

Step-1

Compute the Eigen values and Eigen vectors of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

Solve the following:

$$\frac{du}{dt} = Au$$

Initial conditions are as follows:

$$u(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Step-2

First step is to find the Eigen values and Eigen vectors of matrix A . To calculate the Eigen values do the following calculations;

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 6 \\ 0 & 0 & 4-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(1-\lambda)(3-\lambda)(4-\lambda) = 0$$

After solving following values are obtained:

$$\lambda_1 = 4$$

$$\lambda_2 = 3$$

$$\lambda_3 = 1$$

Step-3

To calculate Eigen vectors do the following calculations:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-4 & 2 & 1 \\ 0 & 3-4 & 6 \\ 0 & 0 & 4-4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 & 1 \\ 0 & -1 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving, values of x, y and z corresponding to $\lambda = 4$ are as follows:

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 13/3 \\ 6 \\ 1 \end{bmatrix}$$

Step-4

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 3$ is as follows:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-3 & 2 & 1 \\ 0 & 3-3 & 6 \\ 0 & 0 & 4-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving values of x, y and z are as follows:

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 1$ is as follows:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-1 & 2 & 1 \\ 0 & 3-1 & 6 \\ 0 & 0 & 4-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving values of x, y and z are as follows:

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Step-6

Recall that $e^{At} = Se^{At}S^{-1}$. Therefore,

$$u(t) = e^{At}u(0)$$

$$= Se^{At}S^{-1}u(0)$$

$$= \begin{bmatrix} 1 & 1 & 13/3 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & -1 & 5/3 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & e^{3t} & (13/3)e^{4t} \\ 0 & e^{3t} & 6e^{4t} \\ 0 & 0 & e^{4t} \end{bmatrix} \begin{bmatrix} (8/3) \\ -6 \\ 1 \end{bmatrix}$$

$$u(t) = \begin{bmatrix} (8/3)e^t - 6e^{3t} + (13/3)e^{4t} \\ -6e^{3t} + 6e^{4t} \\ e^{4t} \end{bmatrix}$$

Step-7

Therefore, solution is as follows:

$$u(t) = \begin{bmatrix} (8/3)e^t - 6e^{3t} + (13/3)e^{4t} \\ -6e^{3t} + 6e^{4t} \\ e^{4t} \end{bmatrix}$$