Step-1

Let $Ax = \lambda(A)x$ and $Bx = \lambda(B)x$.

Consider a vector z, which has n^2 components, as follows:

$$z = (x_1 y, x_2 y, \ldots, x_n y)^{\mathrm{T}}$$

Here, $y = (y_1, y_2, \ldots, y_n)$

Thus, $z = (x_1 y_1, \ldots x_1 y_n, x_2 y, \ldots, x_2 y_n, \ldots, x_n y_1, \ldots, x_n y_n)^{\mathrm{T}}$

Step-2

Consider $(A \otimes I)z$ as shown below:

$$(A \otimes I)z = \begin{bmatrix} a_{11}I & a_{12}I & \cdots & a_{1n}I \\ a_{21}I & a_{22}I & \cdots & a_{2n}I \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}I & a_{n2}I & \cdots & a_{nn}I \end{bmatrix} \begin{bmatrix} x_1 y_1 \\ \vdots \\ \vdots \\ x_n y_n \end{bmatrix}$$

This will be a vector having n^2 components. These can be further classified as first n components, second n components,... n^{th} n components. Consider its i^{th} n components. It is as follows:

$$\begin{aligned}\left[(A \otimes I)z\right]_i = {} & a_{i1}x_1 y_1 + (0)x_1 y_2 + \ldots + (0)x_1 y_n + a_{i2}x_2 y_1 + (0)x_2 y_2 + \ldots + (0)x_2 y_n \\ & + \ldots + a_{in}x_n y_1 + (0)x_n y_2 + \ldots + (0)x_n y_n\end{aligned}$$

Step-3

Since, we have assumed that $Ax = \lambda(A)x$, it is clear that $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = \lambda x_i$.

Therefore, the first component of the i^{th} n components will be as follows:

$$\begin{aligned}\left[(A \otimes I)z\right]_{i1} &= a_{i1}x_1 y_1 + (0)x_1 y_2 + \ldots + (0)x_1 y_n + a_{i2}x_2 y_1 + (0)x_2 y_2 + \ldots + (0)x_2 y_n \\ &\quad + \ldots + a_{in}x_n y_1 + (0)x_n y_2 + \ldots + (0)x_n y_n \\ &= a_{i1}x_1 y_1 + a_{i2}x_2 y_1 + \ldots + a_{in}x_n y_1 \\ &= (a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n)y_1 \\ &= \lambda x_i y_1\end{aligned}$$

Similarly, we get

$$
\begin{aligned}
\left[(A\otimes I)z\right]_{i2} &= (a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n)y_2 \\
&= \lambda x_i y_2
\end{aligned}
$$

And so on!

Step-4

Therefore, we get

$$
\begin{aligned}
\left[(A\otimes I)z\right] &= \begin{bmatrix} \lambda x_1 y_1 \\ \lambda x_1 y_2 \\ \vdots \\ \lambda x_1 y_n \\ \vdots \\ \lambda x_n y_1 \\ \vdots \\ \lambda x_n y_n \end{bmatrix} \\
&= \lambda \begin{bmatrix} x_1 y_1 \\ x_1 y_2 \\ \vdots \\ x_1 y_n \\ \vdots \\ x_n y_1 \\ \vdots \\ x_n y_n \end{bmatrix} \\
&= \boxed{\lambda(A)z}
\end{aligned}
$$

Now consider the following:

$$
(A\otimes B)z = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix} \begin{bmatrix} x_1 y_1 \\ \vdots \\ \vdots \\ x_n y_n \end{bmatrix}
$$

This will be a vector of n^2 components. This too can be split into n components, each having n components.

The i^{th} component containing n components will be as follows:

$$\left[(A\otimes B)z\right]_i = (a_{i1}B, a_{i2}B, \ldots, a_{in}B)\begin{bmatrix} x_1y_1 \\ \vdots \\ \vdots \\ x_ny_n \end{bmatrix}$$
$$= a_{i1}B(x_1y_1+\ldots+x_1y_n)+a_{i2}B(x_2y_1+\ldots+x_2y_n)+\ldots+a_{in}B(x_ny_1+\ldots+x_ny_n)$$
$$= a_{i1}x_1By_1+\ldots+a_{i1}x_1By_n+\ldots+a_{in}x_nBy_1+a_{in}x_nBy_n$$

Step-5

Therefore, $\boxed{(A\otimes B)z = \lambda(A)\lambda(B)z}$.