## Linear Algebra I Final Examination Fall 2018 A

Department: Math Class: Name:

Answer all parts of Questions (1)-(11). Total is 100 points.	
(1) (12 points, 2 points each) True or false. No need to justify.	
(a) The diagonal entries of an $n \times n$ $(n > 1)$ real symmetric positive of are positive.	<del></del>
(b) If A is similar to B, then $A^2$ is similar to $B^2$ .	Trhe
(c) If A and B are diagonalizable, so is $AB$ .	Fallse
(c) If A and B are diagonalizable, so is $\overrightarrow{AB} = A = A = A$ (d) If A is a $\times 3$ skew-symmetric $(A^T = -A)$ , then $ A  = 0$ .	Time
(e) If $A$ is negative definite, then all the upper left submatrices $A_k$ of $A$ determinants.	have negative
(f) Let $A$ be an $\underline{n \times n}$ matrix, then the number of nonzero eigenvalues of the multiplicities) is equal to the rank of $A$ . $X \longrightarrow X \longrightarrow$	of A (counting False)

- (2) (9 points, 3 points each) Fill in the blanks.

  - (b) If  $A \in \mathbb{R}^{3\times 3}$  has eigenvalues 0, 1, 2, then the eigenvalues of A(A-I)(A-2I) are 0, 0, 0
  - (c) A box has edges from (0,0,0) to (3,1,1),(1,3,1),(1,1,3) then its volume is \_\_\_\_\_\_.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$\lambda_{1} = \lambda_{1}, \quad \lambda_{2} = -\lambda_{1}, \quad \lambda_{3} = \lambda_{4} = \lambda_{5}$$

- (i) Find all the eigenvalues of A and their associated eigenvectors.
- (ii) Is A diagonalizable? Explain why.

## (4) (9 points) Let

(i) Verify that 
$$A$$
 is Hermitian.  $\bigvee A \vdash \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix}$ .

(ii) Verify that  $A$  is Hermitian.  $\bigvee A \vdash \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix} = A$ 

(ii) Find a unitary matrix  $U$  that diagonalizes  $A$ .

$$\lambda_1 = -1, \quad \lambda_2 = b$$
(5) (12 points) Let

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

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- (i) Find all the singular values of A.
- (ii) Find the singular value decomposition of A, in other words, find orthogonal matrices U and V, such that  $A = U\Sigma V^T$ .

$$(6)$$
  $(8 points)$  Let

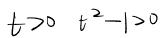
$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\lambda_{1} = 0, \quad \lambda_{2} = 2, \quad \lambda_{3} = 2$$

(i) Find an orthogonal matrix Q and a diagonal matrix  $\Lambda$  such that  $A = Q\Lambda Q^T$ .

(ii) Find 
$$A^k$$
, where  $k$  is a positive integer.

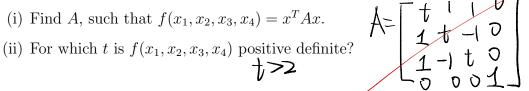
$$A^k \begin{bmatrix} \frac{1}{12} & 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 2^k \end{bmatrix} \begin{bmatrix} \frac{1}{12} & 0 & \frac{1}{12} \\ \frac{1}{12} & 0 & -\frac{1}{12} \\ 0 & 0 & 2^k \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 2^k & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2^{k-1} & 0 & -2^{k-1} \\ 0 & 2^k & 0 \\ -2^{k-1} & 0 & 2^{k-1} \end{bmatrix}$$



(7) (8 points) Consider the following quadratic form

$$f(x_1, x_2, x_3, x_4) = t(x_1^2 + x_2^2 + x_3^2) + x_4^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3.$$



- (8) (10 points) Let N be a normal matrix  $(N^H N = NN^H)$ .
  - (i) Show that  $||Nx|| = ||N^Hx||$  for every vector x.



- (ii) Deduce that the <u>ith</u> row of N has the same length as the <u>ith</u> column.
- (iii) If N is upper triangular, then N must be diagonal.
- $A = Q \wedge Q^T = |Q \wedge Q^T + Q InQ^T|$ (9) (8 points) Prove the following two statements:
  - (i) Suppose A is an  $n \times n$  real symmetric positive definite matrix, then  $|A + I_n| > 1$ .

- (ii) Let A be an  $n \times n$  matrix, then  $A^T A$  is similar to  $AA^T$ .  $A = \left( \begin{array}{c} \bigwedge \\ \searrow \end{array} \right) \bigvee^{\top} A^{\top} = \bigvee \sum_{i} \bigcup_{j=1}^{T} A^{T} = \sum_{i} \bigcup_{j=$  $= |Q| \cdot |\underbrace{\Lambda + I}_{1} \cdot |Q^{T}| > 1$
- (10) (6 points) Let A be an  $n \times n$  real matrix. If  $A^k = O$  for some positive integer k, then
  - (i) Show that all the eigenvalues of a nilpotent matrix must be zero.  $\Rightarrow A^k x = \lambda^k x$ (ii) Prove that a nonzero nilpotent matrix can not be symmetric.  $(A^k \Rightarrow 0, x \neq 0) \Rightarrow \lambda^k x = \lambda^k x$ (if  $A = A^T$  and  $A^k = 0$   $A = Q \land Q^T = D$

(11) (8 points) Let 
$$A$$
 be an  $n \times n$  real symmetric positive definite matrix, and  $\alpha \in \mathbb{R}^n$  be a

nonzero vector. Consider

$$M = \begin{bmatrix} A & \alpha \\ \alpha^T & b \end{bmatrix} := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{d} \mathbf{A}^{\mathsf{T}} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{d} \\ \mathbf{0} & \mathbf{b} - \mathbf{d}^{\mathsf{T}} \mathbf{A}^{\mathsf{T}} \mathbf{d} \end{bmatrix}$$

Here b is a real number.

$$dot(M) = |A| \cdot |b - a^{T}A^{-1}\alpha|$$

- (i) Under what condition on b is M positive definite?  $b > \alpha^{T} A^{-1} A$
- (ii) In the case that M is positive semidefinite (not positive definite), find a basis for  $b = \alpha^T A^{-1} \alpha$ the nullspace of M, N(M).

