1. (a) False

$V=\mathbb{R}^2 \quad W=\{(x,0): x\in\mathbb{R}\} \quad U_1=\{(0,y): y\in\mathbb{R}\} \quad U_2=\{(x,x): x\in\mathbb{R}\}$

(b) False

$V=\mathbb{R}^2 \quad Te_1=e_1 \quad Te_2=2e_2$

If $T(e_1+e_2)=\lambda(e_1+e_2)$, then $\lambda e_1+\lambda e_2=e_1+2e_2 \Rightarrow (\lambda-1)e_1+(\lambda-2)e_2=0$

$\Rightarrow \lambda=1$ and $\lambda=2$

(c) True Theorem

(d) True

$\forall T\in\mathcal{L}(V,\mathbb{F})$ If T is not zero map, $\exists v\in V$, s.t. $Tv\neq 0$

Let $z=\frac{v}{Tv}$, then $Tz=1$

$\forall k\in\mathbb{F}$. $T(kz)=k \Rightarrow T$ is surjective

(e) True.

$v_1+U_1=v_2+U_2 \Rightarrow U_1=(v_2-v_1)+U_2$.

Since U_1, U_2 are subspaces, then $0\in U_1, 0\in U_2$, $\Rightarrow v_1-v_2\in U_2 \Rightarrow U_1=U_2$.

2. (1) $U_1+U_2+U_3$ is a subspace of V

(2) $U_1+U_2+U_3$ is the smallest subspace of V containing U_1, U_2, U_3.

Let W be a subspace of V containing U_1, U_2, U_3. WTS: $U_1+U_2+U_3 \subseteq W$.

$\forall\ u_1+u_2+u_3 \in U_1+U_2+U_3$, $u_1 \in U_1$, $u_2 \in U_2$. $u_3 \in U_3$

And $U_1 \subseteq W$. $U_2 \subseteq W$. $U_3 \subseteq W$, then $u_1 \in W$, $u_2 \in W$. $u_3 \in W$

Since W is a subspace of V, $u_1+u_2+u_3 \in W \Rightarrow U_1+U_2+U_3 \subseteq W$.

3. Suppose $k_1(-1)+k_2\sin x+k_3\cos^2 x=0$ $\forall x \in \mathbb{R}$. $k_1, k_2, k_3 \in \mathbb{R}$

Take $x=0$, $-k_1+k_3=0$

Take $x=\frac{\pi}{2}$, $-k_1+k_2=0$

Take $x=-\frac{\pi}{2}$, $-k_1-k_2=0$

then $\begin{cases} -k_1+k_3=0 \\ -k_1+k_2=0 \\ -k_1-k_2=0 \end{cases} \Rightarrow k_1=k_2=k_3=0$

so $-1, \sin x, \cos^2 x$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

4. (1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$

$\forall A, B \in S$. $A = A^T$, $B = B^T$, then $(A+B)^T = A^T + B^T = A+B \Rightarrow A+B \in S$

$\forall k \in \mathbb{R}$. $\forall A \in S$. $(kA)^T = kA^T = kA \Rightarrow kA \in S$

Hence, S is a subspace of $\mathbb{R}^{2\times 2}$

(2) ① $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is linearly independent

② $\forall A \in S$. $A = (-a_{12})\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

so E is a basis of S

(3)

$T(1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = (-1)\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T(1-x) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = 0\cdot\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T(1+x^2) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = 0\cdot\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\cdot\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathcal{M}(T) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

5. (a)

$T(x, y, z) = \lambda(x, y, z) = (8x,\ 3x+5y,\ y+2z)$

$\Rightarrow (\lambda x, \lambda y, \lambda z) = (8x, 3x+5y, y+2z)$

$$\Rightarrow \begin{cases} \lambda x = 8x \\ \lambda y = 3x+5y \\ \lambda z = y+2z \end{cases} \Rightarrow \begin{cases} (\lambda-8)x = 0 \\ (\lambda-5)y = 3x \\ (\lambda-2)z = y \end{cases}$$

If $x=0$, $(\lambda-5)y=0$.

If $y=0$, $(\lambda-2)z=0$ $z\neq 0$ $\Rightarrow \lambda=2$ $(0,0,1)$ is an eigenvector of $\lambda=2$

If $y\neq 0$, $\lambda=5$ $\Rightarrow 3z=y$ $\Rightarrow (0,3,1)$. is an eigenvector of $\lambda=5$

If $x\neq 0$, $\lambda=8$ $\Rightarrow \begin{cases} 3y=3x \\ 6z=y \end{cases} \Rightarrow \begin{cases} y=x \\ z=\frac{1}{6}x \end{cases} \Rightarrow (6,6,1)$ is an eigenvector of $\lambda=8$

(b) Let $v_1=\begin{pmatrix}0\\0\\1\end{pmatrix}$ $v_2=\begin{pmatrix}0\\3\\1\end{pmatrix}$ $v_3=\begin{pmatrix}6\\6\\1\end{pmatrix}$, since v_1, v_2, v_3 are eigenvectors corresponded to distinct eigenvalues of T, then v_1, v_2, v_3 is linearly independent.

The length of v_1, v_2, v_3 is 3, which is equal to $\dim \mathbb{R}^3$, so v_1, v_2, v_3 is a basis of $\mathbb{R}^3$, and

$$\mathcal{M}(T) = \begin{pmatrix} 2 & & \\ & 5 & \\ & & 8 \end{pmatrix}$$

(c) 方法不唯一!

$T(x,y,z)=(0,0,0) \Rightarrow (8x, 3x+5y, y+2z)=(0,0,0)$

$\Rightarrow 8x=0,\ 3x+5y=0,\ y+2z=0$

$\Rightarrow x=y=z=0$

$\Rightarrow T$ is injective.

And $\mathbb{R}^3$ is finite-dimensional, so T is invertible.

(d) 方法不唯一！

Suppose $T^{-1}(1,2,3)=(x,y,z)$, then

$$T(T^{-1}(1,2,3))=T(x,y,z)$$

$$\Rightarrow T(x,y,z)=(1,2,3)$$

$$\Rightarrow (8x,\ 3x+5y,\ y+2z)=(1,2,3)$$

$$\Rightarrow \begin{cases} 8x=1 \\ 3x+5y=2 \\ y+2z=3 \end{cases} \Rightarrow \begin{cases} x=\frac{1}{8} \\ y=\frac{13}{40} \\ z=\frac{107}{80} \end{cases}$$

6. Let $E_{ij}\in\mathbb{R}^{n,n}$

$$E_{ij}= \begin{array}{c} \\ i \end{array} \begin{array}{c} \begin{array}{ccccccc} & & & j & & & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} \qquad \forall i,j=1,\cdots,n$$

$$E_{ij}\cdot E_{kl}=\begin{cases} E_{il}, & j=k \\ 0, & j\neq k \end{cases} \qquad \forall i,j,k,l=1,\cdots,n.$$

$$T(E_{ii})=T(E_{ij}\cdot E_{ji})=T(E_{ji}\cdot E_{ij})=T(E_{jj}) \qquad \forall i,j=1,\cdots,n$$

If $i \neq j$, $T(E_{ij}) = T(E_{ij} \cdot E_{jj}) = T(E_{jj} \cdot E_{ij}) = T(0) = 0$

$\forall A \in \mathbb{R}^{n,n}$ $A = (a_{ij})_{n \times n}$

$$T(A) = T\left(\sum_{i,j=1}^{n} a_{ij} E_{ij}\right) = \sum_{i,j=1}^{n} a_{ij} T(E_{ij}) = \sum_{i=1}^{n} a_{ii} T(E_{ii}) = T(E_{ii})\, tr(A)$$

Let $\lambda = T(E_{ii})$. $T(A) = \lambda\, tr(A)$.

7. Let $w_i = Tv_i$, $i = 1, \cdots, m$ and let $v_{m+1}, \cdots, v_n$ be a basis of null T

$\Rightarrow v_1, \cdots, v_m, v_{m+1}, \cdots, v_n$ is a basis of V

Let $g_1, \cdots, g_n$ be the dual basis of $v_1, \cdots, v_n$

$\forall v \in V$. we have $v = \sum_{i=1}^{n} g_i(v) v_i$

$$Tv = \sum_{i=1}^{n} g_i(v)\, Tv_i = \sum_{i=1}^{m} g_i(v)\, w_i$$