Matrix Representations of Linear Transformations

V的一组基
$E=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$

W的一组基
$F=\{\mathbf{w}_1,\mathbf{w}_2,\cdots,\mathbf{w}_m\}.$

$$
\begin{tikzcd}
\mathbf{v}\in V \arrow[r, "T"'] \arrow[d, leftrightarrow] & \mathbf{w}=T(\mathbf{v})\in W \arrow[d, leftrightarrow] \\
\mathbf{x}=[\mathbf{v}]_E\in\mathbb{R}^n \arrow[r, "A\in\mathbb{R}^{m\times n}"] & A\mathbf{x}=[\mathbf{w}]_F\in\mathbb{R}^m
\end{tikzcd}
$$

$\mathbf{v}$可由V的基$E=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$线性表出，不妨设

$$\mathbf{v}=x_1\mathbf{v}_1+x_2\mathbf{v}_2+\cdots+x_n\mathbf{v}_n$$

那么$\mathbf{v}$在基E下的坐标向量为

$$[\mathbf{v}]_E=\mathbf{x}=\begin{bmatrix}x_1\\x_2\\\cdots\\x_n\end{bmatrix}$$

$\mathbf{w}=T(\mathbf{v})$可由W的基$F=\{\mathbf{w}_1,\mathbf{w}_2,\cdots,\mathbf{w}_m\}$线性表出，不妨设

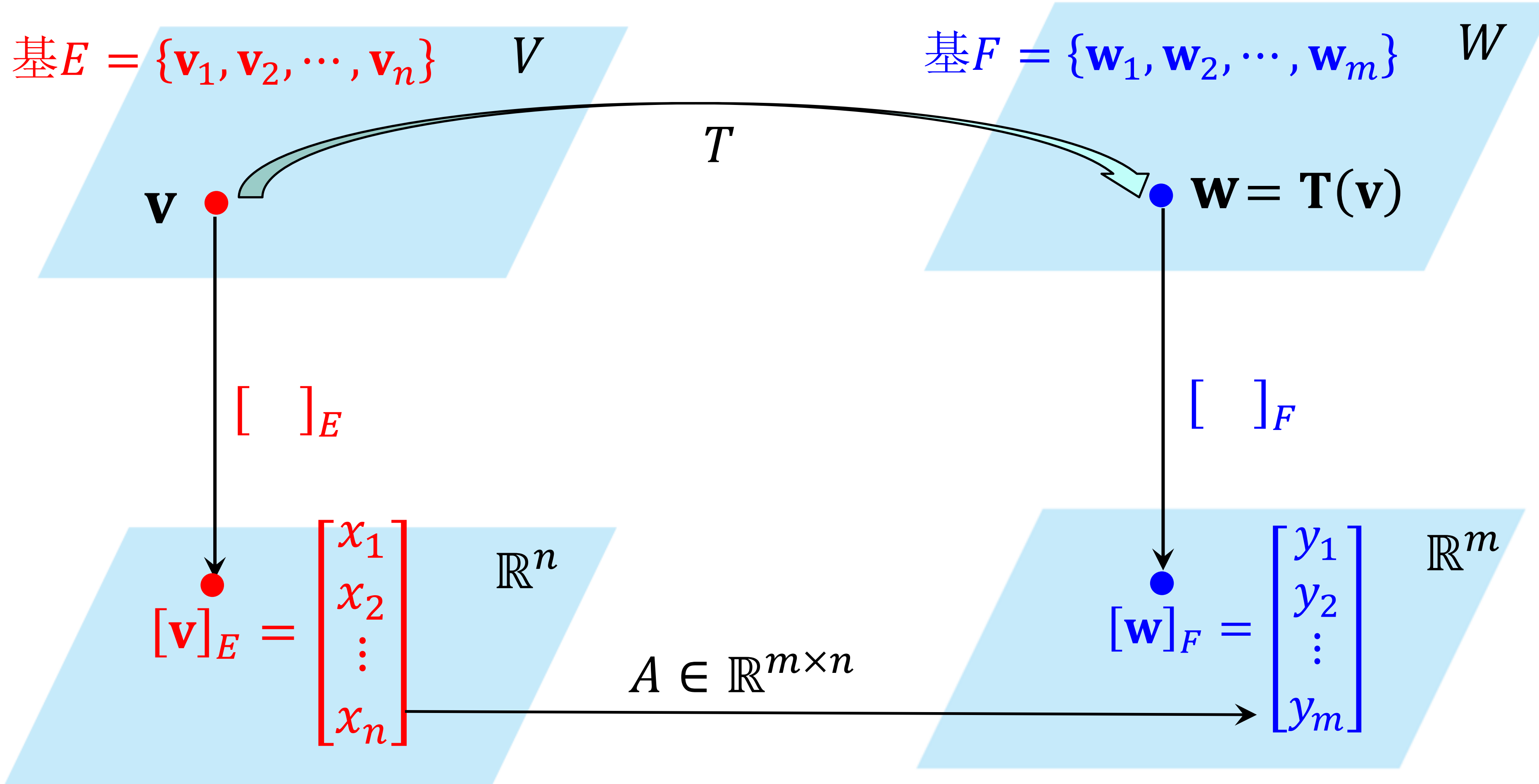
$$\mathbf{w}=y_1\mathbf{w}_1+y_2\mathbf{w}_2+\cdots+y_m\mathbf{w}_m$$

那么$\mathbf{w}$在基F下的坐标向量为

$$[\mathbf{w}]_F=\mathbf{y}=\begin{bmatrix}y_1\\y_2\\\cdots\\y_m\end{bmatrix}$$

问题： 是否存在$m\times n$矩阵A使得$A\mathbf{x}=\mathbf{y}$?

Matrix Representations of Linear Transformations



问题： 是否存在$m \times n$矩阵A使得： $A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$

Matrix Representations of Linear Transformations

❑ Theorem(Matrix Representation Theorem)

If $E = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ and $F = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$ are ordered bases for vector spaces V and W, respectively, then, corresponding to each linear transformation $T : V \to W$, there is an $m \times n$ matrix A such that

$$[T(\mathbf{v})]_F = A[\mathbf{v}]_E \ \ \forall \mathbf{v} \in V$$

A is the matrix representing T relative to the ordered bases E and F. In fact, Column j of A is the coordinate vector of $T(\mathbf{v}_j)$ with respect to F, that is

$$T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + \cdots + a_{mj}\mathbf{w}_m$$

Matrix Representations of Linear Transformations

$$\begin{tikzcd}
\mathbf{v} \in V \arrow[r, "T"'] \arrow[d, leftrightarrow] & \mathbf{w} = T(\mathbf{v}) \in W \arrow[d, leftrightarrow] \\
\mathbf{x} = [\mathbf{v}]_E \in \mathbb{R}^n \arrow[r, "A \in \mathbb{R}^{m\times n}"] & A\mathbf{x} = [\mathbf{w}]_F \in \mathbb{R}^m
\end{tikzcd}$$

V的一组基
$E = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$

W的一组基
$F = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$.

则

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$[T(\mathbf{v}_1)]_F \quad [T(\mathbf{v}_2)]_F \quad [T(\mathbf{v}_n)]_F$$

其中

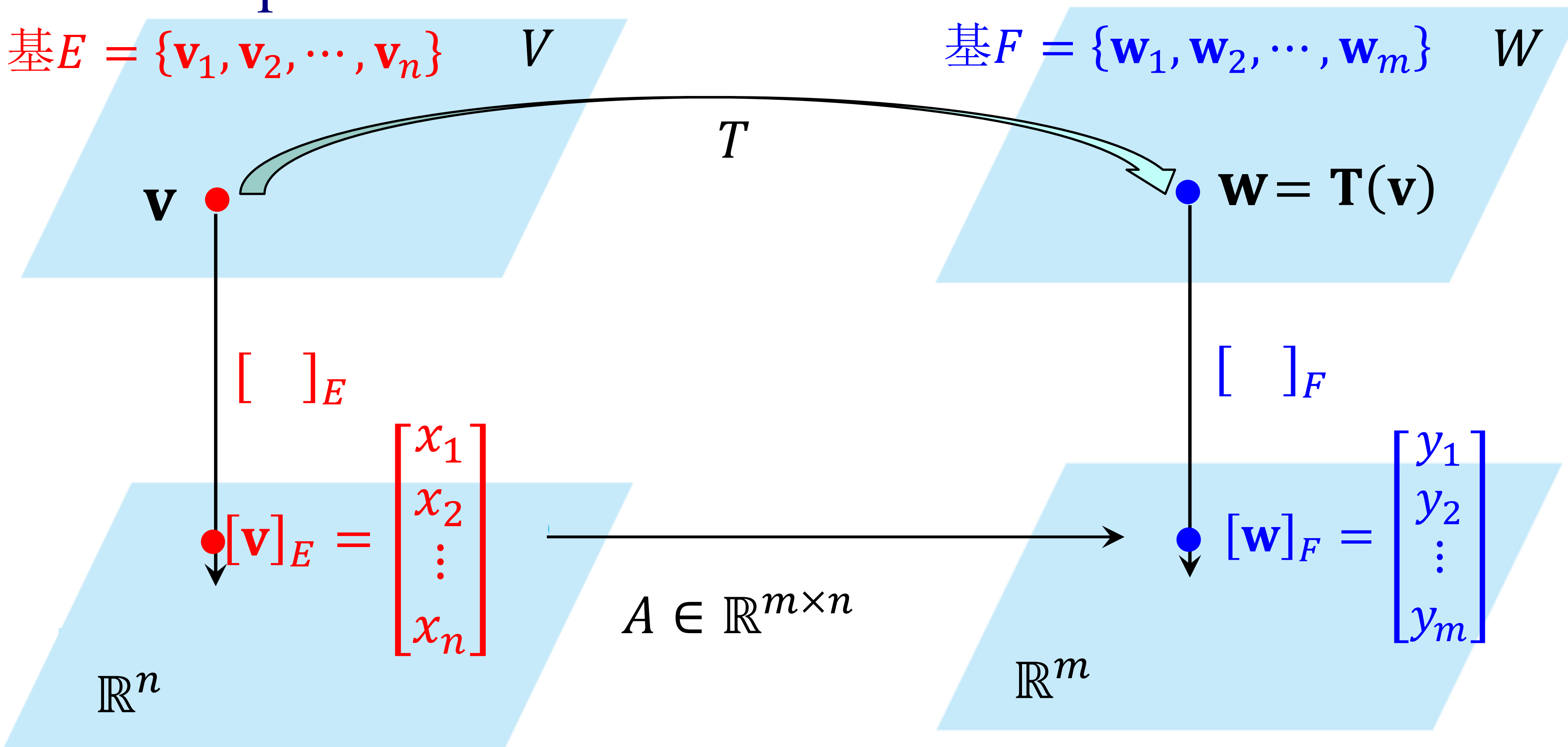
$$T(\mathbf{v}_1) = a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m$$

$$T(\mathbf{v}_2) = a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m$$

$$\vdots$$

$$T(\mathbf{v}_n) = a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m$$

Matrix Representations of Linear Transformations



问题：是否存在$m\times n$矩阵A使得：

$$A\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}y_1\\y_2\\\vdots\\y_m\end{bmatrix}$$

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}$$

$$\begin{matrix}\downarrow&\downarrow&\downarrow\\{[T(\mathbf{v}_1)]_F}&{[T(\mathbf{v}_2)]_F}&{[T(\mathbf{v}_n)]_F}\end{matrix}$$

定理的证明：

设V中任一向量$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$，$\boxed{T(\mathbf{v}) = y_1\mathbf{w}_1 + y_2\mathbf{w}_2 + \cdots + y_m\mathbf{w}_m}$

$$\mathbf{w} = T(\mathbf{v}) = T(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n)$$

$$= x_1T(\mathbf{v}_1) + x_2T(\mathbf{v}_2) + \cdots + x_nT(\mathbf{v}_n)$$

$$\left.\begin{aligned} &= x_1(a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m) \\ &\quad + x_2(a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m) \\ &\quad + \cdots \\ &\quad + x_n(a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m) \end{aligned}\right\} \text{这部分是} \sum_{j=1}^{n} x_j\left(\sum_{i=1}^{m} a_{ij}\mathbf{w}_i\right) \text{的展开式}$$

$$\left.\boxed{\begin{aligned} &= (x_1a_{11} + x_2a_{12} + \cdots + x_na_{1n})\mathbf{w}_1 \\ &\quad + (x_1a_{21} + x_2a_{22} + \cdots + x_na_{2n})\mathbf{w}_2 \\ &\quad + \cdots \\ &\quad + (x_1a_{m1} + x_2a_{m2} + \cdots + x_na_{mn})\mathbf{w}_m \end{aligned}}\right\} \text{这部分是} \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right)\mathbf{w}_i \text{的展开式}$$

比较上面两个等式

因此有

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} x_1a_{11} + x_2a_{12} + \cdots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + \cdots + x_na_{2n} \\ \vdots \\ x_1a_{m1} + x_2a_{m2} + \cdots + x_na_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$