Southern University of Science and Technology

Linear Algebra I Final Examination
Fall 2017 A

Department: Math Class:________

Student ID:____________________Name:____________

Answer all parts of Questions (1)-(10). Total is 100 points.

(1) (10 points, 2 points each) True or false. No need to justify.

(a) Let A be an $n \times n$ matrix ($n > 1$), then $\det(kA) = k\det(A)$. (X)

(b) For any real matrix A and $\delta > 0$, then matrix $\delta I + A^T A$ is positive definite. (√)

(c) Let A be a 2×2 matrix whose eigenvalues are 2 and 3, then the matrix $A^2 - 3A + 6I$ is singular. $\det \neq 0$ $\det(A)\det(A-I)=0$ (X)

(d) Let A be an $n \times n$ matrix satisfying $A^2 = A$ and $A \neq I$, then $\det(A) = 0$. (√)

(e) Let A be a real square matrix, then A and A^T have the same eigenvectors. (X)

(2) (12 points, 3 points each) Fill in the blanks.

(a) Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$. $\boldsymbol{C_{ij} = (-1)^{i+j}\det M_{ij}}$, **Delete row i, column j.** According to the formula **Cofactors along row i**, $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$, then $C_{11} + C_{12} + C_{13} =$ <u>-16</u>.

$A^T_{2\times3}$ $\operatorname{rank}(A)=2$

(b) Let A be a 3×2 real matrix whose column vectors a_1, a_2 are <u>linearly independent.</u> The eigenvalues of $\underset{3\times2}{P = A}(\underset{2\times3}{A^T}\underset{3\times2}{A})^{-1}\underset{2\times3}{A^T}$ (A^TA: 2×2) are <u>0, 1, 1</u>. $\boxed{3\times3}$

0 geometric multiplicity $n-r$
1. geometric multiplicity r

(c) Let A be a 3×3 matrix and its eigenvalues are $-1, 2, 3$, then $\det(A^3 - 2A^2 + A + 2I) =$ <u>$-2, 4, 14$</u>

~~(d)~~ Which of following four assertions are true? They are <u>1, 3</u>.

1. If Q_1 and Q_2 are orthogonal matrices, then Q_1Q_2 is orthogonal.
2. If H_1 and H_2 are positive definite, then H_1H_2 is positive definite. → 反例
3. If A and B are similar, then they have the same eigenvalues.
4. If A and B have same eigenvalues, then A and B are similar.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

$B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} -1 & 3 \\ -3 & 8 \end{bmatrix}$ X

(3) (10 points) Given

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(i) Find the determinant of A. 0

(ii) Decide whether A is positive definite, negative definite, $\boxed{\text{semidefinite,}}$ or indefinite.

(iii) Find all the eigenvalues of A and their associated eigenvectors. $\lambda = 0, 3, 3$

(iv) Is A diagonalizable? If so, diagonalize it. Otherwise, explain why. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(4) (10 points) Consider

$$\frac{du}{dt} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} u = Au.$$

(i) Find e^{At}.

(ii) If $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, solve for $u(t)$.

(5) (10 points) Let

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

(i) Find AA^T and A^TA.

(ii) Find all the singular values of A.

(iii) Find all the eigenvectors of A^TA.

(iv) Find the singular value decomposition of A, in other words, find orthogonal matrices U and V, such that $A = U\Sigma V^T$.

(v) Find the pseudoinverse of A, namely, $A^+ = V\Sigma^+U^T$.

(6) (10 points) Consider

$$A = \begin{bmatrix} 1 & t \\ t & 4 \end{bmatrix}.$$

$4-t^2>0$

(i) For which numbers t is matrix A positive definite? $-2<t<2$

(ii) Factor $A = LDL^T$ when t is in the range for positive definiteness. $A=\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4-t^2 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

(iii) Find the minimum value of $P(x) = \frac{1}{2}(x_1^2 + 2tx_1x_2 + 4x_2^2) - x_1 - x_2$ for t in the range found in (ii). $\triangleq P(x_1, x_2)$

用高数法

$\frac{\partial P(x)}{x_1} = \frac{\partial P(x)}{x_2} = 0$

(iv) What is the minimum if $t = 2$?

(7) (10 points) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}.$$

Prove that $\det(A) = 0$ in the following ways:

(i) Show that the columns of A are dependent. First 3 dependent

(ii) Explain why all 120 terms are zero in the "big formula" for $\det(A)$.

(8) (10 points) Consider a complex matrix $C = A + iB$ with A and B real and

$$D = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

(i) If C is a Hermitian matrix, show that D is symmetric.

(ii) If C is a unitary matrix, show that D is orthogonal.

(9) (10 points) Prove the following statements: $Ax_1=\lambda_1 x_1 \quad Ax_2=\lambda_2 x_2$

(i) If ~~eigenvectors~~ for any $x_1, x_2, \cdots, x_k$ of matrix A correspond to different eigenvalues

$$\lambda_1, \lambda_2, \cdots, \lambda_k,$$

then those eigenvectors are linearly independent.

$a_1x_1 + a_2x_2 = 0$

$a_1\lambda_1x_1 + a_2\lambda_1x_2 = 0$

$a_1Ax_1 + a_2Ax_2 = 0$

$a_1\lambda_1x_1 + a_2\lambda_2x_2 = 0$

$\Rightarrow a_2 = 0$

$\Uparrow$

$a_1 = 0$ ✓

(ii) Two eigenvectors of a real symmetric matrix B, if they come from different eigenvalues, are orthogonal to one another. $Bx_1 = \lambda_1 x_1 \quad Bx_2 = \lambda_2 x_2$

$\lambda_1 x_1^T x_2 = (\lambda_1 x_1)^T x_2 = (Bx_1)^T x_2 = x_1^T B^T x_2$

$= x_1^T B x_2 = x_1^T(\lambda_2 x_2) = \lambda_2 x_1^T x_2$

$\Rightarrow x_1^T x_2 = 0$