Real Spectral Theorem (7.29)

$\mathbb{F}=\mathbb{R}$. T self-adjoint operator

V real inner product space

(7.24) Complex spectral Thm

$\mathbb{F}=\mathbb{C}$, T normal. $TT^*=T^*T$

Operators on Real Inner Product Spaces

eigenvalue ?

$T(\underline{v_1, v_2})=(v_1, v_2)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$

orthonormal

T normal but not self-adjoint

$\dim V=2$ $T\in\mathcal{L}(V)$

Lecture 28

Dept. of Math., SUSTech

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Operators on Real Inner Product Spaces

Introduction

We now switch out focus to the context of inner product spaces. We will give a complete description of normal operators on real inner product spaces; a key step in the proof of this result (9.34) requires the result from the previous section that an operator on a finite-dimensional real vector space has an invariant subspace of dimension 1 or 2 (9.8).

After describing the normal operators on real inner product spaces, we will use that result to give a complete description of isometries on such spaces.

Normal Operators on Real Inner Product Spaces

Theorem $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$

(9.27) Suppose V is a 2-dimensional real inner product space and $T \in \mathscr{L}(V)$. Then the following are equivalent:

(7.20) $\|Tv\| = \|T^*v\|$ T normal

$\|Te_1\|^2 = a^2 + c^2 = \|T^*e_1\|^2 = a^2 + b^2 \Rightarrow$ $b = c$ ✗, $b = -c$ ✓

(a) *T is normal but not self-adjoint.*

(b) *The matrix of T with respect to ~~every~~ orthonormal basis of V has the form*

$(a) \Rightarrow (b)$ $T(\underbrace{e_1, e_2}_{\text{orthonormal basis of } V}) = (e_1\ e_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$T^*(e_1, e_2) = (e_1, e_2)\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$TT^* = T^*T \Rightarrow a = d$

with $b \neq 0$.

(c) *The matrix of T with respect to some orthonormal basis of V has the form*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

with $b > 0$.

Proof.

First suppose (a) holds, so that T is normal but not self-adjoint. Let e_1, e_2 be an orthonormal basis of V. Suppose

$$\mathcal{M}(T,(e_1,e_2)) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Then

$$||Te_1||^2 = a^2 + b^2 \text{ and } ||T^* e_1||^2 = a^2 + c^2.$$

Because T is normal, $||Te_1||^2 = ||T^* e_1||^2$ (See 7.20); thus these equations imply that $b^2 = c^2$. Thus $c = b$ or $c = -b$. But $c \neq b$, because otherwise T would be self-adjoint, as can be seen from the matrix in 9.28. Hence $c = -b$, so

$$\mathcal{M}(T,(e_1,e_2)) = \begin{pmatrix} a & -b \\ b & d \end{pmatrix}.$$

Proof.

The matrix of T^* is the transpose of the matrix above. Use matrix multiplication to compute the matrices of TT^* and T^*T (do it now). Because T is normal, these two matrices are equal. Equating the entries in the upper-right corner of the two matrices you computed, you will discover that $bd = ab$. Now $b \neq 0$, because otherwise T would be self-adjoint, as can be seen from the matrix in 9.29. Thus $d = a$, completing the proof that (a) implies (b).

Now suppose (b) holds. We want to prove that (c) holds. Choose an orthonormal basis e_1, e_2 of V. We know that the matrix of T with respect to this basis has the form give by (b), with $b \neq 0$. If $b > 0$, then (c) holds and we have proved that (b) implies (c).

Proof.

If $b < 0$, then, as you should verify, the matrix of T with respect to the orthonormal basis $e_1, -e_2$ equals $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where $-b > 0$; thus in this case we also see that (b) implies (c).

Now suppose (c) holds, so that the matrix of T with respect to some orthonormal basis has the form given in (c) with $b > 0$. Clearly the matrix of T is not equal to its transpose (because $b \neq 0$). Hence T is not self-adjoint. Now use the matrix multiplication to verify that the matrices of TT^* and T^*T are equal. We conclude that $TT^* = T^*T$. Hence T is normal. Thus (c) implies that (a), completing the proof.

Normal operators and invariant subspaces

(9.8) invariant subspace U. $\dim U = 1, 2$

7.28

9.30 Normal operators and invariant subspaces

Suppose V is an inner product space, $T \in \mathcal{L}(V)$ is normal, and U is a subspace of V that is invariant under T. Then

$\dim V = n$. $V = U \oplus U^\perp$

U orthonormal basis $e_1, e_2, \cdots, e_m$ of V

Extend $e_1, \cdots, e_m$ to a basis $e_1, \cdots, e_m, f_1, \cdots, f_n$ orthonormal

(6.35)

$f_1, \cdots, f_n$ orthonormal basis of $U^\perp$

(a) $\boxed{U^\perp \text{ is invariant under } T}$;

$\Leftrightarrow B = O_{m \times n}$

(b) U is invariant under T^*;

(c) $(T|_U)^* = (T^*)|_U$;

$T(e_1, \cdots, e_m, f_1, \cdots, f_n) = (e_1, \cdots, e_m, f_1, \cdots, f_n)\begin{pmatrix} A_{m\times m} & B_{m\times n} \\ O_{n\times m} & C_{n\times n}\end{pmatrix}$

$\Rightarrow \mathcal{M}(T^*, e_1, \cdots, e_m, f_1, \cdots, f_n) = \begin{pmatrix} A^T & O \\ B^T & C^T \end{pmatrix}$

(d) $T|_U \in \mathcal{L}(U)$ and $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ are normal operators.

$\Leftrightarrow (T|_{U^\perp})(T|_{U^\perp})^* = (T|_{U^\perp})^*(T|_{U^\perp})$

$A = (a_{ij})_{m\times n}$

$\sum_{j=1}^{m} \|Te_j\|^2 = \sum_{j=1}^{m}\sum_{i=1}^{m} a_{ij}^2$

(7.20) $= \sum_{j=1}^{m} \|T^* e_j\|^2 = \sum_{j=1}^{m}\sum_{i=1}^{m} a_{ij}^2 + \sum_{j=1}^{n}\sum_{i=1}^{m} b_{ij}^2$

$\Rightarrow B = O$ (a) $\checkmark$

(c) $S = T|_U$, $v \in U$

$\langle Su, v\rangle = \langle Tu, v\rangle = \langle u, T^* v\rangle$

$u \in U$. $Su = Tu$ $= \langle u, T^*|_U v\rangle$

$= \langle u, S^* v\rangle$

$\Rightarrow (T^*|_U) v = S^* v$ for all $v \in U$.

$\Rightarrow T^*|_U = S^* = (T|_U)^*$

Characterization of normal operators

9.34 Characterization of normal operators when $\mathbf{F} = \mathbf{R}$

Suppose V is a real inner product space and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal. self-adjoint (7.29)

(b) There is an orthonormal basis of V with respect to which T has a block diagonal matrix such that each block is a 1-by-1 matrix or a 2-by-2 matrix of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

with $b > 0$.

$\dim V = 1$ ✓

$\dim V = 2$ (9.27) ✓

$\dim V > 2$. $V = U \oplus U^{\perp}$

U invariant under T 9.8 ($\dim U = 1$ or 2)

(9.30) $U^{\perp}$ is invariant under T

$$T(e_1, \cdots, e_n) = \begin{pmatrix} a_1 & -b_1 & & & \\ b_1 & a_1 & & & \\ & & \ddots & & \\ & & & a_n & -b_n \\ & & & b_n & a_n \end{pmatrix}$$

Isometries on Real Inner Product Spaces

9.35 **Example** Let $\theta \in \mathbf{R}$. Then the operator on $\mathbf{R}^2$ of counterclockwise rotation (centered at the origin) by an angle of θ is an isometry, as is geometrically obvious. The matrix of this operator with respect to the standard basis is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

If θ is not an integer multiple of π, then no nonzero vector of $\mathbf{R}^2$ gets mapped to a scalar multiple of itself, and hence the operator has no eigenvalues.

Description of isometries when $\mathbb{F} = \mathbb{R}$

9.36 Description of isometries when $\mathbf{F} = \mathbf{R}$

Suppose V is a real inner product space and $S \in \mathcal{L}(V)$. Then the following are equivalent:

(a) S is an isometry.

(b) There is an orthonormal basis of V with respect to which S has a block diagonal matrix such that each block on the diagonal is a 1-by-1 matrix containing 1 or -1 or is a 2-by-2 matrix of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

with $\theta \in (0, \pi)$.

Homework Assignment 28

9.B: 1, 2, 3, 4, 8.