Step-1

We have to by example that the following three statements are generally false:

(a) A and A^T have the same null space.

(b) A and A^T have the same free variables.

(c) If R is the row reduced echelon form (rref (A)) then R^T is $\text{rref}\left(A^T\right)$.

Step-2

a) Let $A=\begin{bmatrix}1 & 2 & 0\\ 2 & 4 & 1\end{bmatrix}$

$$\xrightarrow{R_2-2R_1}\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

To find null space of A, consider

$$\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\Rightarrow x_1+2x_2=0, x_3=0$$
$$\Rightarrow x_1=-2x_2$$

Step-3

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-2x_2\\ x_2\\ 0\end{bmatrix}=x_2\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}$$

Null space of A is a line through $(-2,1,0)$

Step-4

Now

$$A^T=\begin{bmatrix}1 & 2\\ 2 & 4\\ 0 & 1\end{bmatrix}$$

To find null space of A^T, take

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-5

$\Rightarrow x_1 + 2x_2 = 0$

$x_2 = 0$

$\Rightarrow x_1 = -2x_2 = 0$

Therefore the null space of A^T is $\{(0,0)\}$

Therefore the statement (a) is false.

Step-6

b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix}$

By part (a), and by its reduced form, x_2 is free variable.

And

$A^T = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, by part (a), and by its reduced form, both x_1, x_2 are pivots.

Therefore A and A^T have the different number of free variables.

Hence the statement (b) is false.

Step-7

c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the rref of $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix}$,and

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is rref of $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Therefore both rref are different and hence the statement (c) is false.