1 Matrices and Gaussian Elimination

1.4

MATRIX OPERATIONS

（矩阵运算）

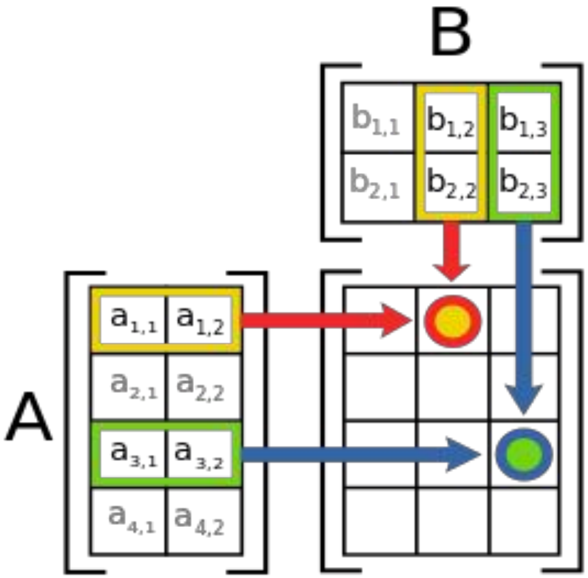
Addition

Scalar multiplication

Multiplication

Power

Elementary Matrices



MATRIX NOTATION

A matrix is an arrangement of mn elements with m rows and n columns, denoted by

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \phantom{a_{11}} & & \begin{array}{c}\text{Column}\\ j\end{array} & & \phantom{a_{1n}} \end{array} \\
\begin{array}{c} \\ \\ \text{Row } i \\ \\ \\ \end{array} & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A
\end{array}
$$

$$\boxed{A = \left[a_{ij} \right]_{m \times n}}$$

If $\boldsymbol{A}$ is an $m \times n$ matrix, that is, a matrix with m rows and n columns, then the scalar entry in the ith row and jth column of $\boldsymbol{A}$ is denoted by a_{ij} and is called the (i, j)-entry of $\boldsymbol{A}$.

THE ORDER IS IMPORTANT: *rows* $\times$ *columns*

- If two matrices have the same number of rows and the same number of columns, then they are called **matrices of the same size (同型矩阵)**.

For example, $\begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 3 & 7 \end{bmatrix}$ and $\begin{bmatrix} 14 & 3 \\ -8 & 4 \\ 3i & 9 \end{bmatrix}$.

- If $\boldsymbol{A} = [a_{ij}]$ and $\boldsymbol{B} = [b_{ij}]$ are matrices of the same size, and the corresponding entries are the same, i.e.,

$$a_{ij} = b_{ij}\ (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n),$$

then $\boldsymbol{A}$ and $\boldsymbol{B}$ are **equal (相等)**, denoted by $\boldsymbol{A} = \boldsymbol{B}$.

Attention! ***equal*** vs ***equivalent*** $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \boldsymbol{B}$

Some special matrices

(1) A matrix with the same number of rows and columns are called a **square matrix(方阵)**. An $n \times n$ matrix is also called a matrix of degree n / order n (n 阶方阵).

For instance, $\begin{bmatrix} 13 & 6 & 2i \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ is a complex matrix of order 3 (3 阶复方阵).

(2) $1 \times n$ matrix $\boldsymbol{A} = [a_1, a_2, \ldots, a_n]$: **行矩阵** (或 **行向量, row vector**).

$$m \times 1 \text{ matrix } \boldsymbol{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$: **列矩阵**(或**列向量, column vector**).

(3) $$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$ is called a **diagonal matrix**, (**对角矩阵**或**对角阵**)

(主) 对角线
(main) diagonal

denoted as $\mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.

A diagonal matrix with the same entry on the diagonal is called a **scalar matrix(数量矩阵)**.

The following diagonal matrix is called an **identity matrix (单位矩阵)**, denoted as $\boldsymbol{I}_n$ or $\boldsymbol{I}$.

$$\boldsymbol{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

(4) The matrix with each entry as zero is called a **zero matrix (零矩阵)**, denoted as **0** .

Zero matrices of different sizes are treated as different matrices.

(5)

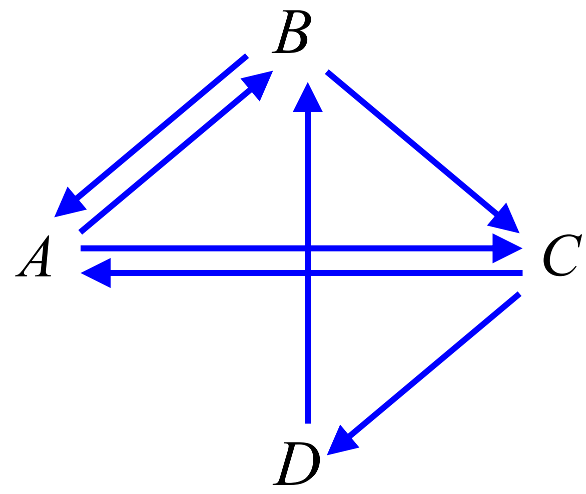
$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}, \qquad B=\begin{bmatrix} b_{11} & & & \\ b_{21} & b_{22} & & \\ \vdots & \vdots & \ddots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

upper triangular matrix (上三角矩阵) **lower triangular matrix (下三角矩阵)**

对角线左下(右上)方的元素都为 0 的方阵称为**上/下三角矩阵 (upper/lower triangular matrix)** .

引例(introductory example):

城市间的航班图



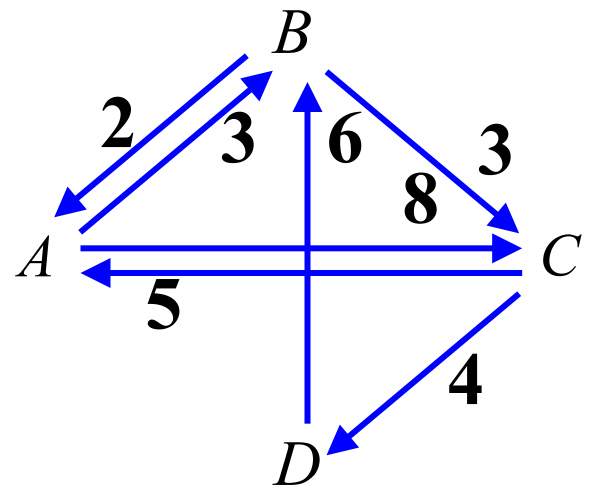
如果从 A 到 B 有航班, 则用带箭头的线连接 A 与 B.

航班图可用矩阵来表示:

$$
\begin{array}{cc}
 & \text{到达城市} \\
 & \begin{array}{cccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \end{array} \\
\text{出发城市} \begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}
$$

引例 (introductory example):

城市间的航班图

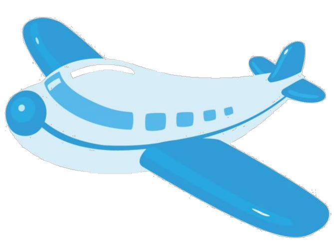


如果从 A 到 B 有航班, 则用带箭头的线连接 A 与 B.

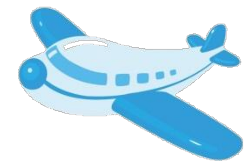
航班量也可用矩阵来表示:

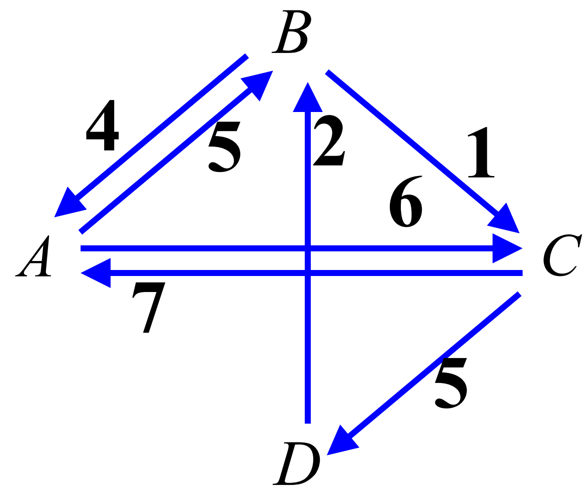
$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

第一天　　　第二天



城市间的航班图





如果从 ***A*** 到 ***B*** 有航班, 则用带箭头的线连接 ***A*** 与 ***B***.

航班量也可用矩阵来表示:

$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 5 & 6 & 0 \\ 4 & 0 & 1 & 0 \\ 7 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

第一天 第二天 第三天…

问题: 各城市2天内发送的航班量?

Matrix operations（矩阵的运算）

Matrix operations
(矩阵的运算)

I. Addition（矩阵的加法）

II. Scalar multiplication（数与矩阵相乘）

III. Multiplication（矩阵的乘法）

IV. Power（方阵的幂）

Elementary Matrices（初等矩阵)

I. 矩阵的加法(Addition)

each entry in $\boldsymbol{A}+\boldsymbol{B}$ is the sum of the corresponding entries in $\boldsymbol{A}$ and $\boldsymbol{B}$.

1. 定义 (Definition)

设有两个 $m\times n$ 矩阵 $\boldsymbol{A}=[a_{ij}]$ 和 $\boldsymbol{B}=[b_{ij}]$, 那么矩阵 $\boldsymbol{A}$ 与 $\boldsymbol{B}$ 的和(sum)记作 $\boldsymbol{A}+\boldsymbol{B}$, 规定为

$$\boldsymbol{A}+\boldsymbol{B}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}.$$

注　只有两个矩阵同型时, 才能进行加法运算;

(The sum $\boldsymbol{A}+\boldsymbol{B}$ is defined only when $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same size.)

For example,

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & -9 & 0 \\ 3 & 6 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 8 & 9 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12+1 & 3+8 & -5+9 \\ 1+6 & -9+5 & 0+4 \\ 3+3 & 6+2 & 8+1 \end{bmatrix} = \begin{bmatrix} 13 & 11 & 4 \\ 7 & -4 & 4 \\ 6 & 8 & 9 \end{bmatrix}.$$

2. 矩阵加法的运算规律

Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be matrices of the same size, then

(1) $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$;

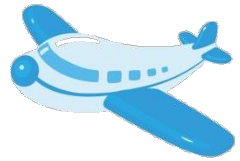
(2) $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$;

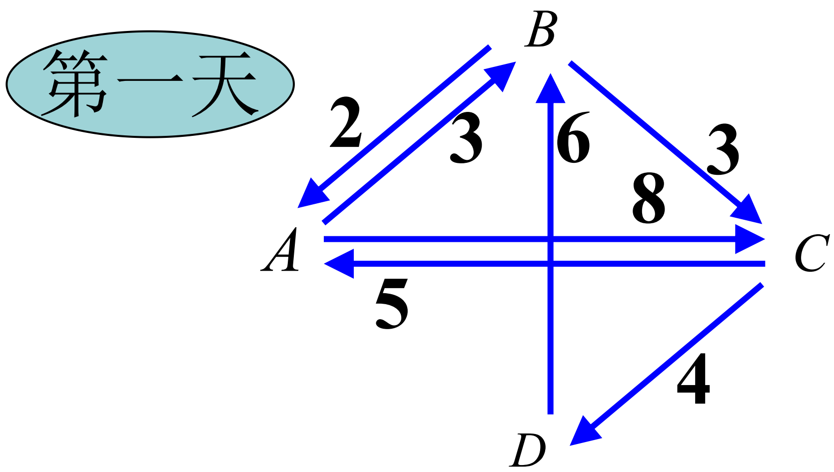
(3)
$$-\boldsymbol{A}=\begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{m1} & -a_{m1} & \cdots & -a_{mn} \end{bmatrix}=[-a_{ij}];$$

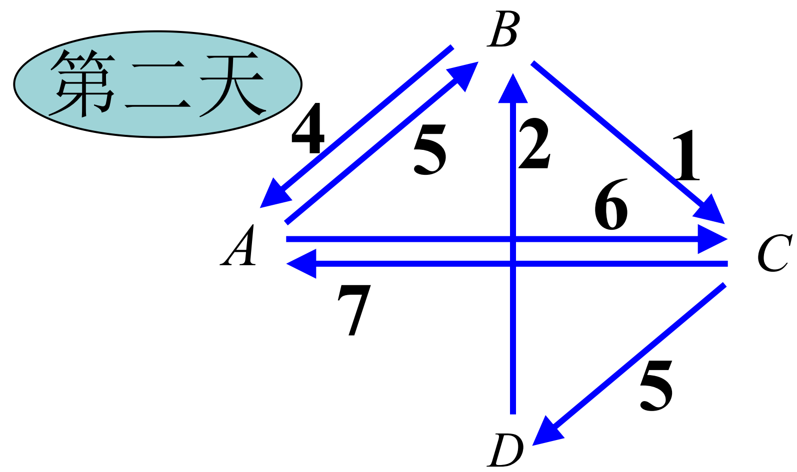
(4) $\boldsymbol{A}+(-\boldsymbol{A})=\boldsymbol{0}$, $\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+(-\boldsymbol{B})$.

定义**矩阵的减法(subtraction)**

思考　城市间航班客流量







$$\begin{bmatrix} 0 & 3 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 5 & 6 & 0 \\ 4 & 0 & 1 & 0 \\ 7 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

问题1：各城市2天内发送的航班量?

问题2：收取的机场建设费(航空基金)有多少?

II. 数与矩阵相乘(Scalar multiplication)

1. 定义

数 λ 与矩阵 $\boldsymbol{A}$ 的乘积 (简称为数乘, **scalar multiple**)

记作 $\lambda\boldsymbol{A}$ 或 $\boldsymbol{A}\lambda$, 规定为

$$\lambda\boldsymbol{A} = \boldsymbol{A}\lambda = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda a_{m1} & \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix}.$$

If λ is a scalar and $\boldsymbol{A}$ is a matrix, then the **scalar multiple** $\lambda\boldsymbol{A}$ is the matrix whose columns are λ times the corresponding columns in $\boldsymbol{A}$.

2. 数乘矩阵的运算规律

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of the same size ($m \times n$), and let λ and μ be scalars, then

(1) $(\lambda\mu)\boldsymbol{A} = \lambda(\mu\boldsymbol{A})$;

(2) $(\lambda + \mu)\boldsymbol{A} = \lambda\boldsymbol{A} + \mu\boldsymbol{A}$;

(3) $\lambda(\boldsymbol{A} + \boldsymbol{B}) = \lambda\boldsymbol{A} + \lambda\boldsymbol{B}$.

矩阵的加法与数乘统称为矩阵的**线性运算**.

Example 1 Let $\boldsymbol{A}-3\boldsymbol{B}=4\boldsymbol{A}-\boldsymbol{C}$, where

$$\boldsymbol{A}=\begin{bmatrix}1 & -1\\ 0 & 2\\ 3 & 1\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}1 & -1\\ 0 & 1\\ -1 & 0\end{bmatrix}.$$

Find $\boldsymbol{C}$.

Solution From $\boldsymbol{A}-3\boldsymbol{B}=4\boldsymbol{A}-\boldsymbol{C}$, we have

$$\boldsymbol{C}=4\boldsymbol{A}-\boldsymbol{A}+3\boldsymbol{B}=3(\boldsymbol{A}+\boldsymbol{B}),$$

so

$$\boldsymbol{C}=\begin{bmatrix}3(1+1) & 3(-1-1)\\ 3(0+0) & 3(2+1)\\ 3(3-1) & 3(1+0)\end{bmatrix}=\begin{bmatrix}6 & -6\\ 0 & 9\\ 6 & 3\end{bmatrix}.$$

III. 矩阵乘法(Multiplication)

引例1 超市购物 =>Product of Matrices

同样的商品在不同的超市内的售价是不尽相同的. 这样, 在一次需要购买多种商品时, 就有到哪一家超市去买花费最少的问题.

这就要用到价格矩阵, 如

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \qquad \begin{bmatrix} 1.7 & 1.1 & 21 & 7 \\ 1.5 & 1.4 & 26 & 9 \\ 1.8 & 1.3 & 28 & 8 \end{bmatrix}$$

可用来表示3家超市里4种商品的“价目表”

第1行的元依次表示超市1里4种商品的售价

III. 矩阵乘法(Multiplication)

引例1 超市购物

$$
\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}
\qquad
\begin{array}{c c}
\begin{array}{cccc} \text{商品1} & \text{商品2} & \text{商品3} & \text{商品4} \end{array} & \\
\begin{bmatrix} 1.7 & 1.1 & 21 & 7 \\ 1.5 & 1.4 & 26 & 9 \\ 1.8 & 1.3 & 28 & 8 \end{bmatrix} & \begin{array}{c} \text{超市1} \\ \text{超市2} \\ \text{超市3} \end{array}
\end{array}
$$

购物者1对4种商品的需求分别为$a_{11}, a_{21}, a_{31}, a_{41}$，则在不同超市去购买所需花费总额为?

若有n名购物者,则可将他们的需求构成<u>**需求矩阵**</u>

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}
$$

那么这n名购物者的采购方案可以用一个数表来表示:

<u>总价矩阵</u> = ?

价格矩阵 × 需求矩阵 = 总价矩阵

$$
\begin{array}{c}
\begin{array}{cccc} \text{商品}1 & \text{商品}2 & \text{商品}3 & \text{商品}4 \end{array} \\
\begin{array}{c} \text{超市}1 \\ \text{超市}2 \\ \text{超市}3 \end{array}
\begin{bmatrix} 1.7 & 1.1 & 21 & 7 \\ 1.5 & 1.4 & 26 & 9 \\ 1.8 & 1.3 & 28 & 8 \end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\begin{array}{cccc} \text{购物者}1 & \text{购物者}2 & \ldots\ldots & \text{购物者}n \end{array} \\
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}
\begin{array}{c} \text{需求}1 \\ \text{需求}2 \\ \text{需求}3 \\ \text{需求}4 \end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccc} \text{购物者}1 & \text{购物者}2 & \ldots\ldots & \text{购物者}n \end{array} \\
\begin{array}{c} \text{超市}1 \\ \text{超市}2 \\ \text{超市}3 \end{array}
\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ * & * & \cdots & * \end{bmatrix}
\end{array}
$$

矩阵乘法(Multiplication)

引例2 数学例子

设 �$_1$,�$_2$,�$_3$ 和 �$_1$,�$_2$ 是两组变量, 它们之间的关系为

�$_1$=�$_{11}$�$_1$+�$_{12}$�$_2$+�$_{13}$�$_3$

�$_2$=�$_{21}$�$_1$+�$_{22}$�$_2$+�$_{23}$�$_3$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}=[a_{ik}]_{2\times 3}$$

又设 �$_1$,�$_2$ 是另一组变量, 它们与�$_1$,�$_2$,�$_3$ 的关系为

�$_1$=�$_{11}$�$_1$+�$_{12}$�$_2$

�$_2$=�$_{21}$�$_1$+�$_{22}$�$_2$

�$_3$=�$_{31}$�$_1$+�$_{32}$�$_2$

$$B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}=[b_{kj}]_{3\times 2}$$

c_{11}

则

�$_2$=(�$_{21}$�$_{11}$+�$_{22}$�$_{21}$+�$_{23}$�$_{31}$)�$_1$ + (�$_{21}$�$_{12}$+�$_{22}$�

矩阵$C=[c_{ij}]_{2\times 2}$是矩阵A与B的一个运算, 定义为矩阵的乘积.

矩阵乘法(Multiplication)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix},$$

其中

$$\boxed{c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj},}$$

*Row-column rule for computing **AB***

上式右边(i, j)元素 c_{ij} 等于左边的第一个矩阵的第 i 行与第二个矩阵的第 j 列对应元素乘积之和.

矩阵运算中具有的特殊规律，主要产生于矩阵的乘法运算.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix},$$

Each entry of ***AB*** is the product (乘积) of a ***row*** and a ***column***:

$(\boldsymbol{AB})_{ij}$=(row i of $\boldsymbol{A}$) times (column j of $\boldsymbol{B}$)

Each column of ***AB*** is the product of a ***matrix*** and a ***column***:

column j of $\boldsymbol{AB}$= $\boldsymbol{A}$ times (column j of $\boldsymbol{B}$)

Each row of ***AB*** is the product of a ***row*** and a ***matrix***:

row i of $\boldsymbol{AB}$=(row i of $\boldsymbol{A}$) times $\boldsymbol{B}$

矩阵乘法(Multiplication)

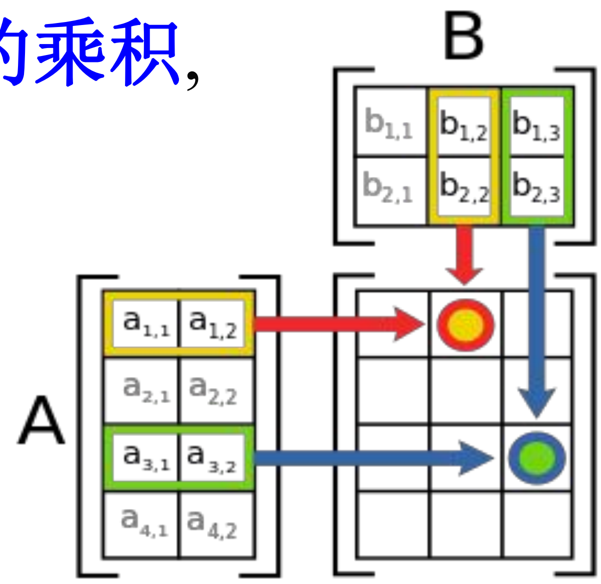
1. Definition

设 $\boldsymbol{A}=[a_{ik}]_{m\times p}$, $\boldsymbol{B}=[b_{kj}]_{p\times n}$ 为两个矩阵，令

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ip}b_{pj}=\sum_{k=1}^{p}a_{ik}b_{kj},$$

$$i=1,2,\ldots m;\ j=1,2,\ldots,n,$$

称 $m\times n$ 矩阵 $\boldsymbol{C}=[c_{ij}]_{m\times n}$ 为 ***A* 与 *B* 的乘积**，记为 $\boldsymbol{C}=\boldsymbol{AB}$.



注 只有当第一个矩阵的列数等于第二个矩阵的行数时，两个矩阵才能相乘.

Example 2 Find $\boldsymbol{AB}$, where

$$\boldsymbol{A}=\begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 3 & 0 \\ 0 & 5 & -1 & 4 \end{bmatrix}, \quad \boldsymbol{B}=\begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}.$$

Solution:

$$\boldsymbol{C}=\boldsymbol{AB}=\underset{3\times 4}{\begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 3 & 0 \\ 0 & 5 & -1 & 4 \end{bmatrix}}\underset{4\times 3}{\begin{bmatrix} 0 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}}=\underset{3\times 3}{\begin{bmatrix} -5 & 6 & 7 \\ 10 & 2 & -6 \\ -2 & 17 & 10 \end{bmatrix}}.$$

Match

Size of ***AB***

Exercises

$$\begin{bmatrix}1&2&3\\3&2&1\\5&8&9\end{bmatrix}\begin{bmatrix}1&6&8\\6&0&1\end{bmatrix}=?$$

注　只有当第一个矩阵的列数等于第二个矩阵的行数时，两个矩阵才能相乘.

$$\begin{bmatrix}1&2&3\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}1\times3+2\times2+3\times1\end{bmatrix}=10$$

$$\begin{bmatrix}3\\2\\1\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix}=\begin{bmatrix}3&6&9\\2&4&6\\1&2&3\end{bmatrix}$$

Example Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times 1$ and $1 \times n$ matrices, and

$$\boldsymbol{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \boldsymbol{B} = [b_1 \quad b_2 \quad \cdots \quad b_n].$$

Compute $\boldsymbol{AB}$ and $\boldsymbol{BA}$.

Solution

$$\boldsymbol{AB} = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{bmatrix}.$$

$$\boldsymbol{BA} = [a_1b_1 + a_2b_2 + \cdots a_nb_n].$$

A system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

The *i*-th equation:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$(a_{i1}, a_{i2}, \ldots, a_{in}) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b_i \quad (i = 1, 2, \ldots, m)$$

Let $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

$$\boxed{\boldsymbol{Ax} = \boldsymbol{b}}$$

System of Linear Equations

$$\begin{cases} \boxed{a_{11}x_1} + \boxed{a_{12}x_2} + \cdots + \boxed{a_{1n}x_n} = \boxed{b_1} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\boxed{\boldsymbol{\alpha}_1 x_1} + \boxed{\boldsymbol{\alpha}_2 x_2} + \cdots + \boxed{\boldsymbol{\alpha}_n x_n} = \boxed{\boldsymbol{b}}$$

Vector Equation

Coefficient Matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$= [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \cdots \ \boldsymbol{\alpha}_n]$$

Augmented Matrix

$$(\boldsymbol{A}, \boldsymbol{b}) = [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \cdots \ \boldsymbol{\alpha}_n \quad \boldsymbol{b}]$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Solution (解向量)

Matrix Equation

$$\boxed{\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.}$$

System of Linear Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

1

Coefficient Matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$= [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \cdots \ \boldsymbol{\alpha}_n]$$

$$\boldsymbol{\alpha}_1 x_1 + \boldsymbol{\alpha}_2 x_2 + \cdots + \boldsymbol{\alpha}_n x_n = \boldsymbol{b}$$

Vector Equation

2

Augmented Matrix

$$(\boldsymbol{A}, \boldsymbol{b}) = [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \cdots \ \boldsymbol{\alpha}_n \quad \boldsymbol{b}]$$

Matrix Equation

$$\boxed{\boldsymbol{Ax} = \boldsymbol{b}.}$$

3

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Solution (解向量)

2. Rules for Matrix Multiplication

(1) Let $\boldsymbol{A}=[a_{ij}]_{m\times n}$, $\boldsymbol{B}=[b_{ij}]_{n\times p}$, $\boldsymbol{C}=[c_{ij}]_{p\times r}$, then

$$(\boldsymbol{AB})\boldsymbol{C}=\boldsymbol{A}(\boldsymbol{BC}), \qquad k(\boldsymbol{AB})=(k\boldsymbol{A})\boldsymbol{B}=\boldsymbol{A}(k\boldsymbol{B}).$$

associative law of multiplication

(2) Let $\boldsymbol{A}=[a_{ij}]_{m\times p}$, $\boldsymbol{B}=[b_{ij}]_{p\times n}$, $\boldsymbol{C}=[c_{ij}]_{p\times n}$, $\boldsymbol{D}=[d_{ij}]_{n\times s}$, then

$$\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{AB}+\boldsymbol{AC}, \quad (\boldsymbol{B}+\boldsymbol{C})\boldsymbol{D}=\boldsymbol{BD}+\boldsymbol{CD}.$$

left distributive law **right distributive law**

(3) Let $\boldsymbol{A}=[a_{ij}]_{m\times n}$, $\boldsymbol{I}_m$, $\boldsymbol{I}_n$ are identity matrices of degree m and n respectively, then

$$\boldsymbol{A}=\boldsymbol{I}_m\boldsymbol{A}=\boldsymbol{A}\boldsymbol{I}_n; \qquad k\boldsymbol{A}=(k\boldsymbol{I}_m)\boldsymbol{A}=\boldsymbol{A}(k\boldsymbol{I}_n).$$

identity for matrix multiplication

证明: $(AB)C=A(BC)$.

设 $A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{n\times p}$, $C=(c_{ij})_{p\times r}$, 则$(AB)C$与$A(BC)$都是$m\times r$矩阵.

只需证明: $\forall i=1,\cdots,m$, $\forall j=1,\cdots,r$, 有

$$[(AB)C]_{ij}=\sum_{k=1}^{p}(AB)_{ik}C_{kj}=\sum_{k=1}^{p}(\sum_{l=1}^{n}a_{il}b_{lk})c_{kj}$$

交换和号顺序

$$=\sum_{l=1}^{n}a_{il}(\sum_{k=1}^{p}b_{lk}c_{kj})$$

$$=\sum_{l=1}^{n}a_{il}(BC)_{lj}=[A(BC)]_{ij}$$

所以 $(AB)C=A(BC)$.

问题： 矩阵乘法是否满足**交换律(commutative law)**，即 $\boldsymbol{AB}=\boldsymbol{BA}$？

$$\begin{bmatrix}1 & 6 & 8\\ 6 & 0 & 1\end{bmatrix},\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 5 & 8 & 9\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix},\begin{bmatrix}3\\ 2\\ 1\end{bmatrix}$$

$$\boldsymbol{A}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\boldsymbol{B}=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$$

$$\boldsymbol{A}=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}.$$

The product of two matrices is not commutative:
$\boldsymbol{AB}$ is not necessarily equal to $\boldsymbol{BA}$.

For example, if $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then

$$\boldsymbol{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{BA} = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix},$$

therefore $\boldsymbol{AB} \neq \boldsymbol{BA}$.

Warnings:

1. In general, $\boldsymbol{AB} \neq \boldsymbol{BA}$.
2. If a product $\boldsymbol{AB}$ is the zero matrix, you *cannot* conclude in general that either $\boldsymbol{A} = \mathbf{0}$ or $\boldsymbol{B} = \mathbf{0}$.
3. The cancellation laws do *not* hold for matrix multiplication. That is, if $\boldsymbol{AB} = \boldsymbol{AC}$, then it is *not* true in general that $\boldsymbol{B} = \boldsymbol{C}$.

投票： Is the product of two upper (lower) triangular matrices still upper (lower) triangular?

（两个上(下)三角阵$\boldsymbol{A}$与$\boldsymbol{B}$的乘积$\boldsymbol{AB}$是否仍是上(下)三角阵?）

$$\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \boldsymbol{B}=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \qquad \boldsymbol{C}=\boldsymbol{AB}=(c_{ij})_{n\times n}$$

What is its diagonal entry? （其主对角元$(\boldsymbol{AB})_{ii}$=?）

证明： 两个上(下)三角阵$\boldsymbol{A}$与$\boldsymbol{B}$的乘积$\boldsymbol{AB}$仍是上(下)三角阵, 且其主对角元$(\boldsymbol{AB})_{ii}=a_{ii}b_{ii}$.

证 设 $\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix}$ $\boldsymbol{C}=\boldsymbol{AB}=(c_{ij})_{n\times n}$

$i>j$时， $c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}=\sum_{k=1}^{i-1} a_{ik}b_{kj}+\sum_{k=i}^{n} a_{ik}b_{kj}$

$\boxed{a_{ik}=0\ (k=1,2,\cdots,i-1)}$

$\boxed{b_{kj}=0\ (k=i,i+1,\cdots,n)}$

即$i>j$时， $c_{ij}=0$ $\boldsymbol{C}$为上三角阵

而$c_{ii}=\sum_{k=1}^{n} a_{ik}b_{ki}=\sum_{k=1}^{i-1} a_{ik}b_{ki}+\sum_{k=i}^{n} a_{ik}b_{ki}$

$=0+a_{ii}b_{ii}=a_{ii}b_{ii}$.

$\boxed{a_{ik}=0\ (k=1,2,\cdots,i-1)}$

$\boxed{b_{ki}=0\ (k=i+1,\cdots,n)}$

IV. 方阵的幂(Power)

1. Definition

Let $\boldsymbol{A}$ be a square matrix of degree n, then we define the power of $\boldsymbol{A}$ ($\boldsymbol{A}$ 的幂) as

$$\boldsymbol{A}^0 = \boldsymbol{I}, \quad \boldsymbol{A}^1 = \boldsymbol{A}, \quad \boldsymbol{A}^2 = \boldsymbol{A}^1\boldsymbol{A}^1, \ \dots, \quad \boldsymbol{A}^{k+1} = \boldsymbol{A}^k\boldsymbol{A}^1.$$

注 只有方阵, 它的幂才有意义.

2. Rules

Let k, l be non-negative integers, then

(1) $\boldsymbol{A}^{k+l} = \boldsymbol{A}^k \boldsymbol{A}^l$;

(2) $(\boldsymbol{A}^k)^l = \boldsymbol{A}^{kl}$.

Usually $(\boldsymbol{AB})^k \neq \boldsymbol{A}^k\boldsymbol{B}^k$.

However, there is exception. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix}, \quad \text{then}$$

$$\boldsymbol{AB} = \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix}, \boldsymbol{BA} = \begin{bmatrix} 5 & 6 \\ 9 & 14 \end{bmatrix}, \Rightarrow \boldsymbol{AB} = \boldsymbol{BA}.$$

Remark When $\boldsymbol{A}$ and $\boldsymbol{B}$ can commute, the following statements hold.

$$(\boldsymbol{AB})^k = \boldsymbol{A}^k\boldsymbol{B}^k,$$

$$(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2,$$

$$(\boldsymbol{A}-\boldsymbol{B})(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{A}^2 - \boldsymbol{B}^2.$$

Let

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

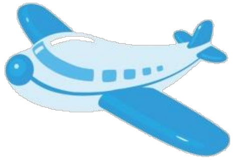
be a polynomial, where a_0, a_1, …, a_n are coefficients.

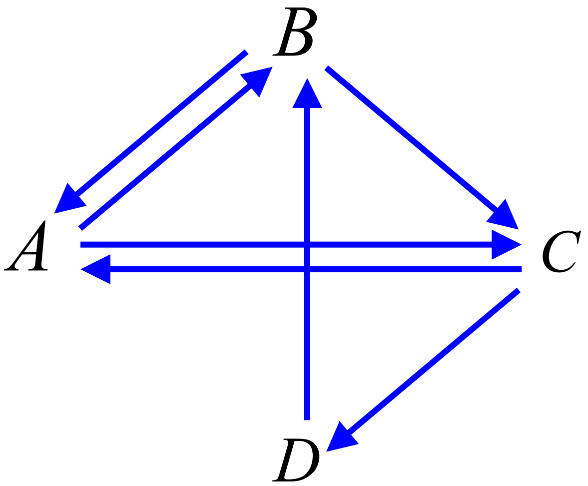
Let $\boldsymbol{A}$ be a square matrix, then

$$p(\boldsymbol{A}) = a_n \boldsymbol{A}^n + a_{n-1} \boldsymbol{A}^{n-1} + \cdots + a_1 \boldsymbol{A} + a_0 \boldsymbol{I}$$

is called the polynomial of the matrix $\boldsymbol{A}$ (方阵 $\boldsymbol{A}$ 的多项式).

城市间航班图





航班图可用矩阵来表示:

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

练习

1: 计算 $\boldsymbol{M}^2$?

2: 思考 $\boldsymbol{M}^2$ 中元素的实际含义.

Example 3 Compute $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n$.

Solution Let

$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then $\boldsymbol{A}$ is a scalar matrix, and $\boldsymbol{AB} = \boldsymbol{BA}$. Therefore,

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n = (\boldsymbol{A} + \boldsymbol{B})^n = \sum_{k=0}^{n} \mathrm{C}_n^k \boldsymbol{A}^{n-k} \boldsymbol{B}^k$$

$$= \mathrm{C}_n^0 \boldsymbol{A}^n \boldsymbol{B}^0 + \mathrm{C}_n^1 \boldsymbol{A}^{n-1} \boldsymbol{B} + \mathrm{C}_n^2 \boldsymbol{A}^{n-2} \boldsymbol{B}^2 + \cdots \mathrm{C}_n^n \boldsymbol{B}^n.$$

Since $\boldsymbol{B}^2 = \mathbf{0}$, we have $\boldsymbol{B}^2 = \boldsymbol{B}^3 = \ldots = \boldsymbol{B}^n = \mathbf{0}$. And

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n = \boldsymbol{A}^n + n\boldsymbol{A}^{n-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix} + \begin{bmatrix} n3^{n-1} & 0 \\ 0 & n3^{n-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix} + \begin{bmatrix} 0 & n3^{n-1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3^n & n3^{n-1} \\ 0 & 3^n \end{bmatrix}.$$

V. Elementary Matrices

定义 由单位矩阵 $\boldsymbol{I}$ 经过一次初等变换得到的方阵称为**初等矩阵**. (An **elementary matrix** is one that is obtained by performing a single elementary operation on an identity matrix.)

三种初等变换对应着三种初等矩阵:

(1) 对调两行或两列——**初等对换矩阵**;

(2) 以数 $k \neq 0$ 乘某行或某列——**初等倍乘矩阵**;

(3) 以数 k 乘某行(列)加到另一行(列)上去——**初等倍加矩阵**.

(1)初等对换矩阵：

将单位矩阵的第 i,j 行(或列)对换

$$\boldsymbol{P}_{ij} = \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 0 & & \cdots & & 1 & & & \\ & & & & 1 & & & & & & \\ & & & \vdots & & \ddots & & \vdots & & & \\ & & & & & & 1 & & & & \\ & & & 1 & & \cdots & & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow 第\ i\ 行 \\ \\ \\ \\ \leftarrow 第\ j\ 行 \\ \\ \\ \\ \end{array}$$

第 i 列　　第 j 列

(2)初等倍乘矩阵：

将单位矩阵第 i 行(或列)乘 $k \neq 0$

$$
\begin{array}{cc}
\boldsymbol{D}_i(k) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & k & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} & \leftarrow \text{第 } i \text{ 行} \\
\text{第 } i \text{ 列} &
\end{array}
$$

(3)初等倍加矩阵：

将单位矩阵第 i 行乘 k 加到第 j 行，

或将第 j 列乘 k 加到第 i 列

$$
\begin{array}{c}
\boldsymbol{E}_{ij}(k) = \begin{bmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & \vdots & \ddots & & & \\
& & k & \cdots & 1 & & \\
& & & & & \ddots & \\
& & & & & & 1
\end{bmatrix}
\begin{array}{l}
\\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \leftarrow \text{第 } j \text{ 行} \\ \\ \\
\end{array} \\
\qquad\qquad \text{第 } i \text{ 列} \qquad \text{第 } j \text{ 列}
\end{array}
$$

- **Example 4** Let

$$\boldsymbol{K}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

$$\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Compute $\boldsymbol{K}_1\boldsymbol{A}$, $\boldsymbol{K}_2\boldsymbol{A}$, and $\boldsymbol{K}_3\boldsymbol{A}$, and describe how these products can be obtained by elementary row operations on $\boldsymbol{A}$.

Solution:

Addition of -4 times row 1 of $\boldsymbol{A}$ to row 3 produces $\boldsymbol{K}_1\boldsymbol{A}$.

$$\boldsymbol{K}_1\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}$$

$$\boldsymbol{K}_2\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}.$$

An interchange of rows 1 and 2 of $\boldsymbol{A}$ produces $\boldsymbol{K}_2\boldsymbol{A}$.

$$\boldsymbol{K}_3\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

Multiplication of row 3 of $\boldsymbol{A}$ by 5 produces $\boldsymbol{K}_3\boldsymbol{A}$.

- ***Left-multiplication*** (左乘， that is, multiplication on the left) by $\boldsymbol{K}_1$ in Example 4 has the same effect on any $3\times n$ matrix.

- Since $\boldsymbol{K}_1\boldsymbol{I}=\boldsymbol{K}_1$, we see that $\boldsymbol{K}_1$ *itself* is produced by this same row operation on the identity. (注：由单位矩阵 $\boldsymbol{I}$ 经过一次初等变换得到的方阵称为**初等矩阵**)

- Example 4 illustrates the following general fact about elementary matrices.
- If an elementary row operation is performed on an $m\times n$ matrix $\boldsymbol{A}$, the resulting matrix can be written as $\boldsymbol{KA}$, where the $m\times m$ matrix $\boldsymbol{K}$ is created by performing the *same row operation* on $\boldsymbol{I}_m$.

What if -- right-multiplication?

Addition of -4 times column 3 of $\boldsymbol{A}$ to column 1 produces $\boldsymbol{AK}_1$.

$$\boldsymbol{AK}_1 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a-4c & b & c \\ d-4f & e & f \\ g-4i & h & i \end{bmatrix}$$

$$\boldsymbol{AK}_2 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix}.$$

An interchange of columns 1 and 2 of $\boldsymbol{A}$ produces $\boldsymbol{AK}_2$.

$$\boldsymbol{AK}_3 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} a & b & 5c \\ d & e & 5f \\ g & h & 5i \end{bmatrix}.$$

Multiplication of column 3 of $\boldsymbol{A}$ by 5 produces $\boldsymbol{AK}_3$.

初等矩阵与矩阵的乘积

1) 用 m 阶初等矩阵 $\boldsymbol{P}_{ij}$ 左乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，得

$$\boldsymbol{P}_{ij}\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{array}{l} \\ \\ \leftarrow 第\ i\ 行 \\ \\ \leftarrow 第\ j\ 行 \\ \\ \\ \end{array}$$

相当于把矩阵 $\boldsymbol{A}$ 第 i 行与第 j 行对调 $(r_i \leftrightarrow r_j)$.

2) 用 m 阶初等矩阵 $\boldsymbol{D}_i(k)$ 左乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，得

$$\boldsymbol{D}_i(k)\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \\ \end{array}$$

相当于以数 k 乘矩阵 $\boldsymbol{A}$ 的第 i 行 (kr_i) $(k\neq 0)$.

3) 用 $\boldsymbol{E}_{ij}(k)$ 左乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，得

$$\boldsymbol{E}_{ij}(k)\boldsymbol{A}=\begin{array}{cl}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1}+ka_{i1} & a_{j2}+ka_{i2} & \cdots & a_{jn}+ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow \text{第 } i \text{ 行} \\ \\ \leftarrow \text{第 } j \text{ 行} \\ \\ \\ \end{array}\end{array}$$

相当于把 $\boldsymbol{A}$ 的第 i 行乘 k 加到第 j 行 (r_j+kr_i).

4) 用 n 阶初等矩阵 $\boldsymbol{P}_{ij}$ 右乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，相当于把矩阵 $\boldsymbol{A}$ 第 i 列与第 j 列对调 $(c_i \leftrightarrow c_j)$.

5) 用 n 阶初等矩阵 $\boldsymbol{D}_i(k)$ 右乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，相当于以数 k 乘矩阵 $\boldsymbol{A}$ 的第 i 列 (kc_i).

6) 用 n 阶初等矩阵 $\boldsymbol{E}_{ij}(k)$ 右乘矩阵 $\boldsymbol{A}=[a_{ij}]_{m\times n}$，相当于将矩阵 $\boldsymbol{A}$ 的第 j 列乘数 k 加到第 i 列 (c_i+kc_j).

三种初等矩阵左乘矩阵 A 是对 A 作相应的初等行变换
三种初等矩阵右乘矩阵 B 是对 B 作相应的初等列变换

Note:
$K_1 = P_{13}$, $K_2 = E_{14}(c)$, $K_3 = D_2(k)$

Example 5 Let

$$K_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ c & 0 & 0 & 1 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Find $K_1K_2K_3$.

Solution We note that $K_1K_2K_3 = P_{13}\,E_{14}(c)\,D_2(k)$,

so

$$K_2K_3 = E_{14}(c)\,D_2(k) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ c & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & k & & \\ & & 1 & \\ c & & & 1 \end{bmatrix}$$

Note:

$\boldsymbol{K}_1=\boldsymbol{P}_{13}$, $\boldsymbol{K}_2=\boldsymbol{E}_{14}(c)$, $\boldsymbol{K}_3=\boldsymbol{D}_2(k)$

$\boldsymbol{K}_1\boldsymbol{K}_2\boldsymbol{K}_3=\boldsymbol{P}_{13}\boldsymbol{K}_2\boldsymbol{K}_3$, and

$$\boldsymbol{K}_1\boldsymbol{K}_2\boldsymbol{K}_3=\boldsymbol{P}_{13}\begin{bmatrix}1 & & & \\ & k & & \\ & & 1 & \\ c & & & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & k & 0 & 0\\ 1 & 0 & 0 & 0\\ c & 0 & 0 & 1\end{bmatrix}$$

Remark We can also use <u>right multiplication</u> and the corresponding ***<u>elementary column operations</u>*** to do the calculation. It leads to the same result.

Example 6 (将初等矩阵概念用于消元：Elimination)

For 3 equations in 3 unknowns:

Suppose $\boldsymbol{E}$ subtracts twice the first equation from the second.

Suppose $\boldsymbol{F}$ is the matrix for the next step, *to add row* 1 *to row* 3.

$$\boldsymbol{E}=\begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}, \qquad \boldsymbol{F}=\begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}.$$

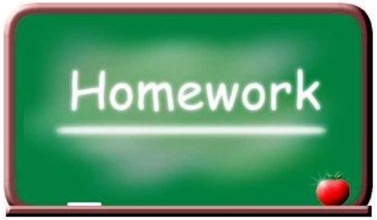
These two matrices do commute and the product does both steps at once:

$$\boldsymbol{EF}=\begin{bmatrix} 1 & & \\ -2 & 1 & \\ 1 & & 1 \end{bmatrix}=\boldsymbol{FE}.$$

In either order, $\boldsymbol{EF}$ or $\boldsymbol{FE}$, this changes rows 2 and 3 using row 1.

What if : $\boldsymbol{E}$ is the same but $\boldsymbol{G}$ *add row* 2 *to row* 3? $\boldsymbol{EG}\neq\boldsymbol{GE}.$

Homework



- See Blackboard announcement
- ***Hardcover*** **textbook + Supplementary problems**
- Pay attention to the notation

In the textbook, the ***elementary matrix*** $\boldsymbol{E}_{ij}$ subtracts l times row j from row i.

$$\boldsymbol{E}_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ -l & & 1 \end{bmatrix},$$

Deadline (DDL):

- See Blackboard