

Step-1

If Q_1, Q_2 are orthogonal, then $Q_1^T Q_1 = I$ and $Q_2^T Q_2 = I$ (1)

We consider $(Q_1 Q_2)^T (Q_1 Q_2)$

$$= (Q_2^T Q_1^T) Q_1 Q_2 \text{ By the properties of transposing matrices}$$

$$= Q_2^T (Q_1^T Q_1) Q_2 \text{ By the associativity of multiplication}$$

$$= Q_2^T (I Q_2) \text{ by (1)}$$

$$= Q_2^T Q_2$$

$$= I$$

Therefore $(Q_1 Q_2)^T (Q_1 Q_2) = I$

Hence $Q_1 Q_2$ is orthogonal

Step-2

Given that Q_1 is the rotation through θ

$$\text{So, } Q_1 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Similarly, Q_2 is a rotation matrix through ϕ

$$\text{So, } Q_2 = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

$$\text{Let us consider } Q_1 Q_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta \cos \phi - \sin \theta \sin \phi & -\cos \theta \sin \phi - \sin \theta \cos \phi \\ \sin \theta \cos \phi + \cos \theta \sin \phi & \cos \theta \cos \phi - \sin \theta \sin \phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

This is nothing but the rotation matrix through an angle α where $\alpha = \theta + \phi$

Therefore, $Q_1 Q_2$ is a rotation matrix through an angle $\theta + \phi$.