Step-1

a)

The matrix M transforms $(1,0)$ to (r,t) and $(0,1)$ to (s,u)

That is $M(1,0)=(r,t)$ and $M(0,1)=(s,u)$

This implies, the matrix M of the linear transformation is $M=\begin{bmatrix} r & s \\ t & u \end{bmatrix}$

The matrix N transforms (a,c) and (b,d) to $(1,0)$ and $(0,1)$.

That is $N(a,c)=(1,0)$ and $N(b,d)=(0,1)$

This implies, the matrix M of the linear transformation is $N=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The objective is to find matrix that transforms (a,c) to (r,t) and (b,d) to (s,u).

Step-2

Since M,N are linear, the product MN also a linear.

Consider the expression,

$$\begin{aligned}(MN)(a,c)&=M(N(a,c)) && \text{Since } (fog)(x)=f[g(x)]\\ &=M(1,0)\\ &=(r,t)\end{aligned}$$

And

$$\begin{aligned}(MN)(b,d)&=M(N(b,d))\\ &=M(0,1)\\ &=(s,u)\end{aligned}$$

Therefore, $\boxed{MN}$ is the matrix that transforms (a,c) to (r,t) and (b,d) to (s,u).

Step-3

b)

Find the matrix that transforms $(2,5)$ and $(1,3)$ to $(1,1)$ and $(0,2)$.

The set $\{(2,5),(1,3)\}$ is a basis for $\mathbf{R}^2$, because $(2,5),(1,3)$ are independent vectors

$$T(2,5)=(1,1)$$
$$T(1,3)=(0,2)$$

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the matrix of the linear transformation .

That is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$2a+5b=1$ â€¦â€¦ (1)

$2c+5d=1$ â€¦â€¦ (2)

And $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}=\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$a+3b=0$ â€¦â€¦ (3)

$c+3d=2$ â€¦â€¦ (4)

Solve the equations (1),(3)

$a=3, b=-1$

Solve the equations (2),(4)

$c=-7, d=3$

Therefore, required matrix is $\boxed{\begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}}$.