Step-1

Consider the following orthogonal matrix:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Diagonalize it to reach $Q = U\Lambda U^H$.

Step-2

First step is to find the Eigen values and Eigen vectors of matrix Q. To calculate the Eigen values do the following calculations;

$$\begin{aligned} Q - \lambda I &= \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} \\ \det(Q - \lambda I) &= 0 \\ (\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta &= 0 \\ (\lambda^2 - 2\lambda\cos\theta + 1) &= 0 \end{aligned}$$

After solving following values are obtained:

$$\lambda_1 = \cos\theta + i\sin\theta$$
$$\lambda_2 = \cos\theta - i\sin\theta$$

Step-3

To calculate Eigen vectors do the following calculations:

$$(Q - \lambda I)x = 0$$

$$\begin{bmatrix} \cos\theta - \cos\theta - i\sin\theta & -\sin\theta \\ \sin\theta & \cos\theta - \cos\theta - i\sin\theta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda = \cos\theta + i\sin\theta$ are as follows:

$$\begin{aligned} x_1 &= \begin{bmatrix} y \\ z \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -i \end{bmatrix} \end{aligned}$$

Step-4

Similarly, Eigen vectors corresponding to Eigen value $\lambda = \cos\theta - i\sin\theta$ is as follows:

$$(Q - \lambda I)x = 0$$

$$\begin{bmatrix} \cos\theta - \cos\theta + i\sin\theta & -\sin\theta \\ \sin\theta & \cos\theta - \cos\theta + i\sin\theta \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$\begin{aligned} x_2 &= \begin{bmatrix} y \\ z \end{bmatrix} \\ &= \begin{bmatrix} -i \\ 1 \end{bmatrix} \end{aligned}$$

Step-5

To put the Eigen vectors in unitary matrix make them orthonormal by dividing the length of the vector.

$$\begin{aligned} \|x\|^2 &= \left|(1)^2\right| + \left|(-i)^2\right| \\ &= |1| + |1| \\ &= 2 \end{aligned}$$

Let the length be L. So $L = \sqrt{2}$.

Step-6

Now, the diagonalization of the matrix can be written as follows:

$$\begin{aligned} Q &= U\Lambda U^H \\ &= \frac{1}{L}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}\frac{1}{L}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \end{aligned}$$

Here, $L = \sqrt{2}$

Step-7

Therefore, orthogonal matrix Q is diagonalize to reach $\boxed{Q = U\Lambda U^H}$. All Eigen values of unitary matrix is $\boxed{|\lambda| = 1}$.