Step-1

Consider the following matrix:

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} \sin \theta & \sin 2\theta & \sin 3\theta \\ \sin 2\theta & \sin 4\theta & \sin 6\theta \\ \sin 3\theta & \sin 6\theta & \sin 9\theta \end{bmatrix}$$

Here, $\theta = \frac{\pi}{4}$. This sine matrix is a variation on Fourier matrix. Verify that $S^T = S^{-1}$.

Step-2

Simplify the matrix:

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} \sin(\pi/4) & \sin 2(\pi/4) & \sin 3(\pi/4) \\ \sin 2(\pi/4) & \sin 4(\pi/4) & \sin 6(\pi/4) \\ \sin 3(\pi/4) & \sin 6(\pi/4) & \sin 9(\pi/4) \end{bmatrix}$$

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ 1 & 0 & -1 \\ 1/\sqrt{2} & -1 & 1/\sqrt{2} \end{bmatrix}$$

Step-3

Compute the inverse and transpose of the matrix:

$$S^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ 1 & 0 & -1 \\ 1/\sqrt{2} & -1 & 1/\sqrt{2} \end{bmatrix}$$
$$S^{T} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ 1 & 0 & -1 \\ 1/\sqrt{2} & -1 & 1/\sqrt{2} \end{bmatrix}$$

It can be seen clearly that $S^T = S^{-1}$.

Step-4

This can also be verified by proving that the matrix is orthogonal matrix. Recall that orthogonal matrix satisfies $Q^T = Q^{-1}$ and $\overline{x}^T y = 0$. So, if it is proved that columns of the matrix are orthogonal to each other then verification will be done.

Step-5

Now do the following calculations:

$$\overline{x}_1^T x_2 = \left(\frac{1}{\sqrt{2}}\right) \cdot 1 + 1 \cdot 0 + \left(\frac{1}{\sqrt{2}}\right) \cdot (-1)$$

$$= 0$$

$$\overline{x}_2^T x_3 = 1 \cdot \left(\frac{1}{\sqrt{2}}\right) + 0 \cdot (-1) + (-1) \cdot \left(\frac{1}{\sqrt{2}}\right)$$

$$= 0$$

$$\overline{x}_1^T x_3 = \left(\frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}\right) + 1 \cdot (-1) + \left(\frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}\right)$$

$$= \left(\frac{1}{2}\right) - 1 + \left(\frac{1}{2}\right)$$

All the column vectors are orthogonal to each other. Therefore, $S^T = S^{-1}$.