Step-1

We have to prove that $\|x\|_\infty \le \|x\| \le \|x\|_1$.

Step-2

We know that the ℓ^1 norm is defined by $\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$ and the ℓ^∞ norm is defined by $\|x\|_\infty = \max\{|x_i| : 1 \le i \le n, x = (x_1, x_2, \ldots, x_n)\}$

Also, the hilbert norm is $\|x\| = \sqrt{x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2}$

Suppose $\max\{|x_i| : 1 \le i \le n, x = (x_1, x_2, \ldots, x_n)\} = x_j$

Then $\|x\|_\infty = |x_j|$

On the other hand, for any scalar x_i, we have $x_i{}^2 \ge 0$ and so, $|x_j| = \left|\sqrt{x_j{}^2}\right| \le \sqrt{x_j{}^2 + \sum_{i=1,j-1}^{j-1,n} x_i{}^2}$ where $\sqrt{x_j{}^2 + \sum_{i=1,j-1}^{j-1,n} x_i{}^2} = \|x\|$

Therefore, $\|x\|_\infty \le \|x\|$ â€¦â€¦ (1)

Step-3

Now

$$x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2 \le \left(|x_1| + |x_2| + |x_3| + \ldots + |x_n|\right)^2$$

Applying the positive square root on both sides, we get

$$\sqrt{x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2} \le |x_1| + |x_2| + \ldots + |x_n|$$

Therefore, $\|x\| \le \|x\|_1$ â€¦â€¦ (2)

From (1) and (2), we get $\boxed{\|x\|_\infty \le \|x\| \le \|x\|_1}$

Step-4

Now we have to show that the ratios $\frac{\|x\|}{\|x\|_\infty}$ and $\frac{\|x\|_1}{\|x\|}$ are never larger than $\sqrt{n}$ from the Schwartz inequality.

If $x = (x_1, x_2, \ldots, x_n)$ is a vector, then we have by Schwarz â€“ inequality that $|x^T x| \le \|x\|^2$

$\sqrt{x_1^2+x_2^2+\ldots+x_n^2} \le \sqrt{x_j^2+x_j^2+x_j^2+\ldots+x_j^2}$ (n times) where $\|x\|_\infty = |x_j|$, the maximum of all x_i's.

$$\Rightarrow \sqrt{x_1^2+x_2^2+\ldots+x_n^2} \le \sqrt{nx_j^2}$$
$$\le \sqrt{n}\,|x_j|$$
$$\Rightarrow \frac{\sqrt{x_1^2+x_2^2+\ldots+x_n^2}}{|x_j|} \le \sqrt{n}$$
$$\Rightarrow \frac{\|x\|}{\|x\|_\infty} \le \sqrt{n}$$

In other words, $\frac{\|x\|}{\|x\|_\infty}$ is not greater than $\sqrt{n}$ â€¦â€¦ (3)

Step-5

If $x=(x_1,x_2)$, then

$$\frac{\|(x_1,x_2)\|_1^2}{\|(x_1,x_2)\|^2} = \frac{\{|x_1|+|x_2|\}^2}{\left(\sqrt{x_1^2+x_2^2}\right)^2}$$

$$= \frac{x_1^2+x_2^2+2x_1x_2}{x_1^2+x_2^2}$$

$\le \frac{4x_j^2}{2x_j^2}$ where $x_j = \max\{|x_1|,|x_2|\}$

Therefore, $\frac{\|(x_1,x_2)\|_1}{\|(x_1,x_2)\|} \le \sqrt{2}$ â€¦â€¦ (4)

Step-6

If $x=(x_1,x_2,x_3)$, then

$$\frac{\|(x_1,x_2,x_3)\|_1^2}{\|(x_1,x_2)\|^2} = \frac{\{|x_1|+|x_2|+|x_3|\}^2}{\left(\sqrt{x_1^2+x_2^2+x_3^2}\right)^2}$$

$$= \frac{x_1^2+x_2^2+x_3^2+2x_1x_2+2x_2x_3+2x_3x_1}{x_1^2+x_2^2+x_3^2}$$

$\leq \frac{9x_j^2}{3x_j^2}$ where $x_j = \max\{|x_1|, |x_2|, |x_3|\}$

Therefore, $\frac{\|(x_1, x_2, x_3)\|_1}{\|(x_1, x_2, x_3)\|} \leq \sqrt{3}$ â€¦â€¦ (5)

Step-7

Continuing (4), (5) with the help of mathematical induction, we confirm that $\frac{\|x\|_1}{\|x\|} \leq \sqrt{n}$

In other words, $\frac{\|x\|_1}{\|x\|}$ is not greater than $\sqrt{n}$ â€¦â€¦ (6)

Hence from (3) and (6), we get the required result.