

Step-1

(a)

The objective is to find the dimension of the column space of matrix A .

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

Step-2

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

Let,

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Let $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That is;

$$c_1 + c_2 = 0$$

$$c_1 + 3c_2 + c_3 = 0$$

$$3c_1 + c_2 - c_3 = 0$$

From the above system of equations,

$$\begin{aligned}(3c_1 + c_2 - c_3) + (c_1 + 3c_2 + c_3) &= 0 \\ 4c_1 + 4c_2 &= 0 \\ c_1 &= -c_2\end{aligned}$$

Now,

$$\begin{aligned}c_3 &= -c_1 - 3c_2 \\ (\text{Since } c_1 &= -c_2) \\ &= c_2 - 3c_2 \\ &= -2c_2\end{aligned}$$

$$\text{Therefore } -c_2v_1 + c_2v_2 - 2c_2v_3 = 0$$

$$\text{If } c_2 = 1, -v_1 + v_2 - 2v_3 = 0$$

Therefore v_1, v_2, v_3 are linearly dependent.

$$\text{If } c_1v_1 + c_2v_2 = 0$$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That is;

$$c_1 + c_2 = 0 \quad \text{--- (1)}$$

$$c_1 + 3c_2 = 0 \quad \text{--- (2)}$$

$$3c_1 + c_2 = 0 \quad \text{--- (3)}$$

$$\text{Apply } (3 \times (1)) - (2);$$

This implies;

$$\begin{aligned}2c_2 &= 0 \\ c_2 &= 0\end{aligned}$$

Plug this value in equation (1)

$$\text{So, } c_2 = 0$$

Therefore, v_1, v_2 are linearly independent and $\{v_1, v_2\}$ spans columns space.

Therefore dimension of column space $\boxed{A=2}$.

Step-3

(b)

The objective is to find the column space of matrix U .

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step-4

Consider the matrix;

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step-5

Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Clearly,

$$v_1 + 2v_3 = v_2$$

$$v_1 - v_2 + 2v_3 = 0$$

Therefore v_1, v_2, v_3 are dependent and $c_1v_1 + c_2v_2 = 0$

$$c_1 + c_2 = 0$$

$$2c_2 = 0$$

Thus,

$$c_1 = 0$$

$$c_2 = 0$$

Hence v_1, v_2 are linearly independent and $\{v_1, v_2\}$ spans column space of U .

Therefore, the dimension of column space of $U = 2$.

Step-6

(c)

The objective is to find the dimension of the row space of matrix A .

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

Step-7

Let

$$v_1 = (1, 1, 0)$$

$$v_2 = (1, 3, 1)$$

$$v_3 = (3, 1, -1)$$

Now,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$c_1 (1, 1, 0) + c_2 (1, 3, 1) + c_3 (3, 1, -1) = 0$$

$$c_1 + c_2 + 3c_3 = 0$$

$$c_1 + 3c_2 + c_3 = 0$$

Solve these two equations.

So,

$$c_2 = c_3 \text{ and}$$

$$c_1 = -c_2 - 3c_3$$

This implies;

$$c_2 = c_3$$

$$c_1 = -4c_3$$

So,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$-4c_3 v_1 + c_3 v_2 + c_3 v_3 = 0$$

If $c_3 = 1$, then;

$$-4v_1 + v_2 + v_3 = 0$$

Therefore, $\{v_1, v_2, v_3\}$ are independent.

But, $\{v_1, v_2\}$ are independent and $\{v_1, v_2\}$ spans row spaces row space of A .

Therefore, the dimension of row space $\boxed{A=2}$.

Step-8

(d)

The objective is to find the row space of matrix U .

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step-9

The row space of U = spanned by $\{(1,1,0), (0,2,1)\}$

If $(1,3,1) = (1,1,0) + (0,2,1)$ and $(1,1,0), (1,3,1)$ are linearly independent $\{(1,1,0), (1,3,1)\}$ space the row space of U .

Therefore, the row space of matrix U is $\boxed{U=2}$

Hence, $\boxed{\text{row space of } U = \text{Row space of } A}$.