Step-1

It is clear that the product of V_i and V_j is zero when $j = i \pm 2$.

Consider V_1V_2. This product is zero when $0 \le x \le \frac{1}{n+1}$ or $\frac{2}{n+1} \le x \le 1$.

Make two equal parts of the interval $\frac{1}{n+1} \le x \le \frac{2}{n+1}$. When $\frac{1}{n+1} \le x \le \frac{3}{2n+2}$, we get

$$\begin{aligned} V_1V_2 &= \left(2-(n+1)x\right)\left((n+1)x-1\right) \\ &= 3(n+1)x - 2 - (n+1)^2 x^2 \end{aligned}$$

Step-2

Consider the following:

$$\begin{aligned} \int_{\frac{1}{n+1}}^{\frac{3}{2n+2}} V_1V_2 dx &= \int_{\frac{1}{n+1}}^{\frac{3}{2n+2}} \left(3(n+1)x - 2 - (n+1)^2 x^2\right) dx \\ &= 3(n+1)\left[\frac{x^2}{2}\right]_{\frac{1}{n+1}}^{\frac{3}{2n+2}} - 2\left[x\right]_{\frac{1}{n+1}}^{\frac{3}{2n+2}} - (n+1)^2\left[\frac{x^3}{3}\right]_{\frac{1}{n+1}}^{\frac{3}{2n+2}} \\ &= \frac{3(n+1)}{2}\left[\frac{9}{(2n+2)^2} - \frac{1}{(n+1)^2}\right] - 2\left[\frac{3}{2n+2} - \frac{1}{n+1}\right] \\ &\qquad - \frac{(n+1)^2}{3}\left[\frac{27}{(2n+2)^3} - \frac{1}{(n+1)^3}\right] \\ &= \frac{3(n+1)}{2}\left[\frac{5}{(2n+2)^2}\right] - 2\left[\frac{1}{2n+2}\right] - \frac{(n+1)^2}{3}\left[\frac{19}{(2n+2)^3}\right] \end{aligned}$$

$$\begin{aligned} \int_{\frac{1}{n+1}}^{\frac{3}{2n+2}} V_1V_2 dx &= \frac{3(n+1)}{2}\left[\frac{5}{4(n+1)^2}\right] - 2\left[\frac{1}{2n+2}\right] - \frac{(n+1)^2}{3}\left[\frac{19}{8(n+1)^3}\right] \\ &= \frac{3}{2}\left[\frac{5}{4(n+1)}\right] - \left[\frac{1}{n+1}\right] - \frac{1}{3}\left[\frac{19}{8(n+1)}\right] \\ &= \frac{15}{8(n+1)} - \frac{1}{n+1} - \frac{19}{24(n+1)} \\ &= \frac{45-24-19}{24(n+1)} \end{aligned}$$

Step-3

Simplifying further, we get

$$\begin{aligned}\int_{\frac{1}{n+1}}^{\frac{3}{2n+2}} V_1V_2dx &= \frac{2}{24(n+1)}\\ &= \frac{1}{12(n+1)}\end{aligned}$$

Therefore, by symmetry, $\int_{\frac{3}{2n+2}}^{\frac{2}{n+1}} V_1V_2dx = \frac{1}{12(n+1)}$. Adding, we get $\int_{\frac{1}{n+1}}^{\frac{2}{n+1}} V_1V_2dx = \frac{1}{6(n+1)}$.

This gives us the idea that $\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} V_iV_{i+1}dx = \frac{1}{6(n+1)}$. Therefore, $\int_0^1 V_iV_{i+1}dx = \frac{1}{6(n+1)}$ and $\int_0^1 V_{i-1}V_idx = \frac{1}{6(n+1)}$, provided $1 < i < n$.

When $0 \le x \le \frac{1}{n+1}$, consider the following:

$$\begin{aligned}V_1^2 &= V_1V_1\\ &= ((n+1)x)((n+1)x)\\ &= (n+1)^2 x^2\end{aligned}$$

Therefore, we get

$$\begin{aligned}\int_0^{\frac{1}{n+1}} V_1^2 &= \int_0^{\frac{1}{n+1}} \left((n+1)^2 x^2\right)dx\\ &= (n+1)^2\left[\frac{x^3}{3}\right]_0^{\frac{1}{n+1}}\\ &= \frac{(n+1)^2}{3(n+1)^3}\\ &= \frac{1}{3(n+1)}\end{aligned}$$

By symmetry, $\int_{\frac{1}{n+1}}^{\frac{2}{n+1}} V_1^2 = \frac{1}{3(n+1)}$. Adding, we get $\int_0^{\frac{2}{n+1}} V_1^2 = \frac{2}{3(n+1)}$. It should be clear that $\int_0^1 V_1^2 = \frac{2}{3(n+1)}$.

Step-4

Now, $h = \frac{1}{n+1}$.

Therefore,

$$\int_0^1 V_i V_{i+1} dx = \frac{h}{6}$$

$$\int_0^1 V_{i-1} V_i dx = \frac{h}{6}$$

$$\int_0^1 V_1^2 = \frac{2h}{3}$$

$$= \frac{4h}{6}$$

Step-5

Thus, the mass matrix M_{ij} has all diagonal entries equal to $\frac{4h}{6}$ and the entries along the diagonals just above and below the main diagonal are $\frac{h}{6}$. If we take $\frac{h}{6}$ outside the matrix then all diagonal entries will be equal to 4 and the entries along the diagonals just above and below the main diagonal will be 1.

Therefore, $\boxed{M_{ij} = \frac{h}{6}\begin{bmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & . & & & \\ & & & . & 1 & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{bmatrix}}$.