

Step-1

$$\|A\|^2 = \max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2}$$

Given inequality is

We have to prove the given inequality.

Step-2

We know that the norm of A is the number $\|A\|$ is $\|A\|^2 = \max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2}$ (1)

Let A, B are square matrices and x is any column matrix.

Then

$$\|(A+B)x\| = \|Ax+Bx\| \quad (\text{Since matrix multiplication is associative})$$

$$\leq \|Ax\| + \|Bx\| \quad (\text{By the triangular inequality})$$

Dividing throughout by $\|x\|$ and applying the maximum limit, we get

$$\max_{x \neq 0} \frac{\|(A+B)x\|}{\|x\|} \leq \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} + \max_{x \neq 0} \frac{\|Bx\|}{\|x\|}$$

$$\Rightarrow \|A+B\| \leq \|A\| + \|B\| \quad (\text{Since by (1)})$$

$$\boxed{\|A+B\| \leq \|A\| + \|B\|}$$

Hence