

## Step-1

$$A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix}, \varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-5}$$

$$(1) \quad A = \begin{bmatrix} 10^{-3} & 1 \\ 1 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 10^3 R_1 \Rightarrow \begin{bmatrix} 10^{-3} & 1 \\ 0 & -999 \end{bmatrix}$$

The resultant is the upper triangular matrix  $U$ .

Applying the same elementary operation on the identity matrix, we get

$$E = \begin{bmatrix} 1 & 0 \\ -1000 & 1 \end{bmatrix}$$

We see that  $EA = U$ .

So,  $A = E^{-1}U$

The inverse of an elementary matrix is elementary.

The inverse of  $E$  is obtained by applying  $R_2 \rightarrow R_2 + 10^3 R_1$  on the identity matrix.

$$\text{i.e., } E^{-1} = \begin{bmatrix} 1 & 0 \\ 1000 & 1 \end{bmatrix} = L \quad \text{the lower triangular matrix such that } A = LU$$

## Step-2

$$(2) \text{ Repeating the above procedure, when } A = \begin{bmatrix} 10^{-6} & 1 \\ 1 & 1 \end{bmatrix}, \text{ we can write it as the product of lower and upper triangular matrices } LU = \begin{bmatrix} 10^{-6} & 1 \\ 0 & -10^6 + 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 10^6 & 1 \end{bmatrix}$$

$$(3) \text{ When } A = \begin{bmatrix} 10^{-9} & 1 \\ 1 & 1 \end{bmatrix}, \text{ we get } LU = \begin{bmatrix} 10^{-9} & 1 \\ 0 & -10^9 + 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 10^9 & 1 \end{bmatrix}$$

$$(4) \text{ When } A = \begin{bmatrix} 10^{-12} & 1 \\ 1 & 1 \end{bmatrix}, \text{ we get } LU = \begin{bmatrix} 10^{-12} & 1 \\ 0 & -10^{12} + 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 10^{12} & 1 \end{bmatrix}$$

$$(5) \text{ When } A = \begin{bmatrix} 10^{-15} & 1 \\ 1 & 1 \end{bmatrix}, \text{ we get } LU = \begin{bmatrix} 10^{-15} & 1 \\ 0 & -10^{15} + 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 10^{15} & 1 \end{bmatrix}$$

### Step-3

$$Ax=b \text{ is given by } \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+\varepsilon \\ 1 \end{bmatrix}$$

This can conveniently be written as  $x = A^{-1}b$

$$A^{-1} = \frac{1}{\varepsilon-1} \begin{bmatrix} 1 & -1 \\ -1 & \varepsilon \end{bmatrix} \text{ and so, } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\varepsilon-1} \begin{bmatrix} 1 & -1 \\ -1 & \varepsilon \end{bmatrix} \begin{bmatrix} 1+\varepsilon \\ 1 \end{bmatrix}$$

$$= \frac{1}{\varepsilon-1} \begin{bmatrix} \varepsilon \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\varepsilon}{\varepsilon-1} \\ \frac{1}{1-\varepsilon} \end{bmatrix} \quad (6)$$

### Step-4

Using  $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$  in (6), we get the solution of the system  $Ax = b$  as

$$(7) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{10^{-3}}{10^{-3}-1} \\ \frac{1}{1-10^{-3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{-999} \\ \frac{1000}{999} \end{bmatrix} = \begin{bmatrix} -0.001 \\ 1.001 \end{bmatrix}$$

$$(8) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{10^{-6}}{10^{-6}-1} \\ \frac{1}{1-10^{-6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{-999999} \\ \frac{1000}{999999} \end{bmatrix} = \begin{bmatrix} -10^{-6} \\ 1+10^{-6} \end{bmatrix}$$

$$(9) \text{ Similarly, when } \varepsilon = 10^{-9}, \text{ we get } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10^{-9} \\ 1+10^{-9} \end{bmatrix}$$

$$(10) \text{ When } \varepsilon = 10^{-12}, \text{ we get } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10^{-12} \\ 1+10^{-12} \end{bmatrix}$$

$$(11) \text{ When } \varepsilon = 10^{-15}, \text{ we get } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10^{-15} \\ 1+10^{-15} \end{bmatrix}$$

Step-5

The true solution of the system is  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and so, the error in each case is

$\varepsilon =$	$10^{-3}$	$10^{-6}$	$10^{-9}$	$10^{-12}$	$10^{-15}$
$\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} =$	$\begin{bmatrix} -1-10^{-3} \\ 10^{-3} \end{bmatrix}$	$\begin{bmatrix} -1-10^{-6} \\ 10^{-6} \end{bmatrix}$	$\begin{bmatrix} -1-10^{-9} \\ 10^{-9} \end{bmatrix}$	$\begin{bmatrix} -1-10^{-12} \\ 10^{-12} \end{bmatrix}$	$\begin{bmatrix} -1-10^{-15} \\ 10^{-15} \end{bmatrix}$