南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 高等数学(下) A 开课单位: 数 学 系
考试时长: 180 分钟 命题教师: 王融、吴纪桃 等

题 号	1	2	3	4	5	6	7	8	9	10
分 值	9 分	15 分	9 分	7 分	7 分	7 分	7 分	7 分	7 分	7 分
题 号	11	12	13							
分 值	7 分	7 分	4 分							

本试卷共 13 大题, 满分 100 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

注意：本试卷里的中文为直译（即完全按英文字面意思直接翻译），所有数学词汇的定义请参照教材（Thomas' Calculus，13th Edition）中的定义。如果其中有些数学词汇的定义不同于中文书籍（比方说同济大学的高等数学教材）里的定义，以教材（Thomas' Calculus，13th Edition）中的定义为准。

1. (9 pts) Determine whether the following statements are **true** or **false**? No justification is necessary.

   (1) If $a_n > 0, \forall n$, and $\lim_{n\to\infty} na_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. F

   (2) The plane $x + y - 2z = 1$ is perpendicular to the plane $x + y + z = 1$. T

   (3) If $f(x, y)$ has two local maxima, then f must have a local minimum. F

一、 （9分）判断题：

   (1) 若$\forall n$, $a_n > 0$, 且 $\lim_{n\to\infty} na_n = 0$, 那么级数$\sum_{n=1}^{\infty} a_n$ 收敛.

   (2) 平面$x + y - 2z = 1$和平面$x + y + z = 1$垂直.

   (3) 如果函数$f(x, y)$有两个局部极大值点，那么f必有局部极小值点.

2. (15pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)

   (1) Which one of the following series diverges? B

   (A) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. (B) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$.

   (C) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$. (D) $\sum_{n=1}^{\infty} \frac{(-1)^n (3 + (-1)^n \cdot 2)^n}{6^n}$.

(2) The iterated integral $\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} f(r\cos\theta, r\sin\theta)r\,drd\theta$ can be written as D

(A) $\int_0^1\int_0^{\sqrt{y-y^2}} f(x,y)\,dxdy.$　　(B) $\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dxdy.$

(C) $\int_0^1\int_0^1 f(x,y)\,dydx.$　　(D) $\int_0^1\int_0^{\sqrt{x-x^2}} f(x,y)\,dydx.$

(3) For the function, $f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0), \end{cases}$ which of the following statements is correct? D

(A) f is not continuous at $(0,0)$.

(B) f is continuous at $(0,0)$, but its partial derivative f_x and f_y do not exist at $(0,0)$.

(C) Both partial derivatives f_x and f_y exist everywhere and are also continuous at $(0,0)$.

(D) f is not differentiable at $(0,0)$.

(4) For the critical points of the function $f(x,y) = 2x^4 + y^4 - 2x^2 - 2y^2$, which one of the following statements is correct? C

$f_x = 8x^3 - 4x$　$x = 0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$

$f_y = 4y^3 - 4y$　$y = 0, 1, -1$

$f_{xx} = 24x^2 - 4$　$f_{xy} = 0$

$f_{yy} = 12y^2 - 4$

(A) $(0,0)$ is a local minima.

(B) $(0,1)$ is a local maxima.

(C) $(0,-1)$ is a saddle point.

(D) There are no local maxima among all the critical points.

(5) If the function $f(x,y)$ has the continuous first partial derivatives $\frac{\partial f}{\partial x} > 0$ and $\frac{\partial f}{\partial y} < 0$, $\forall(x,y)\in\mathbf{R}^2$, which one of the following statements is correct? D

(A) $f(0,0) > f(1,1)$.　　(B) $f(0,0) < f(1,1)$.

(C) $f(0,1) > f(1,0)$.　　(D) $f(0,1) < f(1,0)$.

二、（15分）单项选择题：

(1) 下列哪个级数发散?

(A) $\sum_{n=1}^{\infty}\frac{n^2}{2^n}.$　　(B) $\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{1}{n}}}.$

(C) $\sum_{n=1}^{\infty}\frac{(-1)^n\ln n}{\sqrt{n}}.$　　(D) $\sum_{n=1}^{\infty}\frac{(-1)^n(3+(-1)^n\cdot 2)^n}{6^n}.$

(2) 累次积分$\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} f(r\cos\theta, r\sin\theta)r\,drd\theta$ 可以写成

(A) $\int_0^1\int_0^{\sqrt{y-y^2}} f(x,y)\,dxdy.$　　(B) $\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dxdy.$

(C) $\int_0^1\int_0^1 f(x,y)\,dydx.$　　(D) $\int_0^1\int_0^{\sqrt{x-x^2}} f(x,y)\,dydx.$

(3) 对于函数$f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0); \\ 0, & (x,y)=(0,0), \end{cases}$ 以下哪个说法是正确的?

(A) f在$(0,0)$处不连续.

(B) f在$(0,0)$连续但其偏导数f_x和f_y在$(0,0)$处不存在.

(C) 两个偏导数f_x和f_y都处处存在且在$(0,0)$处连续.

(D) f在$(0,0)$处不可微.

(4) 关于函数$f(x,y)=2x^4+y^4-2x^2-2y^2$的临界点，以下哪种说法正确?

(A) $(0,0)$是局部极小值点.

(B) $(0,1)$是局部极大值点.

(C) $(0,-1)$是鞍点.

(D) 在所有的临界点中不存在局部极大值点.

(5) 设$f(x,y)$在$xy-$ 平面上有一阶连续偏导数，并且$\forall(x,y)\in\mathbf{R}^2$，都有$\frac{\partial f}{\partial x}>0$，和$\frac{\partial f}{\partial y}<0$. 则以下哪种说法正确?

(A) $f(0,0)>f(1,1)$.　　(B) $f(0,0)<f(1,1)$.

(C) $f(0,1)>f(1,0)$.　　(D) $f(0,1)<f(1,0)$.

3. (9 pts) Please fill in the blank for the questions below.

(1) Compute the limit: $\lim\limits_{(x,y)\to(0,0)}\frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2}=$ $\frac{1}{2}$.

(2) The direction (unit vector) in which the function $f(x,y)=x^2+xy+y^2-y$ increases most rapidly at the point $(-1,2)$ is $0\vec{i}+1\vec{j}$.

(3) $\int_0^1\int_y^1\frac{\tan x}{x}\,dxdy=$ $\ln(\sec 1)$.

三、（9分）填空题:

(1) 计算极限: $\lim\limits_{(x,y)\to(0,0)}\frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2}=$ ________.

(2) 如果函数$f(x,y)=x^2+xy+y^2-y$在$(-1,2)$处的方向导数$D_{\mathbf{u}}f(-1,2)$沿单位向量$\mathbf{u}$ 达到最大值，那么$\mathbf{u}=$ ________.

(3) $\int_0^1\int_y^1\frac{\tan x}{x}\,dxdy=$ ________.

4. (7 pts)

(1) Find the interval of convergence of the series $\sum\limits_{n=2}^{\infty}\frac{(-1)^n(x-1)^{2n+1}}{\sqrt{n+9012\ln n}}$.

(2) For what values of x does the series converge absolutely, or conditionally?

四、（7分）

(1) 求级数$\sum\limits_{n=2}^{\infty}\frac{(-1)^n(x-1)^{2n+1}}{\sqrt{n+9012\ln n}}$ 的收敛区间.

(2) x取哪些值时级数绝对收敛，取哪些值时条件收敛?

5. (7 pts) The region D is bounded by $z=\sqrt{x^2+y^2}$ and $z=\sqrt{1-x^2-y^2}$. Consider the following integral

$$\iiint\limits_D (x+z)\,dxdydz,$$

$\int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}}\int_r^{\sqrt{1-r^2}}(r\cos\theta+z)r\,dz\,dr\,d\theta$

(1) Convert the above integral to an equivalent iterated integral in cylindrical coordinates;

(2) Convert the above integral to an equivalent iterated integral in spherical coordinates.

五、 （7分）区域 D由 $z=\sqrt{x^2+y^2}$ 与 $z=\sqrt{1-x^2-y^2}$ 所围成. 考虑积分 $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1(\rho\sin\varphi\cos\theta+\rho\cos\theta)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$$\iiint_D (x+z)\,dxdydz,$$

(1) 将上述积分化为柱坐标下对应的累次积分（要求写出累次积分上下限）.

(2) 将上述积分化为球坐标下对应的累次积分（要求写出累次积分上下限）.

$V=8xyz$

6. (7 pts) Assume we can put a cuboid into the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Use the method of **Lagrange multipliers** to find the length, width and height of the cuboid such that it achieve the maximum volume. $x=\frac{a}{\sqrt{3}},\ y=\frac{b}{\sqrt{3}},\ z=\frac{c}{\sqrt{3}}$ $V=8xyz=\frac{8abc}{3\sqrt{3}}$ $g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

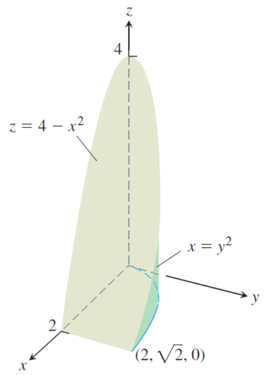
六、 （7分）在椭球$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$内嵌入有最大体积的长方体，请使用拉格朗日乘子法给出这个长方体的长宽高分别等于多少.

7. (7 pts) Find the equation of the osculating circle for the parabola $y=x^2$ at $x=1$.

七、 （7分）写出抛物线$y=x^2$ 当$x=1$ 处的曲率圆的方程. $(x+4)^2+(y-\frac{7}{2})^2=\frac{125}{4}$

8. (7 pts) A solid in the first octant is bounded by the planes $y=0$ and $z=0$ and by the surfaces $z=4-x^2$ and $x=y^2$ (see the figure below). Its density function is $\delta(x,y,z)=xy$. Find the center of the mass for the solid.

八、 （7分）设D是由xy-平面、xz-平面、曲面$z=4-x^2$ 和$x=y^2$ 所围成的闭区域，其密度函数为$\delta(x,y,z)=xy$. 计算D 的质心.



9. (7 pts) Use the **substitution in double integral** (please find the transformation by yourself) to evaluate the integral

$$\iint_D e^{\frac{y-x}{y+x}}\,dxdy,$$

here D is the triangular region bounded by the lines $x=0$, $y=0$, and $x+y=2$.

$$=\int_0^2\int_{-v}^{v}e^{\frac{u}{v}}\frac{1}{2}\,du\,dv$$

$$=\int_0^2\frac{1}{2}ve^{\frac{u}{v}}\Big]_0^v\,dv=e-e^{-1}$$

九、 （7分）用换元法来求二重积分

$$\iint_D e^{\frac{y-x}{y+x}}\,dxdy,$$

其中D 是由x 轴、y 轴和直线$x+y=2$ 所围成的三角形闭区域.

10. (7 pts) Consider the line integral

$$\int_{(1,1,1)}^{(1,3,\pi)} e^x \ln y\,dx + \left(\frac{e^x}{y} + \sin z\right) dy + y\cos z\,dz.$$ Done

(1) Show that the differential form in the integral is exact.

(2) Evaluate the integral.

十、 （7分）考虑曲线积分

$$\int_{(1,1,1)}^{(1,3,\pi)} e^x \ln y\,dx + \left(\frac{e^x}{y} + \sin z\right) dy + y\cos z\,dz.$$

(1) 证明积分中的微分形式是恰当的.

(2) 求积分的值.

11. (7 pts) Evaluate

$$\iint_S \nabla \times (4x\mathbf{j}) \cdot \mathbf{n}\,d\sigma, = \iint 4\,da = 64\pi$$

$S: x^2+y^2=16 \quad r=4$

where S is the hemisphere $x^2+y^2+z^2=16$, $z \geq 0$. Use the normal vectors pointed away from the origin.

$x^2+y^2=16$

十一、 （7分）计算

$$\iint_S \nabla \times (4x\mathbf{j}) \cdot \mathbf{n}\,d\sigma,$$

其中S 是半球面$x^2+y^2+z^2=16$, $z \geq 0$. 法向$\mathbf{n}$ 指向远离原点的方向.

12. (7 pts) Find the outward flux of $\mathbf{F} = (6x+y)\mathbf{i} - (x+z)\mathbf{j} + 4yz\mathbf{k}$ across the boundary of D, where D is the region in the first octant bounded by the cone $z=\sqrt{x^2+y^2}$, the cylinder $x^2+y^2=1$, and the coordinate planes.

$\int_0^{\frac{\pi}{2}}\int_0^1\int_0^r (6+4r\sin\theta)\,dz\,r\,dr\,d\theta = \pi+1$

十二、 （7分）设D是第一卦限中由锥面$z=\sqrt{x^2+y^2}$，柱面$x^2+y^2=1$以及坐标平面围成的区域，求$\mathbf{F} = (6x+y)\mathbf{i} - (x+z)\mathbf{j} + 4yz\mathbf{k}$向外穿过$D$的边界的通量.

13. (4 pts) The sequences $\{a_n\}$ and $\{b_n\}$ satisfy $0 < a_n < \frac{\pi}{2}$, $0 < b_n < \frac{\pi}{2}$, and $\cos a_n - a_n = \cos b_n$, $n = 1, 2, 3, \cdots$. The series $\sum_{n=1}^{\infty} b_n$ converges. Show that $\lim_{n\to\infty} a_n = 0$.

十三、 （4分）设数列$\{a_n\}$，$\{b_n\}$ 满足$0 < a_n < \frac{\pi}{2}$，$0 < b_n < \frac{\pi}{2}$，$\cos a_n - a_n = \cos b_n$，且级数$\sum_{n=1}^{\infty} b_n$收敛. 证明：$\lim_{n\to\infty} a_n = 0$.

$\cos a_n - \cos b_n = a_n > 0$

$0 < a_n < b_n < \frac{\pi}{2}$

$\Rightarrow \sum a_n$ converges.

$\lim_{n\to\infty} a_n = 0$