

Step-1

Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Step-2

Show that $e^A e^B$ is different from $e^B e^A$ and they both are different from e^{A+B} .

Step-3

First step is to find the Eigen values and Eigen vectors of matrix A . To calculate the Eigen values do the following calculations;

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$$
$$\det(A - \lambda I) = 0$$
$$(1-\lambda)(-\lambda) = 0$$
$$\lambda^2 - \lambda = 0$$

After solving following values are obtained:

$$\lambda_1 = 1$$
$$\lambda_2 = 0$$

Step-4

To calculate Eigen vectors do the following calculations:

$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 1-1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, values of y and z corresponding to $\lambda = 1$ are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step-5

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 0$ is as follows:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-6

Therefore Eigen values are:

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-7

Recall that $e^{At} = Se^{At}S^{-1}$. Therefore,

$$\begin{aligned}
e^{At} &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} e^t & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Step-8

For $t = 1$:

$$e^A = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$$

Consider the following matrix

$$\begin{aligned}
B &= \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \\
B \cdot B &= \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Step-9

Substitute B in the expansion of e^{Bt} .

$$\begin{aligned}
e^{Bt} &= I + Bt \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Step-10

For $t = 1$:

$$e^B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Step-11

Now calculate the following:

$$\begin{aligned} e^A e^B &= \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e & -1 \\ 0 & 1 \end{bmatrix} \\ e^B e^A &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e & e-2 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Therefore, $\boxed{e^A e^B \neq e^B e^A}$.

Step-12

Consider the following matrix:

$$\begin{aligned} A+B &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ (A+B) \cdot (A+B) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Thus, $(A+B)^2 = (A+B)$.

Step-13

Substitute $(A+B)$ in the expansion of $e^{(A+B)t}$.

$$\begin{aligned}
e^{(A+B)t} &= I + (A+B)t + \frac{(A+B)^2 t^2}{2!} + \dots \\
&= I + (A+B)t + \frac{(A+B)t^2}{2!} + \dots \\
&= I + (A+B) \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) \\
&= I + (A+B)(e' - 1) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e' - 1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 + (e' - 1) & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} e' & 0 \\ 0 & 1 \end{bmatrix} \\
e^{(A+B)} &= \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Step-14

Therefore, $\boxed{e^A e^B \neq e^B e^A}$ and $e^A e^B, e^B e^A$ are different from $\boxed{e^{A+B}}$.