

## Step-1

Consider the solution for differential equation

$$\begin{aligned}\frac{du}{dt} &= Au \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u\end{aligned}$$

goes around in a circle  $u = (\cos t, \sin t)$ .

Matrix  $I + A$  is given by

$$\begin{aligned}I + A &= \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\end{aligned}$$

## Step-2

To find the eigenvalues of matrix  $I + A$ , solve the equation  $\det((I + A) - \lambda I) = 0$ , for  $\lambda$ .

$$\begin{aligned}\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} &= 0 \\ (\lambda-1)^2 &= -1 \\ \lambda-1 &= \pm i\end{aligned}$$

Therefore, the eigenvalues of matrix  $I + A$  are  $\lambda_1 = \boxed{1-i}$  and  $\lambda_2 = \boxed{1+i}$ .

## Step-3

To calculate eigenvector  $x_1$  do the following calculation:

$$\begin{aligned}\det((I + A) - \lambda_1 I) x_1 &= 0 \\ \begin{bmatrix} 1-(1-i) & -1 \\ 1 & 1-(1-i) \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

On solving values of  $y$  and  $z$  corresponding to  $\lambda_1 = 1-i$  is as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix} \\ = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

### Step-4

To calculate eigenvector  $x_2$  do the following calculation:

$$\det((I + A) - \lambda_2 I) x_2 = 0 \\ \begin{bmatrix} 1 - (1 + i) & -1 \\ 1 & 1 - (1 + i) \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of  $y$  and  $z$  corresponding to  $\lambda_2 = 1 + i$  is as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix} \\ = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

### Step-5

Therefore, the eigenvectors for the matrix  $I + A$  are  $x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$  and  $x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ .

### Step-6

Matrix  $(I - A)^{-1}$  is given by

$$(I - A)^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \\ = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

### Step-7

To find the eigenvalues of matrix  $(I - A)^{-1}$ , solve the equation  $\det((I - A)^{-1} - \lambda I) = 0$ , for  $\lambda$ .

$$\det \begin{bmatrix} 1/2 - \lambda & -1/2 \\ 1/2 & 1/2 - \lambda \end{bmatrix} = 0$$

$$\left(\frac{1}{2} - \lambda\right)^2 + \frac{1}{4} = 0$$

Therefore, the eigenvalues of matrix  $(I - A)^{-1}$  are  $\lambda_1 = \boxed{\frac{1}{2} - \frac{1}{2}i}$  and  $\lambda_2 = \boxed{\frac{1}{2} + \frac{1}{2}i}$ .

To calculate eigenvector  $x_1$  do the following calculation:

$$\det \left( (I - A)^{-1} - \lambda_1 I \right) x_1 = 0$$

$$\begin{bmatrix} 1/2 - (1/2 - 1/2i) & -1/2 \\ 1/2 & 1/2 - (1/2 - 1/2i) \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of  $y$  and  $z$  corresponding to  $\lambda_1 = (1/2 - 1/2i)$  is as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ i \end{bmatrix}$$

## Step-8

To calculate eigenvector  $x_2$  do the following calculation:

$$\det \left( (I - A)^{-1} - \lambda_2 I \right) x_2 = 0$$

$$\begin{bmatrix} 1/2 - (1/2 + 1/2i) & -1/2 \\ 1/2 & 1/2 - (1/2 + 1/2i) \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of  $y$  and  $z$  corresponding to  $\lambda_2 = (1/2 + 1/2i)$  is as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

## Step-9

Therefore, the eigenvectors for the matrix  $(I - A)^{-1}$  are  $x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$  and  $x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ .

## Step-10

Consider matrix  $B = \left(I - \frac{1}{2}A\right)^{-1} \left(I + \frac{1}{2}A\right)$  is given by

$$\begin{aligned} B &= \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1/2 \\ 1/2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4/5 & -2/5 \\ 2/5 & 4/5 \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 1/2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \end{aligned}$$

## Step-11

To find the eigenvalues of matrix  $B$ , solve the equation  $\det(B - \lambda I) = 0$ , for  $\lambda$ .

$$\begin{aligned} \det \begin{bmatrix} 3/5 - \lambda & -4/5 \\ 4/5 & 3/5 - \lambda \end{bmatrix} &= 0 \\ \left(\frac{3}{5} - \lambda\right)^2 + \frac{16}{25} &= 0 \end{aligned}$$

Therefore, the eigenvalues of matrix  $B$  are  $\lambda_1 = \frac{3}{5} - \frac{4}{5}i$  and  $\lambda_2 = \frac{3}{5} + \frac{4}{5}i$ .

## Step-12

To calculate eigenvector  $x_1$  do the following calculation:

$$\begin{aligned} \det(B - \lambda_1 I) x_1 &= 0 \\ \begin{bmatrix} 3/5 - (3/5 - 4/5i) & -4/5 \\ 4/5 & 3/5 - (3/5 - 4/5i) \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

On solving values of  $y$  and  $z$  corresponding to  $\lambda_1 = (3/5 - 4/5i)$  is as follows:

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix} \\ = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

### Step-13

To calculate eigenvector  $x_2$  do the following calculation:

$$\det(B - \lambda_2 I) x_2 = 0 \\ \begin{bmatrix} 3/5 - (3/5 + 4/5i) & -4/5 \\ 4/5 & 3/5 - (3/5 + 4/5i) \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving values of  $y$  and  $z$  corresponding to  $\lambda_2 = (3/5 + 4/5i)$  is as follows:

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix} \\ = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

### Step-14

Therefore, the eigenvectors for the matrix  $(I - A)^{-1}$  are  $x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$  and  $x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ .

### Step-15

Consider  $(x_1, x_2)$  lies on unit circle then  $\sqrt{(x_1)^2 + (x_2)^2} = 1$ , that is

$$(x_1)^2 + (x_2)^2 = 1.$$

Take the forward difference (**F**):

$$u_{n+1} = (I + A)u_n$$

So, solving the following equation, we have

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

If  $(y_1, y_2)$  lies on unit circle then  $y_1^2 + y_2^2 = 1$ .

Solve the following equation:

$$\begin{aligned} y_1^2 + y_2^2 &= (x_1 - x_2)^2 + (x_1 + x_2)^2 \\ &= 2[(x_1)^2 + (x_2)^2] \\ &= 2(1) \\ &\neq 1 \end{aligned}$$

## Step-16

Therefore, for the differential equation  $u_{n+1} = (I + A)u_n$ , solution does not stay on a circle.

## Step-17

Consider the forward difference **(B)**:

$$u_{n+1} = (I - A)^{-1} u_n$$

So, solving the following equation, we have

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ = \begin{bmatrix} 1/2 x_1 - 1/2 x_2 \\ 1/2 x_1 + 1/2 x_2 \end{bmatrix}$$

If  $(z_1, z_2)$  lies on unit circle then  $z_1^2 + z_2^2 = 1$ . For this solve the following equation:

$$\begin{aligned} z_1^2 + z_2^2 &= \left(\frac{1}{2}x_1 - \frac{1}{2}x_2\right)^2 + \left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right)^2 \\ &= \frac{1}{4}[(x_1)^2 + (x_2)^2] \\ &= \frac{1}{4}(1) \\ &\neq 1 \end{aligned}$$

Therefore, for the differential equation  $u_{n+1} = (I - A)^{-1} u_n$ , solution does not stay on a circle.

## Step-18

Consider the forward difference (C):

$$u_{n+1} = \left( I - \frac{1}{2} A \right)^{-1} \left( I + \frac{1}{2} A \right) u_n$$

So, solving the following equation, we have

$$\begin{aligned} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 3/5 x_1 - 4/5 x_2 \\ 4/5 x_1 + 3/5 x_2 \end{bmatrix} \end{aligned}$$

If  $(w_1, w_2)$  lies on unit circle then  $w_1^2 + w_2^2 = 1$ . For this solve the following equation:

$$\begin{aligned} w_1^2 + w_2^2 &= \left( \frac{3}{5} x_1 - \frac{4}{5} x_2 \right)^2 + \left( \frac{4}{5} x_1 + \frac{3}{5} x_2 \right)^2 \\ &= \frac{25}{25} \left[ (x_1)^2 + (x_2)^2 \right] \\ &= 1(1) \\ &= 1 \end{aligned}$$

Therefore, for the differential equation  $u_{n+1} = \left( I - \frac{1}{2} A \right)^{-1} \left( I + \frac{1}{2} A \right) u_n$ , solution does stay on a circle.