2 = x_1^2 + 2x_2^2 + 11x_3^2 - 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ is;

$$A_2 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -2 & 11 \end{pmatrix}.$$

Therefore,

$$\boxed{A_1 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -2 & 11 \end{pmatrix}}.$$

Step-4

(b).

Given that $f_1 = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

$$= (x_1 - x_2 - x_3)^2$$

Therefore, f_1 is a perfect square

For

$$x_1 = 2$$

$$x_2 = x_3$$

$$= 1,$$

$$f_1(x_1, x_2, x_3) = 0$$

Therefore, f_1 is not positive definite

Therefore,

$$\begin{aligned} f_1 &= (x_1 - x_2 - x_3)^2 \\ &= 0 \end{aligned}$$

$$\text{when } x_1 - x_2 - x_3 = 0.$$

Step-5

(c).

Now,

$$\begin{aligned} f_2 &= x_1^2 + 2x_2^2 + 11x_3^2 - 2x_1x_2 - 2x_1x_3 - 4x_2x_3 \\ &= x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3 - 2x_1x_3 + x_2^2 + 10x_3^2 - 6x_2x_3 \\ &= (x_1 - x_2 - x_3)^2 + x_2^2 + 9x_3^2 - 6x_2x_3 + x_3^2 \\ &= (x_1 - x_2 - x_3)^2 + (x_2 - 3x_3)^2 + x_3^2. \end{aligned}$$

Now, write L as;

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$$

So that,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= LL^T.$$

Therefore,
$$f_2 = (x_1 - x_2 - x_3)^2 + (x_1 - 3x_3)^2 + x_3^2; L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}.$$