Southern University of Science and Technology

Advanced Linear Algebra Spring 2023

MA109– Quiz #3

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Name: ______________________________

Student Number: ______________________________

1. Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

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Proof. Since W is infinite-dimensional, then there is a sequence $w_1, w_2, \cdots$ in W such that $w_1, w_2, \cdots, w_m$ is linearly independent for every positive integer m.

Let $v_1, \cdots, v_n$ be a basis for V, we consider $T_i \in \mathcal{L}(V, W)$ such that $T_i(v_1) = w_i$, then we can show $T_1, \cdots, T_m$ is linearly independent for every positive integer m.

Suppose $a_1T_1 + \cdots + a_mT_m = 0$, then

$$(a_1T_1 + \cdots + a_mT_m)v_1 = 0 \Rightarrow a_1T_1v_1 + \cdots + a_mT_mv_1 = 0 \Rightarrow a_1w_1 + \cdots + a_mw_m = 0 \Rightarrow a_1 = \cdots = a_m = 0.$$

Thus $T_1, \cdots, T_m$ is linearly independent, so $\mathcal{L}(V, W)$ is infinite-dimensional.

□

2. Suppose V and W are finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \text{null } T = \{0\}$ and range $T = \{Tu : u \in U\}$.

Proof. Since V is finite-dimensional and null T is a subspace of V, then there exists a subspace U of V such that $V = \text{null } T \oplus U$, so $U \cap \text{null } T = \{0\}$.

WTS: range $T = \{Tu : u \in U\}$.

Let $w \in \text{range } T$, then there exists some $v \in V$ s.t. $Tv = w$. Since $v \in V$, $V = \text{null } T \oplus U$, then we can find $x \in \text{null } T$, $u \in U$, s.t. $v = x + u$, thus $Tv = Tx + Tu = Tu \Rightarrow w = Tu$ for some $u \in U$, which means that $w \in \{Tu : u \in U\}$. Thus range $T \subset \{Tu : u \in U\}$

For any $u \in U$, u is also in V as $U \subset V$, thus $Tu \in \text{range } T$. Therefore $\{Tu : u \in U\} \subset \text{range } T$.

So we have range $T = \{Tu : u \in U\}$. Thus there exists a subspace U of V s.t. $U \cap \text{null } T = \{0\}$ and range $T = \{Tu : u \in U\}$.

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