Step-1

Consider the following system:

$$dx/dt = 0x - 4y$$
$$dy/dt = -2x + 2y$$

Find the solution for $x(t)$ and $y(t)$ that gets large as $t \to \infty$.

Step-2

Let following be the differential equation of matrices:

$$\frac{du}{dt} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 0 & -4 \\ -2 & 2 \end{bmatrix}$$

Step-3

Find the Eigen values and Eigen vectors and then find e^{At} from $Se^{\Lambda t}S^{-1}$. Check $u(t)$ at $t \to \infty$.

Step-4

First step is to find the Eigen values and Eigen vectors of matrix A. To calculate the Eigen values do the following calculations;

$$A - \lambda I = \begin{bmatrix} 0-\lambda & -4 \\ -2 & 2-\lambda \end{bmatrix}$$
$$\det(A - \lambda I) = 0$$
$$(-\lambda)(2-\lambda) - 8 = 0$$
$$\lambda^2 - 2\lambda - 8 = 0$$

After solving following values are obtained:

$$\lambda_1 = 4$$
$$\lambda_2 = -2$$

Step-5

To calculate Eigen vectors do the following calculations:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix}0-4 & -4\\ -2 & 2-4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-4 & -4\\ -2 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

On solving, values of x and y corresponding to $\lambda=4$ are as follows:

$$x_1=\begin{bmatrix}x\\ y\end{bmatrix}$$
$$=\begin{bmatrix}1\\ -1\end{bmatrix}$$

Step-6

Similarly, Eigen vectors corresponding to Eigen value $\lambda=-2$ is as follows:

$$(A-\lambda I)x=0$$

$$\begin{bmatrix}0+2 & -4\\ -2 & 2+2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}2 & -4\\ -2 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

On solving values of x and y are as follows:

$$x_2=\begin{bmatrix}x\\ y\end{bmatrix}$$
$$=\begin{bmatrix}2\\ 1\end{bmatrix}$$

Step-7

Recall that $e^{At}=Se^{\Lambda t}S^{-1}$. Therefore,

$$e^{At}=\frac{1}{3}\begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix}\begin{bmatrix}e^{4t} & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}1 & -2\\ 1 & 1\end{bmatrix}$$
$$=\frac{1}{3}\begin{bmatrix}e^{4t} & 2e^{-2t}\\ -e^{4t} & e^{-2t}\end{bmatrix}\begin{bmatrix}1 & -2\\ 1 & 1\end{bmatrix}$$
$$=\frac{1}{3}\begin{bmatrix}e^{4t}+2e^{-2t} & -2e^{4t}+2e^{-2t}\\ -e^{4t}+e^{-2t} & 2e^{4t}+e^{-2t}\end{bmatrix}$$

Step-8

Recall that $u(t)=e^{At}u(0)$. To cancel the fraction lets consider initial values as $(1,-1)$.

$$\begin{aligned}u(t)&=e^{At}u(0)\\&=\frac{1}{3}\begin{bmatrix}e^{4t}+2e^{-2t} & -2e^{4t}+2e^{-2t}\\ -e^{4t}+e^{-2t} & 2e^{4t}+e^{-2t}\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\\&=\frac{1}{3}\begin{bmatrix}3e^{4t}\\-3e^{4t}\end{bmatrix}\\&=\begin{bmatrix}e^{4t}\\-e^{4t}\end{bmatrix}\end{aligned}$$

Step-9

Therefore, the solution is as follows:

$$\boxed{\begin{aligned}x(t)&=e^{4t}\\y(t)&=-e^{4t}\end{aligned}}$$

As $t\to\infty$ the values of $x(t)$ and $y(t)$ becomes very large and the solution becomes unstable.

Step-10

To avoid this instability the two equations of the system are exchanged as follows:

$$\begin{aligned}dy/dt&=-2x+2y\\dx/dt&=0x-4y\end{aligned}$$

Now the matrix becomes:

$$B=\begin{bmatrix}-2 & 2\\0 & -4\end{bmatrix}$$

Eigen values of changed system are less than zero $\lambda<0$. As per the scientists the system is now stable.

Step-11

Compare the two matrices *A* and *B*.

$$A = \begin{bmatrix} 0 & -4 \\ -2 & 2 \end{bmatrix}$$
$$B = \begin{bmatrix} -2 & 2 \\ 0 & -4 \end{bmatrix}$$

It can be seen that values of x and y in matrix A are exchanged among themselves in matrix B. To change the values of x from values of y column values should be exchanged. Thus, correct matrix is as follows:

$$\boxed{u(y,x) = \begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}}$$

This matrix gives the same Eigen values as the original system matrix A.