Exploring Quarto and Latex

Ryan Lajola

1 Integration by Substitution

1.1 Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is a continuous on I then,

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du$$

1.$\int (1-4x)^{\frac{1}{2}}\, dx$

If we let $u = 1-4x$, then $du = -4dx$. We multiply the integrand $\frac{-4}{-4}$. Thus,

$$\int (1-4x)^{\frac{1}{2}}\, dx = \int (1-4x)^{\frac{1}{2}} \cdot \frac{-4}{-4} dx = \int u^{\frac{1}{2}} \left(-\frac{du}{4}\right) = -\frac{1}{4}\int u^{\frac{1}{2}} du = -\frac{1}{4} \cdot \frac{2u^{\frac{3}{2}}}{3} + C.$$

We put the final answer in terms of x by substituting $u = 1-4x$. Therefore,

$$\int (1-4x)^{\frac{1}{2}}\, dx = \frac{(1-4x)^{\frac{3}{2}}}{6} + C.$$

2.$\int x^2 \left(x^3-1\right)^{10} dx$

Let $u = x^3 - 1$. Then $du = 3x^2 dx$, or $\frac{du}{3} = x^2 dx$. By substitution,

$$\int x^2 \left(x^3-1\right)^{10} dx = \int u^{10} \cdot \frac{du}{3} = \frac{1}{3}\int u^{10} du = \frac{u^{11}}{33} + C = \frac{\left(x^3-1\right)^{11}}{33} + C.$$

3.$\int \frac{x}{\left(x^2+1\right)^3} dx$

Let $u = x^2 + 1$. Then $du = 2xdx$, or $\frac{du}{2} = xdx$. By substitution,

$$\int \frac{x}{\left(x^2+1\right)^3} dx = \frac{1}{2} \int u^{-3} du = \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C = -\frac{1}{4\left(x^2+1\right)^2} + C.$$

4.$\int \cos^4 x \sin x dx$

Let $u = \cos x$. Then $du = -\sin x dx$, or $-du = \sin x dx$. By substitution,

$$\int \cos^4 x \sin x dx = -\int u^4 du = -\frac{u^5}{5} + C = -\frac{\cos^5 x}{5} + C.$$

5.$\int x \sec^3\left(x^2\right) \tan\left(x^2\right) dx$

Let $u = \sec\left(x^2\right)$. Then $du = \sec\left(x^2\right) \tan\left(x^2\right) \cdot 2xdx$, or $\frac{du}{2} = \sec\left(x^2\right) \tan\left(x^2\right) \cdot xdx$. By substitution,

$$\begin{aligned}
\int x \sec^3\left(x^2\right) \tan\left(x^2\right) dx &= \int \sec^2\left(x^2\right) \sec\left(x^2\right) \tan\left(x^2\right) \cdot xdx \\
&= \int u^2 du = \frac{1}{2} \cdot \frac{u^3}{3} + C \\
&= \frac{\sec^3\left(x^2\right)}{6} + C.
\end{aligned}$$

6.$\int \frac{\tan\frac{1}{s} + \tan\frac{1}{s} \sin\frac{1}{s}}{s^2 \cos\frac{1}{s}} ds$ Let $u = \frac{1}{s}$. Then $du = -\frac{1}{s^2} ds$ or $-du = \frac{ds}{s^2}$. By substitution,

$$\begin{aligned}
\int \frac{\tan\frac{1}{s} + \tan\frac{1}{s} \sin\frac{1}{s}}{s^2 \cos\frac{1}{s}} ds &= -\int \frac{\tan u + \tan u \sin u}{\cos u} du \\
&= -\int (\sec u \tan u + \tan^2 u) du \\
&= -\int (\sec u \tan u + \sec^2 u - 1) du \\
&= -(\sec u + \tan u - u) + C \\
&= -\sec\frac{1}{s} - \tan\frac{1}{s} + \frac{1}{s} + C.
\end{aligned}$$

7.$\int t\sqrt{t-1} dt$

Let $u = t - 1$. Then $u = dt$. Also, $t = u + 1$. By substitution,

$$\begin{aligned}
\int t\sqrt{t-1} dt = \int (u+1) u^{\frac{1}{2}} du = \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du = \frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} + C \\
= \frac{2(t-1)^{\frac{5}{2}}}{5} + \frac{2(t-1)^{\frac{3}{2}}}{3} + C.
\end{aligned}$$

8.$\int \frac{t^3}{\sqrt{t^2+3}}dt$

Let $u=t^2+3$. Then $du=2tdt$, or $\frac{du}{2}=tdt$. Also, $t^2=u-3$. By substitution,

$$
\begin{aligned}
\int \frac{t^3}{\sqrt{t^2+3}}dt=\int \frac{t^2\cdot t}{\sqrt{t^2+3}}dt&=\int u^{\frac{-1}{2}}(u-3)\frac{du}{2}\\
&=\frac{1}{2}\int \left(u^{\frac{1}{2}}-3u^{\frac{-1}{2}}\right)du=\frac{1}{2}\left(\frac{2u^{\frac{3}{2}}}{3}-6u^{\frac{1}{2}}\right)+C\\
&=\frac{\left(t^2+3\right)^{\frac{3}{2}}}{3}-3\left(t^2+3\right)^{\frac{1}{2}}+C.
\end{aligned}
$$

9.$\int \sqrt{4+\sqrt{x}}dx$

Let $u=4+\sqrt{x}$. Then $du=\frac{1}{2\sqrt{x}}dx$ or $2du=\frac{dx}{\sqrt{x}}$. By substitution,

$$
\begin{aligned}
\int \sqrt{4+\sqrt{x}}dx&=\int \sqrt{4+\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}}dx\\
&=\int \sqrt{4+\sqrt{x}}\cdot\sqrt{x}\cdot\frac{dx}{\sqrt{x}}(\sqrt{x}=u-4)\\
&=\int u^{\frac{1}{2}}\cdot(u-4)\cdot 2du\\
&=\int (2u^{\frac{3}{2}}-8u^{\frac{1}{2}})du\\
&=\frac{2\cdot 2u^{\frac{5}{2}}}{5}-\frac{2\cdot 8u^{\frac{3}{2}}}{3}+C\\
&=\frac{4\left(4+\sqrt{x}\right)^{\frac{5}{2}}}{5}-\frac{16\left(4+\sqrt{x}\right)^{\frac{3}{2}}}{3}+C.
\end{aligned}
$$

1.1.1 Particular Antiderivatives

Now suppose that given a function $f(x)$, we wish to find a particular antiderivative $F(x)$ of $f(x)$ that satisfies a given condition. Such a condition is called an initial or boundary condition.

::: {#exm:particular_derivatives} 1. Given that $F'(x)=2x$ and $F(2)=6$, find $F(x)$.

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Solution.

Since $F'(x)=2x$, we have $$
F(x)=\int 2xdx=x^2+C.
$$The initial condition $F(2)=6$ implies that $F(2)=2^2+C=6$. We get $C=2$. Therefore, the pa
F(x)=x^2+2.
$$
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2. The slope of the the tangent line at any point (x, y) on a curve is given by $3\sqrt{x}$. Find an equation of the curve if the point $(9, 4)$ is on the curve.

   Solution.

   1.2 Let $y = F(x)$ be an equation of the curve. The slope of the tangent line m_{TL} at a point (x, y)