

Step-1

Consider the following orthogonal matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Eigen vectors of P are as follows:

$$(1,1,1), (1, e^{2\pi i/3}, e^{4\pi i/3}), (1, e^{4\pi i/3}, e^{8\pi i/3})$$

Find the circulant matrix C which has the same Eigen vectors as orthogonal matrix P and satisfies the following relation:

$$C = 2I + 5P + 4P^2$$

Also, find Eigen values of the circulant matrix C .

Step-2

Circulant matrix C can be calculated as follows:

$$\begin{aligned} C &= 2I + 5P + 4P^2 \\ &= 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 5 \\ 5 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} \end{aligned}$$

Therefore, circulant matrix is defined as follows:

$$C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix}$$

Step-3

To calculate the Eigen values of circulant matrix, do the following calculations:

$$C - \lambda I = \begin{bmatrix} 2-\lambda & 5 & 4 \\ 4 & 2-\lambda & 5 \\ 5 & 4 & 2-\lambda \end{bmatrix}$$

$$\det(C - \lambda I) = 0$$

$$(2-\lambda)^3 \left((2-\lambda)^2 - 20 \right) - 5(8-4\lambda-25) + 4(6+\lambda) = 0$$

$$(-\lambda^3 + 6\lambda^2 + 48\lambda + 77) = 0$$

After solving following values are obtained:

$$\lambda_1 = 11$$

$$\lambda_2 = -\frac{5}{2} + i\frac{\sqrt{3}}{2}$$

$$\lambda_3 = -\frac{5}{2} - i\frac{\sqrt{3}}{2}$$

Step-4

These Eigen values can also be written as a combination of Eigen values of matrix P :

$$\lambda_1 = 11$$

$$= 2 + 5 + 4$$

$$\lambda_2 = -\frac{5}{2} + i\frac{\sqrt{3}}{2}$$

$$= 2 \cdot 1 + 5e^{2\pi i/3} + 4e^{4\pi i/3}$$

$$\lambda_3 = -\frac{5}{2} - i\frac{\sqrt{3}}{2}$$

$$= 2 \cdot 1 + 5e^{4\pi i/3} + 4e^{8\pi i/3}$$

Step-5

Therefore, following are the Eigen values of the circulant matrix:

$\lambda_1 = 11$
$\lambda_2 = -\frac{5}{2} + i\frac{\sqrt{3}}{2}$
$\lambda_3 = -\frac{5}{2} - i\frac{\sqrt{3}}{2}$