Step-1

a) The augmented matrix of the system is $B=\left[A|b\right]$

$$=\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 2 & 4 & 0 & 1 & b_3 \end{array}\right]$$

$$\xrightarrow{R_3-2R_1}\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & -5 & b_3-2b_1 \end{array}\right]$$

$$\xrightarrow{R_3/-5\leftrightarrow R_2}\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 1 & (2b_1-b_3)/5 \\ 0 & 0 & 0 & 0 & b_2 \end{array}\right]$$

The portion of the coefficient matrix is reduced and so, the number of non zero rows = 2

If $b_2 \neq 0$, then the number of non zero rows in the augmented matrix *B* will be 3

In such case, by rewriting the bottom equation, we get $0x+0y+0z+0w=b_2$

In other words, $0=b_2$ which is a contradiction to our supposition.

So, the system becomes inconsistent.

In other words, the system $Ax=b$ has no solution.

Therefore, to have a solution to the system, we must have $b_2=0$

Step-2

b) To get the null space of A, we solve the system $Ax=0$ using the above reduced matrix.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_4=0, x_1+2x_2-3x_4=0$$

Consequently, $x_4=0, x_1=-2x_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$$

$= m\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + n\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ where $x_2 = m, x_3 = n$ are the parameters.

So, the basis for null space $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Step-3

c) Putting $b_2 = 0$ in $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, suppose $b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

Then the get the solution for the non homogeneous system, we have

$$Ax = b \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} &\Rightarrow x_1 + 2x_2 + 3x_4 = 1 \\ &\quad -x_4 = 1 \\ &\quad x_4 = \frac{-1}{5} \end{aligned}$$

$$x_1 = 1 - 2x_2 - 3x_4 = 1 + \frac{3}{5} - 2x_2$$
$$= \frac{8}{5} - 2x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (8/5) - 2x_2 \\ x_2 \\ x_3 \\ -1/5 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 8 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 4 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Putting $x_2 = t, x_3 = s$ parameters, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 8 \\ 0 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Step-4

d) $A^T = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

Using row operations on this matrix,

$$\xrightarrow[R_4 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\xrightarrow{R_4 \leftrightarrow R_2 / -5} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the reduced matrix and there are two non zero rows in it. So, we follow that there are two linearly independent rows in the matrix.

It is spanned by any two non zero rows of A^T or any two non zero columns of A.

Step-5

(e) Rank of A^T is the number of non zero rows in the reduced form of A^T

It is nothing but 2.