SUSTech

Midterm II for Calculus II in Spring Semester, 2018 (Solutions)

1. (15 pts) Determine whether the following statements are **true** or **false**? No justification is necessary.

   F (a) If both $f_x(x,y)$ and $f_y(x,y)$ exist at (x_0, y_0), then $f(x,y)$ is continuous at (x_0, y_0).

   F (b) Let

   along $y=0$. along $y=kx$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases} \tag{1}$$

   At the point $(0,0)$, $f(x,y)$ is continuous. [Along $y=x$ & $y=2x$, $f(x,y)$ as different limits as $(x,y) \to (0,0)$]

   Y (c) For the $f(x,y)$ as in (1), both $f_x(0,0)$ and $f_y(0,0)$ exist.

   Y (d) Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. $|\vec{u}|\cdot|\vec{v}|\sin\theta = 0$ ✓

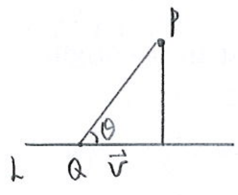
   Y (e) The surface $y^2 - x^2 = z$ is a hyperbolic paraboloid.

   F (f) If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

2. (3 pts) Suppose that the function $f(x,y)$ is differentiable, and $f(0,0) = 1$, $f_x(0,0) = 2$, $f_y(0,0) = 3$. Then $f(x,y) \approx$ <u>$1+2x+3y$</u> when both x and y are small (using the standard linear approximation at $(0,0)$).

3. (10 pts) Find the distance from the point $(1,1,5)$ to the line

$$L: \quad x = 1+t, \quad y = 3-t, \quad z = 2t.$$



Sol: $d = |\vec{QP}|\cdot\sin\theta$

$|\vec{QP}\times\vec{v}| = |\vec{QP}|\cdot|\vec{v}|\cdot\sin\theta$

$\therefore d = \frac{|\vec{QP}\times\vec{v}|}{|\vec{v}|}$

$P = (1,1,5)$, $Q = (1,3,0)$ $(t=0)$

$\vec{QP} = \langle 1,1,5\rangle - \langle 1,3,0\rangle = \langle 0,-2,5\rangle$

$\vec{v} = \langle 1,-1,2\rangle$

$\vec{QP}\times\vec{v} = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = i + 5j + 2k$

$d = \sqrt{1+5^2+2^2}/\sqrt{1+1+2^2} = \sqrt{5}$

4. (10 pts) Find the length of the curve

$$\mathbf{r}(t) = (\sqrt{2}t)\mathbf{i} + (\sqrt{2}t)\mathbf{j} + (1-t^2)\mathbf{k}$$

   from $(0,0,1)$ to $(\sqrt{2}, \sqrt{2}, 0)$.

Sol: $\because r(0) = \langle 0,0,1\rangle$, $r(1) = \langle\sqrt{2},\sqrt{2},0\rangle$, $\therefore 0 \le t \le 1$.

$r'(t) = \sqrt{2}i + \sqrt{2}j - (2t)k$, $|r'(t)| = \sqrt{2+2+4t^2} = 2\sqrt{1+t^2}$

$L = \int_0^1 |r'(t)|\,dt = \int_0^1 2\sqrt{1+t^2}\,dt = \left[t\sqrt{1+t^2} + \ln(t+\sqrt{1+t^2})\right]_0^1 = \sqrt{2} + \ln(1+\sqrt{2})$

$\left(2\int\sqrt{1+t^2}\,dt \overset{t=\tan\theta}{\underset{-\frac{\pi}{2}<\theta<\frac{\pi}{2}}{=\!=}} 2\int\sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta \right.$

$dt = \sec^2\theta\,d\theta$

$\left. = 2\int\sec^3\theta\,d\theta = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| + c \quad \text{P.475 Example6} \right)$

5. (12 pts) Find the normal vector and the curvature for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + (bt)\mathbf{k}, \quad a, b \geq 0, \quad a^2 + b^2 \neq 0.$$

Sol: $r'(t) = (-a\sin t)i + (a\cos t)j + bk = v(t)$

$|r'(t)| = \sqrt{a^2+b^2} = |v(t)|$

Unit tangent vector: $T = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{a^2+b^2}} \langle -a\sin t, a\cos t, b \rangle$

$\frac{dT}{dt} = \frac{1}{\sqrt{a^2+b^2}} \langle -a\cos t, -a\sin t, 0 \rangle$, $k = \left|\frac{dT}{dt}\right| \frac{1}{|v|} = \frac{a}{a^2+b^2}$, $N = \frac{dT}{dt} \Big/ \left|\frac{dT}{dt}\right| = \langle -\cos t, -\sin t, 0 \rangle$

6. (10 pts) Find $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$, if it exists; otherwise give the reason why the limit does not exist.

Sol: Along pathes $y = kx^2$, $k \neq 0$: $f(x,y) = \frac{x^2(kx^2)}{x^4 + k^2x^4} = \frac{k}{1+k^2}$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{k}{1+k^2} = \frac{k}{1+k^2}$.

$\therefore$ By the two-path Test for nonexistence of a limit, we see the limit DNEs.

7. (10 pts) Find $\frac{\partial w}{\partial v}$ when $u = -1$, $v = 2$, if $w = xy + \ln z$, $x = \frac{v^2}{u}$, $y = u + v$, $z = \cos u$.

$$\begin{array}{ccccc} & & w & & \\ & y \swarrow & \downarrow x & \searrow \frac{1}{z} & \\ x & & y & & z \\ \frac{2v}{u} \searrow & & \downarrow 1 & & \swarrow 0 \\ & & v & & \end{array}$$

Sol: $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}$.

$= y\left(\frac{2v}{u}\right) + x$ when $u = -1$, $v = 2$,

$= (u+v)\left(\frac{2v}{u}\right) + \frac{v^2}{u}$ $\quad \frac{\partial w}{\partial v} = -4 + (-4) = -8$.

8. (10 pts) Find the critical points of the function $f(x, y) = x^4 + y^4 + 4xy$, and use the second derivative test to classify each point as one where a saddle, local maximum or local minimum occurs.

Sol: $f_x = 4x^3 + 4y$

$f_y = 4y^3 + 4x$.

Critical pts: $\begin{cases} 4x^3 + 4y = 0 \\ 4y^3 + 4x = 0 \end{cases}$

$\begin{cases} x = 0 \\ y = 0 \end{cases}$, $\begin{cases} x = 1 \\ y = -1 \end{cases}$, $\begin{cases} x = -1 \\ y = 1 \end{cases}$

$f_{xx} = 12x^2$, $f_{xy} = 4$

$f_{yy} = 12y^2$

$H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = 144x^2y^2 - 16$

$(0,0)$: $H = -16 < 0$ Saddle

$(1,-1)$: $H > 0$, $f_{xx} > 0$ loc. min

$(-1,1)$: $H > 0$, $f_{xx} > 0$ loc. min

9. (10 pts) Find the point on the surface $z^2 = xy + 4$ closest to the origin.

Sol: Minimize $f(x,y,z) = x^2 + y^2 + z^2$ s.t. $z^2 = xy + 4$

Substitute $f(x,y) = x^2 + y^2 + xy + 4$. $f_x = 2x + y$, $f_y = 2y + x$.

Critical pt. $2x + y = 2y + x = 0 \Rightarrow x = y = 0. \Rightarrow z^2 = 4, z = \pm 2$.

$\therefore$ The pts $(0,0,\pm 2)$ are the closest on $z^2 = xy + 4$ to $(0,0,0)$

$f(0,0,\pm 2) = 4$

$f(x,y) \geq 2|xy| + xy + 4$

$\geq |xy| + 4 \geq 4$

10. (10 pts) Use Taylor's formula for $f(x, y) = xe^y$ at the origin to find quadratic and cubic approximations of f near the origin.

#9. By Lagrange Multiplier: $g(x,y,z) = xy + 4 - z^2 = 0$

$\nabla f = \langle 2x, 2y, 2z \rangle$, $\nabla g = \langle y, x, -2z \rangle$

$\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda y$, $2y = \lambda x$, $2z = -2\lambda z$

Case 1. $\lambda = -1$: $z = \pm 2$, $x = y = 0$

Case 2. $\lambda \neq -1$, $z = 0$: $x = 2 = -y$ or $x = -2 = -y$

$f(0,0,\pm 2) = 4$, $f(\pm 2, \mp 2, 0) = 8$

$\uparrow$

closest to the origin.

#10. $f_x = e^y$, $f_y = xe^y$, $f_{xx} = 0$, $f_{xy} = e^y$, $f_{yy} = xe^y$

quadratic appr.:

$f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0)$

$+ \frac{1}{2}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$

$= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0]$

$= \boxed{x + xy}$

Cubic appr.: At $(0,0)$: $f_{xxx} = 0$, $f_{xxy} = 0$, $f_{xyy} = e^y|_{y=0} = 1$

$f_{yyy} = xe^y|_{x=0} = 0$.

$f(x,y) \approx$ quadratic $+ \frac{1}{6}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)]$

$f(x,y) \approx \boxed{x + xy + \frac{1}{2}xy^2}$