Step-1

Let A be an m by n matrix

Let r be the rank of A or $r = \dim(C(A))$

(a) Suppose the number of solutions of $Ax = b$ is 0 depending on column b.

That means $Ax = b$ has no solution

So, the number of non zero rows of reduced A < number of non zero rows of reduced augmented matrix [A| b]

Consequently, the columns of A are dependent

In other words, $r < m$

If the number of solutions $Ax = b$ is 1, that is the non homogeneous system has a unique solution depending on b.

So, $r = m = n$.

Step-2

(b) If the number of solutions of the non homogeneous system $Ax = b$ is infinite, then we follow that the number of linearly independent rows of A = $r = m$ = number of linearly independent rows of the augmented matrix [A | b] and $r < n$.

Step-3

(c) If the number of solutions of $Ax = b$ is 0, then we follow that the system is inconsistent or has no solution.

Then we observe that number of non zero rows of A < number of non zero rows of [A | b].

If the number of solutions of $Ax = b$ is infinite, then number of non zero rows of A = number of non zero rows of [A| b] but less than the number of columns (or variables)

That is $r = m < n$

Step-4

(d) If the number of solutions of $Ax = b$ is unique depending on b, then we can write $x = A^{-1}b$ is that unique solution.

In this case $r = m = n$