Dec. 28, 2021 Dr. Y. Chen

Practice Problems Set 1: Answers.

Question 1: (1) False (2) True (3) True (4) True (5) False.

Question 2: (1) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, (2) $T(x,y) = (x+y, -\frac{x}{2}+\frac{y}{2})$, (3) I, (4) 2, (5) $2\sqrt{2}, 1$.

Question 3: (a) $a_3 = 4$, $a_4 = 5$.

(b) $a_n = 2a_{n-1} - a_{n-2}$.

(c) $a_n = n+1$.

Question 4: $y = \frac{1}{2}x + 2$.

Question 5: (a) $\begin{bmatrix} -1 & -2 & -3 \\ -2 & -5 & -4 \\ -3 & -4 & -9 \end{bmatrix}$ (b) No (c) Yes.

Question 6: (a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (b) 2, 2 (c) No. $M^TMv = 0 \Longleftrightarrow Mv = 0$.

Question 7:

$$0 = (c_1\alpha_1 + \cdots + c_n\alpha_n)^T A (c_1\alpha_1 + \cdots + c_n\alpha_n)$$

$$= c_1^2 + c_2^2 + \cdots + c_n^2$$

$$\Rightarrow c_1 = \cdots = c_n = 0.$$

Question 8:

(a) $Av = \lambda v \quad (v \neq 0)$

$\Rightarrow \lambda v = Av = A^2 v = A(\lambda v) = \lambda^2 v \Rightarrow \lambda = \lambda^2 \Rightarrow \lambda = 0$ or $\lambda = 1$

(b) Must have an eigen value of 0.

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Question 9: (a) $Av = \lambda v \Rightarrow Bv = f(\lambda)v \; (= f(A)v)$.

(b) B is symmetric.

Question 10: (a) v^Tv, 0.

(b) $n-1$.

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Practice Problems Set 2. Answers

Question 1: (a) True (b) True (c) False (d) True (e) False (f) False

Question 2: (a) 2. (b) 0, 0, 0. (c) 20.

Question 3: (i). eigenvalues : $1, i, i^2, i^3$.

eigenvectors : $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}, \begin{bmatrix} 1 \\ i^2 \\ i^4 \\ i^6 \end{bmatrix}, \begin{bmatrix} 1 \\ i^3 \\ i^6 \\ i^9 \end{bmatrix}$.

(ii). Diagonalizable. The four eigenvalues are distinct.

Question 4: (i). $A = A^H$.

(ii). $U = \begin{bmatrix} \frac{3-i}{\sqrt{14}} & \frac{3-i}{\sqrt{35}} \\ \frac{2}{\sqrt{14}} & \frac{5}{\sqrt{35}} \end{bmatrix}$, $U^H A U = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}$

Question 5: (i) $\sigma_1 = \sqrt{3}$, $\sigma_2 = 1$.

(ii) $A = U \Sigma V^T = \begin{bmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Question 6: (i) $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$, $Q^T A Q = \Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(ii) $A^k = \begin{bmatrix} 2^{k-1} & 0 & -2^{k-1} \\ 0 & 2^k & 0 \\ -2^{k-1} & 0 & 2^k \end{bmatrix}$

Question 7: (i) $A = \begin{bmatrix} t & 1 & 1 & 0 \\ 1 & t & -1 & 0 \\ 1 & -1 & t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(ii) $t > 2$.

Question 8: (i) $\|Nx\|^2 = (Nx)^H Nx = x^H N^H N x = x^H N N^H x = \|N^H x\|^2$.

(ii) Let x be $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to i^{th}$ component.

(iii)
$$N = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \cdots & \cdots & & \\ 0 & 0 & \cdots & t_{nn} \end{bmatrix}$$

$$N^H = \begin{bmatrix} \bar{t}_{11} & 0 & \cdots & 0 \\ \bar{t}_{12} & \bar{t}_{22} & \cdots & 0 \\ \cdots & \cdots & & \\ \bar{t}_{1n} & \bar{t}_{2n} & \cdots & \bar{t}_{nn} \end{bmatrix}$$

Since $N^H N = N N^H$, Comparing the diagonal entries of both sides, we obtain $t_{ij} = 0$ for $i \neq j$.

Question 9: (i) $A = Q \Lambda Q^T$, diagonal entries of Λ are positive (A is positive definite)

$$|A + I_n| = |Q\Lambda Q^T + Q I Q^T| = |Q|\,|\Lambda + I|\,|Q^T| = |\Lambda + I| > 1.$$

(ii) $A = U \Sigma V^T$, A is $n \times n$.

$$A^T A = (U\Sigma V^T)^T (U \Sigma V^T) = V \Sigma^T \Sigma V^T \Leftrightarrow V^T A^T A V = \Sigma^T \Sigma$$
$$\shortparallel$$
$$A A^T = (U \Sigma V^T)(U\Sigma V^T)^T = U \Sigma \Sigma^T U^T \Leftrightarrow U^T A A^T U = \Sigma \Sigma^T$$

$\Rightarrow A^T A$ is similar to AA^T.

Question 10: (i) $Ax = \lambda x$, $x \neq 0$ $\Rightarrow A^k x = \lambda^k x$ $\quad A^k = 0 \Rightarrow \lambda^k = 0$ $\Rightarrow \lambda = 0$

(ii) A symmetric $\Rightarrow A = Q\Lambda Q^T$, $A^k = Q \underbrace{\Lambda^k}_{=0} Q^T \Rightarrow A = O$ zero matrix.

Question 11: (i) $b > \alpha^T A^{-1} \alpha$.

(ii) $\left\{ \begin{bmatrix} -A^{-1}\alpha \\ 1 \end{bmatrix} \right\}$.

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Practice Problems Set 3 Answers

Question 1: (1) C (2) B (3) B (4) A (5) D

Question 2: (1) CA^{-1}, $D - CA^{-1}B$.

(2) 2.

(3) 1, −3, −3.

(4) $\left[-\frac{1}{9}\ \frac{2}{9}\ -\frac{2}{9}\right]^T$

(5) 2.

Question 3: (a) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{bmatrix}$ (b) No (c) No.

Question 4: (a) Indefinite (b) $\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ (c) C.

Question 5: (a) $\sqrt{3}, \sqrt{2}$ (b) $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{4} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{4} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Question 6: Suppose $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$, then

$$\alpha^T A \alpha = \sum_{i=1}^{n} c_i^2 \|\alpha_i\|^2 \lambda_i = 0 \Rightarrow c_1 = \cdots = c_n = 0.$$

Therefore, $\alpha_1, \cdots, \alpha_n$ are linearly independent.

Question 7: (a) $Bv = \lambda v$, $v^H B v = \lambda \|v\|^2$. On the other hand,

$v^H B v = (Bv)^H v = \bar{\lambda} v^H v = \bar{\lambda}\|v\|^2 \Rightarrow \lambda = \bar{\lambda}$.

(b) $\operatorname{rank}(B) \le m < n$.

(c) No. Consider the SVD of A, $A = U\Sigma V^T$, then $B = A^H A$
$= V\Sigma^2 V^H$.

~~Question 8:~~

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Practice Problems Set 4 Answers

Question 1: (1) A (2) C (3) C (4) B (5) A

Question 2: (1) 3, 2 (2) 2 (3) 1, $1-u^Tv$ (4) r (5) 6.

Question 3: (a) $q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $q_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $q_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

(b) $$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & \frac{5\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

Question 4: $|A| = a^n - a^{n-2}$.

Question 5: (a) $a = 5$, $b = 6$.

(b) $S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$.

Question 6: (a) $3\sqrt{2}$, $\sqrt{2}$.

(b) $$A = U\Sigma V^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Question 7: (a) $$\begin{bmatrix} I_n & 0 \\ -BA^{-1} & I_m \end{bmatrix} \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} I_n & -A^{-1}B^T \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}.$$

(b) $\operatorname{rank}(B) = r$, $-BA^{-1}B^T$ is negative definite

(c) n, r, $m-r$.

Question 8: (a) $A = P^TP$, P is invertible

$B = Q^TQ$. $AB = P^TPQ^TQ = P^TPQ^TQP^T(P^T)^{-1}$

AB is similar to PQ^TQP^T, PQ^TQP^T is positive semidefinite $\Rightarrow$ AB is positive semidefinite.

(b) $C^TAC = I_n$, $C^TAB(C^T)^{-1} = C^TACC^{-1}B(C^T)^{-1}$, $Q^TMQ = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow Q^T\underbrace{C^TAC}_{I_n}\underbrace{C^{-1}B(C^T)^{-1}}_{M}Q = Q^TMQ = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}.$$ Therefore, AB is diagonalizable.

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