题 号	1	2	3	4	5	6	7
分 值	15 分	15 分	15 分	10 分	15 分	10 分	20 分

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

   (共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

   (1) Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors. If the vectors $\mathbf{p} = \mathbf{u} + 5\mathbf{v}$ and $\mathbf{q} = 5\mathbf{u} - 4\mathbf{v}$ are orthogonal, then the angle α between $\mathbf{u}$ and $\mathbf{v}$ is:

   (A) $\alpha = \frac{\pi}{6}$. (B) $\alpha = \frac{\pi}{4}$.
   (C) $\alpha = \frac{\pi}{3}$. (D) $\alpha = \frac{3\pi}{4}$.

   设 $\mathbf{u}, \mathbf{v}$ 为单位向量. 如果向量 $\mathbf{p} = \mathbf{u}+5\mathbf{v}$ 和向量 $\mathbf{q} = 5\mathbf{u}-4\mathbf{v}$ 正交, 则 $\mathbf{u}$ 和 $\mathbf{v}$ 的夹角 α 为:

   (A) $\alpha = \frac{\pi}{6}$. (B) $\alpha = \frac{\pi}{4}$.
   (C) $\alpha = \frac{\pi}{3}$. (D) $\alpha = \frac{3\pi}{4}$.

   (2) Suppose $A = I - 2\alpha^T\alpha$, and $\alpha\alpha^T = 1$, then which of the following statements of A is not correct? ( )

   (A) $A^T = A$ (B) $A^T = A^{-1}$
   (C) $AA^T = I$ (D) $A^2 = A$

   设 $A = I - 2\alpha^T\alpha$, 且 $\alpha\alpha^T = 1$, 则 A 不能满足的结论是（ ）.

   (A) $A^T = A$ (B) $A^T = A^{-1}$
   (C) $AA^T = I$ (D) $A^2 = A$

   (3) For a matrix M we denote by rank(M) the rank of M. Let A and B be two $n \times n$ matrices. Which of the following statements is not true? ( )

   (A) $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.
   (B) $\text{rank}(A) + \text{rank}(B) \leq \text{rank}(AB) + n$.
   (C) $\text{rank}(A + B) \leq \text{rank}\ (A) + \text{rank}(B)$.
   (D) $\text{rank}(AB) = \min\{\text{rank}(A), \text{rank}(B)\}$.

   用 rank(M) 表示矩阵 M 的秩. 假定 A, B 都是 n 阶矩阵. 下列哪个选项是不正确的?

   (A) $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.
   (B) $\text{rank}(A) + \text{rank}(B) \leq \text{rank}(AB) + n$.
   (C) $\text{rank}(A + B) \leq \text{rank}\ (A) + \text{rank}(B)$.
   (D) $\text{rank}(AB) = \min\{\text{rank}(A), \text{rank}(B)\}$.

   (4) Let A be a symmetric matrix of order n, and B be a skew-symmetric matrix of order n. Then which of the following matrices is skew-symmetric? ( )

   (A) $AB - BA$.

(B) $(AB)^2$.

(C) $AB+BA$.

(D) BAB.

设 A 是 n 阶对称矩阵, B 是 n 阶反对称矩阵, 则下列矩阵为反对称矩阵的是（ ）.

(A) $AB-BA$.

(B) $(AB)^2$.

(C) $AB+BA$.

(D) BAB.

(5) If the vectors

$$\alpha_1=\begin{bmatrix}1\\1\\2\end{bmatrix},\alpha_2=\begin{bmatrix}1\\3\\4\end{bmatrix},\alpha_3=\begin{bmatrix}1\\5\\x\end{bmatrix}$$

are linearly dependent, then $x=$ ( )

(A) 2. (B) 4.

(C) 6. (D) 8.

如果 $\alpha_1=\begin{bmatrix}1\\1\\2\end{bmatrix}$, $\alpha_2=\begin{bmatrix}1\\3\\4\end{bmatrix}$, $\alpha_3=\begin{bmatrix}1\\5\\x\end{bmatrix}$ 线性相关. 则 $x=$ ( )

(A) 2. (B) 4.

(C) 6. (D) 8.

2. (15 points, 3 points each) Fill in the blanks. (共 15 分, 每小题 3 分) 填空题.

(1) Let $X = AX + B$, where $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 5 & -3 \end{bmatrix}$, then $X =$ ___________.

已知$X = AX + B$, 其中$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 5 & -3 \end{bmatrix}$, 则$X =$ ___________.

(2) If η_1, η_2, η_3 are solutions to the system of linear equations $Ax = b$, and $\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3$ is another solution to $Ax = b$, then $\lambda_1,\ \lambda_2, \lambda_3$ must satisfy ___________.

已知 η_1, η_2, η_3 均是线性方程组 $Ax = b$ 的解, 若 $\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3$ 也是 $Ax = b$ 的解, 则 $\lambda_1,\ \lambda_2, \lambda_3$ 应满足 ___________.

(3) If

$$A = \begin{bmatrix} 8 & 2 & 2 & -1 & 1 \\ 1 & 7 & 4 & -2 & 5 \\ -2 & 4 & 2 & -1 & 3 \end{bmatrix},$$

then $\operatorname{rank} A =$ _____.

如果

$$A = \begin{bmatrix} 8 & 2 & 2 & -1 & 1 \\ 1 & 7 & 4 & -2 & 5 \\ -2 & 4 & 2 & -1 & 3 \end{bmatrix},$$

则 $\operatorname{rank} A =$ _____.

(4) Let A be an $n \times n$ matrix with n independent eigenvectors corresponding to the eigenvalue λ_0, then $A =$ ___________.

若 n 阶矩阵 A 有 n 个属于特征值 λ_0 的线性无关的特征向量, 则 $A =$ ___________.

(5) Let M be a 2×2 matrix satisfying

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}.$$

Then the determinant of M, $|M| =$___________.

设 M 是一个满足以下条件的 2×2 矩阵

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}.$$

则矩阵 M 的行列式 $|M| =$___________.

3. (15 points 本题共 15 分) Consider the system of linear equations

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ -x_1 + 3x_2 + 2x_3 = b \\ 2x_1 + 4x_2 + ax_3 = 3. \end{cases}$$

Decide a, b so that the above system has

(a) no solution;

(b) has a unique solution;

(c) has infinitely many solutions and find all the solutions.

考虑线性方程组

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ -x_1 + 3x_2 + 2x_3 = b \\ 2x_1 + 4x_2 + ax_3 = 3. \end{cases}$$

求 a, b 的值，使得以上方程组

(a) 没有解;

(b) 有唯一解;

(c) 有无穷多个解, 并求出所有解.

4. (10 points 本题共 10 分) Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & 2 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Please give the bases for the four fundamental subspaces:

$$C(A),\ N(A),\ C(A^T),\ \text{and } N(A^T), \text{respectively.}$$

令

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & 2 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

请找出四个基本子空间

$$C(A),\ N(A),\ C(A^T),\ \text{和 } N(A^T)$$

各自的一组基.

5. (15 points 本题共 15 分) Consider

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(a) Show that $A^T A$ is positive definite.

(b) Find a Singular Value Decomposition of A.

考虑

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(a) 证明：$A^T A$ 是正定矩阵.

(b) 求 A 的一个奇异值分解.

6. (10 points 本题共 10 分) The following 2×2 matrix provides a model for a 2-band non-Hermitian system in condensed matter physics:

$$\begin{bmatrix} f_1 & f_2 \\ -f_2 & -f_1 \end{bmatrix}.$$

Where f_1 and f_2 are real-valued functions of vectors in the momentum space.

(a) Let

$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

be an indefinite Riemannian metric form. Verify that H satisfies the symmetry

$$\eta H \eta^{-1} = \overline{H}^T.$$

(b) Determine points (f_1, f_2) in the parameter space where H has a double eigenvalue, with (1) two linearly independent eigenvectors and (2) only one linearly independent eigenvector. (These are called non-defective/defective degeneracies, or exceptional lines. They help physicists design solid materials that do not exist in nature in order to build optical devices with extraordinary applications such as holography).

下面这个矩阵给出了凝聚态物理中双频非埃尔米特系统的一个模型

$$\begin{bmatrix} f_1 & f_2 \\ -f_2 & -f_1 \end{bmatrix}.$$

这里 f_1 和 f_2 是动量空间中取实值的向量函数.

(a) 设

$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

是一个不定的黎曼度量形式. 验证 H 满足以下对称性：

$$\eta H \eta^{-1} = \overline{H}^T.$$

(b) 在参数空间中决定 (f_1, f_2) 使得 H 具有相同的特征值, 同时满足 (1) 两个线性无关的特征向量 或 (2) 只有一个线性无关的特征向量. (这些称为无缺陷/有缺陷的退化形式, 或者叫异常线。它们可以帮助物理学家设计在自然界不存在的固体材料，这些材料可以用来生产用于全息摄影的光学仪器).

7. (20 points 本题共 20 分)The following two questions are independent.

(a) Let $A,\ B$ be $n \times n$ matrices. If $AB = I$, show that $BA = I$.

(b) Find all values of the parameter λ such that the form

$$Q(x_1, x_2, x_3) = 5x_1^2 + x_2^2 + \lambda x_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$$

is positive definite.

下面两个小题是独立的.

(a) 设 A, B 为 n 阶方阵, 证明: 如果 $AB = I$, 则一定有 $BA = I$.

(b) 求所有的 λ 使得

$$Q(x_1, x_2, x_3) = 5x_1^2 + x_2^2 + \lambda x_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$$

为正定的.