Exploring Quarto and Latex

Ryan Jhon Lee N. Lajola

4.1 ANTIDIFFERENTIATION AND INDIFINITE INTEGRALS

4.1.2 Integration by Substitution

Theorem 4.1.11 (Substitution Rule)

If $u = g(x)$ is a differentiable function whose range is an interval I and f is a continuous on I then,

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du$$

1. $\int (1-4x)^{\frac{1}{2}}\, dx$

If we let $u = 1-4x$, then $du = -4dx$. We multiply the integrand $\frac{-4}{-4}$. Thus,

$$\int (1-4x)^{\frac{1}{2}}\, dx = \int (1-4x)^{\frac{1}{2}} \cdot \frac{-4}{-4}dx = \int u^{\frac{1}{2}} \left(-\frac{du}{4}\right) = -\frac{1}{4}\int u^{\frac{1}{2}}du = -\frac{1}{4} \cdot \frac{2u^{\frac{3}{2}}}{3} + C.$$

We put the final answer in terms of x by substituting $u = 1-4x$. Therefore,

$$\int (1-4x)^{\frac{1}{2}}\, dx = \frac{(1-4x)^{\frac{3}{2}}}{6} + C.$$

2. $\int x^2 \left(x^3-1\right)^{10} dx$

Let $u = x^3 - 1$. Then $du = 3x^2dx$, or $\frac{du}{3} = x^2dx$. By substitution,

$$\int x^2 \left(x^3-1\right)^{10} dx = \int u^{10} \cdot \frac{du}{3} = \frac{1}{3}\int u^{10}du = \frac{u^{11}}{33} + C = \frac{\left(x^3-1\right)^{11}}{33} + C.$$

3. $\int \frac{x}{\left(x^2+1\right)^3}dx$

Let $u = x^2 + 1$. Then $du = 2xdx$, or $\frac{du}{2} = xdx$. By substitution,

$$\int \frac{x}{\left(x^2+1\right)^3} dx = \frac{1}{2} \int u^{-3} du = \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C = -\frac{1}{4\left(x^2+1\right)^2} + C.$$

4. $\int \cos^4 x \sin x dx$

Let $u = \cos x$. Then $du = -\sin x dx$, or $-du = \sin x dx$. By substitution,

$$\int \cos^4 x \sin x dx = -\int u^4 du = -\frac{u^5}{5} + C = -\frac{\cos^5 x}{5} + C.$$

5. $\int x \sec^3\left(x^2\right) \tan\left(x^2\right) dx$

Let $u = \sec\left(x^2\right)$. Then $du = \sec\left(x^2\right)\tan\left(x^2\right) \cdot 2xdx$, or $\frac{du}{2} = \sec\left(x^2\right)\tan\left(x^2\right) \cdot xdx$. By substitution,

$$\begin{aligned}
\int x \sec^3\left(x^2\right) \tan\left(x^2\right) dx &= \int \sec^2\left(x^2\right) \sec\left(x^2\right) \tan\left(x^2\right) \cdot xdx \\
&= \int u^2 du = \frac{1}{2} \cdot \frac{u^3}{3} + C \\
&= \frac{\sec^3\left(x^2\right)}{6} + C.
\end{aligned}$$

6. $\int \frac{\tan\frac{1}{s} + \tan\frac{1}{s}\sin\frac{1}{s}}{s^2 \cos\frac{1}{s}} ds$ Let $u = \frac{1}{s}$. Then $du = -\frac{1}{s^2} ds$ or $-du = \frac{ds}{s^2}$. By substitution,

$$\begin{aligned}
\int \frac{\tan\frac{1}{s} + \tan\frac{1}{s}\sin\frac{1}{s}}{s^2 \cos\frac{1}{s}} ds &= -\int \frac{\tan u + \tan u \sin u}{\cos u} du \\
&= -\int (\sec u \tan u + \tan^2 u) du \\
&= -\int (\sec u \tan u + \sec^2 u - 1) du \\
&= -(\sec u + \tan u - u) + C \\
&= -\sec\frac{1}{s} - \tan\frac{1}{s} + \frac{1}{s} + C.
\end{aligned}$$

7. $\int t\sqrt{t-1} dt$

Let $u = t - 1$. Then $u = dt$. Also, $t = u + 1$. By substitution,

$$\begin{aligned}
\int t\sqrt{t-1} dt = \int (u+1) u^{\frac{1}{2}} du = \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du = \frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} + C \\
= \frac{2(t-1)^{\frac{5}{2}}}{5} + \frac{2(t-1)^{\frac{3}{2}}}{3} + C.
\end{aligned}$$

8. $\int \frac{t^3}{\sqrt{t^2+3}}dt$

Let $u = t^2 + 3$. Then $du = 2tdt$, or $\frac{du}{2} = tdt$. Also, $t^2 = u - 3$. By substitution,

$$\begin{aligned}\int \frac{t^3}{\sqrt{t^2+3}}dt &= \int \frac{t^2 \cdot t}{\sqrt{t^2+3}}dt = \int u^{\frac{-1}{2}}(u-3)\frac{du}{2} \\ &= \frac{1}{2}\int \left(u^{\frac{1}{2}} - 3u^{\frac{-1}{2}}\right)du = \frac{1}{2}\left(\frac{2u^{\frac{3}{2}}}{3} - 6u^{\frac{1}{2}}\right) + C \\ &= \frac{\left(t^2+3\right)^{\frac{3}{2}}}{3} - 3\left(t^2+3\right)^{\frac{1}{2}} + C.\end{aligned}$$

9. $\int \sqrt{4+\sqrt{x}}dx$

Let $u = 4 + \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx$ or $2du = \frac{dx}{\sqrt{x}}$. By substitution,

$$\begin{aligned}\int \sqrt{4+\sqrt{x}}dx &= \int \sqrt{4+\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}dx \\ &= \int \sqrt{4+\sqrt{x}} \cdot \sqrt{x} \cdot \frac{dx}{\sqrt{x}}(\sqrt{x} = u-4) \\ &= \int u^{\frac{1}{2}} \cdot (u-4) \cdot 2du \\ &= \int (2u^{\frac{3}{2}} - 8u^{\frac{1}{2}})du \\ &= \frac{2 \cdot 2u^{\frac{5}{2}}}{5} - \frac{2 \cdot 8u^{\frac{3}{2}}}{3} + C \\ &= \frac{4\left(4+\sqrt{x}\right)^{\frac{5}{2}}}{5} - \frac{16\left(4+\sqrt{x}\right)^{\frac{3}{2}}}{3} + C.\end{aligned}$$

4.1.3 Particular Antiderivatives

Now suppose that given a function $f(x)$, we wish to find a particular antiderivative $F(x)$ of $f(x)$ that satisfies a given condition. Such a condition is called an initial or boundary condition.

1. Given that $F'(x) = 2x$ and $F(2) = 6$, find $F(x)$.

Solution

Since $F'(x) = 2x$, we have

$$F(x) = \int 2xdx = x^2 + C.$$

The initial condition $F(2) = 6$ implies that $F(2) = 2^2 + C = 6$. We get $C = 2$. Therefore, the particular antiderivative that we wish to find is:

$$F(x) = x^2 + 2.$$

2. The slope of the the tangent line at any point (x, y) on a curve is given by $3\sqrt{x}$. Find an equation of the curve if the point $(9, 4)$ is on the curve.

Solution

Let $y = F(x)$ be an equation of the curve. The slope of the tangent line m_{TL} at a point (x, y) on the graph of the curve is given by F'(x)=$3\sqrt{x}$. We have

$F(x) = \int 3x^1/2dx = 2x^3/2 + C$.

The initial condition that $(9, 4)$ is on the curve implies that $F(9) = 2 \cdot 9^3/2 = 4$. We obtain $C = -50$. Thus, an equation of the curve is

$y = 2x^3/2 - 50$

0.1 5.6 Indeterminate Forms and L'Hopital's Rule

5.6.1 Indeterminate Forms of Type $0/0$ and undefined ∞/∞

We began this course with the concept of the limit; the behavior of a function as the independent variable approaches a certain value, or as increases or decreases without bound. We saw tangent, lines, rates of change, and areas about which the entire calculus revolves.

Now, we conclude this course by revising this fundamental idea. We shall see that, with the aid of derivatives, certain limits can be evaluated more conveniently. Before proceeding, we first recall some terminology defined in the early part of this course. We can also here some techniques in evaluating limts we have previously encountered.

CHAPTER 5. NEW CLASSES OF FUNCTIONS AND L'HOSPITAL'S RULE

5.6 Intermediate Forms and L'Hopital's Rule

5.6.1 Intermediate Forms of Type $\frac{0}{0}$ and $\frac{\infty}{\infty}$ We began this course with the concept of the limit: the behavior of a function as the independent variable approaches a certain value, or as it increases or decreases without bound. We saw tangent lines, rates of change, and areas of plane regions, as limits of certain quantities. Indeed, the concept of the limit is the central idea about which the entire calculus revolves.

Now, we conclude this course by revisiting this fundamental idea. We shall see that, with the aid of derivatives, certain limits can be evaluated more conveniently. Before proceeding, we first recall some terminology defined in the early part of this course. We also recall here some techniques in evaluating limits we have previously encountered.

Definition 5.6.1 The $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type

1. $\frac{0}{0}$ if $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$
2. $\frac{\infty}{\infty}$ if $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ are both $+\infty$ or $-\infty$.

Of course, "$x \to a$" may be replaced by "$x \to a^+$", "$x \to a^-$", "$x \to +\infty$" or "$x \to -\infty$".

Example 5.6.2 Evaluate the following limits.

1. $\lim\limits_{x \to 0} \dfrac{x^2 - 3x}{2x^2 + x}$ $\left(\frac{0}{0}\right)$

Solution.

$$\lim_{x \to 0} \frac{x^2 - 3x}{2x^2 + x} = \lim_{x \to 0} \frac{x(x-3)}{x(2x+1)} = \lim_{x \to 0} \frac{x-3}{2x+1} = -3$$

2. $\lim\limits_{x \to 0} \dfrac{\sin 5x}{\sin 3x}$ $\left(\frac{0}{0}\right)$

Solution.

$$\lim_{x \to 0} \frac{\sin 5x}{\sin 3x} = \lim_{x \to 0} \left(\frac{\sin 5x}{5x}\right)\left(\frac{3x}{\sin 3x}\right)\left(\frac{5}{3}\right) = 1 \cdot 1 \cdot \frac{5}{3} = \frac{5}{3}$$

3. $\lim\limits_{x \to 2} \dfrac{x^2 + 3x - 10}{x^2 - 4x + 4}$ $\left(\frac{0}{0}\right)$

Solution.

$$\begin{aligned}\lim_{x \to 2} \frac{x^2 + 3x - 10}{x^2 - 4x + 4} &= \lim_{x \to 2} \frac{(x+5)(x-2)}{(x-2)^2} \\ &= \lim_{x \to 2} \frac{x+5}{x-2} \\ &= -\infty\end{aligned}$$

4. $\lim\limits_{x \to +\infty} \dfrac{3x - 1}{7 - 6x}$ $\left(\frac{\pm\infty}{-\infty}\right)$

Solution.

$$\lim_{x \to +\infty} \frac{3x-1}{7-6x} = \lim_{x \to +\infty} \frac{3x-1}{7-6x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to +\infty} \frac{3 - \frac{1}{x}}{\frac{7}{x} - 6} = -\frac{1}{2}$$

5.6.2 L'Hospital's Rule The following theorem tells us how derivatives can be used to evaluate limits that are indeterminate of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. It usually referred to as L'Hospital's Rule, after the French mathematician Guillaume Francois Marquis de L'Hospital.

Let f and g be functions differentiable on an open interval I containing a except possibly at a and $g'(x) \neq 0$ for all $x \in I \quad a$. If $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is indeterminate of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists or $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = \pm\infty$.

Remarks L'Hospital's Rule, with suitable modifications, is valid if "$x \to a$" is replaced by "$x \to a^+$", "$x \to a^-$", "$x \to +\infty$" or "$x \to -\infty$".

Example Evaluate the following limits.

1. $\lim\limits_{x\to 0} \dfrac{x^2-3x}{2x^2+x}$ $\frac{0}{0}$

Solution.

$$\lim_{x\to 0} \frac{x^2-3x}{2x^2+x} = \lim_{x\to 0} \frac{D_x\left(x^2-3x\right)}{D_y\left(2x^2+x\right)} = \lim_{x\to 0} \frac{2x-3}{4x+1} = -3$$

2. $\lim\limits_{x\to 0} \dfrac{\sin 5x}{\sin 3x}$ $\left(\frac{0}{0}\right)$

Solution.

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 3x} = \lim_{x\to 0} \frac{5\cos 5x}{3\cos 3x} = \frac{5}{3}$$

3. $\lim\limits_{x\to 2^-} \dfrac{x^2+3x-10}{x^2-4x+4}$ $\left(\frac{0}{0}\right)$

Solution.

$$\lim_{x\to 2^-} \frac{x^2+3x-10}{x^2-4x+4} = \lim_{x\to 2^-} \frac{2x+3}{2x-4}$$

$$= -\infty$$

4. $\lim\limits_{x\to +\infty} \dfrac{3x-1}{7-6x}$ $\left(\frac{\pm\infty}{-\infty}\right)$

Solution.

$$\lim_{x\to +\infty} \frac{3x-1}{7-6x} = \lim_{x\to +\infty} \frac{3}{-6} = -\frac{1}{2}$$

5. $\lim\limits_{x\to 1} \dfrac{x^3-3x+2}{1-x+\ln x}$ $\left(\frac{0}{0}\right)$

Solution.

$$\begin{aligned}\lim_{x\to 1} \frac{x^3-3x+2}{1-x+\ln x} &= \lim_{x\to 1} \frac{3x^2-3}{-1+\frac{1}{x}}\\ &= \lim_{x\to 1} \frac{6x}{-\frac{1}{x^2}}\\ &= -6\end{aligned}$$

6. $\lim\limits_{x\to 0^-} \dfrac{\csc x}{1-\cot x}$ $\left(\frac{-\infty}{+\infty}\right)$

Solution.

$$\begin{aligned}
\lim_{x\to 0^-}\frac{\csc x}{1-\cot x} &= \lim_{x\to 0^-}\frac{-\csc x\cot x}{\csc^2 x}\\
&= \lim_{x\to 0^-}\frac{-\cot x}{\csc x}\\
&= \lim_{x\to 0^-}\frac{\csc^2 x}{-\csc x\cot x}\\
&= \lim_{x\to 0^-}\frac{\csc x}{-\cot x}\\
&= \lim_{x\to 0^-}\frac{-\cot x}{\csc x}
\end{aligned}$$

Observe that the expression in the last line above is exactly the same as the expression in the second line. Hence, continued application of L'Hospital's Rule here will just lead to an infinite string of equations and will not help us evaluate the limit. This example should make you realize that L'Hospital's Rule is not always helpful. Sometimes, we just have to use some old-fashioned tricks. For instance, we evaluate this limit by simply manipulating the given expression to obtain a simpler expression:

$$\begin{aligned}
\lim_{x\to 0^-}\frac{\csc x}{1-\cot x} &= \lim_{x\to 0^-}\frac{\frac{1}{\sin x}}{1-\frac{\cos x}{\sin x}}\\
&= \lim_{x\to 0^-}\frac{1}{\sin x-\cos x}\\
&= -1
\end{aligned}$$

It is imperative to remember the behavior of each function introduced in this course; doing so will help us in computing new limits. Recalling the graphs of our new functions will be helpful in remembering their behavior. For instance, using the graph of $f(x)=\log_a x$, where \$ 0< a < 1\$, one sees that $\lim\limits_{x\to 0^+}\log_a x=+\infty$. Thus if $f(x)$ approaches 0 through positive values x approaches k, then

$$\lim_{x\to k}\log_a[f(x)]=\lim_{y\to 0^+}\log_a y=+\infty.$$

Solution

Note that since $x+\sin x \geq x-1$ for any $x \in R$ and $\lim\limits_{x\to+\infty} x-1=+\infty$, then $\lim\limits_{x\to+\infty} x+\sin x=+\infty$ as well. Thus, the limit is indeterminate of type $(\frac{\infty}{\infty})$.

However, $\lim\limits_{x\to+\infty}\dfrac{D_x(x+\sin x)}{D_x(x)}=\lim\limits_{x\to+\infty}\dfrac{1+\cos x}{1}$ does not exist $\cos x$ does not approach any particular value as $x\to+\infty$. Neither does $1+\cos x$ grow without bound. Thus, L'Hospital's Rule does not apply.

We therefore need to employ other techniques. In particular, notice that any $x > 0$, the following hold:

$$\begin{aligned} & x-1 \leq x+\sin x \leq x+1 \\ \Leftrightarrow & \frac{x-1}{x} \leq x+\sin x \leq \frac{x+1}{x} \end{aligned}$$

Also, $\lim\limits_{x\to+\infty} \dfrac{x-1}{x} = 1 = \lim\limits_{x\to+\infty} \dfrac{x+1}{x}$, so by the Squeeze Theorem, $\lim\limits_{x\to+\infty} \dfrac{x+\sin x}{x} = 1$.

Indeterminate Forms of Type $0 \cdot \infty$ and $\infty - \infty$

Definition

The $\lim\limits_{x\to a} f(x)g(x)$ is an indeterminate form of type $0 \cdot \infty$ if either

$$\lim_{x\to a} f(x) = 0$$

and $\lim\limits_{x\to a} g(x) = +\infty$ or $-\infty$, or

$$\lim_{x\to a} f(x) = +\infty$$

or $-\infty$ and $\lim\limits_{x\to a} g(x) = 0$.

The $\lim\limits_{x\to a} f(x)g(x)$ is an indeterminate form of type $\infty - \infty$ if either

$$\lim_{x\to a} f(x) = +\infty$$

and $\lim\limits_{x\to a} g(x) = +\infty$ or $-\infty$, or

$$\lim_{x\to a} f(x) = -\infty$$

and $\lim\limits_{x\to a} g(x) = +\infty$.

Remark L'Hospital's Rule works only for indeterminate forms of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Any other indeterminate form must be expressed equivalently in one of these two forms if we wish to apply L'Hospital's Rule. For the new indeterminate forms described above, these conversions can be performed as described below.

If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = +\infty$ or $-\infty$, write $\lim\limits_{x\to a} f(x)g(x)$ as:

$\lim\limits_{x\to a} \dfrac{f(x)}{\frac{1}{g(x)}}$, which is indeterminate of type $\frac{0}{0}$, or

$\lim\limits_{x\to a} \dfrac{g(x)}{\frac{1}{f(x)}}$, which is indeterminate of type $\frac{\infty}{\infty}$ and apply L'Hospital's Rule

If $\lim\limits_{x\to a} f(x)+g(x)$ is indeterminate of type $\infty-\infty$, rewrite $f(x)+g(x)$ as a single expression to obtain an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and apply L'Hospital's Rule.

Example Evaluate the following limits.

$\lim\limits_{x\to 0^+} \sin^{-1}(2x)\csc x \ (0\cdot+\infty)$

Solution

$$\begin{aligned}
\lim_{x\to 0^+} \sin^{-1}(2x)\csc x &= \lim_{x\to 0^+} \frac{\sin^{-1}(2x)}{\sin x} \\
&= \lim_{x\to 0^+} \frac{1}{\sqrt{1-4x^2}}\cdot(2) \\
&= \frac{2}{1} \\
&= 2
\end{aligned}$$

2. $\lim\limits_{0\to\frac{\pi}{2}^-} \tan\theta\ln(\sin\theta) \ +\infty\cdot 0$

Solution

$$\begin{aligned}
\lim_{\theta\to\frac{\pi}{2}^-} \tan\theta\ln(\sin\theta) &= \lim_{\theta\to\frac{\pi}{2}^-} \frac{\ln(\sin\theta)}{\cot\theta} \\
&= \lim_{\theta\to\frac{\pi}{2}^-} \frac{\left(\frac{1}{\sin\theta}\right)\cos\theta}{-\csc^2\theta} \\
&= \lim_{\theta\to\frac{\pi}{2}^-} -\sin\theta\cos\theta \\
&= -1(0) \\
&= 0
\end{aligned}$$

3. $\lim\limits_{x\to 1^+}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right) \left(\frac{1}{0^+}-\frac{1}{0^+}\right)$

Solution

$$
\begin{aligned}
\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) &= \lim_{x\to 1^+}\frac{x\ln x-(x-1)}{(x-1)\ln x}\\
&= \lim_{x\to 1^+}\frac{x\cdot\frac{1}{x}+\ln x-1}{(x-1)\cdot\frac{1}{x}+\ln x}\\
&= \lim_{x\to 1^+}\frac{\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}}\\
&= \frac{1}{2}
\end{aligned}
$$

Indeterminate Forms of Type

Indeterminate Forms of Type 1^∞, 0^∞ and ∞^0

Definition Let f be a nonconstant function. The $\lim\limits_{x\to a} f(x)^{g(x)}$ is an indeterminate form of type

1. 1^∞ if $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = +\infty$ or $-\infty$.
2. 0^∞ if $\lim\limits_{x\to a} f(x) = 0$, through positive values, and $\lim\limits_{x\to a} g(x) = 0$.
3. ∞^0 if $\lim\limits_{x\to a} f(x) = +\infty$ and $\lim\limits_{x\to a} g(x) = 0$.

Remark If $\lim\limits_{x\to a} f(x)^{g(x)}$ is indeterminate of type 1^∞, 0^∞ and ∞^0, we write

$$
\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln[f(x)]}
$$

and evaluate $\lim\limits_{x\to a} g(x)\ln[f(x)]$ first. Then, if

1. $\lim\limits_{x\to a} g(x)\ln[f(x)] = L \in R$, then $\lim\limits_{x\to a} f(x)^{g(x)} = e^L$.
2. $\lim\limits_{x\to a} g(x)\ln[f(x)] = +\infty$, then $\lim\limits_{x\to a} f(x)^{g(x)} = +\infty$.
3. $\lim\limits_{x\to a} g(x)\ln[f(x)] = -\infty$, then $\lim\limits_{x\to a} f(x)^{g(x)} = 0$.

Example Evaluate the following limits.

1. $\lim\limits_{x\to 0^+} x^{\sin x}$ 0^0

Solution

First, write $x^{\sin x} = e^{\sin x \ln x}$. Evaluate first $\lim\limits_{x\to 0^+} \sin x \ln x$.

$$
\begin{aligned}
&\lim_{x\to 0^+} \sin x \ln x \\
&\qquad = \lim_{x\to 0^+} \frac{\ln x}{\csc x} \\
&\qquad = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x} \\
&\qquad = \lim_{x\to 0^+} \frac{-\sin^2 x}{x \cos x} \\
&\qquad = \lim_{x\to 0^+} \frac{-2\sin x \cos x}{x(-\sin x) + \cos x} \\
&\qquad = 0.
\end{aligned}
$$

Hence, $\lim\limits_{x\to 0^+} x^{\sin x} = e^0 = 1$.

2. $\lim\limits_{x\to +\infty} \left(1 - \dfrac{3}{x}\right)^{2x}$ 1^{∞}

Solution

$\left(1 - \frac{3}{x}\right)^{2x} = e^{2x \ln\left(1 - \frac{3}{x}\right)}$

$$
\begin{aligned}
&\lim_{x\to +\infty} 2xln\left(1 - \frac{3}{2}\right) \\
&\qquad = \lim_{x\to +\infty} \frac{ln\left(1 - \frac{3}{x}\right)}{\frac{1}{2x}} \\
&\qquad = \lim_{x\to +\infty} \frac{\frac{1}{1-\frac{3}{x}} \cdot \left(\frac{3}{x^2}\right)}{-\frac{1}{2x^2}} \\
&\qquad = \lim_{x\to +\infty} \frac{-6}{1 - \frac{3}{x}} \\
&\qquad = -6.
\end{aligned}
$$

Hence, $\lim\limits_{x\to +\infty} \left(1 - \dfrac{3}{x}\right)^{2x} = e^{-6}$.