Midterm I for Calculu II (solutions)

1. (1) $a_n = \dfrac{2^n+4^n}{3^n+4^n} = \dfrac{(\frac{1}{2})^n+1}{(\frac{3}{4})^n+1} \to 1$ as $n\to\infty$

$\therefore a_n \not\to 0 \quad \therefore \sum a_n$ div.

(2) $f(x) = \dfrac{1}{x(\ln x)^2}$, $f(x) > 0$ & $\searrow$ for $x \geq 2$.

$a_n = f(n)$ for $n \geq 2$.

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x(\ln x)^2}\,dx \overset{u=\ln x}{\underset{du=\frac{1}{x}dx}{=\!=\!=}} \int_{\ln 2}^\infty \frac{1}{u^2}\,du \quad \text{Conv. } (p=2>1)$$

$\therefore \sum a_n$ conv. & A.C. $(a_n>0)$

(3) $a_n = \dfrac{1}{n\sqrt[n]{n}}$, $b_n = \dfrac{1}{n}$

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt[n]{n}} = 1 \Rightarrow$ both conv. or both div.

$\because \sum b_n$ div. $\therefore \sum a_n$ div.

(4) $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{\cancel{(n+1)!}\,\cancel{(n+2)!}\,(n+3)!}{(3n+3)!} \cdot \dfrac{(3n)!}{n!\,\cancel{(n+1)!}\,\cancel{(n+2)!}} = \dfrac{(n+3)(n+2)(n+1)}{(3n+3)(3n+2)(3n+1)} \to \dfrac{1}{27} < 1$ as $n\to\infty$

$\Rightarrow \sum a_n$ A.C.

(5) $\sum\limits_{n=1}^{\infty} (-1)^n u_n$, $u_n = \sqrt{n^2+1} - n > 0$,

$u_n = \dfrac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} = \dfrac{1}{\sqrt{n^2+1}+n} \searrow$ in n.

$\lim\limits_{n\to\infty} u_n = 0$. $\therefore$ it is conv. by alternating series test.

$\sum\limits_{n=1}^{\infty} u_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n^2+1}+n}$, $b_n = \dfrac{1}{n}$

$\dfrac{u_n}{b_n} = \dfrac{n}{\sqrt{n^2+1}+n} \to \dfrac{1}{2}$ $\therefore \sum u_n$ & $\sum b_n$ both conv. or both div.

Since $\sum \frac{1}{n}$ div., $\sum u_n$ div. In summary, $\sum (-1)^n u_n$ is C.C.

2. $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{|x|^n} \longrightarrow |x|$

$\therefore$ radius of conv. $r = 1$.

$x = 1$: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n^2+3}}$ conv. by alternating series test

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$ div. Since $\frac{\frac{1}{\sqrt{n^2+3}}}{\frac{1}{n}} \to 1$ & $\sum \frac{1}{n}$ div.

$\therefore$ C.C. at $x=1$.

$x = -1$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$ div.

In summary, radius of conv. is 1 ;
interval of conv. is $(-1, 1]$;
the series is A.C. in $(-1, 1)$ & C.C. at $x=1$.

3. $f(x) = (x+1)e^x = (x+1)\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$

$= 1 + 2x + \left(\frac{1}{2!} + 1\right)x^2 + \left(\frac{1}{3!} + \frac{1}{2!}\right)x^3 + \cdots$

$= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{(n-1)!} + \frac{1}{n!}\right) x^n = \sum_{n=0}^{\infty} \frac{n+1}{n!} x^n$

4. $\left.\begin{aligned} &\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots \\ &[\ln(1+x)]' = \frac{1}{1+x} \end{aligned}\right\}$

$\Rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\Rightarrow \ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\Rightarrow 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots$

4 (Continued)

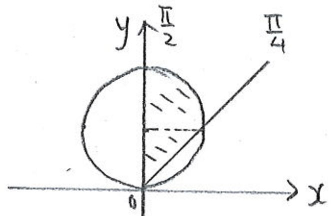
$$\lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim_{x\to 0} \frac{x^2 - \frac{x^4}{2} + \frac{x^6}{3} \cdots}{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} \cdots} = \boxed{2}.$$

5. $L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_0^{2\pi} \frac{3}{2} |\sin 2t| \, dt = (4)\left(\frac{3}{2}\right) \int_0^{\frac{\pi}{2}} \sin 2t \, dt$

$= -3 \cos 2t \Big]_0^{\frac{\pi}{2}} = \boxed{6}.$

6. $r = 2\sin\theta$, $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$.

$r^2 = 2r\sin\theta$, $x^2 + y^2 = 2y$, $x^2 + (y-1)^2 = 1$



$A = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2 \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin^2\theta \, d\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [1 - \cos 2\theta] \, d\theta = \left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \boxed{\frac{\pi}{4} + \frac{1}{2}}.$

Alternative method: $A = \frac{1}{4}$(area of disc) + area of the triangle

$= \frac{\pi}{4} + \frac{1}{2}.$

7. $(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$, $\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$ for $|x| < 1$.

$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$ for $|x| < 1$.

8. First assume $\{x_n\}$ conv. to L. Then $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = L$.

Thus $L = \frac{L}{2} + \frac{1}{L} \Rightarrow L = \sqrt{2}$ (we observe $L \geq 0$).

Now we show $\{x_n\}$ conv. indeed:

- Since $x_n > 0$, we see $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \geq \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$ for all n.
- $x_{n+1} - x_n = \frac{2 - x_n^2}{2x_n} \leq 0$

Thus, $\{x_n\}$ is nonincreasing and bounded from below by $\frac{\sqrt{2}}{2}$, and therefore it must converge.

$\square$.