## Step-1

Consider the matrix $A=\begin{bmatrix}5 & -3\\ 4 & -2\end{bmatrix}$

The objective is to determine a unitary $U$ and triangular $T$ matrices so that $U^{-1}AU=T$

## Step-2

Find the eigen values and eigen vectors of matrix $A$.

The characteristic equation of matrix $A$ is $|A-\lambda I|=0$

Consider $|A-\lambda I|=0$

$$\begin{bmatrix}5 & -3\\ 4 & -2\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=0$$
$$\begin{bmatrix}5-\lambda & -3\\ 4 & -2-\lambda\end{bmatrix}=0$$
$$(5-\lambda)(-2-\lambda)+12=0$$
$$-10-5\lambda+2\lambda+\lambda^2+12=0$$

$$\lambda^2-3\lambda+2=0$$
$$\lambda^2-2\lambda-\lambda+2=0$$
$$\lambda(\lambda-2)-1(\lambda-2)=0$$
$$(\lambda-1)(\lambda-2)=0$$
$$\lambda=1,2$$

Therefore, the eigen values are $\lambda_1=1,\lambda_2=2$

## Step-3

To find eigen vector $x_1$ for the eigenvalue $\lambda_1=1$ calculate $(A-\lambda_1 I)x_1=0$

Consider

$$(A-\lambda_1 I)x_1 = 0$$

$$\left[\begin{pmatrix} 5 & -3 \\ 4 & -2 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad R_2 - R_1 \to R_2$$

$$\begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equation of the above matrix is $4x-3y=0$. Substitute $y=k_1$ then

$$4x - 3k_1 = 0$$
$$4x = 3k_1$$
$$x = \frac{3}{4}k_1$$

## Step-4

Hence,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{4}k_1 \\ k_1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} k_1$$

$$= \begin{bmatrix} \frac{3}{4}\times 4 \\ 1\times 4 \end{bmatrix} k_1$$
$$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} k_1$$

Therefore, the eigen vector corresponding to the eigen value $\lambda_1 = 1$ is $x_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

## Step-5

To find eigen vector $x_1$ for the eigenvalue $\lambda_2 = 2$ calculate $(A-\lambda_2 I)x_2 = 0$

Consider

$$(A-\lambda_2 I)x_2 = 0$$

$$\left[\begin{pmatrix} 5 & -3 \\ 4 & -2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad 3R_2 - 4R_1 \to R_2$$

$$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equation of the above matrix is $3x-3y=0$. Substitute $y=k_2$ then

$$\begin{aligned} 3x - 3k_2 &= 0 \\ 3x &= 3k_2 \\ x &= k_2 \end{aligned}$$

## Step-6

Hence,

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} k_2 \\ k_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} k_2 \end{aligned}$$

## Step-7

Therefore, the eigen vector corresponding to eigen value $\lambda_2 = 2$ is $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The length of eigen vector $x_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is,

$$\begin{aligned} \|x_1\| &= \sqrt{3^2+4^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Consider $\dfrac{x_1}{\|x_1\|} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$

The first column of $U$ is $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$, it needs the columns of $U$ should be orthonormal

By using these normalized vectors as the columns $U = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

## Step-8

Since, the vectors in the matrix are orthogonal.

So,

$$\begin{aligned} U^{-1} &= U^T \\ &= \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}. \end{aligned}$$

Since, $U^{-1}AU = T$.

$$\begin{aligned} U^{-1}AU &= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 5\times\frac{3}{5}-3\times\frac{4}{5} & 5\times\frac{4}{5}+3\times\frac{3}{5} \\ 4\times\frac{3}{5}-2\times\frac{4}{5} & 4\times\frac{4}{5}-2\times-\frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{29}{5} \\ \frac{4}{5} & \frac{22}{5} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{bmatrix} \frac{3}{5}\times\frac{3}{5}+\frac{4}{5}\times\frac{4}{5} & \frac{3}{5}\times\frac{29}{5}+\frac{4}{5}\times\frac{22}{5} \\ \frac{4}{5}\times\frac{3}{5}-\frac{3}{5}\times\frac{4}{5} & \frac{4}{5}\times\frac{29}{5}-\frac{3}{5}\times\frac{22}{5} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 7 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

Therefore, the lower triangular matrix is, $T = \boxed{\begin{bmatrix} 1 & 7 \\ 0 & 2 \end{bmatrix}}$.

## Step-9

The length of eigenvectors $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is

$$\begin{aligned} \|x_2\| &= \sqrt{1^2 + 1^2} \\ &= \sqrt{2} \end{aligned}$$

Consider $\frac{x_2}{\|x_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

The first column of $U$ is $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ it needs the columns of $U$ should be orthonormal

By using these normalized vectors as the columns $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

## Step-10

Since, the vectors in the matrix are orthogonal.

So, $U^{-1} = U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Consider

## Step-11

$$U^{-1}AU=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}5 & -3\\ 4 & -2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{2}} & \frac{8}{\sqrt{2}}\\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}}\times\frac{8}{\sqrt{2}}+\frac{1}{\sqrt{2}}\times\frac{6}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{2}}-\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}}\times\frac{8}{\sqrt{2}}-\frac{1}{\sqrt{2}}\times\frac{6}{\sqrt{2}}\end{bmatrix}$$

$$=\begin{bmatrix}2 & 7\\ 0 & 1\end{bmatrix}$$

Since $U^{-1}AU=T$

Hence $T=\begin{bmatrix}2 & 7\\ 0 & 1\end{bmatrix}$

Therefore, the lower triangular matrix is, $T=\boxed{\begin{bmatrix}2 & 7\\ 0 & 1\end{bmatrix}}$.

# Step-12

Consider the matrix $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}$

The objective is to find unitary $U$ and triangular $T$ so that $U^{-1}AU=T$

Find the eigenvalues and eigenvectors of matrix $A$

The characteristic equation of the matrix $A$ is $\det\left(A-\lambda I\right)=0$.

$$\left|\begin{bmatrix}0&1&0\\0&0&0\\1&0&0\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right|=0$$

$$\left|\begin{bmatrix}-\lambda&1&0\\0&-\lambda&0\\1&0&-\lambda\end{bmatrix}\right|=0$$

$$\lambda^3=0$$

Then $\lambda_1=0,\lambda_2=0,\lambda_3=0$

# Step-13

To calculate eigen vectors consider,

$$(A-\lambda_1 I)x_1=0$$

$$\begin{bmatrix}0&1&0\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Then $x=0,y=0$. Let $z=k$ then,

$$x_1=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\boxed{x_1=\begin{bmatrix}0\\0\\1\end{bmatrix}}$$

Similarly, the eigen vectors corresponding to the eigen values $\boxed{x_2=\begin{bmatrix}0\\0\\1\end{bmatrix}}$ and $\boxed{x_3=\begin{bmatrix}0\\0\\1\end{bmatrix}}$

# Step-14

Consider $\frac{x_1}{\|x_1\|}=\frac{1}{\sqrt{0^2+0^2+1^2}}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$, similarly $\frac{x_2}{\|x_2\|}=\begin{bmatrix}0\\0\\1\end{bmatrix},\frac{x_3}{\|x_3\|}=\begin{bmatrix}0\\0\\1\end{bmatrix}$

The first column of $U$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, it needs the columns of $U$ should be orthonormal.

Therefore, by using these normalized vectors as the columns, $U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Since, the vectors in the matrix are orthogonal.

Consider $U^{-1} = U^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

## Step-15

Consider

$$
\begin{aligned}
U^{-1}AU &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = T
\end{aligned}
$$

Therefore, the lower triangular matrix is $\boxed{T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$