Step-1

Let $x = (x_1, x_2, x_3)$ is orthogonal to both $y = (1, 1, 1)$, and $z = (1,-1, 0)$

By definition of orthogonal, $x^T y = 0, x^T z = 0$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (1 \quad 1 \quad 1) = 0$ and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (1 \quad -1 \quad 0) = 0$

$\Rightarrow x_1 + x_2 + x_3 = 0$, $x_1 - x_2 = 0$

$\Rightarrow x_1 = -x_2 - x_3 = 0$, â€¦â€¦ (1)

$x_1 = x_2$ â€¦â€¦ (2)

Using (2) in (1) $x_3 = -2x_2$

Therefore, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ where $k = x_2$ a parameter takes infinite real values

Put $k = 1$ to obtain $x = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ which is the vector orthogonal to both the given vectors.

Step-2

Ortho normal basis:

Find

1. The length $\|x\|$ of a vector;

2. The test $x^T y = 0$ for perpendicular vectors; and

3. Create perpendicular vectors from linearly independent vectors.

Step-3

The given vectors $y = (1, 1, 1)$, and $z = (1,-1, 0)$ are not the scalar multiples of each other and so, they are linearly independent.

Further, $x = (1, 1, -2)$ is perpendicular to both these and so, x, y, z are linearly independent.

Now, produce three mutually perpendicular unit vectors from $x, y,$ and z.

The unit vector along x is $\hat{x} = \frac{x}{\|x\|}$

$$= \frac{(1,1,-2)}{\sqrt{1^2 + 1^2 + (-2)^2}}$$

$= \frac{(1,1,-2)}{\sqrt{6}}$ is the first orthonormal vector.

$$t = y - \left(\hat{x}^T y\right)\hat{x}$$

$$= (1,1,1) - \frac{1+1-2}{\sqrt{6}}\left(\frac{(1,1,-2)}{\sqrt{6}}\right)$$

$$= (1,1,1)$$

Unit vector along t is the second orthonormal vector required is $\hat{y} = \frac{t}{\|t\|} = \frac{(1,1,1)}{\sqrt{3}}$

Step-4

Similarly, $u = z - \left(\hat{x}^T z\right)\hat{x} - \left(\hat{y}^T z\right)\hat{y}$

$$= (1,-1,0) - \left((1,-1,0)\begin{pmatrix}1\\1\\-2\end{pmatrix}\frac{1}{\sqrt{6}}\right)\frac{(1,1,-2)}{\sqrt{6}} - \left((1,-1,0)\begin{pmatrix}1\\1\\1\end{pmatrix}\frac{1}{\sqrt{3}}\right)\frac{(1,1,1)}{\sqrt{3}}$$

$$= (1,-1,0)$$

The unit vector along u is the third orthonormal vector required.

i.e., $\hat{z} = \frac{u}{\|u\|} = \frac{(1,-1,0)}{\sqrt{2}}$

Thus, the suitable orthonormal basis corresponding to the given x, y, z is

$\hat{x} = \frac{(1,1,-2)}{\sqrt{6}}$, $\hat{y} = \frac{(1,1,1)}{\sqrt{3}}$ and $\hat{z} = \frac{(1,-1,0)}{\sqrt{2}}$.