Step-1

We have to write a 2 by 2 system $Ax = b$ with many solutions x_n but no solution x_p. (Therefore the system has no solution) and we have to find that which bâ€™s allow an x_p.

Step-2

Consider $A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Then $\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\underline{R_2 - R_1}\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\underline{\frac{1}{2}R_1}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\Rightarrow x + 2y = 1$

$0 = -2$, which is wrong

Therefore, this system has no solution x_p

Step-3

For the solutions x_n, consider $Ax = 0$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x + 2y = 0$

$\Rightarrow x = -2y$

Step-4

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix}$$

$$\Rightarrow x_n = y\begin{bmatrix} -2 \\ 1 \end{bmatrix}, y \in R$$

Therefore the system has infinitely many solutions x_n

Step-5

If we take $b=\begin{bmatrix} c \\ c \end{bmatrix}$ then we get the system as follows:

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} c \\ c \end{bmatrix}$$

$$\xrightarrow{R_2-R_1}\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} c \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x+4y=c$$

$$\Rightarrow x+2y=\frac{c}{2}, c\in R$$

$$\Rightarrow x=\frac{c}{2}-2y$$

Step-6

Therefore the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \frac{c}{2}-2y \\ y \end{bmatrix}$$

$$=c\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}+y\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Hence for $b=\boxed{\begin{bmatrix} c \\ c \end{bmatrix}}$, the system has solution.