Step-1

Consider

$$[U \vdots 0] = \begin{bmatrix} 1 & 2 & 3 & \vdots & 0 \\ 0 & 0 & 4 & \vdots & 0 \end{bmatrix} \text{and}$$

$$[U \vdots c] = \begin{bmatrix} 1 & 2 & 3 \vdots 5 \\ 0 & 0 & 4 \vdots 2 \end{bmatrix}$$

To apply Gaussian-Jordan elimination to

$$Ux = 0 \text{and}$$
$$Ux = c$$

To reach

$$Rx = 0 \text{and}$$
$$Rx = d$$

To solve $Rx = 0$ to find x_n and also solve $Rx = d$ to find x_p

Step-2

$$[U \vdots 0] = \begin{bmatrix} 1 & 2 & 3 & \vdots & 0 \\ 0 & 0 & 4 & \vdots & 0 \end{bmatrix}$$

$$\underline{\frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & 3 \vdots 0 \\ 0 & 0 & 1 \vdots 0 \end{bmatrix}$$

$$\underline{R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 \vdots 0 \\ 0 & 0 & 1 \vdots 0 \end{bmatrix}$$

First and third columns are pivot columns, second columns is free column.

Therefore x_1, x_2 are pivot variable, x_3 is free variable.

This is converted to $Rx = 0$.

Step-3

Now,

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1+2x_2&=0 \\ x_1&=-2x_2 \end{aligned}$$

And $x_3=0$

Step-4

Therefore the solution x_n for $Ux=0$ is;

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix}=x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}$$

Step-5

Let ,

$$[U \vdots c]=\begin{bmatrix} 1 & 2 & 3 \vdots 5 \\ 0 & 0 & 4 \vdots 2 \end{bmatrix}$$

$$\underline{\tfrac{1}{4}R_2}\begin{bmatrix} 1 & 2 & 3 \vdots 5 \\ 0 & 0 & 1 \vdots 2 \end{bmatrix}$$

$$\underline{R_1-3R_2}\begin{bmatrix} 1 & 2 & 0 \vdots -1 \\ 0 & 0 & 1 \vdots 2 \end{bmatrix}$$

First and third columns are pivots, second column is free column.

Step-6

Now, $Rx=d$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x_1+2x_2&=-1 \\ x_3&=2 \end{aligned}$$

Step-7

This implies,

$$x_1 = -1 - 2x_2$$
$$x_3 = 2$$

Therefore the solution x_p of the system is;

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1-2x_2 \\ x_2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$