Step-1

In Hilbert space, to find the length of the vector $v=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{4}},\frac{1}{\sqrt{8}},\ldots\right)$ and the length of the function $f(x)=e^{x}$ (over the interval $0\le x\le 1$), and

Also to find the inner product of;

$f(x)=e^{x}$
$g(x)=e^{-x}$

Over this interval

Step-2

In Hilbert space, if $v=(v_1,v_2,v_3,\ldots)$ then

$\|v\|^2=v_1^2+v_2^2+v_3^2+\ldots$

Given, $v=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{4}},\frac{1}{\sqrt{8}},\ldots\right)$

$\|v\|^2=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

$=\frac{a}{1-r}(\text{Sum of infinite G.P})$

$=\frac{1/2}{1-\frac{1}{2}}$

$=\frac{1/2}{1/2}$

$=1$

Length of the vector $v=\|v\|$

$=\sqrt{1}$

$=\boxed{1}$

Step-3

Now,

$$\|f\|^2 = \int_0^1 \left[f(x)\right]^2 dx$$

$$= \int_0^1 \left(e^x\right)^2 dx$$

$$= \int_0^1 e^{2x} dx$$

$$= \left[\frac{e^{2x}}{2}\right]_0^1$$

$$= \frac{e^2}{2} - \frac{e^0}{2}$$

$$= \frac{e^2 - 1}{2}$$

Step-4

Therefore the length of the function $f(x)$;

$$= \|f(x)\|$$

$$= \boxed{\sqrt{\frac{e^2 - 1}{2}}}$$

Step-5

Let

$$f(x) = e^x,$$
$$g(x) = e^{-x}$$

Inner product of f and g;

$$= (f, g)$$

$$= \int_0^1 e^x e^{-x} dx$$

$$= \int_0^1 e^0 dx$$

$$= \int_0^1 1 dx$$
$$= [x]_0^1$$
$$= 1 - 0$$
$$= 1$$

Hence inner product of f and g is 1