Step-1

Let $B = \left\{ u = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix} v = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 2 \end{pmatrix} \right\}$ and S is spanned by B

Suppose $w = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ is in the orthogonal complement of S.

Then we see that $w^T u = 0, w^T v = 0$

Step-2

That is $x + 2y + 2z + 3t = 0$

$x + 3y + 3z + 2t = 0$

We write this system as the product of matrices $Ax = 0$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying row operation $R_2 \to R_2 - R_1$ on the coefficient matrix A, we get $\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

We easily see that all the entries below the principal diagonal are zero and so, this is the row reduced form.

Step-3

We rewrite the homogeneous equations from this as

$$\begin{aligned} x + 2y + 2z + 3t &= 0 \\ y + z - t &= 0 \end{aligned}$$

So, we write from the 2nd equation that $y = t$ â€“ z

Using this in the 1st, we get x = -2(t â€“ z) -2z â€“ 3t

= - 5 t

Using these, the solution set is $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -5t \\ t-z \\ z \\ t \end{pmatrix}$

$$= z\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

When $z = t = 1$, we get the fundamental solutions $\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}$ span $S^{\perp}$