Step-1

Given system is $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We have to use elimination on $\begin{bmatrix} A & I \end{bmatrix}$ to solve $AA^{-1} = I$.

Step-2

Given that $\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

Subtracting b times row 3 from row 1; subtracting c times row 3 from row 2

$$\begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Step-3

Subtracting a times row 2 from row 1

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -a & -b+ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Now $\begin{bmatrix} I & A^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & -a & -b+ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

Step-4

Now

$$AA^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+a(0)+b(0) & 1(-a)+a(1)+b(0) & 1(-b+ac)+a(-c)+b(1) \\ 0(1)+1(0)+c(0) & 0(-a)+1(1)+c(0) & 0(-b+ac)+1(-c)+c(1) \\ 0(1)+0(0)+1(0) & 0(-a)+0(1)+1(0) & 0(-b+ac)+0(-c)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & -a+a & -b+ac-ac+b \\ 0+0+0 & 0+1+0 & -c+c \\ 0+0+0 & 0+0+0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Hence $\boxed{A^{-1} = \begin{bmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}}$