Step-1

Consider the following matrices:

$A=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$, and $B=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$

To find null space, row space, column space

Step-2

First, consider the matrix

$$A=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$$

By definition of null space, $Ax=0$

$$\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Apply $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Step-3

Therefore,

$x_1 + 2x_2 = 0$ If $x_2 = a$, then $x_1 = -2a$

Hence

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-2a\\ a\end{bmatrix}$$

$$=a\begin{bmatrix}-2\\ 1\end{bmatrix}$$

$x=(-2,1)$ Therefore the null space spanned by the vector $(-2,1)$

Step-4

The row space spanned by any row of A

Therefore row space spanned by the vector $(1,2)$

Step-5

The column space spanned by any column of A

Therefore column space spanned by the vector $(1,3)$

Step-6

By definition of left null space $yA = 0$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{aligned} y_1 + 3y_2 &= 0 \\ y_2 &= a \\ y_1 &= -3a \end{aligned}$$

Step-7

Therefore,

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} -3a \\ a \end{bmatrix} \\ &= a\begin{bmatrix} -3 \\ 1 \end{bmatrix} \\ y &= (-3,1) \end{aligned}$$

And $\begin{bmatrix} 1 & 3 \end{bmatrix}^T \begin{bmatrix} 3 & -1 \end{bmatrix} = 0$

Therefore $N\left(A^T\right)$ is the perpendicular line through $(-3,1)$

Step-8

Consider the second matrix

$$B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

By definition of null space, $Bx = 0$

$$\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Step-9

Therefore,

$x_1 = 0$

Above system not depends on x_2,Â therefore x_2 is any arbitrary.

Let $x_2 = a$

Hence,

Step-10

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

$$= a\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$x = (0,1)$ The null space spanned by the vector $(0,1)$

Step-11

The row space spanned by any row of A

Therefore row space spanned by the vector $(1,0)$

Step-12

The column space spanned by any column of A

Therefore column space spanned by the vector $(1,3)$

Step-13

By definition of left null space $yA = 0$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$y_1 + 3y_2 = 0$$

$$\text{Put } y_2 = a$$

$$y_1 = -3a$$

Step-14

Therefore,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -3a \\ a \end{bmatrix}$$

$$= a \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$y = (-3,1)$$

Therefore $N(A^T)$ is the perpendicular line through $(-3,1)$