Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #4

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Name: ____________________

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1. Let V be a n-dimensional vector space, try to construct $T \in \mathcal{L}(V)$ such that $T^n = I$, $T^k \neq I$ for all $1 \leqslant k \leqslant n-1$, where I is identity operator on V.

   Proof. Let $v_1, v_2, \cdots, v_n$ be a basis of V, we define $Tv_i = v_{i+1}$, $i = 1, 2, \cdots, n-1$, $Tv_n = v_1$. Let $v \in V$, $v = a_1 v_1 + \cdots + a_n v_n$, then $Tv = a_1 v_2 + a_2 v_3 + \cdots + a_{n-1} v_n + a_n v_1$.

   It's easy to check $T \in \mathcal{L}(V)$, and $T^k v_i = \begin{cases} v_{i+k}, & \text{if } k \leqslant n-i, \\ v_{i+k-n}, & \text{if } k > n-i \end{cases}$,$1 \leqslant k \leqslant n-1$, so $T^k v_i \neq v_i$, $T^k \neq I$, for all $1 \leqslant k \leqslant n-1$.

   And $T^n v_i = T(T^{n-1} v_i) = \begin{cases} Tv_n, & \text{if } i = 1, \\ Tv_{i-1}, & \text{if } i > 1 \end{cases}$, so $T^n v_i = v_i$, $i = 1, 2, \cdots, n$, thus $T^n = I$. □

2. Suppose V is a n-dimensional vector space, $\mathcal{A} \in \mathcal{L}(V)$ satisfies $\mathcal{A}^2 = \mathcal{I}$. Prove that $V = \text{null }(\mathcal{A}+\mathcal{I}) \oplus \text{null }(\mathcal{A}-\mathcal{I})$.

   Proof. Since $I = \frac{1}{2}(\mathcal{A}+\mathcal{I}) - \frac{1}{2}(\mathcal{A}-\mathcal{I})$, for all $v \in V$, we have $v = \mathcal{I}v = \frac{1}{2}(\mathcal{A}+\mathcal{I})v - \frac{1}{2}(\mathcal{A}-\mathcal{I})v$. Let $v_1 = -\frac{1}{2}(\mathcal{A}-\mathcal{I})v, v_2 = \frac{1}{2}(\mathcal{A}+\mathcal{I})v$, we can get $(\mathcal{A}+\mathcal{I})v_1 = -\frac{1}{2}(\mathcal{A}^2-\mathcal{I})v = 0$, $(\mathcal{A}-\mathcal{I})v_2 = \frac{1}{2}(\mathcal{A}^2-\mathcal{I})v = 0$, then $v_1 \in \text{null }(\mathcal{A}+\mathcal{I})$, $v_2 \in \text{null }(\mathcal{A}-\mathcal{I})$, so $V = \text{null }(\mathcal{A}+\mathcal{I}) + \text{null }(\mathcal{A}-\mathcal{I})$.

   For any $u \in \text{null }(\mathcal{A}+\mathcal{I}) \cap \text{null }(\mathcal{A}-\mathcal{I})$, we have $\mathcal{A}u = -u$ and $\mathcal{A}u = u$, so $u = 0$. Therefore, $V = \text{null }(\mathcal{A}+\mathcal{I}) \oplus \text{null }(\mathcal{A}-\mathcal{I})$.

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