

Step-1

Similar matrices have the same eigenvalues. Consider the following 2 by 2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Its eigenvalues can be obtained by solving $\det(A - \lambda I) = 0$.

Consider

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \end{aligned}$$

Step-2

Since, any of a , b , c , and d can be either 0 or 1, there are only 3 choices (0 or 1 or 2) possible for $a + d$.

Suppose $a + d = 0$. Then $a = 0$ and $d = 0$. Now bc can be equal to 1 or 0. Thus, the following two equations are possible:

$$\begin{aligned} \lambda^2 &= 0 \\ \lambda^2 - 1 &= 0 \end{aligned}$$

Suppose $a + d = 1$. Then one of a and d is 1 and the other is 0. Now bc can be equal to 1 or 0. Thus, the following two equations are possible:

$$\begin{aligned} \lambda^2 - \lambda &= 0 \\ \lambda^2 - \lambda - 1 &= 0 \end{aligned}$$

Suppose $a + d = 2$. Then both a and d are equal to 1. Now bc can be equal to 1 or 0. Thus, the following two equations are possible:

$$\begin{aligned} \lambda^2 - 2\lambda &= 0 \\ \lambda^2 - 2\lambda + 1 &= 0 \end{aligned}$$

Step-3

Let us obtain the roots of the 6 quadratic equations.

For $\lambda^2 = 0$, the roots are 0 and 0. For $\lambda^2 - 1 = 0$, the roots are 1 and -1 . For $\lambda^2 - \lambda = 0$, the roots are 0 and 1. For $\lambda^2 - \lambda - 1 = 0$, the roots are $\frac{1 \pm \sqrt{5}}{2}$. For $\lambda^2 - 2\lambda = 0$, the roots are 0 and 2. Finally, for $\lambda^2 - 2\lambda + 1 = 0$, the roots are 1 and 1.

Each of these roots corresponds to some eigenvalue of a 2 by 2 matrix. Thus, 6 different eigenvalues are possible. These are: $1, 0, 1, 2, \frac{1 \pm \sqrt{5}}{2}$.

We have obtained 6 distinct quadratic equations and each of these have a unique set of 2 eigenvalues.

Thus, it is clear that there are 6 families of similar matrices.

Step-4

Let us enlist all possible matrices in each of the 6 families:

Let the characteristic equation be $\lambda^2 = 0$. Then both a and d should be zero and either one or both of b and c should be zero. Thus, we get the following three matrices in this family:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Let the characteristic equation be $\lambda^2 - 1 = 0$. Then both a and d should be zero and both b and c should be 1. Thus, we get the following (one and only one) matrix in this family:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let the characteristic equation be $\lambda^2 - \lambda = 0$. Then exactly one of a and d should be equal to one and at least one of b and c should be equal to zero. Thus, we get the following six matrices in this family:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Let the characteristic equation be $\lambda^2 - \lambda - 1 = 0$. Then exactly one of a and d should be equal to one and both b and c should be equal to one. Thus, we get the following two matrices in this family:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Step-5

Let the characteristic equation be $\lambda^2 - 2\lambda = 0$. Then both a and d should be equal to one and both b and c should be equal to one. Thus, we get the following one matrix in this family:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Let the characteristic equation be $\lambda^2 - 2\lambda + 1 = 0$. Then both a and d should be equal to one and at least one of b and c should be equal to zero. Thus, we get the following three matrices in this family:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step-6

Thus, we have categorised all the 16 matrices in the 6 families.