Chapter 2

Introduction

- Boolean Algebra
  - Similar to “regular” Algebra
  - Developed by George Boole.
- Used to optimize circuits for lowest cost
  - Why?

Basic Definitions

- Set of Elements (S)
- Set of operators
  - Binary Operator
    - A rule that assigns, to each pair of elements from S, a unique element from S.
- Postulates
  - Form the basic assumptions from which it is possible to deduce the rules, theorems and properties of a mathematical system.

Boolean Algebra

- An algebraic structure defined by
  - A set of elements, B
  - Operators (+, ·)

Huntington postulates

- Closure
- Commutation
- Identity
- Inverse
- Distribution

Closure

- Closed with respect to
  - AND (·)
  - OR (+)

Identity

The element 0 is an identity element WRT +.

The element 1 is an identity element WRT $\cdot$.

Given x an element of the set

Then,

$$x + 0 = 0 + x = x$$

$$x \cdot 1 = 1 \cdot x = x$$

Commutativity

x, y elements of the set

Then,

$$x + y = y + x$$

$$x \cdot y = y \cdot x$$

Distributivity

x, y, z elements of the set

Then,

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

$$x + (y \cdot z) = (x + y) \cdot (x + z)$$

Complement

For every $x \in B$, there exists an element x' such that

$$x + x' = 1$$

$$x \cdot x' = 0$$

Last one

There exist at least 2 elements $x, y \in B$ such that $x \neq y$.

What’s missing?

We’ve talked about

- Closure
- Commutation
- Identity
- Inverse
- Distribution

What about Association?

- Applies to Boolean Algebra
- Can be proven from other postulates

Two-Valued Boolean Algebra

Defined on a set of two elements

$B = \{0, 1\}$

Two Binary Operators

AND

OR

Now must show that the Huntington postulates apply.

By definition

- Can prove the last postulate
  - There exist at least 2 elements $x, y \in B$ such that $x \neq y$.

Truth Tables

Prove Closure; Identity; Commutation; Complement

AND

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

OR

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

NOT

x	x'
0	1
1	0

Distribution

Use a truth table to prove

What have we just done

- Shown that Boolean Algebra can be defined in a formal mathematical manner
- That it is equivalent to the binary logic from Chapter 1.

Basic Theorems and Properties

- Duality Principle
  - Change
    - OR's to AND's and vice versa
    - 1's to 0's and vice versa
  - If first statement held true, then so will the result
    - $x + 0 = x$
    - $x \cdot 1 = z$

Some Postulates and Theorems

Table 2.1
Postulates and Theorems of Boolean Algebra

Postulate 2	(a)	$x + 0 = x$	(b)	$x \cdot 1 = x$
Postulate 5	(a)	$x + x' = 1$	(b)	$x \cdot x' = 0$
Theorem 1	(a)	$x + x = x$	(b)	$x \cdot x = x$
Theorem 2	(a)	$x + 1 = 1$	(b)	$x \cdot 0 = 0$
Theorem 3, involution		$(x')' = x$		
Postulate 3, commutative	(a)	$x + y = y + x$	(b)	$xy = yx$
Theorem 4, associative	(a)	$x + (y + z) = (x + y) + z$	(b)	$x(yz) = (xy)z$
Postulate 4, distributive	(a)	$x(y + z) = xy + xz$	(b)	$x + yz = (x + y)(x + z)$
Theorem 5, DeMorgan	(a)	$(x + y)' = x'y'$	(b)	$(xy)' = x' + y'$
Theorem 6, absorption	(a)	$x + xy = x$	(b)	$x(x + y) = x$

Theorems must be proved

Theorem 1a: $x + x = x$

$$
\begin{aligned}
x + x &= (x + x) \cdot 1 \\
&= (x + x)\,(x + x') \\
&= xx + xx + xx' + xx' \\
&= xx + xx' \\
&= x + xx' \\
&= x + 0 \\
&= x
\end{aligned}
$$

Theorem 1b: $x \cdot x = x$

$$
\begin{aligned}
x \cdot x &= (xx) + 0 \\
&= (xx)\,(xx') \\
&= x(x + x') \\
&= x \cdot 1 \\
&= x
\end{aligned}
$$

Theorem 2a: $x + 1 = 1$

$$
\begin{aligned}
x + 1 &= 1 \cdot (x + 1) \\
&= (x + x')(x + 1) \\
&= xx + xx' + x \cdot 1 + x' \cdot 1 \\
&= x + 0 + x + x' \\
&= x + x' \\
&= 1
\end{aligned}
$$

Theorem 2b: $x \cdot 0 = 0$ by duality

$$
\begin{aligned}
x + 1 &= 1 \cdot (x + 1) \\
&= (x + x')(x + 1) \\
&= xx + xx' + x \cdot 1 + x' \cdot 1 \\
&= x + 0 + x + x' \\
&= x + x' \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
x \cdot 0 &= 0 + (x \cdot 0) \\
&= (x \cdot x') + (x \cdot 0) \\
&= (x + x) \cdot (x + x') \cdot (x + 0) \cdot (x' + 0) \\
&= x + 1 + x + x' \\
&= x + 1 + x' \\
&= x + x' + 1 \\
&= 1
\end{aligned}
$$

Operator Precedence

- Parentheses
- Not
- And
- Or

DeMorgan's Theorems

Very useful.

$$(x + y)' = x' \cdot y'$$

$$xy = x' + y'$$

Boolean Functions

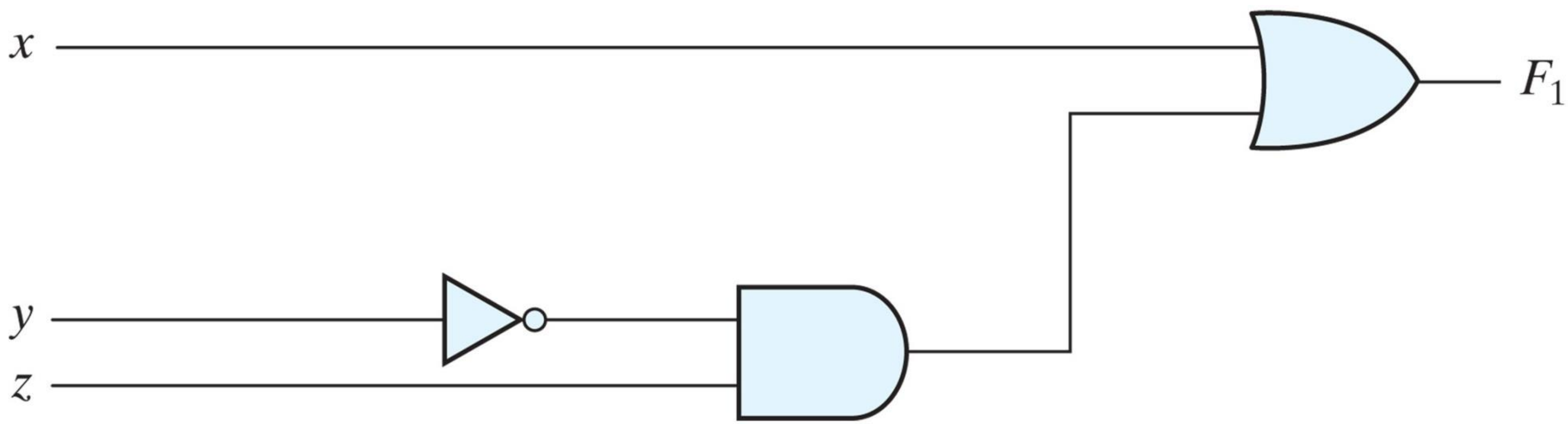
- Described by an algebraic expression
- Consisting of
  - Binary variables
  - Constants
  - Logic operation systems
- Expresses the logical relationship between binary variables

Example

$F_1 = x + y'z$

x	y	z	F_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$F_1 = x + y'z$

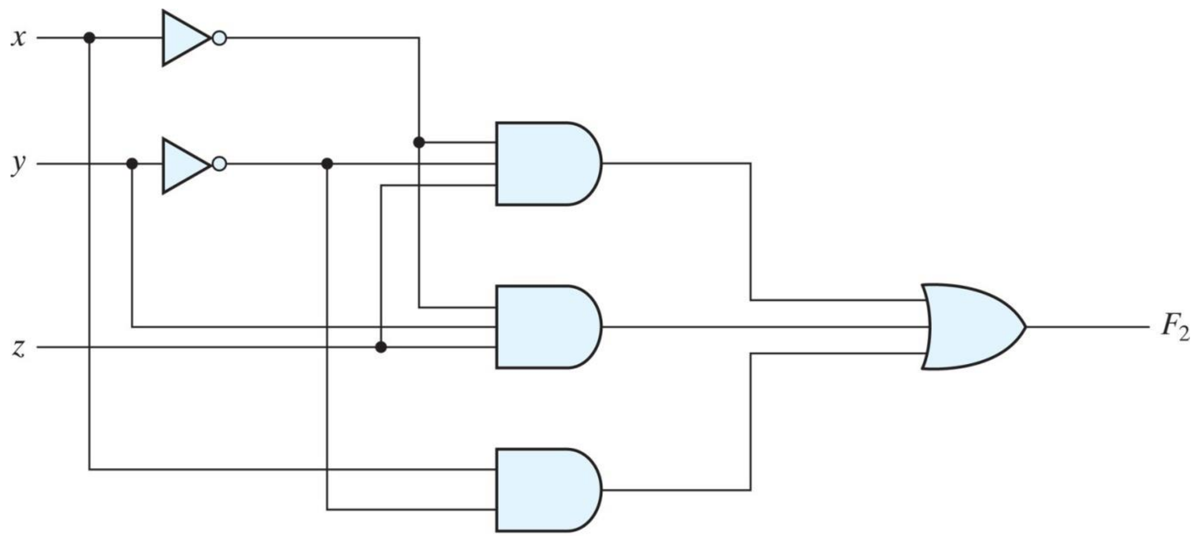


Example

$F_2 = x'y'z + x'yz + xy'$

x	y	z	F_2
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

$F_2 = x'y'z + x'yz + xy'$

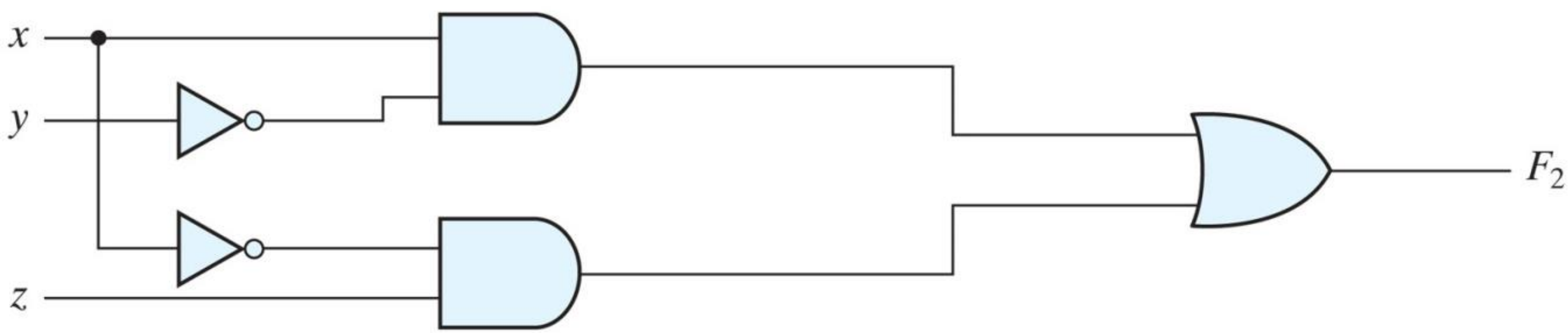


(a) $F_2 = x'y'z + x'yz + xy'$

Optimization

$F_2 = x'y'z' + x'yz + xy'$

$F_2 = xy' + x'z$



(b) $F_2 = xy' + x'z$