

## Step-1

Let  $A$  be  $A = \begin{bmatrix} 1 & 2 \\ 3 & 9 \\ 4 & 8 \end{bmatrix}$  and its rank is 2.

In order to find the null space basis, we need to set it as homogenous equation.

$$Ax = 0$$

Apply  $R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 4R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply  $R_2 \rightarrow \frac{R_2}{3}$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 = 0$$

$$x_2 = 0$$

$$\Rightarrow x_1 = -2x_2$$

$$= 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} \\ = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore,  $(1, 0, 0), (0, 1, 0)$  is in column space.

The basis for  $C(A)$  is  $\{(1, 0, 0), (0, 1, 0)\}$ .

The basis for null space is  $\{(0, 0, 1)\}$ .

## Step-2

The row space contains all multiples of  $(1, 2)$ . Therefore, the basis for row space is  $\{(1, 2)\}$ .

The left null space basis is,

The transpose of the matrix is  $A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 9 & 8 \end{bmatrix}$

The homogeneous equations using this,

$$A^T x = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 2 & 9 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply  $R_2 \rightarrow R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 3x_2 + 4x_3 = 0$$

$$3x_2 = 0$$

$$\Rightarrow x_2 = 0$$

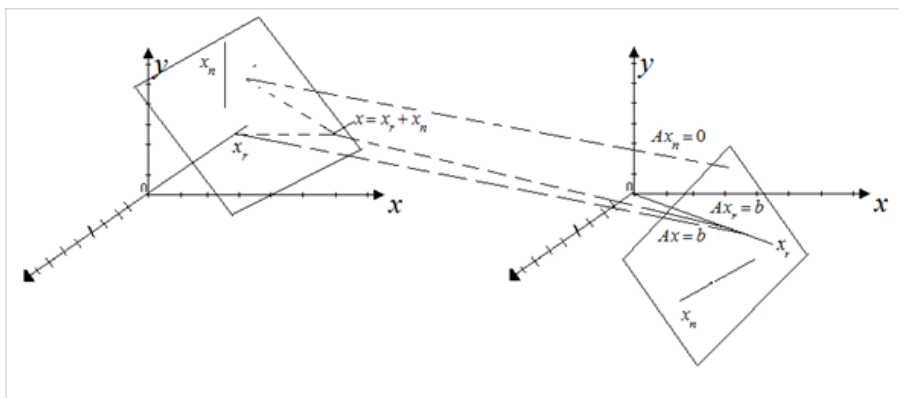
$$x_1 = -4x_3$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 0 \\ x_3 \end{bmatrix} \\ = x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

The basis for null space is  $\{(-4, 0, 1)\}$ .

### Step-3

The four subspaces of this matrix is as shown below.



## Step-4

We know that  $\dim(N(A)) + \dim(C(A)) = \text{number of columns}$

$$\Rightarrow \dim(N(A)) + \dim(C(A)) = 2$$

$$\Rightarrow \dim(N(A)) + 2 = 2$$

$$\Rightarrow \dim(N(A)) = 0$$

So, null space is  $2$  (zero vector only)

Therefore, the null space of any vector  $(x_1, x_2)$  in  $\mathbb{R}^2$  and  $x_n = 0$  for  $n = 1, 2$

Therefore, rank  $r = n$  means nullspace is zero vector and  $x_n = 0$ .