## Step-1

By using Gram-Schmidt, we have to construct an orthonormal pair $q_1$ and $q_2$ from $a_1=(4,5,2,2)$ and $a_2=(1,2,0,0)$. We have to express $a_1$ and $a_2$ as linear combination of $q_1$ and $q_2$, and we have to find the triangular $R$ in $A=QR$

## Step-2

Let $A=\begin{bmatrix} a_1 & a_2 \end{bmatrix}$ where $a_1=\begin{bmatrix}4\\5\\2\\2\end{bmatrix}, a_2=\begin{bmatrix}1\\2\\0\\0\end{bmatrix}$

$$\begin{aligned}\|a_1\| &= \sqrt{4^2+5^2+2^2+2^2}\\ &=\sqrt{49}\\ &=7\end{aligned}$$

## Step-3

$$\begin{aligned}q_1 &= \frac{a_1}{\|a_1\|}\\ &=\frac{1}{7}\begin{bmatrix}4\\5\\2\\2\end{bmatrix}\\ &=\begin{bmatrix}4/7\\5/7\\2/7\\2/7\end{bmatrix}\end{aligned}$$

## Step-4

$$q_2 = \frac{\beta}{\|\beta\|} \text{ where } \beta = a_2 - \left(q_1^T a_2\right) q_1$$

$$q_1^T a_2 = \begin{bmatrix} \frac{4}{7} & \frac{5}{7} & \frac{2}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{4+10+0+0}{7}$$

$$= 2$$

**Step-5**

$$\left(q_1^T a_2\right) q_1 = 2 \begin{bmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{bmatrix}$$

$$= \begin{bmatrix} 8/7 \\ 10/7 \\ 4/7 \\ 4/7 \end{bmatrix}$$

**Step-6**

$$\beta = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 8/7 \\ 10/7 \\ 4/7 \\ 4/7 \end{bmatrix}$$

$$= \begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix}$$

**Step-7**

$$\|\beta\| = \sqrt{\frac{1}{49} + \frac{16}{49} + \frac{16}{49} + \frac{16}{49}}$$

$$= 1$$

Therefore $q_2 = \begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix}$

Hence an orthonormal basis for the column space of $A$ $= \left\{ \begin{bmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{bmatrix}, \begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix} \right\}$

$$a_1 = \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}$$
$$= 7\begin{bmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{bmatrix} + 0\begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix}$$
$$= \boxed{7q_1 + 0q_2}$$

Therefore $a_1$ is a linear combination of $q_1$ and $q_2$

## Step-8

$$a_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$
$$= 2\begin{bmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{bmatrix} + 1\begin{bmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{bmatrix}$$
$$= \boxed{2q_1 + 1q_2}$$

Therefore $a_2$ is a linear combination of $q_1$ and $q_2$

## Step-9

$$q_1^T a_1 = \begin{bmatrix} \frac{4}{7} & \frac{5}{7} & \frac{2}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}$$

$$= \frac{16+25+4+4}{7}$$

$$= 7$$

# Step-10

$$q_1^T a_2 = \begin{bmatrix} \frac{4}{7} & \frac{5}{7} & \frac{2}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{4+10+0+0}{7}$$

$$= 2$$

# Step-11

$$q_2^T a_2 = \begin{bmatrix} \frac{-1}{7} & \frac{4}{7} & \frac{-4}{7} & \frac{-4}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{-1+8+0+0}{7}$$

$$= 1$$

# Step-12

Therefore

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

$$= \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4/7 & -1/7 \\ 5/7 & 4/7 \\ 2/7 & -4/7 \\ 2/7 & -4/7 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 0 & 1 \end{bmatrix} = QR}$$