Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #2

2023/03/05

Name: ____

Student Number: __________________________________

1. Let $V = \{f(\sqrt{2}) : f(x) = a_0 + a_1 x + \cdots + a_n x^n, a_0, a_1, \cdots, a_n \in \mathbf{Q}, n \in \mathbf{Z}^+\}$, prove that V is a vector space over $\mathbf{Q}$, what's the dimension of V?

   Solution $\forall f(x) = a_0 + a_1 x + \cdots + a_n x^n \in P(\mathbf{Q}), \exists\, a, b \in \mathbf{Q}$, s.t.

$$f(\sqrt{2}) = a_0 + a_1\sqrt{2} + \cdots + a_n(\sqrt{2})^n = a + b\sqrt{2}. \tag{1}$$

So $V \subset \{a + b\sqrt{2} : a, b \in \mathbf{Q}\}$. On the other hand, it's obvious that $\{a + b\sqrt{2} : a, b \in \mathbf{Q}\} \subset V$. Therefore, we have

$$V = \{a + b\sqrt{2} : a, b \in \mathbf{Q}\}. \tag{2}$$

It's easy to check that $\{a + b\sqrt{2} : a, b \in \mathbf{Q}\}$ contains 0, and is closed under addition and scalar multiplication. So V is a vector space over $\mathbf{Q}$.

Let $\xi_1 = 1, \xi_2 = \sqrt{2}$. Both of them belong to V. $\forall \eta = a + b\sqrt{2} \in V$, we have

$$\eta = a\xi_1 + b\xi_2. \tag{3}$$

Hence, $V = \text{span}\ \{\xi_1, \xi_2\}$. Obviously, ξ_1, ξ_2 are linearly independent, which can imply ξ_1, ξ_2 is a basis of V. So $\dim V = 2$ □

2. Let $V = \{A \in \mathbf{R}^{n\times n} : A \text{ is symmetric }\}$. It's obvious that V is a vector space over $\mathbf{R}$ corresponding to matrix addition and scalar multiplication. Let $U = \{A \in V : A = (a_{ij})_{n\times n}, \sum_{i=1}^{n} a_{ii} = 0\}$, $W = \{\lambda I : \lambda \in \mathbf{R}\}$, where I is the identity matrix. It's easy to check U and W are subspaces of V.

   1. Find bases for U and W respectively, further compute the dimentsions of U and W.
   2. Try to prove $V = U \oplus W$.

Solution

1. It's obvious that I is a basis of W, so dim $W = 1$.

Let E_{ij} be the n by n matrix with 1 on the ith row and jth coloum, 0 on the other position. $\forall A \in U$, let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn}. \end{bmatrix} \tag{4}$$

Since $a_{nn} = -\sum_{i=1}^{n-1} a_{ii}$, we can get

$$A = \sum_{i\neq j} a_{ij}(E_{ij} + E_{ji}) + \sum_{i=1}^{n-1} a_{ii}(E_{ii} - E_{nn}). \tag{5}$$

So $U = \text{span}\,\{E_{ij} + E_{ji} : i, j = 1, 2, \cdots, n, i \neq j\} \cup \{E_{ii} - E_{nn} : i = 1, 2, \cdots, n\}$. And it's easy to check the matrices above are linearly independent. Therefore, $\{E_{ij} + E_{ji} : i, j = 1, 2, \cdots, n, i \neq j\} \cup \{E_{ii} - E_{nn} : i = 1, 2, \cdots, n\}$ is a basis of U and

$$\dim U = (1 + 2 + \cdots + (n-1)) + (n-1) = \frac{(n+2)(n-1)}{2}. \tag{6}$$

2. It's easy to check $U \cap W = \{0\}$. And dim $U+$ dim $W = \frac{(n+2)(n-1)}{2} + 2 = \frac{(n+1)n}{2} =$ dim V. So $V = U \oplus W$. $\square$