Calculus(I) Final Review 2022 Fall

SUSTech Learning center Calculus Team

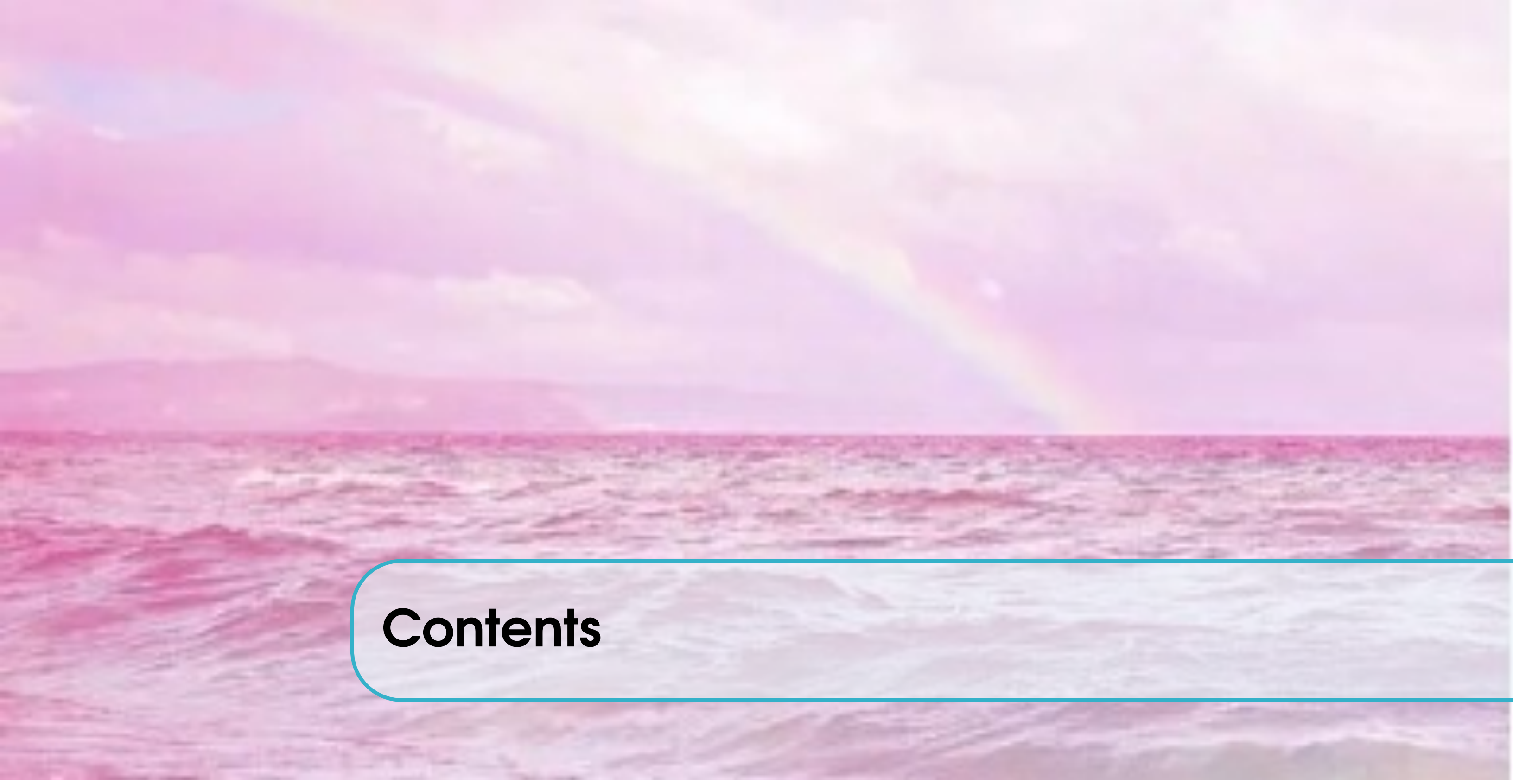
Cover the knowledge after mid-exam

Perface

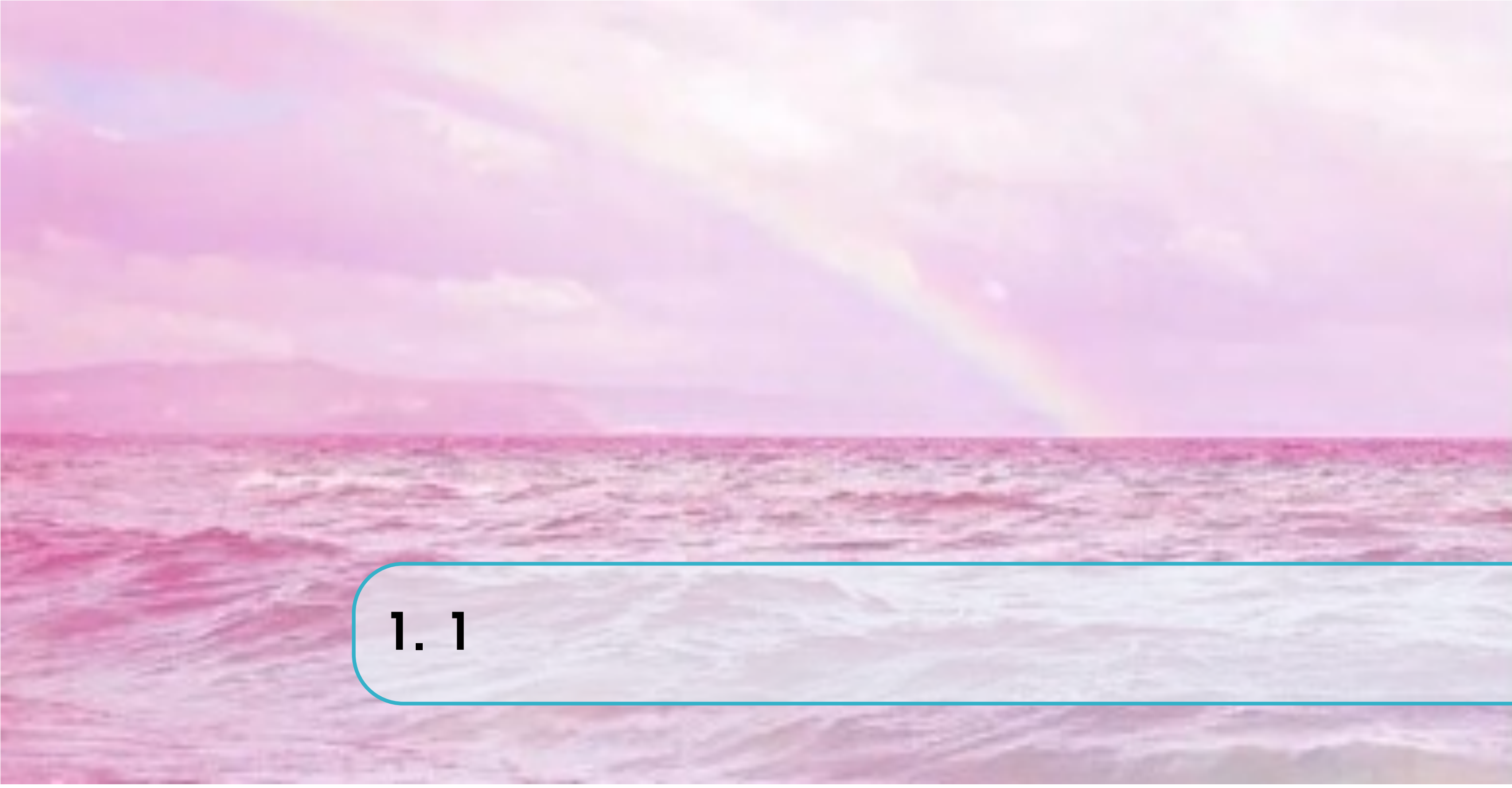
本复习攻略文档由南科大学习中心第十期互助课堂高数小组提供,其中前五章并未罗列，希望同学们可以自行归纳。期中考试后的内容具体分工为：

左子腾 第八章
张羽乐 第七章
张宇哲 第六、九章

另，本学期期末考试受到返乡政策冲击，但提醒同学们：
无论何时考，无论 A 卷 B 卷，实力才是硬道理。希望各位可以好好复习，并预祝期末考得好成绩，顺祝新春快乐！



Contents



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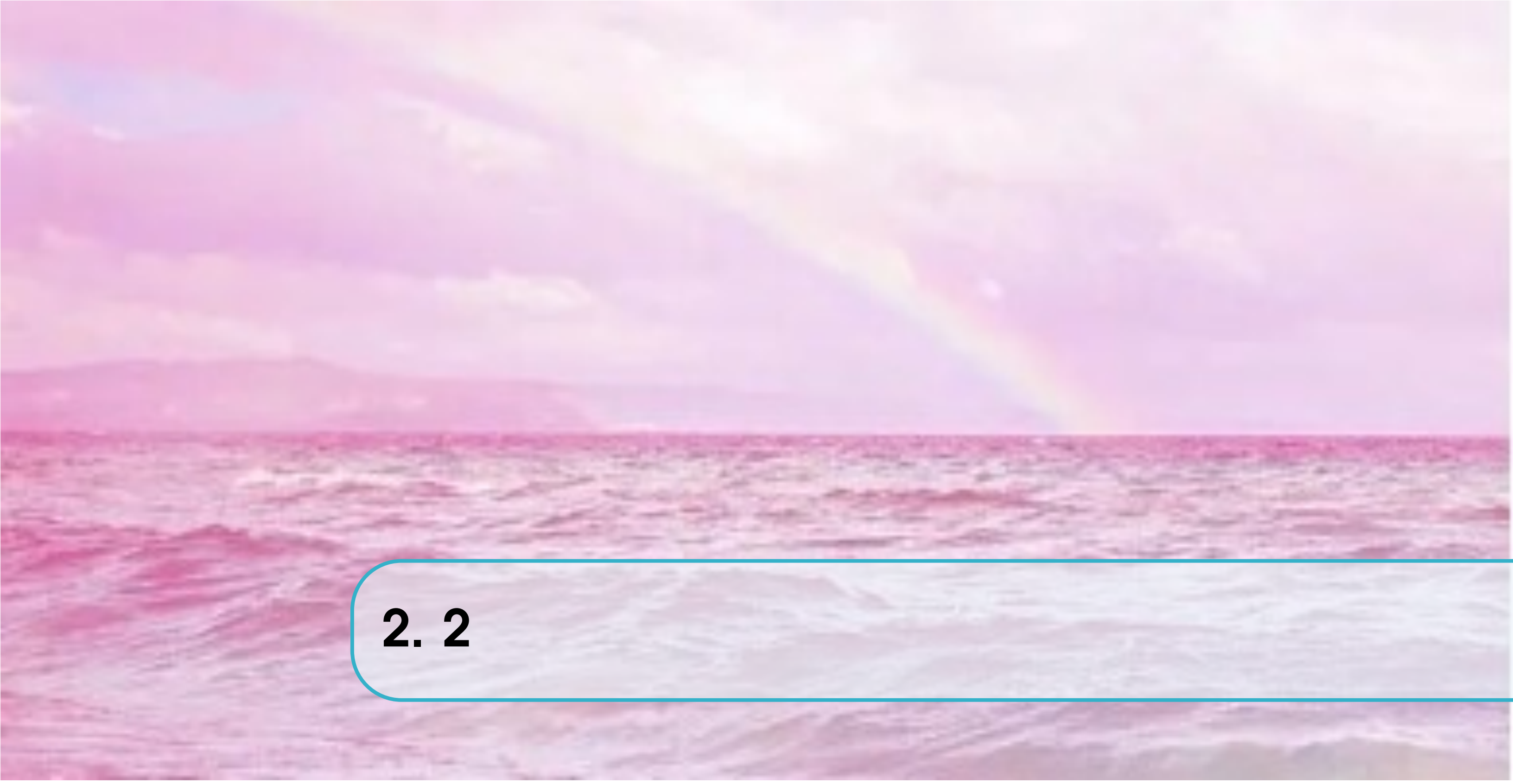

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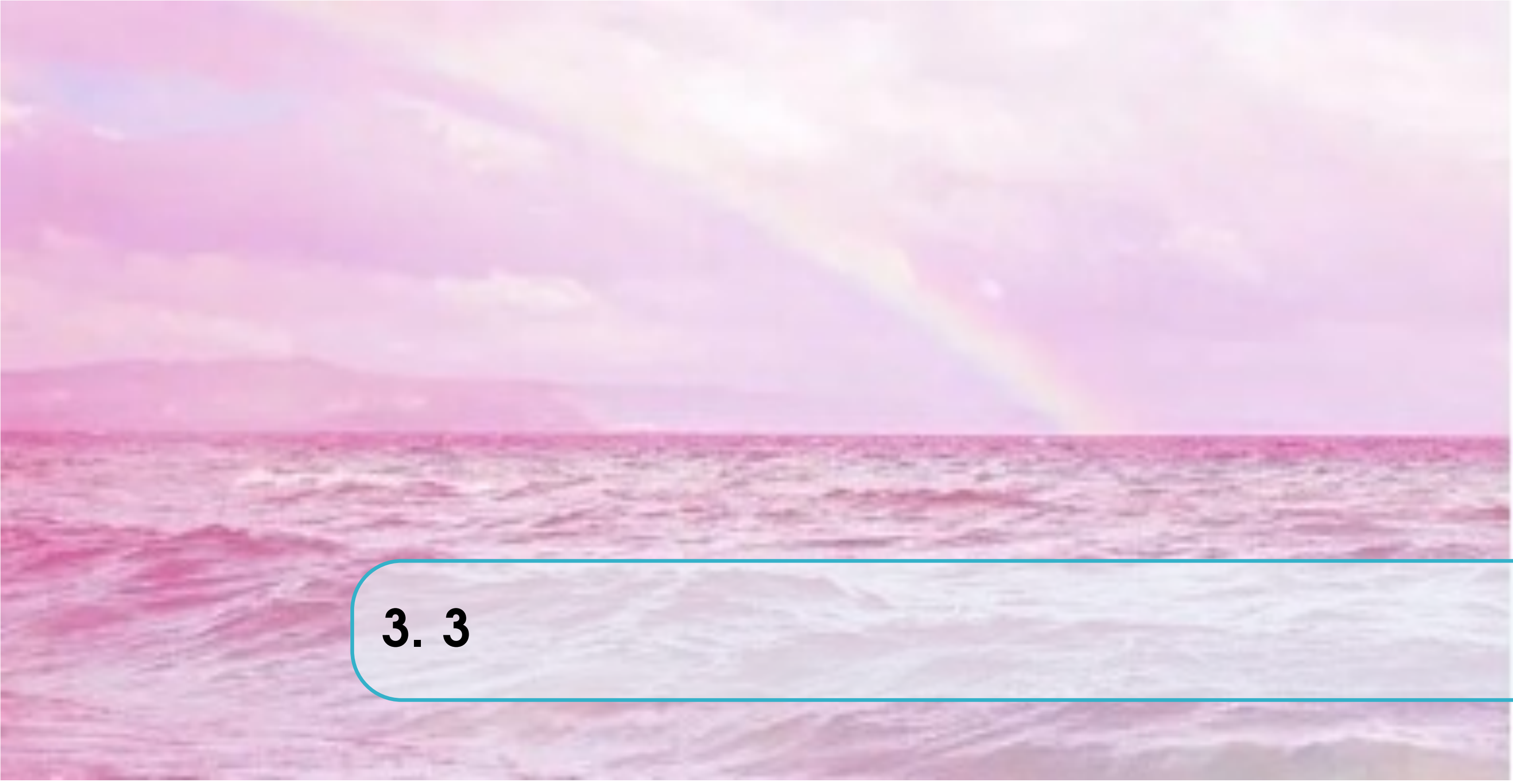

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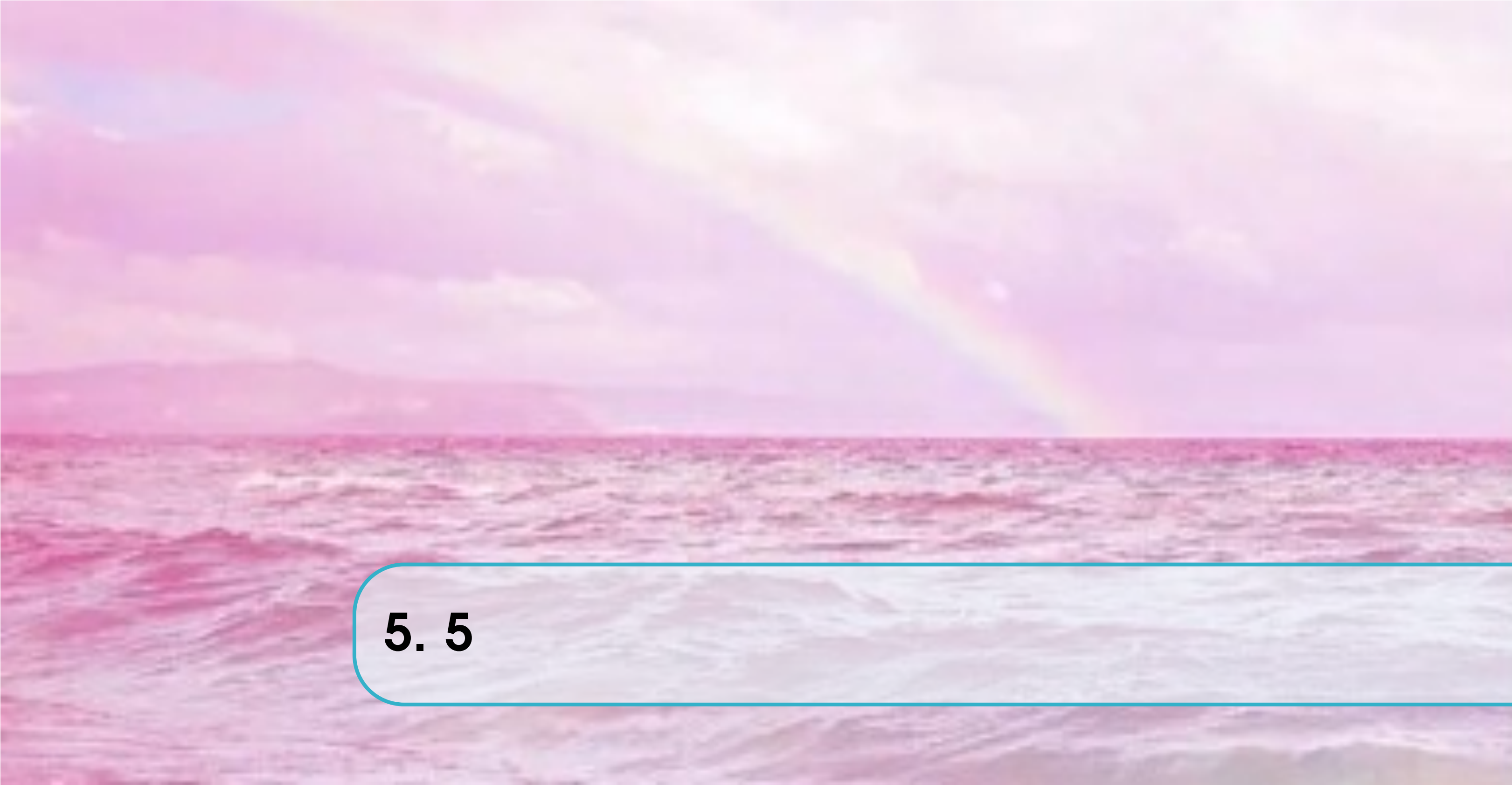


5. 5

6. Applications of Definite Integrals

6.1 一些公式

6.1.1 1维

Theorem 6.1.1 — Arc Length. If f' is continuous on $[a,b]$, then the length (arc length) of the curve $y=f(x)$ from the point $A=(a,f(a))$ to the point $B=(b,f(b))$ is the value of the integral $L=\int_a^b\sqrt{1+[f'(x)]^2}dx=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx.$

R $L=\lim_{n\to\infty}\sum_{k=1}^n L_k=\lim_{n\to\infty}\sum_{k=1}^n\sqrt{(\Delta x_k)^2+(\Delta y_k)^2}$

$$=\lim_{n\to\infty}\sum^n\sqrt{(\Delta x_k)^2+(f'(c_k)\Delta x_k)^2}=\lim_{n\to\infty}\sum_{k=1}^n\sqrt{1+[f'(c_k)]^2}\Delta x_k.$$

6.1.2 2维

Theorem 6.1.2 — Areas of Surfaces of Revolution. 1. If the function $f(x)\geq 0$ is continuously differentiable on $[a,b]$, the area of the surface generated by revolving the graph of $y=f(x)$ about the x-axis is

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\boxed{\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}dx.}$$

2. If the function $x=g(y)\geq 0$ is continuously differentiable on [c, d], the area of the surface

generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \boxed{\int_c^d 2\pi g(y) \sqrt{1+[g'(y)]^2} dy}$$

R 此处记忆其一即可

如果函数值为恒负，需加绝对值。

6.1.3 3维

Theorem 6.1.3 — Areas of Surfaces of Revolution

. Slicing by Parallel Planes:

The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b :

$$\boxed{V = \int_a^b A(x)dx}$$

Solids of Revolution (The Disk Method):

(1) Volume by Disk for Rotation About the x-axis:

$$V = \int_a^b A(x)dx = \int_a^b \pi[R(x)]^2 dx$$

(2) Volume by Disk for Rotation About the y-axis:

$$V = \int_C^d A(y)dy = \int_c^d \pi[R(y)]^2 dy$$

Solids of Revolution(The Washer Method):

(1) Volume by Washers for Rotation About the x-axis:

$$V = \int_a^b A(x)dx = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx.$$

(2) Volume by Washers for Rotation About the y-axis:

$$V = \int_c^d A(y)dy = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy.$$

R 垫圈法是圆盘法的推广。

解题时，千万记住：有二维图的话就在脑海中想一想;无二维图的话就先手画在草稿纸上然后想想空间几何体。

R 这些公式的连续性条件可以和中值定理部分补充记忆。

■ **Example 6.1** (Equivalence of the washer and shell methods for finding volume.)
Let f be differentiable and increasing on the interval $a \le x \le b$, with $a > 0$, and suppose that f has a differentiable inverse, f^{-1}. Revolve about the y-axis the region bounded by the graph of f and the lines $x = a$ and $y = f(b)$ to generate a solid. Then the values of the integrals given by the washer and shell methods for the volume have identical values:

$$\int_{f(a)}^{f(b)} \pi \left(\left(f^{-1}(y)\right)^2 - a^2 \right) dy = \int_a^b 2\pi x (f(b) - f(x)) dx.$$

To prove this equality, define

$$W(t) = \int_{f(a)}^{f(t)} \pi \left(\left(f^{-1}(y)\right)^2 - a^2 \right) dy$$
$$S(t) = \int_a^t 2\pi x (f(t) - f(x)) dx$$

■

Proof. 按照题后instruction做即可。
此题给了我们一种很好的思路，证明同一闭区间上的两个不好直接计算定积分相等，可以考察其导数相等并且存在一点被积函数函数值相同。（一种转化思想） ■

6.2 Work, Fluid Forces, Moments and Centers of Mass

这里的题目可以结合3.7:Implicit Differentiation一起复习。
倘若考，也仅是考察使用公式，不会涉及用微元法等物理思想去构建模型，所以读者不必太担心。

Definition 6.2.1 — work. The work done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x) dx$$

Definition 6.2.2 — Fluid Force (Against a Vertical Flat Plate). Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force is

$$F = \int_a^b w \cdot (\text{ strip depth }) \cdot L(y) dy$$

Definition 6.2.3 — moment and COM. Moment about the x-axis: $M_x = \int \tilde{y} dm$ Moment about the y-axis: $M_y = \int \tilde{x} dm$ Center of mass: $\bar{x} = \frac{M_y}{M}$, $\bar{y} = \frac{M_x}{M}$,(mass: $M = \int dm$)

Definition 6.2.4 — rotational inertia. to be updated

6.3 绕非x、y轴的旋转问题

绕x轴、y轴属于特殊情形，一般情形下，可使用距离公式或者旋转矩阵。

Definition 6.3.1 设两个点 AΔB 以及坐标分别为 $A(x_1,y_1)\Delta B(x_2,y_2)$，则 A 和两点之间的距离为:

$$|AB| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

直线上两点间的距离公式:
设直线 l 的方程为 $y = kx+m$，点 $P_1(x_1,y_1)$ﬁ$P_2(x_2,y_2)$ 为该线上任意两点，则

$$|P_1P_2| = \sqrt{1+k^2}|x_1-x_2| = \sqrt{1+\frac{1}{k^2}}|y_1-y_2|$$

Definition 6.3.2 在二维空间中，旋转可以用一个单一的角 θ 定义。作为约定，正角表示顺时针旋转。把笛卡尔坐标的列向量关于原点逆时针旋转 θ 的矩阵是:

$$M(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \exp\left(\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

把X-Y旋转θ后得到S-T坐标系，那么点$P(x,y)$在S-T坐标系下的坐标为$P_{S-T}(s,t)$为用行列式表达如下：

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

考试时候也可以用“笨”方法，若是二次函数，直接用三点法求出旋转后的解析式。

那definition6.3.1怎么用呢，只需写出函数关于旋转轴的距离表达式，即关于x的函数，此距离作为半径，对应着normal表达式里的$f(x), g(y), R(x), r(x)$

而Definition6.3.2中的行列式表达时，把y的显性表达式代入，比如$y = 3x^2+2x+1$,如果绕着$y = \sqrt{3}x$旋转，那么结果是

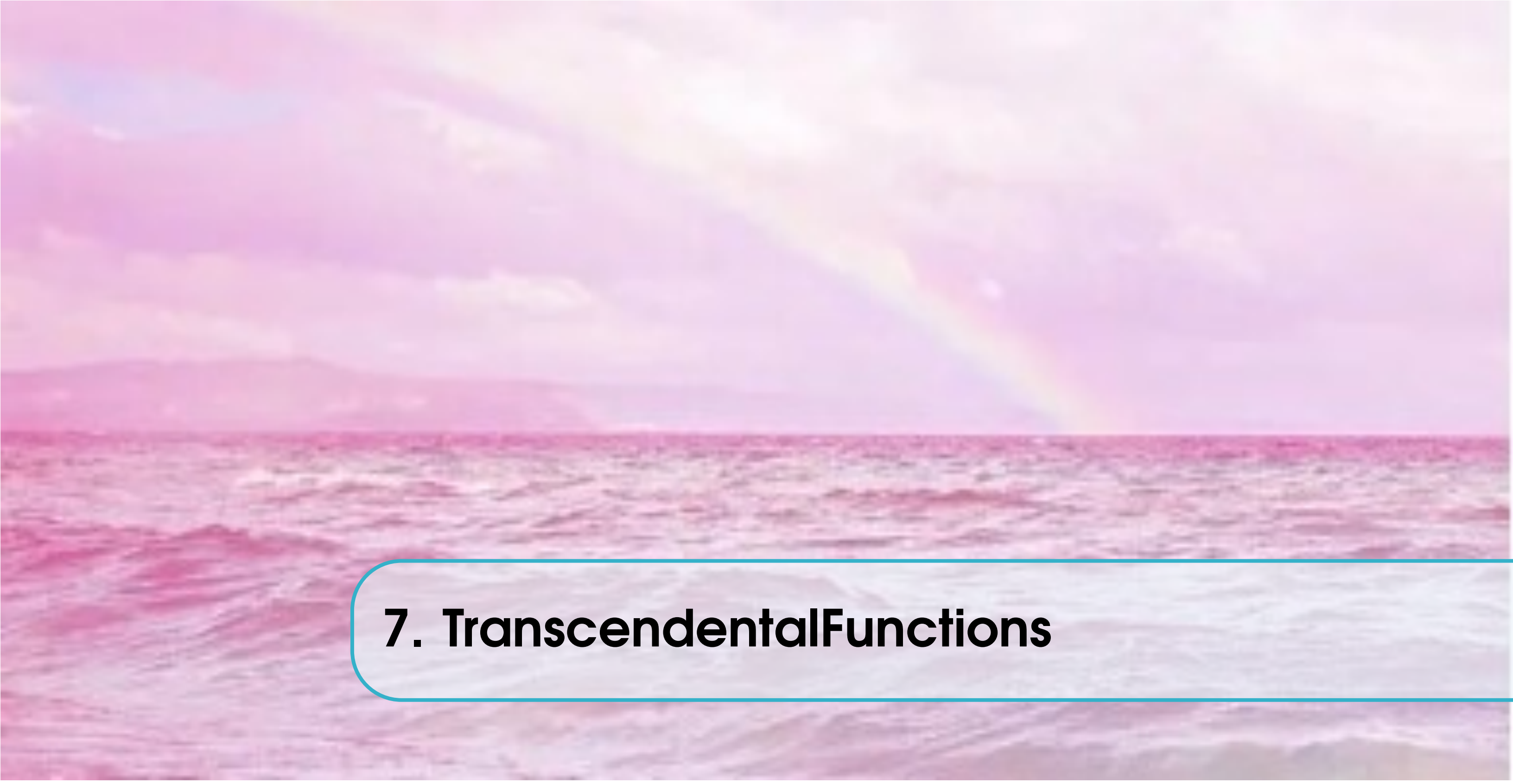
$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} x \\ 3x^2+2x+1 \end{pmatrix}$$

注意此时坐标应是$(s(x),t(x))$，即$(\frac{3\sqrt{3}}{2}x^2+(\sqrt{3}+\frac{1}{2})x+\frac{\sqrt{3}}{2}, \frac{3}{2}x^2+(1-\frac{\sqrt{3}}{2})x+\frac{1}{2})$，不放心的话，

这边建议单位化，就是把t和s的关系找到，会很复杂。这个地方的话直接就可以使用参数方程进行公式的代入求解。（千万注意，旋转之后不一定是函数）

■ **Example 6.2** Find the volume of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about the line $y = x$

■



7. TranscendentalFunctions

7.1 One-to-One Functions

1. A function $f(x)$ is one-to-one on a domain D if $\forall x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, or if $\forall f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
2. The Horizontal Line Test for One-to-One Functions A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

7.1.1 Inverse function

Definition 7.1.1 Suppose that f is a one-to-one function on a domain D with range R. The inverse function f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

3. The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $\left(f^{-1}\right)'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$: $\left(f^{-1}\right)'(b) = \frac{1}{f'(f^{-1}(b))}$ or $\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$.

R yljj温馨提示：请注意范围

7.2 ln x

7.2 lnx

formula:

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}dt, \qquad x > 0.$$

THEOREM 2—Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ For r rational

1. The Derivative of $y = \ln x$: $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, u > 0;$
$\Rightarrow \frac{d}{dx}\ln|x| = \frac{1}{x}, x \neq 0;$ $\quad \frac{d}{dx}\ln(bx) = \frac{1}{x}, \quad bx > 0.$

2. The Integral $\int \frac{1}{u}du$: If u is a differentiable function that is never zero,
$\int \frac{1}{u}du = \ln|u| + C \Rightarrow \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C.$

$\int \tan u du = \ln|\sec u| + C;$ $\quad \int \sec u du = \ln|\sec u + \tan u| + C;$

$\int \cot u du = \ln|\sin u| + C;$ $\quad \int \csc u du = -\ln|\csc u + \cot u| + C.$

$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}.$

Remember!

7.3 exp(x)

formula

THEOREM 3 For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^r = e^{rx_1}$, if r is rational

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0)$$
$$\ln(e^x) = x \qquad (\text{all } x)$$

For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}.$$

$$\int e^u du = e^u + C$$

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}, a > 0$$

$$\int a^u du = \frac{a^u}{\ln a} + C.$$

重要极限 $e = \lim_{x\to 0}(1+x)^{1/x}.$

- $f(x)^{g(x)}$ 的极限求法：

 1. 化到 $(1+h(x))^{\frac{1}{h(x)}}$ 的形式，其中 $h(x) \to 0$

 2. 使用 lnX： $\ln f(x)^{g(x)} = g(x)\ln f(x)$ 后求极限

7.4 可分离变量微分方程

Definition 7.4.1 Separable Differentiable Equations: If the differential equation has the form: $\frac{dy}{dx} = g(x)H(y)$, then let $H(y) = \frac{1}{h(y)}$, $\Rightarrow \frac{dy}{dx} = \frac{g(x)}{h(y)} \Rightarrow \int h(y)dy = \int g(x)dx$.

Application:

(1)Unlimited Population Growth: $\frac{dy}{dt} = ky, y(0) = y_0 \Rightarrow y = y_0 e^{kt}$.

(2)Radioactivity: $\frac{dy}{dt} = -ky, k > 0, y(0) = y_0 \Rightarrow y = y_0 e^{-kt}, k > 0$ and we know the Half-life $= \frac{\ln 2}{k}$.

(3) Heat Transfer: Newton's Law of cooling:

$$\frac{dH}{dt} = -k(H - H_s), H(0) = H_0, \text{ let } y = H - H_s \Rightarrow \frac{dy}{dt} = -ky$$

7.5 洛必达法则

Theorem 7.5.1 — L'Hôpital's Rule. Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

type: $\frac{0}{0}$ $\frac{\infty}{\infty}$ $\infty - \infty$ $\infty \cdot 0$

正常使用洛必达直到无法使用

将一个式子作为分母后变到 $\frac{0}{0}$ 或 $\frac{\infty}{\infty}$ 类型

多为合并为一个分式后变到 $\frac{0}{0}$ 类型

Tips: 洛必达前先处理式子到最简形式

way 1: e.g 原 $= \lim_{x \to a} A \cdot B$

若 A 在 $x=a$ 时为常数则直接代入

way 2: 等价无穷小

$x \to 0$: $\sin x \sim x$ $\tan x \sim x$

$\arcsin x \sim x$ $\arctan x \sim x$

$e^x \sim x+1$ $\ln(x+1) \sim x$

Tips: 洛必达有失效情况、

洛必达后极限不存在 $\nRightarrow$ 极限不存在

7.6 反三角

Definition 7.6.1 — the inverse Trigonometric Functions.. (1) $y=\sin^{-1}x$ is the number in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\sin y=x$. (2) $y=\cos^{-1}x$ is the number in $[0,\pi]$ for which $\cos y=x$. (3) $y=\tan^{-1}x$ is the number in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ for which $\tan y=x$. (4) $y=\cot^{-1}x$ is the number in $(0,\pi)$ for which $\cot y=x$. (5) $y=\sec^{-1}x$ is the number in $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ for which $\sec y=x$. (6) $y=\csc^{-1}x$ is the number in $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ for which $\csc y=x$.

注意反三角函数的范围。

7.7 Big O and Small O

Definition 7.7.1 Let $f(x)$ and $g(x)$ be positive for x sufficiently large. 1. f grows faster than g as $x\to\infty$ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$$

or, equivalently, if

$$\lim_{x\to\infty}\frac{g(x)}{f(x)}=0.$$

We also say that g grows slower than f as $x\to\infty$. 2. f and g grow at the same rate as $x\to\infty$ if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=L$$

where L is finite and positive.

Definition 7.7.2 Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is of at most the order of g as $x\to\infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)}\leq M,$$

for x sufficiently large. We indicate this by writing $f=O(g)$ (" f is big-oh of g ").

Definition 7.7.3 A function f is of smaller order than g as $x\to\infty$ if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=0$. We indicate this by writing $f=o(g)$ (" f is little-oh of g "). If f grows at the same rate as g as $x\to\infty$, and g grows at the same rate as h as $x\to\infty$, then f grows at the same rate as h as $x\to\infty$.

8. Techniques of Integration

8.1 Using Basic Integration Formulas

Theorem 8.1.1 — 常见积分式

.

$$\int kdx = kx + c \quad \int x^n dx = \frac{x^{n+1}}{n+1} + c(n \neq -1) \quad \int \frac{dx}{x} = \ln|x| + c \quad \int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c(a > 0) \quad \int \sin x dx = -\cos x + c \quad \int \cos x dx = \sin x + c$$

$$\int \sec^2 x dx = \tan x + c \quad \int \sec x \tan x dx = \sec x + c$$

$$\int \tan x dx = \ln|\sec x| + c \quad \int \sec x dx = \ln|\sec x + \tan x| + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(\frac{x}{a}) + c \quad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c \quad \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}(\frac{x}{a}) + c$$

Theorem 8.1.2 — 常见基本积分方法

. 凑微分，三角代换（二倍角公式，积化和差，三角函数平方和1关系转换）
换元，列项……

8.2 Integration by Parts

$$\int udv = uv - \int vdu \quad \int_a^b udv = uv|_a^b - \int_a^b vdu$$

主要类型（适用形式）：降次和循环。

■ **Example 8.1** $\int x^2e^xdx = \int x^2de^x = x^2e^x - 2\int xe^xdx = x^2e^x - 2\int xde^x = x^2e^x - 2(xe^x - e^x) + C$
（这里的分部积分的作用是对x^2降次） ■

■ **Example 8.2** $\int e^x\sin xdx = \int \sin xde^x = e^x\sin x - \int e^x\cos xdx = e^x\sin x - \int \cos xde^x$

$$= e^x\sin x - (e^x\cos x - \int e^xd\cos x) = e^x(\sin x - \cos x) - \int e^x\sin xdx$$

$$\int e^x sinxdx = \frac{e^x}{2}(sinx - cosx) + C$$

(这里分部积分的作用是出现与原式相同的循环式)

■

必须记住的结论

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x\,\mathrm{d}x \qquad \text{（J.Wallis公式）}$$

$$= \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2} & \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \quad n \text{ 为正偶数} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2} & \frac{4}{5}\cdot\frac{2}{3}, \quad n \text{ 为大于 1 的正奇数} \end{cases}$$

8.3 Trigonometric Integrals

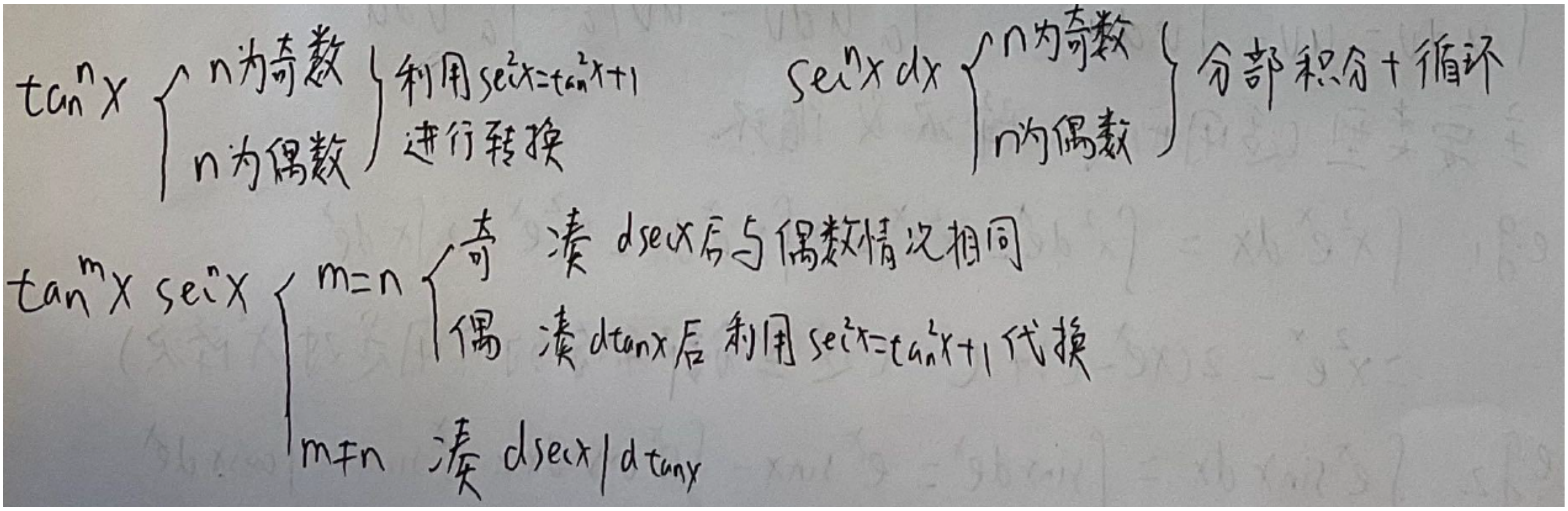
主要方法:二倍角公式,积化和差,代换式如$sin^2x + cos^2x = 1$, $sec^2x = tan^2x + 1$,凑微分如：$secxdx = dtanx, sectanxdx = dsecx$

左子腾同学满绩笔记一览

主要类型：$\sin^n x$
- n为奇数 => 凑 $d\cos x$，后将整个式子变化与 $\cos x$ 相关的函数
- n为偶数 => 二倍角降次

$\cos^n x$
- n为奇数 => 凑 $d\sin x$，与上同理
- n为偶数 => 二倍角降次

$\sin^m x \cos^n x$
- $m=n$
  - 奇
  - 偶
  - 转化成 $(\sin 2x)^p$ 形式后与 $\sin^n x$ 的情况同理
- $m \neq n$
  - 都为奇
  - 都为偶
  - 先用 $\sin 2x = 2\sin x\cos x$，再用 $\cos 2x = 2\cos^2 x - 1$
  - 奇+偶 凑 $d\sin x$/$\cos x$进行转换



$\tan^n x$ { n为奇数 / n为偶数 } 利用 $\sec^2 x=\tan^2 x+1$ 进行转换

$\sec^n x\,dx$ { n为奇数 / n为偶数 } 分部积分+循环

$\tan^m x\sec^n x$
- $m=n$
  - 奇 凑 $d\sec x$ 后与偶数情况相同
  - 偶 凑 $d\tan x$ 后 利用 $\sec^2 x=\tan^2 x+1$ 代换
- $m\neq n$ 凑 $d\sec x \,/\, d\tan x$

■ **Example 8.3**

$$
\begin{aligned}
&\int \sin^2 x\cos^4 x dx\\
&=\frac{1}{4}\sin^2 2x\cos^2 x dx=\frac{1}{4}\int \sin^2 2x\frac{1+\cos 2x}{2}dx\\
&=\frac{1}{8}\int \sin^2 2x dx+\frac{1}{8}\int \sin^2 2x\cos 2x dx\\
&=\frac{1}{16}\int (1-\cos 4x)dx+\frac{1}{16}\int \sin^2 2x d(\sin 2x)\\
&=\frac{1}{16}x-\frac{\sin 4x}{64}+\frac{\sin^3 2x}{48}+C
\end{aligned}
$$

■

■ **Example 8.4**

$$
\begin{aligned}
&\int \sin^2 x\cos^3 x dx\\
&=\int \sin^2 x\cos^2 x d(\sin x)\\
&=\int \sin^2 x\left(1-\sin^2 x\right)d(\sin x)\\
&=\int \left(\sin^2 x-\sin^4 x\right)d(\sin x)\\
&=\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C
\end{aligned}
$$

■

Example 8.5

$$
\begin{aligned}
&\int \tan^4 x \sec^4 x dx \\
&= \int \tan^4 x \sec^2 x d(\tan x) \\
&= \int \tan^4 x \left(1+\tan^2 x\right) d \tan x \\
&= \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^2 x + C
\end{aligned}
$$

■

R 总结方法为基本息路,具体题目仍应具体分析.

8.4 Trigonometric Substitutions

基本思路：利用$sec^2x = tan^2x+1, Sin^2x+cos^2x = 1$进行代换.
如:$\sqrt{x^2+a^2}$,令$x = atanu$;$\sqrt{a^2-x^2}$,令$x = aSinu/aCosu$; $\sqrt{x^2-a^2}$ 令 $x = aSecu$

R 注：进行三角换元或任何形式的换元时，务必注意自变量的取值范围。

8.5 Integration of Rational Functions by Partial Fractions

分母中若有因式$(x-r)^k$,则拆项后有其中$A_1,A_2,...,A_k$ 都是常数。
分母中若有因式$(x+px+q)^k$,其中$p^2-4q<0$,则拆项后有

$$\frac{B_1x+C_1}{(x^2+px+q)^k}+\frac{B_2x+C_2}{(x^2+px+q)^k-1}+...+\frac{B_kx+C_k}{(x^2+px+q)^k}$$

其中B_i, C_i都是常数(i=1,2,...,k)

R 若分子的最高次幂大于或等于分母的最高次幂,则可进行多项式除法后再进行因式裂项。

8.6 Integral Tables and Computer Algebra Systems

略略略

8.7 Numerical Integration

Theorem 8.7.1 — The Trapezoidal Rule. To approximate $\int_a^b f(x)dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y 's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$$

where $\Delta x = (b-a)/n$.

Theorem 8.7.2 — Simpson's Rule. To approximate $\int_a^b f(x)dx$, use

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right).$$

The y 's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b-a)/n$.

Theorem 8.7.3 — Error Estimates in the Trapezoidal and Simpson's Rules. If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a,b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \le \frac{M(b-a)^3}{12n^2}.$$

(Trapezoidal Rule)

If $f^{(4)}$ is continuous and M is any upper bound for the values of $\left|f^{(4)}\right|$ on $[a,b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \le \frac{M(b-a)^5}{180n^4}.$$

(Simpson's Rule)

8.8 Improper Integrals

Definition 8.8.1 Integrals with infinite limits of integration are improper integrals of Type I.

1. If f(x) is continuous on $[a,+\infty)$, then $\int_a^\infty f(x)dx = lim_{b\to\infty}\int_a^b f(x)dx$.

2. If f(x) is Continuous on $(-\infty,b]$, then $\int_{-\infty}^b f(x)dx = lim_{a\to\infty}\int_a^b f(x)dx$.

3. If f(x) is cuntinuous on $(-\infty,+\infty)$, then $\int_{-\infty}^{+\infty} f(x)dx = \int_{-infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx$, whare c is any real nvmber.

Limit is finite,then Converge.

Limit fails to exilt,then diverge.

Example 8.6 $\int_1^{\infty} \frac{1}{x^p} d_x = \begin{cases} \frac{1}{p-1}, & p>1 \\ diverge, & p \le 1 \end{cases}$ ■

Definition 8.8.2 Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.

1. If $f(x)$ is continuous on $(a,b]$ and discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx.$$

2. If $f(x)$ is continuous on $[a,b)$ and discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{c\to b^b} \int_a^c f(x)dx.$$

3. If $f(x)$ is discontinuous at c, where $a<c<b$, and continuous on $[a,c)\cup(c,b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

If the limit is finite we say the improper integral converges and that the limit is the value of the improper integral. If the limit does not exist, the integral diverges.

Example 8.7 $\int_0^1 \frac{1}{x^q} d_x = \begin{cases} \frac{1}{1-q}, & p<1 \\ d, & p \neq 1 \end{cases}$

■

Theorem 8.8.1 — Direct Comparison Test. Let f and g be continuous on $[a,\infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\int_a^{\infty} f(x)dx$ converges if $\int_a^{\infty} g(x)dx$ converges.
2. $\int_a^{\infty} g(x)dx$ diverges if $\int_a^{\infty} f(x)dx$ diverges.

R

类似于放缩法,可利用函数的上下界进行比较。

Theorem 8.8.2 — Limit Comparison Test. If the positive functions f and g are continuous on $[a, \infty)$, and if

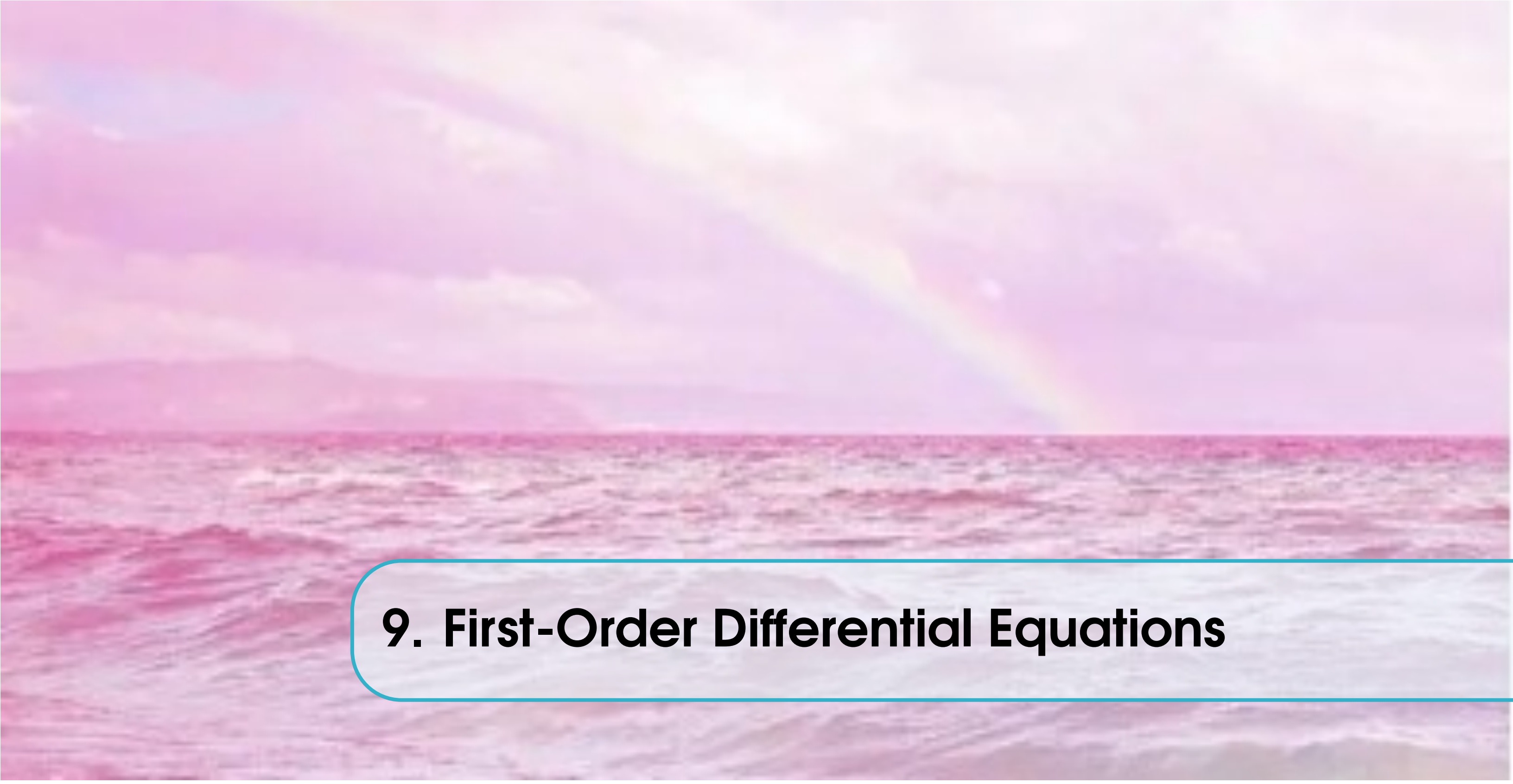
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,$$

then

$$\int_a^{\infty} f(x)dx \quad \text{and} \quad \int_a^{\infty} g(x)dx$$

both converge or both diverge.

R 一般是与初等函数，且其收敛性已知进行比值判断。



9. First-Order Differential Equations

9.1 可分离变量的一阶微分方程

Definition 9.1.1 — General First-Order Differential Equations and Solutions. A first-order differential equation(ODE) is an equation

$$\frac{dy}{dx} = f(x,y)$$

in which $f(x,y)$ is a function of two variables defined on a region in the xy-plane.
The equation is of first order because it involves only the first derivative dy/dx (或者说只有一阶微分出现).
The solutions to equation are:

$$y' = f(x,y) \quad \text{and} \quad \frac{d}{dx}y = f(x,y)$$

Definition 9.1.2 — Separable Differential Equations 一阶、可分离变量

. Differential equation is separable if f can be expressed as a product of a function of x and a function of y.
The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y). \qquad \begin{array}{l} g \text{ is a function of } x \\ H \text{ is a function of } y \end{array}$$

How to solve?

等价于

$$\frac{dy}{dx}=\frac{g(x)}{h(y)}, \quad H(y)=\frac{1}{h(y)}$$

左右同除：

$$h(y)dy=g(x)dx.$$

左右同积：

$$\int h(y)dy=\int g(x)dx.$$

千万注意，一侧加C!

9.2 算法

Step1:标准形式要记牢

$$y'+P(x)y=Q(x)$$

Step2:integrating factor要找到

$$v(x)=e^{\int P(x)dx}$$

Step3:左右同时积起来

Step4:初值条件确定C

9.3 补充

形如 $\frac{dy}{dx}+P(x)y=Q(x)$ 的线性方程应该是常见的一种一阶常微分方程.

它分为两种情况，齐次 (Homogeneous) 和非齐次 (Nonhomogeneous)。齐次即 $Q(x)=0$ ，非齐次即 $Q(x)\neq 0$ 。通过解齐次方程，我们只会得到一个通解 (general solution) 即 y_c ； 而通过解非齐次方程，我们可以得到一个具体的解(particular solution) 即 y_p 。最后该方程的解 $y=y_c+y_p$ 。最后方程的解y值中的常数C需要额外信息求出。

值得注意的是:无论式子长啥样，我们都要先把式子转为一般式，即如 $\frac{dy}{dx}+P(x)y=Q(x)$ 的式子以后再求解。

对于 y_c ， 我们可通过可分离变量方程的方法求解 $\frac{dy}{dx}+P(x)y=0$ ， 我们会得到 $y_c=e^{-\int P(x)dx}$ ， 这里的 P(x) 一定要是转为一般式以后的 P(x) 。

根据线性代数的知识， y_c 和 y_p 不能是在同一直线上的解，所以在 $y_c=cy_1$ 的时候 $y_p=u(x)y_1(x)$ ，我们的目标是让 $\frac{dy}{dx}+P(x)y=Q(x)$ 左右同时乘以一个 $v(x)$ 以后，左边可以凑出乘积法则的样子，即

$$\frac{d}{dx}\left(e^{\int P(x)dx}y\right)=e^{\int P(x)dx}\frac{dy}{dx}+P(x)e^{\int P(x)dx}=e^{\int P(x)dx}Q(x)$$

求导的时候左边直接等于 $\left(e^{\int P(x)dx}y\right)$

■ **Example 9.1** 求 $xy' + 3y = 4x^2 - 3x, y(1) = 3, x > 0$ ■

$$y' + \frac{3}{x}y = 4x - 3 \quad x^3$$

$$x^3y' + 3x^2y = 4x^4 - 3x^3$$

$$(x^3y)' = 4x^4 - 3x^3$$

$$x^3y = \frac{4}{5}x^5 - \frac{3}{4}x^4 + C$$

$$y = \frac{4}{5}x^2 - \frac{3}{4}x + \frac{59}{20x^3}$$