Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #7

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Name: ____

Student Number: __________________________________

1. Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

   Proof. $\forall v_1, v_2 \in V$, $Tv_1 = \lambda_1 v_1$, $Tv_2 = \lambda_2 v_2$. Since $v_1 + v_2 \in V$, $v_1 + v_2$ is also an eigenvector of T, then $\lambda_1 = \lambda_2$. We have $\forall v \in V$, $Tv = \lambda v$. i.e. $T = \lambda I$.

   $\square$

2. Suppose W is a complex vector space and $T \in \mathcal{L}(W)$ has no eigenvalues. Prove that every subspace of W invariant under T is either $\{0\}$ or infinite-dimensional.

Proof. Assume U is an invariant subspace of T, and $U \neq \{0\}$, $W \neq \{0\}$. Take $u_1 \in U, u_1 \neq 0$, since T doesn't have eigenvalues, $Tu_1 \notin span\{u_1\}$, i.e. u_1, Tu_1 are linearly independent. Let $u_2 = Tu_1$, then $Tu_1 \in span\{u_1, u_2\}$.

Claim that $Tu_2 \notin span\{u_1, u_2\}$. If not, $span\{u_1, u_2\}$ is a finite-dimensional invariant subspace of T. According to 5.21, T must have an eigenvalue in $span\{u_1, u_2\}$, which is a contradiction. Then $Tu_2 \notin span\{u_1, u_2\}$. Let $u_3 = Tu_2$, we have u_1, u_2, u_3 are linearly independent. Continue the above process, we can get a consequence of vectors in U: $u_1, u_2, \cdots$ such that $\forall m \in \mathbf{Z}^+$, $u_1, u_2, \cdots, u_m$ are linearly independent, so U is infinite-dimensional.

$\square$