Southern University of Science and Technology

Advanced Linear Algebra Spring 2023

MA109– Quiz #6

2023/04/02

Name: ____

Student Number: ____________________________________

1. Does $f(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$ have rational roots?

   设 $f(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$, 问 $f(x)$ 是否有有理根?

   Proof. Suppose $\frac{p}{q}$ ($p, q$are relatively prime) is a rational zero of $f(x)$, then $p|5$,$q|1$, so all possible rational zeros of $f(x)$ are $1, -1, 5, -5$. And $f(1) = 15 \neq 0$, $f(-1) = 3 \neq 0$, $f(5) = 975 \neq 0$, $f(-5) = 435 \neq 0$, so $f(x)$ doesn't have rational zeros. □

2. Suppose $T \in \mathcal{L}(V)$, and $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ are bases of V. Prove that T is invertible if and only if the rows of $\mathcal{M}(T)$ spans $\mathbf{F}^{1,n}$. Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \cdots, u_n), (v_1, \cdots, v_n))$.

   设 $T \in \mathcal{L}(V)$, $u_1, \cdots, u_n$ 和 $v_1, \cdots, v_n$ 是 V 的两组基. 证明 T 是可逆线性映射当且仅当 $\mathcal{M}(T)$ 的行向量组可以张成 $\mathbf{F}^{1,n}$. 此处 Here $\mathcal{M}(T)$ 是 $\mathcal{M}(T, (u_1, \cdots, u_n), (v_1, \cdots, v_n))$.

   Proof. "$\Rightarrow$" : Suppose

$$\mathcal{M}(T) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

   we only need to show $(a_{11}, a_{12}, \cdots, a_{1n}), \cdots, (a_{n1}, a_{n2}, \cdots, a_{nn})$ is linearly independent.

   If

$$\begin{aligned} & k_1(a_{11}, a_{12}, \cdots, a_{1n}) + \cdots + k_n(a_{n1}, a_{n2}, \cdots, a_{nn}) = 0 \\ \Rightarrow\ & k_1 a_{11} + \cdots + k_n a_{n1} = 0, \cdots, k_1 a_{11} + \cdots + k_n a_{nn} = 0 \\ \Rightarrow\ & (k_1 a_{11} + \cdots + k_n a_{n1}) v_1 + \cdots + (k_1 a_{1n} + \cdots + k_n a_{nn}) v_n = 0 \\ \Rightarrow\ & k_1(a_{11} v_1 + \cdots + a_{1n} v_n) + \cdots + k_n(a_{n1} v_1 + \cdots + a_{nn} v_n) = 0 \\ \Rightarrow\ & k_1 T' u_1 + \cdots + k_n T' u_n = 0 \end{aligned}$$

   Since T is invertible, then T' is also invertible, so $T'u_1, \cdots, T'u_n$ is linearly independent $\Rightarrow k_1 = \cdots = k_n = 0$, then $(a_{11}, a_{12}, \cdots, a_{1n}), \cdots, (a_{n1}, a_{n2}, \cdots, a_{nn})$ is linearly independent which can span $\mathbf{F}^{1,n}$.

   "$\Leftarrow$" : Since $(a_{11}, a_{12}, \cdots, a_{1n}), \cdots, (a_{n1}, a_{n2}, \cdots, a_{nn})$ is linearly independent, then the columns of $(\mathcal{M}(T))'$ is linearly independent, so the columns of $\mathcal{M}(T')$ is linearly independent, thus $T'u_1, \cdots, T'u_n$ is a basis of V, i.e. T' is invertible, so T is invertible.

   $\square$