Step-1

Given that the solutions to the linear differential equation $\frac{d^2u}{dt^2}=u$ form a vector space.

We have to find what combination of basis vectors solves $u''-u=0$ with the initial values $u=x, \frac{du}{dt}=y$ at $t=0$.

Step-2

We have $\frac{d^2u}{dt^2}=u$

$\Rightarrow u''-u=0$

The auxiliary equation is $m^2-1=0$

$\Rightarrow m=\pm 1$

Therefore, $u=e^t, u=e^{-t}$ are solutions of $\frac{d^2u}{dt^2}=u$.

And e^t, e^{-t} are independent solutions of the basis for the solution space is $\boxed{\{e^t, e^{-t}\}}$.

The general solution of the given differential equation is $u=c_1 e^t+c_2 e^{-t}$, where c_1, c_2 are constants.

Step-3

If $u=e^t \Rightarrow x=e^t \quad (\text{Since } u=x)$,

Then $\frac{du}{dt}=e^t \Rightarrow y=e^t \quad \left(\text{Since } \frac{du}{dt}=y\right)$

At $t=0, u=1, y=1$,

If $u=e^{-t} \Rightarrow x=e^{-t}$

And $\frac{du}{dt}=-e^{-t} \Rightarrow y=-e^{-t}$

Step-4

We have the general solution of the given differential equation is $u = c_1 e^t + c_2 e^{-t}$.

If $u = e^t$

Then

$$\frac{d^2u}{dt^2} = e^t$$
$$= 1.e^t + 0.e^{-t}$$

Step-5

If $u = e^{-t}$

Then

$$\frac{d^2u}{dt^2} = e^{-t}$$
$$= 0.e^t + 1.e^{-t}$$

Therefore, the required 2 by 2 matrix is $\boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$.