Step-1

Suppose $K^H = -K$ is a skew-Hermitian matrix, and *K* has imaginary eigenvalues and its the eigenvectors are orthogonal.

Consider a matrix iK, so

$$\begin{aligned}(iK)^H &= (i)^H (K)^H \\ &= (-i)(-K) \\ &= iK\end{aligned}$$

Since $(iK)^H = iK$, so iK is a Hermitian matrix.

Step-2

(a) To show $K - I$ is invertible; we have to first find the eigenvalues of $K - I$.

For that just subtract 1 from the eigenvalues of *K*.

Suppose λ is an eigenvalue of *K* and *x* is the corresponding eigenvector.

Then we have

$$\begin{aligned}Kx &= \lambda x \\ (K-1)x &= (\lambda - 1)x\end{aligned}$$

We know that in the matrix *K*, eigenvalues are imaginary and the eigenvectors are orthogonal.

The absolute values of eigenvalues of $K - I$ are greater than 1, so $|\lambda - 1| \neq 0$.

Therefore, $\boxed{K - I \text{ is invertible}}$.

Step-3

(b) We know that *iK* is a Hermitian matrix, then there exist a unitary matrix *U* and a

diagonal matrix Λ, such that

$$iK = U\Lambda U^H$$

This implies that

$$\begin{aligned}K &= U(-i)\Lambda U^H \\ K &= U\Lambda U^H\end{aligned}$$

Therefore, $\boxed{K = U\Lambda U^H}$, for a unitary U.

Step-4

(c) Suppose $\lambda_1 \dots \lambda_n$ are imaginary eigenvalues of K, the we have

$$\begin{aligned}
e^{\Lambda t} &= \sum_{j=1}^{\infty} \frac{\Lambda^j t^j}{j!} \\
&= \begin{pmatrix} \sum_{j=1}^{\infty} \frac{\lambda_1^j t^j}{j!} & & \\ & \ddots & \\ & & \sum_{j=1}^{\infty} \frac{\lambda_n^j t^j}{j!} \end{pmatrix} \\
&= \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix}
\end{aligned}$$

Since λ_n Eigen values are imaginary, then e^{λ_n} show a complex number and its magnitude will be1, so we have

$$e^{\Lambda t} \cdot e^{\Lambda t} = I$$

Therefore, $\boxed{e^{\Lambda t} \text{ is unitary}}$.

Step-5

(d) Suppose $\lambda_1 \dots \lambda_n$ are imaginary eigenvalues of K, the we have

$$\begin{aligned}
e^{Kt} &= \sum_{j=1}^{\infty} \frac{K^j t^j}{j!} \\
&= \sum_{j=1}^{\infty} \frac{\left(U\Lambda U^H\right)^j t^j}{j!} \\
&= U\left(\sum_{j=1}^{\infty} \frac{(\Lambda)^j t^j}{j!}\right)U^H \\
&= U\begin{pmatrix} \sum_{j=1}^{\infty} \frac{\lambda_1^j t^j}{j!} & & \\ & \ddots & \\ & & \sum_{j=1}^{\infty} \frac{\lambda_n^j t^j}{j!} \end{pmatrix}U^H
\end{aligned}$$

$$e^{Kt} = U \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} U^H$$

Since λ_n eigenvalues are imaginary, then e^{λ_n} show a complex number and its magnitude will be 1, so we have

$$e^{Kt} \cdot e^{Kt} = I$$

Therefore, $\boxed{e^{Kt} \text{ is unitary}}$.