Step-1

Thus, we have

$$
\begin{aligned}
0 &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 - 1 \\
&= \lambda^2 - 2\lambda
\end{aligned}
$$

Thus, one eigenvalue of A is positive and the other is zero.

Step-2

Consider the matrix $C^{\mathrm{T}}AC$.

We have

$$
\begin{aligned}
C^{\mathrm{T}}AC &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}
\end{aligned}
$$

Step-3

To obtain the eigenvalues of $C^{\mathrm{T}}AC$, we solve $\det\left(C^{\mathrm{T}}AC - \lambda I\right) = 0$. This gives,

$$
\begin{aligned}
0 &= \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} \\
&= (4-\lambda)(1-\lambda) - 4 \\
&= \lambda^2 - 5\lambda + 4 - 4 \\
&= \lambda^2 - 5\lambda
\end{aligned}
$$

Thus, here also one eigenvalue of $C^{\mathrm{T}}AC$ is positive and the other is zero.

Step-4

Let $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Consider the following chain of matrices:

$$
\begin{aligned}
C(t) &= tQ + (1-t)C \\
&= t\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (1-t)\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} + \begin{bmatrix} 2(1-t) & 0 \\ 0 & t-1 \end{bmatrix} \\
&= \begin{bmatrix} 2-t & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}
$$

Step-5

Note that

$$
\begin{aligned}
C(0) &= \begin{bmatrix} 2-0 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \\
C(1) &= \begin{bmatrix} 2-1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}
$$

Step-6

Thus, $\boxed{C(t) = tQ + (1-t)C}$ is the required chain of non-singular matrices.

Since C has one positive and one zero eigenvalue and the identity matrix I has both positive eigenvalues, it is impossible to have such a chain of matrices from C to I.