Quiz 1: 2022/9/17

Name: SID: Advanced Linear Algebra

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1. (5pt) Let V be a vector space. Prove that $-(-v) = v$ for every $v \in V$. (Recall that $-v$ denotes the additive inverse of v.)

Proof. Since

$$-v + (-(-v)) = 0,$$

add a v on both sides and

$$-v + (-(-v)) + v = v.$$

Using the commutativity and associativity, we obtain

$$-(-v) = (-(-v)) + (-v + v) = v.$$

□

A subset U of V is called a subspace of V if U is also a vector space.

2. (a) (2pt) Write the definition of subspace.
(b) (3pt) Prove that the intersection of every collection of subspaces of V is a subspace of V.
(c) (Bonus 2pt) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. (a) See textbook. □

(b) Let $\{U_\alpha\}_{\alpha \in I}$ be an arbitrary collection of subspaces of V, then by definition

$$0 \in U_\alpha, \quad \forall \alpha \in I,$$

$$u, w \in U_\alpha \quad \Rightarrow \quad u + w \in U_\alpha, \quad \forall \alpha \in I,$$

$$a \in \mathbb{F},\ u \in U_\alpha \quad \Rightarrow \quad au \in U_\alpha, \quad \forall \alpha \in I.$$

Since

$$0 \in U_\alpha, \quad \forall \alpha \in I \quad \Rightarrow \quad 0 \in \bigcap_{\alpha \in I} U_\alpha,$$

$$u, w \in \bigcap_{\alpha \in I} U_\alpha \quad \Rightarrow \quad u, w \in U_\alpha, \quad \forall \alpha \in I \quad \Rightarrow \quad u + w \in U_\alpha, \quad \forall \alpha \in I \quad \Rightarrow \quad u + w \in \bigcap_{\alpha \in I} U_\alpha,$$

$$a \in \mathbb{F},\ u \in \bigcap_{\alpha \in I} U_\alpha \quad \Rightarrow \quad a \in \mathbb{F},\ u \in U_\alpha, \quad \forall \alpha \in I \quad \Rightarrow \quad au \in U_\alpha, \quad \forall \alpha \in I \quad \Rightarrow \quad au \in \bigcap_{\alpha \in I} U_\alpha,$$

$\bigcap_{\alpha \in I} U_\alpha$ is a subspace of V by definition. □

(c) If one of the subspaces is contained in the other, then the union is equal to one of the subspaces, hence is a subspace.

Conversely, if the two subspaces U_1 and U_2 do not contain each other, then $U_1 \setminus U_2$ and $U_2 \setminus U_1$ are both non-empty, thus there exists $u \in U_1 \setminus U_2$ and $w \in U_2 \setminus U_1$. Note that $u + w$ cannot be in $U_1 \cup U_2$, because if $u + w \in U_1 \cup U_2$, then either $u + w \in U_1$ or $u + w \in U_2$, and it follows that either $w = (u + w) - u \in U_1$ or $u = (u + w) - w \in U_2$, while $w \notin U_1$ and $u \notin U_2$ by definition. Therefore $U_1 \cup U_2$ is not closed under addition, hence is not a subspace. □

Quiz 1: 2022/9/18
Name: SID: Advanced Linear Algebra

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1. (5pt) Show that the additive inverse of a certain element in a vector space is unique.

Proof. Let $v \in V$ be the certain element and $u, u' \in V$ be two additive inverses of v. We have

$$\underline{u = 0 + u = (v + u') + u = (v + u) + u' = 0 + u' = u'.}$$

Therefore the additive inverse must be unique. □

2. (a) (2pt) Write the definition of that the sum $U_1 + \cdots + U_m$ of subspaces is a direct sum.
(b) (3pt) Give an example of subspaces U_1, U_2 and U_3 such that $U_1 \cap U_2 = U_2 \cap U_3 = U_1 \cap U_3 = \{0\}$ but $U_1 + U_2 + U_3$ is not a direct sum. Verify that your example does satisfy the required property.
(c) (Bonus 2pt) Prove or give a counterexample: If U_1, U_2 and W are subspaces of V such that

$$V = U_1 \oplus W = U_2 \oplus W,$$

then $U_1 = U_2$.

Proof. (a) See textbook. If each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum of $u_1 + \cdots + u_m$, where each u_j is in U_j □
(b) Let

$$U_1 = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\},$$
$$U_2 = \{(0, x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\},$$
$$U_3 = \{(x, x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}.$$

Clearly $U_1 \cap U_2 = U_2 \cap U_3 = U_1 \cap U_3 = \{0\}$; $U_1 + U_2 + U_3$ is not a direct sum since

$$0 = (1, 0) + (0, 1) + (-1, -1).$$ □

(c) Counterexample: Take $V = \mathbb{R}^2$, U_1 and U_2 and $W = U_3$ as above. □

Quiz 2: 2022/9/24
Name: SID: Advanced Linear Algebra

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1. (5pt) Admitting that the span of a list of vectors in V is a subspace of V, show that it is the smallest subspace of V containing all the vectors in the list.
(Remark. With the result, which is proved in quiz 1, that the intersection of every collection of subspaces of V is a subspace of V, it follows that the span of a list of vectors is equivalent to the smallest subspace containing all the vectors in the list.)

Proof. See the last paragraph of the proof of Theorem 2.7 in the textbook. $\square$

2. (a) (3pt) Suppose that V is a finite-dimensional vector space and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.
(b) (2pt) Suppose that U and V are both n-dimensional subspaces of a vector space of dimension $2n-1$. Prove that $U \cap V \neq \{0\}$.
(c) (Bonus 1pt) Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Proof. (a) Let $u_1, \cdots, u_m$ be a basis of U, then $m = \dim U = \dim V$. Since $u_1, \cdots, u_m$ is a linearly independent list of length $\dim V$, by Theorem 2.39 it is a basis of V, in particular $\operatorname{span}(u_1, \cdots, u_m) = V$. Therefore $U = \operatorname{span}(u_1, \cdots, u_m) = V$. $\square$
(b) By theorem 2.43,

$$2n - 1 = \dim U + \dim V - \dim(U \cap V) = n + n - \dim(U \cap V),$$

hence $\dim(U \cap V) = 1$. Since $\dim\{0\} = 0$, $U \cap V \neq \{0\}$. $\square$
(c) Counterexample: consider $U \coloneqq \operatorname{span}(v_1, v_2, v_3 + v_4)$. Clearly v_1, v_2 is not a basis of U since $\dim U = 3$ (note that the fact that v_1, v_2, v_3, v_4 is linearly independent implies that $v_1, v_2, v_3 + v_4$ is also linearly independent). Either v_3 or $v_4 \in U$ implies that $v_3 = (v_3 + v_4) - v_4$ and $v_4 = (v_3 + v_4) - v_3$ are both in U, which would give that $U = V$, consequently $\dim U = \dim V = 4$, contradicting to that $\dim U = 3$. Therefore $v_1, v_2 \in U$, $v_3, v_4 \notin U$ while v_1, v_2 is not a basis of U. $\square$

Quiz 2: 2022/9/25
Name: SID: Advanced Linear Algebra

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1. (5pt) You have proved the following fact in your homework:
Suppose that $v_1, \cdots, v_m$ is linearly independent in V and $w \in V$, then $v_1, \cdots, v_m, w$ is linearly independent if and only if

$$w \notin \operatorname{span}(v_1, \cdots, v_m).$$

Use the above fact and Theorem 2.23 (the length of a linearly independent list is no more than that of any span list), prove that if V is a finite-dimensional linear space, then every linearly independent list of vectors in V of length $\dim V$ is a basis of V.

Proof. Let $\dim V = n$ and $v_1, \cdots, v_n$ be a linearly independent list of vectors in V. If $\operatorname{span}(v_1, \cdots, v_n) \neq V$, then there exists $w \in V \setminus \operatorname{span}(v_1, \cdots, v_n)$, hence, by the fact, $v_1, \cdots, v_n, w$ is a linearly independent list of length $n+1$ in V. By the definition of the dimension, there exists a span list of V of length n, thus that the length $n+1$ list $v_1, \cdots, v_n, w$ is linearly independent contradicts Theorem 2.23. Therefore $\operatorname{span}(v_1, \cdots, v_n) = V$, concluding that $v_1, \cdots, v_n$ is a basis of V. □

2. (a) (3pt) Prove that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, $\mathbb{R}^2$ and all lines in $\mathbb{R}^2$ through the origin. (Hint: classify all the subspaces of $\mathbb{R}^2$ according to their dimensions)
(b) (2pt) Let $U := \{f \in \mathbb{R}^{\mathbb{R}} \mid f(0) = 0\}$, we know that U is a subspace of $\mathbb{R}^{\mathbb{R}}$. Find a subspace W of $\mathbb{R}^{\mathbb{R}}$ such that

$$\mathbb{R}^{\mathbb{R}} = U \oplus W,$$

and verify your claim.

Proof. (a) Clearly, $\{0\}$, $\mathbb{R}^2$ and all lines in $\mathbb{R}^2$ through the origin are subspaces of $\mathbb{R}^2$.
Conversely, since $\dim \mathbb{R}^2 = 2$, the possible dimensions of its subspaces are $0, 1, 2$. The only subspaces of dimension 0 and 2 are $\{0\}$ and $\mathbb{R}^2$ respectively. If a subspace U is of dimension 1, then $U = \operatorname{span}(v)$ for some $0 \neq v = (x, y) \in \mathbb{R}^2$, thereby $U = \{k(x, y) \in \mathbb{R}^2 \mid k \in \mathbb{R}\}$ is a line in $\mathbb{R}^2$ through the origin. □
(b) Define $W := \{f \in \mathbb{R}^{\mathbb{R}} \mid f \text{ is constant}\}$. Clearly $U \cap W = \{0\}$. To see that $\mathbb{R}^{\mathbb{R}} = U + W$, note that for any $f \in \mathbb{R}^{\mathbb{R}}$, let $g(x) \equiv f(0)$, then $f = (f - g) + g$ and $f - g \in U$, $g \in W$. □

Quiz 3: 2022/10/8
Name: SID: Advanced Linear Algebra

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1. (5pt) Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \cdots, v_m$ is a list of vectors in V such that $Tv_1, \cdots, Tv_m$ is linearly independent in W. Prove that $v_1, \cdots, v_m$ is linearly independent.

Proof. For any $a_1, \cdots, a_m \in \mathbb{F}$ such that

$$a_1 v_1 + \cdots + a_m v_m = 0,$$

we have

$$a_1 T v_1 + \cdots + a_m T v_m = T(a_1 v_1 + \cdots + a_m v_m) = T0 = 0.$$

Since $Tv_1, \cdots, Tv_m$ is linearly independent, there must be $a_1 = \cdots = a_m = 0$, concluding that $v_1, \cdots, v_m$ is linearly independent. $\square$

2. (a) (3pt) Prove or disprove: there does not exist a linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

(b) (2pt) Let V be a vector space. Suppose that there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Proof. (a) By the fundamental theorem,

$$5 = \dim \mathbb{R}^5 = \dim \operatorname{range} T + \dim \operatorname{null} T.$$

Since 5 is odd, the dimension of $\operatorname{range} T$ and $\operatorname{null} T$ cannot be equal, consequently they themselves cannot be equal. $\square$
(b) Let $T\colon V \to W$ be a linear map such that $\operatorname{null} T$ and $\operatorname{range} T$ are both finite-dimensional. Let $v_1, \cdots, v_n$ be a basis of $\operatorname{null} T$ and $w_1, \cdots, w_m$ be a basis of $\operatorname{range} T$, then since $w_i \in \operatorname{range} T$, we can choose $u_i \in V$ such that $Tu_i = w_i$ for each $i = 1, \cdots, m$. The claim is that V is spanned by $v_1, \cdots, v_n, u_1, \cdots, u_m$ (in fact, one can show that $v_1, \cdots, v_n, u_1, \cdots, u_m$ is a basis of V). Indeed, given any $v \in V$, then

$$Tv = a_1 T u_1 + \cdots + a_m T u_m,$$

for some $a_1, \cdots, a_m \in \mathbb{F}$. Thus $v - (a_1 u_1 + \cdots + a_m u_m) \in \operatorname{null} T$ since

$$T(v - (a_1 u_1 + \cdots + a_m u_m)) = Tv - (a_1 T u_1 + \cdots + a_m T u_m) = 0.$$

Hence $v - (a_1 u_1 + \cdots + a_m u_m) = b_1 v_1 + \cdots + b_n v_n$ for some $b_1, \cdots, b_n \in \mathbb{F}$. Therefore

$$v = b_1 v_1 + \cdots + b_n v_n + a_1 u_1 + \cdots + a_m u_m,$$

concluding that $v \in \operatorname{span}(v_1, \cdots, v_n, u_1, \cdots, u_m)$ for any $v \in V$, so V is finite-dimensional by definition. $\square$

3. (Bonus 2pt) Let V and W be two vector spaces over a same field $\mathbb{F}$. Show that the inverse of a bijective linear map from V to W is also linear. Note that V and W are not assumed to be finite-dimensional.
(Remark. This tells that bijective linear maps are exactly the isomorphisms in the category of vector spaces, if you know about some category theory.)

Proof. Let $T\colon V \to W$ be a bijective linear map, we need to prove that its inverse $T^{-1}\colon W \to V$ is also linear. Indeed, for any $w, u \in W$ and $\lambda \in \mathbb{F}$,

$$T^{-1}(\lambda w + u) \xlongequal{TT^{-1} = 1_W} T^{-1}(\lambda T T^{-1} w + T T^{-1} u) \xlongequal{\text{linearity of } T} T^{-1} T(\lambda T^{-1} w + T^{-1} u) \xlongequal{T^{-1}T = 1_V} \lambda T^{-1} w + T^{-1} u.$$

Therefore T^{-1} is linear. $\square$

1. (5pt) Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \cdots, v_n$ is linearly independent in V. Show that $Tv_1, \cdots, Tv_n$ is linearly independent in W.

Proof. Let $a_1, \cdots, a_n \in \mathbb{F}$ be such that

$$0 = a_1Tv_1 + \cdots + a_nTv_n = T(a_1v_1 + \cdots + a_nv_n),$$

then since T is injective, $a_1v_1 + \cdots + a_nv_n = 0$ by Theorem 3.16. Since $v_1, \cdots, v_n$ is linearly independent, there must be $a_1 = \cdots = a_n = 0$, concluding that $Tv_1, \cdots, Tv_n$ is linearly independent. ☐

2. (a) (3pt) Suppose that $\varphi \in \mathcal{L}(V, \mathbb{F})$. Suppose that $u \in V$ is not in $\operatorname{null}\varphi$. Prove that

$$V = \operatorname{null}\varphi \oplus \operatorname{span}(u).$$

Note that V is not required to be finite-dimensional.
(b) (2pt) Suppose φ_1 and φ_2 are both in $\mathcal{L}(V, \mathbb{F})$ and $\operatorname{null}\varphi_2 \subset \operatorname{null}\varphi_1$. Show that there exists a scalar $c \in \mathbb{F}$ such that $\varphi_1 = c\varphi_2$. (Hint: the conclusion of (a) may be helpful.)

Proof. (a) Clearly $\operatorname{null}\varphi \cap \operatorname{span}(u) = \{0\}$. For any $v \in V$, since $\varphi(u) \neq 0$, we have

$$\boxed{v = \left(v - \frac{\varphi(v)}{\varphi(u)}u\right) + \frac{\varphi(v)}{\varphi(u)}u}$$

where $v - \frac{\varphi(v)}{\varphi(u)}u \in \operatorname{null}\varphi$ because

$$\varphi\left(v - \frac{\varphi(v)}{\varphi(u)}u\right) = \varphi(v) - \frac{\varphi(v)}{\varphi(u)}\varphi(u) = \varphi(v) - \varphi(v) = 0.$$

Therefore $V = \operatorname{null}\varphi \oplus \operatorname{span}(u)$. ☐
(b) If $\operatorname{null}\varphi_1 = V$, take $c = 0$. If $\operatorname{null}\varphi_1 \neq V$, then there exists $u \in V \setminus \operatorname{null}\varphi_1$. Since $\operatorname{null}\varphi_2 \subset \operatorname{null}\varphi_1$, we also have $u \notin \operatorname{null}\varphi_2$. By (a), we see that

$$V = \operatorname{null}\varphi_2 \oplus \operatorname{span}(u).$$

Thus every $v \in V$ admits a unique decomposition $v = w + au$ for some $w \in \operatorname{null}\varphi_2 \subset \operatorname{null}\varphi_1$ and $a \in \mathbb{F}$. Since $\varphi_2(u) \neq 0$, we see that

$$\varphi_1(v) = a\varphi_1(u) = a\frac{\varphi_1(u)}{\varphi_2(u)}\varphi_2(u) = \frac{\varphi_1(u)}{\varphi_2(u)}\varphi_2(w + au) = \frac{\varphi_1(u)}{\varphi_2(u)}\varphi_2(v),$$

for each $v \in V$. Therefore $c = \frac{\varphi_1(u)}{\varphi_2(u)} \in \mathbb{F}$ satisfies $\varphi_1 = c\varphi_2$. ☐

3. (Bonus 2pt) Let V and W be two finite-dimensional vector spaces over a same field $\mathbb{F}$. A bijective linear map from V to W is called an isomorphism between V and W. If there exists an isomorphism between V and W, we say that V and W are isomorphic. Prove that V and W are isomorphic if and only if $\dim V = \dim W$.

Proof. If V and W are isomorphic, then there exists a bijective linear map $T: V \to W$. Since $\operatorname{null} T = \{0\}$ and $\operatorname{range} T = W$ by the bijectivity, by the fundamental theorem,

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = 0 + \dim W = \dim W.$$

Conversely, if $\dim V = \dim W = n$, then let $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ be bases of V and W respectively, and define via Theorem 3.5 a linear map $T: V \to W$ such that

$$Tv_i = w_i,$$

for each $i = 1, \cdots, n$. Such T is surjective since $\operatorname{range} T = \operatorname{span}(Tv_1, \cdots, Tv_n) = \operatorname{span}(w_1, \cdots, w_n) = W$. For any $v = a_1v_1 + \cdots + a_nv_n \in V$ such that $Tv = 0$, we have

$$0 = Tv = a_1Tv_1 + \cdots + a_nTv_n = a_1w_1 + \cdots + a_nw_n,$$

hence $a_1 = \cdots = a_n$ since $w_1, \cdots, w_n$ is linearly independent, consequently $v = 0$, concluding that $\operatorname{null} T = \{0\}$, so T is injective. ☐

Quiz 4: 2022/10/15
Name: SID: Advanced Linear Algebra

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1. (5pt) Let $T : \mathcal{P}_2(\mathbb{F}) \to \mathcal{P}_1(\mathbb{F})$ be defined by

$$\begin{array}{rccc} T\colon & \mathcal{P}_2(\mathbb{F}) & \longrightarrow & \mathcal{P}_1(\mathbb{F}) \\ & p & \longmapsto & p' + 2p'' \end{array}$$

then we know that T is linear. Compute $\mathcal{M}(T, (1, x-1, x^2), (1, x+1))$.

Proof. We have

$$\begin{aligned} T(1) = 0 \quad & = 0 \cdot 1 + 0 \cdot (x+1), \\ T(x-1) = 1 \quad & = 1 \cdot 1 + 0 \cdot (x+1), \\ Tx^2 = 2x + 4 \quad & = 2 \cdot 1 + 2 \cdot (x+1). \end{aligned}$$

Therefore

$$\mathcal{M}(T, (1, x, x^2), (1, x)) = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

□

2. Let U, V and $W \neq \{0\}$ be three vector spaces and $T \in \mathcal{L}(U, V)$. The precomposition of T gives a map $(-) \circ T$ from $\mathcal{L}(V, W)$ to $\mathcal{L}(U, W)$, by

$$\begin{array}{rccc} (-) \circ T\colon & \mathcal{L}(V, W) & \longrightarrow & \mathcal{L}(U, W) \\ & S & \longmapsto & ST \end{array}$$

i.e. $(-) \circ T$ sends a linear map $S \in \mathcal{L}(V, W)$ to $ST \in \mathcal{L}(U, W)$. You should have seen that $(-) \circ T$ is a linear map (verify it yourself if you haven't!).
(a) (2pt) Suppose that V is finite-dimensional. Prove that T is surjective if and only if $(-) \circ T$ is injective.
(b) (3pt) Suppose that U and V are both finite-dimensional. Prove that T is injective if and only if $(-) \circ T$ is surjective.
(*Remark.* These conclude that, if U and V are both finite-dimensional, then T is an isomorphism if and only if $(-) \circ T$ is an isomorphism. In fact, if we admit the axiom of choice, then the results are true even if U and V are not assumed to be finite-dimensional because the extensions of linear maps would always exist.)

Proof. (a) If T is surjective, then for any $S, S' \in \mathcal{L}(V, W)$ such that $ST = S'T$, for any $v \in V$, there exists $u \in U$ such that $Tu = v$, thus

$$Sv = STu = S'Tu = S'v,$$

concluding that $S = S'$, so that $(-) \circ T$ is injective.
If T is not surjective, then let $v_1, \cdots, v_m$ be a basis of $\operatorname{range} T$ and extend it to $v_1, \cdots, v_m, v_{m+1}, \cdots, v_n$ a basis of V. We have $n > m$ since $\operatorname{range} T \neq V$. Let $w \in W \setminus \{0\}$. Define $S \in \mathcal{L}(V, W)$ by

$$Sv_n = w, \quad Sv_i = 0, \quad \forall i < n,$$

then $ST = 0 = 0T$ while $S \neq 0$, so $(-) \circ T$ is not injective. □
(b) If T is injective, let $u_1, \cdots, u_m$ be a basis of U, then $Tu_1, \cdots, Tu_m$ is linearly independent in V, thus we can extend it to $Tu_1, \cdots, Tu_m, v_1, \cdots, v_n$ a basis of V. For any $F \in \mathcal{L}(U, W)$, define $S \in \mathcal{L}(V, W)$ by

$$S(Tu_i) = Fu_i, \quad \forall i = 1, \cdots, m,$$

and $Sv_j = 0$ for each $j = 1, \cdots, n$, then $ST = F$ since they agree on each of the basis elements u_i's of U, concluding that $(-) \circ T$ is surjective.
If T is not injective, then there exists $u \in \operatorname{null} T \setminus \{0\}$; extend u to a basis of U, $u, u_1, \cdots, u_m$. Let $w \in W \setminus \{0\}$, and define $F \in \mathcal{L}(U, W)$ by

$$Fu = w, \quad Fu_i = 0, \quad \forall i = 1, \cdots, m,$$

then F cannot be in the range of $(-) \circ T$, since $STu = 0 \neq w = Fu$ for all $S \in \mathcal{L}(V, W)$. □

3. (Bonus 2pt) Let V be a finite-dimensional vector space. Show that the linear map φ given by the following (you don't need to check its linearity here, but do check it yourself if you are not sure about that it is linear)

$$\begin{array}{rccc} \varphi\colon & V & \longrightarrow & \mathcal{L}(\mathcal{L}(V, \mathbb{F}), \mathbb{F}) \\ & v & \longmapsto & (\varphi(v)\colon f \mapsto f(v)) \end{array}$$

i.e. φ sends a vector $v \in V$ to the function $\varphi(v)$ in $\mathcal{L}(\mathcal{L}(V, \mathbb{F}), \mathbb{F})$ that sends an element $f \in \mathcal{L}(V, \mathbb{F})$ to $f(v) \in \mathbb{F}$, is an isomorphism of vector spaces between V and $\mathcal{L}(\mathcal{L}(V, \mathbb{F}), \mathbb{F})$.
(*Remark.* This is not true if V is infinite-dimensional. In fact, we have that φ is an isomorphism if and only if V is finite-dimensional.)

Proof. Since V is finite dimensional, we know by Theorem 3.61 that $\mathcal{L}(V,\mathbb{F})$ and $\mathcal{L}(\mathcal{L}(V,\mathbb{F}),\mathbb{F})$ are both finite dimensional and

$$\dim \mathcal{L}(\mathcal{L}(V,\mathbb{F}),\mathbb{F}) = (\dim \mathcal{L}(V,\mathbb{F})) \cdot \dim \mathbb{F} = \dim V \cdot \dim \mathbb{F} \cdot 1 = \dim V,$$

hence it suffices to verify that φ is injective as a consequence of the Fundamental Theorem of Linear Maps (see the proof of Theorem 3.69 if you don't see how). Indeed, for $v \in V \setminus \{0\}$, we can extend it to a basis $v, v_1, \cdots, v_n$ of V and define $f \in \mathcal{L}(V,\mathbb{F})$ by

$$f(v) = 1, \quad f(v_i) = 0, \quad \forall i = 1, \cdots, n,$$

so that there exists $f \in \mathcal{L}(V,\mathbb{F})$ such that $f(v) = 1 \neq 0$. Since $\varphi(v) = 0$ means exactly that $f(v) = 0$ for all $f \in \mathcal{L}(V,\mathbb{F})$, we conclude that $\varphi(v) = 0$ only if $v = 0$, therefore φ is injective. $\square$

1. (5pt) Let $T : \mathcal{P}_1(\mathbb{F}) \to \mathcal{P}_2(\mathbb{F})$ be defined by

$$\begin{array}{cccc} T\colon & \mathcal{P}_1(\mathbb{F}) & \longrightarrow & \mathcal{P}_2(\mathbb{F}) \\ & p & \longmapsto & xp - 2p' \end{array}$$

then we know that T is linear. Compute $\mathcal{M}(T, (1, x-2), (1, x+1, x^2))$.

Proof. We have

$$\begin{aligned} T(1) = x \qquad\qquad &= -1 \cdot 1 + 1 \cdot (x+1) + 0 \cdot x^2, \\ T(x-2) = x^2 - 2x - 2 \quad &= 0 \cdot 1 + (-2) \cdot (x+1) + 1 \cdot x^2, \end{aligned}$$

Therefore

$$\mathcal{M}(T, (1, x-2), (1, x+1, x^2)) = \begin{pmatrix} -1 & 0 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}$$

□

2. Let U, V and $W \neq \{0\}$ be three vector spaces and $T \in \mathcal{L}(U, V)$. The post-composition of T gives a map $(-) \circ T$ from $\mathcal{L}(W, U)$ to $\mathcal{L}(W, V)$, by

$$\begin{array}{cccc} T \circ (-)\colon & \mathcal{L}(W, U) & \longrightarrow & \mathcal{L}(W, V) \\ & S & \longmapsto & TS \end{array}$$

i.e. $T \circ (-)$ sends a linear map $S \in \mathcal{L}(W, U)$ to $TS \in \mathcal{L}(W, V)$. You should have seen that $T \circ (-)$ is a linear map (verify it yourself if you haven't!).
(a) (2pt) Suppose that W is finite-dimensional. Prove that T is injective if and only if $T \circ (-)$ is injective.
(b) (3pt) Suppose that W and V are both finite-dimensional. Prove that T is surjective if and only if $T \circ (-)$ is surjective.
(*Remark.* These conclude that, if W and V are both finite-dimensional, then T is an isomorphism if and only if $T \circ (-)$ is an isomorphism. In fact, if we admit the axiom of choice, then the results are true even if W and V are not assumed to be finite-dimensional.)

Proof. (a) If T is injective, then for any $S, S' \in \mathcal{L}(W, U)$ such that $TS = TS'$, we have for any $w \in W$,

$$T(Sw) = TSw = TS'w = T(S'w) \quad \Rightarrow \quad Sw = S'w,$$

concluding that $S = S'$, thereby $T \circ (-)$ is injective.
If T is not injective, then there exists $u \in \operatorname{null} T \setminus \{0\}$. Let $w_1, \cdots, w_m$ be a basis of W and define $S \in \mathcal{L}(W, U)$ by

$$Sw_i = u, \quad \forall i = 1, \cdots, m,$$

then $S \neq 0$ while $TS = 0$ since $\operatorname{range} S = \operatorname{span}(u) \subset \operatorname{null} T$, concluding that $T \circ (-)$ is not injective. □
(b) If T is surjective, then for any $F \in \mathcal{L}(W, V)$, let $v_1, \cdots, v_n$ be a basis of $\operatorname{range} F$ and $w_1, \cdots, w_n \in W$ be such that $Fw_i = v_i$ for each $i = 1, \cdots, n$. Let $w_{n+1}, \cdots, w_m$ be a basis of $\operatorname{null} F$, then it is known that $w_1, \cdots, w_m$ is a basis of W (you should have proved this result for several times; if you have forgotten it, you should reprove it by yourself). Since T is surjective, we can select $u_i \in U$ such that $Tu_i = v_i$ for each $i = 1, \cdots, n$. Define $S \in \mathcal{L}(W, U)$ by

$$Sw_i = u_i, \quad \forall i = 1, \cdots, n,$$

and $Sw_j = 0$ for all $j > n$, then $TS = F$ since they agree on the basis elements w_i's of W. Therefore $T \circ (-)$ is surjective.
If T is not surjective, then there exists $v \in V \setminus \operatorname{range} T$. Let $w_1, \cdots, w_m$ be a basis of W and define $F \in \mathcal{L}(W, V)$ by

$$Fw_i = v, \quad \forall i = 1, \cdots, m,$$

then $\operatorname{range} F = \operatorname{span}(v)$ and there does not exist $S \in \mathcal{L}(W, U)$ such that $TS = F$ since $\operatorname{range} TS \subset \operatorname{range} T$ for all $S \in \mathcal{L}(W, U)$ and $v \notin \operatorname{range} T$, concluding that $T \circ (-)$ is not surjective. □

3. Let V be an n-dimensional vector space and $T \in \mathcal{L}(V)$ satisfy $T^n = 0$ and $T^{n-1} \neq 0$.
(a) (Bonus 1pt) Let $v_0 \in V \setminus \operatorname{null} T^{n-1}$, prove that $v_0, Tv_0, \cdots, T^{n-1}v_0$ is a basis of V.
(b) (Bonus 1pt) Define $\Sigma \coloneqq \{S \in \mathcal{L}(V) \mid ST = TS\}$, then Σ is a subspace of $\mathcal{L}(V)$ (you don't need to verify it here, but again, if you do not see this, verify it yourself). Define the linear map $\varphi \in \mathcal{L}(\Sigma, V)$ by

$$\begin{array}{cccc} \varphi\colon & \Sigma & \longrightarrow & V \\ & S & \longmapsto & Sv_0 \end{array}$$

show that φ is an isomorphism, hence conclude that $\dim \Sigma = n$.

Proof. (a) Since $v_0, Tv_0, \cdots, T^{n-1}v_0$ has length $n = \dim V$, it suffices to show that it is linearly independent by Theorem 2.39. For any $a_1, \cdots, a_n \in \mathbb{F}$ such that

$$a_1 v_0 + a_2 T v_0 + \cdots + a_n T^{n-1} v_0 = 0,$$

apply T^{n-1} to both sides, since $T^n = 0$, we have

$$a_1 T^{n-1} v_0 = T^{n-1} a_1 v_0 = 0.$$

Since $v_0 \notin \operatorname{null} T^{n-1}$, $T^{n-1} v_0 \neq 0$, hence there must be $a_1 = 0$. Thereby

$$a_2 T v_0 + \cdots + a_n T^{n-1} v_0 = 0.$$

Again, apply T^{n-2} and we see that $a_2 = 0$. Do this repeatedly, applying $T^{n-3}, T^{n-4}, \cdots, T, T^0 = I$ for each time, and we see that $a_3 = a_4 = \cdots = a_n = 0$, concluding that $v_0, Tv_0, \cdots, T^{n-1}v_0$ is linearly independent. □

(b) Since $I = T^0, T, \cdots, T^{n-1}$ are all in Σ, we see that $\{v_0, Tv_0, \cdots, T^{n-1}v_0\} \subset \operatorname{range} \varphi$, hence by (a) we know that φ is surjective. For the injectivity, notice that if $Sv_0 = 0$ for some $S \in \Sigma$, then

$$0 = TSv_0 = STv_0 = S(Tv_0).$$

Similarly we obtain that $S(T^i v_0) = 0$ for each $i = 0, 1, \cdots, n-1$. Since $v_0, Tv_0, \cdots, T^{n-1}v_0$ spans V by (a), we conclude that $S = 0$, therefore φ is injective. □

Quiz 5: 2022/10/22
Name: SID: Advanced Linear Algebra

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1. (5pt) Let $V = \mathbb{F}^{\mathbb{F}}$ (recall that $\mathbb{F}^{\mathbb{F}}$ is ~~the set of all functions from $\mathbb{F}$ to $\mathbb{F}$~~). Let $U = \{f \in V \mid f(0) = f(1) = 0\}$, then U is a subspace of V and both V and U are infinite-dimensional (you don't need to prove these here). Find a basis of V/U, and conclude that $\dim(V/U) = 2$.

$$f_0(\lambda) = \begin{cases} 1, & \lambda = 0 \\ 0, & \lambda \neq 0 \end{cases} \qquad f_1(\lambda) = \begin{cases} 1, & \lambda = 1 \\ 0, & \lambda \neq 1 \end{cases}$$

Proof. Let $f_0, f_1 \in \mathbb{F}^{\mathbb{F}}$ be defined by

$$f_i(\lambda) = \begin{cases} 1 & \lambda = i \\ 0 & \lambda \neq i \end{cases}$$

for any $\lambda \in \mathbb{F}$ and each $i = 0, 1$. For any $f + U \in V/U$, we have $f - (\underbrace{f(0)}_{=0} f_0 + f\underbrace{(1)f_1}_{=0}) \in U$, hence

$$f + U = (f(0)f_0 + f(1)f_1) + U = f(0)(f_0 + U) + f(1)(f_1 + U),$$

by Theorem 3.85. Thus $f_0 + U, f_1 + U$ spans V/U. For any $a_1, a_2 \in \mathbb{F}$ such that

$$(a_1 f_0 + a_2 f_1) + U = a_1(f_0 + U) + a_2(f_1 + U) = 0 + U,$$

we have by Theorem 3.85 $a_1 f_0 + a_2 f_1 \in U$, hence

$$0 = (a_1 f_0 + a_2 f_1)(0) = a_1 f_0(0) + a_2 f_1(0) = a_1, \quad 0 = (a_1 f_0 + a_2 f_1)(1) = a_1 f_0(1) + a_2 f_1(1) = a_2,$$

concluding that $f_0 + U, f_1 + U$ is linearly independent. Therefore $f_0 + U, f_1 + U$ is a basis of V/U and $\dim(V/U) = 2$. $\square$

2. Suppose U is a subspace of V. Let $\pi : V \to V/U$ be the usual quotient map, i.e. $\pi(v) = v + U$ for each $v \in V$. Thus $\pi' \in \mathcal{L}((V/U)', V')$.
(a) (2pt) Show that π' is injective.
(b) (3pt) Show that $\operatorname{range} \pi' = U^0$. Thus conclude that π' is an isomorphism between $(V/U)'$ and U^0.

Proof. (a) If $\varphi \in (V/U)'$ is not zero, then there exists $v + U \in V/U$ such that $\varphi(v + U) \neq 0$. Thus $\pi'(\varphi)(v) = (\varphi \circ \pi)(v) = \varphi(v + U) \neq 0$, hence $\pi'(\varphi) \neq 0$, concluding that π' is injective. $\square$
(b) For any $\varphi \in (V/U)'$ and any $u \in U$, since $u + U = 0 + U$, we have

$$\pi'(\varphi)(u) = \varphi(u + U) = \varphi(0 + U) = 0,$$

hence $\operatorname{range} \pi' \subset U^0$. For any $\varphi \in U^0 \subset V'$, define $\widetilde{\varphi} \in (V/U)'$ by

$$\widetilde{\varphi}(v + U) \coloneqq \varphi(v),$$

for each $v + U \in V/U$, then $\widetilde{\varphi}$ is well-defined, because if $v + U = w + U$ for some $v, w \in V$, then $v - w \in U$ by Theorem 3.85, hence $\varphi(v - w) = 0$ since $\varphi \in U^0$, and thereby

$$\widetilde{\varphi}(w + U) = \varphi(w) = \varphi(w) + \varphi(v - w) = \varphi(v) = \widetilde{\varphi}(v + U).$$

The verification of the linearity of $\widetilde{\varphi}$ is routine, hence $\widetilde{\varphi}$ is indeed in $(V/U)'$. Now that for any $v \in V$,

$$\pi'(\widetilde{\varphi})(v) = \widetilde{\varphi}(v + U) = \varphi(v),$$

hence $\pi'(\widetilde{\varphi}) = \varphi$, concluding that $\operatorname{range} \pi' \supset U^0$. Therefore we obtain $\operatorname{range} \pi' = U^0$.
Since π' is injective by (a), we see that π', seen as a map from $(V/U)'$ to U^0, is bijective and linear, concluding that π' is an isomorphism between $(V/U)'$ and U^0, $\square$

3. (Bonus 2pt) Let V and W both be finite-dimensional vector spaces. Define $\varphi \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(W', V'))$ by

$$\begin{array}{cccc} \varphi\colon & \mathcal{L}(V, W) & \longrightarrow & \mathcal{L}(W', V') = \mathcal{L}(\mathcal{L}(W, \mathbb{F}), \mathcal{L}(V, \mathbb{F})) \\ & T & \longmapsto & T' = (-) \circ T \end{array}$$

then we know that φ is linear by Theorem 3.101. Show that φ is an isomorphism of vector spaces.

Proof. Since $\dim \mathcal{L}(V, W) = (\dim V) \cdot (\dim W) = (\dim V') \cdot (\dim W') = \dim \mathcal{L}(W', V')$, it suffices to show that φ is injective. If $T \neq 0$, then $Tv = w \neq 0$ for certain $v \in V$ and $w \in W$. Extend w to a basis $w, w_1, \cdots, w_m$ of W, and define $f \in \mathcal{L}(W, \mathbb{F})$ by

$$f(w) = 1, \quad f(w_i) = 0, \quad \forall i = 1, \cdots, m.$$

Then $\varphi(T)(f) = T'(f) = f \circ T \neq 0$ since $f \circ T(v) = f(w) = 1$, hence $\varphi(T) \neq 0$. Therefore φ is injective. $\square$

1. (5pt) Suppose that V is finite-dimensional and U and W are two subspaces of V. Prove that

$$(U \cap W)^0 = U^0 + W^0.$$

Proof. For any $\varphi \in U^0$, we have $\varphi(u) = 0$ for all $u \in U$, hence $\varphi(u) = 0$ for all $u \in U \cap W$, thus $\varphi \in (U \cap W)^0$, concluding that $U^0 \subset (U \cap W)^0$. Similarly we have $W^0 \subset (U \cap W)^0$.
Since V is finite-dimensional, so is U, W and $U \cap W$. Let $v_1, \cdots, v_n$ be a basis of $U \cap W$ and extend it to a basis $v_1, \cdots, v_n, u_1, \cdots, u_k$ of U and a basis $v_1, \cdots, v_n, w_1, \cdots, w_r$ of W. It is clear that $v_1, \cdots, v_n, u_1, \cdots, u_k, w_1 \cdots, w_r$ spans $U + W$. Since $\dim U \cap W = n$, $\dim U = n + k$, $\dim W = n + r$ and

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = n + k + r,$$

we see that $v_1, \cdots, v_n, u_1, \cdots, u_k, w_1 \cdots, w_r$ is a basis of $U + W$, in particular it is linearly independent in V. Extend $v_1, \cdots, v_n, u_1, \cdots, u_k, w_1 \cdots, w_r$ to a basis $v_1, \cdots, v_n, u_1, \cdots, u_k, w_1 \cdots, w_r, v_{n+1}, \cdots, v_m$ of V. Now, let $\varphi \in (U \cap W)^0$ be arbitrary, define $\varphi_U \in U^0$ and $\varphi_W \in W^0$ by Theorem 3.5 that

$$\varphi_U(v_i) = \begin{cases} 0 & i \leq n \\ \varphi(v_i) & i \geq n+1 \end{cases}, \quad \varphi_U(u_j) = 0, \forall j = 1, \cdots, k, \quad \varphi_U(w_s) = \varphi(w_s), \forall s = 1, \cdots, r,$$

and

$$\varphi_W(v_i) = 0, \forall i = 1, \cdots, m, \quad \varphi_W(u_j) = \varphi(u_j), \forall j = 1, \cdots, k, \quad \varphi_W(w_s) = 0, \forall s = 1, \cdots, r,$$

then $\varphi = \varphi_U + \varphi_W$ since they agree on each of the basis elements of V. Therefore $(U \cap W)^0 = U^0 + W^0$ as desired. $\square$

2. Let V be a vector space and U a subspace of V such that V/U is finite-dimensional. Let $v_1 + U, \cdots, v_n + U$ be a basis of V/U.
(a) (2pt) Prove that $v_1, \cdots, v_n$ is a linearly independent list in V.
(b) (3pt) Let $W = \operatorname{span}(v_1, \cdots, v_n)$, prove that $V = U \oplus W$.
(c) (Bonus 2pt) Show that every linear map on U can be extended to a linear map on V, i.e. for any vector space X and any linear map $T \in \mathcal{L}(U, X)$, there exists a linear map $\overline{T} \in \mathcal{L}(V, X)$ such that $\overline{T}u = Tu$ for all $u \in U$. (Hint: define $S \in \mathcal{L}(U \times W, X)$ such that $S(u, 0) = T(u)$ for all $u \in U$, and then use the canonical isomorphism $U \times W \cong U \oplus W$.)

Proof. (a) Let $a_1, \cdots, a_n \in \mathbb{F}$ be such that

$$a_1 v_1 + \cdots + a_n v_n = 0,$$

then $a_1 v_1 + \cdots + a_n v_n = 0 \in U$ implies by Theorem 3.85 that

$$a_1(v_1 + U) + \cdots + a_n(v_n + U) = (a_1 v_1 + \cdots + a_n v_n) + U = 0 + U.$$

Since $v_1 + U, \cdots, v_n + U$ is a basis of V/U, it is linearly independent in V/U, hence $a_1 = \cdots = a_n = 0$, concluding that $v_1, \cdots, v_n$ is linearly independent in V. $\square$
(b) By definition, every element in W is of the form $a_1 v_1 + \cdots + a_n v_n$ for some $a_1, \cdots, a_n \in \mathbb{F}$. If $a_1 v_1 + \cdots + a_n v_n \in U$, then

$$a_1(v_1 + U) + \cdots + a_n(v_n + U) = (a_1 v_1 + \cdots + a_n v_n) + U = 0 + U,$$

hence $a_1 = \cdots = a_n = 0$ since $v_1 + U, \cdots, v_n + U$ is linearly independent in V/U, thus $a_1 v_1 + \cdots + a_n v_n = 0$, concluding that $W \cap U = \{0\}$.
To see that $V = U + W$, let $v \in V$ be an arbitrary vector, then there exists $a_1, \cdots, a_n \in \mathbb{F}$ such that

$$v + U = a_1(v_1 + U) + \cdots + a_n(v_n + U) = (a_1 v_1 + \cdots + a_n v_n) + U,$$

since $v_1 + U, \cdots, v_n + U$ is a basis of V/U. Hence $v - (a_1 v_1 + \cdots + a_n v_n) \in U$ by Theorem 3.85. Therefore

$$v = (a_1 v_1 + \cdots + a_n v_n) + v - (a_1 v_1 + \cdots + a_n v_n) \in W + U,$$

concluding that $V = W + U$. $\square$
(c) Define $S \in \mathcal{L}(U \times W, X)$ by

$$\begin{array}{rccc} S: & U \times W & \longrightarrow & X \\ & (u, w) & \longmapsto & T(u) \end{array}$$

then S is indeed linear since

$$S(\lambda(u, w) + (u', w')) = S(\lambda u + u', \lambda w + w') = T(\lambda u + u') = \lambda T u + T u' = \lambda S(u, w) + S(u', w'),$$

for any $(u, w), (u', w') \in U \times W$ and $\lambda \in \mathbb{F}$.
Define $\Gamma : U \times W \to U \oplus W = V$ by $\Gamma(u, w) = u + w$ for any $(u, w) \in U \times W$, then Γ is an injective linear map by Theorem 3.77. Since Γ is clearly surjective, Γ is an isomorphism between $U \times W$ and $U \oplus W$. Define $\overline{T} := S\Gamma^{-1} \in \mathcal{L}(V, X)$, then for any $u \in U$,

$$\overline{T}(u) = S\Gamma^{-1}(u) = S(u, 0) = T(u),$$

as desired. $\square$

Quiz 6: 2022/10/29
Name: SID: Advanced Linear Algebra

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1. (5pt) Prove or give a counterexample: if V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Proof. We show the contrapositive, that if $U \neq \{0\}$ and $U \neq V$, then there exists an operator on V such that U is not invariant under that operator.
If $U \neq \{0\}$ and $U \neq V$, let $u_1, \cdots, u_n$ be a basis of U and extend it to a basis $u_1, \cdots, u_n, w_1, \cdots, w_m$ of V then $n, m \geq 1$ and in particular $w_1 \notin U$. Define $T \in \mathcal{L}(V)$ by

$$T(u_i) = w_1, \forall i = 1, \cdots, n, \quad T(w_j) = w_j, \forall j = 1, \cdots, m,$$

then U is not invariant under T since $Tu_1 = w_1 \notin U$. $\square$

2. Let V be finite-dimensional vector space and $T \in \mathcal{L}(V)$. Define $\varphi \in \mathcal{L}(\mathcal{L}(V))$ by

$$\begin{array}{rccc} \varphi\colon & \mathcal{L}(V) & \longrightarrow & \mathcal{L}(V) \\ & S & \longmapsto & ST \end{array}$$

then we know that φ is linear. Prove that
(a) (5pt) $\lambda \in \mathbb{F}$ is an eigenvalue of T if λ is an eigenvalue of φ.
(b) (Bonus 2pt) $\lambda \in \mathbb{F}$ is an eigenvalue of T only if λ is an eigenvalue of φ.

Proof. (a) If $\lambda \in \mathbb{F}$ is an eigenvalue of φ, then

$$ST = \varphi(S) = \lambda S$$

for some nonzero $S \in \mathcal{L}(V)$. Hence

$$S(T - \lambda I) = 0.$$

If $T - \lambda I$ is injective, then $T - \lambda I$ is invertible by Theorem 3.69, hence $S = 0$, contradicting to our choice of S. Therefore $T - \lambda I$ cannot be injective, hence λ is an eigenvalue of T by Theorem 5.6. $\square$
(b) If $\lambda \in \mathbb{F}$ is an eigenvalue of T, then $T - \lambda I$ is not injective by Theorem 5.6, hence $T - \lambda I$ is not surjective by Theorem 3.69. Let $v_1, \cdots, v_m$ be a basis of range$(T - \lambda I)$ and extend it to a basis $v_1, \cdots, v_m, v_{m+1}, \cdots, v_n$ of V, then $n > m$. Define $S \in \mathcal{L}(V)$ by

$$S(v_i) = \begin{cases} 0 & i \leq m \\ v_i & i > m \end{cases}$$

then $S \neq 0$ since $n > m$. Since range$(T - \lambda I) \subset \operatorname{null} S$, We have

$$0 = S(T - \lambda I) = ST - \lambda S,$$

hence

$$\varphi(S) = ST = \lambda S,$$

concluding that λ is an eigenvalue of φ. $\square$

1. (5pt) Suppose V is a finite dimensional complex vector space, $T \in \mathcal{L}(V)$ and the matrix of T with respect to some basis of V contains only real entries. Show that if λ is an eigenvalue of T, then so is $\bar{\lambda}$.

Proof. Write $\dim V = n$. Let $M(T)$ be the matrix of T with respect to the basis of V such that $M(T)$ contains only real entries. Then $\overline{M(T)} = M(T)$. If λ is an eigenvalue of T, then $T - \lambda I$ is not injective, hence is not surjective by Theorem 3.69. By Theorem 3.117, $M(T - \lambda I)$ is not of full rank, hence the columns $C_1, \cdots, C_n \in \mathbb{C}^n$ of $M(T - \lambda I)$ are linearly dependent over $\mathbb{C}$. Thus there exists $a_1, \cdots, a_n \in \mathbb{C}$ that are not all zeros such that

$$a_1 C_1 + \cdots + a_n C_n = 0.$$

Take conjugate of the both sides and we obtain

$$\overline{a_1}\,\overline{C_1} + \cdots + \overline{a_n}\,\overline{C_n} = 0.$$

Since $\overline{a_1}, \cdots, \overline{a_n} \in \mathbb{C}$ are not all zeros and $\overline{C_1}, \cdots, \overline{C_n}$ are the columns of $\overline{M(T - \lambda I)} = \overline{M(T)} - \overline{\lambda M(I)} = M(T) - \bar{\lambda} M(I) = M(T - \bar{\lambda} I)$, we see that $M(T - \bar{\lambda} I)$ is not of full rank, hence $T - \bar{\lambda} I$ is not surjective by Theorem 3.117, therefore $T - \bar{\lambda} I$ is not injective by Theorem 3.69, concluding that $\bar{\lambda}$ is an eigenvalue of T by Theorem 5.6. □

2. Suppose that V is a vector space, $T, S \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$.
(a) (5pt) If $TS = ST$, prove that $\operatorname{null}(S - \lambda I)$ and $\operatorname{range}(S - \lambda I)$ are both invariant under T.
(b) (Bonus 2pt) If V is of finite dimensional and $\dim \operatorname{range}(ST - TS) = 1$, prove that either $\operatorname{null}(S - \lambda I)$ or $\operatorname{range}(S - \lambda I)$ must be invariant under T.
(Hint: If $\operatorname{null}(S - \lambda I)$ is not invariant under T, then there exists $v \in \operatorname{null}(S - \lambda I)$ such that $\operatorname{range}(ST - TS) = \operatorname{span}((S - \lambda I)Tv) \subset \operatorname{range}(S - \lambda I)$, and then...)

Proof. (a) For any $v \in \operatorname{null}(S - \lambda I)$, we have

$$(S - \lambda I)(Tv) = (ST - \lambda T)v = (TS - T(\lambda I))v = T(S - \lambda I)v = T0 = 0,$$

hence $Tv \in \operatorname{null}(S - \lambda I)$, concluding that $\operatorname{null}(S - \lambda I)$ is invariant under T.
For any $v \in \operatorname{range}(S - \lambda I)$, there exists $u \in V$ such that $v = (S - \lambda I)u$, hence we have

$$Tv = T(S - \lambda I)u = (TS - \lambda T)u = (ST - \lambda T)u = (S - \lambda I)(Tu) \in \operatorname{range}(S - \lambda I),$$

concluding that $\operatorname{range}(S - \lambda I)$ is invariant under T. □
(b) If $\operatorname{null}(S - \lambda I)$ is not invariant under T, then there exists $v \in \operatorname{null}(S - \lambda I)$ such that $(S - \lambda I)(Tv) \neq 0$ by Definition 5.2. Since $v \in \operatorname{null}(S - \lambda I)$, we have $Sv = \lambda v$, hence

$$\operatorname{range}(ST - TS) \ni (ST - TS)v = STv - T(\lambda v) = (S - \lambda I)(Tv) \neq 0.$$

Since $\dim \operatorname{range}(ST - TS) = 1$, it follows that $\operatorname{range}(ST - TS) = \operatorname{span}((S - \lambda I)Tv)$. Now for any $w \in \operatorname{range}(S - \lambda I)$, there exists $u \in V$ such that $(S - \lambda I)u = w$ and we have since $(ST - TS)u \in \operatorname{range}(ST - TS) = \operatorname{span}((S - \lambda I)Tv)$ that

$$(ST - TS)u = \mu(S - \lambda I)Tv \quad \Rightarrow \quad TSu = STu - (S - \lambda I)(\mu Tv)$$

for some $\mu \in \mathbb{F}$. Hence

$$Tw = T(S - \lambda I)u = TSu - \lambda Tu = STu - (S - \lambda I)(\mu Tv) - \lambda Tu = (S - \lambda I)(Tu - \mu Tv) \in \operatorname{range}(S - \lambda I),$$

concluding that $\operatorname{range}(S - \lambda I)$ is invariant under T. □

Quiz 7: 2022/11/5
Name: SID: Advanced Linear Algebra

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1. Let V be a vector space and $T \in \mathcal{L}(V)$ such that $T^2 = T$ and $T \neq 0$. You have seen in your homework that $V = \operatorname{null} T \oplus \operatorname{range} T$. Suppose that $\operatorname{null} T \neq \{0\}$.
Question:

(a) (2pt) Find all the eigenvalues of T;

(b) (2pt) For each eigenvalue λ of T, find $E(\lambda, T)$;

(c) (1pt) Conclude that T is diagonalizable if V is finite dimensional.

Proof. (a) Since $\operatorname{null} T \neq \{0\}$, 0 is an eigenvalue of T; since $T \neq 0$, there exists $v \in V$ such that $Tv \neq 0$, and $T(Tv) = T^2 v = Tv$ tells that 1 is an eigenvalue of T. Conversely, if λ is an eigenvalue of T, then there exists $v \in V$ such that $Tv = \lambda v$. Since $\lambda v = Tv = T^2 v = \lambda^2 v$, we have $\lambda^2 - \lambda = 0$, hence $\lambda = 0$ or 1. ■
(b) $E(0, T) = \operatorname{null} T$ is clear. We have $E(1, T) = \operatorname{range} T$: $E(1, T) \subset \operatorname{range} T$ is easy; for any $v \in \operatorname{range} T$, there exists $u \in V$ such that $Tu = v$, hence

$$Tv = T(Tu) = T^2 u = Tu = v,$$

thus $v \in E(1, T)$. ■
(c) By (b), since $V = \operatorname{null} T \oplus \operatorname{range} T$, we have $V = E(0, T) \oplus E(1, T)$, hence by Theorem 5.41 we know that T is diagonalizable. □

2. (5pt) Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V and that $\lambda \in \mathbb{F}$. Prove that λ appears on the diagonal of A precisely $\dim E(\lambda, T)$ times.

Proof. Clearly, if λ is not an eigenvalue of T, then λ cannot appear in the diagonal of A and $\dim E(\lambda, T) = 0$.
Let $\lambda_1, \cdots, \lambda_m \in \mathbb{F}$ be all the distinct eigenvalues of T and $v_1, \cdots, v_n$ be the basis of V with respect to which the matrix of T is A. Since A is diagonal, each v_i is an eigenvector of T, hence if we define the set $S_i := \{v_j \mid Tv_j = \lambda_i v_j\}$ for each $i = 1, \cdots, m$, then

$$\{v_1, \cdots, v_n\} = \bigcup_{i=1}^{m} S_i.$$

Since $v_1, \cdots, v_n$ is a basis of V, $|S_i| = \dim \operatorname{span} S_i$ (where $|S_i|$ is the number of elements in S_i) and

$$V = \operatorname{span}(v_1, \cdots, v_n) = \operatorname{span} S_1 + \cdots + \operatorname{span} S_m,$$

in particular $\dim \operatorname{span} S_1 + \cdots + \dim \operatorname{span} S_m \geq \dim V$. Since T is diagonalizable, $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$, hence

$$\dim \operatorname{span} S_1 + \cdots + \dim \operatorname{span} S_m \geq \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T).$$

On the other hand, we have $\dim \operatorname{span} S_i \leq \dim E(\lambda_i, T)$ since $S_i \subset E(\lambda_i, T)$ gives $\operatorname{span} S_i \subset E(\lambda_i, T)$ for each $i = 1, \cdots, m$, therefore there must be $\dim \operatorname{span} S_i = \dim E(\lambda_i, T)$ for each $i = 1, \cdots, m$, hence $\dim E(\lambda_i, T) = \dim \operatorname{span} S_i = |S_i|$. By the construction of S_i, $|S_i|$ is the number of times of that λ_i appears on the diagonal of A, concluding the proof. □

Alternative Proof (Sketch). Since $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, take a basis for each $E(\lambda_i, T)$ and put them together, we obtain a basis $v_1, \cdots, v_n$ of V, with respect to which $B := M(T, (v_1, \cdots, v_n))$ is diagonal and $\lambda \in \mathbb{F}$ appears on the diagonal of B precisely $\dim E(\lambda, T)$ times.
Since with respect to any basis, we always have $\dim \operatorname{null}(M(T) - \lambda I) = \dim \operatorname{null} M(T - \lambda I) = \dim \operatorname{null}(T - \lambda I)$, we have $\dim \operatorname{null}(B - \lambda I) = \dim \operatorname{null}(T - \lambda I) = \dim \operatorname{null}(A - \lambda I)$ for any $\lambda \in \mathbb{F}$, and it follows that λ appears on the diagonal of A precisely the times it appears on the diagonal of B, which is $\dim E(\lambda, T)$. □

3. (Bonus 2pt) Suppose V is finite-dimensional with $\dim V > 1$ and $T \in \mathcal{L}(V)$. Prove that

$$\{p(T) \mid p \in \mathcal{P}(\mathbb{F})\} \neq \mathcal{L}(V)$$

Proof. By Theorem 5.20, the multiplication of elements in $\{p(T) \mid p \in \mathcal{P}(\mathbb{F})\}$ is commutative. However, for $\dim V > 1$, we have proved in Exercise 3.A.18 that there exists $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$. Therefore the two sets cannot be equal. □

1. (5pt) Prove or give a counterexample: Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$. If $V = \operatorname{null} T \oplus \operatorname{range} T$, then T is diagonalizable.

Proof. Counterexample I: Let $V = \mathbb{R}^2$ and

$$\begin{aligned} T\colon \quad \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \boldsymbol{v} &\longmapsto \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \boldsymbol{v} \end{aligned}$$

then T is invertible, hence $V = \mathbb{R}^2 = \operatorname{range} T = \{0\} \oplus \operatorname{range} T = \operatorname{null} T \oplus \operatorname{range} T$. It is easy to see that T does not have any eigenvalue, hence is not diagonalizable. ■
Counterexample II: Let $V = \mathbb{C}^2$ and

$$\begin{aligned} T\colon \quad \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ \boldsymbol{v} &\longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \boldsymbol{v} \end{aligned}$$

again T is invertible. The only eigenvalue of T is 1 (Theorem 5.32) and $E(1, T) = \operatorname{null} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\{(1, 0)^T\}$, hence T is not diagonalizable by Theorem 5.41. □

2. (4pt) Prove or give a counterexample: Let $T \in \mathcal{L}(V)$ and there exists a positive integer n such that $T^n = 0$, then $I - T$ is invertible.

Proof. It is easy to verify that $I + T + \cdots + T^{n-1}$ is the inverse of $I - T$. Therefore $I - T$ is invertible. □

Remark: The idea is that we have the expansion that

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots .$$

3. (1pt+Bonus 2pt) Let V be a finite dimensional nonzero real vector space and $T \in \mathcal{L}(V)$. Prove that there exists an invariant subspace of V under T of dimension 1 or 2.
(Hint: Recall Theorem 4.17 that every non-constant polynomial $p \in \mathcal{P}(\mathbb{R})$ has a unique factorization of the form

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M)$$

where $c, \lambda_1, \cdots, \lambda_m, b_1, \cdots, b_M, c_1, \cdots, c_M \in \mathbb{R}$. Try imitating the proof of Theorem 5.21 that every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.)

Proof. Write $\dim V = n$. Let $v \in V$ with $v \neq 0$, then the list of $n + 1$ vectors,

$$v, Tv, T^2 v, \cdots, T^n v,$$

is not linearly independent, hence there exists $a_0, \cdots, a_n \in \mathbb{R}$ that are not all zeros, such that

$$a_0 v + a_1 T v + \cdots + a_n T^n v = 0.$$

Note that $a_1, \cdots, a_n$ cannot be all zeros, otherwise $a_0 v = 0$ would give also $a_0 = 0$ since $v \neq 0$. Define $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{P}(\mathbb{R})$, then p is non-constant, hence by Theorem 4.17,

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M),$$

where $0 \neq c, \lambda_1, \cdots, \lambda_m, b_1, \cdots, b_M, c_1, \cdots, c_M \in \mathbb{R}$, and either m or M must be no less than 1. Since

$$p(T)v = a_0 + a_1 T v + \cdots + a_n T^n v = 0,$$

we see that $p(T) = c(T - \lambda_1 I) \cdots (T - \lambda_m I)(T^2 + b_1 T + c_1 I) \cdots (T^2 + b_M T + c_M I)$ is not injective. Since $c \neq 0$, there must be either $T - \lambda_i I$ or $T^2 + b_j T + c_j I$ is not injective for some $1 \leq i \leq m$ or $1 \leq j \leq M$ respectively.
If $T - \lambda_i I$ is not injective, then λ_i is an eigenvalue of T, hence T has an invariant subspace of dimension 1.
If $T^2 + b_j T + c_j I$ is not injective, then there exists a non-zero vector $u \in V$ such that $T^2 u + b_j T u + c_j u = 0$. Let $U = \operatorname{span}(u, Tu)$, then for any $w \in U$, $w = au + bTu$ for some $a, b \in \mathbb{R}$, and

$$Tw = T(au + bTu) = aTu + bT^2 u = aTu + b(-b_j T u - c_j u) = (a - bb_j)T_u - c_j u \in U,$$

hence U is an invariant subspace under T and $1 \leq \dim U \leq 2$, concluding the proof. (In fact, you can show that $\dim U = 2$ if you assume that $b_j^2 < c_j$ honestly as in Theorem 4.17) □

Quiz 8: 2022/11/19
Name: SID: Advanced Linear Algebra

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1. (5pt) Prove or disprove: There exists a polynomial $q \in \mathcal{P}_1(\mathbb{R})$ such that

$$\int_0^1 p(x)\cos(\pi x)\,\mathrm{d}x = \int_0^1 p(x)q(x)\,\mathrm{d}x$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

Proof. Take $q(x) = -\frac{12}{\pi^2}(2x-1)$ as in Exercise 6.B.8. Since $q \in \mathcal{P}_1(\mathbb{R})$, such q does exist. $\square$

Remark. For any $n, m \in \mathbb{N}$, does there exist $q \in \mathcal{P}_n(\mathbb{R})$ such that

$$\int_0^1 p(x)\cos(\pi x)\,\mathrm{d}x = \int_0^1 p(x)q(x)\,\mathrm{d}x$$

for every $p \in \mathcal{P}_m(\mathbb{R})$?

2. (5pt) Let n be a positive integer. Recall that the *trace* of an $n \times n$ real matrix A is defined by

$$\operatorname{tr}(A) \coloneqq \sum_{i=1}^n A_{i,i}.$$

Define $\langle \cdot,\cdot \rangle \colon \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ by

$$\langle A, B \rangle \coloneqq \operatorname{tr}(AB^T),$$

show that $\langle \cdot,\cdot \rangle$ is an inner product on $\mathbb{R}^{n\times n}$.

Proof. Since $(AB^T)_{i,j} = \sum_{k=1}^n A_{i,k}B_{j,k}$,

$$\langle A, B \rangle = \operatorname{tr}(AB^T) = \sum_{l=1}^n \left(\sum_{k=1}^n A_{l,k}B_{l,k}\right) = \left(\sum_{k=1}^n B_{l,k}A_{l,k}\right) = \langle B, A \rangle,$$

so we have the (conjugate) symmetry. The positive definiteness follows from:

$$\langle A, A \rangle = \sum_{l=1}^n \left(\sum_{k=1}^n A_{l,k}^2\right) \geq 0,$$

and $\langle A, A \rangle = 0$ if and only if $A_{l,k} = 0$ for all $l, k \in \{1, \cdots, n\}$, if and only if $A = 0$. For any $n \times n$ real matrix C and $\lambda \in \mathbb{R}$, we have

$$\langle \lambda A + C, B \rangle = \sum_{l=1}^n \left(\sum_{k=1}^n (\lambda A_{l,k} + C_{l,k})B_{l,k}\right) = \lambda\left(\sum_{k=1}^n A_{l,k}B_{l,k}\right) + \left(\sum_{k=1}^n C_{l,k}B_{l,k}\right) = \lambda\langle A, B \rangle + \langle C, B \rangle,$$

so the additivity and homogeneity in first slot are both clear. $\square$

3. (Bonus 2pt) Suppose $e_1, \cdots, e_n$ is an orthonormal basis of V and $v_1, \cdots, v_n$ are vectors in V such that

$$\|e_j - v_j\| < \frac{1}{\sqrt{n}}$$

for each j. Prove that $v_1, \cdots, v_n$ is a basis of V.
(Hint: Define $T \in \mathcal{L}(V)$ by $Te_i = e_i - v_i$ for each $i = 1, \cdots, n$. Show that if $I - T$ is invertible, then $v_1, \cdots, v_n$ is a basis of V. Then show that 1 is not an eigenvalue of T using the inequality that

$$\frac{\sum_{i=1}^n x_i}{n} \leq \sqrt{\frac{\sum_{i=1}^n x_i^2}{n}}$$

whenever $x_1, \cdots, x_n$ are positive real numbers.)

Proof. Since $(I-T)e_i = v_i$ for each $i = 1, \cdots, n$, if $I - T$ is invertible, then it is injective (since here V is f.d.), hence for any $a_1, \cdots, a_n \in \mathbb{F}$, $a_1v_1 + \cdots + a_nv_n = 0$ only if $a_1e_1 + \cdots + a_ne_n = 0$, only if $a_1 = \cdots = a_n = 0$ since $e_1, \cdots, e_n$ is a basis. Thus $v_1, \cdots, v_n$ is linearly independent of length $\dim V = n$, so it is a basis.
Thus it suffices to show that $I - T$ is injective. If there exists a nonzero $v \in V$ such that $Tv = v$, then by Theorem 6.30,

$$\begin{aligned}
\|v\|^2 = \|Tv\|^2 = \left\| \sum_{i=1}^{n} \langle v, e_i \rangle (e_i - v_i) \right\|^2 &\leq \left(\sum_{i=1}^{n} |\langle v, e_i \rangle| \|e_i - v_i\| \right)^2 \\
&< \frac{1}{n} \left(\sum_{i=1}^{n} |\langle v, e_i \rangle| \right)^2 = n \left(\frac{\sum_{i=1}^{n} |\langle v, e_i \rangle|}{n} \right)^2 \\
&\leq n \frac{\sum_{i=1}^{n} |\langle v, e_i \rangle|^2}{n} = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2 = \|v\|^2,
\end{aligned}$$

so we obtain $\|v\|^2 < \|v\|^2$, which is impossible. Therefore $I - T$ is injective, hence invertible (as V is f.d.), concluding the proof. $\square$

1. (5pt) Prove or give a counterexample: For every finite dimensional vector space, there exists an inner product on it.

Proof. Let V be finite dimensional and $v_1, \cdots, v_n$ be a basis of V, then the assignment $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ defined by, for any two vectors $v = a_1 v_1 + \cdots + a_n v_n, u = b_1 v_1 + \cdots + b_n v_n$ in V,

$$\langle v, u \rangle := \sum_{i=1}^{n} a_i \overline{b_i},$$

is an inner product on V (the verification is omitted here, but you need to verify it when answering). Therefore the statement is true. $\square$

2. (5pt) Let n be a positive integer. Recall that the *trace* of an $n \times n$ real matrix $A = (a_{ij})$ is defined by

$$\operatorname{tr}(A) := \sum_{i=1}^{n} a_{ii}.$$

Define $\langle \cdot, \cdot \rangle : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ by

$$\langle A, B \rangle := \operatorname{tr}(AB^T),$$

then it is easy to see that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^{n\times n}$ (you don't need to show it here). Show that the $n \times n$ matrices E_{ij}, whose entries are all zeros except an 1 in row i, column j, give an orthonormal basis of $(\mathbb{R}^{n\times n}, \langle \cdot, \cdot \rangle)$.

Proof. It is clear that E_{ij}'s form a basis. To show that it is orthonormal, note that there is

$$E_{ij} E_{kl}^T = \begin{cases} E_{ik} & l = j \\ 0 & l \neq j \end{cases}$$

for any $i, j, k, l \in \{1, \cdots, n\}$ (This follows from the fact that $(AB^T)_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{j,k}$). Hence

$$\langle E_{ij}, E_{kl} \rangle = \begin{cases} 1 & i = k,\ l = j \\ 0 & \text{otherwise} \end{cases}$$

concluding that E_{ij}'s are orthonormal. $\square$

3. (Bonus 2pt) Suppose V is finite-dimensional and $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ are inner products on V with corresponding norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that there exists a positive number c such that

$$\|v\|_1 \leq c\|v\|_2$$

for every $v \in V$.
(Hint: Write $\dim V = n$. Let $e_1, \cdots, e_n$ be an orthonormal basis of $(V, \langle \cdot, \cdot \rangle_2)$, try showing that

$$\|v\|_1 \leq n \left(\max_{1\leq i\leq n} \|e_i\|_1 \right) \left(\max_{1\leq j\leq n} |\langle v, e_j \rangle_2| \right)$$

and $\max_{1\leq j\leq n} |\langle v, e_j \rangle_2| \leq \|v\|_2$.)

Proof. Write $\dim V = n$. The statement is trivial if $n = 0$, hence we may assume that $n \geq 1$. Let $e_1, \cdots, e_n$ be an orthonormal basis of $(V, \langle \cdot, \cdot \rangle_2)$, then by Theorem 6.30,

$$v = \langle v, e_1 \rangle_2 e_1 + \cdots + \langle v, e_n \rangle_2 e_n.$$

Hence

$$\|v\|_1 = \|\langle v, e_1 \rangle_2 e_1 + \cdots + \langle v, e_n \rangle_2 e_n\|_1 \leq \sum_{i=1}^{n} |\langle v, e_i \rangle_2| \|e_i\|_1 \leq n \left(\max_{1\leq i\leq n} \|e_i\|_1 \right) \left(\max_{1\leq j\leq n} |\langle v, e_j \rangle_2| \right).$$

Since by Theorem 6.30,

$$\|v\|_2 = \sqrt{|\langle v, e_1 \rangle_2|^2 + \cdots + |\langle v, e_n \rangle_2|^2} \geq \max_{1\leq j\leq n} |\langle v, e_j \rangle_2|,$$

let $c = n \left(\max_{1\leq i\leq n} \|e_i\|_1 \right) > 0$ and we are done. $\square$

Quiz 9: 2022/11/26

Name: SID: Advanced Linear Algebra

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1. (10pt) Suppose U and W are finite-dimensional subspaces of V. Prove that $P_U P_W = 0$ if and only if $\langle u, w \rangle = 0$ for all $u \in U$ and all $w \in W$.

Proof. If $\langle u, w \rangle = 0$ for all $u \in U$ and all $w \in W$, then $W \subset U^\perp$. Since $\operatorname{range} P_W = W \subset U^\perp = \operatorname{null} P_U$, we see that $P_U P_W = 0$.

Conversely, if $\langle u_0, w_0 \rangle \neq 0$ for some $u_0 \in U$ and $w_0 \in W$, then $w_0 \notin U^\perp = \operatorname{null} P_U$, hence $P_U w_0 \neq 0$. Since $w_0 \in W = \operatorname{range} P_W$, we thus see that $P_U P_W \neq 0$. □

Quiz 9: 2022/11/27

Name: SID: Advanced Linear Algebra

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1. (10pt) Suppose that V is finite-dimensional. $T \in \mathcal{L}(V)$, and U is a subspace of V. Prove that U and $U^\perp$ are both invariant under T if and only if $P_U T = T P_U$.

Proof. If $P_U T = T P_U$, then for any $u \in U$,

$$Tu = TP_U(u) = P_U(Tu),$$

hence $Tu \in U$, because if not, then $Tu = u' + w$ with nonzero $u' \in U$ and $w \in U^\perp$, hence $P_U(Tu) \neq Tu$.

For any $w \in U^\perp$, we have $P_U w = 0$, and

$$P_U T w = T P_U w = T0 = 0,$$

thus $Tw \in \operatorname{null} P_U = U^\perp$. Therefore both U and $U^\perp$ are invariant under T.

Conversely, if both U and $U^\perp$ are invariant under T, then for any $v \in V$, write $v = u + w$ with $u \in U$ and $w \in U^\perp$, then $Tw \in U^\perp = \operatorname{null} P_U$, hence

$$P_U T v = P_U(Tu + Tw) = P_U T u + 0 = Tu = T P_U u = T P_U(u + w) = T P_U v,$$

concluding that $P_U T = T P_U$.

□

Quiz 10: 2022/12/3
Name: SID: Advanced Linear Algebra

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1. (8pt) Prove or disprove: There exists $T \in \mathcal{L}(\mathbb{R}^2)$ such that T is not self-adjoint (with respect to the usual inner product) and such that there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

Proof. Consider the basis $(1,0),(1,1)$ of $\mathbb{R}^2$. Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(1,0) = (1,0)$ and $T(1,1) = (2,2)$, then $(1,0),(1,1)$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors of T. T is not self-adjoint, because $\operatorname{span}\{(1,0)\}$ is invariant under T while

$$T(0,1) = T(1,1) - T(1,0) = (2,2) - (1,0) = (1,2) \notin \operatorname{span}\{(0,1)\} = (\operatorname{span}\{(1,0)\})^{\perp},$$

contradicting to Theorem 7.28(a) if T is self-adjoint. $\square$

2. Suppose $T \in \mathcal{L}(V, W)$. Prove that

(a) (2pt) T is injective if and only if T^* is surjective;

(b) (2pt) T is surjective if and only if T^* is injective.

Proof. (a) By Theorem 7.7(b), we have $\operatorname{range} T^* = (\operatorname{null} T)^{\perp}$, hence $\operatorname{range} T^* = V$ if and only if $\operatorname{null} T = \{0\}$. $\square$
(b) By Theorem 7.7(d), we have $\operatorname{range} T = (\operatorname{null} T^*)^{\perp}$, hence $\operatorname{range} T = V$ if and only if $\operatorname{null} T^* = \{0\}$. $\square$

Quiz 10: 2022/12/4
Name: SID: Advanced Linear Algebra

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1. (8pt) Prove or disprove: Suppose V is a complex inner product space with $V \neq \{0\}$, then the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.

Disproof. If $T \in \mathcal{L}(V)$ is self-adjoint, then by Theorem 7.10 its matrix with respect to every orthonormal basis of V must always be real. However, let I be the identity operator on V, then I is self-adjoint by definition, but $\mathcal{M}(\mathrm{i}\, I)$ is not real with respect to any orthonormal basis, with i's on its diagonal. These conclude that the set of self-adjoint operators on V is not closed under scalar multiplication when $\mathbb{F} = \mathbb{C}$, hence it is not a subspace. $\square$

2. (4pt) Prove that every self-adjoint operator on V has a self-adjoint cube root, i.e. for any self-adjoint $T \in \mathcal{L}(V)$, there exists a self-adjoint $S \in \mathcal{L}(V)$ such that $S^3 = T$.

Proof. By the Spectral Theorem and Theorem 7.13 there exists an orthonormal basis $e_1, \cdots, e_n$ of V such that $Te_i = \lambda_i e_i$ with $\lambda_i \in \mathbb{R}$ for each $i = 1, \cdots, n$. There exists a (unique) real cube root s_i for each λ_i, and if we define $S \in \mathcal{L}(V)$ by $Se_i = s_i e_i$ for each $i = 1, \cdots, n$, then for any $v = \sum_{i=1}^{n} a_i e_i \in V$, we have

$$S^3 v = S^3 \left(\sum_{i=1}^{n} a_i e_i \right) = \sum_{i=1}^{n} a_i S^3 e_i = \sum_{i=1}^{n} a_i s_i^3 e_i = \sum_{i=1}^{n} a_i \lambda_i e_i = \sum_{i=1}^{n} a_i T e_i = T \left(\sum_{i=1}^{n} a_i e_i \right) = Tv.$$

If $\mathbb{F} = \mathbb{R}$, then S is self-adjoint by the Real Spectral Theorem. If $\mathbb{F} = \mathbb{C}$, then S is self-adjoint since under the basis $e_1, \cdots, e_n$, we have by Theorem 7.10 $\mathcal{M}(S^*) = \mathcal{M}(S)^* = \mathcal{M}(S)$, which implies $S^* = S$ because $\mathcal{M} \colon \mathcal{L}(V) \to \mathbb{F}^{n \times n}$ is injective. (Also, one can compute and find that $\langle Sv, v \rangle \in \mathbb{R}$ for all $v \in V$ and then use Theorem 7.15) $\square$

Quiz 11: 2022/12/10
Name: SID: Advanced Linear Algebra

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1. (4pt) Prove or disprove: Let V be finite-dimensional inner product space and $T \in \mathcal{L}(V)$, then $\operatorname{null} T^*T = \operatorname{null} T$.

Proof. It is clear that $\operatorname{null} T \subset \operatorname{null} T^*T$.
Conversely, for any $v \in \operatorname{null} T^*T$, we have

$$\|Tv\|^2 = \langle Tv, Tv \rangle = \langle v, T^*Tv \rangle = \langle v, 0 \rangle = 0,$$

hence $Tv = 0$, concluding that $v \in \operatorname{null} T$. □

2. (4pt) Find an isometry matrix S for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

such that $A = S\sqrt{A^*A}$.

Proof. We have

$$A^*A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence

$$\sqrt{A^*A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence

$$S := A(\sqrt{A^*A})^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is the desired isometry. □

3. (4pt) Find an isometry matrix S for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

such that $A = S\sqrt{A^*A}$.

Proof. We have

$$\sqrt{A^*A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Inspired by Question 2, define $S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and we find that

$$S\sqrt{A^*A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A,$$

as desired. □

1. (4pt) Admitting the fact that $\operatorname{null} T^*T = \operatorname{null} T$ for any $T \in \mathcal{L}(V)$, prove or disprove: Let V be finite-dimensional inner product space and $T \in \mathcal{L}(V)$, then $\operatorname{range} TT^* = \operatorname{range} T$.

Proof. It is clear that $\operatorname{range} TT^* \subset \operatorname{range} T$. By the given fact, we have $\operatorname{null} TT^* = \operatorname{null} T^*$, hence

$$\dim \operatorname{range} TT^* = \dim V - \dim \operatorname{null} TT^* = \dim V - \dim \operatorname{null} T^* \xlongequal{\text{Theorem 7.7(a)}} \dim \operatorname{range} T,$$

concluding that $\operatorname{range} TT^* = \operatorname{range} T$. $\square$

2. (4pt) Find an isometry matrix S for the matrix

$$A = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

such that $A = S\sqrt{A^*A}$.

Proof. We have

$$A^*A = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix},$$

hence

$$\sqrt{A^*A} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix},$$

hence

$$S := A(\sqrt{A^*A})^{-1} = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

is the desired isometry. $\square$

3. Find an isometry matrix S for the matrix

$$A = \begin{pmatrix} 3 & 0 \\ -4 & 0 \end{pmatrix}$$

such that $A = S\sqrt{A^*A}$.

Proof. We have

$$\sqrt{A^*A} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}.$$

Inspired by Question 2, define $S = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$ and we find that

$$S\sqrt{A^*A} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ -4 & 0 \end{pmatrix} = A,$$

as desired. $\square$

Remark. Does the above S still work for $A = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}$? Can you come up with an explanation for this phenomenon?

Quiz 12: 2022/12/17
Name: SID: Advanced Linear Algebra

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1. (4pt) Prove or give a counterexample: The set of nilpotent operators on V is a subspace of $\mathcal{L}(V)$.

Counterexample. Let $V = \mathbb{R}^2$ and consider the two operators

$$N_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then clearly N_1 and N_2 are nilpotents. However, $N_1 + N_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is invertible, hence is not a nilpotent. $\square$

2. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be defined by

$$T(x, y, z) = (11x - 4y - 5z, 21x - 8y - 11z, 3x - y),$$

for any $(x, y, z) \in \mathbb{R}^3$.

(a) (2pt) Find all the eigenvalues of T;

(b) (4pt) Find all the generalized eigenspaces of T, and hence tell the multiplicity of each eigenvalue of T.

Answers. (a) Eigenvalues: 2 and -1. $\square$
(b) Since there are two eigenvalues, the maximum of the multiplicity is 2. Therefore Theorem 8.3 implies that $G(\lambda, T) = \operatorname{null}(T - \lambda I)^2$. A computation thus gives:

$$G(2, T) = \operatorname{span}\{(1, 1, 1), (0, -1, 1)\}, \quad G(1, T) = \operatorname{span}\{(1, 3, 0)\}.$$

Therefore the eigenvalue $\lambda = 2$ has multiplicity 2 and $\lambda = -1$ has multiplicity 1.
(Of course you can compute $\operatorname{null}(T - \lambda I)^3$ directly, but the argument above would save you some time) $\square$

3. (Bonus 2pt) Let V be a finite-dimensional inner product space and $N \in \mathcal{L}(V)$ be a nilpotent. Prove that there exists an orthonormal basis of V with respect to which N has an upper-triangular matrix.

Proof. Since we have

$$\operatorname{null} N \subset \operatorname{null} N^2 \subset \cdots \subset \operatorname{null} N^{\dim V} = V,$$

choose an orthonormal basis $e_1^{(1)}, \cdots, e_{l_1}^{(1)}$ of $\operatorname{null} N$, and then extend it to an orthonormal basis $e_1^{(1)}, \cdots, e_{l_1}^{(1)}, e_1^{(2)}, \cdots, e_{l_2}^{(2)}$ of $\operatorname{null} N^2$, and then extend it to an orthonormal basis $e_1^{(1)}, \cdots, e_{l_1}^{(1)}, e_1^{(2)}, \cdots, e_{l_2}^{(2)}, e_1^{(3)}, \cdots, e_{l_3}^{(3)}$ of $\operatorname{null} N^3$... after $\dim V$ steps we obtain an orthonormal basis $e_1^{(1)}, \cdots, e_{l_{\dim V}}^{(\dim V)}$ of $\operatorname{null} N^{\dim V} = V$. Since for any $k \in \mathbb{N}^*$, $N(\operatorname{null} N^k) \subset \operatorname{null} N^{k-1}$, it is easy to see that the matrix of N with respect to this basis is upper-triangular. $\square$

1. (6pt) Give an example of an operator T on a finite-dimensional real vector space such that 0 is the only eigenvalue of T but T is not a nilpotent.

(A Family of) Examples. Consider $\theta \in (0, 2\pi) \setminus \{\pi\}$. Note that the counterclockwise rotation by θ, i.e. $R_\theta \in \mathcal{L}(\mathbb{R}^2)$ given by $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ has no eigenvalue, since for any nonzero vector $v \in \mathbb{R}^2$, v and $R_\theta(v)$ never lie on the same line.
Let $V = \mathbb{R}^3$, $P\colon V \to \mathbb{R}^2\colon (x, y, z) \mapsto (x, y, 0)$ be the projection to the xy-plane, $i\colon \mathbb{R}^2 \hookrightarrow V\colon (x, y) \mapsto (x, y, 0)$ be the inclusion of the xy-plane into $\mathbb{R}^3$, and $T \in \mathcal{L}(V)$ be the composition of maps

$$T\colon V \xrightarrow{P} \mathbb{R}^2 \xrightarrow{R_\theta} \mathbb{R}^2 \xrightarrow{i} V,$$

i.e. T brings a vector $v \in \mathbb{R}^3$ to its projection to the xy-plane and then rotate it counterclockwisely by θ. To be more explicit, the matrix of T under the canonical basis of $\mathbb{R}^3 = V$ is

$$\mathcal{M}(T) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since v and Tv lie on the same line only if Pv and PTv lie on the same line, and $PTv = Tv = R_\theta(Pv)$, by the argument in the previous paragraph we see that v is an eigenvector of T only if v lies on the z-axis. Therefore the only eigenvalue of T is 0. However, T is not a nilpotent, since its restriction to the xy-plane is R_θ and the rotation R_θ is not a nilpotent. □

2. (4pt) Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $G(\lambda, T) = G(\frac{1}{\lambda}, T^{-1})$ for every $\lambda \in \mathbb{F} \setminus \{0\}$.

Proof. Write $n \coloneqq \dim V$. For any $v \in V$, since

$$(T - \lambda I)^n v = 0 \quad \Leftrightarrow \quad \left(T^{-1} - \frac{1}{\lambda} I\right)^n v = \left(-\frac{1}{\lambda} T^{-1}\right)^n (T - \lambda I)^n v = 0,$$

because T^{-1} is injective, the conclusion follows. □

3. (Bonus 2pt) Give an example of vector space W and $T \in \mathcal{L}(W)$ such that $\operatorname{null} T^k \subsetneq \operatorname{null} T^{k+1}$ and $\operatorname{range} T^k \supsetneq \operatorname{range} T^{k+1}$ for every positive integer k.

Example. Let $W = \mathcal{P}(\mathbb{F}) \times \mathcal{P}(\mathbb{F})$ and $T \in \mathcal{L}(W)$ be defined by:

$$\begin{array}{rccc} T\colon & \mathcal{P}(\mathbb{F}) \times \mathcal{P}(\mathbb{F}) & \longrightarrow & \mathcal{P}(\mathbb{F}) \times \mathcal{P}(\mathbb{F}) \\ & (p, q) & \longmapsto & (x \cdot p, q') \end{array}$$

then $\operatorname{null} T^k = \{0\} \times \mathcal{P}_{k-1}(\mathbb{F})$ and $(x^k, 0) \in \operatorname{range} T^k$ while $(x^k, 0) \notin \operatorname{range} T^{k+1}$ for every positive integer k, therefore the desired property follows. □

Quiz 13: 2022/12/24
Name: SID: Advanced Linear Algebra

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1. Let $T \in \mathcal{L}(\mathbb{C}^3)$ be defined by

$$T(x, y, z) = (11x - 4y - 5z, 21x - 8y - 11z, 3x - y),$$

for any $(x, y, z) \in \mathbb{C}^3$. Recall that we have computed last week that the eigenvalues of T are 2 and -1 with $\dim G(2, T) = 2$ and $\dim G(-1, T) = 1$.

(a) (4pt) Find the characteristic polynomial and the Jordan form of T.

(b) (4pt) Find the minimal polynomial of T.

(c) (2pt) Give an example of an operator on $\mathbb{C}^3$ whose characteristic polynomial equals $(z+1)(z-2)^2$ and whose minimal polynomial equals $(z+1)(z-2)$.

Answer. (a) It is immediate by definition that the characteristic polynomial of T is $(z+1)(z-2)^2$.
Since $\dim \operatorname{null}(T - 2I) = 1$ while $\dim \operatorname{null}(T - 2I)^2 = 2$, $(T - 2I)|_{G(2,T)}$ is a non-zero nilpotent, hence the Jordan form of T is

$$J(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(Read 8.D thoroughly if you don't see how this form comes out) □

(b) Since $(J(T) - 2I)(J(T) + I) = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$, the minimal polynomial of T cannot be $(z-2)(z+1)$, hence by Theorem 8.48, the minimal polynomial of T must be $(z+1)(z-2)^2$. □

(c) Consider the operator given by the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is easy (hence is left to you) to verify that it is the desired example. □

1. Let $T \in \mathcal{L}(\mathbb{C}^3)$ be defined by

$$T(x, y, z) = (-3x + 3y - 2z, -7x + 6y - 3z, x - y + 2z),$$

for any $(x, y, z) \in \mathbb{C}^3$.

(a) (2pt) Find all the eigenvalues of T.

(b) (2pt) Find all the generalized eigenspaces of T, and hence tell the multiplicity of each eigenvalue of T and the characteristic polynomial of T.

(c) (3pt) Find the Jordan form of T.

(d) (3pt) Find the minimal polynomial of T.

(e) (Bonus 1pt) Give an example of an operator on $\mathbb{C}^3$ whose characteristic polynomial equals $(z-1)(z-2)^2$ and whose minimal polynomial equals $(z-1)(z-2)$.

Answer. (a) The eigenvalues are 2 and 1. □

(b) Since there are two eigenvalues, the maximum of the multiplicity is 2. Therefore Theorem 8.3 implies that $G(\lambda, T) = \text{null}(T - \lambda I)^2$. A computation thus gives:

$$G(2, T) = \text{span}\{(1, 1, -1), (1, 2, 0)\}, \quad G(1, T) = \text{span}\{(1, 2, 1)\}.$$

Therefore the eigenvalue $\lambda = 2$ has multiplicity 2 and $\lambda = 1$ has multiplicity 1, and the characteristic polynomial of T is $(z-1)(z-2)^2$ □

(c) Since $\dim \text{null}(T - 2I) = 1$ while $\dim \text{null}(T - 2I)^2 = 2$, $(T - 2I)|_{G(2,T)}$ is a non-zero nilpotent, hence the Jordan form of T is

$$J(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Read 8.D thoroughly if you don't see how this form comes out) □

(d) Since $(J(T) - 2I)(J(T) - I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$, the minimal polynomial of T cannot be $(z-2)(z-1)$, hence by Theorem 8.48, the minimal polynomial of T must be $(z-1)(z-2)^2$. □

(e) Consider the operator given by the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy (hence is left to you) to verify that it is the desired example. □