Step-1

Consider the following orthogonal matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find the circulant matrix C that satisfies the following relation:

$$C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3$$

Also, find the four components of product of vector x and circulant matrix C.

Step-2

Circulant matrix C can be calculated as follows:

$$C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3$$

$$= c_0 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + c_1 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_0 & 0 & 0 & 0 \\ 0 & c_0 & 0 & 0 \\ 0 & 0 & c_0 & 0 \\ 0 & 0 & 0 & c_0 \end{bmatrix} + \begin{bmatrix} 0 & c_1 & 0 & 0 \\ 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & c_1 \\ c_1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_2 \\ c_2 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & c_3 \\ c_3 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ 0 & 0 & c_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}$$

Therefore, circulant matrix is defined as follows:

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}$$

Step-3

Let $x = (x_0, x_1, x_2, x_3)$ be the vector. Product of circulant matrix and vector x will be as follows:

$$Cx = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} c_0 x_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 \\ c_3 x_0 + c_0 x_1 + c_1 x_2 + c_2 x_3 \\ c_2 x_0 + c_3 x_1 + c_0 x_2 + c_1 x_3 \\ c_1 x_0 + c_2 x_1 + c_3 x_2 + c_0 x_3 \end{bmatrix}$$

Step-4

Therefore, the four components of product of vector x and circulant matrix C is:

$$\boxed{\begin{aligned} & \left(c_0 x_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 \right), \left(c_3 x_0 + c_0 x_1 + c_1 x_2 + c_2 x_3 \right) \\ & \left(c_2 x_0 + c_3 x_1 + c_0 x_2 + c_1 x_3 \right), \left(c_1 x_0 + c_2 x_1 + c_3 x_2 + c_0 x_3 \right) \end{aligned}}$$