Step-1

We have to prove that the condition number $\|A\|\|A^{-1}\|$ is atleast 1.

Step-2

Def: 1: *conditional number* of a matrix A is $c = \|A\|\|A^{-1}\|$

Def: 2: $\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

Observation: 1: $Ax = \lambda x$ where λ is the eigenvalue and x is the corresponding eigenvector.

Consequently, $\|A\| = \max_{x \neq 0} \frac{\|\lambda x\|}{\|x\|}$ by def.2

While λ is a scalar, we get $\|A\| = \max_{x \neq 0} \frac{|\lambda|\|x\|}{\|x\|}$

$= \lambda_{\max}(A)$

Step-3

Observation: 2: $\|A\|^2 = \max \frac{\|Ax\|^2}{\|x\|^2}$

$= \max \frac{x^T A^T A x}{x^T x}$

$= \lambda_{\max}(A^T A)$ (By the observation 1)

Observation: 3: $\|A^{-1}\|^2 = \frac{1}{\lambda_{\min}(A^T A)}$

Step-4

Using definition 1 and observations 2, 3, we get $c^2 = \|A\|^2 \|A^{-1}\|^2$

$$= \frac{\lambda_{\max}\left(A^T A\right)}{\lambda_{\min}\left(A^T A\right)}$$

While the numerator is greater than the denominator, the fraction is greater than or equal to 1. Also, we are considering the non negative values of the eigenvalues from the observation 1.

So, its square root is greater than or equal to 1.

That is ***conditional number*** c is greater than or equal to 1.

Further, if $\lambda_{\min}\left(A^T A\right)$ is 0, then the c is infinite and the singular value and obviously greater than 1.

If $\lambda_{\max}\left(A^T A\right)$ is zero, then A is a zero matrix which is not our case.

Hence the condition number is atleast 1.