Step-1

We know that Singular Value Decomposition for any m by n matrix A is given by

$$\begin{aligned}A &= U\Sigma V^T \\ &= \begin{pmatrix}\text{orthogonal} \\ U \text{ is } m\times m\end{pmatrix}\begin{pmatrix}m\times n \text{ matrix } \Sigma \\ \sigma_1\cdots\sigma_r \text{ on diagonal}\end{pmatrix}\begin{pmatrix}\text{orthogonal} \\ V \text{ is } n\times n\end{pmatrix}\end{aligned}$$

Here eigenvectors of AA^T are in U, eigenvectors of A^TA are in V and the diagonal matrix Σ has square roots of the nonzero eigenvalues of both AA^T and A^TA.

Step-2

Consider the following matrices:

$$A = \begin{bmatrix}1 & 1 \\ 0 & 0\end{bmatrix}$$

$$B = \begin{bmatrix}0 & 0 \\ 1 & 1\end{bmatrix}$$

Then we get

$$\begin{aligned}AB &= C \\ &= \begin{bmatrix}1 & 1 \\ 0 & 0\end{bmatrix}\end{aligned}$$

So, we have

$$C^T = \begin{bmatrix}1 & 0 \\ 1 & 0\end{bmatrix}$$

Calculate CC^T.

$$\begin{aligned}CC^T &= \begin{bmatrix}1 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 \\ 1 & 0\end{bmatrix} \\ &= \begin{bmatrix}2 & 0 \\ 0 & 0\end{bmatrix}\end{aligned}$$

Step-3

To find the eigenvalues of CC^T, solve the following equation for λ.

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 0-\lambda \end{bmatrix} = 0$$
$$(2-\lambda)(-\lambda) = 0$$

Therefore, the eigenvalues for CC^T are:

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

The eigenvectors of CC^T are given by

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Step-4

The unit eigenvectors of CC^T are given by,

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We know that eigenvectors of CC^T are in *U*, so

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step-5

Calculate $C^T C$.

$$C^T C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues of $C^T C$ are given by

$$\det\begin{bmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{bmatrix}=0$$
$$(1-\lambda)^2-1=0$$
$$\lambda^2-2\lambda=0$$

Step-6

Therefore, the eigenvalues for C^TC are:

$$\lambda_1=2$$
$$\lambda_2=0$$

The eigenvectors of C^TC are given by

$$x_1=\begin{bmatrix}1\\1\end{bmatrix}$$
$$x_2=\begin{bmatrix}-1\\1\end{bmatrix}$$

The unit eigenvectors of C^TC are given by

$$v_1=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$
$$v_2=\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

We know that eigenvectors of C^TC are in V, so

$$V=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$$

Step-7

The diagonal matrix has Σ eigenvalue from CC^T.

$$\Sigma=\begin{bmatrix}\sigma_1 & 0\\0 & 0\end{bmatrix}$$
$$=\begin{bmatrix}\sqrt{2} & 0\\0 & 0\end{bmatrix}$$

Therefore, we get,

$$C = U\Sigma V^T$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

Step-8

We know that if $A = U\Sigma V^T$, then its pseudoinverse is give by

$$A^+ = V\Sigma^+ U^T$$

Hence pseudoinverse of C is given by

$$C^+ = V\Sigma^+ U^T$$
$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$$

Therefore,

$$C^+ = (AB)^+$$
$$= \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$$

Step-9

Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Similarly we can find the pseudoinverse of A as

$$A^+ = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$$

Consider the matrix $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

So, we have

$$B^T = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Calculate BB^T.

$$\begin{aligned} BB^T &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

Step-10

To find the eigenvalues of BB^T, solve the following equation for λ.

$$\begin{bmatrix} 0-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = 0$$
$$(2-\lambda)(-\lambda) = 0$$

Therefore, the eigenvalues for BB^T are:

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$

The eigenvectors of CC^T are given by

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Step-11

The unit eigenvectors of BB^T are given by,

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We know that eigenvectors of BB^T are in *U*, so

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step-12

Calculate $B^T B$.

$$B^T B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues of $B^T B$ are given by

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$
$$(1-\lambda)^2 - 1 = 0$$
$$\lambda^2 - 2\lambda = 0$$

Therefore, the eigenvalues for $B^T B$ are:

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

The eigenvectors of $B^T B$ are given by

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The unit eigenvectors of $B^T B$ are given by

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
$$v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

We know that eigenvectors of $C^T C$ are in V, so

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Step-13

The diagonal matrix has Σ eigenvalue from BB^T.

$$\begin{aligned} \Sigma &= \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Therefore, we get,

$$\begin{aligned} B &= U\Sigma V^T \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T \end{aligned}$$

Step-14

We know that if $A = U\Sigma V^T$, then its pseudoinverse is give by

$$A^+ = V\Sigma^+ U^T$$

Hence pseudoinverse of B is given by

$$\begin{aligned} B^+ &= V\Sigma^+ U^T \\ &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix} \end{aligned}$$

Therefore,

$$B^+ = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$$

Step-15

We have

$$A^{+}=\begin{bmatrix}1/2 & 0\\ 1/2 & 0\end{bmatrix}$$

$$B^{+}=\begin{bmatrix}1/2 & 0\\ 1/2 & 0\end{bmatrix}$$

Calculate $B^{+}A^{+}$.

$$\begin{aligned}B^{+}A^{+}&=\begin{bmatrix}1/2 & 0\\ 1/2 & 0\end{bmatrix}\begin{bmatrix}1/2 & 0\\ 1/2 & 0\end{bmatrix}\\ &=\begin{bmatrix}1/4 & 0\\ 1/4 & 0\end{bmatrix}\end{aligned}$$

Since $(AB)^{+}=\begin{bmatrix}1/2 & 0\\ 1/2 & 0\end{bmatrix}$ and $B^{+}A^{+}=\begin{bmatrix}1/4 & 0\\ 1/4 & 0\end{bmatrix}$, so

$$\boxed{(AB)^{+}\neq B^{+}A^{+}}$$