Step-1

Given $a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Letâ€™s do Gram â€“Schmidt on $\{\vec{a}_1, \vec{a}_2\}$.

The Gram-Schmidt process starts with independent vectors $a_1, ..., a_n$ and ends with orthonormal vectors $q_1, ..., q_n$. At step j it subtracts form a_j its components in the direction $q_1, ..., q_{j-1}$ that are already settled:

$A_j = a_j - (q_1^T a_j) - ... - (q_{j-1}^T a_j) q_{j-1}$.

Then q_j is the unit vector $A_j / \| A_j \|$

Step-2

First,

$$\begin{aligned} q_1 &= \frac{a_1}{\|a_1\|} \\ &= \frac{1}{\sqrt{1^2 + 1^2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \end{aligned}$$

Next,

$q_2 = \frac{\beta}{\|\beta\|}$ where $\beta = a_2 - (q_1^T a_2) q_1$

So,

$$\begin{aligned} q_1^T a_2 &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \\ &= \frac{4}{\sqrt{2}} + 0 \\ &= \frac{4}{\sqrt{2}} \end{aligned}$$

And,

$$\left(q_1^T a_2\right) q_1 = \frac{4}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Therefore,

$$\beta = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

By construction, β is orthogonal to a_1, so we see that we needed to subtract $\boxed{2\text{ times}}$ a_1 from a_2 to get a vector perpendicular to a_2

Step-3

Now, continuing with Gram-Schmidt, So

$$\|\beta\| = \sqrt{4+4} = \sqrt{8}$$

Therefore $q_2 = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/2\sqrt{2} \\ -2/2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

Therefore, if $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$ and $Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}$ then $A = QR$ where $R = \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ 0 & a_2^T q_2 \end{bmatrix}$.

Step-4

So,

$$a_1^T q_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}}$$
$$= \sqrt{2}$$

$$a_2^T q_1 = [4 \quad 0]\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
$$= \frac{4}{\sqrt{2}} + 0$$
$$= 2\sqrt{2}$$

$$a_2^T q_2 = [4 \quad 0]\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$
$$= \frac{4}{\sqrt{2}} + 0$$
$$= 2\sqrt{2}$$

Plug these values in R, so

$$R = \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ 0 & a_2^T q_2 \end{bmatrix}$$

Step-5

$$= \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}.$$

Therefore, $\boxed{R == \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}}$.