



南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 高等数学(上) A

开课单位: 数学系

考试时长: 120 分钟

命题教师: _____

题号	1	2	3	4	5	6	7	8	9
分值	15 分	15 分	10 分	10 分	10 分	10 分	10 分	16 分	4 分

1. (15pts) Multiple Choice Questions: (only one correct answer for each of the following questions.)

(1) The number of the real roots for the equation $x^3 - 3x + 3 = 0$ is

- (A) 0. (B) 1. (C) 2. (D) 3.

(2) If $f(x)$ is continuous on $(-\infty, +\infty)$, which of the following statements is **wrong** ?

- (A) $\int_0^1 f(x) dx = \int_0^1 f(t) dt$. (B) $\int_0^1 f(x) dx = \int_0^1 f(\sin x) d(\sin x)$.
(C) $d\left(\int_0^x f(t) dt\right) = f(x) dx$. (D) $d\left(\int_0^{x^2} f(t) dt\right) = f(x^2) d(x^2)$.

(3) Let

$$f(x) = \begin{cases} x^4 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Then the largest positive integer n , for which $f^{(n)}(0)$ exists, is

- (A) 1. (B) 2. (C) 3. (D) 4.

(4) If $f(x)$ is twice-differentiable on $(-\infty, +\infty)$, and $g(x) = (1-x)f(0) + xf(1)$, then which of the following statements is **correct** on $(0, 1)$?

- (A) $f(x) > g(x)$ if $f'(x) > 0$. (B) $f(x) > g(x)$ if $f'(x) < 0$.
(C) $f(x) > g(x)$ if $f''(x) > 0$. (D) $f(x) > g(x)$ if $f''(x) < 0$.

(5) If the improper integral $\int_0^{+\infty} \frac{\tan^{-1}(x^2)}{x^k} dx$ converges, then the constant k must satisfy

- (A) $k < 1$. (B) $k > 3$.
(C) $1 < k < 2$. (D) $1 < k < 3$.

Solution: (1) B; (2) B; (3) B; (4) D; (5) D.

2. (15 pts) Fill in the blanks.

(1) Function $f(x) = x^2$ has a tangent line $y = Kx - 1$ if $K =$ _____, or _____.

(2) Assume that $f'(0) = 3$, $f''(0) = 5$, $f'(1) = -4$, and $f''(1) = -7$. Let $g(x) = f(\ln x)$. Then $g''(1) =$ _____.

(3) The average value for $f(x) = \sin^3 x$ on $[0, \pi]$ is _____.

(4) Let $y = (\cos x)^x$ for $0 < x < \frac{\pi}{2}$, then $y'(x) = \underline{\hspace{2cm}}$.

(5) If $f''(a)$ exists, and $f'(a) \neq 0$, then $\lim_{x \rightarrow a} \left(\frac{1}{f'(a)(x-a)} - \frac{1}{f(x) - f(a)} \right) = \underline{\hspace{2cm}}$.

Solution: (1) 2, -2; (2) 2; (3) $\frac{4}{3\pi}$; (4) $(\cos x)^x (\ln(\cos x) - x \tan x)$; (5) $\frac{f''(a)}{2(f'(a))^2}$.

3. (10 pts) The region D is enclosed by the curve $y = \ln \sqrt{x-1}$, the straight line $x = 5$, and the x -axis.

(1) Find the area of the region D .

(2) Find the volumes generated by revolving the region D about the line $x = 5$.

Solution:

(1)

$$\int_2^5 \ln \sqrt{x-1} dx = \frac{1}{2} \int_1^4 \ln t dt = \frac{1}{2} (t(\ln t - 1)) \Big|_1^4 = 4 \ln 2 - \frac{3}{2}.$$

(2)

$$\int_0^{\ln 2} \pi (5 - (e^{2y} + 1))^2 dy = \pi \int_0^{\ln 2} (4 - e^{2y})^2 dy = \pi \left(16 \ln 2 - \frac{33}{4} \right).$$

4. (10 pts) Find the particular solution of

$$xy' + (x-2)y = 3x^3 e^{-x}, \quad x > 0,$$

satisfying $y(1) = 0$.

Solution: The integrating factor is $\frac{1}{x^2} e^x$.

$$\frac{d}{dx} \left(\frac{1}{x^2} e^x y \right) = 3$$

$$y = (3x + C)x^2 e^{-x}$$

Because $y(1) = 0$, we have $c = -3$. Therefore,

$$y = 3(x-1)x^2 e^{-x}$$

5. (10 pts) Evaluate the following limits.

$$(1) \lim_{n \rightarrow +\infty} \left(\frac{n}{2n^2 + 3n + 1^2} + \frac{n}{2n^2 + 6n + 2^2} + \cdots + \frac{n}{2n^2 + 3nk + k^2} + \cdots + \frac{n}{2n^2 + 3n^2 + n^2} \right).$$

$$(2) \lim_{x \rightarrow 0} \left(\frac{\ln(1+x)}{x} \right)^{\frac{1}{e^x - 1}}.$$

Solution:

(1)

$$= \int_0^1 \frac{1}{(x+1)(x+2)} dx = 2 \ln 2 - \ln 3$$

(2) Note $\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{e^x-1}} = e^{\frac{\ln(\ln(1+x)) - \ln x}{e^x-1}}$.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln(\ln(1+x)) - \ln x}{e^x - 1} &= \lim_{x \rightarrow 0} \frac{\frac{1}{(1+x)\ln(1+x)} - \frac{1}{x}}{e^x} = \lim_{x \rightarrow 0} \frac{x - (1+x)\ln(1+x)}{x \ln(1+x)} \\ &= \lim_{x \rightarrow 0} \frac{-\ln(1+x)}{\ln(1+x) + \frac{x}{1+x}} = -\frac{1}{2}. \end{aligned}$$

The final result is $= \frac{1}{\sqrt{e}}$.

6. (10 pts)

- (1) For $y = \frac{x^2+1}{x+1}$, identify the coordinates of any local and absolute extreme points and inflection points that may exist.
- (2) Sketch the graph of the above function. (Please identify all the asymptotes and some specific points, such as local maximum and minimum points, inflection points, and intercepts.)

Solution:

(1) $y' = 1 - \frac{2}{(x+1)^2}$, $y'' = \frac{4}{(x+1)^3}$.

local maximum point $(-1 - \sqrt{2}, -2 - 2\sqrt{2})$, local minimum point $(-1 + \sqrt{2}, 2\sqrt{2} - 2)$.

(2) oblique asymptote is $y = x - 1$.

7. (10 pts) Find $\frac{dy}{dx}$ if

$$y = \int_{x^2+1}^{2x^2+3} t \tan \sqrt{x+t} dt.$$

Solution: Let $u = x + t$.

$$y = \int_{x^2+x+1}^{2x^2+x+3} (u-x) \tan \sqrt{u} du$$

$$\frac{dy}{dx} = - \int_{x^2+x+1}^{2x^2+x+3} \tan \sqrt{u} du + (4x+1)(2x^2+3) \tan \sqrt{2x^2+x+3}$$

$$- \cancel{0} (2x+1)(x^2+1) \tan \sqrt{x^2+x+1}.$$

8. (16 pts) Evaluate the integrals.

(1) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc^3 x dx.$

(2) $\int \sqrt{\frac{x}{x-2}} dx$, where $x > 2$.

(3) $\int_1^e \ln^3 x dx.$

(4) $\int_1^{+\infty} \frac{(x+2)\ln(x^2+1)}{x^3} dx.$

Solution:

(1)

$$\begin{aligned} &= -\frac{1}{2} \csc x \cot x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc x \, dx = -\left[\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln(\csc x + \cot x) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \sqrt{3} - \frac{1}{3} + \frac{1}{2} \ln \frac{2+\sqrt{3}}{\sqrt{3}} \end{aligned}$$

(2)

$$= \int \frac{x}{\sqrt{x^2-2x}} \, dx$$

Let $u = x - 1$, we have

$$\begin{aligned} &= \int \frac{u+1}{\sqrt{u^2-1}} \, du = \sqrt{u^2-1} + \ln(u + \sqrt{u^2-1}) + C \\ &= \sqrt{x^2-2x} + \ln(x-1 + \sqrt{x^2-2x}) + C \end{aligned}$$

(3) Let $u = \ln x$, we have

$$= \int_0^1 u^3 e^u \, du = (u^3 - 3u^2 + 6u - 6)e^u \Big|_0^1 = 6 - 2e.$$

(4)

$$\begin{aligned} &\int \frac{(x+2) \ln(x^2+1)}{x^3} \, dx = -\frac{1+x}{x^2} \ln(x^2+1) + 2 \int \frac{x+1}{x(x^2+1)} \, dx \\ &\int \frac{x+1}{x(x^2+1)} \, dx = \int \left(\frac{1}{x} - \frac{x-1}{x^2+1} \right) \, dx = \ln x - \frac{1}{2} \ln(x^2+1) + \tan^{-1} x + C \end{aligned}$$

Therefore,

$$\int_1^{+\infty} \frac{(x+2) \ln(x^2+1)}{x^3} \, dx = 3 \ln 2 + \frac{\pi}{2}.$$

9. (4 pts) Let $f(n) = \sum_{m=1}^n \int_0^m \cos \frac{2\pi n \lfloor x+1 \rfloor}{m} \, dx$, here $\lfloor x+1 \rfloor$ is the largest integer which is less than or equal to $x+1$. Evaluate $f(2021)$.

Solution:

$$\int_0^m \cos \frac{2\pi n \lfloor x+1 \rfloor}{m} \, dx = \sum_{k=1}^m \int_{k-1}^k \cos \frac{2\pi nk}{m} \, dx = \sum_{k=1}^m \cos k \frac{2n\pi}{m}$$

If $m|n$, then $\int_0^m \cos \frac{2\pi n \lfloor x+1 \rfloor}{m} \, dx = m$; otherwise, because

$$\sum_{k=1}^m \cos kt = \frac{\cos\left(\frac{m+1}{2}t\right) \sin\left(\frac{m}{2}t\right)}{\sin \frac{t}{2}},$$

when $t = \frac{2n\pi}{m}$, $\sin\left(\frac{m}{2}t\right) = 0$. Thus $\int_0^m \cos \frac{2\pi n \lfloor x+1 \rfloor}{m} \, dx = 0$.

Note $2021 = 43 * 47$, therefore $f(2021) = 1 + 43 + 47 + 2021 = 2112$.