1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

   (共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

   (1) Let $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ and $f(t) = 1 + 2t + t^2 + t^4 - 5t^8$. Then $f(A) =$ A

   (A) $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

   (B) $\begin{bmatrix} 1 & 8 \\ 1 & 0 \end{bmatrix}$

   (C) $\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$

   (D) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

   设 $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$, 且 $f(t) = 1 + 2t + t^2 + t^4 - 5t^8$, 则 $f(A) =$

   (A) $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

   (B) $\begin{bmatrix} 1 & 8 \\ 1 & 0 \end{bmatrix}$

   (C) $\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$

   (D) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

   (2) Let A and B be invertible matrices. If A is similar to B, which of the following statements is **NOT** correct? C

   (A) A^T is similar to B^T.

   (B) A^{-1} is similar to B^{-1}.

   (C) $A + A^T$ is similar to $B + B^T$.

   (D) $A + A^{-1}$ is similar to $B + B^{-1}$.

   假定 A 和 B 都是可逆矩阵，且 A 和 B 相似, 下列陈述中哪个是**不正确**的？

   (A) A^T 和 B^T 相似.

   (B) A^{-1} 和 B^{-1} 相似.

   (C) $A + A^T$ 和 $B + B^T$ 相似.

   (D) $A + A^{-1}$ 和 $B + B^{-1}$ 相似.

(3) Let $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 6 & 8 \\ 3 & 8 & 5 \end{bmatrix}$. Then the number of positive eigenvalues of A is C

-1　-10　-34

(A) 0.

(B) 1.

(C) 2.

(D) 3.

设 $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 6 & 8 \\ 3 & 8 & 5 \end{bmatrix}$, 则矩阵 A 的正的特征值的个数为

(A) 0.

(B) 1.

(C) 2.

(D) 3.

(4) The equation $2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1$ represents a graph of B

$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$　$\lambda=1$, $\lambda=1$, $\lambda=-2$　$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(A) An ellipsoid.

(B) Hyperboloid of one sheet.

(C) Hyperboloid of two sheets.

(D) Hyperbolic paraboloid.

$2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1$ 表示的曲面是

(A) 椭球面.

(B) 单叶双曲面.

(C) 双叶双曲面.

(D) 双曲抛物面.

(5) Which of the following statements is correct? A

(A) If A is a Hermitian matrix, and $x^H Ax = 0$ for all complex vectors x, then $A = O$, where O denotes the zero matrix.

(B) An $n \times n$ matrix with real eigenvalues and n linearly independent real eigenvectors is symmetric.

(C) If A is a complex matrix, and $A^T = A$, then A is diagonalizable.

(D) Let A, B be $n \times n$ real matrices, then $\det(A + B) = \det A + \det B$.

下面的哪个陈述是正确的？

(A) 如果 A 是厄密特矩阵, 而且对所有的复向量 x 都有 $x^H Ax = 0$, 那么 $A = O$, 这里 O 表示零矩阵.

(B) 一个 n 阶的方阵的所有特征值和 n 个线性无关的特征向量都是实的, 则这个矩阵是对称的.

(C) 如果 A 是一个复矩阵, 且满足 $A^T = A$, 则 A 是可对角化的.

(D) 设 A, B 都是 n 阶实方阵, 则 $\det(A+B) = \det A + \det B$.

ANS: (1) A　(2) C　(3) C　(4) B　(5) A

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Suppose A is a 5×4 real matrix with 3 linearly independent columns. The dimension of the row space of A is $\underline{\quad 3 \quad}$, and the dimension of the left nullspace of A is $\underline{\quad 2 \quad}$.

设一个 5×4 的实矩阵 A 有三个线性无关的列向量, 则 A 的行空间的维数为 $\underline{\qquad\qquad}$, A 的左零空间的维数为 $\underline{\qquad\qquad}$.

(2) If the real quadratic form $f(x_1, x_2, x_3) = a(x_1^2 + x_2^2 + x_3^2) + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$ is changed to standard form $f = 6y_1^2$ by orthogonal transformation $x = Qy$, then $a = \underline{\quad 2 \quad}$.

$$\begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

如果实二次型 $f(x_1, x_2, x_3) = a(x_1^2 + x_2^2 + x_3^2) + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$ 经过正交变换 $x = Qy$ 化为标准形 $f = 6y_1^2$, 则 $a = \underline{\qquad\qquad}$.

(3) The eigenvalues of $I_3 - uv^T$ are $\underline{1 - v^T u,\ 1}$. Where I_3 is the 3×3 identity matrix, and u and v are nonzero vectors in $\mathbb{R}^3$.

矩阵 $I_3 - uv^T$ 的特征值为 $\underline{\qquad\qquad}$. 这里 I_3 表示 3 阶单位阵, u 和 v 是 $\mathbb{R}^3$ 中的非零向量.

r, $n-r$

1, 0

(4) If $A^2 = A$ and rank $(A) = r$, then trace $(A) = \underline{\quad r \quad}$.

如果 $A^2 = A$ 且 rank $(A) = r$, 则 trace $(A) = \underline{\qquad\qquad}$.

(5) Let $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ a \end{bmatrix}$.

If the dimension of the vector space generated by $\alpha_1, \alpha_2, \alpha_3$ is 2, then $a = \underline{\quad 6 \quad}$.

设 $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ a \end{bmatrix}$.　$\alpha_3 = 3\alpha_2 - \alpha_1$

如果由 $\alpha_1, \alpha_2, \alpha_3$ 生成的子空间的维数为 2, 则 $a = \underline{\qquad\qquad}$.

ANS: (1) 3,2　(2) 2　(3) $1, 1 - u^T v$　(4) r　(5) 6

3. (12 points) Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad q_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

(a) Find an orthonormal basis for the column space of A;

(b) Write A as QR, where Q has orthonormal columns and R is upper triangular.

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \quad R = \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & 5/\sqrt{2} \\ 0 & \sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

(12 分) 设

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) 求 A 的列空间的一组标准正交基;

(b) 将 A 分解成 QR, 其中 Q 的列是标准正交的向量, R 是上三角矩阵.

Solution.

(a) An orthonormal basis for the column space of A is:

$$q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

(b) The QR factorization of A is as follows:

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & \frac{5\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

4. (10 points) Compute the determinant of an $n \times n$ matrix A:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2. = a^n - a^{n-2}$$

(10 分) 计算 n 阶矩阵 A 的行列式:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2.$$

Solution. $a^n - a^{n-2}$.

5. (10 points) Suppose $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, and A is similar to B.

(a) Find a and b;

(b) Find an invertible matrix S, such that $S^{-1}AS = B$.

(10 分) 假定 $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$, 并且 A 和 B 相似.

(a) 求 a 和 b 的值;

(b) 求一个可逆矩阵 S, 使得 $S^{-1}AS = B$.

Solution.

(a) $a = 5$ and $b = 6$;

(b) S can be chosen to be

$$S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}.$$

6. (12 points) Let

$$A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

(a) Find all the singular values of A; $\sqrt{2}, 3\sqrt{2}$

(b) Find a singular value decomposition of A.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 3\sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(12 分) 设

$$A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

(a) 求 A 的所有奇异值;

(b) 求矩阵 A 的一个奇异值分解.

Solution.

(a) The singular values of A are $3\sqrt{2}, \sqrt{2}$;

(b) A singular value decomposition of A is as follows:

$$A = U\Sigma V^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

7. (8 points) Let A be a real symmetric $n \times n$ positive definite matrix and B be an $m \times n$ real matrix.

(a) Show that the matrix $M = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$ is congruent to the matrix $\begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}$.

(b) Suppose $\text{rank}(B) = r$. Find the number of positive eigenvalues, the number of negative eigenvalues, and the number of zero eigenvalues of M (counted with multiplicities).

n r $m-r$

(8 分) 假定 A 是一个 $n \times n$ 实对称正定矩阵， B 为一个 $m \times n$ 实矩阵.

(a) 证明：矩阵 $M = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$ 和矩阵 $\begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}$ 相合.

(b) 假定 $\text{rank}(B) = r$. 求矩阵 M 的正的特征值的个数, 负的特征值的个数, 以及零特征值的个数 (重根按重复的次数计).

Solution.

(a) Since

$$\boxed{\begin{bmatrix} I_n & 0 \\ -BA^{-1} & I_m \end{bmatrix}} \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} I_n & -A^{-1}B^T \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}$$

Therefore, M is congruent to the matrix $\begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}$.

(b) Since $\text{rank}(B) = r$ and $-BA^{-1}B^T$ is negative definite, the number of positive eigenvalues is n, the number of negative eigenvalues r, and the number of zero eigenvalues of M is $m - r$. (counted with multiplicities).

$A = P^T P \Leftarrow P$ is invertible

8. (8 points) Let A be an $n \times n$ real symmetric positive definite matrix, and B be an $n \times n$ real symmetric positive semidefinite matrix.

$B = Q^T Q$ $A = S^T S$

(a) Prove that the eigenvalues of AB are all nonnegative real numbers.

(b) Prove that AB is diagonalizable.

$AB = S^T S B = S^T (S B S^T)(S^T)^{-1}$

(8 分) 设 A 为 n 阶正定实对称矩阵, B 为 n 阶半正定实对称矩阵.

(a) 证明: AB 的所有特征值都是非负实数.

(b) 证明: AB 可对角化.

$AB = P^T P Q^T Q = P^T P Q^T Q P^T (P^T)^{-1}$

$\sim (QP^T)^T Q P^T \Rightarrow$ positive semidefinite

Solution.

(a) Since A is positive definite, and B is positive semidefinite, then we can find P and Q such that

$$A = P^T P, B = Q^T Q,$$

where P is an invertible matrix. It follows that

$$AB = P^T P Q^T Q = P^T P Q^T Q P^T (P^T)^{-1}.$$

This means that AB is similar to $PQ^T QP^T$. This together with the fact that $PQ^T QP^T$ is a positive semidefinite matrix complete the proof.

(b) Since A is positive definite, then there is an invertible matrix C such that

$$C^T AC = I_n, \; C^T AB(C^T)^{-1} = C^T AC\cancel{C^{-1}B(C^T)^{-1}}.$$

$M = C^{-1}B(C^T)^{-1}$ is a positive semidefinite matrix, therefore we can find an orthogonal matrix Q such that

$$Q^T MQ = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore

$$Q^T C^T AB(C^T)^{-1}Q = Q^T C^T ACC^{-1}B(C^T)^{-1}Q = Q^T MQ = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that AB is diagonalizable.