Step-1

Consider the following projection matrix P:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The objective is to solve $\frac{du}{dt} = Pu$.

Where P is projection matrix with initial condition $u(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Here, part of $u(0)$ increases exponentially while the null space part stays fixed.

Step-2

Solve the following differential equation $\frac{du}{dt} = Pu$ with $u = u(0)$ at $t = 0$ using the pure exponential solution.

$$u(t) = e^{a \cdot t} u(0)$$

Let the Eigen value equation be $Px = \lambda x$.

Where $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $x = \begin{bmatrix} y \\ z \end{bmatrix}$

Then,

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix}$$

$$\frac{1}{2}y + \frac{1}{2}z = \lambda y$$

$$\frac{1}{2}y + \frac{1}{2}z = \lambda z$$

$$\frac{1}{2}y+\frac{1}{2}z=\lambda y$$
$$\frac{1}{2}y+\frac{1}{2}z=\lambda z$$

Step-3

Find the Eigen values and Eigen vectors of the projection matrix.

The characteristic form of the projection matrix is, $[P-\lambda I]$.

$$|P-\lambda \mathrm{I}|=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{2}-\lambda & \frac{1}{2}-0\\ \frac{1}{2}-0 & \frac{1}{2}-\lambda\end{bmatrix}$$
$$=\begin{bmatrix}\frac{1}{2}-\lambda & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}-\lambda\end{bmatrix}$$

So, the characteristic polynomial of the matrix is $\begin{bmatrix}\frac{1}{2}-\lambda & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}-\lambda\end{bmatrix}$

Step-4

Find the characteristic equation of the matrix is $|P-\lambda \mathrm{I}|=0$

$$\begin{vmatrix}\frac{1}{2}-\lambda & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}-\lambda\end{vmatrix}=0$$
$$\left(\frac{1}{2}-\lambda\right)^2-\left(\frac{1}{2}\right)^2=0$$

$$\frac{1}{4}+\lambda^2-\lambda-\frac{1}{4}=0$$
$$\lambda^2-\lambda=0$$
$$\lambda(\lambda-1)=0$$
$$\lambda=0 \text{ or } \lambda=1$$

Therefore, the Eigen values of the matrix are $\lambda=0 \text{ or } \lambda=1$.

Step-5

Find the Eigen vectors of the projection matrix.

The Eigen vectors corresponding to the Eigen value of the matrix is $[P-\lambda_1 I]x_1=0$.

$\lambda_1=0$ and the Eigen vector $x_1=\begin{bmatrix} y \\ z \end{bmatrix}$.

Continue the above calculation.

$$\begin{bmatrix} \frac{1}{2}-0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}-0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the row operation $R_2 \to R_2-R_1$.

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\frac{1}{2}y+\frac{1}{2}z=0$$

Step-6

Choose $z=k$ is an arbitrary constant.

$$\frac{1}{2}y+\frac{1}{2}k=0$$
$$\frac{1}{2}y=\frac{-1}{2}k$$
$$y=-k$$

$$x_1=\begin{bmatrix} y \\ z \end{bmatrix}$$
$$=\begin{bmatrix} -k \\ k \end{bmatrix}$$
$$x_1=-k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, the Eigen vector is $x_1=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step-7

The Eigen vectors corresponding to the Eigen value of the matrix is $\left[P-\lambda_2 I\right]x_2=0$.

$\lambda_2=1$ and the Eigen vector $x_2=\begin{bmatrix} y \\ z \end{bmatrix}$.

$$\begin{bmatrix} \frac{1}{2}-1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}-1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the row operation $R_2 \to R_2+R_1$.

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-\frac{1}{2}y+\frac{1}{2}z=0$$
$$\frac{1}{2}y=\frac{1}{2}z$$
$$y=z$$

Step-8

Choose $z = k$ is an arbitrary constant.

Then, $y = k$

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} k \\ k \end{bmatrix}$$
$$= k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence, the Eigen vector $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step-9

The Solutions corresponding to these Eigen vectors are as follows:

$$u_1(t) = e^{\lambda_1 t} x_1$$
$$= e^{0 \cdot t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u_2(t) = e^{\lambda_2 t} x_2$$
$$= e^{1 \cdot t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If $u_1(t)$ and $u_2(t)$ satisfy the linear differential equation $\frac{du}{dt} = Pu$, so will their sum $u_1(t) + u_2(t)$.

So, complete solution is $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

Here, c_1 and c_2 are constants and can be chosen in a manner that it satisfies the initial condition $u = u(0)$ at $t = 0$.

Step-10

At $t=0$, $u(t)$ becomes:

$$c_1x_1+c_2x_2=u(0)$$
$$\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}5\\ 3\end{bmatrix}$$

$$\begin{aligned}c_1+c_2&=5\\ -c_1+c_2&=3\end{aligned}$$

Solving these two equations to get c_1 and c_2.

Adding the above two equations are,

$$\begin{aligned}2c_2&=8\\ c_2&=4\end{aligned}$$

Substitute $c_2=4$ in the equation $c_1+c_2=5$.

$$\begin{aligned}c_1+4&=5\\ c_1&=1\end{aligned}$$

Thus, the constants are, $c_1=1$ and $c_2=4$.

Step-11

Substitute $c_1=1$ and $c_2=4$ and the Eigen vectors in the complete solution $u(t)$.

$$\begin{aligned}u(t)&=c_1e^{\lambda_1 t}x_1+c_2e^{\lambda_2 t}x_2\\ u(t)&=1e^{0\cdot t}\begin{bmatrix}1\\ -1\end{bmatrix}+4e^{1\cdot t}\begin{bmatrix}1\\ 1\end{bmatrix}\\ &=1\cdot\begin{bmatrix}1\\ -1\end{bmatrix}+4e^{t}\begin{bmatrix}1\\ 1\end{bmatrix}\end{aligned}$$

Hence, the solution to the original equation is $\boxed{u(t)=1\begin{bmatrix}1\\ -1\end{bmatrix}+4e^{t}\begin{bmatrix}1\\ 1\end{bmatrix}}$.