Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #2

2023/03/02

Name: ______________________________

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1. Let W be a subspace of a vector space V over $\mathbf{F}$, $\dim V = n < \infty$, $0 < \dim W < \dim V$. Show that there exist infinite subspaces U such that $V = U \oplus W$.

   Proof Let $\dim W = r$, $\xi_1, \xi_2, ..., \xi_r$ be a basis of W, expand it to a basis of V: $\xi_1, ..., \xi_r, \xi_{r+1}, ..., \xi_n$.

   子空间U_k的构造

   Let $U_k = \text{span}\,\{k\xi_1 + \xi_{r+1}, \xi_{r+2}, ..., \xi_n\}$, $k = 1, 2, ...$ Obviously, U_k is a subspace of V, $U_k \cap W = \{\mathbf{0}\}$, $\dim U_k = n - r$. $\Rightarrow \dim U_k \oplus W = n$. $\Rightarrow V = U_k \oplus W$.

   Next we will prove if $k \neq s$, then $U_k \neq U_s$ by contradiction. Assume $k \neq s$, but $U_k = U_s$. $\Rightarrow k\xi_1 + \xi_{r+1} \in \text{span}\,\{s\xi_1 + \xi_{r+1}, \xi_{r+2}, ..., \xi_n\}$.

   $\Rightarrow \exists\, l_{r+1}, l_{r+2}, ..., l_n$, s.t. $k\xi_1 + \xi_{r+1} = l_{r+1}(s\xi_1 + \xi_{r+1}) + l_{r+2}\xi_{r+2} + ... + l_n\xi n$.

   $\Rightarrow (sl_{r+1} - k)\xi_1 + (l_{r+1} - 1)\xi_{r+1} + l_{r+2}\xi_{r+2} + ... + l_n\xi_n = 0$.

   Since $\xi_1, \xi_2, ..., \xi_n$ are linearly independent, we have $sl_{r+1} - k = 0, l_{r+1} - 1 = 0$. $\Rightarrow l_{r+1} = 1, s = k$, which is contradict with $s \neq k$.

   All in all, $U_1, U_2, ...$ are infinite subspaces such that $V = U_k \oplus W$. $\square$

2. Prove the following set is a subspace of $\mathbf{R}^3$ and compute the dimension of W:

$$W = \{(x, y, z) \in \mathbf{R}^3 : x + 2y + 3z = 0, 4x + 5y + 6z = 0, x + y + z = 0\}$$

Solution It's easy to find that $\mathbf{0} = (0, 0, 0) \in W$. $\forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in W$, we can check $(x_1, y_1, z_1) + (x_2, y_2, z_2) \in W$ and $a(x_1, y_1, z_1) \in W$ holds for any $a \in \mathbf{R}$. So W is a subspace of $\mathbf{R}^3$.

Ones can discover that

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ x + y + z = 0 \end{cases} \iff \begin{cases} \boxed{\begin{array}{l} x + y + z = 0 \\ y + 2z = 0 \end{array}} \end{cases}. \tag{1}$$

Therefore, we have $(x, y, z) \in W \iff x = z, y = -2z, z \in \mathbf{R}$. In other words, $\forall (x, y, z) \in W, (x, y, z) = z(1, -2, 1)$, i.e., $W = \text{span}\{(1, -2, 1)\}$. $(1, -2, 1)$ is certainly linearly independent, so $(1, -2, 1)$ is a basis of W, which shows that $\dim W = 1$. $\square$