Step-1

(a) If $y=(1,1,1,1)$, we have to show that $c=(1,0,0,0)$ satisfies $F_4c=y$.

Given $c=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, y=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$$F_4c=\begin{bmatrix}1 & 1 & 1 & 1\\1 & i & i^2 & i^3\\1 & i^2 & i^4 & i^6\\1 & i^3 & i^6 & i^9\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

Step-2

$$=\begin{bmatrix}1+0+0+0\\1+0+0+0\\1+0+0+0\\1+0+0+0\end{bmatrix}$$

$$=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$=y$$

Hence $F_4c=y$

Step-3

(b) If $y=(1,0,0,0)$, then we have to find c.

Let $c=\begin{bmatrix}c_0\\c_1\\c_2\\c_3\end{bmatrix}, y=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

Step-4

Fourier matrix

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$\begin{pmatrix} \text{since } i^2 = -1, i^3 = -i, \\ i^4 = 1, i^6 = i^4 i^2 = -1, \\ i^9 = i^6 i^3 = i \end{pmatrix}$$

Step-5

Now

$$F_4 c = y$$
$$\Rightarrow c = F_4^{-1} y$$

But $F_n^{-1} = \frac{\overline{F_n}}{n} y, (n \text{ is the order of } F_n)$

Therefore $c = \frac{\overline{F_4}}{4} y$

Step-6

$$\Rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

Hence $c = (1/4, 1/4, 1/4, 1/4)$