Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #6

2023/04/02

Name: ____

Student Number: ____________________________________

1. Is $f(x) = x^4 + x^3 + x^2 + x + 1$ reducible over $\mathbf{Q}$?

   Proof. Let $g(x) = f(x+1) = x^4 + 5x^3 + 10x^2 + 10x + 5$, then $f(x)$ is reducible over $\mathbf{Q}$ which is equivalent to $g(x)$ is reducible. Let $p = 5$, then $p \nmid 1, p|5, p|10, p^2 \nmid 5$, by Eisentein's irreducibility criterion, we have $g(x)$ is reducible over $\mathbf{Q}$, then $f(x)$ is also reducible over $\mathbf{Q}$. $\square$

2. Suppose $T \in \mathcal{L}(V)$, and $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ are bases of V. Prove that the columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$ if and only if the rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$.

   Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \cdots, u_n), (v_1, \cdots, v_n))$.

   Proof. The columns of $\mathcal{M}(T)$ are linear independent $\Leftrightarrow$ $Tu_1, \cdots, Tu_n$ are linearly independent $\Leftrightarrow$ T is invertible $\Leftrightarrow$ T' is invertible $\Leftrightarrow$ $T'u_1, \cdots, T'u_n$ are linearly independent $\Leftrightarrow$ The columns of $\mathcal{M}(T')$ are linearly independent $\Leftrightarrow$ The rows of $\mathcal{M}(T)$ are linearly independent. $\square$