Step-1

The objective is to prove that $A^2 = 0$ is possible but $A^T A = 0$ is not possible.

Step-2

Let $A = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$

Then,

$$\begin{aligned} A^2 &= A.A \\ &= \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 2(2)+2(-2) & 2(2)+2(-2) \\ (-2)(2)+(-2)(-2) & (-2)(2)+(-2)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 & 4-4 \\ -4+4 & -4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore, $A^2 = 0$ if $A \neq 0$.

Step-3

Consider the matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Where, $a_{ij} \neq 0$.

The transpose of the matrix A is,

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

The product of A^T, and A is,

$$A^T A = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{m} a_{k1}^2 & \sum_{k=1}^{m} a_{k1}a_{k2} & \dots & \sum_{k=1}^{m} a_{k1}a_{kn} \\ \sum_{k=1}^{m} a_{k2}a_{k1} & \sum_{k=1}^{m} a_{k2}^2 & \dots & \sum_{k=1}^{m} a_{k2}a_{kn} \\ \dots & \dots & \dots & \dots \\ \sum_{k=1}^{m} a_{kn}a_{k1} & \sum_{k=1}^{m} a_{kn}a_{k2} & \dots & \sum_{k=1}^{m} a_{kn}^2 \end{bmatrix}$$

In the matrix AA^T, the diagonal elements are sum of squares of elements. So, the diagonal elements are non-negative and $a_{ij} \neq 0.$

Therefore,

$$\sum_{k=1}^{m} a_{kl}^2 = a_{1l}^2 + a_{2l}^2 + a_{3l}^2 + \dots + a_{ml}^2 > 0.$$

Step-4

For example:

$$A = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$$

The transpose of the matrix A is,

$$A^T = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} 2(2)+(-2)(-2) & 2(2)+(-2)(-2) \\ (2)(2)+(-2)(-2) & (2)(2)+(-2)(-2) \end{bmatrix} \\
&= \begin{bmatrix} 4+4 & 4+4 \\ 4+4 & 4+4 \end{bmatrix} \\
&= \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \\
&\neq O
\end{aligned}
$$

Hence, it is clear that if $A \neq 0$ then $\boxed{A^2 = 0 \text{ but } A^T A \neq 0}$.