Southern University of Science and Technology
Advanced Linear Algebra Spring 2023
MA109– Quiz #9

2023/04/22

Name: ____

Student Number: ___________________________________

1. Prove or disprove: there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by

$$\|(x,y)\| = \max\{|x|,|y|\}$$

for all $(x,y) \in \mathbf{R}^2$.

证明或反证：存在 $\mathbf{R}^2$ 上的内积，使得相应的内积是

$$\|(x,y)\| = \max\{|x|,|y|\}$$

对任意 $(x,y) \in \mathbf{R}^2$.

Proof. Suppose there exists an inner product $\langle \cdot, \cdot \rangle$ such that $\forall (x,y) \in \mathbf{R}^2$,

$$\langle (x,y),(x,y) \rangle = \|(x,y)\| = \max\{|x|^2,|y|^2\}.$$

According to 6.22 (Parallelogram Equality), we have $\forall (x_1,y_1),(x_2,y_2) \in \mathbf{R}^2$,

$$\|(x_1,y_1)+(x_2,y_2)\|^2 + \|(x_1,y_1)-(x_2,y_2)\|^2 = 2(\|(x_1,y_1)\|^2 + \|(x_2,y_2)\|^2)$$

then

$$\max\{|x_1+x_2|^2,|y_1+y_2|^2\} + \max\{|x_1-x_2|^2,|y_1-y_2|^2\} = 2(\max\{|x_1|^2,|y_1|^2\} + \max\{|x_2|^2+|y_2|^2\}).$$

Take $x_1 = x_2 = y_1 = 0$, $y_2 = 1$, we have LHS$=2$, RHS$=4$, which is a contradiction! $\square$

2. Suppose $V_1, \cdots, V_m$ are inner product spaces. Show that the equation

$$\langle (u_1, \cdots, u_m), (v_1, \cdots, v_m) \rangle = \langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle$$

defines an inner product on $V_1 \times \cdots \times V_m$.

设 $V_1, \cdots, V_m$ 是内积空间. 证明

$$\langle (u_1, \cdots, u_m), (v_1, \cdots, v_m) \rangle = \langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle$$

是 $V_1 \times \cdots \times V_m$ 上的内积.

Proof. **positivity**: $\forall (v_1, \cdots, v_m) \in V_1 \times \cdots \times V_m$

$$\langle (v_1, \cdots, v_m), (v_1, \cdots, v_m) \rangle = \langle v_1, v_1 \rangle + \cdots + \langle v_m, v_m \rangle \geqslant 0$$

definiteness

$$\langle (v_1, \cdots, v_m), (v_1, \cdots, v_m) \rangle = \langle v_1, v_1 \rangle + \cdots + \langle v_m, v_m \rangle = 0 \Leftrightarrow \langle v_i, v_i \rangle = 0 \forall i \ \Leftrightarrow v_i = 0 \forall i \ \Leftrightarrow (v_1, \cdots, v_m) = 0$$

additivity in the first slot: $\forall (v_1, \cdots, v_m), (u_1, \cdots, u_m), (w_1, \cdots, w_m) \in V_1 \times \cdots \times V_m$

$$\begin{aligned} \langle (u_1, \cdots, u_m) + (v_1, \cdots, v_m), (w_1, \cdots, w_m) \rangle = & \ \langle (u_1 + v_1, \cdots, u_m + v_m), (w_1, \cdots, w_m) \rangle \\ & = \langle u_1 + v_1, w_1 \rangle + \cdots + \langle u_m + v_1, w_m \rangle \\ & = \langle u_1, w_1 \rangle + \langle v_1, w_1 \rangle + \cdots + \langle u_m, w_m \rangle + \langle v_m, w_m \rangle \\ & = (\langle u_1, w_1 \rangle + \cdots + \langle u_m, w_m \rangle) + (\langle v_1, w_1 \rangle + \cdots + \langle v_m, w_m \rangle) \\ & = \langle (u_1, \cdots, u_m), (w_1, \cdots, w_m) \rangle + \langle (v_1, \cdots, v_m), (w_1, \cdots, w_m) \rangle \end{aligned}$$

homogeneity in the first slot: $\forall (v_1, \cdots, v_m), (u_1, \cdots, u_m) \in V_1 \times \cdots \times V_m$, $\forall \lambda \in \mathbf{F}$

$$\begin{aligned} \langle \lambda (u_1, \cdots, u_m), (v_1, \cdots, v_m) \rangle = & \ \langle (\lambda u_1, \cdots, \lambda u_m), (v_1, \cdots, v_m) \rangle \\ & = \langle \lambda u_1, v_1 \rangle + \cdots + \langle \lambda u_m, v_m \rangle \\ & = \lambda \langle u_1, v_1 \rangle + \cdots + \lambda \langle u_m, v_m \rangle \\ & = \lambda (\langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle) \\ & = \lambda \langle (u_1, \cdots, u_m), (v_1, \cdots, v_m) \rangle \end{aligned}$$

conjugate symmetry: $\forall (v_1, \cdots, v_m), (u_1, \cdots, u_m) \in V_1 \times \cdots \times V_m$

$$\begin{aligned} \langle (v_1, \cdots, v_m), (u_1, \cdots, u_m) \rangle = & \ \langle v_1, u_1 \rangle + \cdots + \langle v_m, u_m \rangle \\ & = \overline{\langle u_1, v_1 \rangle} + \cdots + \overline{\langle u_m, v_m \rangle} \\ & = \overline{\langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle} \\ & = \overline{\langle (u_1, \cdots, u_m), (v_1, \cdots, v_m) \rangle} \end{aligned}$$

Hence, $\langle (u_1, \cdots, u_m), (v_1, \cdots, v_m) \rangle = \langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle$ defines an inner product on $V_1 \times \cdots \times V_m$. $\square$