Linear Algebra Practice Problems By Dr. Y. Chen

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1. Start with the matrix
$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 7 & 0 \end{bmatrix}.$$

   a. Find a basis for the column space $C(A)$.

   b. Find a basis for the nullspace $N(A)$.

   c. Find a basis for the row space $C(A^T)$.

   d. Write the complete solution to $Ax = b$.

$$A = \begin{bmatrix} 1 & -2 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 7 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

2. Suppose the matrices A and B have the same column space. Give an example where A and B have different nullspaces—-or say why this is impossible.

3. Find a 3 by 3 matrix A whose column space is the plane $x + y + z = 0$ in $\mathbb{R}^3$.

4. Does there exist a matrix B whose column space is spanned by $(1, 2, 3), (1, 0, 1)$ and whose nullspace is spanned by $(1, 2, 3, 6)$. If so, construct B. If not, explain why not.

5. Is the set of matrices a vector space or not? All 3 by 3 matrices with $(1, 1, 1)$ in their column space. YES or NO with a reason.

6. Suppose A is an $m \times n$ matrix of rank r.

   a. If $Ax = b$ has a solution for every right side b, what is the column space of A.

   b. In part (a), what are all equations or inequalities that must hold between the numbers m, n, r.

   c. Give a specific example of rank 1 with first row $[2 \quad 5]$. Describe the column space $C(A)$ and the nullspace $N(A)$ completely.

   d. Suppose the right side b is the same as the first column in your example (part c). Find the complete solution to $Ax = b$.

7. Suppose that row operations (elimination) reduce the matrices A and B to the same row echelon form
$$R = \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   a. Which of the four subspaces are sure to be the same for A and B.

   b. Each time the subspaces in part (a) are the same for A and B, find a basis for the subspace.

8. Let

$$b_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, b_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and let T be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ defined by

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = x_1 b_1 + x_2 b_2 + (x_1 + x_2) b_3$$

Find the matrix A representing T with respect to the ordered bases $\{e_1, e_2\}$ and $\{b_1, b_2, b_3\}$.

9. Suppose T is reflection across the x-axis and S is the reflection across the y-axis. The domain V is the $x-y$ plane. If $v = (x, y)$ what is $S(T(v))$? Find a simple description of the product ST.

10. Suppose T is reflection across the 45° line, and S is a reflection across the y-axis. If $v = (1, 2)$ then $T(v) = (1, 2)$. Find $S(T(v))$ and $T(S(v))$.

11. Show that the product ST of two reflections is a rotation.

12. Let $E = \{u_1, u_2, u_3\}$ and $F = \{b_1, b_2\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

For the following linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$, find the matrix representing T with respect to the ordered bases E and F:

1. $T(x) = (x_3, x_1)^T$.
2. $T(x) = (x_1 + x_2, x_1 - x_3)^T$.
3. $T(x) = (2x_2, -x_1)^T$.

13. Find a matrix whose row space contains $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and whose nullspace contains $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, or prove that there is no such matrix.

14. Find the matrix that projects every point in the plane onto the line $x + 2y = 0$.

15. What matrix P projects every point in $\mathbb{R}^3$ onto the line of intersection of the planes $x + y + t = 0$ and $x - t = 0$?

16. Give a vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ makes

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 11 \\ -8 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

an orthogonal basis for the vector space $\mathbb{R}^3$.

17. Can you find a 3×3 matrix A such that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the left-nullspace of A and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for nullspace of A?

18. The system $Ax = b$ has a solution if and only if b is orthogonal to what subspace?

19. Prove that the trace of $P = aa^T/a^Ta$–which is the sum of its diagonal entries–always equals 1.

20. Let S be the subspace of $\mathbb{R}^4$ containing all vectors $x_1 + x_2 + x_3 + x_4 = 0$. Find a basis for the space $S^\perp$, containing all vectors orthogonal to S.

21. Prove that if A is symmetric, then the column space of A is orthogonal to the nullspace of A.

22. Let P be the plane in $\mathbb{R}^2$ with equation $x + 2y - z = 0$. Find a vector perpendicular to P. What matrix has the plane P as its nullspace, and what matrix has P as its row space?

23. Find an orthonormal set q_1, q_2, q_3 for which q_1, q_2 span the column space of

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$$

Which fundamental subspace contains q_3? What is the least-squares solution of $Ax = b$ if $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

24. Show that an orthogonal matrix that is upper triangular must be diagonal.

25. (a) Find a basis for the subspace S in $\mathbb{R}^4$ spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0.$$

(b) Find a basis for the orthogonal complement $S^\perp$.

(c) Find b_1 in S and b_2 in $S^\perp$ so that $b_1 + b_2 = b = (1, 1, 1, 1)$.

26. Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbb{R}^6$. Under what condition on the vector v will there be a fourth orthonormal vector q_4 that is a combination of v, q_1, q_2, q_3. Give a formula for that fourth orthonormal vector q_4.

27. Find an orthonormal basis for the subspace S of $\mathbb{R}^4$ spanned by these three vectors:

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}, a_3 = a_1 + a_2.$$

Find the closest vector p in that subspace S to the vector

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

28. Given that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}.$$

Find det A.

29. Find an orthonormal basis for the column space of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 4 & 6 \\ 1 & 4 & 6 \end{bmatrix}$$

30. Give an orthonormal basis for the nullspace of

$$A = \begin{bmatrix} 1 & -2 & -5 & 1 \\ 1 & -4 & -10 & 3 \end{bmatrix}.$$

31. At $t = 1, 2, 3$ we are given values b_1, b_2, b_3. The idea is to fit the best straight line $b = C + Dt$ to those three points.

(a) Find the best line $\bar{C} + \bar{D}t$ if the values are

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) What 3×3 matrix P projects every vector onto the plane containing the column vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

32. Find the determinant of

$$A = \begin{bmatrix} -2 & 5 & -1 & 3 \\ 1 & -9 & 13 & 7 \\ 3 & -1 & 5 & -5 \\ 2 & 8 & -7 & -10 \end{bmatrix}.$$

33. Find the determinant of

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

34. Find the determinant of

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 5 & 0 \\ 1 & 3 & 9 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

35. (a) If Q is an orthogonal matrix (square with orthonormal columns), show that $\det Q = 1$ or -1.

(b) How many of the 24 terms in $\det A$ are nozero, and what is $\det A$?

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

36. (a) Suppose A is a 4 by 4 matrix. If you add 1 to the entry a_{14} in the northeast corner, how much will the determinant change?

(b) Explain why the determinant of every projection matrix is either 0 or 1.

(c) Find the determinant of the "circulant matrix"

$$\begin{bmatrix} 0 & b & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & b \\ b & 0 & a & 0 \end{bmatrix}.$$

37. Compute the determinant of

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

38. Using Cramer's rule, find b_3 such that $x_3 = 0$ for the solution of

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ b_3 \end{bmatrix}$$

39. Using rules for the determinant (so do not compute it with any of the 3 formulas), show the steps and rules that lead to

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

40. If you know that $\det A = 6$, what is the determinant of B?

$$\det A = \begin{vmatrix} row\ 1 \\ row\ 2 \\ row\ 3 \end{vmatrix} = 6 \quad \det A = \begin{vmatrix} row\ 3 + row\ 2 + row\ 1 \\ row\ 2 + row\ 1 \\ row\ 1 \end{vmatrix}$$

41. Prove $\det A = 0$ for the 5 by 5 *all-ones matrix* (all $a_{ij} = 1$) in two ways:

   (1) Using Properties 1-10 for determinants.

   (2) Using the "big formula" $=$ sum of 120 terms.

42. Compute the determinant of the following matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 & 2 \end{bmatrix}.$$

   Mention the method used for each step in the calculation.

43. Show that the following determinant is zero for any values of a, b, and c:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

44. For the following 3×3 matrix A, compute its determinant by using the cofactor formula and expanding along the third column. Show that values of the 3 cofactors you compute.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -2 \\ 1 & -4 & 1 \end{bmatrix}$$

45. The matrix A has varying $1 - x$ in the (1,2) position:

$$\begin{bmatrix} 2 & 1-x & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix}$$

   (a) When $x = 1$ compute $\det A$. What is the (1,1) entry in the inverse when $x = 1$?

   (b) When $x = 0$ compute $\det A$.

   (c) How do the properties of the determinant say that $\det A$ is a linear function of x? For any x compute $\det A$. For which x's is the matrix singular?

46. Find the determinant of A and A^{-1} and the (1,2) entry of A^{-1} if

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 1 & 7 \end{bmatrix}$$

47. (a) Find the area of the triangle on the plane $\mathbb{R}^2$ with the vertices $(1,1), (2,3), (3,2)$.

   (b) Calculate the determinant of the 4×4 matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

(c) Find the inverse of the matrix A from part (b). Check your answer by multiplying it with A.

48. If A is the 4 by 4 matrix of ones, find the eigenvalues and the determinant of $A - I$.

49. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

50. Find the eigenvalues for the following two permutation matrices:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

51. If A has eigenvalues 0 and 1, corresponding to the eigenvectors

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

How can you tell in advance that A is symmetric? What are its trace and determinant? What is A?

52. Find a complete set of eigenvalues and eigenvectors for the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Find A^{100}.

53. Suppose A has eigenvalues $\lambda_1 = 3$,$\lambda_2 = 1$, $\lambda_3 = 0$ with corresponding eigenvectors

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

(a) How do you know that the third column of A contains all zeros?

(b) Find the matrix A.

(c) By transposing $S^{-1}AS = \Lambda$, find the eigenvectors y_1, y_2, y_3 of A^T.

54. The Fibonacci numbers $F_0, F_1, F_2, F_3, \cdots$ are $0, 1, 2, 3, \cdots$ and they obey the rule $F_{k+2} = F_{k+1} + F_k$. In matrix form this is

$$\begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} \quad \text{or} \quad u_{k+1} = Au_k.$$

The eigenvalues of this particular matrix A will be called λ_1 and λ_2.

(a) Find a matrix that has eigenvalues λ_1^2 and λ_2^2.

(b) Find A^k.

(c) What is the determinant of A^k?

55. Find the eigenvalues of

$$\begin{bmatrix} -3 & 2 & 4 \\ 2 & -6 & 2 \\ 4 & 2 & -3 \end{bmatrix}.$$

56. (a) Find the matrix A (fill in the two blank entries) so that A has eigenvectors $x_1 = (3, 1)$ and $x_2 = (2, 1)$:

$$\begin{bmatrix} 2 & 6 \\ & \end{bmatrix}$$

(b) Find a different matrix B with those same x_1 and x_2, and with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$. What is B^{10}?

57. Let

$$A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Find all the eigenvalues and their corresponding eigenvectors of A.

58. Suppose u is a unit vector in $\mathbb{R}^n$, so $u^T u = 1$. This problem is about the n by n symmetric matrix $H = I - 2uu^T$. Find all the eigenvalues and eigenvectors of H.

59. There are six 3 by 3 permutation matrices. What numbers can be the determinants of P? What numbers can be pivots? What numbers can be the trace of P? What four numbers can be eigenvalues of P?

60. If A is the n by n matrix and B is n by n, show that $\text{Trace}(AB) = \text{Trace}(BA)$.

61. Consider the following matrix

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 0 & -3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$$

(a). Show that A is diagonalizable.

(b). Find A^k, where k is a positive integer.

62. Prove that there do not exist $n \times n$ matrices A and B such that

$$AB - BA = I.$$

63. Let A and B be $n \times n$ matrices. Show that

(a) If λ is a nonzero eigenvalue of AB, then it is also an eigenvalue of BA.

(b) If 0 is an eigenvalue of AB, then it is also an eigenvalue of BA.

64. Let $p(\lambda) = (-1)^n(\lambda^n - a_{n-1}\lambda^{n-1} - \cdots - a_1\lambda - a_0)$ be a polynomial of degree $n \geq 1$, and let

$$C = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

(a) Show that if λ_i is a root of $p(\lambda) = 0$, then λ_i is an eigenvalue of C with eigenvector $x = (\lambda_i^{n-1}, \lambda_i^{n-2}, \cdots, \lambda_i, 1)$.

(b) Use part (a) to show that if $p(\lambda)$ has n distinct roots then $p(\lambda)$ is the characteristic polynomial of C.

65. Let A be a matrix whose columns all add up to a fixed constant δ. Show that δ is an eigenvalue of A.

66. Let Q be a 3×3 orthogonal matrix whose determinant is equal to 1.

    (a) If the eigenvalues of Q are all real and if they are ordered so that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, determine the values of all possible triples of eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$.

    (b) In the case that the eigenvalues λ_2 and λ_3 are complex, what are the possible values for λ_1? Explain.

    (c) Explain why $\lambda = 1$ must be an eigenvalue of Q.

67. For the following matrix

$$A = \begin{bmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Find a matrix B such that $B^2 = A$.

68. Let x, y be nonzero vectors in $\mathbb{R}^n, n \geq 2$, and let $A = xy^T$. Show that

    (a) $\lambda = 0$ is an eigenvalue of A with $n - 1$ linearly independent eigenvectors and consequently has multiplicity at least $n - 1$.

    (b) the remaining eigenvalue of A is $\lambda_n = \text{tr}(A) = x^T y$ and x is an eigenvector belonging to λ_n.

    (c) if $\lambda_n = x^T y \neq 0$, then A is diagonalizable.

69. Let A be diagonalizable matrix whose eigenvalues are all either 1 or -1. Show that $A^{-1} = A$.

70. Let

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

Find the eigenvalues and the corresponding eigenvectors.

71. Let

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

Find the eigenvalues and the corresponding eigenvectors.

72. Show that if U and V are unitary, so is UV.

73. Diagonalize A (real λ's) and K (imaginary λ's) to reach $U\Lambda U^H$.

$$A = \begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}$$

74. Diagonalize this matrix by constructing its eigenvalue matrix Λ and its eigenvector matrix S:
$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}.$$

75. (a) What matrix M changes the basis $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ to the basis $v_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$?

(b) For the same two bases, express the vector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as a combination $c_1V_1 + c_2V_2$ also as $d_1v_1 + d_2v_2$. Check numerically that M connects c to d: $Md = c$.

76. On the space of 2×2 matrices, let T be the transformation that transposes every matrix. Find the eigenvalues and "eigenvectors" for $A^T = \lambda A$.

77. If A and B have the exactly the same eigenvalues and eigenvectors, does $A = B$?

78. Find the eigenvalues and eigenvectors of
$$A = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix}.$$

79. (a) Find the matrix $P = aa^T/a^Ta$ that projects any vector onto the line through $a = (2, 1, 2)$.

(b) What is the only nonzero eigenvalue of P, and what is the corresponding eigenvector?

(c) Find P^k, where k is a positive integer.

80. Suppose the first row of A is $7, 6$ and its eigenvalues are $i, -i$. Find A.

81. (a) For which numbers c and d does A have real eigenvalues and orthogonal eigenvectors?
$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & d & c \\ 0 & 5 & 3 \end{bmatrix}.$$

(b) For which c and d can we find three orthonormal vectors that are combinations of the columns (don't to it!)?

82. Explain why A is never similar to $A + I$.

83. Describe in words all matrices that are similar to
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
and find two of them.

84. (i) Find a nonzero matrix N such that $N^3 = 0$.

(ii) If $Nx = \lambda x$, show that λ must be zero.

(iii) Prove that N can not be symmetric.

85. If $A^2 = -I$, what are the eigenvalues of A? If A is a real n by n matrix show that n must be even, and give an example.

86. Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

87. Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

88. Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2.$$

89. Show that if A is positive definite, so are A^2 and A^{-1}.

90. Write down the five conditions for a 3 by 3 matrix to be negative definite ($-A$ is positive definite) with special attention to condition III: How is $\det(-A)$ related to $\det A$?

91. If A has eigenvalues $1, 2, 3$, what are the eigenvalues of $(A - I)(A - 2I)(A - 3I)$?

92. The matrix A has independent columns. The matrix C is square, diagonal, and has positive diagonal entries. Why is the matrix $K = A^T CA$ positive definite?

93. Show that if A is a diagonalizable and has orthonormal eigenvectors and real eigenvalues, then A must be symmetric.

94. Suppose A is a positive definite symmetric matrix.

    (i) How do you know that A^{-1} is also positive definite?

    (ii) Suppose Q is any orthogonal n by n matrix. How do you know that QAQ^T is positive definite?

    (iii) Show that the block matrix
    $$B = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$$
    is positive semidefinite. How do you know B is not positive definite?

95. Let A be an $m \times n$ matrix with rank n. Show that the matrix $A^T A$ is symmetric positive definite.

96. Let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times n}$.

(i) Show that

$$\det(I_n + AB) = \det(I_m + BA).$$

(ii) Let x, y, u, v be given vectors in R^n. Please give

$$\left|I_n - xy^T - uv^T\right|$$

in the form of the inner product of the given vectors.

97. And all the homework assignments$\cdots\cdots$

98. Happy New Year!