- self-adjoint $T=T^*$ $\longleftrightarrow$ $\bar{z}=z$
- normal $TT^*=T^*T$
- positive $\begin{cases} \text{self-adjoint} \\ \langle Tv,v\rangle \geq 0 \text{ for all } v\in V \end{cases}$ $\longleftrightarrow$ nonnegative

Positive Operators and Isometries (正算子和等距同构)

- $\|Sv\|=\|v\|$ $\longleftrightarrow$ unit circle
  isometry

Lecture 21

Dept. of Math., SUSTech

2022.04

Operators on Inner Product Spaces

Positive Operators

7.31 Definition ***positive operator***

An operator $T \in \mathcal{L}(V)$ is called ***positive*** if T is self-adjoint and

$$\langle Tv, v\rangle \geq 0$$

for all $v \in V$.

If V is a complex vector space, then the requirement that T is self-adjoint can be dropped from the definition above (by 7.15).

$\langle Tv, v\rangle \in \mathbb{R} \Rightarrow$ self-adjoint

7.32 Example ***positive operators***

U subspace of V $\quad v, w \in V$

$P_U \in \mathcal{L}(V)$. positive operator

- $\langle P_U(v), v\rangle = \langle v_1, v_1 + v_2\rangle = \langle v_1, v_1\rangle \geq 0$

(a) If U is a subspace of V, then the orthogonal projection P_U is a positive operator, as you should verify.

$v = \underset{\in U}{v_1} + \underset{\in U^\perp}{v_2}$

$w = \underset{\in U}{w_1} + \underset{\in U^\perp}{w_2}$

- $\langle P_U(v), w\rangle = \langle v_1, w_1 + w_2\rangle = \langle v_1, w_1\rangle = \langle v_1 + v_2, w_1\rangle = \langle v, P_U(w)\rangle \Rightarrow P_U$ self-adjoint

(b) If $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbf{R}$ are such that $b^2 < 4c$, then $T^2 + bT + cI$ is a positive operator, as shown by the proof of 7.26.

Square Root

7.33 Definition ***square root***

An operator R is called a ***square root*** of an operator T if $R^2 = T$.

7.34 Example If $T \in \mathcal{L}(\mathbf{F}^3)$ is defined by $T(z_1, z_2, z_3) = (z_3, 0, 0)$, then the operator $R \in \mathcal{L}(\mathbf{F}^3)$ defined by $R(z_1, z_2, z_3) = (z_2, z_3, 0)$ is a square root of T.

Characterizations of the Positive Operators

1. The characterizations of the positive operators in the next result correspond to characterizations of the nonnegative numbers among $\mathbb{C}$.
2. Specifically, a complex number z is nonnegative if and only if it has a nonnegative square root, corresponding to condition (c).
3. Also, z is nonnegative if and only if it has a real square root, corresponding to condition (d).
4. Finally, z is nonnegative if and only if there exists a complex number w such that $z = \bar{w}w$, corresponding to condition (e).

Characterizations of positive operators

7.35 Characterization of positive operators

Let $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is positive;

$(a) \Rightarrow (b).\ 0 \leq \langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda \langle v, v\rangle$

(b) T is self-adjoint and all the eigenvalues of T are nonnegative;

$(b) \Rightarrow (c)$ T self-adjoint. all eigenvalues ≥ 0

$\Rightarrow T = R^2$, R positive V complex $\to$ 7.24, real $\to$ 7.29

(c) T has a positive square root;

$T \in \mathcal{L}(V)$.

$T(e_1, \cdots, e_n) = (e_1, \cdots, e_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

(d) T has a self-adjoint square root;

$Te_1 = \lambda_1 e_1, \cdots, Te_n = \lambda_n e_n$

$(c) \Rightarrow (d)$

$Re_j = \sqrt{\lambda_j}\, e_j \quad j = 1, \cdots, n \qquad \lambda_1, \cdots, \lambda_n \geq 0$

(e) there exists an operator $R \in \mathcal{L}(V)$ such that $T = R^* R$.

- R positive: self adjoint, $\langle Rv, v\rangle \geq 0$
- $R^2 = T$

Proof.

We will prove that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a)$. □

Each positive operator has only one positive square root

Each nonnegative number has a unique nonnegative square root. The next result shows that positive operators enjoy a similar property.

$T \in \mathcal{L}(V)$ positive $R^2 = T$. R positive (unique)

7.36 Each positive operator has only one positive square root

Every positive operator on V has a $\boxed{\text{unique}}$ positive square root.

T self-adjoint

7.24 $\Downarrow$ 7.29

$v \in V$ eigenvector of T R square root of T $v_1, v_2, \cdots, v_n$ orthonormal basis

$Tv = \lambda v$, $\lambda \geq 0$. $Rv = \sqrt{\lambda} v$. (positive) consisting of eigenvectors of T

$\Downarrow$

Proof. Suppose $T \in \mathscr{L}(V)$ is positive. Suppose $v \in V$ is an eigenvector of T. Thus there exists $\lambda \geq 0$ such that $Tv = \lambda v$.

$\Rightarrow Tv_1 = \lambda_1 v_1, \cdots, Tv_n = \lambda_n v_n$

$\Rightarrow Rv_1 = \sqrt{\lambda_1} v_1, \cdots, Rv_n = \sqrt{\lambda_n} v_n$

Let R be a positive square root of T. We will prove that $Rv = \sqrt{\lambda} v$. This will imply that the behavior of R on the eigenvectors of T is uniquely determined. Because there is a basis of V consisting of eigenvectors of T (by the Spectral Theorem), this will imply that R is uniquely determined.

Proof

- To prove that $Rv = \sqrt{\lambda}v$, note that the Spectral Theorem asserts that there is an orthonormal basis $e_1, e_2, \cdots, e_n$ of V consisting of eigenvectors of R. Because R is a positive operator, all its eigenvalues are nonnegative. Thus there exist nonnegative numbers $\lambda_1, \cdots, \lambda_n$ such that

$$\underline{Re_j = \sqrt{\lambda_j}e_j}$$

for $j = 1, \cdots, n$.

- Because $e_1, e_2, \cdots, e_n$ is an orthonormal basis of V, we can write

$$\underline{v = a_1e_1 + \cdots + a_ne_n}$$

for some numbers $a_1, \cdots, a_n \in \mathbb{F}$. Thus

$$Rv = a_1\sqrt{\lambda_1}e_1 + \cdots + a_n\sqrt{\lambda_n}e_n$$

and hence

$$Tv = R^2v = a_1\lambda_1e_1 + \cdots + a_n\lambda_ne_n. = \lambda v$$
$$= \lambda(a_1e_1 + \cdots + a_ne_n)$$

Proof

But $R^2 = T$, and $Tv = \lambda v$. Thus the equation above implies

$$a_1\lambda e_1 + \cdots + a_n\lambda e_n = a_1\lambda_1 e_1 + \cdots + a_n\lambda_n e_n.$$

The equation above implies that $a_j(\lambda - \lambda_j) = 0$ for $j = 1, 2, \cdots, n$. Hence

$$v = \sum_{\{j:\lambda_j=\lambda\}} a_j e_j,$$

and thus

$$Rv = \sum_{\{j:\lambda_j=\lambda\}} a_j\sqrt{\lambda}e_j = \sqrt{\lambda}v, \text{ as desired.}$$

Remarks

1. Some mathematicians also use the term positive semidefinite operator, which means the same as positive operator.
2. A positive operator can have infinitely many square roots(although only one of them can be positive). For example, the identity operator on V has infinitely many roots if $\dim V > 1$.

Isometries

Operators that preserve norms are sufficiently important to deserve a name:

7.37 Definition ***isometry***

- An operator $S \in \mathcal{L}(V)$ is called an ***isometry*** if
$$\|Sv\| = \|v\|$$
for all $v \in V$.
- In other words, an operator is an isometry if it preserves norms.

For example, λI is an isometry whenever $\lambda \in \mathbb{F}$ satisfies $|\lambda| = 1$. We will see soon that if $\mathbb{F} = \mathbb{C}$, then the next example includes all isometries.

$\lambda_1, \cdots, \lambda_n.\quad \|\lambda_j\| = 1.$

7.38 Example Suppose $\lambda_1, \ldots, \lambda_n$ are scalars with absolute value 1 and $S \in \mathcal{L}(V)$ satisfies $Se_j = \lambda_j e_j$ for some orthonormal basis $e_1, \ldots, e_n$ of V. Show that S is an isometry.

$v = <v, e_1>e_1 + \cdots + <v, e_n>e_n$

$Sv = <v, e_1> Se_1 + \cdots + <v, e_n> Se_n$

$= \lambda_1 <v, e_1> e_1 + \cdots + \lambda_n <v, e_n> e_n$

Solution Suppose $v \in V$. Then

7.39 $$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$ 6.30

and

7.40 $$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2,$$

where we have used 6.30. Applying S to both sides of 7.39 gives

$$\begin{aligned} Sv &= \langle v, e_1 \rangle Se_1 + \cdots + \langle v, e_n \rangle Se_n \\ &= \lambda_1 \langle v, e_1 \rangle e_1 + \cdots + \lambda_n \langle v, e_n \rangle e_n. \end{aligned}$$

The last equation, along with the equation $|\lambda_j| = 1$, shows that

7.41 $$\|Sv\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2.$$

Comparing 7.40 and 7.41 shows that $\|v\| = \|Sv\|$. In other words, S is an isometry.

Several Remarks

1. The Greek word isos means equal; the Greek word metron means measure. Thus isometry literally means equal measure.
2. The next result provides several conditions that are equivalent to being an isometry. The equivalence of (a) and (b) shows that an operator is an isometry if and only if it preserves inner products. The equivalence of (a) and (c) [or (d)] shows that an operator is an isometry if and only if the list of columns of its matrix with respect to every [or some] basis is orthonormal. Exercise 10 implies that in the previous sentence we can replace "columns " with " rows".

Characterization of Isometries

7.42 Characterization of isometries

Suppose $S \in \mathcal{L}(V)$. Then the following are equivalent:

(a) S is an isometry;

(b) $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$;

(c) $Se_1, \dots, Se_n$ is orthonormal for every orthonormal list of vectors $e_1, \dots, e_n$ in V;

(d) there exists an orthonormal basis $e_1, \dots, e_n$ of V such that $Se_1, \dots, Se_n$ is orthonormal;

(e) $S^*S = I$;

(f) $SS^* = I$;

(g) S^* is an isometry;

(h) S is invertible and $S^{-1} = S^*$.

Proof and Remarks

Proof.

$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (a)$. $\square$

Remarks:

1. An isometry on a real inner product space is often called an orthogonal operator.
2. An isometry on a complex inner product space is often called a unitary operator.
3. We use the term isometry so that our results can apply to both real and complex inner product spaces.

Description of isometries when $\mathbb{F} = \mathbb{C}$

The previous result shows that every isometry is normal[see (a), (c), and (f) of 7.42]. Thus characterizations of normal operators can be used to give descriptions of isometries. We do this in the next result in the complex case and in Chapter 9 in the real case (see 9.36).

7.43 Description of isometries when $\mathbf{F} = \mathbf{C}$

Suppose V is a complex inner product space and $S \in \mathcal{L}(V)$. Then the following are equivalent:

(a) S is an isometry.

(b) There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.

Proof

Proof We have already shown (see Example 7.38) that (b) implies (a).

To prove the other direction, suppose (a) holds, so S is an isometry. By the Complex Spectral Theorem (7.24), there is an orthonormal basis $e_1, \dots, e_n$ of V consisting of eigenvectors of S. For $j \in \{1, \dots, n\}$, let λ_j be the eigenvalue corresponding to e_j. Then

$$|\lambda_j| = \|\lambda_j e_j\| = \|S e_j\| = \|e_j\| = 1.$$

Thus each eigenvalue of S has absolute value 1, completing the proof. ■

Homework Assignment 21

7.C: ~~1, 2~~, 3, 7, 8, 9, 10, 11, 13, 14.