Step-1

Given that $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We have to find the weighted least-squares solution $\hat{x}_w$ to $Ax = b$.

Step-2

We know that the least squares solution to $WAx = Wb$ is $\hat{x}_w$.

We know that the weighted normal equations are obtained by $A^T W^T W A \hat{x}_w = A^T W^T W b$.

Now

$$A^T W^T W A \hat{x}_w = A^T W^T W b$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \hat{x}_w = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \hat{x}_w = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 4 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \hat{x}_w = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Step-3

Applying $R_2 \to 2R_2 - R_1$, we get

$$\begin{bmatrix} 6 & 3 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$\Rightarrow 7x_2 = 4$ and $6x_1 + 3x_2 = 2$

$$\Rightarrow x_2 = \frac{4}{7} \text{ and } 6x_1 = 2 - 3x_2 = \frac{2}{7}$$

$$\Rightarrow x_2 = \frac{4}{7} \text{ and } x_1 = \frac{1}{21}$$

Hence the weighted least-squares solution to $Ax = b$ is $\boxed{\hat{x}_w = \begin{bmatrix} \frac{1}{21} \\ \frac{4}{7} \end{bmatrix}}$.

Step-4

We have to check that the projection $A\hat{x}_w$ is perpendicular to the error $b - A\hat{x}_w$.

Now

$$A\hat{x}_w = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{21} \\ \frac{4}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 1\left(\frac{1}{21}\right) + 0\left(\frac{4}{7}\right) \\ 1\left(\frac{1}{21}\right) + 1\left(\frac{4}{7}\right) \\ 1\left(\frac{1}{21}\right) + 4\left(\frac{4}{7}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{21} \\ \frac{13}{21} \\ \frac{25}{21} \end{bmatrix}$$

Step-5

And

$$b - A\hat{x}_w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{21} \\ \frac{13}{21} \\ \frac{25}{21} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{21} \\ \frac{8}{21} \\ -\frac{4}{21} \end{bmatrix}$$

Now

$$A\hat{x}_w \cdot \left[W^T W \left(b - A\hat{x}_w\right)\right] = \begin{bmatrix} \frac{1}{21} \\ \frac{13}{21} \\ \frac{25}{21} \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{21} \\ \frac{8}{21} \\ -\frac{4}{21} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{21} \\ \frac{13}{21} \\ \frac{25}{21} \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{21} \\ \frac{8}{21} \\ -\frac{4}{21} \end{bmatrix}$$

Step-6

Continuation to the above

$$= \begin{bmatrix} \frac{1}{21} \\ \frac{13}{21} \\ \frac{25}{21} \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{21} \\ \frac{8}{21} \\ -\frac{4}{21} \end{bmatrix}$$

$$= \frac{1}{21^2}\left[-4 + 104 - 100\right]$$

$$= 0$$

Hence $A\hat{x}_w$ is perpendicular to $W^T W\left(b - A\hat{x}_w\right)$.