

Step-1

We have to find the special solutions to $Rx=0$ and $R^T y=0$ for the following R .

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Step-2

$$\text{Given } R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Let } x = \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix}$$

Step-3

$$Rx = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow u + 2w + 3t = 0$$

$$v + 4w + 5t = 0$$

Step-4

First, second columns are pivot columns.

u, v are pivot variables.

w, t are free variables.

$$u = -2w - 3t$$

$$v = -4w - 5t$$

Therefore

$$\begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix} = \begin{bmatrix} -2w-3t \\ -4w-5t \\ w \\ t \end{bmatrix} = w \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

The columns in the above solution are the special solutions for $Rx = 0$

Step-5

$$R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix}$$

Now

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Let

Step-6

Therefore

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix}$$

$$\begin{array}{l} R_3 - 2R_1, \\ R_4 - 3R_1 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\begin{array}{l} R_3 - 4R_2, \\ R_4 - 5R_2 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step-7

Now

$$R^T y = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 = 0, y_2 = 0$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The column in the above solution is the special solution for $R^T y = 0$

Given

$$R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Let $Rx = 0$, where

$$v + 2w = 0$$

$$\Rightarrow v = -2w$$

Second column is pivot column; v is pivot variable and w is free variable.

Step-8

Therefore

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ -2w \\ w \end{bmatrix}$$

$$= u \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

The columns in the above solution are the special solutions for $Rx = 0$

Step-9

Now

$$R^T y = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 = 0$$

Step-10

Therefore

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ y_2 \\ y_3 \end{bmatrix}$$
$$= y_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The columns in the above solution are the special solutions for $R^T y = 0$