Step-1

Consider a 4 by 4 matrix in which $a_{ij} = 1$ above the diagonal and $a_{ij} = 0$ elsewhere. Find the Jordan form by finding the Eigen vectors.

Step-2

Let the matrix be *A* defined as follows:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Calculate Eigen values of matrix *A*:

$$A - \lambda I = \begin{bmatrix} 0-\lambda & 1 & 1 & 1 \\ 0 & 0-\lambda & 1 & 1 \\ 0 & 0 & 0-\lambda & 1 \\ 0 & 0 & 0 & 0-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$
$$(-\lambda)(-\lambda^3) = 0$$
$$\lambda^4 = 0$$

On solving above equation Eigen values obtained are $\lambda = (0,0,0,0)$

Step-3

Clearly it can be seen that $\lambda = 0$ is the only repeated Eigen value of the matrix *A*. Eigen vector corresponding to this will be $v_1 = (1,0,0,0)$. Generalised Eigen vectors will be as follows:

$$v_2 = (0,1,0,0)$$
$$v_3 = (0,-1,1,0)$$
$$v_4 = (0,1,-2,1)$$

Step-4

Put these vectors in a matrix *M*. Therefore matrix *M* and its inverse can be defined as follows:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step-5

Recall that Jordan form is given by $M^{-1}AM = J$. Therefore,

$$\begin{aligned} M^{-1}AM &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= J \end{aligned}$$

Step-6

As matrix A has only one independent Eigen vector, so Jordan form will have only one block. In Jordan matrix three 1â€™s represents absence of three Eigen vectors. Therefore, Jordan form of matrix A is :

$$\boxed{J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$