Step-1

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a) We have to find a 2 by 2 matrix with $a_{12}=\frac{1}{2}$ for which $A^2=I$.

Let $A=\begin{bmatrix}1 & \frac{1}{2}\\ 0 & -1\end{bmatrix}$

Then

$$\begin{aligned}A^2&=\begin{bmatrix}1 & \frac{1}{2}\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2}\\ 0 & -1\end{bmatrix}\\&=\begin{bmatrix}1(1)+\frac{1}{2}(0) & 1\left(\frac{1}{2}\right)+\frac{1}{2}(-1)\\ 0(1)+(-1)(0) & 0\left(\frac{1}{2}\right)+(-1)(-1)\end{bmatrix}\end{aligned}$$

Step-2

Continuation to the above

$$\begin{aligned}&=\begin{bmatrix}1+0 & \frac{1}{2}-\frac{1}{2}\\ 0-0 & 0+1\end{bmatrix}\\&=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\&=I\end{aligned}$$

Hence the required matrix with $a_{12}=\frac{1}{2}$ for which $A^2=I$ is $\boxed{A=\begin{bmatrix}1 & \frac{1}{2}\\ 0 & -1\end{bmatrix}}$.

Step-3

b) We have to find a matrix with $a_{12}=\frac{1}{2}$ for which $A^{-1}=A^T$.

Let $A=\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Then $A^T=\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Step-4

Now by inverse formula of 2 by 2 matrix, we have $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Therefore,

$$A^{-1}=\frac{1}{\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}-\left(-\frac{1}{2}\right).\left(\frac{1}{2}\right)}\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$=\frac{1}{\frac{3}{4}+\frac{1}{4}}\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Step-5

Continuation to the above

$$=\frac{\frac{1}{4}}{\frac{4}{4}}\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

$$=\frac{1}{1}\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

Hence the required matrix with $a_{12}=\frac{1}{2}$ for which $A^{-1}=A^{T}$ is $\boxed{A=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}}$.

Step-6

c) We have to find a matrix with $a_{12}=\frac{1}{2}$ for which $A^{2}=A$.

Step-7

Let $A=\begin{bmatrix}1 & \frac{1}{2}\\ 0 & 0\end{bmatrix}$

Then

$$A^{2}=\begin{bmatrix}1 & \frac{1}{2}\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2}\\ 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1(1)+\frac{1}{2}(0) & 1\left(\frac{1}{2}\right)+\frac{1}{2}(0)\\ 0(1)+0(0) & 0\left(\frac{1}{2}\right)+0(0)\end{bmatrix}$$

Step-8

Continuation to the above

$$=\begin{bmatrix} 1+0 & \frac{1}{2}+0 \\ 0+0 & 0+0 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$$= A$$

Hence the required matrix with $a_{12} = \frac{1}{2}$ for which $A^2 = A$ is $\boxed{A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}}$.