Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Step-2

Find the Eigen values and Eigen vectors and write the solution in the form of $S\Lambda S^{-1}$. Then find e^{At} from $Se^{\Lambda t}S^{-1}$.

Step-3

First step is to find the Eigen values and Eigen vectors of matrix A. To calculate the Eigen values do the following calculations;

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{bmatrix} \\ \det\left(A - \lambda I\right) &= 0 \\ (1-\lambda)(-\lambda) &= 0 \\ \lambda^2 - \lambda &= 0 \end{aligned}$$

After solving following values are obtained:

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 0 \end{aligned}$$

Step-4

To calculate Eigen vectors do the following calculations:

$$\begin{aligned} \left(A - \lambda I\right)x &= 0 \\ \begin{bmatrix} 1-1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

On solving, values of y and z corresponding to $\lambda = 1$ are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 0$ is as follows:

$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 1-0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore Eigen values are:

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-6

Recall that: $A = S\Lambda S^{-1}$.Therefore,

$$A = \boxed{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}$$

Step-7

Recall that $e^{At} = Se^{\Lambda t}S^{-1}$. Therefore,

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} e^t & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Step-8

Therefore,

$$
e^{At} = \boxed{\begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}}.
$$