Step-1

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The matrix A of size 4 by 4 from the referred equation is $A=\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ by $a_{11}=1$

Let us consider the system $A = IAI$ where I is the identity matrix of order 4.

We apply row operations on A left hand side, the same row operation must be applied on the pre multiple I on the right hand side, immediately, the congruent column operation on A as well as on the post multiple I.

This maintains the symmetry of pre and post multiples so that they can be seen as L, L^T.

Step-2

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 1: $R_2 \to R_2 + R_1$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Immediately, we do $C_2 \to C_2 + C_1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & \boxed{1} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step-3

Step 2: $R_3 \to R_3 + R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{1} & 1 & 0 & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & \boxed{1} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Immediately, we perform $C_3 \to C_3 + C_2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{1} & 1 & 0 & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & \boxed{1} & \boxed{1} & 0 \\ 0 & 1 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step-4

Step 3: $R_4 \to R_4 + R_3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{1} & 1 & 0 & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \end{pmatrix} A \begin{pmatrix} 1 & \boxed{1} & \boxed{1} & 0 \\ 0 & 1 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Immediately, $C_4 \to C_4 + C_3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{1} & 1 & 0 & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \end{pmatrix} A \begin{pmatrix} 1 & \boxed{1} & \boxed{1} & \boxed{1} \\ 0 & 1 & \boxed{1} & \boxed{1} \\ 0 & 0 & \boxed{1} & \boxed{1} \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

Step-5

Thus, we have written $D = BAB^T$ where $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{1} & 1 & 0 & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & 0 \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \end{pmatrix}$

This can now be written as $B^{-1} D \left(B^T\right)^{-1} = A$

So, considering B^{-1} as L, we get $\left(B^T\right)^{-1} = L^T$ and thus $A = LDL^T$

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ and $L^T = \left(B^T\right)^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Step-6

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$