Step-1

Here, $A = 3\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Let us obtain *M*. We have, $M_{ij} = \int V_i V_j dx$.

Therefore,

$$M_{12} = \int_0^1 V_1 V_2 dx$$

Now, it should be clear that $V_1 V_2 = 0$, when $0 \le x \le \frac{1}{3}$ or $\frac{2}{3} \le x \le 1$.

Further note that, when $\frac{1}{3} \le x \le \frac{2}{3}$, we have

$$\begin{aligned} V_1 V_2 &= (3x-1)(2-3x) \\ &= 9x - 2 - 9x^2 \end{aligned}$$

Step-2

Therefore,

$$\begin{aligned} \int_0^1 V_1 V_2 dx &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left(9x - 2 - 9x^2\right) dx \\ &= 9\left[\frac{x^2}{2}\right]_{\frac{1}{3}}^{\frac{2}{3}} - 2[x]_{\frac{1}{3}}^{\frac{2}{3}} - 9\left[\frac{x^3}{3}\right]_{\frac{1}{3}}^{\frac{2}{3}} \\ &= \frac{9}{2}\left(\frac{1}{3}\right) - 2\left(\frac{1}{3}\right) - \frac{9}{3}\left(\frac{7}{27}\right) \\ &= \frac{1}{18} \end{aligned}$$

Step-3

Thus, $M_{12} = \frac{1}{18}$.

It should be clear that $M_{21} = \frac{1}{18}$.

Step-4

Now obtain M_{11}. Note that $M_{11} = M_{22}$.

$$M_{11} = \int_0^1 (V_1)^2 \, dx$$

Consider

$$\begin{aligned}\int_0^{\frac{1}{3}} (V_1)^2 \, dx &= \int_0^{\frac{1}{3}} 9x^2 dx \\ &= 9\left[\frac{x^3}{3}\right]_0^{\frac{1}{3}} \\ &= 3\left[\frac{1}{27}\right] \\ &= \frac{1}{9}\end{aligned}$$

Similarly, $\int_{\frac{1}{3}}^{\frac{2}{3}} (V_1)^2 \, dx = \frac{1}{9}$, by symmetry.

Therefore, $M_{11} = \frac{2}{9}$ and hence, $M_{22} = \frac{2}{9}$.

Therefore,

$$M = \begin{bmatrix} \frac{2}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{2}{9} \end{bmatrix}$$

Step-5

Consider the equation $Ax = \lambda Mx$

This is same as $(A - \lambda M)x = 0$.

Therefore, $\det(A - \lambda M) = 0$

Step-6

Thus, we get

$$0=\begin{vmatrix} 6-\frac{2\lambda}{9} & -3-\frac{\lambda}{18} \\ -3-\frac{\lambda}{18} & 6-\frac{2\lambda}{9} \end{vmatrix}$$

$$=\left(6-\frac{2\lambda}{9}\right)^2-\left(3+\frac{\lambda}{18}\right)^2$$

$$=36-\frac{24\lambda}{9}+\frac{4\lambda^2}{81}-\left(9+\frac{6\lambda}{18}+\frac{\lambda^2}{324}\right)$$

$$0=27-\frac{54\lambda}{18}+\frac{15\lambda^2}{324}$$

$$=\frac{5\lambda^2}{108}-3\lambda+27$$

Step-7

Therefore,

$$\lambda=\frac{9\pm\sqrt{9-5}}{\frac{5}{54}}$$

$$=\frac{54}{5}(9\pm 2)$$

$$=\frac{378}{5}\text{ or }\frac{594}{5}$$

Step-8

Therefore, the eigenvalues are $\boxed{\frac{378}{5}}$ and $\boxed{\frac{594}{5}}$.