Step-1

Consider the following first order differential equations:

$$\frac{dv}{dt} = w - v$$
$$\frac{dw}{dt} = v - w$$

To show that $v + w$ is constant. Also, write the equations in the following form:

$$\frac{du}{dt} = Au$$

Find the matrix A its Eigen values and Eigen vectors.

Step-2

Initial condition:

$$v(0) = 30$$
$$w(0) = 10$$

Find the values of v and w at $t = 1$.

Step-3

To show that $v + w$ is constant, do the following calculations:

$$\begin{aligned}\frac{d(v+w)}{dt} &= \frac{dv}{dt} + \frac{dw}{dt} \\ &= w - v + v - w \\ &= 0\end{aligned}$$

This shows that $v + w$ is constant and its value is:

$$\begin{aligned}v + w &= 30 + 10 \\ &= \boxed{40}\end{aligned}$$

Step-4

Above differential equations can be written as follows:

$$\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}$$

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Matrix $u(t)$ is: $u(t) = (v, w)$

Therefore, A is defined as follows:

$$\boxed{A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}}$$

Step-5

First step is to find the Eigen values and Eigen vectors of matrix A. To calculate the Eigen values do the following calculations;

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} \\ \det(A - \lambda I) &= 0 \\ (-1-\lambda)(-1-\lambda) - 1 &= 0 \\ \lambda^2 + 2\lambda &= 0 \end{aligned}$$

After solving following values are obtained:

$$\begin{aligned} \lambda_1 &= -2 \\ \lambda_2 &= 0 \end{aligned}$$

Step-6

Therefore, Eigen values are $\boxed{-2, 0}$

Step-7

To calculate Eigen vectors do the following calculations:

$$\begin{aligned} (A - \lambda I)x &= 0 \\ \begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1+2 & 1 \\ 1 & -1+2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Step-8

On solving, values of y and z corresponding to $\lambda = -2$ are as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step-9

Similarly, Eigen vectors corresponding to Eigen value $\lambda = 0$ is as follows:

$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1-0 & 1 \\ 1 & -1-0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of y and z are as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore Eigen values are:

$$\boxed{\begin{array}{l} x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}}$$

Step-10

Recall that: $e^{At} = Se^{\Lambda t}S^{-1}$

Here, Eigen value matrix is given as follows:

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore, the general solution of the differential equation is:

$$\begin{aligned} u(t) &= c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

Here, c_1 and c_2 are constants. Their values are determined by the following values:

$$c = S^{-1} u(0)$$

Step-11

So, the solution for differential equation can be written as follows:

$$\begin{aligned} u(t) &= S e^{\Lambda t} S^{-1} u(0) \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 30 \\ 10 \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} & 1 \\ -e^{-2t} & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} 10e^{-2t} + 20 \\ -10e^{-2t} + 20 \end{bmatrix}$$

Step-12

Therefore, specific solution of the differential equation is:

$$\begin{aligned} v(t) &= 10e^{-2t} + 20 \\ w(t) &= -10e^{-2t} + 20 \end{aligned}$$

The values of v and w at $t = 1$.

$$\boxed{\begin{aligned} v(1) &= 10e^{-2} + 20 \\ w(1) &= -10e^{-2} + 20 \end{aligned}}.$$