Step-1

Given that $b=(b_1,\dots,b_m)$ projected onto the line through $a=(1,\dots,1)$

(a) We have to solve $a^T a\,\hat{x}=a^T b$ to show that $\hat{x}$ is the mean of *bâ€™*s.

Now

$$a^T a\,\hat{x}=a^T b$$

$$\Rightarrow \begin{pmatrix}1 & 1 & . & . & . & 1\end{pmatrix}\begin{pmatrix}1\\1\\.\\.\\.\\1\end{pmatrix}\hat{x}=\begin{pmatrix}1 & 1 & . & . & . & 1\end{pmatrix}\begin{pmatrix}b_1\\b_2\\.\\.\\.\\b_m\end{pmatrix}$$

$$\Rightarrow \left[1+1+1+\dots(m\text{ times})\right]\hat{x}=b_1+b_2+\dots+b_m$$

$$\Rightarrow (m.1)\hat{x}=b_1+b_2+\dots+b_m$$

$$\Rightarrow \hat{x}=\frac{b_1+b_2+\dots+b_m}{m}$$

Hence $\hat{x}$ is the mean of $b_1,b_2,\dots,b_m$.

Step-2

(b) We have to find $e=b-a\hat{x}$

Now

$$e=b-a\hat{x}$$

Â Â Â Â Â Â Â $=(b_1,b_2,\dots,b_m)-(1,1,\dots,1)\left(\frac{b_1+b_2+\dots+b_m}{m}\right)$

$$=\left(b_1-\frac{b_1+b_2+\dots+b_m}{m},\dots,b_m-\frac{b_1+b_2+\dots+b_m}{m}\right)$$

$$=\left(\frac{mb_1-b_1+b_2+\dots+b_m}{m},\dots,\frac{mb_m-b_1+b_2+\dots+b_m}{m}\right)$$

$$=\left(\frac{(m-1)b_1+b_2+...+b_m}{m},...,\frac{b_1+b_2+...+(m-1)b_m}{m}\right)$$

Hence $\boxed{e=\left(\frac{(m-1)b_1+b_2+...+b_m}{m},...,\frac{b_1+b_2+...+(m-1)b_m}{m}\right)}$

Step-3

Now we have to find the variance $\|e\|^2$.

The variance $\|e\|^2=\sum_{i=1}^{m}\left(b_i-\hat{x}\right)^2$

$$=\sum_{i=1}^{m}\left(b_i-\frac{b_1+b_2+...+b_m}{m}\right)^2$$
$$=\sum_{i=1}^{m}\left(b_i-\frac{b_1+b_2+...+b_m}{m}\right)^2$$

Â Â Â Â Â Â Â Â Â Â Â Â Â Â Â Â Â $=\sum_{i=1}^{m}\left(\frac{(m-1)b_i-(b_1+b_2+...+b_{i-1}+b_{i+1}+...+b_m)}{m}\right)^2$

Hence the variance is $\boxed{\|e\|^2=\sum_{i=1}^{m}\left(\frac{(m-1)b_i-(b_1+b_2+...+b_{i-1}+b_{i+1}+...+b_m)}{m}\right)^2}$.

Step-4

Now we have to find the standard deviation $\|e\|$

Standard deviation

$$\|e\|=\sqrt{\|e\|^2}$$
$$=\sqrt{\sum_{i=1}^{m}\left(\frac{(m-1)b_i-(b_1+b_2+...+b_{i-1}+b_{i+1}+...+b_m)}{m}\right)^2}$$
$$=\sum_{i=1}^{m}\left(\frac{(m-1)b_i-(b_1+b_2+...+b_{i-1}+b_{i+1}+...+b_m)}{m}\right)$$

Hence the standard deviation is $\boxed{\|e\|=\sum_{i=1}^{m}\left(\frac{(m-1)b_i-(b_1+b_2+...+b_{i-1}+b_{i+1}+...+b_m)}{m}\right)}$

Step-5

(c) Given that the horizontal line is $\hat{b} = 3$ is closest to $b = (1,2,6)$.

We have to check that $p = (3,3,3)$ is perpendicular to *e*.

Step-6

We have $b = (1,2,6)$ Â

So $b_1 = 1, b_2 = 2, b_3 = 6$

Since $\hat{x}$ is the mean of $b_1, b_2, \ldots, b_m$

So

Step-7

$$\begin{aligned}\hat{x} &= \frac{b_1 + b_2 + b_3}{3} \\ &= \frac{1+2+6}{3} \\ &= \frac{9}{3} \\ &= 3\end{aligned}$$

Step-8

Now

$$\begin{aligned}e &= b - a\hat{x} \\ &= (1,2,6) - 3(1,1,1) \\ &= (1-3, 2-3, 6-3) \\ &= (-2,-1,3)\end{aligned}$$

Step-9

Now

$$p^T e = (3,3,3)\begin{pmatrix}-2\\-1\\3\end{pmatrix}$$
$$= 3(-2)+3(-1)+3(3)$$
$$= -6-3+9$$
$$= 0$$

Since $p^T e = 0$

Therefore p is perpendicular to e.

Step-10

Now we have to find the projection matrix P.

We know that the Projection matrix is $P = \frac{aa^T}{a^T a}$

Now

$$aa^T = \begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$$
$$= \begin{bmatrix}1(1) & 1(1) & 1(1)\\1(1) & 1(1) & 1(1)\\1(1) & 1(1) & 1(1)\end{bmatrix}$$
$$= \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

Step-11

And

$$a^T a = \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$= 1(1)+1(1)+1(1)$$
$$= 1+1+1$$
$$= 3$$

Therefore the Projection matrix is $\boxed{P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$.