Step-1

For every c, we have to find R and the special solutions to $Ax=0$:

$$A=\begin{bmatrix}1&1&2&2\\2&2&4&4\\1&c&2&2\end{bmatrix}$$, and $$A=\begin{bmatrix}1-c&2\\0&2-c\end{bmatrix}$$

Step-2

Now $$A=\begin{bmatrix}1&1&2&2\\2&2&4&4\\1&c&2&2\end{bmatrix}$$

Let $$x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$ for $Ax=0$

Step-3

Case I: if $c\neq 1$,

$$\begin{bmatrix}1&1&2&2\\2&2&4&4\\1&c&2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\xrightarrow{\substack{R_2-2R_1,\\ R_3-R_1}}\begin{bmatrix}1&1&2&2\\0&0&0&0\\0&(c-1)&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\xrightarrow{R_{23}}\begin{bmatrix}1&1&2&2\\0&c-1&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Step-4

$$\xrightarrow{\frac{1}{c-1}R_2}\begin{bmatrix}1&1&2&2\\0&1&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\xrightarrow{R_1-R_2}\begin{bmatrix}1&0&2&2\\0&1&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Therefore $\boxed{R=\begin{bmatrix}1&0&2&2\\0&1&0&0\\0&0&0&0\end{bmatrix}}$

Step-5

Therefore x_3, x_4 are free variables, x_1, x_2 are pivot variables.

$$\Rightarrow x_1+2x_3+2x_4=0$$
$$x_2=0$$

$$\Rightarrow x_1=-2x_3-2x_4$$

$$x_2=0$$

Step-6

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-2x_3-2x_4\\0\\x_3\\x_4\end{bmatrix}$$
$$=x_3\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}$$

The special solutions are $\boxed{\begin{bmatrix}-2\\0\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}}$

Step-7

Case ii :if $c=1$

$$A=\begin{bmatrix}1&1&2&2\\2&2&4&4\\1&1&2&2\end{bmatrix}$$

$$\xrightarrow[R_3-R_1]{R_2-2R_1,}\boxed{\begin{bmatrix}1&1&2&2\\0&0&0&0\\0&0&0&0\end{bmatrix}}=R$$

Therefore x_2, x_3, x_4 are free variables.

x_1 is the pivot variable.

$$\begin{aligned}&Ax=0\\&\Rightarrow x_1+x_2+2x_3+2x_4=0\\&\Rightarrow x_1=-x_2-2x_3-2x_4\end{aligned}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-x_2-2x_3-2x_4\\x_2\\x_3\\x_4\end{bmatrix}$$

$$=x_2\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}+x_3\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}$$

Therefore the special solutions are $\boxed{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}}$

Step-8

Next we consider,

$$A=\begin{bmatrix}1-c&2\\0&2-c\end{bmatrix}$$

case (i): if $c=1$,then

$$A=\begin{bmatrix}0&2\\0&1\end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1,\,R_2-\frac{1}{2}R_1}\boxed{\begin{bmatrix}0&1\\0&0\end{bmatrix}}=R$$

Let $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ be the solution for the system $Ax=0$

Solution x_1 is free variable.

x_2 is pivot variable.

Step-9

Here $x_2=0$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\0\end{bmatrix}$$
$$=x_1\begin{bmatrix}1\\0\end{bmatrix}$$

Therefore the special solution is $\begin{bmatrix}1\\0\end{bmatrix}$

Step-10

Case (ii): if $c=2$,

$$A=\begin{bmatrix}-1&2\\0&0\end{bmatrix}$$
$$\boxed{\begin{bmatrix}1&-2\\0&0\end{bmatrix}}=R$$

$$\Rightarrow -x_1+2x_2=0$$
$$\Rightarrow x_1=2x_2$$

Step-11

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix}$$
$$= x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence the special solution is $\boxed{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

Step-12

Case (iii): if $c \neq 1, c \neq 2$

Then

$$A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$
$$\xrightarrow{\frac{1}{2-c} R_2} \begin{bmatrix} 1-c & 2 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \boxed{\begin{bmatrix} 1-c & 0 \\ 0 & 1 \end{bmatrix}} = R$$

Step-13

Now

$$\begin{bmatrix} 1-c & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (1-c)x_1 = 0$$
$$x_2 = 0$$
$$\Rightarrow x_1 = 0$$
$$x_2 = 0$$

Therefore the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Hence the special solution is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$