$T \in \mathcal{L}(V)$. V complex vector space

$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

$T = T|_{G(\lambda_1,T)} + \cdots + T|_{G(\lambda_m,T)}$

$T|_{G(\lambda_k,T)} = (T - \lambda_k I)|_{G(\lambda_k,T)} + \lambda_k I|_{G(\lambda_k,T)}$

Jordan Form

Lecture 26

Dept. of Math., SUSTech

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Jordan Form

Examples

Example

Let $\underline{N \in \mathscr{L}(\mathbb{F}^4)}$ be the nilpotent operator defined by

$$N(z_1, z_2, z_3, z_4) = (0, z_1, z_2, z_3).$$

If $v = (1, 0, 0, 0)$, then $N^3 v, N^2 v, Nv, v$ is a basis of $\mathbb{F}^4$. The matrix of N with respect to this basis is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example

Example

Let $N \in \mathcal{L}(\mathbb{F}^6)$ be the nilpotent operator defined by

$$N(z_1, z_2, z_3, z_4, z_5, z_6) = (0, z_1, z_2, 0, z_4, 0).$$

- Unlike the nice behavior of the nilpotent operator of the previous example, for this nilpotent operator there does not exist a vector such that $N^5v, N^4v, N^3v, N^2v, Nv, v$ is a basis of $\mathbb{F}^6$.
- However, if we take $v_1 = (1,0,0,0,0,0), v_2 = (0,0,0,1,0,0)$, and $v_3 = (0,0,0,0,0,1)$, then $\underbrace{N^2v_1, Nv_1, v_1}_{\text{cycle length 3}}, \underbrace{Nv_2, v_2}_{\text{cycle length2}}, \underbrace{v_3}_{\text{cycle}}$ is a basis of $\mathbb{F}^6$.

3x3, 2x2, 1x1

3 Jordan Blocks

$$N(N^2v_1, Nv_1, v_1, Nv_2, v_2, v_3) = (N^2v_1, Nv_1, v_1, Nv_2, v_2, v_3)\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N(u_1, u_2, u_3, u_4, u_5, u_6) = (u_1, u_2, u_3, u_4, u_5, u_6) B$$

Similar

$$M^{-1} B M = J$$

The Matrix Representation

The matrix of N with respect to this basis is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This 6-by-6 block diagonal matrix consists of a 3-by-3 block with 1's on the line above the diagonal and 0's elsewhere, a 2-by-2 block with 1 above the diagonal and 0's elsewhere, and a 1-by-1 block containing 0.

Basis corresponding to a nilpotent operator

The next result shows that every nilpotent operator $N \in \mathscr{L}(V)$ behaves similarly to the previous example. $T|_{G(\lambda_k,T)} = \underbrace{(T-\lambda_k I)|_{G(\lambda_k,T)}}_{\text{nilpotent}} + \lambda I|_{G(\lambda_k,T)}$

8.55 Basis corresponding to a nilpotent operator

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then there exist vectors $v_1, \ldots, v_n \in V$ and $\boxed{\text{nonnegative integers } m_1, \ldots, m_n}$ such that

(a) $\underbrace{N^{m_1}v_1, \ldots, Nv_1, v_1}_{\text{cycle}}, \ldots, \underbrace{N^{m_n}v_n, \ldots, Nv_n, v_n}_{\text{cycle}}$ is a basis of V;

(b) $N^{m_1+1}v_1 = \cdots = N^{m_n+1}v_n = 0.$

$N(\underbrace{N^{m_1}v_1}_{\text{eigenvalue}}) = 0 N^{m_1}v_1$

Proof. We will prove this result by induction on $\dim V$. To get started, note that the desired result obviously holds if $\dim V = 1$. Now assume that $\dim V > 1$ and that the desired result holds on all vector spaces of smaller dimension.

Proof

· range N · dim range N < dim V
· range N invariant under N
· N|range N

Because N is nilpotent, N is not injective. Thus N is not surjective and hence range N is a subspace of V that has a smaller dimension than V. Thus we can apply our induction hypothesis to the restriction operator $N|_{\text{range } N} \in \mathscr{L}(\text{range } N)$.

By our induction hypothesis applied to $N|_{\text{range } N}$, there exists vectors $v_1, v_2, \cdots, v_n \in$ range N and nonnegative integers $m_1, \cdots, m_n$ such that

$$N^{m_1}v_1, \cdots, Nv_1, v_1, \cdots, N^{m_n}v_n, \cdots, Nv_n, v_n \cdots\cdots (2)$$

is a basis of range N and

$$N^{m_1+1}v_1 = \cdots = N^{m_n+1}v_n = 0.$$

Because each v_j is in range N, for each j there exists $u_j \in V$ such that $\boxed{v_j = Nu_j}$. Thus $N^{k+1}u_j = N^k v_j$ for each j and each nonnegative integer k.

Proof

linearly independent ?

$N(a_1 N^{m_1+1} u_1 + \cdots + a_{m_1+2} u_1 + \cdots + b_1 N^{m_n+1} u_1 + \cdots + b_{m_n+2} u_n) = N(0)$

We now claim that $0 + a_2 N^{m_1+1} u_2 + \cdots + a_{m_1+2} N u_1 + \cdots + 0 + b_2 N^{m_n+1} u_2 + \cdots + b_{m_n+2} N u_n = 0$

$\overset{\|}{0} \quad \overset{\|}{N^{m_1} v_1} \quad \cdots \qquad a_1 N^{m_1+1} v_1 + \cdots + b_1 N^{m_n+1} v_n = 0$

$$N^{m_1+1}u_1, \cdots, Nu_1, u_1, \cdots, N^{m_n+1}u_n, \cdots, Nu_n, u_n \cdots\cdots (3)$$

is linearly independent list of vectors in V. To verify this claim, suppose that some linear combination of (3) equals 0. Applying N to that linear combination, we get a linear combination of (2). However, the list (2) is linearly independent, and hence all the coefficients in our original linear combination of (3) equal 0 except possibly the coefficients of the vectors

$$N^{m_1+1}u_1, \cdots, N^{m_n+1}u_n,$$

which equal the vectors

$$N^{m_1}v_1, \cdots, N^{m_n}v_n.$$

Proof.

Again using the linear independence of the list (2), we conclude that those coefficients also equal 0, completing our proof that the list (3) is linearly independent.

Now we extend (3) to a basis

$Nw_1 \in$ range N

u_{n+1}

$N^1 x_1 \Rightarrow N(\underbrace{w_1 - x_1}) = 0$

$$N^{m_1+1}u_1, \cdots, Nu_1, u_1, \cdots, N^{m_n+1}u_n, \cdots, Nu_n, u_n, w_1, w_2, \cdots, w_p \cdots\cdots (4)$$

$N u_{n+1} = 0$

of V. Each Nw_j is in range N and hence is in the span of (2). Each vector in the list (2) equals N applied to some vector in the list of (3).

Proof

Thus there exists x_j in the span of (3) such that $Nw_j = Nx_j$. Now let

$$u_{n+j} = w_j - x_j.$$

Then $Nu_{n+j} = 0$. Furthermore,

$$\underbrace{N^{m_1+1}u_1,\cdots,Nu_1,u_1}_{\text{1st cycle}},\cdots,\underbrace{N^{m_n+1}u_n,\cdots,Nu_n,u_n}_{\text{nth cycle}},\overset{\text{n+1th cycle}}{u_{n+1}},u_{n+2},\cdots,\overset{\text{--- n+pth cycle}}{u_{n+p}}$$

spans V because its span contains each x_j and each u_{n+j} and hence each w_j.

Thus the spanning list above is a basis of V because it has the same length as the basis (4). This basis has the required form, completing the proof.

Remarks

1. There is a finite collection of vectors $v_1, v_2, \cdots, v_n \in V$ such that there is a basis of V consisting of the vectors of the form $N^k v_j$, as j varies from 1 to n and k varies (in reverse order) from 0 to the largest nonnegative integer m_j such that $N^{m_j} v_j \neq 0$.
2. For the matrix interpretation of the next result, see the first part of the proof of 8.60.

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}}_{(m_1+2)\times(m_1+2)} & & \\ & \ddots & \\ & & \underbrace{\begin{matrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{matrix}}_{p} \end{pmatrix}$$

Jordan Basis

- In the next definition, the diagonal of each A_j is filled with some eigenvalue λ_j of T, the line directly above the the diagonal of A_j is filled with the 1's, and all other entries in A_j are 0.
- The λ_j's need not be distinct.
- A_j may be a 1-by-1 matrix (λ_j) containing just an eigenvalue of T.
- French mathematician Camille Jordan first published a proof of 8.60 in 1870.

$$T|_{G(\lambda_k,T)} = \underline{(T-\lambda_k I)|_{G(\lambda_k,T)}} + \lambda I|_{G(\lambda_k,T)}$$

$$\begin{bmatrix} B_1 & & \\ & B_2 & \\ & & \ddots & \\ & & & B_3 \end{bmatrix} + \begin{bmatrix} \lambda_k & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} = \begin{bmatrix} \underset{A_1}{\boxed{\overset{(m_1+2)\times(m_1+2)}{\begin{matrix} \lambda_k & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_k \end{matrix}}}} & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}_{(m_1+2)\times(m_1+2)}$$

Jordan Basis

8.59 Definition ***Jordan basis***

Suppose $T \in \mathcal{L}(V)$. A basis of V is called a ***Jordan basis*** for T if with respect to this basis T has a block diagonal matrix

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix},$$

where each A_j is an upper-triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}.$$

Jordan Basis

8.60 Jordan Form

Suppose V is a complex vector space. If $T \in \mathcal{L}(V)$, then there is a basis of V that is a Jordan basis for T.

Proof. First consider a nilpotent operator $N \in \mathscr{L}(V)$ and the vectors $v_1, v_2, \cdots, v_n$ given by 8.55. For each j, note that N sends the first vector in the list $N^{m_j} v_j, \cdots, Nv_j, v_j$ to 0 and that N sends each vector in this list other than the first vector to the previous vector.

Proof.

In other words, 8.55 gives a basis of V with respect to which N has a block diagonal matrix, where each matrix on the diagonal has the form

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Proof

(a) Thus the desired result holds for nilpotent operators.

(b) Now suppose $T \in \mathscr{L}(V)$. Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be the distinct eigenvalues of T. We have the generalized eigenspace decomposition

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T),$$

where each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent.

(c) Thus some basis of each $G(\lambda_j, T)$ is a Jordan basis for $(T - \lambda_j I)|_{G(\lambda_j, T)}$.

(d) Put these bases together to get a basis of V that is a Jordan basis for T.

Homework Assignment 26

8.D: 2, 3, 4, 5, 6, 7.