Step-1

Consider the two matrices,

$A=\begin{bmatrix}3&4&2\\0&1&2\\0&0&0\end{bmatrix}$ And $B=\begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}.$

The objective is to find the eigenvalues and eigenvectors of the given matrices.

Now, to find the eigenvalues of the matrix $A=\begin{bmatrix}3&4&2\\0&1&2\\0&0&0\end{bmatrix}.$

It is an upper triangular matrix.

So the diagonal elements are the eigenvalues.

That is the eigenvalues are $\lambda_1=0, \lambda_2=3, \text{and } \lambda_3=1.$

Step-2

To find the eigenvector corresponding to the eigenvalue $\lambda_1=0$,

Case (i):

Let $\lambda_1=0$

Eigen vectors X corresponding to the Eigen value 0 are given by,

$(A-0I)X=0$

$$\begin{bmatrix}3&4&2\\0&1&2\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$3x_1+4x_2+2x_3=0$

$x_2+2x_3=0$

Step-3

There are 3 variables and two equations in the above system.

So, it has so many solutions.

Let $x_3 = k(\neq 0)$

From second equations,

$x_2 = -2x_3$

$x_2 = -2k$ Substitute $x_3 = k(\neq 0)$

Step-4

From the first equation $3x_1 + 4x_2 + 2x_3 = 0$,

$3x_1 = -4x_2 - 2x_3$

$3x_1 = -4(-2k) - 2(k)$

$3x_1 = 8k - 2k$

$x_1 = 2k$

Therefore, eigenvectors corresponding to eigenvalue 0 are given by $k\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Here k is a non-zero parameter.

Step-5

Case (ii):

To find the eigenvector corresponding to the eigenvalue $\lambda_2 = 3$,

Let $\lambda_2 = 3$

Eigen vectors X corresponding to the Eigen value 3 are given by,

$(A\text{-}3I)X = 0$

$$\left(\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 4x_2 + 2x_3 &= 0 \\ -2x_2 + 2x_3 &= 0 \\ -3x_3 &= 0 \end{aligned}$$

Step-6

From the last equation above, $x_3 = 0.$

Substitute $x_3 = 0$ in the first and second equations to get $x_2.$

$x_2 = 0$

Let $x_1 = k(\neq 0)$

Therefore, eigenvectors corresponding to Eigen value 3 are given by $k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$

Here k is a non-zero parameter.

Step-7

Case (iii):

To find the eigenvector corresponding to the eigenvalue $\lambda_3 = 1$,

Let $\lambda_3 = 1$

Eigen vectors X corresponding to the Eigen value 1 are given by,

$(A - I)X = 0$

Continue the calculation,

Step-8

$$\left(\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_3 &= 0 \\ -x_3 &= 0 \end{aligned}$$

Step-9

From the second and third equations above, $x_3 = 0.$

Substitute $x_3 = 0$ in the first equation $2x_1 + 4x_2 + 2x_3 = 0.$

$2x_1 + 4x_2 = 0$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

Let $x_2 = k.$

Then $x_1 = -2k$

Therefore, eigenvectors corresponding to Eigen value 1 are given by $k\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$

Here k is a non-zero parameter.

Step-10

To find the eigenvalues of the matrix $B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$

$$B-\lambda I=\begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$
$$=\begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$$
$$=\begin{bmatrix}-\lambda&0&2\\0&2-\lambda&0\\2&0&-\lambda\end{bmatrix}$$

Step-11

To find the eigenvalues, take the determinant.

$$|B-\lambda I|=\begin{vmatrix}-\lambda&0&2\\0&2-\lambda&0\\2&0&-\lambda\end{vmatrix}$$
$$=(-\lambda)(2-\lambda)(-\lambda)+0+2(0-(2-\lambda)2)$$
$$=\lambda^2(2-\lambda)+2(-4+2\lambda)$$
$$=2\lambda^2-\lambda^3-8+4\lambda$$

$$=-\lambda^3+2\lambda^2+4\lambda-8$$

Step-12

To find the eigenvalues, set $|B-\lambda I|=0$.

$$\lambda^3-2\lambda^2-4\lambda+8=0$$

$$\begin{array}{r|rrrr}2&1&-2&-4&8\\&0&2&0&-8\\\hline&1&0&-4&0\end{array}$$

$$(\lambda-2)(\lambda^2-4)=0$$
$$\lambda=2,\lambda^2=4$$
$$\lambda=2,2,-2$$

Therefore, the eigenvalues are 2, 2, and -2.

Step-13

Case (i):

Let $\lambda_1 = 2$

Eigen vectors X corresponding to the Eigen value 2 are given by,

$(B - 2I)X = 0$

$$\left(\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -2x_1 + 2x_3 &= 0 \\ 2x_1 - 2x_3 &= 0 \\ x_2 &= 0 \end{aligned}$$

From the above two equations, $x_1 = x_3$.

Let $x_3 = x_1 = k$

Step-14

Therefore, eigenvectors corresponding to Eigen value 2 are given by $k\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Here k is a non-zero parameter.

Step-15

Case (ii):

Let $\lambda_2 = 2$

Eigen vectors X corresponding to the Eigen value 2 are given by,

$(B - 2I)X = 0$

$$\left(\begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}-2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-2&0&2\\0&0&0\\2&0&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$-2x_1+2x_3=0$$
$$2x_1-2x_3=0$$
$$x_2=0$$

From the above two equations, $x_1=x_3$.

Let $x_3=x_1=k$

Therefore, eigenvectors corresponding to Eigen value 2 are given by $k\begin{bmatrix}1\\0\\1\end{bmatrix}$.

Here k is a non-zero parameter.

Step-16

Case (iii):

Let $\lambda_3=-2$

Eigen vectors X corresponding to Eigen value -2 are given by,

$$(B+2I)X=0$$

$$\left(\begin{bmatrix}0&0&2\\0&2&0\\2&0&0\end{bmatrix}+2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}2&0&2\\0&4&0\\2&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{aligned} 2x_1 + 2x_3 &= 0 \\ 4x_2 &= 0 \\ 2x_1 + 2x_3 &= 0 \end{aligned}$$

Step-17

From first and third equations,

$$2x_1 + 2x_3 = 0$$

$$x_1 = -x_3$$

From second equation above,

$$4x_2 = 0$$

$$x_2 = 0$$

Put $x_3 = k$

$$\begin{aligned} 2x_1 + 2k &= 0 \\ 2x_1 &= -2k \\ x_1 &= -k, x_2 = 0 \end{aligned}$$

Therefore, eigenvectors corresponding to Eigen value -2 are given by $k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Here *k* is a non-zero parameter.

Step-18

Next, the objective is to check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace.

In *A* matrix the eigenvalues are 0, 3, and 1.

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 0 + 3 + 1 \\ &= 4 \end{aligned}$$

And the trace of the matrix *A*,

Add the diagonal elements of matrix *A*.

$$\begin{aligned} \text{trace } A &= 3 + 1 + 0 \\ &= 4. \end{aligned}$$

Hence, $\lambda_1 + \lambda_2 + \lambda_3$ is equal to the trace of *A*.

Step-19

Next, the objective is to check $\lambda_1\lambda_2\lambda_3$ equals the determinant.

Consider the matrix,

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \lambda_1\lambda_2\lambda_3 &= (3)(1)(0) \\ &= 0 \end{aligned}$$

Step-20

The determinant of the matrix *A*,

$$\begin{aligned} \begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} &= 3(0-0) - 4(0-0) + 2(0-0) \\ &= \boxed{0} \end{aligned}$$

Hence, $\lambda_1\lambda_2\lambda_3$ is equal to the determinant of A.

Step-21

In matrix *B*, the eigenvalues are $\lambda_1 = 2, \lambda_2 = 2, \text{ and } \lambda_3 = -2.$

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 2 + 2 - 2 \\ &= 2 \end{aligned}$$

And the trace of the matrix *B*,

Add the diagonal elements of matrix *B*.

$$\begin{aligned} \text{trace}\, B &= 0 + 2 + 0 \\ &= 2. \end{aligned}$$

Hence, $\lambda_1 + \lambda_2 + \lambda_3$ is equal to the trace of *B*.

Step-22

Next, the objective is to check $\lambda_1\lambda_2\lambda_3$ equals the determinant.

Consider the matrix,

$$B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} \lambda_1\lambda_2\lambda_3 &= (2)(2)(-2) \\ &= -8 \end{aligned}$$

Step-23

The determinant of the matrix *B*,

$$\begin{aligned} \begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{vmatrix} &= 0(0-0)-0(0-0)+2(0-4) \\ &= \boxed{-8} \end{aligned}$$

Hence, $\lambda_1\lambda_2\lambda_3$ is equal to the determinant of *B*.