Step-1

If A is $n \times m$ matrix, and x is $m \times 1$ matrix, then Ax is $n \times 1$ matrix

Also, A is $n \times m$ matrix says A^T is $m \times n$

Consequently, $A^T A$ is $m \times m$

A^T is $m \times n$, Ax is $n \times 1$ implies $A^T(Ax)$ is $m \times 1$

So the null spaces of A and $A^T A$ are sets of vectors of different lengths if $m \neq n$

So, A is a square matrix.

Step-2

Now, A is square matrix of order n, if y is null space of A

Now $(A^T A)y = A^T(Ay) = A^T 0 = 0$

So the null space of A is a subset of A is a subset of the null space of $A^T A$

But if x is in the null space of $A^T A$

We get $(A^T A)x = A^T(Ax)$

$= A^T 0$
$= 0$

Applying x^T on both sides, we get $x^T\left[A^T(Ax)\right] = x^T 0$

$= 0$ â€¦â€¦ (1)

Step-3

On the other hand, $x^T\left[A^T(Ax)\right] = (x^T A^T)(Ax)$

$= (Ax)^T(Ax)$ â€¦â€¦ (2)

In view of (1) and (2), we get Ax must have length 0

That is, $(A^T A)x = A^T(Ax)$

$= 0$

This implies $Ax = 0$

So the null space of $A^T A$ is also a subspace of the null space of A

Hence for a square matrix A, the null spaces of $A^T A$ and A are one and the same.