Step-1

Given that if A has r pivot columns then A^T has r pivot columns. We have to give a 3 by 3 example for which the column numbers are different for A and A^T.

Step-2

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

$$\xrightarrow[R_3 - 2R_1]{R_2 - 3R_1,} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step-3

$$\xrightarrow{-R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore 1,3 columns are pivot columns.

Hence the number of pivot columns in A is 2

Step-4

Now

$$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 4 \\ 3 & 5 & 6 \end{bmatrix}$$

$$\xrightarrow{\substack{R_2-2R_1,\\ R_3-3R_1}} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step-5

$$\xrightarrow{\frac{-1}{2}R_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then A^T has first and second columns as pivot columns.

Therefore the number of pivot columns in A^T is 2

Hence number of pivot columns in A is same as the number of pivot columns in A^T