Step-1

Consider the matrices,

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}, \text{ and } \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

The objective is to find the complete solution of $Mx = b$.

Step-2

Consider the system,

$$Mx = b$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Apply the row operation, $R_4 \to R_4 - 3R_2$.

$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 - 3b_2 \end{bmatrix}$$

Apply the row operation, $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \\ b_3 \\ b_4 - 3b_2 \end{bmatrix}$$

Therefore, the upper triangular matrix is,

$$U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The system $Mx = b$ is consistent if,

$b_1 = 0, b_3 = 0$

$b_4 - 3b_2 = 0$

$b_4 = 3b_2$

Step-3

Consider the system,

$$Mx = 0$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives, $x_1 + 2x_2 = 0.$

Thus x_2 is a free variable.

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_1 &= -2x_2 \end{aligned}$$

The solution is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So free variable is, x_2.

Therefore, the special solution is, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Step-4

The system $Mx = b$ is consistent if,

$b_1 = 0, b_3 = 0, b_4 = 3b_2$

Consider,

$$\begin{bmatrix} b_2 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \text{Since, } b_4 = 3b_2.$$

Choose $b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix},$ which has $b_4 - 3b_2 = 0.$

Elimination takes $Ax = b$ to $Ux = c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

Then,

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-5

This gives,

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ x_1 &= 1 - 2x_2 \end{aligned}$$

Hence, the complete solution is,

$$\begin{aligned} x &= x_p + x_n \\ &= x_{\text{particular}} + x_{\text{nullspace}} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 - 2x_2 \\ x_2 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}} \end{aligned}$$