

Step-1

(a) Consider the following:

$$\begin{aligned} A &= uv^T \\ Au &= uv^T u \\ &= u(v^T u) \\ &= (v^T u)u \end{aligned}$$

Thus, we have shown that $Au = (v^T u)u$.

Step-2

When u is a column and v is a row, $v^T u$ is simply a number.

Thus, we have shown that Au is equal to a number multiplied by u . Therefore, u is an eigenvector.

Since, $Au = (v^T u)u$, the corresponding eigenvalue is $\boxed{\lambda = v^T u}$.

Step-3

(b) Since $A = uv^T$ is a rank 1 matrix, it has only one independent row. Therefore, it has only one nonzero pivot, rest all are zero pivots.

Therefore, the other eigenvalues of A are zeros.

Step-4

(c) Let u and v be as follows: $u = (u_1, u_2, \dots, u_n)^T$ and $v = (v_1, v_2, \dots, v_n)^T$.

Therefore, we get

$$\begin{aligned} A &= uv^T \\ &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} (v_1, v_2, \dots, v_n)^T \\ &= \begin{bmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_n \\ \dots & \dots & \dots & \dots \\ u_n v_1 & u_n v_2 & \dots & u_n v_n \end{bmatrix} \end{aligned}$$

Step-5

Therefore, the trace $(A) = u_1v_1 + u_2v_2 + \dots + u_nv_n$. Now consider $v^T u$.

$$\begin{aligned} v^T u &= (v_1, v_2, \dots, v_n)^T \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \\ &= v_1 u_1 + v_2 u_2 + \dots + v_n u_n \\ &= u_1 v_1 + u_2 v_2 + \dots + u_n v_n \end{aligned}$$

This is equal to sum of the eigenvalues.