Step-1

The objective is to prove that $\operatorname{rank}(AB) \le \operatorname{rank}(A)$ and $\operatorname{rank}(AB) \le \operatorname{rank}(B)$.

Step-2

Consider matrix A and B, each column of matrix AB be the combination of the column of matrix A.

Let A be $m \times n$ matrix, B matrix be $n \times p$ then, the combination of matrix AB be $m \times p$ matrix.

The j^{th} column of matrix AB is:

$$\begin{bmatrix} (AB)_{1j} \\ (AB)_{2j} \\ \vdots \\ (AB)_{mj} \end{bmatrix}$$

Since, $(AB)_{1j} = \sum_{k=1}^{n} A_{1k} \cdot B_{kj}$

Now, the j^{th} column of matrix AB is:

$$\begin{bmatrix} \sum_{k=1}^{n} A_{1k} \cdot B_{kj} \\ \sum_{k=1}^{n} A_{2k} \cdot B_{kj} \\ \vdots \\ \sum_{k=1}^{n} A_{mk} \cdot B_{kj} \end{bmatrix} = \sum_{k=1}^{n} \begin{bmatrix} A_{1k} B_{kj} \\ A_{2k} B_{kj} \\ \vdots \\ A_{mk} B_{kj} \end{bmatrix}$$

$$= \sum_{k=1}^{n} \begin{bmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{mk} \end{bmatrix} \begin{bmatrix} B_{kj} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{bmatrix} \begin{bmatrix} B_{1j} \end{bmatrix} + \begin{bmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{bmatrix} \begin{bmatrix} B_{2j} \end{bmatrix} + \cdots + \begin{bmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{bmatrix} \begin{bmatrix} B_{nj} \end{bmatrix}$$

Therefore, this is the linear combination of matrix A.

Step-3

Now, each column of matrix AB be the combination of the column of matrix A.

Any column of matrix (AB) is:

$$\begin{aligned}(AB) &\subseteq \text{span}\{\text{columns of } A\}\\ \text{span}\{\text{columns of } (AB)\} &\subseteq \text{span}\{\text{columns of } A\}\\ \dim\{\text{span}\{\text{columns of } (AB)\}\} &\le \dim\{\text{span}\{\text{columns of } A\}\}\end{aligned}$$

Thus, $\text{rank}(AB) \le \text{rank}(A)$.

Hence, **Proved.**

Step-4

Now, prove that $\text{rank}(AB) \le \text{rank}(B)$. Let each column of matrix AB be the combination of the column of matrix B.

The i^{th} row of matrix AB is:

$$\begin{bmatrix}(AB)_{i1} & (AB)_{i2} & \cdots & (AB)_{ip}\end{bmatrix}$$

Since,

$$\begin{bmatrix}\sum_{k=1}^{n}(A_{ik}B_{k1}) & \sum_{k=1}^{n}(A_{ik}B_{k2}) & \cdots & \sum_{k=1}^{n}(A_{ik}B_{k3})\end{bmatrix} = \sum_{k=1}^{n} A_{ik}\begin{bmatrix}B_{k1} & B_{k2} & \cdots & B_{kp}\end{bmatrix}$$

Thus, the i^{th} row of matrix (AB) be a linear combination of rows of matrix B.

Each row of matrix (AB) is:

$$\begin{aligned}(AB) &\subseteq \text{span}\{\text{rows of } B\}\\ \text{span}\{\text{rows of } (AB)\} &\subseteq \text{span}\{\text{rows of } B\}\\ \dim\{\text{span}\{\text{rows of } (AB)\}\} &\le \dim\{\text{span}\{\text{rows of } B\}\}\end{aligned}$$

Thus, $\text{rank}(AB) \le \text{rank}(B)$.

Hence, **Proved.**