Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #2

2023/03/05

Name: ______________________________

Student Number: ______________________________

1. Let $V = \{A \in \mathbf{R}^{n\times n} : A \text{ is symmetric } \}$. It's obvious that V is a subspace of $\mathbf{R}^{n\times n}$. Try to find another subspace of $\mathbf{R}^{n\times n}$, denoted as W such that $\mathbf{R}^{n\times n} = V \oplus W$.

   Proof It's easy to find that $A \in V$ if and only if $A^T = A$. Let $W = \{A \in \mathbf{R}^{n\times n} : A^T = -A\}$. It's obvious that W is a subspace of $\mathbf{R}^{n\times n}$. So $V + W \subset \mathbf{R}^{n\times n}$.

   $\forall A \in \mathbf{R}^{n\times n}$, we have
   $$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}, \tag{1}$$
   where $\frac{A+A^T}{2} \in V$ and $\frac{A-A^T}{2} \in W$. Hence $\mathbf{R}^{n\times n} \subset V + W$.

   Finally, we will prove $V + W$ is a direct sum. $\forall B \in V \cap W$, since $B \in V$, we have
   $$B^T = B. \tag{2}$$
   And $B \in W$, we can get
   $$B^T = -B. \tag{3}$$
   Combine with (2) and (3), we imply $B = 0$. Therefore, $V \cap W = \{0\}$, i.e., $V + W$ is a direct sum, which has finished the proof. $\square$

2. Let $U = \{A \in \mathbf{R}^{2\times 2} : AB = BA, \forall B \in \mathbf{R}^{2\times 2}\}$. U is a subspace of $\mathbf{R}^{2\times 2}$, try to compute its dimension.

Solution $\forall A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U$, take $B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. According to the definition of U, we have

$$\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = AB_1 = B_1A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}. \tag{4}$$

So $b = c = 0$. Similarly, take $B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we have

$$\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = AB_2 = B_2A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}. \tag{5}$$

Therefore, we can get $a = d$, further more $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and

$$U \subset \{A \in \mathbf{R}^{2\times 2} : A = aI,\ a \in \mathbf{R}\}. \tag{6}$$

On the other hand, it's obvious that

$$\{A \in \mathbf{R}^{2\times 2} : A = aI,\ a \in \mathbf{R}\} \subset U. \tag{7}$$

Hence, $U = \{A \in \mathbf{R}^{2\times 2} : A = aI,\ a \in \mathbf{R}\}$ and I is a basis of U. So we can get

$$\dim\ U = 1. \tag{8}$$

□