Exploring Quarto and Latex

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4.1 ANTIDIFFERENTIATION AND INDIFINITE INTEGRALS

4.1.2 Integration by Substitution

Theorem 4.1.11 (Substitution Rule)

If $u = g(x)$ is a differentiable function whose range is an interval I and f is a continuous on I then,

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du$$

::: {exm:subrule} 1. $\int (1-4x)^{\frac{1}{2}}\, dx$

If we let $u = 1-4x$, then $du = -4dx$. We multiply the integrand $\frac{-4}{-4}$. Thus,

$$\int (1-4x)^{\frac{1}{2}}\, dx = \int (1-4x)^{\frac{1}{2}} \cdot \frac{-4}{-4} dx = \int u^{\frac{1}{2}} \left(-\frac{du}{4}\right) = -\frac{1}{4} \int u^{\frac{1}{2}} du = -\frac{1}{4} \cdot \frac{2u^{\frac{3}{2}}}{3} + C.$$

We put the final answer in terms of x by substituting $u = 1-4x$. Therefore,

$$\int (1-4x)^{\frac{1}{2}}\, dx = \frac{(1-4x)^{\frac{3}{2}}}{6} + C.$$

2. $\int x^2 \left(x^3-1\right)^{10} dx$

Let $u = x^3 - 1$. Then $du = 3x^2 dx$, or $\frac{du}{3} = x^2 dx$. By substitution,

$$\int x^2 \left(x^3-1\right)^{10} dx = \int u^{10} \cdot \frac{du}{3} = \frac{1}{3} \int u^{10} du = \frac{u^{11}}{33} + C = \frac{\left(x^3-1\right)^{11}}{33} + C.$$

3. $\int \frac{x}{\left(x^2+1\right)^3} dx$

Let $u = x^2 + 1$. Then $du = 2xdx$, or $\frac{du}{2} = xdx$. By substitution,

$$\int \frac{x}{\left(x^2+1\right)^3} dx = \frac{1}{2} \int u^{-3} du = \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C = -\frac{1}{4\left(x^2+1\right)^2} + C.$$

4. $\int \cos^4 x \sin x dx$

Let $u = \cos x$. Then $du = -\sin x dx$, or $-du = \sin x dx$. By substitution,

$$\int \cos^4 x \sin x dx = -\int u^4 du = -\frac{u^5}{5} + C = -\frac{\cos^5 x}{5} + C.$$

5. $\int x \sec^3\left(x^2\right) \tan\left(x^2\right) dx$

Let $u = \sec\left(x^2\right)$. Then $du = \sec\left(x^2\right)\tan\left(x^2\right) \cdot 2xdx$, or $\frac{du}{2} = \sec\left(x^2\right)\tan\left(x^2\right) \cdot xdx$. By substitution,

$$\begin{aligned}
\int x \sec^3\left(x^2\right) \tan\left(x^2\right) dx &= \int \sec^2\left(x^2\right) \sec\left(x^2\right) \tan\left(x^2\right) \cdot xdx \\
&= \int u^2 du = \frac{1}{2} \cdot \frac{u^3}{3} + C \\
&= \frac{\sec^3\left(x^2\right)}{6} + C.
\end{aligned}$$

6. $\int \frac{\tan\frac{1}{s} + \tan\frac{1}{s}\sin\frac{1}{s}}{s^2\cos\frac{1}{s}} ds$ Let $u = \frac{1}{s}$. Then $du = -\frac{1}{s^2} ds$ or $-du = \frac{ds}{s^2}$. By substitution,

$$\begin{aligned}
\int \frac{\tan\frac{1}{s} + \tan\frac{1}{s}\sin\frac{1}{s}}{s^2\cos\frac{1}{s}} ds &= -\int \frac{\tan u + \tan u \sin u}{\cos u} du \\
&= -\int (\sec u \tan u + \tan^2 u) du \\
&= -\int (\sec u \tan u + \sec^2 u - 1) du \\
&= -(\sec u + \tan u - u) + C \\
&= -\sec\frac{1}{s} - \tan\frac{1}{s} + \frac{1}{s} + C.
\end{aligned}$$

7. $\int t\sqrt{t-1} dt$

Let $u = t - 1$. Then $u = dt$. Also, $t = u + 1$. By substitution,

$$\begin{aligned}
\int t\sqrt{t-1} dt = \int (u+1) u^{\frac{1}{2}} du = \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du = \frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} + C \\
= \frac{2\left(t-1\right)^{\frac{5}{2}}}{5} + \frac{2\left(t-1\right)^{\frac{3}{2}}}{3} + C.
\end{aligned}$$

8. $\int \frac{t^3}{\sqrt{t^2+3}} dt$

Let $u = t^2 + 3$. Then $du = 2tdt$, or $\frac{du}{2} = tdt$. Also, $t^2 = u - 3$. By substitution,

$$\begin{aligned}
\int \frac{t^3}{\sqrt{t^2+3}} dt &= \int \frac{t^2 \cdot t}{\sqrt{t^2+3}} dt = \int u^{\frac{-1}{2}} (u-3) \frac{du}{2} \\
&= \frac{1}{2} \int \left(u^{\frac{1}{2}} - 3u^{\frac{-1}{2}} \right) du = \frac{1}{2} \left(\frac{2u^{\frac{3}{2}}}{3} - 6u^{\frac{1}{2}} \right) + C \\
&= \frac{\left(t^2+3\right)^{\frac{3}{2}}}{3} - 3\left(t^2+3\right)^{\frac{1}{2}} + C.
\end{aligned}$$

9. $\int \sqrt{4+\sqrt{x}} dx$

Let $u = 4 + \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} dx$ or $2du = \frac{dx}{\sqrt{x}}$. By substitution,

$$\begin{aligned}
\int \sqrt{4+\sqrt{x}} dx &= \int \sqrt{4+\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} dx \\
&= \int \sqrt{4+\sqrt{x}} \cdot \sqrt{x} \cdot \frac{dx}{\sqrt{x}} (\sqrt{x} = u - 4) \\
&= \int u^{\frac{1}{2}} \cdot (u-4) \cdot 2du \\
&= \int (2u^{\frac{3}{2}} - 8u^{\frac{1}{2}}) du \\
&= \frac{2 \cdot 2u^{\frac{5}{2}}}{5} - \frac{2 \cdot 8u^{\frac{3}{2}}}{3} + C \\
&= \frac{4\left(4+\sqrt{x}\right)^{\frac{5}{2}}}{5} - \frac{16\left(4+\sqrt{x}\right)^{\frac{3}{2}}}{3} + C.
\end{aligned}$$

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4.1.3 Particular Antiderivatives

Now suppose that given a function $f(x)$, we wish to find a particular antiderivative $F(x)$ of $f(x)$ that satisfies a given condition. Such a condition is called an initial or boundary condition.

1. Given that $F'(x) = 2x$ and $F(2) = 6$, find $F(x)$.

Solution

Since $F'(x) = 2x$, we have

$$F(x) = \int 2x dx = x^2 + C.$$

The initial condition $F(2) = 6$ implies that $F(2) = 2^2 + C = 6$. We get $C = 2$. Therefore, the particular antiderivative that we wish to find is:

$$F(x) = x^2 + 2.$$

2. The slope of the the tangent line at any point (x, y) on a curve is given by $3\sqrt{x}$. Find an equation of the curve if the point $(9, 4)$ is on the curve.

Solution

Let $y = F(x)$ be an equation of the curve. The slope of the tangent line m_{TL} at a point (x, y) on the graph of the curve is given by F'(x)=$3\sqrt{x}$. We have