

Step-1

Consider the second-order equation

$$\frac{d^2 u}{dt^2} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u$$

With the following conditions

$$u(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We know that if A has negative eigenvalues $\lambda_1, \dots, \lambda_n$ and if $\omega_j = \sqrt{-\lambda_j}$, then solution of $\frac{d^2 u}{dt^2} = Au$ is given by

$$u(t) = (a_1 \cos \omega_1 t + b_1 \sin \omega_1 t)x_1 + \dots + (a_n \cos \omega_n t + b_n \sin \omega_n t)x_n. \quad (1)$$

Step-2

To find the eigenvalues for matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, by using the equation

$$\det(A - \lambda I) = 0.$$

$$\det \begin{bmatrix} -2 - \lambda & 1 \\ 1 & -2 - \lambda \end{bmatrix} = 0$$

$$(-2 - \lambda)(-2 - \lambda) - (1) = 0$$

$$(-2 - \lambda)^2 - 1 = 0$$

Step-3

Solve the $(-2 - \lambda)^2 - 1$, to find the value of λ .

$$(-2 - \lambda)^2 = 1$$

$$-2 - \lambda = \pm \sqrt{1}$$

$$-2 - \lambda = \pm 1$$

$$-2 - \lambda = -1 \text{ or } -2 - \lambda = 1$$

So, we get λ_1 and λ_2 as

$$\lambda_1 = -1$$

$$\lambda_2 = -3$$

Step-4

The eigenvector for $\lambda_1 = -1$ is given by

$$(A - \lambda_1 I)x_1 = \begin{bmatrix} -2+1 & 1 \\ 1 & -2+1 \end{bmatrix} x_1$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x_1$$

$$= 0$$

The eigenvector for $\lambda_1 = -1$ is given by $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step-5

The eigenvector for $\lambda_2 = -3$ is given by

$$(A - \lambda_2 I)x_2 = \begin{bmatrix} -2+3 & 1 \\ 1 & -2+3 \end{bmatrix} x_2$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_2$$

$$= 0$$

The eigenvector for $\lambda_2 = -3$ is given by $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Step-6

We know that $\omega_j = \sqrt{-\lambda_j}$.

Substituting -1 for λ_1 in equation $\omega_1 = \sqrt{-\lambda_1}$, we get

$$\omega_1 = \sqrt{-(-1)}$$

$$= 1$$

Similarly substituting -3 for λ_2 in equation $\omega_2 = \sqrt{-\lambda_2}$, we get

$$\begin{aligned}\omega_2 &= \sqrt{-(-3)} \\ &= \sqrt{3}\end{aligned}$$

To find the values of a 's, substitute $t = 0$ into the equation (1).

$$u(0) = (a_1 \cos(0) + b_1 \sin(0))x_1 + \cdots + (a_n \cos(0) + b_n \sin(0))x_n$$

so all $b_n \sin(0) = 0$ and $a_n \cos(0) = 1$, this gives:

$$a = S^{-1}u(0)$$

We have:

$$u(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

So, the values of a 's are:

$$\begin{aligned}a &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

Step-7

Thus, $a_1 = 0$ and $a_2 = 0$.

Step-8

To find the values of b 's, first differentiating $u(t)$ and then substituting $t = 0$ we get:

$$u'(t) = b_1 \omega_1 x_1 + \cdots + b_n \omega_n x_n$$

We have:

$$u'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

And

$$\omega_1 = 1$$

$$\omega_2 = \sqrt{3}$$

So, the values of $b\hat{a}\epsilon^{\text{TM}}$ s are:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = b_1 \times 1 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_2 \times \sqrt{3} \times \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 - \sqrt{3}b_2 \\ b_1 + \sqrt{3}b_2 \end{bmatrix}$$

Step-9

Solving the following system of equation, we get the values of $b\hat{a}\epsilon^{\text{TM}}$ s.

$$b_1 - \sqrt{3}b_2 = 1$$

$$b_1 + \sqrt{3}b_2 = 0$$

Thus, $b_1 = \frac{1}{2}$ and $b_2 = -\frac{1}{2\sqrt{3}}$.

Step-10

Substituting the values of $a\hat{a}\epsilon^{\text{TM}}$ s and $b\hat{a}\epsilon^{\text{TM}}$ s into the equation (1), we get

$$u(t) = \left(0 \times \cos(1)t + \frac{1}{2} \sin(1)t \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(0 \times \cos \sqrt{3}t + \left(\frac{1}{-2\sqrt{3}} \right) \sin \sqrt{3}t \right) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \sin t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2\sqrt{3}} \sin \sqrt{3}t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore, the motion $\boxed{u(t) = \frac{1}{2} \sin t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2\sqrt{3}} \sin \sqrt{3}t \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$.