

## Step-1

Multiply  $Aq_j = b_{j-1}q_{j-1} + a_jq_j + b_jq_{j+1}$  throughout by  $q_j^T$ . Since, the vectors  $q_i$  are orthogonal,

$$q_i^T q_j = 1 \text{ if } i = j$$

$$q_i^T q_j = 0 \text{ if } i \neq j$$

Thus, we get

$$\begin{aligned} q_j^T (Aq_j) &= q_j^T (b_{j-1}q_{j-1} + a_jq_j + b_jq_{j+1}) \\ &= q_j^T b_{j-1}q_{j-1} + q_j^T a_jq_j + q_j^T b_jq_{j+1} \\ &= b_{j-1}q_j^T q_{j-1} + a_jq_j^T q_j + b_jq_j^T q_{j+1} \\ &= a_j \end{aligned}$$

Therefore, the formula for  $a_j$  is  $\boxed{a_j = q_j^T Aq_j}$ .

## Step-2

Consider again  $Aq_j = b_{j-1}q_{j-1} + a_jq_j + b_jq_{j+1}$ . Let us express this equation in the matrix form. As  $q_0 = 0$ , we need not consider  $a_0$  and  $b_0$ .

We get

$$A \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & 0 & \dots & 0 \\ 0 & b_2 & a_3 & b_3 & \dots & 0 \\ 0 & 0 & b_3 & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 0 & . & . & . & a_n \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{bmatrix}$$

## Step-3

$$T = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & 0 & \dots & 0 \\ 0 & b_2 & a_3 & b_3 & \dots & 0 \\ 0 & 0 & b_3 & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 0 & . & . & . & a_n \end{bmatrix}$$

If we write  $\begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & 0 & \dots & 0 \\ 0 & b_2 & a_3 & b_3 & \dots & 0 \\ 0 & 0 & b_3 & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 0 & . & . & . & a_n \end{bmatrix}$ , then we get the following:

The equation  $Aq_j = b_{j-1}q_{j-1} + a_jq_j + b_jq_{j+1}$  can be expressed as  $AQ = QT$ , where  $T$  is a symmetric matrix.

