Step-1

Consider the second order equation,

$\frac{d^2u}{dt^2}=\begin{bmatrix}-5 & -1\\ -1 & -5\end{bmatrix}u$ with $u(0)=\begin{bmatrix}1\\0\end{bmatrix}$ and $u'(0)=\begin{bmatrix}0\\0\end{bmatrix}$

The objective is to find the solution of it.

Step-2

By comparing the equation $\frac{d^2u}{dt^2}=\begin{bmatrix}-5 & -1\\ -1 & -5\end{bmatrix}u$ with $\frac{d^2u}{dt^2}=Au$ get $A=\begin{bmatrix}-5 & -1\\ -1 & -5\end{bmatrix}$

Find eigenvalues and eigenvectors of matrix A.

Characteristic equation of the matrix A is $|A-\lambda I|=0$

$$\begin{vmatrix}-5-\lambda & -1\\ -1 & -5-\lambda\end{vmatrix}=0$$
$$(-5-\lambda)(-5-\lambda)-(-1)\cdot(-1)=0$$
$$(-5-\lambda)^2-1=0$$
$$\lambda^2+10\lambda+25-1=0$$

$$\lambda^2+10\lambda+24=0$$
$$\lambda^2+6\lambda+4\lambda+24=0$$
$$\lambda(\lambda+6)+4(\lambda+6)=0$$

$$(\lambda+4)(\lambda+6)=0$$
$$\lambda=-4,-6$$

So, eigenvalues of matrix A are $\lambda=-4,-6$

Step-3

Let $\mathbf{x}_1=(m_1,m_2)^T$ be the eigenvector corresponding to $\lambda_1=-4$. Then,

$$\left(A-(-4)\cdot I\right)\mathbf{x}_1 = 0$$
$$\begin{pmatrix} -5+4 & -1 \\ -1 & -5+4 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\underbrace{\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}}_{\mathbf{P}}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mathbf{0}}$$

Step-4

Augmented matrix associated with the above notation is,

$$[\mathbf{P} \mid \mathbf{0}] = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$
$$R_2 \to R_2 - R_1$$
$$\approx \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$
$$R_1 \to (-1)R_1$$
$$\approx \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

From the last matrix, get the equation,

$$m_1 + m_2 = 0$$

Note that we have two unknowns (m_1, m_2) and one equation. So, there must be 2-1=1 free variable.

Step-5

Let $m_2 = s,\ s \in \mathbb{R}$

Then from the equation $m_1 + m_2 = 0$ get $m_1 = -s$

Therefore, eigenvector corresponding to the eigenvalue $\lambda_1 = -4$ is,

$$\mathbf{x}_1 = (m_1, m_2)^{\mathrm{T}}$$
$$= \left\{\begin{pmatrix} -s \\ s \end{pmatrix} : s \in \mathbb{R}\right\}$$
$$= \left\{s\begin{pmatrix} -1 \\ 1 \end{pmatrix} : s \in \mathbb{R}\right\}$$

Step-6

Let $\mathbf{x}_2 = (n_1, n_2)^{\mathrm{T}}$ be the eigenvector corresponding to $\lambda_2 = -6$. Then,

$$\begin{aligned}
(A-(-6)\cdot I)\mathbf{x}_2 &= 0 \\
\begin{pmatrix} -5+6 & -1 \\ -1 & -5+6 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\mathbf{Q}}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mathbf{0}}
\end{aligned}$$

Augmented matrix associated with the above notation is,

$$\begin{aligned}
[\mathbf{Q} \mid \mathbf{0}] &= \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \\
&R_2 \to R_2 + R_1 \\
&\approx \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

From the last matrix, get the equation,

$$n_1 - n_2 = 0$$

Note that we have two unknowns (n_1, n_2) and one equation. So, there must be 2-1=1 free variable.

Step-7

Let $n_2 = t,\ t \in \mathbb{R}$

Then from the equation $n_1 - n_2 = 0$ get $n_1 = t$

Therefore, eigenvector corresponding to the eigenvalue $\lambda_2 = -6$ is,

$$\begin{aligned}
\mathbf{x}_2 &= (n_1, n_2)^{\mathrm{T}} \\
&= \left\{\begin{pmatrix} t \\ t \end{pmatrix} : t \in \mathbb{R}\right\} \\
&= \left\{t\begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \in \mathbb{R}\right\}
\end{aligned}$$

Step-8

Hence, eigenvalues and eigenvectors of the matrix $A=\begin{bmatrix}-5 & -1\\ -1 & -5\end{bmatrix}$ are,

$\lambda_1=-4$; $\mathbf{x}_1=\begin{bmatrix}-1\\ 1\end{bmatrix}$

$\lambda_2=-6$; $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$

Step-9

If the matrix A has negative eigenvalues $\lambda_1,\ldots,\lambda_n$ and if $\omega_j=\sqrt{-\lambda_j}$, then general solution of $\frac{d^2u}{dt^2}=Au$ is,

$$u(t)=(a_1\cos\omega_1 t+b_1\sin\omega_1 t)\mathbf{x}_1+\cdots+(a_n\cos\omega_n t+b_n\sin\omega_n t)\mathbf{x}_n$$

Step-10

Substitute -4 for λ_1 in the equation $\omega_1=\sqrt{-\lambda_1}$

$$\begin{aligned}\omega_1&=\sqrt{-(-4)}\\&=\sqrt{4}\\&=2\end{aligned}$$

Substitute -6 for λ_2 in the equation $\omega_2=\sqrt{-\lambda_2}$

$$\begin{aligned}\omega_2&=\sqrt{-(-6)}\\&=\sqrt{6}\end{aligned}$$

Therefore, the frequencies are $\omega_1=2$ and $\omega_2=\sqrt{6}$.

Step-11

Put the values $\mathbf{x}_1=\begin{bmatrix}-1\\ 1\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$, $\omega_1=2$, $\omega_2=\sqrt{6}$ in the following equation to get the general solution.

$$\begin{aligned}u(t)&=(a_1\cos\omega_1 t+b_1\sin\omega_1 t)\mathbf{x}_1+(a_2\cos\omega_2 t+b_2\sin\omega_2 t)\mathbf{x}_2\\&=(a_1\cos 2t+b_1\sin 2t)\begin{bmatrix}-1\\ 1\end{bmatrix}+\left(a_2\cos\sqrt{6}t+b_2\sin\sqrt{6}t\right)\begin{bmatrix}1\\ 1\end{bmatrix}\end{aligned}$$

Step-12

Apply initial conditions $u(0)=\begin{bmatrix}1\\0\end{bmatrix}$ and $u'(0)=\begin{bmatrix}0\\0\end{bmatrix}$ to obtain the values of $a_1, b_1, a_2,$ and b_2 in the solution $u(t)$.

As $u(t)=\left(a_1\cos 2t+b_1\sin 2t\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(a_2\cos\sqrt{6}t+b_2\sin\sqrt{6}t\right)\begin{bmatrix}1\\1\end{bmatrix}$ get,

$$u'(t)=\left(-2a_1\sin 2t+2b_1\cos 2t\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(-\sqrt{6}a_2\sin\sqrt{6}t+\sqrt{6}b_2\cos\sqrt{6}t\right)\begin{bmatrix}1\\1\end{bmatrix}$$

Step-13

From the condition $u(0)=\begin{bmatrix}1\\0\end{bmatrix}$ get,

$$\begin{bmatrix}1\\0\end{bmatrix}=\left(a_1\cos 2(0)+b_1\sin 2(0)\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(a_2\cos\sqrt{6}(0)+b_2\sin\sqrt{6}(0)\right)\begin{bmatrix}1\\1\end{bmatrix}$$
$$\begin{bmatrix}1\\0\end{bmatrix}=\left(a_1\cos(0)+b_1\sin(0)\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(a_2\cos(0)+b_2\sin(0)\right)\begin{bmatrix}1\\1\end{bmatrix}$$
$$\begin{bmatrix}1\\0\end{bmatrix}=\left(a_1\cdot 1+b_1\cdot 0\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(a_2\cdot 1+b_2\cdot 0\right)\begin{bmatrix}1\\1\end{bmatrix}$$
$$\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-a_1+a_2\\a_1+a_2\end{bmatrix}$$

Equating the corresponding positions get,

$$\begin{aligned}-a_1+a_2&=1\\a_1+a_2&=0\end{aligned}$$

By solving the system, get, $a_1=-\frac{1}{2}, a_2=\frac{1}{2}$

Step-14

From the condition $u'(0)=\begin{bmatrix}0\\0\end{bmatrix}$ get,

$$\begin{bmatrix}0\\0\end{bmatrix}=\left(-2a_1\sin 2(0)+2b_1\cos 2(0)\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(-\sqrt{6}a_2\sin\sqrt{6}(0)+\sqrt{6}b_2\cos\sqrt{6}(0)\right)\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix}=\left(-2a_1\sin(0)+2b_1\cos(0)\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(-\sqrt{6}a_2\sin(0)+\sqrt{6}b_2\cos(0)\right)\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix}=\left(-2a_1\cdot 0+2b_1\cdot 1\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(-\sqrt{6}a_2\cdot 0+\sqrt{6}b_2\cdot 1\right)\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}-2b_1+\sqrt{6}b_2\\2b_1+\sqrt{6}b_2\end{bmatrix}$$

Equating the corresponding positions get,

$$-2b_1+\sqrt{6}b_2=0$$
$$2b_1+\sqrt{6}b_2=0$$

By solving the system, get, $b_1=0, b_2=0$

Step-15

Finally, solution of the given problem is,

$$\begin{aligned}u(t)&=\left(\left(-\frac{1}{2}\right)\cos 2t+(0)\sin 2t\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(\left(\frac{1}{2}\right)\cos\sqrt{6}t+(0)\sin\sqrt{6}t\right)\begin{bmatrix}1\\1\end{bmatrix}\\&=\left(-\frac{1}{2}\cos 2t+0\right)\begin{bmatrix}-1\\1\end{bmatrix}+\left(\frac{1}{2}\cos\sqrt{6}t+0\right)\begin{bmatrix}1\\1\end{bmatrix}\\&=\boxed{\frac{1}{2}\cos 2t\begin{bmatrix}1\\-1\end{bmatrix}+\frac{1}{2}\cos\sqrt{6}t\begin{bmatrix}1\\1\end{bmatrix}}\end{aligned}$$