

Step-1

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The matrix that rotates the $x-y$ plane by an angle θ is

$$A(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

We have to verify that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$ from the identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$. Also we have to find $A(\theta).A(-\theta)$.

Step-2

Given
$$A(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Hence
$$A(\theta_1) = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix} \text{ and}$$

$$A(\theta_2) = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{bmatrix}$$

Now
$$A(\theta_1).A(\theta_2) = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix} \cdot \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix} \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \end{bmatrix} + \begin{bmatrix} -\sin \theta_1 \\ \cos \theta_1 \end{bmatrix} \begin{bmatrix} \sin \theta_2 & \cos \theta_2 \end{bmatrix}$$

Step-3

Continuation to the above

$$= \begin{pmatrix} \cos \theta_1 \cdot \cos \theta_2 & -\cos \theta_1 \cdot \sin \theta_2 \\ \sin \theta_1 \cdot \cos \theta_2 & -\sin \theta_1 \cdot \sin \theta_2 \end{pmatrix} + \begin{pmatrix} -\sin \theta_1 \cdot \sin \theta_2 & -\sin \theta_1 \cdot \cos \theta_2 \\ \cos \theta_1 \cdot \sin \theta_2 & \cos \theta_1 \cdot \cos \theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos \theta_1 \cdot \cos \theta_2 - \sin \theta_1 \cdot \sin \theta_2 & -\cos \theta_1 \cdot \sin \theta_2 - \sin \theta_1 \cdot \cos \theta_2 \\ \sin \theta_1 \cdot \cos \theta_2 + \cos \theta_1 \cdot \sin \theta_2 & -\sin \theta_1 \cdot \sin \theta_2 + \cos \theta_1 \cdot \cos \theta_2 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix} \\
&= A(\theta_1 + \theta_2)
\end{aligned}$$

Therefore $\boxed{A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)}$

Step-4

$$A(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$A(\theta).A(-\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$(\because \sin(-\theta) = -\sin \theta \text{ and } \cos(-\theta) = \cos \theta)$$

Step-5

Continuation to the above

$$= \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \cos \theta \sin \theta & \cos^2 \theta + \sin^2 \theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

Therefore $A(\theta).A(-\theta) = \boxed{I}$