Step-1

Given matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

Now $A^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$

$$\begin{aligned} A^T A &= \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 13 \end{pmatrix} \end{aligned}$$

Step-2

Compare this with $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$,

So, $a = 1,\ b = 2,\ c = 13.$

Clearly $a = 1 > 0$ and $ac - b^2 = 13 - 4 = 9 > 0$

Thus the matrix $\boxed{A^T A \text{ is positive definite.}}$

Step-3

Given second matrix is $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\begin{aligned} A^T A &= \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix} \end{aligned}$$

Step-4

Compare this with $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$,

So, $a=6,\ b=5,\ c=6.$

Clearly $a=6>0$ and $ac-b^2=6(6)-25=11>0$

Thus the matrix $\boxed{A^T A \text{ is positive definite.}}$

Step-5

Given third matrix is $A=\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

$$\begin{aligned} A^T A &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{pmatrix} \end{aligned}$$

Step-6

Let $X^T=\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$, then

$$\begin{aligned} X^T AX &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}\begin{pmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= 2x_1^2+5x_2^2+5x_3^2+6x_1x_2+6x_1x_3+8x_2x_3 \end{aligned}$$

Step-7

$$A=\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Apply this operation $R_2 \to R_2 - R_1$

$$=\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

Clearly A have independent columns.

So $A^T A$ is square and symmetric and invertible.

Therefore, $\boxed{A^T A \text{ is positive definite.}}$