Step-1

Any m by n matrix A can be factor into

$$A = U\Sigma V^T$$
$$= \begin{pmatrix} \text{orthogonal} \\ U \text{ is } m\times m \end{pmatrix}\begin{pmatrix} m\times n \text{ matrix } \Sigma \\ \sigma_1\cdots\sigma_r \text{ on diagonal} \end{pmatrix}\begin{pmatrix} \text{orthogonal} \\ V \text{ is } n\times n \end{pmatrix}$$

Here eigenvectors of AA^T are in U, eigenvectors of $A^T A$ are in V.

The r singular-values on the diagonal of Σ are the square roots of the nonzero eigenvalues of both AA^T and $A^T A$.

Step-2

To show AA^+ is projection matrix, we have to show

$$\left(AA^+\right)^2 = AA^+$$

Consider the matrix factorization for m by n matrix A as

$$A = Q_1\Sigma Q_2^T$$

So, its pseudoinverse is given by

$$A^+ = Q_2\Sigma^+ Q_1^T$$

Step-3

Find AA^+ as follows:

$$AA^+ = \left(Q_1\Sigma Q_2^T\right)\left(Q_2\Sigma^+ Q_1^T\right)$$
$$= Q_1\Sigma\Sigma^+ Q_1^T$$

Squaring both sides of above equation we get

$$\left(AA^+\right)^2 = \left(Q_1\Sigma\Sigma^+ Q_1^T\right)^2$$
$$= Q_1\Sigma\Sigma^+ Q_1^T Q_1\Sigma\Sigma^+ Q_1^T$$
$$= Q_1\Sigma\Sigma^+\Sigma\Sigma^+ Q_1^T$$
$$= Q_1\Sigma\Sigma^+ Q_1^T$$

Step-4

So, we have

$$\begin{aligned}\left(AA^+\right)^2 &= Q_1 \Sigma \Sigma^+ Q_1^T \\ &= AA^+\end{aligned}$$

Therefore, $\boxed{AA^+ \text{ is projection matrix}}$.

We know that every projection matrix is symmetric.

Therefore, $\boxed{AA^+ \text{ is symmetric matrix}}$.

Step-5

Now find A^+A as follows:

$$\begin{aligned}A^+A &= \left(Q_2 \Sigma^+ Q_1^T\right)\left(Q_1 \Sigma Q_2^T\right) \\ &= Q_2 \Sigma^+ \Sigma Q_2^T\end{aligned}$$

Squaring both sides of above equation we get

$$\begin{aligned}\left(A^+A\right)^2 &= \left(Q_2 \Sigma^+ \Sigma Q_2^T\right)^2 \\ &= Q_2 \Sigma^+ \Sigma Q_2^T Q_2 \Sigma^+ \Sigma Q_2^T \\ &= Q_2 \Sigma^+ \Sigma \Sigma^+ \Sigma Q_2^T \\ &= Q_2 \Sigma^+ \Sigma Q_2^T\end{aligned}$$

Step-6

So, we have

$$\begin{aligned}\left(A^+A\right)^2 &= Q_2 \Sigma^+ \Sigma Q_2^T \\ &= A^+A\end{aligned}$$

Therefore, $\boxed{A^+A \text{ is projection matrix}}$.

We know that every projection matrix is symmetric.

Therefore, $\boxed{A^+A \text{ is symmetric matrix}}$.

We know that for $A = U \Sigma V^T$, U and V gives orthonormal bases for all four fundamental subspaces as follows:

First r columns of U : column space of A

Last $m-r$ columns of U : left null space of A

First r columns of V : row space of A

Last $n-r$ columns of V : null space of A

We have

$$AA^+ = Q_1 \Sigma \Sigma^+ Q_1^T$$
$$A^+ A = Q_2 \Sigma^+ \Sigma Q_2^T$$

So, AA^+ and $A^+ A$ project onto the $\boxed{\text{column space and row space}}$ of A.