Step-1

Given that $$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4\left(x_1 - x_2 + 2x_3\right)^2$$

So,

$$\begin{aligned}\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= 4\left(x_1 - x_2 + 2x_3\right)^2 \\ &= 4\left(x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 - 4x_2x_3 + 4x_1x_3\right)\end{aligned}$$

$\left(\text{Using } (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca\right)$

$(\text{ here, } a = x_1, b = -x_2, c = 2x_3)$

$$\begin{aligned}&= 4x_1^2 + 4x_2^2 + 16x_3^2 - 8x_1x_2 - 8x_2x_3 + 16x_1x_3 \\ &= Ax_1^2 + Bx_2^2 + Cx_3^2 + 2Hx_1x_2 + 2Gx_2x_3 + 2Fx_1x_3\end{aligned}$$

Step-2

The corresponding matrix A is,

$$\begin{aligned}A &= \begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix} \\ &= \begin{pmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{pmatrix}\end{aligned}$$

Apply row operations,

$$\begin{aligned}R_2 &\to R_2 + R_1 \\ R_3 &\to R_3 + (-2)R_1\end{aligned}$$

$$= \begin{pmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, A has only one point $= 4$

and $Rank(A) =$ number of non-zero rows

$= 1$.

Also, $\det A = 0$ (since A has zero rows)

Step-3

Eigen values of A are,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & -4 & 8 \\ -4 & 4-\lambda & -8 \\ 8 & -8 & 16-\lambda \end{vmatrix} = 0$$

Now applying the transformation as follows:

$$\sim \begin{vmatrix} -\lambda & -\lambda & 0 \\ -4 & 4-\lambda & -8 \\ 8 & -8 & 16-\lambda \end{vmatrix} = 0 \quad (R_1 \to R_1 + R_2)$$

$$\sim \begin{vmatrix} -\lambda & -\lambda & 0 \\ 0-4 & 4-\lambda & -8 \\ 8 & -2\lambda & -\lambda \end{vmatrix} = 0 \quad (R_3 \to R_3 + 2R_2)$$

$$(-\lambda)(-\lambda) \begin{vmatrix} 1 & 1 & 0 \\ -4 & 4-\lambda & -8 \\ 8 & -2 & 1 \end{vmatrix} = 0$$

$$\lambda^2 \left((4-\lambda) + 16 + 1(+4) + 0 \right) = 0$$

$$\lambda^2 (24 - \lambda) = 0$$

$$\lambda = 0, 0, 24.$$

Thus, the Eigen values are $24, 0, 0$.

So, $\boxed{A \text{ has only one point, rank } = 1\text{, determinant is 0 and Eigen values are } 24, 0, 0.}$