Step-1

We have to mark all the sixth roots of 1 in the complex plane, we have to find the primitive root w_6, and we have to find that which power of w_6 is equal to $\frac{1}{w_6}$. Also we have to find the value of $1+w+w^2+w^3+w^4+w^5$.

Step-2

The equation $z^6=1$ denotes sixth roots of unity in complex plane.

$$z^6=1$$
$$\Rightarrow z=1^{1/6}$$
$$\Rightarrow z=\left(\cos 2k\pi+i\sin 2k\pi\right)^{1/6} \text{ for } k=0,1,2,3,4,5.$$
$$\Rightarrow z=\left(e^{i2k\pi}\right)^{\frac{1}{6}}$$
$$=\left(e^{\frac{2i\pi}{6}}\right)^k \text{ for } k=0,1,2,3,4,5.$$

Step-3

$$\Rightarrow z=1,e^{\frac{i\pi}{3}},e^{\frac{2i\pi}{3}},e^{\frac{3i\pi}{3}},e^{\frac{4i\pi}{3}},e^{\frac{5i\pi}{3}}$$ Â

Â $\Rightarrow z=1,w,w^2,w^3,w^4,w^5.$ â€¦â€¦ (1)

Here $w=e^{\frac{2i\pi}{6}}$

Step-4

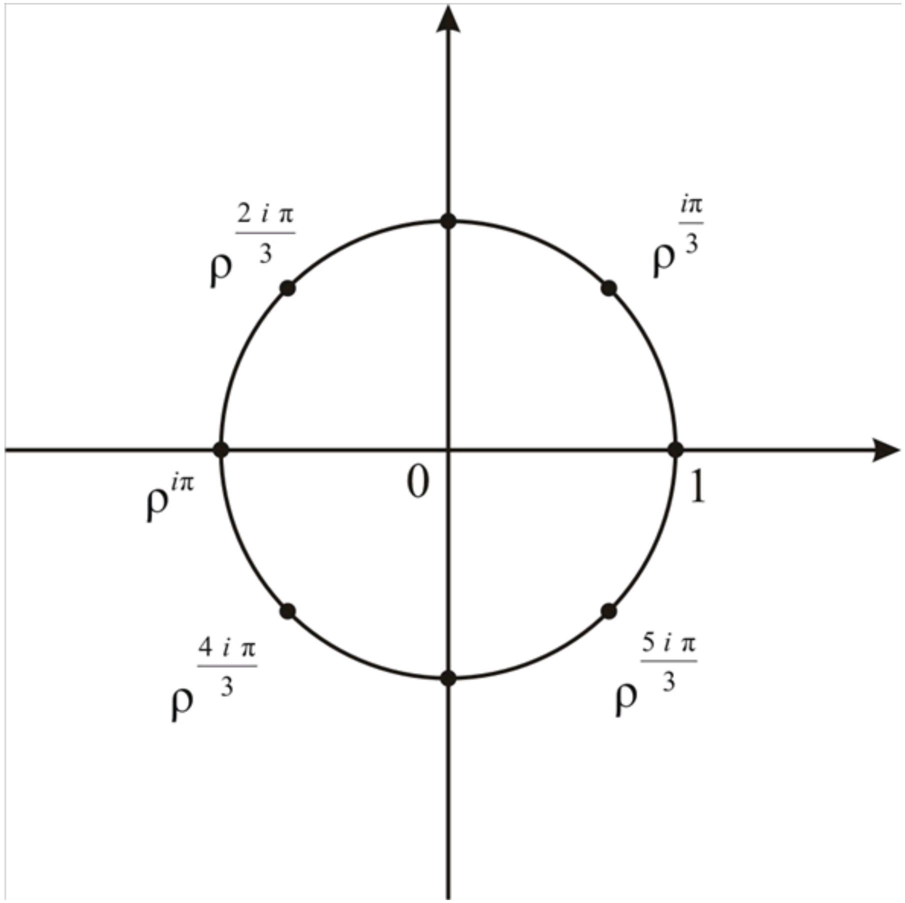
Â Â Â Â Â Â Â

Hence

$$z=1,\frac{1+i\sqrt{3}}{2},\frac{-1+i\sqrt{3}}{2},-1,\frac{-1-i\sqrt{3}}{2},\frac{1-i\sqrt{3}}{2}$$ Â â€¦â€¦ (2)

Step-5

The sixth roots of 1 in the complex plane are shown as follows:



Step-6

The primitive root w_6 is $\cos\frac{2\pi}{6}+i\sin\frac{2\pi}{6}$

$=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

$=\boxed{\frac{1+i\sqrt{3}}{2}}$

Step-7

Now

$$w^5 = e^{\frac{5i\pi}{3}}$$
$$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$
$$= \cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right)$$

Step-8

$$= \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$$
$$= \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$$
$$= e^{\frac{-i\pi}{3}}$$
$$= w^{-1}$$
$$= \frac{1}{w}$$

Hence 5th power of w is equal to $\frac{1}{w}$

Step-9

From (1) and (2),

$$w = \frac{1 + i\sqrt{3}}{2}$$
$$w^2 = \frac{-1 + i\sqrt{3}}{2}$$
$$w^3 = -1$$
$$w^4 = \frac{-1 - i\sqrt{3}}{2}$$
$$w^5 = \frac{1 - i\sqrt{3}}{2}$$

Step-10

$$1 + w + w^2 + w^3 + w^4 + w^5$$
$$= 1 + \frac{1 + i\sqrt{3}}{2} + \frac{-1 + i\sqrt{3}}{2} - 1 + \frac{-1 - i\sqrt{3}}{2} + \frac{1 - i\sqrt{3}}{2}$$
$$= 0$$

Hence $\boxed{1+w+w^2+w^3+w^4+w^5=0}$