Step-1

The complete sentence is:

If $A = QR$ then $A^T A = R^T R$ $=$ lower triangular times upper triangular.

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \ A^T A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

It is known that, pivots for $A^T A$ are $2, \frac{3}{2}, \frac{4}{3}$ and the multiplier are $-\frac{1}{2}, -\frac{2}{3}$.

Step-2

(a)

Objective is to prove that column 1 of *A*, *B* equals $\text{column } 2 - \frac{1}{2}(\text{column } 1)$ and $C = \text{column } 3 - \frac{2}{3}(\text{column } 2)$ are orthogonal.

First find the columns *B* and *C* when positive multiplier are $\frac{1}{2}, \frac{2}{3}$:

$$\begin{aligned} \text{B} &= \text{column } 2 + \tfrac{1}{2}(\text{column } 1) \\ &= \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

and

$C = \text{column } 3 + \frac{2}{3}\text{B}$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Step-3

Let the new matrix is E given by:

$$E = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ -1 & \frac{1}{2} & \frac{1}{3} \\ 0 & -1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}, \; E^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix}$$

According to the definition of orthogonality, columns of E are said to be an orthogonal if $E^T E$ is a diagonal matrix.

Then

$$E^T E = \begin{bmatrix} 1 & -1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ -1 & \frac{1}{2} & \frac{1}{3} \\ 0 & -1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

Hence, column 1 of A, B, and C are orthogonal columns.

Step-4

(b)

Using the pivot elements show that $\|\text{column } 1\|^2 = 2, \|\text{B}\|^2 = \frac{3}{2}, \|\text{C}\|^2 = \frac{4}{3}$.

Note that the length of the column vectors of matrix E are the diagonal entries of the matrix $E^T E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$.

So, the length of column 1 is 2, length of column B is $\frac{3}{2}$ and the length of column C is $\frac{4}{3}$.

Step-5

Hence, $\|\text{column 1}\|^2 = 2, \|\text{B}\|^2 = \frac{3}{2}, \|\text{C}\|^2 = \frac{4}{3}$.