Step-1

The matrix A is of size 5 by 4 and rank 4

That means we can make the 5th row completely zero using the elementary row operations on A.in other words, 5th row is linearly dependent of the previous rows. â€¦â€¦ (1)

On the other hand, the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ of the system of non homogeneous equations

$Ax = b$ is of order 5 by 5 and is invertible.

We have the following facts.

(i) A matrix is invertible if and only if it is non singular

(ii) A matrix is non singular if and only if the determinant is not zero.

(iii) The determinant of a matrix is not zero if and only if its rank = size

Step-2

Now, The size of $\begin{bmatrix} A & b \end{bmatrix}$is 5 by 5 and so, rank $\begin{bmatrix} A & b \end{bmatrix}$is 5.

That means by elementary row operations we cannot make any row completely zero.

So, the 5th entry of the column b in the matrix $\begin{bmatrix} A & b \end{bmatrix}$cannot be made zero while the 5th row in A is already made zero by (1).

If we write non homogeneous equation using this row, we get

$0x_1 + 0x_2 + 0x_3 + 0x_4 = k$ where k is non zero.

$\Rightarrow 0 = k$ and k is non zero.

This is an absurdity.

Step-3

In view of the entire discussion, we must have rank A = rank $\begin{bmatrix} A & b \end{bmatrix}$ = 4

While $\begin{bmatrix} A & b \end{bmatrix}$is of size 5, its rank is 4 confirms that $\begin{bmatrix} A & b \end{bmatrix}$is a singular matrix.

Thus, $Ax = b$ is solvable only when $\begin{bmatrix} A & b \end{bmatrix}$is a singular matrix.