Step-1

(a).

Given matrix is, $A=\begin{pmatrix}1 & b\\ b & 9\end{pmatrix}$

We know that the matrix $\begin{pmatrix}a & b\\ b & c\end{pmatrix}$ is positive definite if and only if $ac-b^2>0$ and $a>0$.

Therefore A is positive definite

Step-2

If $(1)(9)-b^2>0$

$\Rightarrow(b+3)(3-b)<0$ â€¦â€¦ (1)

If $b^2-9<0$

$(b+3)(b-3)<0$ ...... (2)

From the both equations, the range is $-3<b<3$

Therefore, $A$is $\boxed{\text{positive definite when } -3<b<3}$.

Step-3

(b).

$A=\begin{pmatrix}1 & b\\ b & 9\end{pmatrix}$

So, $L=\begin{pmatrix}1 & 0\\ 6 & 1\end{pmatrix}$

$A=\begin{pmatrix}1 & b\\ b & 9\end{pmatrix}$

$=\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 9-b^2\end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 9-b^2 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$= LDL^T$$

Where L have is on the diagonal and D is the diagonal matrix of points.

Here b is in the range of positive definiteness.

Therefore, $\boxed{A = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 9-b^2 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}}.$

Step-4

(c).

Let $F(x,y) = \frac{1}{2}(x^2 + 2bxy + 9y^2) - y$

The first and second derivatives with respect to x are,

$$F_x = \frac{1}{2}(2x + 2by)$$

$$= x + by$$

$$F_{xx} = 1 > 0$$

$$a + c > 2b,\ ac < b^2$$

$$(a-c)^2 = (a+c)^2 - 4ab$$

$$= 4b^2 - 4b^2$$

Step-5

Now,

The first and second derivatives are,

$$F_y = \frac{1}{2}(2bx + 18y) - 1$$

$$F_{yy} = 9$$

$$F_{xy} = b$$

Now,

$$\left(F_{xx}\right)\left(F_{yy}\right)=(1)(9)$$
$$=9$$

$$\left(F_{xy}\right)^2=b^2$$

$$\Rightarrow\left(F_{xx}\right)\left(F_{yy}\right)-\left(F_{xy}\right)^2>0$$

$$\Rightarrow 9>b^2$$

$$\Rightarrow b^2-9<0$$

$$\Rightarrow -3<b<3$$

Thus $F_{xx}>0$ and $\left(F_{xx}\right)\left(F_{yy}\right)-\left(F_{xy}\right)^2>0$

Step-6

For b in this range.

Stationary points are given by,

$F_x=0$ and $F_y=0$

$$\Rightarrow x+by=0 \text{ and } bx+9y-1=0$$

$$\Rightarrow x=-by \text{ and } bx+9y-1=0$$

$$\Rightarrow\ b(-by)+9y-1=0$$

$$\Rightarrow\ \left(9-b^2\right)y=1$$

$$\Rightarrow\ y=\frac{1}{\left(9-b^2\right)}$$

Therefore, $x=\dfrac{-b}{\left(9-b^2\right)}$ and $y=\dfrac{1}{\left(9-b^2\right)}$

Step-7

So the minimum value of F is

$$=\frac{1}{2}\left(\frac{b^2}{\left(9-b^2\right)^2}+2b\left(\frac{-b}{9-b^2}\right)\left(\frac{1}{9-b^2}\right)+9\left(\frac{1}{9-b^2}\right)^2\right)-\frac{1}{9-b^2}$$

$$=\frac{1}{2}\left(\frac{b^2+9}{\left(9-b^2\right)^2}\right)-\frac{1}{9-b^2}$$

$$=-\frac{1}{2\left(9-b^2\right)}$$

Thus the stationary point is $=\left(\frac{-b}{9-b^2},\frac{1}{9-b^2}\right)$.

Therefore, the minimum value of F is $\boxed{-\frac{1}{2\left(9-b^2\right)} \text{ when } \begin{bmatrix}1 & b\\ b & 9\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix} \text{ which is } \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{\left(9-b^2\right)}\begin{bmatrix}-b\\ 1\end{bmatrix}}$.

Step-8

(d).

Given $b=3$.

Then minimum value is $-\frac{1}{0}=-\infty$.

Thus there is no minimum when $b=3$.

Step-9

$$x+by=0$$
$$\Rightarrow x+3y=0$$

$$\Rightarrow x=-3y$$

Let $y\to\infty,\ x=-3y,$ then $x-y\to-\infty.$

Therefore, $\boxed{\text{No minimum.let } y\to\infty,\ x=-3y,\ \text{then } x-y\to-\infty}$.