Step-1

Given that the column space of AB is contained in (possibly equal to) the column space of A.

To give an example where the column spaces of A and AB are not equal.

The system $Ax = b$ is **solvable** if and only if the vector b can be expressed as a combination of the column of A.

Then b is in the column space;

Step-2

Let a vector $\alpha \in \text{col}AB$

That is α is linear combination of columns of AB

(Since each element in AB is linear combination of elements in A and columns of B)

Let us consider,

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$$

The column space of A is,

Since the system A have 2 equations in 2 unknowns.

The combination of columns equal to b is,

$$Ax = b$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$u\begin{bmatrix} 2 \\ 0 \end{bmatrix} + v\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

So, the column space of A is,

$$colA = \left\{ d\begin{bmatrix} 1 \\ 0 \end{bmatrix} / d \in R \right\}$$

$$colA = \left\{ d\begin{bmatrix} 1 \\ 0 \end{bmatrix} / d \in R \right\}$$

Therefore, the column space of A is

Step-3

The column space of AB is,

Since the system of AB have two equations in two unknowns.

The combination of columns equal to b is,

$$(AB)x = b$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$u\begin{bmatrix} 0 \\ 0 \end{bmatrix} + v\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

So, the column space of AB is,

$$colAB = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

Therefore, the column space of AB is $colAB = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

Therefore, the columns spaces of AB and A are not equal. i.e. $colA \neq colAB$