Step-1

(a) Suppose Î»$_j$ is positive. We need to show that every S_j contains a vector x so that $R(x) > 0$.

Let if possible, there exists S_j such that for every vector $x \in S_j$, $R(x) \le 0$.

Then, $\max\limits_{x \in S_j} R(x) \le 0$. Therefore, $\min\limits_{S_j}\left[\max\limits_{x \in S_j} R(x)\right] \le 0$. Thus, Î»$_j$ cannot be positive. This is a contradiction. Thus, our assumption is wrong.

This shows that every S_j contains a vector x so that $R(x) > 0$.

Step-2

(b) We know that $R(x) = \dfrac{x^T A x}{x^T x}$.

Let $y = C^{-1}x$. This gives $x = Cy$.

Therefore,

$$
\begin{aligned}
R(x) &= \frac{(Cy)^T A (Cy)}{(Cy)^T (Cy)} \\
&= \frac{y^T C^T A C y}{y^T C^T C y} \\
&= \frac{y^T C^T A C y}{y^T y}
\end{aligned}
$$

Since $R(x) > 0$, it is clear that $\dfrac{y^T C^T A C y}{y^T y} > 0$.

Step-3

(c) Consider the j^{th} eigenvalue of $C^T A C$.

We have

$$\lambda_j = \min_{S_j}\left[\max_{x \in S_j} R(x)\right]$$

We have shown that $R(x) = \dfrac{y^T C^T A C y}{y^T y}$.

Step-4

Therefore, $\lambda_j = \min_{S_j}\left[\max_{Cy \in S_j} \frac{y^T C^T ACy}{y^T y}\right]$.

Therefore, it is clear that the j^{th} eigenvalue of $C^T AC$ is also positive.