Step-1

Consider the system,

$$u+v+2w=2$$

$$2u + 3v - w = 5$$

$$3u + 4v + w = c$$

The objective is to determine the value of c that makes to solve the system and solve it.

Step-2

The above system can be written as,

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ c \end{bmatrix}$$

This is in the form of Ax = b

$$\begin{array}{cccc}
R_2 - 2R_1, & 1 & 1 & 2 \\
0 & 1 & -5 \\
R_3 - 3R_1, & 0 & 1 & -5
\end{array}
\begin{bmatrix}
u \\ v \\ w
\end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ c - 6 \end{bmatrix}$$

$$\underbrace{R_3 - R_2}_{0} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ c - 7 \end{bmatrix}$$

$$\underline{R_1 - R_2} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ c - 7 \end{bmatrix}$$

The system is consistent if c-7=0

That is, c = 7

Step-3

If c = 7, then Ax = b becomes,

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Here, w is a free variable.

$$u + 7w = 1$$

$$v-5w=1$$

It follows that,

$$u=1-7w$$

$$v = 1 + 5w$$

Step-4

Therefore,

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 - 7w \\ 1 + 5w \\ w \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}$$



Hence, the solution is,

The column space is a plane.