Step-1

Given that $A=\begin{pmatrix}1 & 0\\ 9 & 3\end{pmatrix}$

We have $A=\begin{pmatrix}1 & 0\\ 9 & 3\end{pmatrix}$

$$\Rightarrow A^T=\begin{pmatrix}1 & 9\\ 0 & 3\end{pmatrix}$$

We know that $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

$$\Rightarrow A^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

$$A=\begin{pmatrix}1 & 0\\ 9 & 3\end{pmatrix}$$

$$\Rightarrow A^{-1}=\frac{1}{3-0}\begin{pmatrix}3 & 0\\ -9 & 1\end{pmatrix}$$

$$\Rightarrow A^{-1}=\begin{pmatrix}1 & 0\\ -3 & \frac{1}{3}\end{pmatrix}$$

$$\Rightarrow \left(A^{-1}\right)^T=\boxed{\begin{pmatrix}1 & -3\\ 0 & \frac{1}{3}\end{pmatrix}}$$

Step-2

We have

$$A^T=\begin{pmatrix}1 & 9\\ 0 & 3\end{pmatrix}$$

$$\Rightarrow \left(A^T\right)^{-1}=\frac{1}{3-0}\begin{pmatrix}3 & -9\\ 0 & 1\end{pmatrix}$$

$$\Rightarrow \left(A^T\right)^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$= \left(A^{-1}\right)^T$$

Therefore $\boxed{\left(A^T\right)^{-1} = \left(A^{-1}\right)^T}$

Step-3

$$A = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{0-c^2}\begin{pmatrix} 0 & -c \\ -c & 1 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{c^2}\begin{pmatrix} 0 & c \\ c & -1 \end{pmatrix}$$

$$\Rightarrow \left(A^{-1}\right)^T = \boxed{\frac{1}{c^2}\begin{pmatrix} 0 & c \\ c & -1 \end{pmatrix}}$$

Step-4

We have

$$A = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix}$$

$$\Rightarrow A^T = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix}$$

$$\Rightarrow \left(A^T\right)^{-1} = \frac{1}{0-c^2}\begin{pmatrix} 0 & -c \\ -c & 1 \end{pmatrix}$$

$$\Rightarrow \left(A^T\right)^{-1} = \frac{1}{c^2}\begin{pmatrix} 0 & c \\ c & -1 \end{pmatrix}$$

$$= \left(A^{-1}\right)^T$$

Therefore $\boxed{\left(A^T\right)^{-1} = \left(A^{-1}\right)^T}$