Linear Algebra-A

Assignments - Week 10

Supplementary Problem Set

1. Calculate the following determinants:

   a)

$$D = \begin{vmatrix} 0 & a_1 & b_1 & 0 \\ a_2 & 0 & 0 & b_2 \\ a_3 & 0 & 0 & b_3 \\ x & a_4 & b_4 & y \end{vmatrix}.$$

   b)

$$D_n = \begin{vmatrix} \cos\theta & 1 & & & \\ 1 & 2\cos\theta & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2\cos\theta & 1 \\ & & & 1 & 2\cos\theta \end{vmatrix}.$$

$$D' = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & & x_n & x_{n+1} \\ \vdots & \vdots & & & \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} & x_{n+1}^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} & x_{n+1}^{n-1} \\ x_1^{n} & x_2^{n} & \cdots & x_n^{n} & x_{n+1}^{n} \end{vmatrix}$$

2. Calculate the following determinant:

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^{n} & x_2^{n} & \cdots & x_n^{n} \end{vmatrix}.$$

   [**Note**: This is ***not*** “Vandermonde determinant”. But you can get some idea from that determinant.]

$$B = [\alpha_1, \alpha_2, \alpha_3] \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 1 & 9 \end{bmatrix}$$

3. （抽象型行列式）

   a) Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ be 3-dimensional column vectors which are linearly independent. If $\boldsymbol{A} = [\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3]$, $\boldsymbol{B} = [\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_1 + 3\boldsymbol{\alpha}_2 + 9\boldsymbol{\alpha}_3]$, and $|\boldsymbol{A}| = 1$. Find $|\boldsymbol{B}|$.

   b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices, and $|\boldsymbol{A}| = 3$, $|\boldsymbol{B}| = 2$, $|\boldsymbol{A}^{-1} + \boldsymbol{B}| = 2$. Find $|\boldsymbol{A} + \boldsymbol{B}^{-1}|$.

4. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices. Please prove that

$$|\boldsymbol{I}_n - \boldsymbol{AB}| = |\boldsymbol{I}_n - \boldsymbol{BA}|.$$

As a corollary, $\boldsymbol{I}_n - \boldsymbol{AB}$ is invertible if and only if $\boldsymbol{I}_n - \boldsymbol{BA}$ is invertible.

(**Hint**: start from the block matrix: $\begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{I}_n \end{bmatrix}$.)

5. Please show that: $rank(\boldsymbol{A}) = r$ if and only if the highest order of the nonzero minors of $\boldsymbol{A}$ is r. (This is an equivalent definition of ***the rank of a matrix***.)
即证明：$rank(\boldsymbol{A}) = r$的充要条件是$\boldsymbol{A}$的非零子式的最高阶数为r. (这也可以作为**矩阵的秩**的等价定义.)

注：定义　对于矩阵$\boldsymbol{A} = [a_{ij}]_{m\times n}$，取其任意$k$行（第$i_1, i_2, \cdots, i_k$行）和任意$k$列（第$j_1, j_2, \cdots, j_k$列），其中$k \le n$，将这些行与列交叉处的$k^2$个元素按原来相对位置构成的$k$阶行列式

$$\begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_k} \\ a_{i_2 j_1} & a_{i_2 j_2} & \cdots & a_{i_2 j_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k j_1} & a_{i_k j_2} & \cdots & a_{i_k j_k} \end{vmatrix}$$

称为$\boldsymbol{A}$的一个k阶子行列式，简称k阶子式(the minor of the k-th order). 当上述行列式等于零（不等于零）时，称为k阶零子式（非零子式）.

显然，如果矩阵$\boldsymbol{A}$存在r阶非零子式，而所有$r+1$阶子式（如果有$r+1$阶子式）都等于零，则矩阵$\boldsymbol{A}$的非零子式的最高阶数为r，因为由所有$r+1$阶子式都等于零可推出所有更高阶的子式都等于零.

假设 $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ 经过高斯消去得到 $\hat{A} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} & c_{1,r+1} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2r} & c_{2,r+1} & \cdots & c_{2n} \\ 0 & & \ddots & & & & \\ 0 & \cdots & 0 & c_{rr} & c_{r,r+1} & & c_{rn} \\ 0 & & & 0 & & & 0 \end{bmatrix}$

$\because rank(A)=r \Rightarrow \exists\ c_1, c_2, \cdots, c_r$ s.t. $c_1, c_2, \cdots, c_r$ 线性无关 且 $\forall c_i \notin \{c_1, c_2, \cdots, c_r\}$
$c_1, c_2, \cdots, c_r, c_i$ 线性相关　$\because$ 高斯消去不改变矩阵行列式的值
$\Rightarrow$ 存在 r 阶非零子式

存在 r 阶非零子式 $\Rightarrow rank(A)=r$. WLOG 假设 r 阶子式是 $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{vmatrix}$
$\therefore \alpha_1, \cdots, \alpha_r$ 线性无关 $\Rightarrow rank(A) \ge r$
$\because \forall\ r+1$ 阶子式都为0 $\Rightarrow$ 任意的 $r+1$ 个列向量都是线性相关的
$\Rightarrow rank(A) < r+1 \Rightarrow rank(A) = r$

※ 而且 $\forall (r+1)$ 阶子式都为0 因为 $\hat{A}(i,j)=0,\ i>r$.
$\Rightarrow$ 非零子式的最高阶为 r.