

Step-1

Consider the following Hadamard matrix H :

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

It is known that H has orthogonal rows and the box is hypercube. Objective is to find $\det H$ which is same as the volume of hypercube in R^4 .

Step-2

For the determinant of H , use the cofactor expansion along with the first row and get,

$$\begin{aligned} \det H &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \end{aligned}$$

Now again use the cofactor expansion along with the first row for all these four determinants:

$$\begin{aligned} \det H &= 1 \begin{vmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \\ &= \left(1 \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} - 1 \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} \right) - \left(1 \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \right) \\ &\quad + \left(1 \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} \right) - \left(1 \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \right) \end{aligned}$$

And then

$$\begin{aligned} \det H &= (0 + 2 + 2) - (0 - 2 - 2) + (2 + 2 + 0) - (-2 - 2 - 0) \\ &= 4 + 4 + 4 + 4 \\ &= 16 \end{aligned}$$

Step-3

Note that in R^4 , the volume of the hypercube is $2^4 = 16$. Since $\det H = 16$, therefore the determinant of Hadamard matrix will be same as the volume of the hypercube in R^4 .

