Southern University of Science and Technology
Advanced Linear Algebra Spring 2023

MA109– Quiz #5

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Name: ______________________________

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1. Suppose T is a function from V to W. The graph of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

设 T 是 V 到 W 的函数，定义 T 的图为 $V \times W$ 的如下子集：

$$T\text{的图} = \{(v, Tv) \in V \times W : v \in V\}$$

证明 T 是线性映射当且仅当 T 的图是 $V \times W$ 的子空间.

Proof. "$\Rightarrow$" Since T is linear, $T0 = 0 \Rightarrow (0, 0) = (0, T0) \in$ graph of T.

For any $v_1, v_2 \in V$, $(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, Tv_1 + Tv_2) = (v_1 + v_2, T(v_1 + v_2)) \in$ graph of T.

For any $v \in V$, $a \in \mathbf{F}$, $a(v, Tv) = (av, aTv) = (av, T(av)) \in$ graph of T.

Thus the graph of T is a subspace.

"$\Leftarrow$" $\forall v_1, v_2 \in V$, $(v_1 + v_2, T(v_1 + v_2)) \in$ graph of T, $(v_1, Tv_1), (v_2, Tv_2) \in$ graph of Tsince the graph of T is a subspace of $V \times W$, then $(v_1 + v_2, T(v_1 + v_2)) - (v_1, Tv_1) - (v_2, Tv_2) = (0, T(v_1 + v_2) - Tv_1 - Tv_2) \in$ graph of T, so $T(v_1 + v_2) - Tv_1 - Tv_2 = T0 = 0 \Rightarrow Tv_1 + Tv_2 = T(v_1 + v_2)$.

For any $v \in V$, $a \in \mathbf{F}$, $(av_1, T(av_1)) \in$ graph of T, $(av_1, aTv_1) = a(v_1, Tv_1) \in$ graph of T, then $(av_1, T(av_1)) - (av_1, aTv_1) = (0, T(av_1) - aTv_1) \in$ graph of T, so $T(av_1) - aTv_1 = T0 = 0 \Rightarrow T(av_1) = aTv_1$. $\square$

2. Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \cdots, c_m) \in \mathbf{F}^m$ and $(d_1, \cdots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, .., m$ and every $k = 1, \cdots, n$.

   设 A 是 $m \times n$ 矩阵且 $A \neq 0$. 证明 A 的秩是 1 当且仅当存在 $(c_1, \cdots, c_m) \in \mathbf{F}^m$ 和 $(d_1, \cdots, d_n) \in \mathbf{F}^n$ 使得对任意 $j = 1, .., m$ 和 $k = 1, \cdots, n$ 有 $A_{j,k} = c_j d_k$.

Proof. "$\Leftarrow$" $\exists c = (c_1, \cdots, c_m) \in \mathbf{F}^m, d = (d_1, \cdots, d_n) \in \mathbf{F}^n$ such that $A = c'd$, rank $A \leqslant \min\{\text{rank } c, \text{rank } d\} = 1$. And since $A \neq 0$, then rank $A = 1$.

"$\Rightarrow$" Assume rank $A = 1$, then $\exists$ invertible matrices $P \in \mathbf{F}^{m \times m}, Q \in \mathbf{F}^{n \times n}$, s.t.

$$A = P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q = P \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} Q.$$

Let $c' = P \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,$d = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} Q$, we have $A = c'd$. □