Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #9

2023/04/20

Name: ______________________________

Student Number: ______________________________

1. Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x)dx$$

$\boxed{\varphi(p) = p(\tfrac{1}{2})} = \langle p, q\rangle$

$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

求 $q \in \mathcal{P}_2(\mathbf{R})$ 使得

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x)dx$$

对任意的 $p \in \mathcal{P}_2(\mathbf{R})$ 都成立.

$q = \overline{\varphi(e_1)}e_1 + \overline{\varphi(e_2)}e_2 + \overline{\varphi(e_3)}e_3$

Proof. Let $p_1 = 1$,$p_2 = 2\sqrt{3}x - \sqrt{3}$, $p_3 = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$ be an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$. Define $\varphi : \mathcal{P}_2(\mathbf{R}) \to \mathbf{R}$, $\forall p \in \mathcal{P}_2(\mathbf{R})$, $\varphi(p) = p\left(\frac{1}{2}\right)$. It's easy to check $\varphi \in (\mathcal{P}_2(\mathbf{R}))'$. By 6.43, we have

$$q = \overline{\varphi(p_1)}p_1 + \overline{\varphi(p_2)}p_2 + \overline{\varphi(p_3)}p_3 = 1 + 0 + \left(-\frac{\sqrt{5}}{2}\right)(6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}) = -15x^2 + 15x - \frac{3}{2}.$$

$\square$

2. Let $\mathbf{R}^{n\times n}$ be a vector space over $\mathbf{R}$. Define $\langle\,\cdot\,,\cdot\,\rangle : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, $\forall A, B \in \mathbf{R}^{n\times n}$,

$$\langle A, B\rangle = \mathrm{Tr}(B^T A)$$

Show that $\langle\,\cdot\,,\cdot\,\rangle$is an inner product on $\mathbf{R}^{n\times n}$.

设 $\mathbf{R}^{n\times n}$ 是 $\mathbf{R}$ 上的向量空间. 定义 $\langle\,\cdot\,,\cdot\,\rangle : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, 对任意的 $A, B \in \mathbf{R}^{n\times n}$,

$$\langle A, B\rangle = \mathrm{Tr}(B^T A)$$

证明 $\langle\,\cdot\,,\cdot\,\rangle$ 是 $\mathbf{R}^{n\times n}$ 上的内积.

Proof. **positivity**

$$\langle A, A\rangle = \mathrm{Tr}(A^T A) = \sum_{i,j=1}^{n} a_{ij}^2 \geqslant 0, \quad \forall A = (a_{ij})_{n\times n} \in \mathbf{R}^{n\times n}.$$

definiteness

$$\langle A, A\rangle = 0 \Leftrightarrow \sum_{i,j=1}^{n} a_{ij}^2 = 0 \Leftrightarrow a_{ij} = 0,\ \forall i, j = 1, \cdots n \Leftrightarrow A = 0.$$

additivity in first slot

$$\langle A+B, C\rangle = \mathrm{Tr}((A+B)^T C) = \mathrm{Tr}(A^T C + B^T C) = \mathrm{Tr}(A^T C) + \mathrm{Tr}(B^T C) = \langle A, C\rangle + \langle B, C\rangle,\ \forall A, B, C \in \mathbf{R}^{n\times n}.$$

homogeneity in first slot

$$\langle \lambda A, B\rangle = \mathrm{Tr}((\lambda A)^T B) = \mathrm{Tr}(\lambda A^T B) = \lambda \mathrm{Tr}(A^T B) = \lambda\langle A, B\rangle,\ \forall \lambda \in \mathbf{R},\ \forall A, B \in \mathbf{R}^{n\times n}.$$

conjugate symmetry

$$\langle A, B\rangle = \mathrm{Tr}(A^T B) = \mathrm{Tr}((A^T B)^T) = \mathrm{Tr}(B^T A) = \langle B, A\rangle,\ \forall A, B \in \mathbf{R}^{n\times n}.$$

Since $\langle B, A\rangle \in \mathbf{R}$, $\overline{\langle B, A\rangle} = \langle B, A\rangle$, so $\langle A, B\rangle = \overline{\langle B, A\rangle}$.

Hence, $\langle\,\cdot\,,\cdot\,\rangle$is an inner product on $\mathbf{R}^{n\times n}$.

$\square$