FINAL EXAMINATION
SPRING 2018

Linear Algebra I A

This three-hour long test has 10 problems in total. Write ***all your answers*** on the examination book.

(1) (10 points, 1 point each) True or false. No need to justify.

(a) If P is a permutation matrix, then $P^{-1} = P^T$.

(b) Suppose A is an $m \times n$ matrix, and rank $(A) = r$, then $\mathsf{dim}N(A) = m - r$.

(c) Symmetric matrices have orthogonal eigenvectors.

(d) Every invertible matrix can be diagonalized.

(e) The eigenvalues of A equal the eigenvalues of A^T.

(f) Suppose A is an $n \times n$ matrix, then $\det(kA) = k\det(A)$, $k \in \mathbb{R}$.

(g) The quadratic form $2x^2 + 4xy + y^2$ is positive definite.

(h) For any symmetric matrix A, the signs of the pivots agree with the signs of the eigenvalues.

(i) Every real symmetric A can be diagonalized by an orthogonal matrix Q.

(j) The difference equation $u_{k+1} = Au_k$ is stable if all eigenvalues satisfy $|\lambda_i| \leq 1$.

(2) (12 points, 3 points each) Fill in the blanks.

(a) Suppose A has eigenvalues 0 and 1, corresponding to the eigenvectors $(1, 2)^T$ and $(2, -1)^T$, then $A =$ __________.

(b) The conditions on a, b, c ensure that the quadratic $f(x, y) = ax^2 + 2bxy + cy^2$ is positive definite are __________.

(c) The 2×2 matrix that projects every vector onto the "θ-line" containing all the multiples of $a = (\cos\theta, \sin\theta)$ is __________.

(d) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 1 & 2 \end{pmatrix}$, $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$, then

$C_{21} + C_{22} + C_{23} =$ __________.

(3) (8 points)Consider the following system of linear equations:

$$\begin{cases} x_1 + 3x_2 + x_3 + 2x_4 = 1 \\ 2x_1 + 6x_2 + 4x_3 + 8x_4 = 3 \\ 2x_3 + 4x_4 = c \end{cases}.$$

(a) (4 pts) Let A be the coefficient matrix of the above system. What condition on $b = (1, 3, c)^T$ makes the system $Ax = b$ solvable?

(b) (4 pts) Find the complete solution to $Ax = b$ in the case it is solvable.

(4) (10 points) Suppose

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Explain why $Ax = b$ is inconsistent.
(b) Find a solution to $Ax = b$ in the sense of least squares.

(5) (10 points) Consider the following matrix:

$$A = \begin{bmatrix} x & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

(a) Let $f(x) = \det A$, find $f(x)$.
(b) Find

$$\det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

(6) (10 pints) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.
(b) Explain why A is diagonalizable, and find an invertible matrix S, such that $\Lambda = S^{-1}AS$.
(c) Find A^k, where k is a positive integer.

(7) (10 points) Let A be the following matrix.

$$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$$

(a) Find e^{At}.
(b) Solve the following system of differential equations:

$$\frac{du}{dt} = Au.$$

(8) (10 points) Consider the following matrix

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \\ 1 & 2 \end{bmatrix}$$

(a) Find all the eigenvalues of AA^T and A^TA.
(b) Find a Singular Value Decomposition of A.

(9) (10 points) For which numbers c is this matrix positive definite?

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & c & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(10) (10 points) Prove:
(a) (4 pts) Let A be a real symmetric matrix. Suppose all the pivots (without row exchanges) of A satisfy $d_k > 0$, then $x^T Ax > 0$ for all nonzero real vectors x.
(b) (3 pts) Suppose A has independent columns, then A^TA is invertible.
(c) (3 pts) Prove or give a counterexample: Suppose A has independent columns, then the projection matrix

$$P = A(A^TA)^{-1}A^T$$

has only 0 or 1 as its eigenvalues.