

## Step-1

The objective is to determine the left inverse or right inverse.

## Step-2

Consider the matrix  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ ,  $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ , and  $T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ .

First, consider the matrix  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ . This matrix is an echelon form. So, the rank of matrix  $A$  is the number of its non-zero rows

Therefore, the rank is  $r(A) = 2$ .

Therefore, a right inverse of the matrix has a dimension  $(3 \times 2)$ . It does not have a left inverse.

Now, compute the right inverse matrix  $[B]$ .

$$AB = I$$

Where the term  $I$  is the identity matrix.

So,

$$AB = I$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ b_{21} + b_{31} & b_{22} + b_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then,

$$b_{11} + b_{21} = 1$$

$$b_{21} + b_{31} = 0$$

$$b_{12} + b_{22} = 0$$

$$b_{22} + b_{32} = 1$$

There are four equation and six unknown variables.

## Step-3

Now, evaluate the value of all six variables.

$$b_{11} = 1$$

$$b_{21} = 0$$

$$b_{31} = 0$$

$$b_{32} = 1$$

And

$$b_{22} = 0$$

$$b_{12} = 0$$

The matrix  $B$  is:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Verified the values,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, it's verified and the right-inverse of  $A$  is  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ .

## Step-4

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Now, consider the matrix  $M$ , to reduce the matrix by use of echelon form.

Replace  $R_2 \rightarrow R_2 - R_1$

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Now, replace  $R_3 \rightarrow R_3 - R_1$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, the rank is  $r(M) = 2$  that is equal to the number of the column that is  $r = n$ . It consists of left inverse matrix, not a right inverse matrix.

## Step-5

Now, compute the left inverse matrix by  $CM = I$  and the matrix  $M$  is the transpose form of a matrix  $A$ .

$$M = A^T$$

## Step-6

So,

$$CM = I$$

$$CA^T = I \text{ here, } CA^T = AC^T$$

Then,

$$AC^T = AB$$

$$C^T = B$$

Thus, the left inverse matrix is the transpose form of a matrix  $B$ .

$$C^T = B$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the left inverse matrix is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

## Step-7

Now, verify the result.

$$\begin{aligned} CM &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

Therefore, it's verified.

## Step-8

Now consider the third matrix,  $T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ . If  $a = 0$  then  $T$  has neither left inverse nor right inverse. If  $a = b$  then,

$$\begin{aligned} T &= \begin{bmatrix} a & a \\ 0 & a \end{bmatrix} \\ \underline{R_2 - R_1} &\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} \text{rank of } T &= 1 \\ &\neq m, n \end{aligned}$$

So that  $T$  has neither left inverse nor right inverse.

If  $a \neq b$  then,

$$T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

The invertible matrix is:

$$T^{-1} = \frac{1}{a^2} \begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}$$

Hence, the invertible matrix is  $T^{-1} = \frac{1}{a^2} \begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}$ .