Step-1

Given system is $ax+2y+3z=b_1$

$ax+ay+4z=b_2$
$ax+ay+az=b_3$

We have to find for which three numbersâ€™aâ€™will elimination fail to give three pivots.

Step-2

Given system can be written as

$$\begin{bmatrix} a & 2 & 3 & b_1 \\ a & a & 4 & b_2 \\ a & a & a & b_3 \end{bmatrix}$$

apply $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\begin{bmatrix} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2-b_1 \\ 0 & a-2 & a-3 & b_3-b_1 \end{bmatrix}$$

apply $R_2 \to R_2 - R_1$

$$\begin{bmatrix} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2-b_1 \\ 0 & 0 & a-4 & b_3-b_2 \end{bmatrix}$$

Step-3

The pivots are circled in $\begin{bmatrix} \boxed{a} & 2 & 3 & b_1 \\ 0 & \boxed{a-2} & 1 & b_2-b_1 \\ 0 & 0 & \boxed{a-4} & b_3-b_2 \end{bmatrix}$

That is $a, a-2, a-4$.

Hence it is clear that the elimination will fail when

$a=0$
$a-2=0$ or
$a-4=0$

i.e. for $a=0, a=2, a=4$ the elimination will fail.

Step-4

Case (i):- If $a=0$

The system becomes

$$\begin{bmatrix} 0 & 2 & 3 & b_1 \\ 0 & 0 & 4 & b_2 \\ 0 & 0 & 0 & b_3 \end{bmatrix}$$

In the above matrix, all the entries in column 1(row 3) are zero.

Hence elimination fails for $a=0$ (zero column).

Step-5

Case (ii):- If $a=2$

The system becomes

$$\begin{bmatrix} 2 & 2 & 3 & b_1 \\ 2 & 2 & 4 & b_2 \\ 2 & 2 & 2 & b_3 \end{bmatrix}$$

In the above matrix, all the entries in column 1 are equal to all the entries in

column 2.Hence elimination fails for $a=2$ (equal columns).

Step-6

Case (iii):- If $a=4$

The system becomes

$$\begin{bmatrix} 4 & 2 & 3 & b_1 \\ 4 & 4 & 4 & b_2 \\ 4 & 4 & 4 & b_3 \end{bmatrix}$$

In the above matrix, upto the coefficient matrix all the entries in row 2 are

equal to all the entries in row3.Hence elimination fails for $a=4$ (equal rows).