Step-1

Consider the matrix,

$$A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}.$$

The objective is to find the four subspaces in three-dimensional space associated with the matrix *A*.

Step-2

The four subspaces in three-dimensional space associated with the matrix *A* are column space $\boldsymbol{C}(A)$ of *A*, the null space $\boldsymbol{N}(A)$ of *A*, the column space $\boldsymbol{C}(A^T)$ of A^T and the null space $\boldsymbol{N}(A^T)$ of A^T.

Note that the matrix *A* is in reduced row echelon form.

The leading 1 sâ€™ are in the second and third columns. Therefore, the corresponding columns in the matrix *A* form a basis for the column space of *A*.

So the basis for $\boldsymbol{C}(A)$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}.$

Thus, the column space of *A* is defined as $\boxed{\boldsymbol{C}(A)=\left\{x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}\middle|\ x,y\in\mathbf{R}\right\}}.$

Step-3

The null space $\boldsymbol{N}(A)$ of the matrix *A* is the solution space of the system $A\mathbf{x}=\mathbf{0}$.

Thus, the system $A\mathbf{x}=\mathbf{0}$ is equivalent to $\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$

The equations obtained from this matrix equation are,

$$x_2=0,x_3=0.$$

Here, x_1 is a free variable.

So choose $x_1=t$, where *t* is a parameter.

Therefore, the vector **x** can be written as,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$
$$= t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the basis for the null space of A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

And the null space of A is defined as $N(A) = \left\{ x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \middle| \; x \in \mathbf{R} \right\}$.

Step-4

The transpose of the matrix A is $A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Now find the column space and the null space of the matrix A^T.

The leading 1 sâ€™ in the matrix A^T are in the first and second columns. So the corresponding columns in the matrix A^T form the basis for the column space of A^T.

Thus, the basis for the column space of A is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Thus, the column space of A^T is defined as $\boxed{C(A^T) = \left\{ x\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| \; x, y \in \mathbf{R} \right\}}$.

Step-5

The null space $N\left(A^T\right)$ of the matrix A is the solution space of the system $A^T\mathbf{x}=\mathbf{0}$.

Thus, the system $A^T\mathbf{x}=\mathbf{0}$ is equivalent to $\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

The equations obtained from this matrix equation are,

$x_1=0, x_2=0.$

Step-6

Here, x_3 is a free variable.

So choose $x_3=t$, where t is a parameter.

Therefore, the vector **x** can be written as,

$$\begin{aligned}\mathbf{x}&=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\&=\begin{bmatrix}0\\0\\t\end{bmatrix}\\&=t\begin{bmatrix}0\\0\\1\end{bmatrix}\end{aligned}$$

Thus, the basis for the null space of A^T is $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

And the null space of A^T is defined as $\boxed{N\left(A^T\right)=\left\{x\begin{bmatrix}0\\0\\1\end{bmatrix}\middle|\, x\in\mathbf{R}\right\}}$.