Step-1

Given that for a positive definite matrix *A,* the Cholesky decomposition is $A = LDL^T = R^T R$, where $R = \sqrt{D}L^T$.

We have to show that the condition number of $c(R)$ is the square root of $c(A)$.

Step-2

We know that

Def: 1: The ***conditional number*** of A is $c = \|A\|\|A^{-1}\|$

Def: 2: The ***norm*** of a square matrix A is defined by $\|A\| = \lambda_{\max}(A)$ and is the square root of the largest eigenvalue of $A^T A$; in other words, $\|A\|^2 = \lambda_{\max}(A^T A)$

Def: 3: $\|A^{-1}\| = \dfrac{1}{\lambda_{\min}(A)}$

We know that if A is a positive definite matrix, then all the eigenvalues are positive.

The square root of a positive real number is a real number. â€¦â€¦ (1)

Step-3

Given that A is a positive definite matrix and by the process of Cholesky decomposition, it can be written as $LDL^T = R^T R$ where $R = \sqrt{D}L^T$, D is the diagonal matrix, L is the lower triangular matrix.

Consequently, $R^T = \left(\sqrt{D}L^T\right)^T$

$= \left(L^T\right)^T \sqrt{D}^T$

$= L\sqrt{D}^T$ $\qquad \left(\text{Since } \left(L^T\right)^T = L\right)$

Since D is the diagonal matrix, the diagonal entries are the eigenvalues of A whose roots are real numbers and other entries are zero.

So, we follow that $\sqrt{D}^T = \sqrt{D}$ and thus, $R^T = L\sqrt{D}$

Step-4

Now, in view of definition2, we write $\|R\|^2 = \lambda_{\max}\left(R^T R\right)$

$$
\begin{aligned}
&= \lambda_{\max}\left(L\sqrt{D}\right)\left(\sqrt{D}L^T\right) \\
&= \lambda_{\max} LDL^T \\
&= \lambda_{\max}\left(A\right)
\end{aligned}
$$

Since norm is a non negative quantity, we get $\|R\| = \sqrt{\lambda_{\max}\left(A\right)}$

By definition 3, we get $\left\|R^{-1}\right\| = \sqrt{\dfrac{1}{\lambda_{\min}\left(A\right)}}$

Step-5

Multiplying the corresponding sides of these equations, we get

$$
\|R\|\left\|R^{-1}\right\| = \sqrt{\lambda_{\max}\left(A\right)}\sqrt{\frac{1}{\lambda_{\min}\left(A\right)}}
$$

$$
\begin{aligned}
&\Rightarrow \|R\|\left\|R^{-1}\right\| = \sqrt{\|A\|\left\|A^{-1}\right\|} \\
&\Rightarrow c\left(R\right) = \sqrt{c\left(A\right)} \qquad \text{(Since by def.1)}
\end{aligned}
$$

Therefore, the ***conditional number*** of R is nothing but the square root of the ***conditional number*** of A.