Step-1

Now suppose only one vector $(1,1,-1)$ is given. We find two more vectors, say *a* and *b*, so that *a*, *b*, and $(1,1-1)$ will be linearly independent vectors.

Let us consider the following:

$$a = (1,1,0)$$
$$b = (1,0,0)$$

It is obvious that $(1,1,-1)$, $(1,1,0)$, and $(1,0,0)$ are independent vectors.

Step-2

Thus, we have $a = (1,1,0)$, $b = (1,0,0)$, and $c = (1,1-1)$.

Therefore,

$$\begin{aligned} q_1 &= \frac{a}{\|a\|} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \\ &= \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{bmatrix}\end{aligned}$$

Step-3

Thus, we get

$$B = b - \left(q_1^{\mathrm{T}} b\right) q_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

Step-4

Thus,

$$q_2 = \sqrt{2} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Finally, we get

$$C = c - \left(q_1^{\mathrm{T}} c\right) q_1 - \left(q_2^{\mathrm{T}} c\right) q_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} - \left(\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Step-5

Note that $\|C\| = 1$.

Thus,

$$q_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Step-6

Therefore, an orthonormal basis of $\mathbf{R}^3$ is: $\boxed{\left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right), (0, 0, -1) \right\}}$.