

## 1

## Matrices and Gaussian Elimination

## 1.5

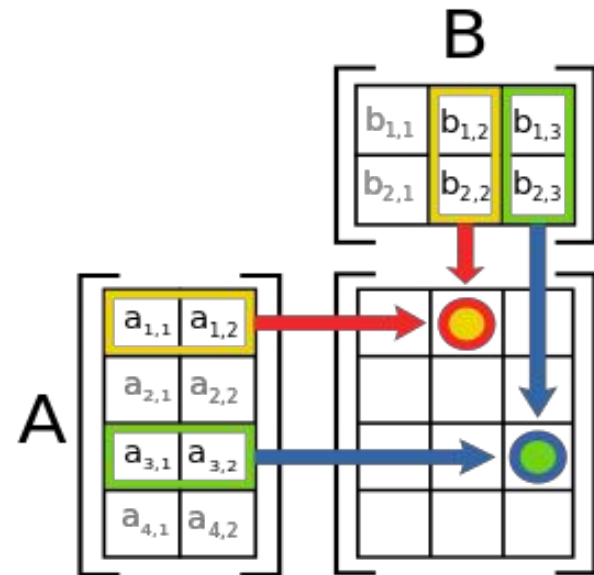
INVERSES AND  
TRANSPPOSES

(矩阵的逆和转置)

Definitions

Properties

Algorithms



\* Textbook: Section 1.6

# I. Transpose of a Matrix (转置矩阵)

## 1. Definition (定义)

Given an  $m \times n$  matrix  $A$ , the **transpose** of  $A$  is the  $n \times m$  matrix, denoted by  $A^T$ , whose columns are formed from the corresponding rows of  $A$ : the  $i$ th row of  $A$  becomes the  $i$ th column of  $A^T$ . (把  $m \times n$  矩阵  $A$  的行换成同序数的列所得到的  $n \times m$  矩阵称为矩阵  $A$  的**转置矩阵**, 记作  $A^T$  (或  $A'$ ) .)

Let  $A = [a_{ij}]_{m \times n}$ ,  $A^T = [b_{ij}]_{n \times m}$ , then  $a_{ij} = b_{ji}$ .

**For example,**

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 5 & 8 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 2 & 8 \end{bmatrix};$$

$$B = (18, 6), \quad B^T = \begin{bmatrix} 18 \\ 6 \end{bmatrix}.$$

## 2、Rules (转置矩阵的运算性质)

Let  $A$ ,  $B$ ,  $A_1, \dots$ , and  $A_n$  denote matrices whose sizes are appropriate for the following sums and products.

$$(1) \quad (A^T)^T = A;$$

$$(2) \quad (A + B)^T = A^T + B^T;$$

$$(A_1 + A_2 + \cdots + A_n)^T = A_1^T + A_2^T + \cdots + A_n^T;$$

$$(3) \quad (kA)^T = kA^T;$$

$$(4) \quad \underline{(AB)^T = B^T A^T};$$

The transpose of a product of matrices equals the product of their transposes in the *reverse* order.

$$(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 15 \end{bmatrix}, \quad \mathbf{B}^T \mathbf{A}^T = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 7 & 15 \end{bmatrix}.$$

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T.$$

**“Proof”.**

$$(j, i)\text{-entry of } (\mathbf{AB})^T = (\mathbf{AB})_{ij} = (\text{row } i \text{ of } \mathbf{A}) \text{ times } (\text{column } j \text{ of } \mathbf{B})$$

$$= [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$(j, i)\text{-entry of } \mathbf{B}^T \mathbf{A}^T = (\text{row } j \text{ of } \mathbf{B}^T) \text{ times } (\text{column } i \text{ of } \mathbf{A}^T)$$

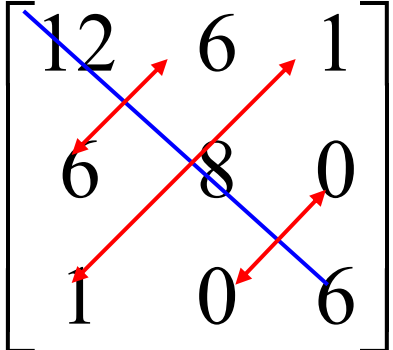
$$= [b_{1j} \quad b_{2j} \quad \cdots \quad b_{nj}] \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} = b_{1j}a_{i1} + b_{2j}a_{i2} + \cdots + b_{nj}a_{in}$$

## 对称(symmetric)与反对称(skew-symmetric)

**Definition** Let  $A$  be a square matrix of order  $n$ . If  $A^T = A$ , i.e.,

$$a_{ij} = a_{ji} \quad (i, j = 1, 2, \dots, n),$$

then  $A$  is called a **symmetric matrix** (对称矩阵).

$$A = \begin{bmatrix} 12 & 6 & 1 \\ 6 & 8 & 0 \\ 1 & 0 & 6 \end{bmatrix}. \quad A = \begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & -9 \\ -1 & 9 & 0 \end{bmatrix}.$$


If  $A^T = -A$ , then  $A$  is called a **skew-symmetric matrix** (反对称矩阵), i.e.,

$$a_{ij} = -a_{ji} \quad (i, j = 1, 2, \dots, n).$$

For a skew-symmetric matrix  $A$ , we have  $a_{ii} = 0$ .

**Example 1** Suppose  $A, B$  are  $n$  by  $n$  matrices, try to verify that  $AB^T + BA^T$  is symmetric.

**Proof** Since

$$\begin{aligned}(AB^T + BA^T)^T &= (AB^T)^T + (BA^T)^T \\ &= (B^T)^T A^T + (A^T)^T B^T \\ &= BA^T + AB^T = AB^T + BA^T,\end{aligned}$$

therefore  $AB^T + BA^T$  is symmetric.

Similary, if  $A$  is an  $m \times n$  matrix, then it is easy to show that,  $A^T A$  are both symmetric matrices.

e.g.,  $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ . What are  $AA^T$  and  $A^T A$ ?

# Inverse

## 引例 *Coding & Decoding; Encrypting & Decrypting*

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Space  
0

考虑加解密方案: 明文信息经编码后分成三个一组 (空格也是一种明文信息, 不足3个时可加空格). 对明文  $(p_1, p_2, p_3)^T$ , 相应的密文为:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \mathbf{b}, \quad \text{其中 } A = \begin{bmatrix} 0 & 0 & -2 \\ -1 & -4 & -3 \\ -1 & -3 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix}.$$

已知明文DIG, 求密文; 已知密文 $(-8, -103, -86)^T$ , 求明文.

## II. Inverse of a Matrix (矩阵的逆)-- Definition

- An  $n \times n$  matrix  $A$  is said to be **invertible** (可逆的) if there is an  $n \times n$  matrix  $C$  such that

$$CA = I \quad \text{and} \quad AC = I,$$

where  $I = I_n$ , the  $n \times n$  identity matrix.

- In this case,  $C$  is an **inverse of  $A$**  ( $A$ 的逆).
- In fact,  $C$  is uniquely determined by  $A$ , because if  $B$  were another inverse of  $A$ , then

$$B = BI = B(AC) = (BA)C = IC = C.$$

- This unique inverse is denoted by  $A^{-1}$  (' $A$  inverse'), so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$



**For example,**  $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, M = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix},$

Since  $AM = MA = I$ ,  $M$  is the inverse of  $A$ .

- A matrix that is *not* invertible is also called a **singular matrix** (奇异矩阵), and an invertible matrix is called a **nonsingular matrix** (非奇异矩阵).

*Not all matrices have inverses.* (并非所有矩阵都可逆)

An inverse is impossible when  $A\mathbf{x}$  is zero and  $\mathbf{x}$  is nonzero.

## Application:

If  $A$  is invertible, the *one and only* solution to  $A\mathbf{x} = \mathbf{0}$  is  $\mathbf{x} = \mathbf{0}$ .

当  $A$  可逆(即: 非奇异)时,  $A\mathbf{x} = \mathbf{0}$  只有零解  $\mathbf{x} = \mathbf{0}$ .

- **Theorem 1** Let  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ . If  $a_{11}a_{22} - a_{12}a_{21} \neq 0$ , then

$A$  is invertible and

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

If  $a_{11}a_{22} - a_{12}a_{21} = 0$ , then  $A$  is not invertible.

- The quantity  $a_{11}a_{22} - a_{12}a_{21}$  is called the **determinant (行列式)** of  $A$ , and we write  $\det A = a_{11}a_{22} - a_{12}a_{21}$ .
- This theorem says that a  $2 \times 2$  matrix  $A$  is invertible if and only if  $\det A \neq 0$ .

## Proof: (by definition, or 待定系数法)

Let  $\mathbf{M} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ , which satisfies that  $\mathbf{A}\mathbf{M} = \mathbf{I}$ ,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Rightarrow \begin{bmatrix} a_{11}x_1 + a_{12}x_3 & a_{11}x_2 + a_{12}x_4 \\ a_{21}x_1 + a_{22}x_3 & a_{21}x_2 + a_{22}x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_3 = 1, \\ a_{21}x_1 + a_{22}x_3 = 0, \\ a_{11}x_2 + a_{12}x_4 = 0, \\ a_{21}x_2 + a_{22}x_4 = 1, \end{cases} \Rightarrow \begin{cases} x_1 = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_2 = \frac{-a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_3 = \frac{-a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, \\ x_4 = \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}}. \end{cases} \Rightarrow \mathbf{M} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

( $a_{11}a_{22} - a_{12}a_{21} \neq 0$ )

We can also check that  $\mathbf{A}\mathbf{M} = \mathbf{M}\mathbf{A} = \mathbf{I}$ . Therefore

$$\mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

求二阶可逆矩阵 $\mathbf{A}$ 的逆的“**两调一除**”方法：

先将矩阵 $\mathbf{A}$ 的主对角元素互换位置, 再将次对角元素反号, 最后用  $\det \mathbf{A}$  去除 $\mathbf{A}$ 的每一个元素.

For example,

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{\begin{matrix} 0 & -3 \\ 3 & 2 \end{matrix}} = \begin{bmatrix} 0 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

**Example 2** A diagonal matrix has an inverse provided no diagonal entries are zero. (主对角元都是非零数的对角阵是可逆的.)

$$\begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix}^{-1} = \begin{bmatrix} a_1^{-1} & & & \\ & a_2^{-1} & & \\ & & \ddots & \\ & & & a_n^{-1} \end{bmatrix}$$

**Note:**

If  $ab \neq 0$ , then  $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

**Note:**  $A+B$  is not necessarily invertible for invertible matrices  $A, B$ .

And we can easily give an example that shows

$$(A+B)^{-1} \neq A^{-1} + B^{-1}, \text{ even if } A+B \text{ is invertible.}$$

**For example,**  $A = \text{diag}(2, -1)$ ,  $B = I_2$ ,  $C = \text{diag}(1, 2)$ .

$$A+B = \text{diag}(3, 0) : \text{ not invertible}$$

$$A+C = \text{diag}(3, 1) : \text{ invertible}$$

$$(A+C)^{-1} = \text{diag}(1/3, 1) \neq A^{-1} + C^{-1} = \text{diag}(3/2, -1/2).$$

**Example 3** Let a square matrix  $B$  be idempotent (幂等, i.e.,  $B^2=B$ ), and  $A=I+B$ . Show that  $A$  is invertible, and  $A^{-1}=(3I-A)/2$ .

**Proof** By  $B=A-I$ ,  $B^2=(A-I)^2=A^2-2A+I$ ,  
and  $B^2=B$ , we can get

$$A^2-2A+I=A-I,$$

$$A^2-3A=A(A-3I)=-2I,$$

i.e.,  $A[(3I-A)/2]=I.$

Similarly,  $[(3I-A)/2]A=I.$

Therefore,  $A$  is invertible, and  $A^{-1}=(3I-A)/2$ .

### III. Inverse -- Properties

- **Theorem 2** If  $A$  is an invertible  $n \times n$  matrix, then for each  $\mathbf{b}$  in  $\mathbb{R}^n$ , the equation  $\mathbf{A} \mathbf{x} = \mathbf{b}$  has the unique (one and only one) solution  $\mathbf{x} = \mathbf{A}^{-1} \mathbf{b}$ .

- **Proof:** Take any  $\mathbf{b}$  in  $\mathbb{R}^n$ .

A solution exists because if  $\mathbf{A}^{-1} \mathbf{b}$  is substituted for  $\mathbf{x}$ , then  $\mathbf{A} \mathbf{x} = \mathbf{A}(\mathbf{A}^{-1} \mathbf{b}) = (\mathbf{A} \mathbf{A}^{-1}) \mathbf{b} = \mathbf{I} \mathbf{b} = \mathbf{b}$ .

So  $\mathbf{A}^{-1} \mathbf{b}$  is a solution.

To prove that the solution is unique, show that if  $\mathbf{u}$  is any solution, then  $\mathbf{u}$  must be  $\mathbf{A}^{-1} \mathbf{b}$ .

If  $\mathbf{A} \mathbf{u} = \mathbf{b}$ , we can multiply both sides by  $\mathbf{A}^{-1}$  and obtain  $\mathbf{A}^{-1} \mathbf{A} \mathbf{u} = \mathbf{A}^{-1} \mathbf{b}$ ,  $\mathbf{I} \mathbf{u} = \mathbf{A}^{-1} \mathbf{b}$ , and  $\mathbf{u} = \mathbf{A}^{-1} \mathbf{b}$ .



### ■ Theorem 3

- a. If  $A$  is an invertible matrix, then  $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$ . (若  $A$  可逆, 则  $A^{-1}$  亦可逆)
- b. If  $A$  and  $B$  are  $n \times n$  invertible matrices, then so is  $AB$  (若  $A, B$  为同阶可逆方阵, 则  $AB$  也可逆), and the inverse of  $AB$  is the product of the inverses of  $A$  and  $B$  in the *reverse* order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}. \quad (A^k)^{-1} = (A^{-1})^k = A^{-k}$$

- c. If  $A$  is an invertible matrix, then so is  $A^T$  (若  $A$  可逆, 则  $A^T$  亦可逆), and the inverse of  $A^T$  is the transpose of  $A^{-1}$ . That is,

$$(A^T)^{-1} = (A^{-1})^T.$$

- **Proof:** To verify statement (a), find a matrix  $\mathbf{C}$  such that

$$\mathbf{A}^{-1}\mathbf{C} = \mathbf{I} \quad \text{and} \quad \mathbf{C}\mathbf{A}^{-1} = \mathbf{I}.$$

These equations are satisfied with  $\mathbf{A}$  in place of  $\mathbf{C}$ .

Hence  $\mathbf{A}^{-1}$  is invertible, and  $\mathbf{A}$  is its inverse.

- Next, to prove statement (b), compute:

$$(\mathbf{A}\mathbf{B})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{B}\mathbf{B}^{-1})\mathbf{A}^{-1} = \mathbf{A}\mathbf{I}\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}.$$

A similar calculation shows that  $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{A}\mathbf{B}) = \mathbf{I}$ .

- For statement (c), use Theorem of transpose matrix, read from right to left,

$$(\mathbf{A}^{-1})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = (\mathbf{A}\mathbf{A}^{-1})^{\mathrm{T}} = \mathbf{I}^{\mathrm{T}} = \mathbf{I}.$$

Similarly,  $\mathbf{A}^{\mathrm{T}}(\mathbf{A}^{-1})^{\mathrm{T}} = \mathbf{I}^{\mathrm{T}} = \mathbf{I}$ .

# The inverses of elementary matrices (初等矩阵的逆矩阵)

**Recall: (1)初等对换矩阵:**

将单位矩阵的第  $i, j$  行(或列)对换

$$P_{ij} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 & & & \\ & & & & & 0 & \cdots & 1 & \\ & & & & & \vdots & & \vdots & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 & \\ & & & & & & & & & 0 & \cdots & 1 & \\ & & & & & & & & & & \ddots & & \\ & & & & & & & & & & & 1 & \end{bmatrix}$$

第  $i$  列      第  $j$  列

← 第  $i$  行  
← 第  $j$  行

## (2)初等倍乘矩阵:

将单位矩阵第  $i$  行(或列)乘  $k \neq 0$

$$D_i(k) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & k & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{第 } i \text{ 行} \\ \text{第 } i \text{ 列} \end{array}$$

### (3)初等倍加矩阵:

将单位矩阵第  $i$  行乘  $k$  加到第  $j$  行,  
 或将第  $j$  列乘  $k$  加到第  $i$  列

$$E_{ij}(k) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & k & \cdots & 1 \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix}$$

← 第  $i$  行

← 第  $j$  行

第  $i$  列    第  $j$  列

## The inverses of elementary matrices (初等矩阵的逆矩阵)

变换  $r_i \leftrightarrow r_j$  的逆变换是其本身, 则

$$\mathbf{P}_{ij}^{-1} = \mathbf{P}_{ij};$$

变换  $kr_i$  ( $k \neq 0$ ) 的逆变换是  $\frac{1}{k}r_i$ , 则

$$\mathbf{D}_i^{-1}(k) = \mathbf{D}_i\left(\frac{1}{k}\right);$$

变换  $r_j + kr_i$  的逆变换是  $r_j + (-k)r_i$ , 则

$$\mathbf{E}_{ij}^{-1}(k) = \mathbf{E}_{ij}(-k).$$

初等矩阵的逆矩阵仍为同类型的初等矩阵.

Each elementary matrix  $\mathbf{E}$  is invertible. The inverse of  $\mathbf{E}$  is the elementary matrix of the same type that transforms  $\mathbf{E}$  back into  $\mathbf{I}$ .

## IV. Algorithm (初等变换法求逆矩阵)

**定理4** 可逆矩阵可以经过若干次初等变换化为单位矩阵.

**析** 任何矩阵 $A$ , 都可经初等行变换将其化为行简化阶梯形矩阵.

任何方阵 $A$ , 都可经初等行变换将其化为上三角形矩阵.

任何可逆矩阵 $A$ , 都可经初等行变换将其化为单位矩阵 $I$ .

即  $P_s \dots P_2 P_1 A = I$ . ( $P_1, \dots, P_s$ 均为初等矩阵)

**Hint:** Suppose that  $A$  is invertible.

Then, since the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b}$  (Theorem 2),  $A$  has a pivot position in every row.

Because  $A$  is square, the  $n$  pivot positions must be on the diagonal, which implies that the reduced echelon form of  $A$  is  $I_n$ .

Then, since each step of the row reduction of  $A$  corresponds to left-multiplication by an elementary matrix, there exist elementary matrices  $P_1, \dots, P_s$  such that  $P_s \dots P_2 P_1 A = I$ .

由  $\underline{P_s \cdots P_2 P_1} A = I$  得  $A = P_1^{-1} P_2^{-1} \cdots P_s^{-1} I$  和

$$A^{-1} = P_s \cdots P_2 P_1 = \underline{P_s \cdots P_2 P_1} I$$

初等矩阵的逆矩阵  
仍然是初等矩阵

( $A^{-1}$  results from applying  $P_1, \dots, P_s$  successively to  $I$ .)

This is the same sequence that reduced  $A$  to  $I$ .

- 结论:**
- (1) 可逆矩阵可以表示为若干初等矩阵的乘积;
  - (2) 对  $A$  作若干初等变换, 将  $A$  化为单位矩阵  $I$  时, 同样的这些初等变换将单位矩阵  $I$  化为  $A^{-1}$ .

$$P_s \cdots P_2 P_1 [A \ I] = [I \ A^{-1}]$$

Row reduce the matrix  $[A \ I]$ . If  $A$  is row equivalent to  $I$ , then  $[A \ I]$  is row equivalent to  $[I \ A^{-1}]$ .

Otherwise,  $A$  does not have an inverse.



## 用初等行变换求 $A$ 的逆矩阵 (the Gauss-Jordan method for calculating $A^{-1}$ )

即对  $n \times 2n$  矩阵  $[A \ I_n]$  实施一系列初等行变换，把矩阵  $A$  变成  $I_n$  时，原来的  $I_n$  就变成了  $A^{-1}$ .

$$[A, I_n] \xrightarrow{\text{ERO}} [I_n, A^{-1}]$$

- **Theorem 5** An  $n \times n$  matrix  $A$  is invertible if and only if  $A$  is row equivalent to  $I_n$ , and in this case, any sequence of elementary row operations that reduces  $A$  to  $I_n$  also transforms  $I_n$  into  $A^{-1}$ .

**Example 4** Write the matrix  $A$  as the product of elementary matrices, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

**Solution** The matrix  $A$  can be obtained from the 3 by 3 identity matrix  $I$  by 4 elementary operations

$$r_2 \leftrightarrow r_3, \quad c_1 + 2c_3, \quad (-1)r_3, \quad (-1)c_3$$

therefore  $A = P_3 P_1 I P_2 P_4 = P_3 P_1 P_2 P_4,$

where

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

**Example 5** Use ERO to find the inverse of  $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$ .

**Solution**

$$[A, I] = \left[ \begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{r1 \leftrightarrow r2} \left[ \begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{r3+r1} \left[ \begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right] \xrightarrow[r2+r3]{r1+r3 \times (-2)} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow[r1+r2 \times (-1)]{r2 \times \frac{1}{2}} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right] \quad A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}.$$

**Remark: 1.** *Can we use elementary **column** operations on  $A$  to find its inverse?*

也可以对  $\begin{bmatrix} A \\ I \end{bmatrix}$  施行初等**列**变换, 当  $A$  变成单位矩阵时,  $I$  被化为了  $A^{-1}$ .



**Remark: 2.** *Can we use elementary operations to solve system of linear equations?*

初等行变换求逆矩阵的方法, 还可用于求矩阵  $A^{-1}b$ .

$$A^{-1}[A \quad b] = [I \quad A^{-1}b]$$



$$\begin{array}{ccc} [A & b] & \\ \downarrow & \downarrow & \text{初等行变换} \\ I & A^{-1}b & \end{array}$$

**Example 6** Find the matrix  $X$ , such that  $AX = B$ , where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 3 & 1 \\ 4 & 3 \end{bmatrix}.$$

**Solution** If  $A$  is invertible, then  $X = A^{-1}B$ .

$$[A \ B] = \begin{bmatrix} 1 & 2 & 3 & 2 & 5 \\ 2 & 2 & 1 & 3 & 1 \\ 3 & 4 & 3 & 4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 5 \\ 0 & -2 & -5 & -1 & -9 \\ 0 & -2 & -6 & -2 & -12 \end{bmatrix}$$

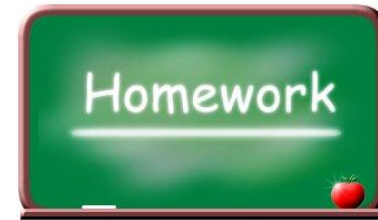
$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 5 \\ 0 & -2 & -5 & -1 & -9 \\ 0 & -2 & -6 & -2 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & -4 \\ 0 & -2 & -5 & -1 & -9 \\ 0 & 0 & -1 & -1 & -3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & -2 & 0 & 4 & 6 \\ 0 & 0 & -1 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix},$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 3 & 2 \\ -2 & -3 \\ 1 & 3 \end{bmatrix}.$$

What if -- we want  
to solve  $\mathbf{XA}=\mathbf{B}$  for  $\mathbf{X}$ ?

# Homework



- See Blackboard announcement
- ***Hardcover* textbook + Supplementary problems**

## Deadline (DDL):

- Submit on Blackboard before DDL

