

Step-1

Let p_1 is the projection of b onto the line through $a_1 = \frac{a_1^T b}{a_1^T a_1} a_1$ (1)

$$\begin{aligned} a_1^T b &= \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \\ &= 0 + 2 + 0 \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_1^T a_1 &= \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix} \\ &= \frac{4}{9} + \frac{4}{9} + \frac{1}{9} \\ &= 1 \end{aligned}$$

$$p_1 = \frac{2}{1} \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix}$$

Use these in (1), to get

$$= \begin{pmatrix} 4/3 \\ 4/3 \\ -2/3 \end{pmatrix}$$

Step-2

Let p_2 is the projection of b onto the line through $a_2 = \frac{a_2^T b}{a_2^T a_2} a_2$ (2)

$$\begin{aligned} a_2^T b &= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \\ &= 0 + 2 + 0 \\ &= 2 \end{aligned}$$

$$\begin{aligned}
 a_2^T a_2 &= \begin{pmatrix} -1 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix} \\
 &= \frac{1}{9} + \frac{4}{9} + \frac{4}{9} \\
 &= 1
 \end{aligned}$$

Use these in (2), and get;

$$\begin{aligned}
 P_2 &= \frac{2}{1} \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix} \\
 &= \begin{pmatrix} -2/3 \\ 4/3 \\ 4/3 \end{pmatrix}
 \end{aligned}$$

Step-3

Let P_3 is the projection of b onto the line through $a_3 = \frac{a_3^T b}{a_3^T a_3} a_3 \in \mathbb{R}^3$

$$\begin{aligned}
 a_3^T b &= \begin{pmatrix} 2 & -1 & 2 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \\
 &= 0 - 1 + 0 \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 a_3^T a_3 &= \begin{pmatrix} 2 & -1 & 2 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix} \\
 &= \frac{4}{9} + \frac{1}{9} + \frac{4}{9} \\
 &= 1
 \end{aligned}$$

Use these in (3), and get;

$$\begin{aligned}
 P_3 &= \frac{-1}{1} \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix} \\
 &= \begin{pmatrix} -2/3 \\ 1/3 \\ -2/3 \end{pmatrix}
 \end{aligned}$$

Step-4

Using all the results above, and get;

$$\begin{aligned} p &= p_1 + p_2 + p_3 \\ &= \left(\frac{4}{3}, \frac{4}{3}, \frac{-2}{3} \right) + \left(\frac{-2}{3}, \frac{4}{3}, \frac{4}{3} \right) + \left(\frac{-2}{3}, \frac{1}{3}, \frac{-2}{3} \right) \\ &= \left(\frac{4-2-2}{3}, \frac{4+4+1}{3}, \frac{-2+4-2}{3} \right) \\ &= (0, 3, 0) \\ &= b \end{aligned}$$

Step-5

Further, consider;

$$\begin{aligned} a_1 a_1^T &= \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix} \begin{pmatrix} 2 & 2 & -1 \\ 3 & 3 & 3 \end{pmatrix} \\ &= \begin{bmatrix} 4/9 & 4/9 & -2/9 \\ 4/9 & 4/9 & -2/9 \\ -2/9 & -2/9 & 1/9 \end{bmatrix} \\ a_2 a_2^T &= \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix} \begin{pmatrix} -1 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \\ &= \begin{bmatrix} 1/9 & -2/9 & -2/9 \\ -2/9 & 4/9 & 4/9 \\ -2/9 & 4/9 & 4/9 \end{bmatrix} \\ a_3 a_3^T &= \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ 3 & 3 & 3 \end{pmatrix} \\ &= \begin{bmatrix} 4/9 & -2/9 & 4/9 \\ -2/9 & 1/9 & -2/9 \\ 4/9 & -2/9 & 4/9 \end{bmatrix} \end{aligned}$$

Step-6

$$\begin{aligned}
a_1 a_1^T + a_2 a_2^T + a_3 a_3^T &= \begin{bmatrix} 4/9 & 4/9 & -2/9 \\ 4/9 & 4/9 & -2/9 \\ -2/9 & -2/9 & 1/9 \end{bmatrix} + \begin{bmatrix} 1/9 & -2/9 & -2/9 \\ -2/9 & 4/9 & 4/9 \\ -2/9 & 4/9 & 4/9 \end{bmatrix} + \begin{bmatrix} 4/9 & -2/9 & 4/9 \\ -2/9 & 1/9 & -2/9 \\ 4/9 & -2/9 & 4/9 \end{bmatrix} \\
&= \begin{bmatrix} \frac{4+1+4}{9} & \frac{4-2-2}{9} & \frac{-2-2+4}{9} \\ \frac{4-2-2}{9} & \frac{4+4+1}{9} & \frac{-2+4-2}{9} \\ \frac{-2-2+4}{9} & \frac{-2+4-2}{9} & \frac{1+4+4}{9} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Follow;

$$\begin{aligned}
P &= a_1 a_1^T + a_2 a_2^T + a_3 a_3^T \\
&= I
\end{aligned}$$

while $\{a_1, a_2, a_3\}$ form an orthonormal basis.