Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #9

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Name: ____

Student Number: ____________________________________

1. Suppose $e_1, \cdots, e_m$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, \cdots, e_m)$.

Proof. "$\Rightarrow$" : Suppose $v \notin \operatorname{span}\{e_1, \cdots, e_n\}$, let $v_0 = \langle v, e_1\rangle e_1 + \cdots \langle v, e_m\rangle e_m \in \operatorname{span}\{e_1, \cdots, e_m\}$, we have $v - v_0 \neq 0$, and $\forall e_i$, $\langle v - v_0, e_i\rangle = 0 \Rightarrow \langle v - v_0, v_0\rangle = 0$ (since $v_0 \in \operatorname{span}\{e_1, \cdots, e_m\}$).

By 6.13 (Pythagorean Theorem), we have

$$\begin{aligned}
\|v\|^2 &= \|(v - v_0) + v_0\|^2 = \|v - v_0\|^2 + \|v_0\|^2 \\
&= \|v - v_0\|^2 + |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2 \\
&> |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2
\end{aligned}$$

which is a contradiction!

"$\Leftarrow$" : If $v \in \operatorname{span}\{e_1, \cdots, e_m\}$, $v = k_1 e_1 + \cdots k_m e_m$, then $\langle v, e_i\rangle = k_i \Rightarrow v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$.

Take the square of the norm on both sides and use Pythagorean Theorem, we have

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2.$$

□

2. Suppose $e_1, \cdots, e_n$ is an orthonormal basis of V and $v_1, \cdots, v_n$ are vectors in V such that

$$\|e_j - v_j\| < \frac{1}{\sqrt{n}}$$

for each j. Prove that $v_1, \cdots, v_n$ is a basis of V.

Proof. Since $n = \dim V$, we only need to show $v_1, \cdots, v_n$ is linearly independent.

Suppose $v_1, \cdots, v_n$ is linearly dependent, $\exists j \in \{1, 2, \cdots, n\}$, s.t. $v_j \in \operatorname{span}\{v_1, \cdots, v_{j-1}\}$, i.e. $\exists k_1, \cdots, k_{j-1} \in \mathbf{F}$, s.t. $v_j = k_1 v_1 + \cdots + k_{j-1} v_{j-1}$, then

$$\begin{aligned}
& v_j - (k_1 v_1 + \cdots + k_{j-1} v_{j-1}) && = k_1(v_1 - e_1) + \cdots + k_{j-1}(v_{j-1} - e_{j-1}) \\
\Rightarrow\ & \|v_j - (k_1 v_1 + \cdots + k_{j-1} v_{j-1})\| && = \|k_1(v_1 - e_1) + \cdots + k_{j-1}(v_{j-1} - e_{j-1})\| \\
& && \leqslant |k_1| \cdot \|v_1 - e_1\| + \cdots + |k_{j-1}| \cdot \|v_{j-1} - e_{j-1}\| \\
& && < \tfrac{|k_1| + \cdots + |k_{j-1}|}{\sqrt{n}} \leqslant \tfrac{|k_1| + \cdots + |k_{j-1}|}{\sqrt{j}}
\end{aligned}$$

Since $\|v_j - e_j\| < \frac{1}{\sqrt{n}} \leqslant \frac{1}{\sqrt{j}}$, by using 6.18 (Triangle Inequality), we have

$$\begin{aligned}
\|e_j - (k_1 e_1 + \cdots + k_{j-1} e_{j-1})\| & \leqslant \|e_j - v_j\| + \|v_j - (k_1 e_1 + \cdots + k_{j-1} e_{j-1})\| \\
& < \tfrac{1 + |k_1| + \cdots + |k_{j-1}|}{\sqrt{j}} \quad (*)
\end{aligned}$$

But $e_1, \cdots, e_j$ is orthonormal list, we have

$$\|e_j - (k_1 e_1 + \cdots + k_{j-1} e_{j-1})\| = \sqrt{1 + k_1^2 + \cdots + k_{j-1}^2} \geqslant \frac{1 + |k_1| + \cdots + |k_{j-1}|}{\sqrt{j}} \quad (**)$$

Combine (*) and (**), we have

$$\frac{1 + |k_1| + \cdots + |k_{j-1}|}{\sqrt{j}} < \frac{1 + |k_1| + \cdots + |k_{j-1}|}{\sqrt{j}}$$

which is a contradiction! Thus $v_1, \cdots, v_n$ is a basis of V.

□