Quiz 4

本人郑重承诺将秉承诚信原则，自觉遵守考场纪律，并承担违纪或作弊带来的后果。

学号：________________ 姓名：________________

题号	1	2	3	4	5	合计
分值	10 分	10 分	10 分	10 分	10 分	50 分
得分						

This 50-minute long test includes 5 questions. Write ***all your answers*** on this examination paper.

1. (1) The area of the triangle on the plane $\mathbf{R}^2$ with the vertices $(2,1),(3,4),(0,5)$ is __5__.

   (2) Let $\boldsymbol{A},\boldsymbol{B}$ be $n\times n$ matrices, and $|\boldsymbol{A}|=-2$, $|\boldsymbol{B}|=-3$, and $\boldsymbol{A}^*$ is the adjoint matrix (伴随矩阵) of $\boldsymbol{A}$, then $|2\boldsymbol{B}^{-1}\boldsymbol{A}^*|=$ __$\frac{1}{3}\times(-2)^n$__

   (3) A matrix $\boldsymbol{A}$ is diagonalizable if and only if it is a normal matrix. (✗)

   (4) A matrix $\boldsymbol{A}$ is invertible if and only if it has no zero eigenvalue. (✓) $\Leftrightarrow \det(A)\neq 0$

   (5) If $\boldsymbol{A}$ is a 3×3 matrix, and none of $\boldsymbol{A}-\boldsymbol{I},\boldsymbol{A}+2\boldsymbol{I},5\boldsymbol{A}-3\boldsymbol{I}$ is invertible, then $\boldsymbol{A}$ is diagonalizable. (✓)

2. Suppose that a 3×3 real symmetric matrix $\boldsymbol{A}$ has the eigenvalues $\lambda_1=1,\lambda_2=-1,\lambda_3=0$. The eigenvectors corresponding to λ_1,λ_2 are $\boldsymbol{p}_1=(1,2,2)^{\mathrm{T}},\boldsymbol{p}_2=(2,1,-2)^{\mathrm{T}}$. Find the matrix $\boldsymbol{A}$.

$$q_1=\frac{p_1}{\|p_1\|}=\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}\quad q_2=\frac{p_2}{\|p_2\|}=\begin{bmatrix}2/3\\1/3\\-2/3\end{bmatrix}\quad q_3 \text{ is orthogonal to } q_1 \text{ and } q_2$$

$$\text{then }\begin{pmatrix}p_1^T\\p_2^T\end{pmatrix}x=\vec{0}\rightarrow\begin{bmatrix}1&2&2\\2&1&-2\end{bmatrix}x=\begin{bmatrix}0\\0\end{bmatrix}\Rightarrow q_3=\begin{bmatrix}2/3\\-2/3\\1/3\end{bmatrix}$$

$$Q=[q_1\ q_2\ q_3]=\begin{bmatrix}1/3&2/3&2/3\\2/3&1/3&-2/3\\2/3&-2/3&1/3\end{bmatrix}$$

$$A=Q\cdot\operatorname{diag}(1,-1,0)\cdot Q^T=\frac{1}{3}\begin{bmatrix}-1&0&2\\0&1&2\\2&2&0\end{bmatrix}$$

3. Find an orthogonal diagonalizing matrix for the following matrix: $\boldsymbol{A}=\begin{bmatrix}2&2&-2\\2&5&-4\\-2&-4&5\end{bmatrix}$.

$$\det(A-\lambda I)=0=\begin{vmatrix}2-\lambda&2&-2\\2&5-\lambda&-4\\-2&4&5-\lambda\end{vmatrix}\Rightarrow\lambda_1=1,\ \lambda_2=1,\ \lambda_3=10$$

$$x_1=\begin{bmatrix}2\\0\\1\end{bmatrix}\quad x_2=\begin{bmatrix}0\\1\\1\end{bmatrix}\quad x_3=\begin{bmatrix}-1\\-2\\2\end{bmatrix}$$

$$q_1=\frac{x_1}{\|x_1\|}=\frac{1}{\sqrt5}\begin{bmatrix}2\\0\\1\end{bmatrix}\qquad b=x_2-q_1^Tx_2q_1=\begin{bmatrix}0\\1\\1\end{bmatrix}-\frac{1}{5}\times1\times\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}-\frac{2}{5}\\1\\\frac{4}{5}\end{bmatrix}$$

$$q_2=\frac{b}{\|b\|}=\frac{\sqrt5}{3}\begin{bmatrix}-\frac{2}{5}\\1\\\frac{4}{5}\end{bmatrix}\qquad q_3=\frac{x_3}{\|x_3\|}=\frac{1}{3}\begin{bmatrix}-1\\-2\\2\end{bmatrix}$$

$$Q=\begin{bmatrix}2/\sqrt5&-2/3\sqrt5&-\frac{1}{3}\\0&\sqrt5/3&-\frac{2}{3}\\1/\sqrt5&4/3\sqrt5&\frac{2}{3}\end{bmatrix}$$

4. Find a unitary diagonalizing matrix for the following matrix: $\boldsymbol{B} = \begin{bmatrix} 0 & 2i \\ -2i & 1 \end{bmatrix}$.

$$\det(B-\lambda I) = 0 = \begin{vmatrix} -\lambda & 2i \\ -2i & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 4$$

$$\lambda_1 = \frac{1-\sqrt{17}}{2} \quad \lambda_2 = \frac{1+\sqrt{17}}{2}$$

$$x_1 = \begin{bmatrix} 4 \\ (\sqrt{17}-1)i \end{bmatrix} \quad x_2 = \begin{bmatrix} 4 \\ (\sqrt{17}+1)i \end{bmatrix} \Rightarrow U = \begin{bmatrix} 4 & 4 \\ (\sqrt{17}-1)i & (\sqrt{17}+1)i \end{bmatrix}$$

$$/\sqrt{34-2\sqrt{17}} \quad /\sqrt{34+2\sqrt{17}}$$

5. Suppose the determinant $\begin{vmatrix} a & -5 & 8 \\ 0 & a+1 & 8 \\ 0 & 3a+3 & 25 \end{vmatrix} = 0$. The 3×3 matrix $\boldsymbol{A}$ has three eigenvalues $1, -1, 0$, with the corresponding eigenvectors $\boldsymbol{p}_1 = (1, 2a, -1)^{\mathrm{T}}, \boldsymbol{p}_2 = (a, a+3, a+2)^{\mathrm{T}}, \boldsymbol{p}_3 = (a-2, -1, a+1)^{\mathrm{T}}$. Try to determine the parameter a, and find the matrix $\boldsymbol{A}$.

Since $\begin{vmatrix} a & -5 & 8 \\ 0 & a+1 & 8 \\ 0 & 3a+3 & 25 \end{vmatrix} = 0$, then $a=-1$, or $a=0$

if $a=-1$, p_1, p_2, p_3 are linearly dependent.

so $a=0$

$$S = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} -5 & 4 & -6 \\ -1 & 1 & -1 \\ -3 & 2 & -3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = S\Lambda S^{-1} = \begin{bmatrix} -5 & 4 & 6 \\ 3 & -3 & 3 \\ 7 & -6 & 8 \end{bmatrix}$$