Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #5

2023/03/26

Name: ____________________

Student Number: ____________________

1. Suppose $W_1, \cdots, W_m$ are vector space. Prove that $\mathcal{L}(V, W_1 \times \cdots W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \mathcal{L}(V, W_m)$ are isomorphic vector spaces.

   Proof. Construct a map $\mathcal{A} : \mathcal{L}(V, W_1) \times \cdots \mathcal{L}(V, W_m) \to \mathcal{L}(V, W_1 \times \cdots W_m)$. $\forall (f_1, \cdots, f_m) \in \mathcal{L}(V, W_1) \times \cdots \mathcal{L}(V, W_m)$, we define $\mathcal{A}((f_1, \cdots, f_m)) = f$, $f \in \mathcal{L}(V, W_1 \times \cdots W_m)$. $\forall v \in V$, $f(v) = (f_1 v, \cdots, f_m v) \in W_1 \times \cdots W_m$. Obviously, $\mathcal{A}$ is well-defined.

   $\forall (f_1, \cdots, f_m), (g_1, \cdots, g_m) \in \mathcal{L}(V, W_1) \times \cdots \mathcal{L}(V, W_m)$, $\forall v \in V$,

$$\begin{aligned}
\mathcal{A}((f_1, \cdots, f_m) + (g_1, \cdots, g_m))(v) &= \mathcal{A}((f_1, \cdots, f_m))(v) + \mathcal{A}((g_1, \cdots, g_m))(v) \\
&= ((f_1 + g_1)v, (f_2 + g_2)v, \cdots, (f_m + g_m)v) \\
&= (f_1 v, \cdots, f_m v) + (g_1 v, \cdots, g_m v) \\
&= \mathcal{A}((f_1, \cdots, f_m))v + \mathcal{A}((g_1, \cdots, g_m))v
\end{aligned}$$

   so $\mathcal{A}((f_1, \cdots, f_m) + (g_1, \cdots, g_m)) = \mathcal{A}((f_1, \cdots, f_m)) + \mathcal{A}((g_1, \cdots, g_m))$.

   $\forall a \in \mathbf{F}$,

$$\begin{aligned}
\mathcal{A}(a(f_1, \cdots, f_m))(v) &= \mathcal{A}((af_1, \cdots, af_m))(v) \\
&= (af_1 v, \cdots, af_m v) \\
&= a(f_1 v, \cdots, f_m v) \\
&= a\mathcal{A}((f_1, \cdots, f_m))(v)
\end{aligned}$$

   so $\mathcal{A}(a(f_1, \cdots, f_m)) = a\mathcal{A}((f_1, \cdots, f_m))$. Thus $\mathcal{A}$ is a linear map.

If $\mathcal{A}((f_1,\cdots,f_m)) = \theta$, θ is zero map, then $\forall v \in V$, $\mathcal{A}((f_1,\cdots,f_m))(v) = \theta v = 0$, so $(f_1 v,\cdots,f_m v) = (0,\cdots,0) \Rightarrow f_i v = 0$ hold for any $v \in V$, $i = 1,\cdots,m$, thus $f_i = \theta \Rightarrow (f_1,\cdots,f_m) = (\theta,\cdots,\theta) \Rightarrow \mathcal{A}$ is injective.

$\forall f \in \mathcal{L}(V,W_1) \times \cdots \mathcal{L}(V,W_m)$, $\forall v \in V$, $f(v) = (w_{V1},\cdots,w_{V_m}) \in W_1 \times \cdots \times W_m$. Define $f_i : V \to W_i$, $f_i(v) = w_{V_i}$. Since f is linear, f_i is also linear, and $\mathcal{A}((f_1,\cdots,f_m)) = f$, so $\mathcal{A}$ is surjective.

Therefore, $\mathcal{A}$ is an isomorphism from $\mathcal{L}(V,W_1) \times \cdots \mathcal{L}(V,W_m)$ to $\mathcal{L}(V,W_1 \times \cdots W_m)$. □

2. Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for $x \in \mathbf{R}$.

   1. Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbf{R})$.
   2. Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\varphi))(x^3)$.

Proof. 1. $T'(\varphi) = \varphi \circ T \in \mathcal{L}(\mathcal{P}(\mathbf{R}),\mathbf{R})$, $\forall p(x) \in \mathcal{P}(\mathbf{R})$, $T'(\varphi)(p(x)) = \varphi \circ T(p(x)) = \varphi \circ (x^2 p(x) + p''(x)) = (2xp(x) + x^2 p'(x) + p'''(x))|_{x=4} = 8p(4) + 16p'(4) + p'''(4)$.

2. $T'(\varphi)(x^3) = \varphi \circ T(x^3) = \varphi(x^2 \cdot x^3 + 6x) = \int_0^1 x^5 + 6xdx = \frac{1}{6} + 3 = \frac{19}{6}$. □