Step-1

(a) Consider $A=\begin{bmatrix}0 & 4\\ \frac{1}{4} & 0\end{bmatrix}$, we have to find eigenvalues of matrix A.

Solve $\det(A-\lambda I)=0$, to find the eigenvalues λ.

$$\det(A-\lambda I)=0$$
$$\det\begin{bmatrix}0-\lambda & 4\\ \frac{1}{4} & 0-\lambda\end{bmatrix}=0$$
$$\lambda^2-1=0$$
$$\lambda=\pm 1$$

Therefore, the eigenvalues of matrix A are $\lambda_1=\boxed{1}$ and $\lambda_2=\boxed{-1}$.

Step-2

The eigenvector for $\lambda_1=1$ is given by

$$(A-\lambda_1 I)x_1=\begin{bmatrix}-1 & 4\\ \frac{1}{4} & -1\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The solution (the first eigenvector) is any nonzero multiple of x_1, thus eigenvector for $\lambda_1=1$ is $\boxed{x_1=\begin{bmatrix}4\\ 1\end{bmatrix}}$.

The eigenvector for $\lambda_2=-1$ is given by

$$(A-\lambda_2 I)x_2=\begin{bmatrix}1 & 4\\ \frac{1}{4} & 1\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

The solution (the second eigenvector) is any nonzero multiple of x_2, thus eigenvector for $\lambda_2=-1$ is $\boxed{x_2=\begin{bmatrix}-4\\ 1\end{bmatrix}}$.

Step-3

(b) We know that, if A can be diagonalized, $A=S\Lambda S^{-1}$, then $du/dt=Au$ has a solution

$$\begin{aligned} u(t) &= e^{At}u(0) \\ &= Se^{\Lambda t}S^{-1}u(0) \end{aligned}$$

The eigenvector matrix of A is given by

$$S = \begin{bmatrix} 4 & -4 \\ 1 & 1 \end{bmatrix}$$

And we have

$$S^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{1}{2} \\ -\frac{1}{8} & \frac{1}{2} \end{bmatrix}$$

The Eigen values matrix of A is given by

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Step-4

Using the above matices we have to find $A = S\Lambda S^{-1}$.

$$\begin{aligned} A &= S\Lambda S^{-1} \\ &= \begin{bmatrix} 4 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{8} & \frac{1}{2} \\ -\frac{1}{8} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

Hence,

$$\begin{aligned} e^{At} &= Se^{\Lambda t}S^{-1} \\ &= \begin{bmatrix} 4 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} \frac{1}{8} & \frac{1}{2} \\ -\frac{1}{8} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

By using the initial condition $u(0) = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, we have

$$u(t)=e^{At}u(0)$$

$$=\begin{bmatrix}4 & -4\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{t} & 0\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}\frac{1}{8} & \frac{1}{2}\\ -\frac{1}{8} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}100\\ 100\end{bmatrix}$$

$$=\begin{bmatrix}4 & -4\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{t} & 0\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}\frac{125}{2}\\ \frac{75}{2}\end{bmatrix}$$

$$=\begin{bmatrix}4 & -4\\ 1 & 1\end{bmatrix}\begin{bmatrix}\frac{125}{2}e^{t}\\ \frac{75}{2}e^{-t}\end{bmatrix}$$

$$u(t)=\begin{bmatrix}250e^{t}-150e^{-t}\\ \frac{125}{2}e^{t}+\frac{75}{2}e^{-t}\end{bmatrix}$$

Therefore, the income to stockbrokers is $\boxed{v(t)=250e^{t}-150e^{-t}}$ and the income to client is $\boxed{w(t)=\frac{125}{2}e^{t}+\frac{75}{2}e^{-t}}$.

Step-5

(c) We know that, the income to stockbrokers is $v(t)=250e^{t}-150e^{-t}$ and the income to client is $w(t)=\frac{125}{2}e^{t}+\frac{75}{2}e^{-t}$.

So the ratio of v by w is given by

$$\frac{v(t)}{w(t)}=\frac{250e^{t}-150e^{-t}}{\frac{125}{2}e^{t}+\frac{75}{2}e^{-t}}$$

Step-6

If $t\to\infty$ then $150e^{-t}\to 0$ and $\frac{75}{2}e^{-t}\to 0$, so we get

$$\frac{v(t)}{w(t)}=\frac{250e^{t}}{\frac{125}{2}e^{t}}$$
$$=4$$

This implies $v(t)=4w(t)$.

Therefore, the income to stockbrokers is $\boxed{\text{four times}}$ the income to client.