Step-1

The objective is to find that under what conditions on b_1, b_2, b_3, b_4, the following systems are solvable, and we have to find x for the systems:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Step-2

Consider the following system,

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Consider

$$A_1 = \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{bmatrix}$$

$$\xrightarrow[R_4 - 3R_1]{\substack{R_2 - 2R_1, \\ R_3 - 2R_1,}} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{bmatrix}$$

Step-3

Simplify further:

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{bmatrix}$$

$$\xrightarrow[R_4-3R_2]{R_1-2R_2,}\begin{bmatrix}1 & 0 & 5b_1-2b_3\\0 & 1 & b_3-2b_1\\0 & 0 & b_2-2b_1\\0 & 0 & b_4-3b_3+3b_1\end{bmatrix}$$

The system is consistent if

$$b_2-2b_1=0$$
$$b_4-3b_3+3b_1=0$$

Step-4

Next, we consider the following system,

$$\begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\\2 & 5 & 7\\3 & 9 & 12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

$$A_2=\begin{bmatrix}1 & 2 & 3 & b_1\\2 & 4 & 6 & b_2\\2 & 5 & 7 & b_3\\3 & 9 & 12 & b_4\end{bmatrix}$$

$$\xrightarrow[R_4-3R_1]{R_2-2R_1,\ R_3-2R_1,}\begin{bmatrix}1 & 2 & 3 & b_1\\0 & 0 & 0 & b_2-2b_1\\0 & 1 & 1 & b_3-2b_1\\0 & 3 & 3 & b_4-3b_1\end{bmatrix}$$

Step-5

$$\xrightarrow[R_4-3R_3]{R_1-2R_3,}\begin{bmatrix}1 & 0 & 1 & 5b_1-2b_3\\0 & 0 & 0 & b_2-2b_1\\0 & 1 & 1 & b_3-2b_1\\0 & 0 & 0 & b_4-3b_3+3b_1\end{bmatrix}$$

$$\xrightarrow{R_{23}}\begin{bmatrix}1 & 0 & 1 & (5b_1-2b_3)\\0 & 1 & 1 & (b_3-2b_1)\\0 & 0 & 0 & b_2-2b_1\\0 & 0 & 0 & b_4-3b_3+3b_1\end{bmatrix}$$

There are two pivot columns. Therefore there are two free columns.

Step-6

Therefore, the system is consistent if $b_3 - 2b_1 = 0$, and $b_4 - 3b_3 + 3b_1 = 0$.

Step-7

Or

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix}$ is the general solution of the first system.

For the solution of the second system,

$$x_1 + x_3 = 5b_1 - 2b_3$$
$$x_2 + x_3 = b_3 - 2b_1$$

$$b_2 - 2b_1 = 0, b_4 - 3b_3 + 3b_1 = 0$$

This implies:

$$x_1 = -x_3 + 5b_1 - 2b_3$$

$$x_2 = -x_3 + b_3 - 2b_1$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 + 5b_1 - 2b_3 \\ -x_3 + b_3 - 2b_1 \\ x_3 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix}}$$

This is the general solution of the second system.