

Step-1

Given that 3 by 3 matrix A is invertible.

The matrix A is invertible, which means that the matrix A is non-singular. So, the rank of A is 3.

The dimension of row space and column space is,

$$\dim(C(A)^T) = 3 \text{ and } \dim(C(A)) = 3$$

Row basis is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and its dimension is 3.

Column basis is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and its dimension is 3.

Step-2

Null space basis is,

Let's write in $AX = 0$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

Null basis is empty and its dimension is 0.

$$\text{i.e. } \dim(C(B)) + \dim(N(B)) = 3$$

$$\Rightarrow \dim(N(B)) = 3 - 3$$

$$\Rightarrow \dim(N(B)) = 0$$

Step-3

In order to find the left null space basis, we need to find the transpose of the matrix is,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transpose of the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Let's write in $A^T X = 0$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

Left null basis is empty and its dimension is 0.

$$\text{i.e. } \dim(C(B^T)) + \dim(N(B^T)) = 3$$

$$\Rightarrow \dim(N(B^T)) = 3 - 3$$

$$\Rightarrow \dim(N(B^T)) = 0$$

Therefore, row space basis and column space is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and its dimension is 3. Null space and left null space dimension is 0.

Step-4

The basis for 3 by 6 matrix is,

$$B = \begin{bmatrix} A \\ A \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Row space basis is $\{(1, 0, 0, 1, 0, 0), (0, 1, 0, 0, 1, 0), (0, 0, 1, 0, 0, 1)\}$ and its dimension is 3.

Column space basis is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and its dimension is 3.

Step-5

Null space basis is,

Let's write in $AX = 0$,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = 0$$

$$\Rightarrow x_1 + x_4 = 0$$

$$x_2 + x_5 = 0$$

$$x_3 + x_6 = 0$$

$$\Rightarrow x_1 = -x_4$$

$$x_2 = -x_5$$

$$x_3 = -x_6$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -x_4 \\ -x_5 \\ -x_6 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$= x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The null space basis is $\{(-1, 0, 0, 1, 0, 0), (0, -1, 0, 0, 1, 0), (0, 0, -1, 0, 0, 1)\}$ and its dimension 3.

$$\text{i.e. } \dim(C(B)) + \dim(N(B)) = 6$$

$$\Rightarrow \dim(N(B)) = 6 - 3$$

$$\Rightarrow \dim(N(B)) = 3$$

Step-6

In order to find the left null space basis, we need to find the transpose of the matrix is,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transpose of the matrix is

Let $\hat{x} \in \mathbb{R}^3$ write in $A^T X = 0$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

The left null space basis is empty and its dimension 0.

$$\text{i.e. } \dim(C(B^T)) + \dim(N(B^T)) = 6$$

$$\Rightarrow \dim(N(B^T)) = 6 - 3$$

$$\Rightarrow \dim(N(B^T)) = 3$$

Therefore, row space basis is $\{(1, 0, 0, 1, 0, 0), (0, 1, 0, 0, 1, 0), (0, 0, 1, 0, 0, 1)\}$ and its dimension is 3, column space basis is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and its dimension is 3 and null space basis is $\{(-1, 0, 0, 1, 0, 0), (0, -1, 0, 0, 1, 0), (0, 0, -1, 0, 0, 1)\}$ and its dimension is 3 and left null space basis is empty and its dimension is 3