Step-1

Given that B is a square matrix. To show that $A = B + B^T$ is always symmetric and $K = B - B^T$ is always skew-symmetric. Also have to find A, K when $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ and have to write B as the sum of a symmetric and skew-symmetric matrix.

Step-2

Let, B be an $n \times n$ square matrix.

Then $B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n2} & b_{n2} & \dots & b_{nn} \end{bmatrix}$.

Now, $B^T = \begin{bmatrix} b_{11} & b_{21} & \dots & b_{n1} \\ b_{12} & b_{22} & \dots & b_{n2} \\ \dots & \dots & \dots & \dots \\ b_{1n} & b_{2n} & \dots & b_{nn} \end{bmatrix}$.

So, $A = B + B^T = \begin{bmatrix} 2b_{11} & b_{12} + b_{21} & \dots & b_{1n} + b_{n1} \\ b_{21} + b_{12} & 2b_{22} & \dots & b_{n2} + b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} + b_{1n} & b_{n2} + b_{2n} & \dots & 2b_{nn} \end{bmatrix}$.

So, ij-th term of $A = B + B^T$ is $b_{ij} + b_{ji}$.

And, ji-th term of $A = B + B^T$ is $b_{ji} + b_{ij}$.

And $b_{ij} + b_{ji} = b_{ji} + b_{ij}$.

So, ij-th term of $A = ji$-th term of A. So, $A = A^T$.

Hence $A = B + B^T$ is always symmetric.

Again, $K = B - B^T = \begin{bmatrix} 0 & b_{12} - b_{21} & \dots & b_{1n} - b_{n1} \\ b_{21} - b_{12} & 0 & \dots & b_{n2} - b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} - b_{1n} & b_{n2} - b_{2n} & \dots & 0 \end{bmatrix}$.

So, ij-th term of $K = B - B^T$ is $b_{ij} - b_{ji}$.

And, ji-th term of $K = B - B^T$ is $b_{ji} - b_{ij}$.

And $b_{ij} - b_{ji} = -\left(b_{ji} - b_{ij}\right)$

So, ij-th term of K = -(ji-th term of K). So, $K = -K^T$.

Hence, $K = B - B^T$ is always skew-symmetric.

Step-3

Now, if $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ then $B^T = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$.

Then $A = B + B^T = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ and $K = B - B^T = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

And $\frac{1}{2}A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\frac{1}{2}K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

And $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.