

## Step-1

Given that  $A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.8 \end{bmatrix}$  and  $A^\infty = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

We have to find the eigenvalues and eigenvectors for both the given Markov matrices.

## Step-2

Now we find the eigenvalues of  $A$ .

The characteristic equation of  $A$  is  $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 0.6 - \lambda & 0.2 \\ 0.4 & 0.8 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (0.6 - \lambda)(0.8 - \lambda) - 0.08 = 0$$

$$\Rightarrow \lambda^2 - 1.4\lambda + 0.4 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 0.4\lambda + 0.4 = 0$$

$$\Rightarrow \lambda(\lambda - 1) - 0.4(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 0.4, 1$$

Therefore, eigenvalues are  $\boxed{\lambda = 0.4 \text{ and } 1}$ .

## Step-3

We know that  $x$  is the eigenvector of  $A$  if and only if  $x$  is the nonzero solution of  $(A - \lambda I)x = 0$

That is  $\begin{bmatrix} 0.6 - \lambda & 0.2 \\ 0.4 & 0.8 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$   $\in (1)$

For  $\lambda = 0.4$ , (1) becomes

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

## Step-4

The augmented matrix is

$$\begin{bmatrix} 0.2 & 0.2 & 0 \\ 0.4 & 0.4 & 0 \end{bmatrix}$$

Add  $(-2)$  times of row 1 to row 2, we get

$$\begin{bmatrix} 0.2 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this, we get  $0.2x_1 + 0.2x_2 = 0$

Here  $x_2$  is free variable.

## Step-5

Let  $x_2 = k$ , where  $k$  is a parameter

Then  $x_1 = -k$

Therefore,

$$\begin{aligned} x &= \begin{bmatrix} -k \\ k \end{bmatrix} \\ &= k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, the eigenvector corresponding to  $\lambda = 0.4$  is  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

## Step-6

For  $\lambda = 1$ , (1) becomes

$$\begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

## Step-7

The augmented matrix is

$$\begin{bmatrix} -0.4 & 0.2 & 0 \\ 0.4 & -0.2 & 0 \end{bmatrix}$$

Add row 1 to row 2, we get

$$\begin{bmatrix} -0.4 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this, we get  $-0.4x_1 + 0.2x_2 = 0$

Here  $x_1$  is free variable.

## Step-8

Let  $x_1 = k$ , where  $k$  is a free variable.

Then  $-0.4x_1 = -0.2x_2$

$$\Rightarrow x_2 = 2k$$

Therefore,

$$\begin{aligned} x &= \begin{bmatrix} k \\ 2k \end{bmatrix} \\ &= k \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

Therefore, the eigenvector corresponding to  $\lambda = 1$  is  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

## Step-9

Let  $S = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$  and

Then

$$A = SAS^{-1}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

## Step-10

Now

$$\begin{aligned}
A^\infty &= S\Lambda^\infty S^{-1} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (.4)^\infty & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (.4)^\infty & 0 \\ 0 & 1^\infty \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}
\end{aligned}$$

## Step-11

Since  $|\lambda| < 1$

So as  $n \rightarrow \infty$ ,  $\lambda^n \rightarrow 0$

Hence the eigenvalues of  $A^\infty$  are 0 and 1

And the eigenvectors of  $A^\infty$  are  $\boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$ .

## Step-12

We have to explain why  $A^{100}$  is close to  $A^\infty$ .

Now

$$\begin{aligned}
A^{100} &= S\Lambda^{100}S^{-1} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix}^{100} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (.4)^{100} & 0 \\ 0 & (1)^{100} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}
\end{aligned}$$

Since  $|\lambda| < 1$

So  $n \rightarrow 100$ ,  $\lambda^{100} \rightarrow 0$

Therefore  $A^{100}$  close to  $A^\infty$ .