

Step-1

Consider the following orthogonal matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the unit Eigen vectors and unitary matrix. Determine the property of P that makes these Eigen vectors orthogonal.

Step-2

First step is to find the Eigen values of matrix P . To calculate the Eigen values do the following calculations:

$$P - \lambda I = \begin{bmatrix} 0 - \lambda & 1 & 0 \\ 0 & 0 - \lambda & 1 \\ 1 & 0 & 0 - \lambda \end{bmatrix}$$

$$\det(P - \lambda I) = 0$$

$$(-\lambda)(\lambda^2) + 1 = 0$$

$$(\lambda^3 - 1) = 0$$

After solving following values are obtained:

$$\lambda_1 = 1$$

$$\lambda_2 = e^{2\pi i/3}$$

$$\lambda_3 = e^{4\pi i/3}$$

Step-3

To calculate Eigen vectors do the following calculations:

$$(P - \lambda I)x = 0$$

$$\begin{bmatrix} 0 - \lambda & 1 & 0 \\ 0 & 0 - \lambda & 1 \\ 1 & 0 & 0 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving, values of x , y and z corresponding to $\lambda = 1$ are as follows:

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Step-4

Similarly, Eigen vector corresponding to Eigen value $\lambda = e^{2\pi i/3}$ is as follows:

$$(P - \lambda I)x = 0$$

$$\begin{bmatrix} -e^{2\pi i/3} & 1 & 0 \\ 0 & -e^{2\pi i/3} & 1 \\ 1 & 0 & -e^{2\pi i/3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving values of x, y and z are as follows:

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ e^{2\pi i/3} \\ e^{4\pi i/3} \end{bmatrix}$$

Step-5

Similarly, Eigen vector corresponding to Eigen value $\lambda = e^{4\pi i/3}$ will be:

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ e^{4\pi i/3} \\ e^{2\pi i/3} \end{bmatrix}$$

Step-6

Therefore, Eigen vectors of matrix P are as follows:

$$\left((1,1,1), (1, e^{2\pi i/3}, e^{4\pi i/3}), (1, e^{4\pi i/3}, e^{2\pi i/3}) \right)$$

Step-7

To put the Eigen vectors in unitary matrix make them orthonormal by dividing the length of the vector.

$$\begin{aligned} \|x\|^2 &= \left| (1)^2 \right| + \left| \left(e^{2\pi i/3} \right)^2 \right| + \left| \left(e^{4\pi i/3} \right)^2 \right| \\ &= |1| + |1| + |1| \\ &= 3 \end{aligned}$$

Let the length be L . So $L = \sqrt{3}$.

Step-8

Therefore, unitary matrix will be as follows:

$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}$$

Step-9

The property of P that makes these Eigen vectors orthogonal is the orthogonality of matrix P and Eigen value absolute value is $|\lambda|=1$