Step-1

Given vectors are $b=\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $a=\begin{bmatrix}1\\0\end{bmatrix}$

The projection of *b* onto a is $\hat{x}a=\frac{a^Tb}{a^Ta}a$ â€¦â€¦ (1)

$$a^Tb=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$
$$=\cos\theta$$

$$a^Ta=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$
$$=1+0$$
$$=1$$

So,

$$\hat{x}=\frac{\cos\theta}{1}$$
$=\cos\theta$ â€¦â€¦ (2)

Step-2

Use (2) in (1), to get $p_1=\hat{x}\,a$

$$=\cos\theta\begin{bmatrix}1\\0\end{bmatrix}$$
$$=\begin{bmatrix}\cos\theta\\0\end{bmatrix}$$

The projection matrix suitable is

$$P_1 = \frac{aa^T}{a^T a}$$

$$= \frac{\begin{bmatrix}1\\0\end{bmatrix}[1\ 0]}{[1\ 0]\begin{bmatrix}1\\0\end{bmatrix}}$$

$$= \frac{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}}{1}$$

$$= \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$ â€¦â€¦ (3)

Step-3

Given vectors are $b = \begin{bmatrix}1\\1\end{bmatrix}$ and $a = \begin{bmatrix}1\\-1\end{bmatrix}$

The projection of b on to a is $\hat{x}a = \frac{a^T b}{a^T a} a$

$$a^T b = \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= 1-1$$

$$= 0$$

And $a^T a = \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 1+1 = 2$

So,

$$\hat{x} = \frac{0}{2}$$

$$= 0$$

Therefore $p_2 = \hat{x}a$

$$= 0\begin{bmatrix}1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\end{bmatrix}$$

Step-4

The matrix suitable to this projection is $P_1 = \dfrac{aa^T}{a^T a}$

$$= \frac{\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix}}{\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}}$$

$$= \frac{\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}}{2}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$ â€¦â€¦ (4)

Step-5

Use (3) and (4), and get $P_1 + P_2 = \dfrac{1}{2}\begin{bmatrix}3 & -1\\-1 & 1\end{bmatrix}$

Step-6

$$P_1 P_2 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$$

$$(P_1 + P_2)^2 = \frac{1}{2}\begin{bmatrix}5 & -2\\-2 & 1\end{bmatrix}$$

$$P_1^2 + P_2^2 = \frac{1}{2}\begin{bmatrix}5 & -1\\-1 & 1\end{bmatrix}$$

From the above observations, it follows that $\boxed{(P_1 + P_2)^2 \neq P_1^2 + P_2^2}$ while $P_1 P_2 \neq 0$