Step-1

Consider a 4 by 6 matrix A, whose LU decomposition is as follows:

$$A = LU$$
$$= \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 2 & 1 & 1 & \\ 3 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-2

(a)

The objective is to find rank of the matrix A and basis of its null space.

Since A is a 4 by 6 matrix, its rank cannot be greater than four.

The matrix U has three pivot rows or three pivot columns.

Therefore, the rank of U is 3 and so the rank of matrix A is $\boxed{3}$.

The rank of the matrix A or U is 3 so the dimension of the null space is $6-3=3$.

Step-3

The basis of the null space is the solution of the system $Ax = 0$ or $Ux = 0$.

So,

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives,

$$\begin{aligned} x_1 + 2x_2 + x_4 + 2x_5 + x_6 &= 0 \\ 2x_3 + 2x_4 &= 0 \\ x_6 &= 0 \end{aligned}$$

Step-4

The solutions of these equations are;

$$x_6 = 0$$
$$x_3 = -x_4$$
$$x_1 = -2x_2 - x_4 - 2x_5$$

So the solution vector becomes;

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 - 2x_5 \\ x_2 \\ -x_4 \\ x_4 \\ x_5 \\ 0 \end{bmatrix}$$
$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence the basis of the null space is;

$$\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

(b)

The first three rows of *U* are a basis for the row space of *A*.

This statement is $\boxed{\text{true}}$, because the rank of *A* is 3 and the first three rows of *U* are linearly independent.

Consider the following statement:

The three columns â€“ first, third, and sixth â€“ are a basis for the column space of *A*.

This statement is $\boxed{\text{true}}$, because these three columns are linearly independent and the rank of *A* is also 3.

Consider the following statement:

The four rows of *A* form a basis for the row space of *A*.

Note that the rows of A will certainly span the row space of A but the four rows of A wonâ€™t be independent, because the rank of A is 3.

Therefore, the statement is $\boxed{\text{false}}$.

Step-5

(c)

Consider the system $Ax = b$. Since A is decomposed as LU, write $LUx = b$.

The three columns â€“ first, third, and sixth â€“ columns of U form a basis for the column space of A.

Step-6

So consider the following;

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$$

Therefore, for all those bâ€™s, which can be written as $\begin{bmatrix} x_1 + x_3 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$, the system $Ax = b$ will have a solution. Fix x_2 and x_3 to 1 and change x_1 from 1, 2, and 3, the three linearly independent bâ€™s will come, for which $Ax = b$ has a solution.

Thus, $\boxed{\{(2,1,1,0),(3,1,1,0),(4,1,1,0)\}}$ is a required set of vectors.

Step-7

(d)

Consider $A = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 2 & 1 & 1 & \\ 3 & 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Observe that in the fourth row, third column of L, there is an entry of 4.

Therefore, while eliminating the third row was multiplied by $\boxed{4}$ to knock out the fourth row.