Step-1

Suppose $x=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The corresponding $Ax=0$ is given as,

$$\begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply the row operation $R_2 \to R_2 - iR_1$ to get,

$$\begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve the above system as,

$$\begin{aligned} x+iy&=0 \\ y+z&=0 \end{aligned}$$

This can be rewritten as,

$$x=-iy, z=-y$$

Therefore, the null space of A is given as,

$$\boxed{N(A)=(-iy, y, -y \mid y \in \mathbb{C})}$$

Step-2

(b)

The objective is to verify that the null space calculated in last part is orthogonal to $C(A^{\mathrm{H}})$ and is not orthogonal to $C(A^{\mathrm{T}})$.

First determine A^{H}

Since, A^{H} is computed by taking the conjugate transpose of the matrix A ,therefore A^{H} is,

$$A^{\mathrm{H}} = \left(\overline{A}\right)^T$$
$$= \begin{bmatrix} 1 & -i & 0 \\ -i & 0 & 1 \end{bmatrix}^T$$
$$= \begin{bmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{bmatrix}$$

Assume an arbitrary vector $X \in N(A)$.

Now, the aim is to verify that $N(A)$ is orthogonal to columns of A^{H} that is find $X^H \cdot c_i = 0$ where $c_i, i = 1,2$ denote the columns of A^{H}

First compute $X^H \cdot c_1$

$$X^H \cdot c_1 = \begin{bmatrix} -iy \\ y \\ -y \end{bmatrix}^H \cdot \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} iy & y & -y \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \quad (\text{Apply conjugate transpose})$$
$$= iy - iy + 0$$
$$= 0$$

Step-3

Now, compute $X^H \cdot c_2$

$$X^H c_2 = \begin{bmatrix} -iy \\ y \\ -y \end{bmatrix}^H \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} iy & y & -y \end{bmatrix} \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} \quad (\text{Apply conjugate transpose})$$
$$= -i^2 y + 0 - y \quad \left(i^2 = -1\right)$$
$$= y - y$$

$$= 0$$

As, $X^H \cdot c_1 = 0$, $X^H \cdot c_2 = 0$, $N(A)$ is orthogonal to $C\left(A^{\mathrm{H}}\right)$

Hence, $N(A)$ is orthogonal to $C(A^H)$

Step-4

Now, check whether null space is orthogonal to $C(A^T)$.

Determine A^T

Since, A^T is computed by taking the transpose of the matrix A, therefore A^T is,

$$A^T = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}^T$$
$$A^T = \begin{bmatrix} 1 & i \\ i & 0 \\ 0 & 1 \end{bmatrix}$$

First compute $X^H \cdot d_i$ where $d_i, i = 1,2$ denote the columns of A^T.

$$\begin{aligned}
X^H \cdot d_1 &= \begin{bmatrix} -iy \\ y \\ -y \end{bmatrix}^H \cdot \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} iy & y & -y \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \quad (\text{Apply conjugate transpose}) \\
&= iy + iy + 0 \\
&= 2iy \neq 0 \;\; (\text{for } y \neq 0)
\end{aligned}$$

Now, compute $X^H \cdot d_2$

$$\begin{aligned}
X^H d_2 &= \begin{bmatrix} -iy \\ y \\ -y \end{bmatrix}^H \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} iy & y & -y \end{bmatrix} \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \quad (\text{Apply conjugate transpose}) \\
&= i^2 y + 0 - y \quad (i^2 = -1) \\
&= -y - y
\end{aligned}$$

$$= -2y \neq 0 \;\; (\text{for } y \neq 0)$$

As, the arbitrary vector $X \in N(A)$ is not orthogonal for any of the column vectors of A^{T}.

Hence, $N(A)$ is not orthogonal to $C(A^T)$