Step-1

Consider the following matrix:

$$A=\begin{bmatrix}0.2 & 0.4 & 0.3\\ 0.4 & 0.2 & 0.3\\ 0.4 & 0.4 & 0.4\end{bmatrix}$$

Here, columns are linearly dependent as column 1 + column 2 = 2(column 3).

Step-2

(a) Find one Eigen value and Eigen vectors of matrix A.

To find the Eigen values calculate the following:

$$\det\left(A-\lambda I\right)=0$$

$$\begin{bmatrix}\frac{1}{5}-\lambda & \frac{2}{5} & \frac{3}{10}\\ \frac{2}{5} & \frac{1}{5}-\lambda & \frac{3}{10}\\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5}-\lambda\end{bmatrix}=0$$

$$-5\lambda^3+4\lambda^2+\lambda=0$$

$$\lambda\left(5\lambda+1\right)\left(-\lambda+1\right)=0$$

After solving following values are obtained:

$$\lambda=0$$
$$\lambda=1$$
$$\lambda=\frac{-1}{5}$$

Step-3

Lets take one Eigen value to calculate Eigen vectors:

$$\lambda = 0$$
$$(A - 0I)x = 0$$
$$\begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{3}{10} \\ \frac{2}{5} & \frac{1}{5} & \frac{3}{10} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-4

Solving it using Gaussians method:

$$\begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{3}{10} \vdots & 0 \\ \frac{2}{5} & \frac{1}{5} & \frac{3}{10} \vdots & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \vdots & 0 \end{pmatrix} \square \begin{pmatrix} 1 & 2 & \frac{3}{2} \vdots & 0 \\ 0 & \frac{-3}{5} & \frac{-3}{10} \vdots & 0 \\ 0 & \frac{-2}{5} & \frac{-1}{5} \vdots & 0 \end{pmatrix}$$
$$\square \begin{pmatrix} 1 & 0 & \frac{1}{2} \vdots & 0 \\ 0 & 1 & \frac{1}{2} \vdots & 0 \\ 0 & 0 & 0 \vdots & 0 \end{pmatrix}$$

Step-5

Thus, Eigen vectors are as follows:

$$x = \begin{bmatrix} -\frac{1}{2}c_1 \\ -\frac{1}{2}c_1 \\ c_1 \end{bmatrix}$$

Lets take the value of constant c_1 as follows and then solve:

$$c_1 = -2$$
$$x = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Step-6

Therefore, Eigen vectors corresponding to Eigen value $\lambda = 0$ are as follows:

$$x = \boxed{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}$$

(b)

Step-7

Find the other Eigen values of matrix A.

As solved in part (a) other Eigen values are as follows:

$$\lambda = \boxed{1}$$
$$\lambda = \boxed{\frac{-1}{5}}$$

Step-8

(c) Find the limit of $A^k u_0$ when $k \to \infty$ for the following value:

$$u_0 = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

Step-9

Difference equation can be written as follows:

$$\begin{aligned} u_{k+1} &= A^k u_0 \\ &= S\Lambda^k S^{-1} \cdot u_0 \end{aligned}$$

Following is the Eigen vector matrix.

$$S = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 3 & 1 \\ -2 & 4 & 0 \end{bmatrix}$$

Step-10

Now, calculate the following:

$$u_{k+1} = S\Lambda^k S^{-1} \cdot u_0$$

$$= \begin{bmatrix} 1 & 3 & -1 \\ 1 & 3 & 1 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & \left(-\frac{1}{5}\right)^k \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{-3}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{-1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

Step-11

Taking the limit $k \to \infty$ makes the element $(-1/5)^k$ very small, so neglect it.

$$u_\infty = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 3 & 1 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{-3}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{-1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{-3}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{-1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{10} & \frac{3}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{3}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

Step-12

Therefore, the limit of $A^k u_0$ is as follows:

$$\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$