| 题 号 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 分 值 | 15 分 | 25 分 | 15 分 | 15 分 | 10 分 | 12 分 | 8 分 |

本试卷共 ( 7 ) 大题, 满分 ( 100 ) 分. **请将所有答案写在答题本上.**

This exam includes **7** questions and the score is 100 in total. **Write all your answers on the examination book.**

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

   (共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

   (1) Let $A$ be an $m \times n$ matrix and suppose $Ax = 0$ has a nonzero solution. Which of the following must be true? ( C )

   (A) The row vectors of $A$ are linearly dependent.

   (B) The column vectors of $A$ are linearly ~~independent~~. dependent

   (C) The rank of $A$ is $< n$.

   (D) $m = n$ and $\det(A) = 0$.

   设 $A$ 为 $m \times n$ 矩阵. 假设 $Ax = 0$ 有非零解. 下列哪一项一定是正确的? ( )

   (A) $A$ 的行向量线性相关.

   (B) $A$ 的列向量线性无关.

   (C) $A$ 的秩 $< n$.

   (D) $m = n$ 且 $\det(A) = 0$.

   (2) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and let $\alpha_1, \alpha_2, \alpha_3$ be linearly independent column vectors in $\mathbb{R}^3$. Then the rank of the vector system $A\alpha_1, A\alpha_2, A\alpha_3$ ( B )

   rank(A)=2

   (A) must be 1.

   (B) must be 2.

   (C) must be 3.

   (D) can be 1 or 2.

   设 $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $\alpha_1, \alpha_2, \alpha_3$ 为 $\mathbb{R}^3$ 中线性无关的向量组. 则向量组 $A\alpha_1, A\alpha_2, A\alpha_3$ 的秩 ( )

   (A) 一定是 1.

   (B) 一定是 2.

   (C) 一定是 3.

   (D) 可能是 1 也可能是 2.

(3) Let $A$ and $P$ be square matrices of order 3 with $P$ invertible. Suppose $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \cancel{2}\,0 \\ 0 & 0 & 2 \end{bmatrix}$. If $P = (\alpha_1, \alpha_2, \alpha_3)$ and $Q = (\alpha_1 + \alpha_2, \alpha_2, \alpha_3)$, then $Q^{-1}AQ =$ ( B )

$$Q = (\alpha_1\ \alpha_2\ \alpha_3)\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= P\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = PM$$

$$Q^{-1}AQ = (PM)^{-1}APM$$

$$= M^{-1}(P^{-1}AP)M$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

设 $A$ 和 $P$ 为 3 阶方阵, $P$ 可逆. 假设 $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. 若 $P = (\alpha_1, \alpha_2, \alpha_3)$, $Q = (\alpha_1 + \alpha_2, \alpha_2, \alpha_3)$, 则 $Q^{-1}AQ =$ (　　)

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(4) Let $A$ and $B$ be real symmetric matrices of order $n$. Suppose $A$ and $B$ are congruent. Then ( A )

(A) The null spaces $N(A)$ and $N(B)$ have the same dimension

(B) $A$ and $B$ have the same eigenvalues

(C) $A$ and $B$ have the same column space

(D) $A$ and $B$ have the same determinant

设 $A$ 与 $B$ 均为 $n$ 阶实对称矩阵. 假设 $A$ 与 $B$ 合同 (也称相合). 则 (　　)

(A) 零空间 $N(A)$ 与 $N(B)$ 有相同的维数

(B) $A$ 与 $B$ 有相同的特征值

(C) $A$ 与 $B$ 有相同的列空间

(D) $A$ 与 $B$ 有相同的行列式

(5) Let $Q$ be a real orthogonal matrix of order 3. Which of the following is false? ( D )

(A) For every real symmetric matrix $A$ of order 3, $Q^{-1}AQ$ is symmetric.

(B) For every column vector $v \in \mathbb{R}^3$, the vectors $Qv$ and $v$ have the same length.

(C) There is a nonzero column vector $v \in \mathbb{R}^3$ such that $Qv = v$ or $Qv = -v$.

(D) There is an invertible real matrix $P$ of order 3 such that $P^{-1}QP$ is diagonal.

设 $Q$ 为 3 阶实正交矩阵. 下列哪一项论断是错误的? (　　)

(A) 对任何 3 阶实对称阵 $A$, $Q^{-1}AQ$ 仍为对称阵.

(B) 对任何列向量 $v \in \mathbb{R}^3$, 向量 $Qv$ 和 $v$ 的长度相同.

(C) 存在非零列向量 $v \in \mathbb{R}^3$ 使得 $Qv = v$ 或 $Qv = -v$.

(D) 存在 3 阶可逆实矩阵 $P$ 使得 $P^{-1}QP$ 为对角阵. P不一定为实矩阵

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let $A, B, C$ and $D$ be square matrices of order $n$. Suppose $A$ is invertible. Find two square matrices $X, Y$ such that $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix} \begin{bmatrix} A & B \\ 0 & Y \end{bmatrix}$. (We denote by $I_n$ the identity matrix of order $n$.)

Answer: $X =$ $CA^{-1}$, $Y =$ $D - CA^{-1}B$

设 $A, B, C, D$ 均为 $n$ 阶方阵. 假设 $A$ 可逆. 写出两个方阵 $X, Y$ 使得 $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & Y \end{bmatrix}$. (我们用 $I_n$ 表示 $n$ 阶单位矩阵.)

答案: $X =$ ________, $Y =$ ________.

$2^3 \times \frac{1}{4}$

(2) Let $A$ be a $3 \times 3$ matrix with determinant $|A| = 4$. Then $|2A^{-1}| =$ 2

设 $A$ 为 $3 \times 3$ 矩阵, 行列式 $|A| = 4$. 则 $|2A^{-1}| =$ ________

(3) Let $A$ be a $3 \times 3$ matrix. Suppose that the sum of the diagonal entries of $A$ is $-5$, and $A^2 + 2A - 3I = 0$, then the three eigenvalues of $A$ are $-3, -3, 1$

设 $A$ 是 3 阶矩阵. 假设 $A$ 的主对角线元素之和为 $-5$, 且满足 $A^2 + 2A - 3I = 0$. 则矩阵 $A$ 的三个特征值是 ________.

$(A+3I)(A-I)=0$

(4) Let $L \subseteq \mathbb{R}^3$ be the line through the vector $\beta = (1, -2, 2)^T$ (and the origin). Then the projection of the vector $\alpha = (1, 0, -1)^T$ onto the line $L$ is $(-1/9, 2/9, -2/9)^T$

设 $L \subseteq \mathbb{R}^3$ 为经过 (原点和) 向量 $\beta = (1, -2, 2)^T$ 的直线. 则向量 $\alpha = (1, 0, -1)^T$ 在直线 $L$ 上的投影是 ________

$$\frac{\beta\beta^T}{\beta^T\beta}\alpha$$

$$\frac{1}{9}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(5) Suppose that the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & a & b \\ 0 & 2 & 3 \end{bmatrix}$ is similar to the matrix $B = \begin{bmatrix} 3 & & \\ & 4 & \\ & & -1 \end{bmatrix}$.

Then $b =$ 2.

假设矩阵 $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & a & b \\ 0 & 2 & 3 \end{bmatrix}$ 和 $B = \begin{bmatrix} 3 & & \\ & 4 & \\ & & -1 \end{bmatrix}$ 相似. 则 $b =$ ________.

3. (15 points) Let $V = \mathbf{M}_2(\mathbb{R})$ be the space of real square matrices of order 2. Let $T$ be the linear transformation

$$T : V \longrightarrow V; \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} 0 & c \\ b & a \end{bmatrix}.$$

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find the matrix $A$ of $T$ in the ordered basis $v_1, v_2, v_3, v_4$, where

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

(b) Is $T$ invertible? Why? No

(c) Investigate whether the matrix $A$ is diagonalizable. No

(15 分) 设 $V = \mathbf{M}_2(\mathbb{R})$ 为 2 阶实方阵构成的向量空间. 令 $T$ 表示如下线性变换

$$T : V \longrightarrow V; \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} 0 & c \\ b & a \end{bmatrix}.$$

(a) 求 $T$ 在有序基 $v_1, v_2, v_3, v_4$ 下的矩阵 $A$, 其中

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

(b) $T$ 是否是可逆的? 为什么?

(c) 判定矩阵 $A$ 是否可对角化.

$$\Lambda = \begin{bmatrix} -2 & & \\ & 4 & \\ & & 1 \end{bmatrix}$$

4. (15 points) Let $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix}$.

Indefinite

(a) Decide whether $A$ is positive (or negative) definite, or positive (or negative) semidefinite.

(b) Find an orthogonal matrix $Q$ such that $Q^{-1}AQ$ is a diagonal matrix. $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$

(c) Let $S$ be the surface in $\mathbb{R}^3$ defined by the equation $2x^2 - 4xy + y^2 - 4yz + 1 = 0$.

Which of the following graphs best illustrates the shape of the surface $S$ (when the coordinate axes are suitably chosen)? (A), (B) or (C)?

C

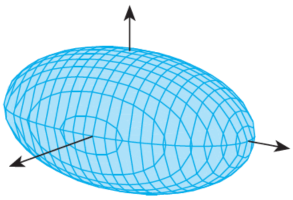
$$A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} -2 & & \\ & 4 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

(15 分) 设 $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix}$.

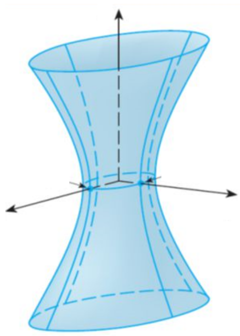
(a) 判定 $A$ 是否正定或负定、是否半正定或半负定.

(b) 找出一个正交矩阵 $Q$ 使 $Q^{-1}AQ$ 为对角阵.

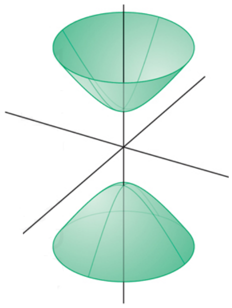
(c) 设 $S$ 为 $\mathbb{R}^3$ 中由方程 $2x^2 - 4xy + y^2 - 4yz + 1 = 0$ 定义的曲面.
(当坐标轴适当选取时) 以下那个图最适合描述曲面 $S$ 的形状？(A), (B) 还是 (C)？



(A) An ellipsoid
椭球面



(B) A hyperboloid of one sheet
单叶双曲面



(C) A hyperboloid of two sheets
双叶双曲面

5. (10 points) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$. $A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(a) Find all the singular values of $A$. $\sqrt{2}, \sqrt{3}$

(b) Find the singular value decomposition of $A$. That is, find two orthogonal matrices $U$ and $V$ (of suitable size) such that $A = U\Sigma V^T$.

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T$$

(10 分) 令 $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) 求 $A$ 的所有奇异值.

(b) 求 $A$ 的奇异值分解. 即, 找出两个 (适当大小的) 正交矩阵 $U$ 和 $V$ 使得 $A = U\Sigma V^T$.

6. (12 points) Let $A$ be an $m \times n$ complex matrix and set $B = A^HA$ (where $A^H = \overline{A}^T$ denotes the conjugate transpose of $A$).

$B^H=B$

(a) Prove that the eigenvalues of $B$ in $\mathbb{C}$ are all real numbers. $\lambda=\bar{\lambda} \Rightarrow$ real number

$\text{rank}(B) \leq m < n$

(b) Suppose $m < n$. Show that 0 is an eigenvalue of $B$. $Ax=0$ ✓ $\Rightarrow A^H A x = 0 \Rightarrow Bx=0$ ✓

(c) Suppose $m = n > 1$. Is it possible that $-1$ is an eigenvalue of $B$? If yes, write down explicitly a matrix $A$ with this property and justify your answer. Otherwise explain why such a phenomenon is impossible.

$A^H A x = \lambda x \Leftrightarrow x^H A^H A x = \lambda x^H x$

(12 分) 设 $A$ 为 $m \times n$ 复矩阵, $B = A^H A$ (其中 $A^H = \overline{A}^T$ 表示 $A$ 的共轭转置).

$\|Ax\| = \lambda \|x\|$

$\lambda \geq 0$

(a) 证明 $B$ 在 $\mathbb{C}$ 中的特征值都是实数.

(b) 假设 $m < n$. 证明 0 是 $B$ 的一个特征值.

(c) 假设 $m = n > 1$. 是否有可能 $-1$ 是 $B$ 的一个特征值？若是, 请具体写出一个满足此条件的矩阵 $A$ 并且解释你给的答案为何满足要求. 若否, 请解释为何此现象不可能出现.

7. (8 points) Let $A$ be a real (symmetric) positive definite matrix of order $n$ and let $\alpha_1, \cdots, \alpha_n$ be column nonzero vectors in $\mathbb{R}^n$ such that for all distinct indices $i, j \in \{1, 2, \ldots, n\}$, $\alpha_i^T A \alpha_j = 0$. Prove that the vectors $\alpha_1, \cdots, \alpha_n$ are linearly independent.

(8 分) 设 $A$ 为 $n$ 阶实 (对称) 正定矩阵. 设 $\alpha_1, \cdots, \alpha_n$ 为 $\mathbb{R}^n$ 中的非零列向量. 假设对任意不同的指标 $i, j \in \{1, 2, \ldots, n\}$ 均有 $\alpha_i^T A \alpha_j = 0$.

证明向量组 $\alpha_1, \cdots, \alpha_n$ 是线性无关的. Suppose $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$

$0 = (c_1\alpha_1 + \cdots + c_n\alpha_n)^T A (c_1\alpha_1 + \cdots + c_n\alpha_n)$

$= c_1^2 \alpha_1^T A \alpha_1 + c_2^2 \alpha_2^T A \alpha_2 + \cdots + c_n^2 \alpha_n^T A \alpha_n$

$\alpha_i^T A \alpha_i > 0 \Rightarrow c_i = 0 \Rightarrow$ linearly independent