Step-1

Consider the matrices,

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, the matrix U is obtained from A by subtracting row 1 from row 3.

The objective is to find the bases for the column spaces of A and U, the bases for the row spaces of A and U and the bases for the null spaces of A and U.

Step-2

Reduce the matrix A to the reduced row echelon form.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step-3

Reduce the matrix U to the reduced row echelon form.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The reduced row echelon forms of the matrices A and U represent the same matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Step-4

Observe that the pivot positions in the reduced row echelon form of the matrix A are in the first and second columns.

Therefore, the corresponding columns in the matrix A form a basis for the column space of A.

Hence, the basis for the column space of the matrix A is $\boxed{\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}3\\1\\3\end{bmatrix}\right\}}$.

The pivot positions in the matrix U are in the first and second columns. Therefore, the corresponding columns in the matrix U form a basis for the column space of U.

Therefore, the basis for the column space of the matrix U is $\boxed{\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}}$.

Step-5

Observe that the pivot positions in the reduced row echelon form of the matrix A are in the first and second rows.

Therefore, the basis for the row space of the matrix A is $\boxed{\{(1,0,-1),(0,1,1)\}}$.

The pivot positions in the reduced row echelon form of the matrix U are in the first and second rows.

Therefore, the basis for the row space of the matrix U is $\boxed{\{(1,0,-1),(0,1,1)\}}$.

Step-6

Now find the bases for the null spaces of the A and U.

From the first and second rows of the reduced row echelon form, the obtained equations are,

$x_1 - x_3 = 0$ and $x_2 + x_3 = 0$.

Here, x_3 is a free variable.

So choose $x_3 = t$, where t is a parameter.

Then $x_1 = t,\ x_2 = -t$.

Therefore, the vector $\mathbf{x} = (x_1, x_2, x_3)$ can be written as,

$$\begin{aligned}\mathbf{x} &= (x_1, x_2, x_3)\\ &= (t, -t, t)\\ &= t(1, -1, 1)\end{aligned}$$

Hence, the basis for the null spaces of the matrices A and U is $\boxed{\{(1, -1, 1)\}}$.