Linear Algebra-A

Assignments - Week 8

Supplementary Problem Set

1. Let $E = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $F = \{\boldsymbol{b}_1,\ \boldsymbol{b}_2\}$, where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

and $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \boldsymbol{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

For each of the following linear transformations $L: \mathbf{R}^3 \to \mathbf{R}^2$, find the matrix representing L with respect to the ordered bases E and F:

(a) $L(\boldsymbol{x}) = (x_3, x_1)^{\mathrm{T}}$;

(b) $L(\boldsymbol{x}) = (x_1 + x_2, x_1 - x_3)^{\mathrm{T}}$;

(c) $L(\boldsymbol{x}) = (2x_2, -x_1)^{\mathrm{T}}$.

2. Let $\mathbb{R}^{2\times 2}$ be the vector space of all 2×2 real matrices, and define

$$T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$$

by $T(\boldsymbol{A}) = \boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}$, where $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Show that T is a linear transformation.

(b) Find its matrix with respect to the basis $\left\{\boldsymbol{b}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{b}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{b}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \boldsymbol{b}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

3. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$ be linearly independent vectors in $\mathbf{R}^n$ $(n > m)$, and

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v}_1^T \\ \boldsymbol{v}_2^T \\ \vdots \\ \boldsymbol{v}_m^T \end{bmatrix}.$$

It follows that $\boldsymbol{A}$ is an $m \times n$ matrix with rank m.

Let $\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_{n-m}$ be a set of linearly independent vectors in $\mathbf{R}^n$ satisfying

$$\boldsymbol{A}\boldsymbol{w}_j = \boldsymbol{0}, \qquad j = 1,2,\cdots,n-m.$$

Show that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m, \boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_{n-m}$ are linearly independent.

(Note: This is to say, *the basis for the row space* $C(\boldsymbol{A}^{\mathrm{T}})$ *and the basis for the nullspace* $N(\boldsymbol{A})$ *together form a basis for* $\mathbf{R}^n$.)

4. Let $\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$.

   a) Explain why $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.
   b) Find the least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$.
   c) Split $\boldsymbol{b}$ into a column space component $\boldsymbol{b}_c$ and a left nullspace component $\boldsymbol{b}_l$, i.e., $\boldsymbol{b} = \boldsymbol{b}_c + \boldsymbol{b}_l$.

5. Let $\boldsymbol{A}$ be an $m \times n$ real matrix and $\boldsymbol{A}^T$ be its transpose. Show that the column spaces of $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}^T$ are the same, i.e., $C(\boldsymbol{A}^T\boldsymbol{A}) = C(\boldsymbol{A}^T)$.
   (Note: This is a way to prove that for the least square method, the normal equation $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$ is always solvable. Another proof is by the properties of "***Rank***".)