Step-1

Let Q be an orthogonal matrix.

We have to show $\|Q\|=1, c(Q)=1$ where Q is the orthogonal matrix.

Step-2

We know that

Def 1: conditional number c of a matrix A is

$c(A) = \|A\|\|A^{-1}\|$ where A is square matrix

$=\frac{\lambda_{\max}}{\lambda_{\min}}$ where A is a positive definite matrix.

Def 2: $\|A\|^2 = \max_{x\neq 0} \frac{\|Ax\|^2}{\|x\|^2}$

$= \max_{x\neq 0} \frac{x^T A^T Ax}{x^T x}$

Step-3

We know that if Q is an orthogonal matrix, then $Q^T = Q^{-1}$ â€¦â€¦ (1)

In view of the above definition 2, we get $\|Q\|^2 = \max_{x\neq 0} \frac{x^T Q^T Qx}{x^T x}$

Using (1) in this, we get $\|Q\|^2 = \max_{x\neq 0} \frac{x^T \left(Q^{-1}Q\right)x}{x^T x}$

$= \max_{x\neq 0} \frac{x^T Ix}{x^T x}$ $\left(\text{Since } Q^{-1}Q = QQ^{-1} = I\right)$

$= \max_{x\neq 0} \frac{x^T x}{x^T x}$

$=1$ Since x is a non zero vector.

Since norm is a non negative quantity, by applying the square root on both sides, we get

$\|Q\| = 1$ â€¦â€¦ (2)

Therefore, $\boxed{\|Q\| = 1}$

Step-4

By definition 1, we have $c(Q) = \|Q\|\|Q^{-1}\|$

By (2), we have $\|Q\| = 1$ and consequently, we get $\|Q^{-1}\| = 1$

Using these in the above equation, we get $c(Q) = 1 \times 1$

= 1.

Step-5

Suppose α is any scalar, then by the above result, we can write

$$c(\alpha Q) = \|\alpha Q\|\|(\alpha Q)^{-1}\|$$

$$= |\alpha|\|Q\|\frac{1}{|\alpha|}\|Q^{-1}\|$$

$$= |\alpha|\frac{1}{|\alpha|}\|Q\|\|Q^{-1}\|$$

$$= 1$$

(When the condition number of a matrix and its scalar multiples is 1, then that matrix is perfectly conditioned matrix. More precisely, the orthogonal matrices are perfectly conditioned.)

Hence $\boxed{C(Q) = 1}$