

Step-1

Given that $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^3 & \omega \\ 1 & \omega^3 & \omega & \omega^2 \end{bmatrix}$

We have to show that the columns of the matrix U are the eigenvectors of P .

Step-2

We have to find the eigenvalues of P .

The characteristic equation of P is

$$|P - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

Step-3

Continuation to the above

$$\Rightarrow -\lambda \{-\lambda(\lambda^2)\} - 1\{-1(-1)\} = 0$$

$$\Rightarrow \lambda^4 - 1 = 0$$

$$\Rightarrow \lambda^4 = 1$$

The λ values are 4th roots of unity they are $\lambda = 1, \omega, \omega^2, \omega^3$

Hence the eigenvalues of P are $\lambda = 1, \omega, \omega^2, \omega^3$

Step-4

We know that $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is an eigenvector of P if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is the nonzero solution of the system $(P - \lambda I)x = 0$.

That is $\begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$ (1)

Step-5

For $\lambda = 1$, (1) becomes

$$(P - I)x = 0$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

The Augmented matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix} = 0$$

Step-6

Add row 1 to row 4

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

Add row 2 to row 4.

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

Step-7

Add row 3 to row 4

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this, we get the equations

$$-x_1 + x_2 = 0$$

$$-x_2 + x_3 = 0$$

$$-x_3 + x_4 = 0$$

Here x_4 is free variable.

Step-8

Let $x_4 = k$, where k is a parameter.

Then $x_3 = k, x_2 = k, x_1 = k$

Therefore,

$$\begin{aligned}
 x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\
 &= \begin{bmatrix} k \\ k \\ k \\ k \end{bmatrix} \\
 &= k \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
 \end{aligned}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = 1$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Step-9

For $\lambda = \omega$, (1) becomes

$$(P - \omega I)x = 0$$

$$\begin{bmatrix} -\omega & 1 & 0 & 0 \\ 0 & -\omega & 1 & 0 \\ 0 & 0 & -\omega & 1 \\ 1 & 0 & 0 & -\omega \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

The Augmented matrix

$$\begin{bmatrix} -\omega & 1 & 0 & 0 & 0 \\ 0 & -\omega & 1 & 0 & 0 \\ 0 & 0 & -\omega & 1 & 0 \\ 1 & 0 & 0 & -\omega & 0 \end{bmatrix} = 0$$

Step-10

Add row 1 to ω times of row 4

$$\begin{bmatrix} -\omega & 1 & 0 & 0 & 0 \\ 0 & -\omega & 1 & 0 & 0 \\ 0 & 0 & -\omega & 1 & 0 \\ 0 & 1 & 0 & -\omega^2 & 0 \end{bmatrix} = 0$$

Add row 2 to ω times of row 4

$$\begin{bmatrix} -\omega & 1 & 0 & 0 & 0 \\ 0 & -\omega & 1 & 0 & 0 \\ 0 & 0 & -\omega & 1 & 0 \\ 0 & 0 & 1 & -\omega^3 & 0 \end{bmatrix} = 0$$

Step-11

Add row 3 to ω times of row 4

$$\begin{bmatrix} -\omega & 1 & 0 & 0 & 0 \\ 0 & -\omega & 1 & 0 & 0 \\ 0 & 0 & -\omega & 1 & 0 \\ 0 & 0 & 0 & -\omega^4 + 1 & 0 \end{bmatrix} = 0$$

From this, we get the equations

$$-\omega x_1 + x_2 = 0$$

$$-\omega x_2 + x_3 = 0$$

$$-\omega x_3 + x_4 = 0$$

$$(-\omega^4 + 1)x_4 = 0$$

Step-12

Choose $x_4 = \omega^3 k$, where k is a parameter.

Then we get $x_3 = \omega^2 k, x_2 = \omega k, x_1 = k$

Therefore,

$$\begin{aligned}
 x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\
 &= \begin{bmatrix} k \\ \omega k \\ \omega^2 k \\ \omega^3 k \end{bmatrix} \\
 &= k \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{bmatrix}
 \end{aligned}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = \omega$ is $\begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{bmatrix}$.

For $\lambda = \omega^2$, (1) becomes

$$(P - \omega^2 I)x = 0$$

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 \\ 0 & 0 & -\omega^2 & 1 \\ 1 & 0 & 0 & -\omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

The Augmented matrix

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 1 & 0 & 0 & -\omega^2 & 0 \end{bmatrix} = 0$$

Step-13

Add row 1 to ω^2 times of row 4

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 0 & 1 & 0 & -\omega^4 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 0 & 1 & 0 & -\omega & 0 \end{bmatrix} = 0 \quad (\text{Since } \omega^3 = 1)$$

Step-14

Add row 2 to ω^2 times of row 4

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 0 & 0 & 0 & -\omega^3 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} = 0$$

Step-15

Add row 3 to ω^2 times of row 4

$$\begin{bmatrix} -\omega^2 & 1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 1 & 0 \\ 0 & 0 & 0 & -\omega^2 + 1 & 0 \end{bmatrix} = 0$$

From this, we get the equations

$$-\omega^2 x_1 + x_2 = 0$$

$$-\omega^2 x_2 + x_3 = 0$$

$$-\omega^2 x_3 + x_4 = 0$$

$$(-\omega^2 + 1)x_4 = 0$$

Step-16

Choose $x_4 = \omega k$, where k is a parameter.

Then we get $x_3 = \omega^3 k, x_2 = \omega^2 k, x_1 = k$

Therefore,

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} k \\ \omega^2 k \\ \omega^3 k \\ \omega k \end{bmatrix} \\ &= k \begin{bmatrix} 1 \\ \omega^2 \\ \omega^3 \\ \omega \end{bmatrix} \end{aligned}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = \omega^2$ is $\begin{bmatrix} 1 \\ \omega^2 \\ \omega^3 \\ \omega \end{bmatrix}$.

Step-17

For $\lambda = \omega^3$, (1) becomes

$$(P - \omega^3 I)x = 0$$

$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 \\ 0 & 0 & -\omega^3 & 1 \\ 1 & 0 & 0 & -\omega^3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

The Augmented matrix

$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 & 0 \\ 0 & 0 & -\omega^3 & 1 & 0 \\ 1 & 0 & 0 & -\omega^3 & 0 \end{bmatrix} = 0$$

Step-18

Add row 1 to ω^3 times of row 4

$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 & 0 \\ 0 & 0 & -\omega^3 & 1 & 0 \\ 0 & 1 & 0 & -\omega^6 & 0 \end{bmatrix} = 0$$
$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 & 0 \\ 0 & 0 & -\omega^3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} = 0 \quad \left(\text{Since } \omega^6 = (\omega^3)^2 = 1 \right)$$

Step-19

Add row 2 to ω^3 times of row 4

$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 & 0 \\ 0 & 0 & -\omega^3 & 1 & 0 \\ 0 & 0 & 1 & -\omega^3 & 0 \end{bmatrix} = 0$$

Add row 3 to ω^3 times of row 4

$$\begin{bmatrix} -\omega^3 & 1 & 0 & 0 & 0 \\ 0 & -\omega^3 & 1 & 0 & 0 \\ 0 & 0 & -\omega^3 & 1 & 0 \\ 0 & 0 & 0 & -\omega^6 + 1 & 0 \end{bmatrix} = 0$$

From this, we get the equations

$$-\omega^3 x_1 + x_2 = 0$$

$$-\omega^3 x_2 + x_3 = 0$$

$$-\omega^3 x_3 + x_4 = 0$$

$$(-\omega^6 + 1)x_4 = 0$$

Step-20

Choose $x_4 = \omega^2 k$, where k is a parameter.

Then we get $x_3 = \omega k, x_2 = \omega^3 k, x_1 = k$

Therefore,

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} k \\ \omega^3 k \\ \omega k \\ \omega^2 k \end{bmatrix} \\ &= k \begin{bmatrix} 1 \\ \omega^3 \\ \omega \\ \omega^2 \end{bmatrix} \end{aligned}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = \omega^3$ is $\begin{bmatrix} 1 \\ \omega^3 \\ \omega \\ \omega^2 \end{bmatrix}$.

Hence the columns of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^3 & \omega \\ 1 & \omega^3 & \omega & \omega^2 \end{bmatrix}$ are the eigenvectors of the given permutation matrix.