考试科目: 线 性 代 数 A　　**开课单位:** 数 学 系

考试时长: 120 分钟　　**命题教师:** 线性代数教学团队

题 号	1	2	3	4	5	6	7
分 值	15 分	15 分	15 分	15 分	20 分	8 分	12 分

本试卷共 (7) 大题, 满分 (100) 分. **请将所有答案写在答题卡上**.

This exam includes **7** questions and the score is 100 in total. **Write all your answers on the examination book**.

本套试卷为 A 卷　Version A

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

   (共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

   (1) Let A be an $m \times n$ real matrix. Which one of the following statements must be true? ( )

   (A) If $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for any $\mathbf{b} \in \mathbb{R}^m$, then $A^T\mathbf{y} = \mathbf{c}$ has infinitely many solutions for any $\mathbf{c} \in \mathbb{R}^n$.

   (B) If $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for any $\mathbf{b} \in \mathbb{R}^m$, then $A^T\mathbf{y} = \mathbf{c}$ has unique solution for any $\mathbf{c} \in \mathbb{R}^n$.

   (C) If $A\mathbf{x} = \mathbf{b}$ has unique solution for any $\mathbf{b} \in \mathbb{R}^m$, then $A^T\mathbf{y} = \mathbf{c}$ has unique solution for any $\mathbf{c} \in \mathbb{R}^n$.

   (D) If $A\mathbf{x} = \mathbf{b}$ has no solutions for some $\mathbf{b} \in \mathbb{R}^m$, then $A^T\mathbf{y} = \mathbf{c}$ has no solutions for some $\mathbf{c} \in \mathbb{R}^n$.

   设 A 为 $m \times n$ 实矩阵. 下列陈述一定正确的是?　( )

   (A) 如果对于任意 $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ 都有无穷多解, 则对于任意 $\mathbf{c} \in \mathbb{R}^n$, $A^T\mathbf{y} = \mathbf{c}$ 有无穷多解.

   (B) 如果对于任意 $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ 都有无穷多解, 则对于任意 $\mathbf{c} \in \mathbb{R}^n$, $A^T\mathbf{y} = \mathbf{c}$ 有唯一解.

   (C) 如果对于任意 $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ 有唯一解, 则对于任意 $\mathbf{c} \in \mathbb{R}^n$, $A^T\mathbf{y} = \mathbf{c}$ 有唯一解.

   (D) 如果对于某个 $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ 无解, 则对于某个 $\mathbf{c} \in \mathbb{R}^n$, $A^T\mathbf{y} = \mathbf{c}$ 也无解.

   (2) The matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 2 & 2 & 0 & a \\ 3 & -4 & 7 & -8 \end{bmatrix}$$

   is NOT right invertible for　( )

   (A) $a = 1$.　(B) $a = 2$.

   (C) $a = 3$.　(D) $a = 4$.

   考虑矩阵

$$A = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 2 & 2 & 0 & a \\ 3 & -4 & 7 & -8 \end{bmatrix}.$$

在下列哪种情形下, A 不存在右逆? ( )

(A) $a = 1$. (B) $a = 2$.

(C) $a = 3$. (D) $a = 4$.

(3) Let A and B be $n \times n$ real symmetric matrices. If A and B are congruent, then ( )

(A) A and B have the same eigenvalues. (B) A and B have the same rank.

(C) A and B have the same eigenvectors. (D) A and B have the same determinant.

设 A 与 B 均为 n 阶实对称矩阵, 若 A 与 B 合同, 则 ( )

(A) A 与 B 有相同的特征值. (B) A 与 B 有相同的秩.

(C) A 与 B 有相同的特征向量. (D) A 与 B 有相同的行列式.

(4) Let A be a 6×7 real matrix of rank 2. Then $\dim N(A^T) =$ ( )

(A) 2. (B) 3.

(C) 4. (D) 5.

设 A 为一个 6×7 的实矩阵. 如果 $\operatorname{rank}(A) = 2$, 则 $\dim N(A^T) =$ ( )

(A) 2. (B) 3.

(C) 4. (D) 5.

(5) Which of the following matrices is similar to $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$? ( )

(A) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. (B) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(C) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (D) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

下列矩阵中, 与矩阵 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ 相似的为 ( )

(A) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. (B) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(C) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (D) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. (15 points, 3 points each) Fill in the blanks. (共 15 分, 每小题 3 分) 填空题.

(1) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, then $\operatorname{rank}\left(A^2 - A\right) =$ ___________.

设 $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, 则 $\operatorname{rank}\left(A^2 - A\right) =$ ___________.

(2) Let A be

$$\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $A^{-1} =$___________.

设 A 为

$$\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}.$$

则 $A^{-1} =$___________.

(3) Let $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and let $B = QR$ be its QR decomposition. Then the orthogonal matrix $Q =$ ___________.

设 $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = QR$ 是矩阵 B 的 QR 分解. 则正交矩阵 $Q =$ ___________.

(4) If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $A^{2022} =$___________.

如果 $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, 则 $A^{2022} =$___________.

(5) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$, then all the singular values of A are ___________.

设 $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$, 则 A 的所有奇异值为 ___________.

3. (15 points 本题共 15 分) Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 5 & 8 \end{bmatrix}.$$

(a) Find the determinant of A, $|A|$.

(b) Find elementary matrices, $E_1,\ E_2,\ E_3$ such that

$$E_3E_2E_1A = U,$$

where U is an upper triangular matrix.

(c) Write A in the form $A = LU$, where L is a lower triangular matrix with 1s on the diagonal and U is an upper triangular matrix.

设

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 5 & 8 \end{bmatrix}.$$

(a) 求矩阵 A 行列式, $|A|$.

(b) 求初等矩阵, $E_1,\ E_2,\ E_3$, 使得

$$E_3E_2E_1A = U,$$

其中 U 是一个上三角矩阵.

(c) 把 A 写成 $A = LU$ 的形式, 其中 L 为一对角元为 1 的下三角矩阵, U 为一上三角矩阵.

4. (15 points 本题共 15 分) Let

$$A = \begin{bmatrix} 1 & a \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & b \end{bmatrix}.$$

If there exists a 2×2 matrix $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ such that:

$$AX - XA = B.$$

(a) Find the values of $a,\ b$.

(b) Find all possible matrices for X.

设

$$A = \begin{bmatrix} 1 & a \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & b \end{bmatrix}.$$

如果存在 2×2 的矩阵 $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ 使得:

$$AX - XA = B.$$

(a) 求 $a,\ b$ 的值.

(b) 求所有满足条件的矩阵 X.

5. (20 points 本题共 20 分) Consider the following quadratic form

$$f(x_1, x_2, x_3, x_4) = 2x_1x_2 + kx_3^2 + 2x_3x_4 + 2x_4^2,\ k \in \mathbb{R}.$$

(a) Find the real symmetric matrix A such that $f(x_1, x_2, x_3, x_4) = \mathbf{x}^T A\mathbf{x}$, where $\mathbf{x} = [x_1\ x_2\ x_3\ x_4]^T$.

(b) If 3 is an eigenvalue of A, find the value for k and find an orthogonal matrix Q such that $Q^{-1}AQ = \Lambda$, where Λ is a diagonal matrix.

(c) Decide whether $f(x_1, x_2, x_3, x_4)$ is positive (or negative) definite, or positive (or negative) semidefinite.

考虑下面的二次型

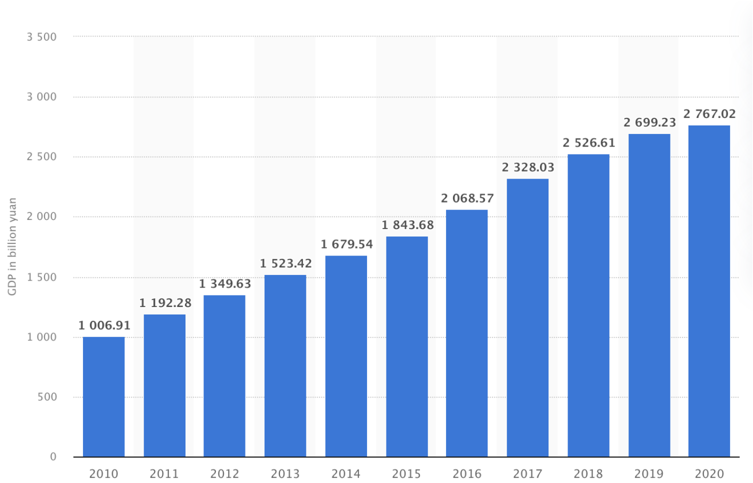
$$f(x_1, x_2, x_3, x_4) = 2x_1x_2 + kx_3^2 + 2x_3x_4 + 2x_4^2,\ k \in \mathbb{R}.$$

(a) 求二次型 $f(x_1, x_2, x_3, x_4)$ 的矩阵 A. 换言之, 求实对称矩阵 A 使得 $f(x_1, x_2, x_3, x_4) = \mathbf{x}^T A\mathbf{x}$, $\mathbf{x} = [x_1\ x_2\ x_3\ x_4]^T$.

(b) 如果 3 是矩阵 A 的一个特征值, 求 k 的值, 并求一个正交矩阵 Q 使得 $Q^{-1}AQ = \Lambda$, 其中 Λ 是一个对角矩阵.

(c) 判断 $f(x_1, x_2, x_3, x_4)$ 是否正定或负定、是否半正定或半负定.

6. (8 points 本题共 8 分) In 2020, the gross domestic product (GDP) at current prices of Shenzhen amounted to around 2.77 trillion yuan. Shenzhen has become one of the most economically active Chinese cities. Benefiting from the opening and reforming policies, the city has witnessed great economic development, which is also reflected by its GDP growth.



Data from Statistica.com, 2022

Question: Apply the method of least-squares for linear regression to predict our city's GDP this year (2022) by sampling every other year's value since 2010 (2010, 2012, 2014, 2016, 2018, 2020), in trillion yuan and rounded off to 2 decimal places.

2020 年, 深圳的 GDP 达到了 2.77 万亿元. 深圳已经成为了中国最具经济活力的城市之一. 受益于城市的开放性和改革政策, 从GDP数据可以清晰地看到深圳的经济经历了快速的发展.

问题: 请根据图表中的数据, 从 2010 年开始考虑隔年的数据 (2010, 2012, 2014, 2016, 2018, 2020), 采用最小二乘的方法得到拟合直线, 并用拟合直线来预测一下今年的 GDP 数据 (以万亿元为单位), 结果请保留两位小数 (相关数据请查看英文题干).

7. (12 points 本题共 12 分) Prove the following two independent statements:

(a) Let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be linearly independent column vectors in $\mathbb{R}^m$, and $\beta_1, \beta_2, \cdots, \beta_s$ be linearly independent column vectors in $\mathbb{R}^n$. If there exist real numbers $c_{ij}, 1 \le i \le r,\ 1 \le j \le s$ such that

$$\sum_{i=1}^{r}\sum_{j=1}^{s} c_{ij}\alpha_i\beta_j^T = 0,$$

show that $c_{ij} = 0$ for all $1 \le i \le r,\ 1 \le j \le s$.

(b) Suppose A and B are two real symmetric positive definite matrices. Show that AB is a real symmetric positive definite matrix if and only if $AB = BA$.

证明下面两个相互独立的结论:

(a) 设 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 是 $\mathbb{R}^m$ 中线性无关的列向量, $\beta_1, \beta_2, \cdots, \beta_s$ 是 $\mathbb{R}^n$ 中线性无关的列向量. 如果存在实数 $c_{ij}, 1 \le i \le r,\ 1 \le j \le s$ 使得

$$\sum_{i=1}^{r}\sum_{j=1}^{s} c_{ij}\alpha_i\beta_j^T = 0.$$

证明对任意的 $1 \le i \le r,\ 1 \le j \le s$ 都有 $c_{ij} = 0$.

(b) 设 A 和 B 都是实对称正定矩阵. 证明: AB 是实对称正定矩阵当且仅当 $AB = BA$.