

Step-1

Given x_p (free variable = 0) is the particular solution and x_n is all special solutions for $Ax = b$.

We have to find x_p (free variable = 0) and all special solutions x_n for the following systems:

$$Ax = 2b, \begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x \\ X \end{bmatrix} = b, \text{ and } \begin{bmatrix} A \\ A \end{bmatrix} [x] = \begin{bmatrix} b \\ b \end{bmatrix}.$$

Step-2

First we consider the system, $Ax = 2b$

The special solutions x_n are the solutions of $Ax = 0$ and the particular solutions x_p are the solutions of $Ax = b$

Given x_p is the solution of $Ax = b$

$$\Rightarrow Ax_p = b$$

$$\Rightarrow Ax_p + Ax_p = b + b$$

$$\Rightarrow A(2x_p) = 2b$$

$$\Rightarrow 2x_p \text{ is the particular solution of } Ax = 2b.$$

Hence $2x_p$ is the particular, and it is clear that x_n is the special solution of $Ax = 2b$.

Step-3

$$\begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x \\ X \end{bmatrix} = b$$

Next we consider the system,

If A is $n \times n$ matrix, then $A \times A$ is $n \times 2n$ matrix. The second n columns are free columns therefore; there are n extra free variables in $\begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x \\ X \end{bmatrix} = b$

$$\begin{bmatrix} x \\ X \end{bmatrix}_p = \begin{bmatrix} x_p \\ 0 \end{bmatrix}$$

Step-4

There are n extra free variables in the system

$$\begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x \\ X \end{bmatrix} = b$$

These n free variables give for the special solutions $\begin{bmatrix} -I \\ I \end{bmatrix}$.

The special solutions of

$\begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x \\ X \end{bmatrix} = b$ is x_n include the columns of $\begin{bmatrix} -I \\ I \end{bmatrix}$, where I is n by n identity matrix.

That is $\begin{bmatrix} x \\ X \end{bmatrix}_n = \begin{bmatrix} x_n & -I \\ 0 & I \end{bmatrix}$

Step-5

Finally we consider the system, $\begin{bmatrix} A \\ A \end{bmatrix} [x] = \begin{bmatrix} b \\ b \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} A \\ 0 \end{bmatrix} [x] = \begin{bmatrix} b \\ 0 \end{bmatrix} \text{ (by using row operations)}$$

Then the particular solution is $[x]_p = x_p$

And special solution $[x]_n = x_n$

Hence the special solutions, particular solutions are unchanged.