

Step-1

Given that

Polar coordinates satisfy $x = r \cos \theta$ and $y = r \sin \theta$. Polar area $\int r dr d\theta$ includes J:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

Step-2

$$\frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

$$J = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

Step-3

The two columns are orthogonal their lengths are $\sqrt{\cos^2 \theta + \sin^2 \theta} = 1$

$$\text{And } \sqrt{r^2 (\cos^2 \theta + \sin^2 \theta)} = \sqrt{r^2}$$

$$= r$$

Thus $J = r$