Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #5

2023/03/26

Name: ____________________________________

Student Number: ________________________________

1. Suppose $V_1, \cdots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that V_j is finite-dimensional for each $j = 1, 2, \cdots, m$.

   设 $V_1, \cdots, V_m$ 均为向量空间使得 $V_1 \times \cdots \times V_m$ 是有限维的. 证明对每个 $j = 1, 2, \cdots, m$ 来说 V_j 都是有限维的.

   proof by contradiction

   Proof. Suppose V_i is infinite dimensional, then $\exists$ a sequence of vectors $\xi_1, \xi_2, \cdots \in V_i$, s.t. $\xi_1, \xi_2, \cdots, \xi_n$ is linearly independent for any integer n.

   Let $\eta_j = (0, \cdots, 0, \xi_j, 0, \cdots, 0)$, we have a sequence $\{\eta_j\}_{j=1}^{\infty}$ in $V_1 \times \cdots \times V_m$, s.t. $\forall n$, $\eta_1, \cdots, \eta_n$ is linearly independent, then $V_1 \times \cdots \times V_m$ is infinite dimensional, which is a contradiction! $\square$

2. Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. Suppose φ_1, φ_2 denotes the dual basis of the standard basis of $\mathbf{R}^2$ and $\psi_1, \psi_2\ \psi_3$ denotes the dual basis of the standard basis of $\mathbf{R}^3$.

   1. Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.
   2. Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of $\psi_1, \psi_2\ \psi_3$.

   定义 $T : \mathbf{R}^3 \to \mathbf{R}^2$ 为 $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. 设 φ_1, φ_2 是 $\mathbf{R}^2$ 的标准基的对偶基，$\psi_1, \psi_2\ \psi_3$ 是 $\mathbf{R}^3$ 的标准基的对偶基.

   1. 描述线性泛函 $T'(\varphi_1)$ and $T'(\varphi_2)$.
   2. 将 $T'(\varphi_1)$ 和 $T'(\varphi_2)$ 写成 $\psi_1, \psi_2\ \psi_3$ 的线性组合.

Proof. 1. $T'(\varphi_1) = \varphi_1 \circ T : \mathbf{R}^3 \to \mathbf{R}$, $T'(\varphi_2) = \varphi_2 \circ T : \mathbf{R}^3 \to \mathbf{R}$. For all $(x, y, z) \in \mathbf{R}^3$,

$$\varphi_1 \circ T(x, y, z) = \varphi_1(4x + 5y + 6z, 7x + 8y + 9z) = (4x + 5y + 6z)\varphi_1(e_1) + (7x + 8y + 9z)\varphi_1(e_2) = 4x + 5y + 6z$$

$$\varphi_2 \circ T(x, y, z) = \varphi_2(4x + 5y + 6z, 7x + 8y + 9z) = (4x + 5y + 6z)\varphi_2(e_1) + (7x + 8y + 9z)\varphi_2(e_2) = 7x + 8y + 9z$$

2. $\psi_1(x, y, z) = x$, $\psi_2(x, y, z) = y$, $\psi_3(x, y, z) = z$, then

$$T'(\phi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3, \quad T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3$$

□