Step-1

Consider the boundary value problem,

$$-\frac{d^2u}{dx^2}+u=x, \quad u(0)=u(1)=0$$

Approximate second derivative by,

$$\frac{d^2u}{dx^2}\approx\frac{\Delta^2 u}{\Delta x^2}$$
$$=\frac{u(x+h)-2u(x)+u(x-h)}{h^2}$$

With the above substitution given differential equation becomes,

$$-\frac{u(x+h)-2u(x)+u(x-h)}{h^2}+u(x)=x$$

Step-2

Measure u at the mesh points $x=jh$ by substituting x by jh.

$$-\frac{u(jh+h)-2u(jh)+u(jh-h)}{h^2}+u(jh)=jh$$
$$-\left[u(jh+h)-2u(jh)+u(jh-h)\right]+h^2u(jh)=h^2jh$$

$$-u(jh+h)+2u(jh)-u(jh-h)+h^2u(jh)=h^3j$$

$$-u(j+1)h+2u(jh)-u(j-1)h+h^2u(jh)=h^3j$$

Use the notation $u_k=u(kh)$

$$-u_{j+1}+2u_j-u_{j-1}+h^2u_j=h^3j$$

$$-u_{j+1}+\left(2+h^2\right)u_j-u_{j-1}=h^3j$$

Let $h=\frac{1}{4}$.

$$-u_{j+1}+\left(2+\left(\frac{1}{4}\right)^2\right)u_j-u_{j-1}=\left(\frac{1}{4}\right)^3j$$
$$-u_{j+1}+\left(2+\frac{1}{16}\right)u_j-u_{j-1}=\frac{j}{64}$$

$$-u_{j+1}+\left(\frac{33}{16}\right)u_j-u_{j-1}=\frac{j}{64}$$ â€¦â€¦ (1)

Step-3

For $j=1$ difference equation (1) becomes,

$$-u_{1+1}+\left(\frac{33}{16}\right)u_1-u_{1-1}=\frac{1}{64}$$
$$-u_2+\left(\frac{33}{16}\right)u_1-u_0=\frac{1}{64}$$

From the data $u_0=u(0)=0$, this simplifies to,

$$-u_2+\left(\frac{33}{16}\right)u_1-0=\frac{1}{64}$$

$$-u_2+\frac{33}{16}u_1=\frac{1}{64}$$ â€¦â€¦ (i)

Step-4

For $j=2$ difference equation (1) becomes,

$$-u_{2+1}+\left(\frac{33}{16}\right)u_2-u_{2-1}=\frac{2}{64}$$

$$-u_3+\frac{33}{16}u_2-u_1=\frac{2}{64}$$ â€¦â€¦ (ii)

Step-5

For $j=3$ difference equation (1) becomes,

$$-u_{3+1}+\left(\frac{33}{16}\right)u_3-u_{3-1}=\frac{3}{64}$$

$$-u_4+\frac{33}{16}u_3-u_2=\frac{3}{64}$$

Since $u_4=u\left(4\times\frac{1}{4}\right)=u(1)=0$, the above equation simplified to,

$$-0+\frac{33}{16}u_3-u_2=\frac{3}{64}$$

$$\frac{33}{16}u_3-u_2=\frac{3}{64}$$ â€¦â€¦ (iii)

Step-6

From equations (i), (ii), and (iii) get the system,

$$\begin{aligned}
-u_2+\frac{33}{16}u_1&=\frac{1}{64}\\
-u_3+\frac{33}{16}u_2-u_1&=\frac{2}{64}\\
\frac{33}{16}u_3-u_2&=\frac{3}{64}
\end{aligned}$$

Rearrange it as shown.

$$\begin{aligned}
\frac{33}{16}u_1-u_2+0\cdot u_3&=\frac{1}{64}\\
-u_1+\frac{33}{16}u_2-u_3&=\frac{2}{64}\\
0\cdot u_1-u_2+\frac{33}{16}u_3&=\frac{3}{64}
\end{aligned}$$

Step-7

Write it in matrix notation to get the required matrix equation.

$$\begin{bmatrix}\frac{33}{16} & -1 & 0\\ -1 & \frac{33}{16} & -1\\ 0 & -1 & \frac{33}{16}\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}\frac{1}{64}\\ \frac{2}{64}\\ \frac{3}{64}\end{bmatrix}$$

It is of the form $Au=b$, where A is 3×3 matrix.