# Step-1

Consider the following matrix:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$$

Then,

$$A^T = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$$

Now,

$$\begin{aligned} A^T A &= \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1+4 & 4+16 \\ 4+16 & 16+64 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 20 \\ 20 & 80 \end{pmatrix} \end{aligned}$$

# Step-2

Now the characteristic equation is,

$$|A - I\lambda| = 0$$

$$\begin{vmatrix} 5-\lambda & 20 \\ 20 & 80-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(80-\lambda) - (20)(20) = 0$$

$$400 - 85\lambda + \lambda^2 - 400 = 0$$

$$\lambda^2 - 85\lambda = 0$$

$$\lambda(\lambda - 85) = 0$$

$$\lambda = 0 \text{ or } 85$$

Therefore, the Eigen values are 0 and 85.

Now,

$$\sigma_1^2 = 85$$

$$\sigma_1 = \sqrt{85}$$

Hence, $A^T A = \begin{pmatrix} 5 & 20 \\ 20 & 80 \end{pmatrix}$ has only $\boxed{\sigma_1^2 = 85}$.

## Step-3

Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be Eigen vector corresponding to $\lambda_2 = 85.$

Then,

$$\begin{pmatrix} 5-85 & 20 \\ 20 & 80-85 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -80 & 20 \\ 20 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -80 & 20 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (\text{Apply } R_2 \to R_1 + 4R_2)$$
$$-4y_1 + y_2 = 0$$

Hence, the Eigen vector is $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Now the unit Eigen vector corresponding to $\lambda_2 = 85$ is $\boxed{v_2 = \begin{pmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{pmatrix}}$.

## Step-4

Let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be Eigen vector corresponding to Eigen value $\lambda_1 = 0.$

So,

$$\begin{pmatrix} 5-0 & 20 \\ 20 & 80-0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 5 & 20 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (\text{Apply } R_2 \to R_2 - 4R_1)$$
$$5x_1 + 20x_2 = 0$$
$$x_1 + 4x_2 = 0$$

Therefore, the Eigen vector is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

## Step-5

Thus, the unit Eigen vector corresponding to $\lambda_1 = 0$ is $\boxed{v_1 = \begin{pmatrix} \frac{4}{\sqrt{17}} \\ \frac{-1}{\sqrt{17}} \end{pmatrix}}$.

Hence, $\boxed{A^T A = \begin{pmatrix} 5 & 20 \\ 20 & 80 \end{pmatrix} \text{has only } \sigma_1^2 = 85 \text{ with } v_2 = \begin{pmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{pmatrix}, \text{so } v_2 = \begin{pmatrix} \frac{4}{\sqrt{17}} \\ \frac{-1}{\sqrt{17}} \end{pmatrix}}$.