

## Step-1

(a)

The objective is to compute  $AA^T$  and its Eigen values  $\sigma_1^2, 0$  and its unit Eigen vector  $u_1, u_2$

## Step-2

Consider the matrix:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$

Thus,

$$A^T = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

Therefore,  $AA^T$  is given by,

$$\begin{aligned} AA^T &= \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1+16 & 2+32 \\ 2+32 & 4+64 \end{bmatrix} \\ &= \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix} \end{aligned}$$

The Eigen values are given by,

$$\begin{vmatrix} 17-\lambda & 34 \\ 34 & 68-\lambda \end{vmatrix} = 0$$
$$(17-\lambda)(68-\lambda) - 34 \times 34 = 0$$

This implies;

$$\begin{aligned} 17 \times 68 - 17\lambda - 68\lambda + \lambda^2 &= 34 \times 34 \\ \lambda^2 - 85\lambda + 17 \times 17(4-4) &= 0 \\ \lambda(\lambda - 85) &= 0 \\ \lambda &= 85, 0 \end{aligned}$$

## Step-3

Therefore, the Eigen values are;

$$\sigma_1^2 = 85 \text{ And } 0$$

The eigenvectors are given by,

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The unit eigenvectors are given by,

$$u_1 = \left( \frac{1}{\sqrt{1^2 + 2^2}} \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \left( \frac{1}{\sqrt{5}} \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$u_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

And,

$$u_2 = \left( \frac{1}{\sqrt{(-2)^2 + 1^2}} \right) \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \left( \frac{1}{\sqrt{5}} \right) \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$u_2 = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

**Step-4**

(b)

The objective is to show the signs so that  $Av_1 = \sigma_1 u_1$  and verify the SVD.

## Step-5

Consider the equation:

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$$

## Step-6

Now take the unit Eigen vector  $v_1, v_2$  of the matrix  $A^T A$ ,

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \frac{4}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} \end{bmatrix}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{17}} & -\frac{4}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 4\sqrt{5} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \end{aligned}$$

Hence verified

Thus,  $AA^T = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$ ,  $v_1 = \begin{bmatrix} \frac{1}{\sqrt{17}} \\ 4 \\ \frac{1}{\sqrt{17}} \end{bmatrix}$ ,  $v_2 = \begin{bmatrix} -\frac{4}{\sqrt{17}} \\ 1 \\ \frac{1}{\sqrt{17}} \end{bmatrix}$ , and  $u_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 2 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ ,  $u_2 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ 1 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

## Step-7

(c)

The objective is to determine the four vectors which give orthonormal bases for  $C(A), N(A), C(A^T), N(A^T)$ .

## Step-8

Vector give Orthonormal bases for  $C(A)$  is  $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 2 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

Vector give Orthonormal bases for  $C(A^T)$  is  $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 4 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

Vector give Orthonormal bases for  $N(A)$  is  $\begin{bmatrix} -\frac{4}{\sqrt{17}} \\ 1 \\ \frac{1}{\sqrt{17}} \end{bmatrix}$

Vector give Orthonormal bases for  $N(A^T)$  is  $\begin{bmatrix} -\frac{2}{\sqrt{5}} \\ 1 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$