

## Step-1

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

The given Hilbert matrix is

Suppose  $A = [A \mid I]$

$$= \left[ \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array} \right]$$

## Step-2

Applying the row operations on both parts of the matrix, to reduce the square matrix on the left to identity matrix, parallelly, the matrix on the right changes to a new matrix which is the inverse of  $A$ .

$$\left. \begin{array}{l} R_2 \rightarrow R_2 - \frac{1}{2}R_1 \\ R_3 \rightarrow R_3 - \frac{1}{3}R_1 \end{array} \right\} \rightarrow \left[ \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{array} \right]$$

## Step-3

Continuation to the above

$$R_3 \rightarrow R_3 - R_2 \Rightarrow \left[ \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array} \right]$$

$$\left. \begin{array}{l} R_2 \rightarrow R_2 - 15R_3 \\ R_1 \rightarrow R_1 - 60R_3 \end{array} \right\} \rightarrow \left[ \begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & -9 & 60 & -60 \\ 0 & \frac{1}{12} & 0 & -3 & 16 & -15 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array} \right]$$

## Step-4

Continuation to the above

$$R_1 \rightarrow R_1 - 6R_2 \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & \frac{1}{12} & 0 & -3 & 16 & -15 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{array} \right]$$

$$R_2(12), R_3(180) \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array} \right] \approx [I \quad A^{-1}]$$

Thus, the given matrix is reduced as  $A = [I | A^{-1}]$  where  $A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$

Hence the inverse of the given Hilbert matrix is  $\boxed{A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}}$ .