1. (20 points, 4 points each) Label the following statements as **True** or **False**. **Along with your answer, provide an informal proof, counterexample, or other explanation.** positive

   (a) The sum of two positive operators on a finite-dimensional complex inner product space is positive. True $\langle Tv, v\rangle \geq 0$ $\langle Sv, v\rangle \geq 0$

   (b) Let V be a 5-dimensional vector space and $T \in \mathcal{L}(V)$. Then there exists a 3-dimensional subspace U of V invariant under T. False

   (c) Any polynomial of degree n with leading coefficients $(-1)^n$ is the characteristic polynomial of some linear operator. True

   (d) If x, y, and z are vectors in an inner product space such that $\langle x, y\rangle = \langle x, z\rangle$, then $y = z$. False

   (e) Every normal operator is diagonalizable. False

2. (20 points) Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ is defined by

$$T(x_1, x_2, x_3) = (2x_1,\ x_2 - x_3, x_2 + x_3). = (x_1, x_2, x_3)\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

   (a) Determine the eigenspace of T corresponding to each eigenvalue. $\lambda = 2$

   (b) Find the Jordan form and a Jordan basis of T.

   (c) Find the minimal polynomial of T.

   (d) Find the trace of T, trace T.

   (e) Find the determinant of T, det T.

3. (10 points) Suppose V is a finite-dimensional inner product space, $T \in \mathcal{L}(V)$ is normal, and U is a subspace of V that is invariant under T. Show that $U^{\perp}$ is invariant under T. 9.30

4. (20 points) Let T be a linear operator on a finite-dimensional vector space V, and let v be a nonzero vector in V. The subspace

$$U = \text{span}\left(\{v, Tv, T^2v, \cdots\}\right)$$

is called the T-cyclic subspace of V generated by v.

   (a) Show that U is a finite-dimensional invariant subspace of V.

   (b) Let $k = \dim U$. Show that $\{v, Tv, T^2v, \cdots, T^{k-1}v\}$ is a basis for U.

   (c) If $a_0v + a_1Tv + a_2T^2v + \cdots + a_{k-1}T^{k-1}(v) + T^k(v) = 0$, show that the characteristic polynomial of $T|_U$ is

$$f(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k).$$

   (d) Let $g(t)$ be the characteristic polynomial of T, show that $g(T) = 0$, where 0 is the zero operator. That is, T "satisfies" it characteristic equation.

5. (10 points) If $\mathbb{F} = \mathbb{C}$, show that T is an isometry if and only if T is normal and $|\lambda| = 1$ for every eigenvalue λ of T.

6. (20 points) Let $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_1(\mathbb{R})$ be the polynomial spaces with inner products defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx,\ \ f, g \in \mathcal{P}_2(\mathbb{R}).$$

Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$ be the linear operator defined by

$$T(f(x)) = f'(x).$$

   (a) Find orthonormal bases $\{v_1,\ v_2,\ v_3\}$ for $\mathcal{P}_2(\mathbb{R})$ and $\{u_1,\ u_2\}$ for $\mathcal{P}_1(\mathbb{R})$.

   (b) Find $p \in \mathcal{P}_1(\mathbb{R})$ that makes

$$\int_{-1}^{1} |x^5 - p(x)|^2 dx$$

   as small as possible.

   (c) Find the singular values σ_1, σ_2 of T such that $T(v_i) = \sigma_i u_i,\ i = 1, 2$, and $T(v_3) = 0$.

7. (Bonus Question) Let V be a real inner product space. A function $f : V \to V$ is called a **rigid motion** if

$$||f(x) - f(y)|| = ||x - y||$$

for all $x,\ y \in V$. For example, any **isometry** on a finite-dimensional real inner product space is a **rigid motion**. Another class of rigid motions are the translations. A function $g : V \to V$, where V is a real inner product space, is called a **translation** if there exists a vector $v_0 \in V$ such that $g(v) = v + v_0$ for all $v \in V$. Let $f : V \to V$ be a rigid motion on a finite-dimensional real inner product space V, show that there exists a unique isometry T on V and a unique translation g on V such that $f = g \circ T$.