Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #10-11

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Name: ______________________________

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1. Suppose that T is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

   设 T 是 V 上的正规算子且 3 和 4 是 T 的本征值. 证明存在向量 $v \in V$ 使得 $\|v\| = \sqrt{2}$ 且 $\|Tv\| = 5$.

   Proof. Let v_1, v_2 be the eigenvectors of T corresponding to eigenvalues 3, 4 respectively, and $\|v_1\| = \|v_2\| = 1$, since T is normal, v_1 and v_2 are orthogonal.

   Let $v = v_1 + v_2$, then $\|v\| = \|v_1 + v_2\| = \sqrt{1+1} = \sqrt{2}$, $\|Tv\| = \|Tv_1 + Tv_2\| = \|3v_1 + 4v_2\| = \sqrt{3^2 + 4^2} = 5$.

   choose $\|v_1\| = \|v_2\| = 1$ ✓ □

2. Suppose U and W are finite-dimensional subspaces of V. Prove that $P_U P_W = 0$ if and only if $\langle u, w \rangle = 0$ for all $u \in U$ and $w \in W$.

$\Leftrightarrow W \subseteq U^\perp$

设 U 和 W 均为 V 的有限维子空间. 证明 $P_U P_W = 0$ 当且仅当对所有 $u \in U$ 和 $w \in W$ 均有 $\langle u, w \rangle = 0$.

Proof. "$\Rightarrow$" : $\forall w \in W$, $0 = P_U P_W w = P_U w \Rightarrow w \in U^\perp \Rightarrow \boxed{W \subseteq U^\perp}$, so $\forall u \in U, w \in W$, $\langle u, w \rangle = 0$.

"$\Leftarrow$" : $\forall v \in V$, $\exists w_1 \in W$, $w_2 \in W^\perp$, s.t. $v = w_1 + w_2$, then $P_U P_W v = P_U P_W (w_1 + w_2) = P_U w_1$. And since $\forall u \in U$, $w \in W$, $\langle u, w \rangle = 0 \Rightarrow W \subseteq U^\perp \Rightarrow P_U w_1 = 0 \Rightarrow P_U P_W v = 0$, $\forall v \in V$, so $P_U P_W = 0$.

□

3. Give an example of an operator T on complex vector space such that $T^9 = T^8$ but $T^2 \neq T$.

   找出复向量空间的一个算子 T 使得 $T^9 = T^8$ 但 $T^2 \neq T$.

   Proof. Consider $V = \mathbf{F}^8$, let $T \in \mathcal{L}(V)$ satisfying $Te_i = e_{i+1}, i = 1, 2, \cdots, 7$, $Te_8 = 0$, then $T^9 = T^8 = 0$, $T^2 e_1 = e_3 \neq e_2 = Te_1$. $\square$

4. Suppose $T \in \mathcal{L}(V)$. Let $\hat{s}$ denote the smallest singular of T, and let s denote the largest singular value of T.

   1. Prove that $\hat{s}\|v\| \leqslant \|Tv\| \leqslant s\|v\|$ for every $v \in V$.

   2. Suppose λ is an eigenvalue of T. Prove that $\hat{s} \leqslant |\lambda| \leqslant s$.

   设 $T \in \mathcal{L}(V)$. 设 $\hat{s}$ 表示 T 的最小奇异值, s 表示 T 的最大奇异值.

   1. 证明对每个 $v \in V$ 均有 $\hat{s}\|v\| \leqslant \|Tv\| \leqslant s\|v\|$.

   2. 设 λ 是 T 的一个本征值. 证明 $\hat{s} \leqslant |\lambda| \leqslant s$.

Proof. Assume all singular values of T are $s_1, \cdots, s_n$, and $\hat{s} = s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n = s$, the singular value decomposition of T is $Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$, $\forall v \in V$, then

$$\|Tv\|^2 = s_1^2\langle v, e_1\rangle^2 + \cdots + s_n^2\langle v, e_n\rangle^2 \leqslant s^2(\langle v, e_1\rangle^2 + \cdots + \langle v, e_n\rangle^2) = s^2\|v\|^2 \Rightarrow \|Tv\| \leqslant s\|v\|$$

Similarly, $\|Tv\| \geqslant \hat{s}\|v\|$, so $\hat{s}\|v\| \leqslant \|Tv\| \leqslant s\|v\|$ for every $v \in V$.

Let v be the eigenvector of λ, then $\|Tv\| = |\lambda|\|v\|$, then $\hat{s}\|v\| \leqslant |\lambda|\|v\| \leqslant s\|v\| \Rightarrow \hat{s} \leqslant |\lambda| \leqslant s$. □