

## Step-1

(a)

Consider the matrix  $A$ ,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The objective is to find the conditions if the matrix  $A$  is non-singular.

## Step-2

The matrix  $A$  is nonsingular if  $\det(A) \neq 0$ .

Here, the matrix  $\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}$  is a diagonal matrix.

If any one of  $d_1, d_2, d_3$  is 0 then  $\det(A) = 0$

Therefore, the matrix  $A$  is nonsingular when,  $d_1 d_2 d_3 \neq 0$ .

## Step-3

(b)

Consider the system  $Lc = b$ :

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b$$

The objective is to solve the system  $Ax = b$ , starts with  $Lc = b$ .

## Step-4

Consider the matrix,

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & -d_1 & 0 \\ 0 & d_2 & -d_2 \\ 0 & 0 & d_3 \end{bmatrix} \\
&= \begin{bmatrix} d_1 & -d_1 & 0 \\ -d_1 & d_1 + d_2 & -d_2 \\ 0 & -d_2 & d_2 + d_3 \end{bmatrix}
\end{aligned}$$

## Step-5

Consider the system,

$$\begin{aligned}
& \quad \quad \quad Ax = b \\
& \begin{bmatrix} d_1 & -d_1 & 0 \\ -d_1 & d_1 + d_2 & -d_2 \\ 0 & -d_2 & d_2 + d_3 \end{bmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
& \begin{array}{l} R_3 \rightarrow R_1 + R_2 + R_3, \\ R_2 \rightarrow R_2 + R_1 \end{array} \begin{bmatrix} d_1 & -d_1 & 0 \\ 0 & d_2 & -d_2 \\ 0 & 0 & d_3 \end{bmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

The equations form is,

$$\begin{aligned}
d_1 u - d_1 v &= 0 \\
d_2 v - d_2 w &= 0 \\
d_3 w &= 1
\end{aligned}$$

## Step-6

Solve by back substitution,

$$d_3 w = 1$$

From part (a),  $d_3 \neq 0$ .

$$w = \frac{1}{d_3}.$$

From second equation,

$$\begin{aligned}
 d_2(v-w) &= 0 \\
 v-w &= 0, \\
 v &= w \\
 &= \frac{1}{d_3}
 \end{aligned}$$

Since,  $d_3 \neq 0$ , and  $d_2 \neq 0$ .

From first equation,

$$\begin{aligned}
 d_1(u-v) &= 0 \\
 u-v &= 0 \\
 u &= v \\
 &= \frac{1}{d_3}
 \end{aligned}$$

Since,  $d_3 \neq 0$ , and  $d_1 \neq 0$ .

## Step-7

Therefore,

$$u = v = w = \frac{1}{d_3}, d_3 \neq 0.$$

Hence, the solution of the system is,

$$\boxed{x = \begin{bmatrix} 1/d_3 \\ 1/d_3 \\ 1/d_3 \end{bmatrix}, d_3 \neq 0.}$$