Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #8

2023/04/13

Name: ____________________

Student Number: ____________________

1. Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \text{null } T \oplus \text{range } T$.

   假设 $T \in \mathcal{L}(V)$ 是可对角化的, 证明 $V = \text{null } T \oplus \text{range } T$.

   Proof. Since T is diagonalizable, then n linearly independent eigenvectors of T can be a basis V.

   If 0 is not an eigenvalue of T, $Tv_i = \lambda_i v_i$, $\lambda_i \neq 0 \Rightarrow v_i = \frac{Tv_i}{\lambda_i} \in$ range T, so $V = \text{range } T \oplus \{0\} = \text{range } T \oplus \text{null } T$.

   If 0 is an eigenvalue of T, $v_1, \cdots, v_k$ are eigenvectors of 0, null $T = span(v_1, \cdots, v_k)$, $\dim \text{span } (v_{k+1}, \cdots, v_n) = \dim \text{range } T$ and if $Tv_i = \lambda_i v_i$, $i \geqslant k+1, \lambda_i \neq 0$, $v_i = \frac{Tv_i}{\lambda_i} \in \text{range } T \Rightarrow \text{span } (v_{k+1}, \cdots, v_n) = \text{range } T$. And since $V = \text{span } (v_1, \cdots, v_k) \oplus \text{span } (v_{k+1}, \cdots, v_n) \Rightarrow V = \text{null } T \oplus \text{range } T$. □

2. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, and $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

   设 V 是有限维向量空间，$T \in \mathcal{L}(V)$ 有 $\dim V$ 个互异特征值，$S \in \mathcal{L}(V)$ 和 T 有相同的特征向量 (特征值不一定相同). 证明 $ST = TS$.

   Proof. Assume $\dim V = n$, $\lambda_1, \cdots, \lambda_n$ be n distinct eigenvalues of T and $\xi_1, \cdots, \xi_n$ be the corresponding eigenvectors of T. And let $S\xi_i = \mu_i \xi_i$, $i = 1, \cdots, n$,

$$ST\xi_i = \lambda_i S\xi_i = \lambda_i \mu_i \xi_i, \quad TS\xi_i = \mu_i T\xi_i = \mu_i \lambda_i \xi_i$$

   Since $\xi_1, \cdots, \xi_n$ is a basis of V, we have $\forall v \in V$, $STv = TSv$, so $ST = TS$.

   □