Step-1

The matrix S is a diagonal matrix. Hence, its inverse is obtained by taking reciprocals of its diagonal elements and keeping remaining zeros as they are.

Therefore, $S^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

This gives,

$$S^{-1}T = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}$$

Step-2

To obtain the eigenvalues of $S^{-1}T$, we solve the equation $\det\left(S^{-1}T - \lambda I\right) = 0$, where Î» is an eigenvalue of $S^{-1}T$.

Therefore,

$$0 = \begin{vmatrix} 0-\lambda & \frac{1}{3} \\ \frac{1}{3} & 0-\lambda \end{vmatrix}$$
$$= \begin{vmatrix} -\lambda & \frac{1}{3} \\ \frac{1}{3} & -\lambda \end{vmatrix}$$
$$= \lambda^2 - \frac{1}{9}$$

Thus, we have $\lambda^2 - \frac{1}{9} = 0$. Therefore, $\boxed{\lambda = \pm\frac{1}{3}}$.

Step-3

For the Gauss-Seidel method, consider the following equation:

$$\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + b$$.

Thus, here we have

$$S = \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix}$$
$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Step-4

Let us obtain the inverse of *S*. The determinant of *S* is 9.

Therefore, $$S^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{9} & \frac{1}{3} \end{bmatrix}$$

This gives,

$$S^{-1}T = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{9} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$$

To obtain the eigenvalues of $S^{-1}T$, we solve the equation $\det\left(S^{-1}T - \mu I\right) = 0$, where Î¼ is an eigenvalue of $S^{-1}T$.

Therefore,

$$0 = \begin{vmatrix} 0-\mu & \frac{1}{3} \\ 0 & \frac{1}{9}-\mu \end{vmatrix}$$
$$= \begin{vmatrix} -\mu & \frac{1}{3} \\ 0 & \frac{1}{9}-\mu \end{vmatrix}$$
$$= \mu^2 - \frac{\mu}{9}$$

Thus, we have $\mu^2 - \frac{\mu}{9} = 0$. Therefore, $\boxed{\mu = 0 \text{ or } \frac{1}{9}}$.

Step-5

The maximum eigenvalue in Gauss-Seidel method is $\frac{1}{9}$ and in Jacobi method, it is $\frac{1}{3}$. Since, $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$, we can say that $|\lambda|_{\max}$ for Gauss-Seidel is equal to the $|\lambda|_{\max}^2$ for Jacobi.