Step-1

Consider the following matrix:

$$S=\frac{1}{\sqrt{2}}\begin{bmatrix}\sin\theta & \sin 2\theta & \sin 3\theta\\ \sin 2\theta & \sin 4\theta & \sin 6\theta\\ \sin 3\theta & \sin 6\theta & \sin 9\theta\end{bmatrix}$$

Here, $\theta=\frac{\pi}{4}$. This sine matrix is a variation on Fourier matrix. Verify that $S^T=S^{-1}$.

Step-2

Simplify the matrix:

$$S=\frac{1}{\sqrt{2}}\begin{bmatrix}\sin(\pi/4) & \sin 2(\pi/4) & \sin 3(\pi/4)\\ \sin 2(\pi/4) & \sin 4(\pi/4) & \sin 6(\pi/4)\\ \sin 3(\pi/4) & \sin 6(\pi/4) & \sin 9(\pi/4)\end{bmatrix}$$

$$S=\frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2} & 1 & 1/\sqrt{2}\\ 1 & 0 & -1\\ 1/\sqrt{2} & -1 & 1/\sqrt{2}\end{bmatrix}$$

Step-3

Compute the inverse and transpose of the matrix:

$$S^{-1}=\frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2} & 1 & 1/\sqrt{2}\\ 1 & 0 & -1\\ 1/\sqrt{2} & -1 & 1/\sqrt{2}\end{bmatrix}$$

$$S^{T}=\frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2} & 1 & 1/\sqrt{2}\\ 1 & 0 & -1\\ 1/\sqrt{2} & -1 & 1/\sqrt{2}\end{bmatrix}$$

It can be seen clearly that $S^T=S^{-1}$.

Step-4

This can also be verified by proving that the matrix is orthogonal matrix. Recall that orthogonal matrix satisfies $Q^T=Q^{-1}$ and $\overline{x}^T y=0$. So, if it is proved that columns of the matrix are orthogonal to each other then verification will be done.

Step-5

Now do the following calculations:

$$\begin{aligned}\bar{x}_1^T x_2 &= \left(\frac{1}{\sqrt{2}}\right)\cdot 1 + 1\cdot 0 + \left(\frac{1}{\sqrt{2}}\right)\cdot(-1)\\ &= 0\end{aligned}$$

$$\begin{aligned}\bar{x}_2^T x_3 &= 1\cdot\left(\frac{1}{\sqrt{2}}\right) + 0\cdot(-1) + (-1)\cdot\left(\frac{1}{\sqrt{2}}\right)\\ &= 0\end{aligned}$$

$$\begin{aligned}\bar{x}_1^T x_3 &= \left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}}\right) + 1\cdot(-1) + \left(\frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}}\right)\\ &= \left(\frac{1}{2}\right) - 1 + \left(\frac{1}{2}\right)\\ &= 0\end{aligned}$$

All the column vectors are orthogonal to each other. Therefore, $\boxed{S^T = S^{-1}}$.