Step-1

Objective is to prove that every unitary matrix A is diagonalizable. For this prove the following parts:

(a)

Suppose that A, U both are unitary matrix. Show that $T = U^{-1}AU$ is also unitary.

Since A, U are unitary matrix, therefore

$$\begin{aligned} UU^H &= U^H U \\ &= I \qquad \text{......(1)} \\ AA^H &= A^H A \\ &= I \qquad \text{......(2)} \end{aligned}$$

The matrix T will be unitary if

$$\begin{aligned} TT^H &= T^H T \\ &= I \end{aligned}$$

Step-2

Consider TT^H and use the above equations and solve as follows:

$$\begin{aligned} TT^H &= \left(U^{-1}AU\right)\left(U^{-1}AU\right)^H \\ &= \left(U^{-1}AU\right)\left(U^H A^H \left(U^{-1}\right)^H\right) \\ &= U^{-1}A\left(UU^H\right)A^H\left(U^{-1}\right)^H \\ &= U^{-1}AA^H\left(U^{-1}\right)^H \qquad \left[UU^H = I\right] \end{aligned}$$

$$\begin{aligned} TT^H &= U^{-1}\left(U^{-1}\right)^H \qquad \left[AA^H = I\right] \\ &= I \end{aligned}$$

Similarly, solve the right side as:

$$\begin{aligned} T^H T &= \left(U^{-1}AU\right)^H\left(U^{-1}AU\right) \\ &= \left(U^H A^H \left(U^{-1}\right)^H\right)\left(U^{-1}AU\right) \\ &= U^H A^H \left(\left(U^{-1}\right)^H U^{-1}\right)AU \\ &= U^H A^H AU \qquad \left[U^H = U^{-1}\right] \end{aligned}$$

$$\begin{aligned} T^H T &= U^H U \\ &= I \end{aligned}$$

Hence, $T = U^{-1}AU$ is an unitary matrix.

Step-3

(b)

Prove that upper triangular unitary matrix T is diagonal.

Note that the columns of an unitary matrices are always orthonormal. Let

$T = [a_{ij}]_{n\times n}$ where $1 \le i, j \le n$.

Since T is upper triangular unitary matrix, therefore the absolute value of its first entry is 1. The entry $a_{12} = 0$ and $|a_{22}| = 1$.

Similarly, the entry $a_{13} = 0, a_{23} = 0$ and $|a_{33}| = 1$ in T, in order to make the first three columns orthonormal.

Step-4

Proceed in the same manner and observe that the matrix T is nothing but the diagonal matrix where the absolute values of the diagonal entries are 1.

Thus, upper triangular unitary matrix $T = U^{-1}AU$ is diagonal and hence every unitary matrix A is diagonalizable.