Step-1

Let following be the differential equation of matrices:

$$\frac{du}{dT} = Au$$

Here, matrix A is defined as follows:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Step-2

Find the general solution of the above differential equation. Also, find the value of T at which the solution $u(T)$ is guaranteed to return to the initial value $u(0)$.

Step-3

Recall that $du/dt = Au$ has the following solution:

$$u(t) = e^{At}u(0)$$

General solution will be:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + \ldots + c_n e^{\lambda_n t} x_n$$

Here, $c_1, c_2, \ldots, c_n$ are constants.

Step-4

Firstly find the Eigen values and Eigen vectors of matrix A.

$$A - \lambda I = \begin{bmatrix} 0-\lambda & -1 & 0 \\ 1 & 0-\lambda & -1 \\ 0 & 1 & 0-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(-\lambda)(\lambda^2 + 1) - \lambda = 0$$

$$-\lambda^3 - 2\lambda = 0$$

After solving following values are obtained:

$$\lambda_1 = 0$$
$$\lambda_2 = \sqrt{2}i$$
$$\lambda_3 = -\sqrt{2}i$$

Step-5

To calculate Eigen vectors do the following calculations:

$$(A - \lambda_1 I)x = 0$$
$$\begin{bmatrix} 0-\lambda & -1 & 0 \\ 1 & 0-\lambda & -1 \\ 0 & 1 & 0-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving values of x, y and z corresponding to $\lambda = 0$ is as follows:

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Step-6

Similarly, Eigen vectors corresponding to Eigen value $\lambda = \sqrt{2}i$ is as follows:

$$(A - \lambda_1 I)x = 0$$
$$\begin{bmatrix} 0-\sqrt{2}i & -1 & 0 \\ 1 & 0-\sqrt{2}i & -1 \\ 0 & 1 & 0-\sqrt{2}i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -\sqrt{2}i & -1 & 0 \\ 1 & -\sqrt{2}i & -1 \\ 0 & 1 & -\sqrt{2}i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step-7

On solving the above matrix equation following values of x, y and z are obtained:

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ \sqrt{2}i \\ 1 \end{bmatrix}$$

Similarly, Eigen vectors corresponding to Eigen value $\lambda = -\sqrt{2}i$ is as follows:

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{bmatrix}$$

Step-8

General solution will be:

$$u(T) = c_1 e^{\lambda_1 T} x_1 + \ldots + c_n e^{\lambda_n T} x_n$$
$$= c_1 e^{0T} x_1 + c_2 e^{\sqrt{2}iT} x_2 + c_3 e^{\sqrt{2}iT} x_3$$
$$u(T) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{\sqrt{2}iT} \begin{bmatrix} -1 \\ \sqrt{2}i \\ 1 \end{bmatrix} + c_3 e^{-\sqrt{2}iT} \begin{bmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{bmatrix}$$

Step-9

Therefore at time T general solution is as follows:

$$\boxed{u(T) = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{\sqrt{2}iT} \begin{bmatrix} -1 \\ \sqrt{2}i \\ 1 \end{bmatrix} + c_3 e^{-\sqrt{2}iT} \begin{bmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{bmatrix}}$$

Step-10

Put $T = 0$ in the general solution and use $e^{\lambda \cdot 0} = 1$:

$$u(0)=c_1\begin{bmatrix}1\\0\\1\end{bmatrix}+c_2\begin{bmatrix}-1\\\sqrt{2}i\\1\end{bmatrix}+c_3\begin{bmatrix}-1\\-\sqrt{2}i\\1\end{bmatrix}$$

Therefore, at $\boxed{T=0}$ solution $u(T)$ will return to the initial value $u(0)$.