Step-1

The lengths $L = 5,6 \text{ and } 7$ feet corresponding to the applied forces $F = 1,2 \text{ and } 4$ respectively.

Hookeâ€™s Law states that, $L = a + bF$.

Substitute the $F = 1,2 \text{ and } 4$, $L = 5,6 \text{ and } 7$, to get the system of equations:

$$a + b(1) = 5$$
$$a + b(2) = 6$$
$$a + b(3) = 7$$

Write the system of equations in the matrix form.

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$, $\hat{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

Step-2

The least-squares solution is given by,

$A^T A\hat{x} = A^T B$, that is $\hat{x} = \left(A^T A\right)^{-1} A^T B$.

Find the transpose of the matrix A.

$$\begin{aligned} A^T &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \end{aligned}$$

Find the product of the matrices $A \text{ and } A^T$.

$$\begin{aligned} A^T A &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1+1+1 & 1+2+4 \\ 1+2+4 & 1+4+16 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \end{aligned}$$

Step-3

Use the following formula: Let $X=\begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then the inverse of the matrix X is defined by, $X^{-1}=\frac{1}{wz-xy}\begin{bmatrix} z & -x \\ -y & w \end{bmatrix}$, where $wz-xy\neq 0$.

Find the inverse of the matrix $A^T A$.

$$\left(A^T A\right)^{-1}=\frac{1}{3(21)-7(7)}\begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix}$$
$$=\frac{1}{14}\begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix}$$

Now, find the product of matrices A^T and B.

$$A^T B=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$
$$=\begin{bmatrix} 5+6+7 \\ 5+12+28 \end{bmatrix}$$
$$=\begin{bmatrix} 18 \\ 45 \end{bmatrix}$$

Step-4

To determine the $\hat{x}=\left(A^T A\right)^{-1} A^T B$, find the product of matrices $\left(A^T A\right)^{-1}$ and $A^T B$:

$$\hat{x}=\left(A^T A\right)^{-1} A^T B$$
$$\begin{bmatrix} a \\ b \end{bmatrix}=\frac{1}{14}\begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 18 \\ 45 \end{bmatrix}$$
$$=\frac{1}{14}\begin{bmatrix} 378-315 \\ -126+135 \end{bmatrix}$$
$$=\frac{1}{14}\begin{bmatrix} 63 \\ 9 \end{bmatrix}$$
$$=\begin{bmatrix} \frac{9}{2} \\ \frac{9}{14} \end{bmatrix}$$

Therefore, the required normal length is $\boxed{a=\frac{9}{2}}$.