Step-1

The given Hilbert matrix is $A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$

Suppose $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

We have to compute $b = Ax$

Step-2

Now

$$b = Ax$$
$$= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Step-3

Continuation to the above

$$= \begin{bmatrix} 1+\frac{1}{2}+\frac{1}{3} \\ \frac{1}{2}+\frac{1}{3}+\frac{1}{4} \\ \frac{1}{3}+\frac{1}{4}+\frac{1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 11/6 \\ 13/12 \\ 47/60 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8333 \\ 1.0833 \\ 0.7833 \end{bmatrix}$$

Therefore, $b \approx \begin{bmatrix} 1.8333 \\ 1.0833 \\ 0.7833 \end{bmatrix}$ â€¦â€¦ (1)

Step-4

Suppose $x = \begin{bmatrix} 0 \\ 6 \\ -3.6 \end{bmatrix}$

Then

$$b = Ax$$

$$= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ -3.6 \end{bmatrix}$$

Step-5

Continuation to the above

$$= \begin{bmatrix} 0+3-1.2 \\ 0+2-0.9 \\ 0+1.5-0.612 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8 \\ 1.1 \\ 0.888 \end{bmatrix}$$

Therefore, $b \approx \begin{bmatrix} 1.8 \\ 1.1 \\ 0.888 \end{bmatrix}$ â€¦â€¦ (2)

From (1) and (2), we easily see that the difference in the product vector b is very small while the factor x is having lot of difference.

In other words, a small change Δb is caused by a large change Δx.