

Step-1

(a).

Consider the matrix $A = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}$.

The objective is to decide the positive definiteness of these matrices and write out the corresponding $f = x^T Ax$

Step-2

Now the corresponding $f = x^T Ax$ is,

$$\begin{aligned} f &= x^T Ax \\ &= (x \ y) \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x + 3y \quad 3x + 5y) \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x + 3y)x + (3x + 5y)y \\ &= x^2 + 3xy + 3xy + 5y^2 \\ &= x^2 + 6xy + 5y^2 \end{aligned}$$

Step-3

Compare quadratic expression $f = x^2 + 6xy + 5y^2$ with $f = ax^2 + 2bxy + cy^2$

So, $a = 1$, $2b = 6 \Rightarrow b = 3$, $c = 5$.

Now $a > 0$ and $ac - b^2 = (1)(5) - (3)^2 = -4 < 0$.

This implies that $a > 0$ and $ac < b^2$

Notice that if $ax^2 + 2bxy + cy^2$ stays positive, then it necessary $ac > b^2$

Hence, here f is not positive definite.

Step-4

(b).

Consider the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Now the corresponding $f = x^T Ax$ is,

$$\begin{aligned} f &= x^T Ax \\ &= (x \ y) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x - y \ -x + y) \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (x - y)x + (-x + y)y \\ &= x^2 - xy - xy + y^2 \\ &= x^2 - 2xy + y^2 \end{aligned}$$

Step-5

Compare the quadratic expression $f = x^2 - 2xy + y^2$ with $f = ax^2 + 2bxy + cy^2$

We get $a = 1, 2b = -2 \Rightarrow b = -1, c = 1$.

Now $a > 0$ and $ac - b^2 = 1 - (-1)^2 = 0$.

Notice that $a > 0$ and $ac = b^2$ then it is **positive semidefinite**.

Therefore, f is positive semi definite.

Step-6

(c).

Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$

The corresponding $f = x^T Ax$ is,

$$\begin{aligned} f &= x^T Ax \\ &= (x \ y) \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= (2x+3y \quad 3x+5y) \begin{pmatrix} x \\ y \end{pmatrix} \\
&= (2x+3y)x + (3x+5y)y \\
&= 2x^2 + 3xy + 3xy + 5y^2 \\
&= 2x^2 + 6xy + 5y^2
\end{aligned}$$

Step-7

Compare the quadratic expression $f = 2x^2 + 6xy + 5y^2$ with $f = ax^2 + 2bxy + cy^2$

We get $a = 2$, $2b = 6$, $c = 5$.

Now $a > 0$ and $ac - b^2 = 1 > 0$.

Notice that if $ax^2 + 2bxy + cy^2$ stays positive, then it necessary $ac > b^2$

Therefore, f is positive definite.

Step-8

(d).

Consider the matrix $A = \begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix}$

The corresponding $f = x^T Ax$ is,

$$\begin{aligned}
f &= x^T Ax \\
&= (x \quad y) \begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= (-x+2y \quad 2x-8y) \begin{pmatrix} x \\ y \end{pmatrix} \\
&= (-x+2y)x + (2x-8y)y \\
&= -x^2 + 2xy + 2xy - 8y^2 \\
&= -x^2 + 4xy - 8y^2
\end{aligned}$$

Step-9

Compare the quadratic expression $-x^2 + 4xy - 8y^2$ with $f = ax^2 + 2bxy + cy^2$

We get $a = -1$, $2b = 4 \Rightarrow b = 2$ and $c = -8$.

Now $a < 0$ and $ac - b^2 = 4 > 0$

This implies that $a < 0$ and $ac > b^2$

The quadratic form is negative definite if and only if $a < 0$ and $ac > b^2$

Therefore, **f is negative definite.**