Step-1

We have to prove that if $\mathbf{V}$ and $\mathbf{W}$ are 3-dimensional subspaces of $\mathbf{R}^5$,then $\mathbf{V}$ and $\mathbf{W}$ must have a non zero vector in common.

Step-2

Let $\mathbf{V}$ and $\mathbf{W}$ are three dimensional subspace of $\mathbf{R}^5$

Let $\beta_1 = \{v_1, v_2, v_3\}$ is a basis for $\mathbf{V}$

$\beta_2 = \{w_1, w_2, w_3\}$ is a basis for $\mathbf{W}$

Now $S = \{v_1, v_2, v_3, w_1, w_2, w_3\}$ is a subset of $\mathbf{R}^5$.

Sine the dimension of R^5 is 5. The maximally linearly independent set have only 5 vectors.

Therefore the above set S is linearly dependent

$\exists$ not all zero scalars $a_1, a_2, a_3, a_4, a_5, a_6$ such that $a_1v_1 + a_2v_2 + a_3v_3 + a_4w_1 + a_5w_2 + a_6w_6 = 0$

$\Rightarrow a_1v_1 + a_2v_2 + a_3v_3 = -a_4w_1 - a_5w_2 - a_6w_6$

Step-3

Let $\alpha = a_1v_1 + a_2v_2 + a_3v_3$

$\therefore \alpha \in V$ (Since $\{v_1, v_2, v_3\}$ is a basis for V)

Then $\alpha = -a_4w_1 - a_5w_2 - a_6w_6 \in W$ (since $\{w_1, w_2, w_3\}$ is a basis for W)

If possible $\alpha = 0$

$\Rightarrow a_1v_1 + a_2v_2 + a_3v_3 = 0$

$\Rightarrow a_1 = 0, a_2 = 0, a_3 = 0$

And hence, $-a_4w_1 - a_5w_2 - a_6w_6 = 0$

$\Rightarrow a_4 = a_5 = a_6 = 0$

Therefore $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$

This is a contradiction to $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ is linearly dependent.

Hence $\alpha \neq 0$

Therefore α is a non zero vector belongs to both $\mathbf{W}$ and $\mathbf{V}$.