Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #4

2023/03/19

Name: ____________________

Student Number: ____________________

1. Suppose V and W are 2-dimensional vector spaces, try to construct $T \in \mathcal{L}(V, W)$ such that the matrix of T with respect to a basis of V and a basis of W satisfies $(\mathcal{M}(T))^2 = \mathcal{M}(T)$ and $\mathcal{M}(T) \neq 0$, $\mathcal{M}(T) \neq I$.

   设 V 和 W 均是 2 维向量空间，构造 $T \in \mathcal{L}(V, W)$，使得 T 在 V 的一组基和 W 的一组基下的矩阵满足 $(\mathcal{M}(T))^2 = \mathcal{M}(T)$ 且 $\mathcal{M}(T) \neq 0$, $\mathcal{M}(T) \neq I$.

   Proof. Let v_1, v_2 be a basis of V, w_1, w_2 be a basis of W. Suppose a map $T : V \to W$ satisfies $Tv_1 = w_1$, $Tv_2 = 0$. It's easy to check $T \in \mathcal{L}(V, W)$. And $\forall v \in V$, $\exists a_1, a_2 \in \mathbf{F}$, s.t. $v = a_1v_1 + a_2v_2$, then $Tv = a_1w_1$, the matrix of T w.r.t the bases v_1, v_2 and w_1, w_2 is

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

   then $\mathcal{M}(T)$ satisfies the condition above.

   □

2. Suppose V are 2-dimensional vector spaces, $T \in \mathcal{L}(V)$, justify the following statement true or false, if true, please give the proof; if not, please give the counter-example.

$$V = \text{null } T \oplus \text{range } T.$$

设 V 是 2 维向量空间，$T \in \mathcal{L}(V)$, 判断下述说法是否正确。若正确，请给出证明；若不正确，请给出反例：

$$V = \text{null } T \oplus \text{range } T.$$

Proof. False!

Counter-example: suppose v_1, v_2 is a basis of V, $T \in \mathcal{L}(V)$ satisfies $Tv_1 = v_2, Tv_2 = 0$. $\forall v \in V$, $\exists a_1, a_2 \in \mathcal{F}$, s.t. $v = a_1v_1 + a_2v_2$, we have $Tv = a_1v_2$, so range $T = \text{span }\{v_2\}$. But $Tv = 0 \Rightarrow a_1 = 0 \Rightarrow v = a_2v_2$, so null $T = \text{span}\{v_2\}$. Obviously, range $T +$ null $T \neq V$ and it is not a direct sum. $\square$