Step-1

We have to explain $\|ABx\| \le \|A\|\|B\|\|x\|$ and also deduce that $\|AB\| \le \|A\|\|B\|$

We know that the norm of A is the number $\|A\|$ is $\|A\|^2 = \max_{x \ne 0} \dfrac{\|Ax\|^2}{\|x\|^2}$ â€¦â€¦ (1)

From (1), we can write $\|A\|^2 \ge \dfrac{\|Ax\|^2}{\|x\|^2}, x \ne 0$

Since norm is a non negative quantity, we get $\|A\| \ge \dfrac{\|Ax\|}{\|x\|}, x \ne 0$

In other words, $\|Ax\| \le \|A\|\|x\|$ â€¦â€¦ (2)

Step-2

Let x be any nonzero column vector.

Now

$\|ABx\| = \|A(Bx)\|$

$\le \|A\|\|Bx\|$

$\le \|A\|\|B\|\|x\| \quad (\text{Since by } (2))$

Therefore, $\|ABx\| \le \|A\|\|B\|\|x\|$

Step-3

We have $\|ABx\| \le \|A\|\|B\|\|x\|$

$\Rightarrow \dfrac{\|ABx\|}{\|x\|} \le \|A\|\|B\|$ for every non zero column vector x

$\Rightarrow \max_{x \ne 0} \dfrac{\|(AB)x\|}{\|x\|} \le \|A\|\|B\|$

$\Rightarrow \|AB\| \le \|A\|\|B\| \quad (\text{Since by } (1))$

Hence $\boxed{\|AB\| \le \|A\|\|B\|}$