

Step-1

Let A be a matrix.

We know that

Def: 1: The **conditional number** of A is defined to be $c = \|A\| \|A^{-1}\|$

Def: 2: The **norm** of A is the number $\|A\|$ defined by $\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

Step-2

a) We have to verify that A and A^{-1} have same condition number or not.

We know that the norm is a non negative number given by $\|kV\| = |k| \|V\|$, where k is a scalar and V is a vector (matrix is also a vector).

So, the product of *norms* is commutative.

Consequently, we follow that $c = \|A^{-1}\| \|A\|$

Further, by the algebraic properties of matrices, we have $(A^{-1})^{-1} = A$

Step-3

$$c = \|A^{-1}\| \|(A^{-1})^{-1}\|$$

So, replacing A with $(A^{-1})^{-1}$ in the above equation, we get $= \|A^{-1}\| \|A\|$ (Since $(A^{-1})^{-1} = A$)

Comparing this with the definition of the conditional number, we get c is the conditional number of A^{-1} .

Therefore, it follows that A and A^{-1} have the same conditional numbers.

Step-4

b) Suppose the non homogeneous system is $Ax = b$ and $\delta x = A^{-1} \delta b$

We know that $\|b\| \leq \|A\| \|x\|$ and $\|\delta x\| \leq \|A^{-1}\| \|\delta b\|$

Conveniently, we can write $\frac{\|b\|}{\|x\|} \leq \|A\|$ and $\frac{\|\delta x\|}{\|\delta b\|} \leq \|A^{-1}\|$ (1)

$$\Rightarrow \frac{\|b\|}{\|x\|} \frac{\|\delta x\|}{\|\delta b\|} \leq \|A\| \|A^{-1}\|$$

$$\Rightarrow \frac{\|b\|}{\|x\|} \frac{\|\delta x\|}{\|\delta b\|} \leq c \quad \left(\text{Since } \|A\| \|A^{-1}\| = c \right)$$

$$\frac{\|\delta x\|}{\|x\|} \leq c \frac{\|\delta b\|}{\|b\|}$$

Precisely,

Step-5

The equation (1) can otherwise be written as $\frac{\|x\|}{\|b\|} \leq \frac{1}{\|A\|}$ and $\frac{\|\delta b\|}{\|\delta x\|} \leq \frac{1}{\|A^{-1}\|}$

Multiplying the respective sides, we get

$$\frac{\|x\|}{\|b\|} \frac{\|\delta b\|}{\|\delta x\|} \leq \frac{1}{\|A\|} \frac{1}{\|A^{-1}\|}$$

$$\Rightarrow \frac{\|x\|}{\|b\|} \frac{\|\delta b\|}{\|\delta x\|} \leq \frac{1}{c} \quad \left(\text{Since } \|A\| \|A^{-1}\| = c \right)$$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \frac{1}{c} \frac{\|b\|}{\|\delta b\|}$$

Hence $\boxed{\frac{\|\delta x\|}{\|x\|} \leq \frac{1}{c} \frac{\|b\|}{\|\delta b\|}}$