

Step-1

A graph consists of a set of vertices or nodes, and a set of edges that connect them. The edge goes from node j to node k , then that row has -1 in column j and +1 in column k .

The 6 by 4 incidence matrix A for the second graph in the figure is,

$$A = \begin{matrix} & x_1 & x_2 & x_3 & x_4 \\ \begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{matrix} & \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{matrix}$$

Step-2

The dimensions of the four fundamental subspaces for the 6 by 4 incidence matrix and a basis for each subspace

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix}$$

Let

In order to find the dimensions and a basis, we need to set the incidence matrix as $[A \ b]$.

$$[A \ b] = \begin{bmatrix} -1 & 1 & 0 & 0 & b_1 \\ -1 & 0 & 1 & 0 & b_2 \\ 0 & -1 & 1 & 0 & b_3 \\ 0 & -1 & 0 & 1 & b_4 \\ -1 & 0 & 0 & 1 & b_5 \\ 0 & 0 & -1 & 1 & b_6 \end{bmatrix}$$

Apply $R_1 \rightarrow R_1 + R_4, R_2 \rightarrow R_2 - R_5$ and $R_4 \rightarrow R_4 - R_6$

$$= \begin{bmatrix} -1 & 0 & 0 & 1 & b_1 + b_4 \\ 0 & 0 & 1 & -1 & b_2 - b_5 \\ 0 & -1 & 1 & 0 & b_3 \\ 0 & -1 & 1 & 0 & b_4 - b_6 \\ -1 & 0 & 0 & 1 & b_5 \\ 0 & 0 & -1 & 1 & b_6 \end{bmatrix}$$

Step-3

Apply $R_1 \rightarrow R_1 - R_5$ and $R_3 \rightarrow R_3 - R_4$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & b_1 + b_4 - b_5 \\ 0 & 0 & 1 & -1 & b_2 - b_5 \\ 0 & 0 & 0 & 0 & b_3 - b_4 + b_6 \\ 0 & -1 & 1 & 0 & b_4 - b_6 \\ -1 & 0 & 0 & 1 & b_5 \\ 0 & 0 & -1 & 1 & b_6 \end{bmatrix}$$

Apply $R_2 \rightarrow R_2 + R_6$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & b_1 + b_4 - b_5 \\ 0 & 0 & 0 & 0 & b_2 - b_5 + b_6 \\ 0 & 0 & 0 & 0 & b_3 - b_4 + b_6 \\ 0 & -1 & 1 & 0 & b_4 - b_6 \\ -1 & 0 & 0 & 1 & b_5 \\ 0 & 0 & -1 & 1 & b_6 \end{bmatrix}$$

Therefore, $AX = b$ is solvable if,

$$b_1 + b_4 - b_5 = 0$$

$$b_3 - b_4 + b_6 = 0$$

$$b_2 - b_5 + b_6 = 0$$

Therefore, it is possible at;

$\begin{aligned} b_1 + b_4 - b_5 &= 0 \\ b_3 - b_4 + b_6 &= 0 \text{ and} \\ b_2 - b_5 + b_6 &= 0 \end{aligned}$
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Step-4

Now $AX = 0$,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_2 + x_3 = 0$$

$$-x_1 + x_4 = 0$$

$$-x_3 + x_4 = 0$$

Step-5

$$x_1 = x_2$$

$$= x_3$$

$$= x_4$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \\ x_1 \end{bmatrix} \\ = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$N(A) = \left[\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right]$$

Therefore, null space of

Step-6

$$\text{Since, } \dim C(A) + \dim(N(A)) = 4$$

$$\left(\text{As } \dim(N(A)) = 1 \right)$$

$$\dim(C(A)) = 4 - 1$$

$$= 3$$

Since, $\dim(C(A^T)) = \text{rank } r$ and $\dim(N(A^T)) = m - \text{rank}(r)$

So,

[Since, $m = \text{no of rows}(6)$, $\text{rank of } A = 3$]

$$\begin{aligned}\dim(N(A^T)) &= m - r \\ &= 6 - 3 \\ &= 3\end{aligned}$$

And,

(as $\text{rank of } A = 3$)

$$\dim(C(A^T)) = 3$$

Therefore,

$\dim(N(A)) = 1$
$\dim(C(A)) = 3$
$\dim(C(A^T)) = 3$ and
$\dim(N(A^T)) = 3$