Step-1

(a) Consider the following differential equation:

$$y'' = -y$$

Find two familiar functions that solve the equation. Also find the function among them which satisfies the following:

$$y(0) = 1$$
$$y'(0) = 0$$

Step-2

Let the two familiar functions be $(\cos t, \sin t)$. Next step is to check that these functions solve the differential equation, for this do the following calculations:

$$\begin{aligned} y(t) &= \cos t \\ y'(t) &= -\sin t \\ y''(t) &= -\cos t \\ &= -y \end{aligned}$$

Similarly,

$$\begin{aligned} y(t) &= \sin t \\ y'(t) &= \cos t \\ y''(t) &= -\sin t \\ &= -y \end{aligned}$$

These calculation shows that both functions satisfies the differential equation. Therefore the familiar functions are $\boxed{(\cos t, \sin t)}$.

Among these functions following function satisfy the initial condition.

$$\boxed{y(t) = \cos t}$$

Step-3

(b) The second order differential equation $y'' + y = 0$ can be written as a first order system by introducing the velocity y' as unknown factor.

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$$

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Step-4

Initial condition:

$$u(0) = (1,0)$$

Compute the solution that starts from the initial value.

Step-5

First step is to find the Eigen values and Eigen vectors of matrix A. To calculate the Eigen values do the following calculations;

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \\ \det(A - \lambda I) &= 0 \\ (-\lambda)(-\lambda) + 1 &= 0 \\ \lambda^2 + 1 &= 0 \end{aligned}$$

After solving following values are obtained:

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}$$

Step-6

To calculate Eigen vectors do the following calculations:

$$\begin{aligned} (A - \lambda I)x &= 0 \\ \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Step-7

On solving, values of y and z corresponding to $\lambda = i$ are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Step-8

Similarly, Eigen vectors corresponding to Eigen value $\lambda = -i$ is as follows:

$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Step-9

Recall that: $e^{At} = Se^{\Lambda t}S^{-1}$

Here, Eigen value matrix is given as follows:

$$\Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Therefore, the general solution of the differential equation is:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$
$$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here, c_1 and c_2 are constants. Their values are determined by the following values:

$$c = S^{-1}u(0)$$

Step-10

So, the solution for differential equation can be written as follows:

$$
\begin{aligned}
u(t) &= Se^{\Lambda t}S^{-1}u(0) \\
&= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} e^{it}+e^{-it} & -ie^{it}+ie^{-it} \\ ie^{it}-ie^{-it} & e^{it}+e^{-it} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} e^{it}+e^{-it} \\ i\left(e^{it}-e^{-it}\right) \end{bmatrix} \\
u(t) &= \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}
\end{aligned}
$$

Step-11

Therefore, specific solution of the differential equation is:

$$\boxed{u(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}$$

This solution can also be obtained by putting $y(t) = \cos t$ from part (a) in the solution $u(t) = \left(y(t), y'(t)\right)$.