$T \in \mathcal{L}(V)$

$\lambda \in \mathbb{F}$

$E(\lambda, T) = \text{null}(T - \lambda I)$

$\lambda_1, \lambda_2, \cdots, \lambda_m$ distinct eigenvalues

$E(\lambda_1, T) \cdots E(\lambda_m, T)$

$v_1, \cdots, v_m$ linearly independent

Eigenspaces and Diagonal Matrices

$\dim E(\lambda_1, T) = k_1, \cdots, \dim E(\lambda_m, T) = k_m$

$k_1 + \cdots + k_m \leq \dim V = n$

Lecture 15

Dept. of Math., SUSTech

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$T(v_1, \cdots, v_n) = (v_1, \cdots, v_n) \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

T is diagonalizable.

Eigenvalues

Diagonal matrix

5.34 **Definition** ***diagonal matrix***

A ***diagonal matrix*** is a square matrix that is 0 everywhere except possibly along the diagonal.

If an operator has a diagonal matrix with respect to some basis, then the entries along the diagonal are precisely the eigenvalues of the operator; this follows from 5.32.

Example

5.35 Example

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

is a diagonal matrix.

Eigenspace

5.36 Definition ***eigenspace***, $E(\lambda, T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The ***eigenspace*** of T corresponding to λ, denoted $E(\lambda, T)$, is defined by

$$E(\lambda, T) = \operatorname{null}(T - \lambda I).$$

In other words, $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

For $T \in \mathscr{L}(V)$ and $\lambda \in \mathbb{F}$, the eigenspace $E(\lambda, T)$ is a subspace of V (because the null space of each linear map on V is a subspace of V). The definitions imply that λ is an eigenvalue of T if and only if $E(\lambda, T) \neq \{0\}$.

Eigenspace

For $T \in \mathscr{L}(V)$ and $\lambda \in \mathbb{F}$, the eigenspace $E(\lambda, T)$ is a subspace of V.

The definitions imply that λ is an eigenvalue of T if and only if $E(\lambda, T) \neq \{0\}$.

5.37 **Example** Suppose the matrix of an operator $T \in \mathcal{L}(V)$ with respect to a basis v_1, v_2, v_3 of V is the matrix in Example 5.35 above. Then

$$E(8, T) = \operatorname{span}(v_1), \quad E(5, T) = \operatorname{span}(v_2, v_3).$$

If λ is an eigenvalue of an operator $T \in \mathscr{L}(V)$, then T restricted to $E(\lambda, T)$ is just the operator of multiplication by λ.

Sum of eigenspaces is a direct sum

If λ is an eigenvalue of an operator $T \in \mathscr{L}(V)$, then T restricted to $E(\lambda, T)$ is just the operator of multiplication by λ.

5.38 Sum of eigenspaces is a direct sum

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$v \in E(\lambda_1, T) + \cdots + E(\lambda_m, T) \rightarrow \text{Subspace of } V$$

$v = u_1 + \cdots + u_m, \quad u_j \in E(\lambda_j, T) \quad j = 1, \cdots, m$

is a direct sum. Furthermore, $\dim(E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T))$

$$T|_{E(\lambda_1, T)} \qquad \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

$= \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V$

$v \in E(\lambda_1, T) \Rightarrow Tv \in E(\lambda_1, T)$

$(T - \lambda_1 I)v = 0 \Rightarrow (T - \lambda_1 I)(Tv) = 0$

$\Rightarrow T(T - \lambda_1 I)v = 0$

$E(\lambda_1, T)$ invariant under T

Proof

Proof. To show that $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum, suppose

$$u_1 + \cdots + u_m = 0,$$

where each u_j is in $E(\lambda_j, T)$. Because eigenvectors corresponding to distinct eigenvalues are linearly independent (see 5.10), this implies that each u_j equals 0. This implies (using 1.44) that $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum, as desired.

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Now

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) = \dim(E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)) \leq \dim V,$$

where the equality above follows from Exercise 16 in Section 2.C.

diagonalizability

$T^2 = I \to T$ diagonalizable

$\Rightarrow$ eigenvalues $1, -1$.

Claim: $V = E(1,T) \oplus E(-1,T)$

5.39 **Definition** ***diagonalizable***

An operator $T \in \mathcal{L}(V)$ is called ***diagonalizable*** if the operator has a diagonal matrix with respect to some basis of V.

5.40 **Example** Define $T \in \mathcal{L}(\mathbf{R}^2)$ by

$$T(x, y) = (41x + 7y, -20x + 74y).$$

The matrix of T with respect to the standard basis of $\mathbf{R}^2$ is

$$\begin{pmatrix} 41 & 7 \\ -20 & 74 \end{pmatrix},$$

which is not a diagonal matrix. However, T is diagonalizable, because the matrix of T with respect to the basis $(1, 4), (7, 5)$ is

$$\begin{pmatrix} 69 & 0 \\ 0 & 46 \end{pmatrix},$$

as you should verify.

Conditions equivalent to diagonalizability

5.41 Conditions equivalent to diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:

$T(v_1, \cdots, v_n) = (v_1, \cdots, v_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

(a) T is diagonalizable; (a)$\Leftrightarrow$(b)

(b) V has a basis consisting of eigenvectors of T; (b)$\Leftrightarrow$(c)

(c) there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V, each invariant under T, such that

$$V = U_1 \oplus \cdots \oplus U_n;$$

(d)$\Rightarrow$(e)

$\Uparrow \quad \Downarrow$

(a)$\Leftrightarrow$(b)$\Leftrightarrow$c

(d) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$;

(e) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Proof

Proof. An operator $T \in \mathscr{L}(V)$ has a diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with respect to a basis $v_1, v_2, \cdots, v_n$ of V if and only if $Tv_j = \lambda_j v_j$ for each j. Thus (a) and (b) are equivalent.

Suppose (b) holds; thus V has a basis $v_1, v_2, \cdots, v_n$ consisting of eigenvectors of T. For each j, let $U_j = \text{span}\ (v_j)$. Obviously each U_j is a 1-dimensional subspace of V that is invariant under T. Because $v_1, v_2, \cdots, v_n$ is a basis of V, each each vector can be written uniquely as a sum $u_1 + u_2 + \cdots + u_n$ where each u_j is in U_j. Thus $V = U_1 \oplus \cdots \oplus U_n$. Hence (b) implies (c).

$V \supseteq U_1 \oplus \cdots \oplus U_n$
$V \subseteq U_1 \oplus \cdots \oplus U_n$

Proof

Suppose now that (c) holds; thus there are 1-dimensional subspaces $U_1, \cdots, U_n$ of V, each invariant under T, such that $V = U_1 \oplus \cdots \oplus U_n$. For each j, let v_j be a nonzero vector in U_j. Then each v_j is an eigenvector of T. Because each vector in V can be written uniquely as a sum $u_1 + u_2 + \cdots + u_n$, where each u_j is in U_j (so each u_j is a scalar multiple of v_j), we see that $v_1, v_2, \cdots, v_n$ is a basis of V. Thus (c) implies (b).

At this stage of the proof we know that (a),(b), and (c) are all equivalent. We will finish the proof by showing that (b) implies (d), that (d) implies (e), and that (e) implies (b).

Proof

Suppose (b) holds; thus V has a basis consisting of eigenvectors of T. Hence every vector in V is a linear combination of eigenvectors of T, which implies that

$$V = E(\lambda_1, T) + \cdots + E(\lambda_m, T).$$

Now 5,38 shows that (d) holds.

That (d) implies (e) follows immediately from Exercise 16 in Section 2.C.

Proof

basis - linearly independent
spans V $\frac{n \text{ vectors}}{\text{eigenvectors}}$

Finally, suppose (e) holds; thus

$k_1 + \cdots + k_m = n$

$$\dim\ V = \underbrace{\dim E(\lambda_1, T)}_{k_1} + \cdots + \dim E(\lambda_m, T).$$

Choose a basis of each $E(\lambda_j, T)$; put all these bases together to form a list $v_1, v_2, \cdots, v_n$ of eigenvectors of T, where $n = \dim V$ (by 5.42). To show that this list is linearly independent, suppose

$$a_1 v_1 + \cdots + a_n v_n = 0,$$

where $a_1, a_2, \cdots, a_n \in \mathbb{F}$. For each $j = 1, \cdots, m$, let u_j denote the sum of all the terms $a_k v_k$ such that $v_k \in E(\lambda_j, T)$. Thus each u_j is in $E(\lambda_j, T)$, and

$\in E(\lambda_1, T) \quad \cdots \quad \in E(\lambda_m, T)$

$$u_1 + u_2 + \cdots + u_m = 0.$$

Proof.

Because eigenvectors corresponding to distinct eigenvalues are linearly independent (see 5.10), this implies that each u_j equals 0. Because each u_j is a sum of terms $a_k v_k$, where the v_k's were chosen to be a basis of $E(\lambda_j, T)$, this implies that all the a_k's equal 0. Thus $v_1, v_2, \cdots, v_n$ is linearly independent and hence is a basis of V (by 2.39). Thus (e) implies (b), completing the proof.

diagonalizability: Counterexample

Unfortunately not every operator is diagonalizable. This sad state of affairs can arise even on complex vector spaces, as shown by the next example.

5.43 **Example** Show that the operator $T \in \mathcal{L}(\mathbf{C}^2)$ defined by

$$T(w, z) = (z, 0)$$

$\lambda = 0$, $\dim E(\lambda, T) = 1 < \dim \mathbb{C}^2$

is not diagonalizable.

Solution As you should verify, 0 is the only eigenvalue of T and furthermore $E(0, T) = \{(w, 0) \in \mathbf{C}^2 : w \in \mathbf{C}\}$.

Thus conditions (b), (c), (d), and (e) of 5.41 are easily seen to fail (of course, because these conditions are equivalent, it is only necessary to check that one of them fails). Thus condition (a) of 5.41 also fails, and hence T is not diagonalizable.

Enough eigenvalues implies diagonalizability

The next result shows that if an operator has as many distinct eigenvalues as the dimension of its domain, then the operator is diagonalizable.

5.44 Enough eigenvalues implies diagonalizability

If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

Proof.

Proof. Suppose $T \in \mathscr{L}(V)$ has $\dim\ V$ distinct eigenvalues $\lambda_1, \cdots, \lambda_{\dim V}$. For each j, let $v_j \in V$ be an eigenvector corresponding to the eigenvalue λ_j. Because eigenvectors corresponding to distinct eigenvalues are linearly independent (see 5.10), $v_1, \cdots, v_{\dim V}$ is linearly independent. A linearly independent list of $\dim V$ vectors in V is a basis of V (see 2.39); thus $v_1, \cdots, v_{\dim V}$ is a basis of V. With respect to this basis consisting of eigenvectors, T has a diagonal matrix.

Example

5.45 **Example** Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(x, y, z) = (2x + y, 5y + 3z, 8z)$. Find a basis of $\mathbf{F}^3$ with respect to which T has a diagonal matrix.

Solution. $(1,0,0),(1,3,0),(1,6,6)$ is a basis of $\mathbb{F}^3$, and with respect to this basis the matrix of T is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Remark

The converse of 5.44 is not true. For example, the operator T defined on the three-dimensional space $\mathbb{F}^3$ by

$$T(z_1, z_2, z_3) = (4z_1, 4z_2, 5z_3)$$

has only two eigenvalues(4 and 5), but this operator has a diagonal matrix with respect to the standard basis.

In later chapters we will find additional conditions that imply that certain operators are diagonalizable.

Homework Assignment 15

5.C: 3, 5, 9, 10, 12, 13, 16.