Step-1

Given: $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

(1)

$E_1 A = \begin{bmatrix} 1 & 0 \\ -4/5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 0 & 9/5 \end{bmatrix}$ where E_1 is an elementary matrix.

So, $A = \begin{bmatrix} 1 & 0 \\ 4/5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 9/5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4/5 \\ 0 & 1 \end{bmatrix} = LDL^T$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 4/5 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 3/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 3/\sqrt{5} \end{bmatrix} \\ &= \left(L\sqrt{D}\right)\left(\sqrt{D}L^T\right) \\ &= R^T R \qquad \text{, where } R = \sqrt{D}L^T \end{aligned}$$

Step-2

(2)

For $A = \left(Q\sqrt{\Lambda}\right)\left(\sqrt{\Lambda}Q^T\right)$ we first find eigenvalues of A.

Characteristics equation of A is given by

$$\begin{aligned} &\det(A - \lambda I) = 0 \\ &\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0 \\ &(5-\lambda)^2 - 16 = 0 \\ &\lambda^2 - 10\lambda + 9 = 0 \\ &(\lambda - 1)(\lambda - 9) = 0 \end{aligned}$$

Let $\lambda_1 = 1$ and $\lambda_2 = 9$ and $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

For orthonormal matrix Q, we find eigenvectors corresponding to $\lambda_1 = 1$ and $\lambda_2 = 9$, with length scaled 1.

Eigenvectors corresponding to $\lambda_1 = 1$

$$\begin{bmatrix} 5-1 & 4 \\ 4 & 5-1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 + 4x_2 = 0$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

So, eigenvector corresponding to $\lambda_1 = 1$ with length scaled 1 is given by $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Eigenvectors corresponding to $\lambda_1 = 9$

$$\begin{bmatrix} 5-9 & 4 \\ 4 & 5-9 \end{bmatrix} \cdot \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4x_3 + 4x_4 = 0$$

$$-x_3 + x_4 = 0$$

$$x_3 = x_4$$

So, eigenvector corresponding to $\lambda_2 = 9$ with length scaled 1 is given by $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step-3

Let $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

So, A can be written as $A = Q\Lambda Q^T$

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \right) \\ &= \begin{bmatrix} 1/\sqrt{2} & 3/\sqrt{2} \\ -1/\sqrt{2} & 3/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 3/\sqrt{2} & 3/\sqrt{2} \end{bmatrix} = R^T R \end{aligned}$$

where $R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 3/\sqrt{2} & 3/\sqrt{2} \end{bmatrix}$

Step-4

(3)

A can also be written as $A = \left(Q\sqrt{\Lambda}Q^T\right)\left(Q\sqrt{\Lambda}Q^T\right)$

$$\begin{aligned}
A &= \left(\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\right)\left(\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
&= R^T R \qquad \text{where } R = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}
\end{aligned}$$