

Step-1

Given that, P is the projection matrix onto the line through a .

a) To find that, why is the inner product of x with Py equal to the inner product of Px with y .

The inner product of x with Py :

$$\begin{aligned} x^T (Py) &= (x^T P) y \\ &= (x^T P^T) y \quad \text{Since the projection } P \text{ is symmetric, } P^T = P \\ &= (Px)^T y \end{aligned}$$

= inner product of Px with y .

Suppose $x^T (Py) = (Px)^T y \quad \hat{a} \in \hat{a} \in (1)$

That is, the inner product of x with Py equal to the inner product of Px with y .

Step-2

b) From the data, $a = (1, 1, -1)$, $x = (2, 0, 1)$, $y = (2, 1, 2)$,

If θ be the angle between x and Py , then $\cos \theta = \frac{x^T Py}{\|x\| \|Py\|} \Rightarrow \theta = \cos^{-1} \left(\frac{x^T Py}{\|x\| \|Py\|} \right)$.

Here, the projection matrix is $P = \frac{aa^T}{a^T a}$.

Now,

$$\begin{aligned} aa^T &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{aligned}$$

And also,

$$\begin{aligned}
 a^T a &= \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\
 &= 1+1+1 \\
 &= 3
 \end{aligned}$$

So, the projection matrix onto the line through a is,

$$\begin{aligned}
 P &= \frac{aa^T}{a^T a} \\
 &= \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}
 \end{aligned}$$

Step-3

The product of the matrices, P, y is,

$$\begin{aligned}
 Py &= \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\
 &= \frac{1}{3} \begin{bmatrix} 2+1-2 \\ 2+1-2 \\ -2-1+2 \end{bmatrix} \\
 &= \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}
 \end{aligned}$$

And also, the product of the matrices, x^T, Py is,

$$\begin{aligned}
 x^T (Py) &= \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\
 &= \frac{1}{3} \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\
 &= \frac{1}{3} (2+0-1) \\
 &= \frac{1}{3}
 \end{aligned}$$

Step-4

Now, find norm of the vectors as follows.

$$\begin{aligned}\|x\| &= \sqrt{4+0+1} \\ &= \sqrt{5}\end{aligned}$$

$$\begin{aligned}\|Py\| &= \frac{1}{3}\sqrt{1+1+1} \\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

Now, the angle is,

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{x^T Py}{\|x\|\|Py\|}\right) \\ &= \cos^{-1}\left(\frac{\left(\frac{1}{3}\right)}{\left(\sqrt{5}\right)\left(\frac{\sqrt{3}}{3}\right)}\right) \\ &= \cos^{-1}\left(\frac{1}{\sqrt{15}}\right)\end{aligned}$$

So, the angle is $\theta = \cos^{-1}\left(\frac{1}{\sqrt{15}}\right)$.

Step-5

If ϕ be the angle between Px with y , then $\cos \phi = \frac{(Px)^T y}{\|Px\|\|y\|} \Rightarrow \phi = \cos^{-1}\left(\frac{(Px)^T y}{\|Px\|\|y\|}\right)$.

Find the product of the matrices P, x .

$$\begin{aligned}
 Px &= \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \\
 &= \frac{1}{3} \begin{bmatrix} 2+0-1 \\ 2+0-1 \\ -2-0+1 \end{bmatrix} \\
 &= \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}
 \end{aligned}$$

And also,

$$\begin{aligned}
 (Px)^T y &= \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\
 &= \frac{1}{3} (2+1-2) \\
 &= \frac{1}{3}
 \end{aligned}$$

Step-6

Now, find norm of the vectors as follows.

$$\begin{aligned}
 \|y\| &= \sqrt{4+1+4} \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \|Px\| &= \frac{1}{3} \sqrt{1+1+1} \\
 &= \frac{\sqrt{3}}{3}
 \end{aligned}$$

Step-7

Now, the angle is,

$$\begin{aligned}
\phi &= \cos^{-1} \left(\frac{(Px)^T y}{\|Px\| \|y\|} \right) \\
&= \cos^{-1} \left(\frac{\left(\frac{1}{3} \right)}{\left(\frac{\sqrt{3}}{3} \right) (3)} \right) \\
&= \cos^{-1} \left(\frac{1}{3\sqrt{3}} \right)
\end{aligned}$$

So, the angle is $\phi = \cos^{-1} \left(\frac{1}{3\sqrt{3}} \right)$.

Hence, the two angles are $\theta = \frac{1}{\sqrt{15}}$, $\phi = \frac{1}{3\sqrt{3}}$

That means the two angles are not same.

Step-8

c) Verify that why the inner product of Px with Py is same the inner product of x with Py and the inner product of Px with y , and also find that what is the angle between those two.

Step-9

Inner product of Px with Py is,

$$\begin{aligned}
(Px)^T Py &= (x^T P^T) Py \\
&= x^T PPy \quad \text{Since } P^T = P \\
&= x^T P^2 y \\
&= x^T (Py) \quad \text{Since } P^2 = P
\end{aligned}$$

Therefore from (1) and (4), $(Px)^T Py = (Px)^T y = x^T (Py)$

Hence the inner product of Px with Py is same the inner product of x with Py and the inner product of Px with y .

Step-10

Let θ be the angle between Px with Py

Step-11

$$\cos \theta = \frac{(P_x)^T P_y}{\|P_x\| \|P_y\|}$$

Then

Here

$$P_x = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } P_y = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} (P_x)^T P_y &= \frac{1}{3} [1 \quad 1 \quad -1] \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{9} (1+1+1) \\ &= \frac{1}{3} \end{aligned}$$

Step-12

From part (b), $\|P_x\| = \frac{\sqrt{3}}{3}$ and $\|P_y\| = \frac{\sqrt{3}}{3}$.

Now,

$$\begin{aligned} \cos \theta &= \frac{(P_x)^T P_y}{\|P_x\| \cdot \|P_y\|} \\ &= \frac{\left(\frac{1}{3}\right)}{\left(\frac{\sqrt{3}}{3}\right) \left(\frac{\sqrt{3}}{3}\right)} \\ &= \frac{\left(\frac{1}{3}\right)}{\left(\frac{3}{9}\right)} \\ &= 1 \end{aligned}$$

So, $\cos \theta = 1 \Rightarrow \theta = 0$

Hence, the angle between P_x, P_y is $\boxed{\theta = 0}$.