Options

COMP0164 Lecture 6 (Week 12)

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Agenda

Readings

- Hull chapters **10–15, 19–21**
- Brealey Myers Allen chapters **20–21**
- Bodie Kane Marcus chapters **20–21**

Topics

- Types of options
- Put-call parity
- Principal-protected notes
- Option trading strategies
- Binomial trees
- Risk-neutral valuation
- Volatility (and volatility surfaces)
- The Black-Scholes-Merton model
- Hedging strategies

Options

writer: seller of an option contract

call option: derivative security conferring a **right** (but not an obligation) to **purchase** an asset for a specified **exercise** (or **strike**) **price**

- If the exercise price is less than the market price, then the asset can be sold in the market to realise a profit.

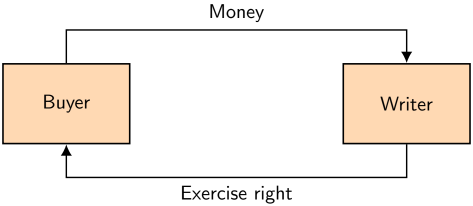
put option: derivative security conferring a **right** (but not an obligation) to **sell** an asset for a specified **exercise** (or **strike**) **price**

- If the exercise price is greater than the market price, then the asset can be purchased in the market and delivered to the put writer to realise a profit.

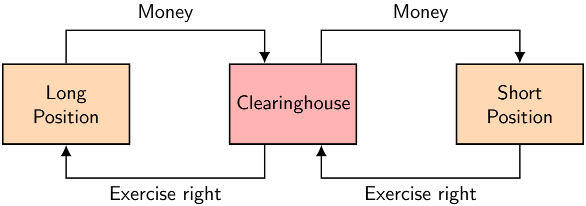
premium: the purchase price of the option

Trading options

Options can be traded **over-the-counter** (**OTC**):



Listed options are traded via an **exchange**, with standardised terms to concentrate the liquidity into a limited number of contracts:



Note: Traders who write listed options are generally required to post **margin**.

Exercising options

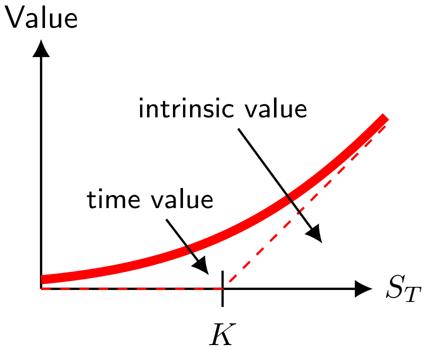
The moneyness of an option is determined by the difference between the **asset price** and the **exercise price**:

- An option is **in the money** if its exercise would produce a **positive** cash flow.
  - for a **call** option, if the exercise price is less than the asset price
  - for a **put** option, if the asset price is less than the exercise price
- An option is **out of the money** if its exercise would produce a **negative** cash flow.
  - for a **call** option, if the asset price is less than the exercise price
  - for a **put** option, if the exercise price is less than the asset price
- An option is **at the money** if the asset price and exercise price are equal.

Valuing options

The value of an option comprises two parts:

- **intrinsic value**: value an option would have if maturity were imminent
- **time value**: value of an option in excess of its intrinsic value



- K = exercise price
- S_T = asset price at maturity

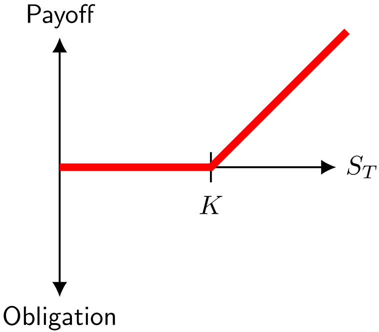
Styles of options

European options allow holders to exercise the option **on** the expiration date **only**.

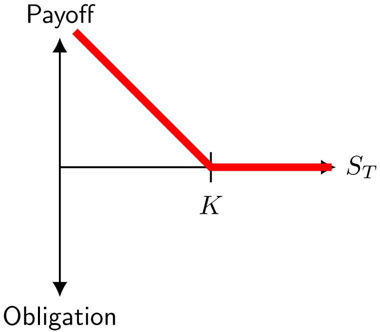
American options allow holders to exercise the option **on or before** the expiration date.

- Most listed options traded in the US are American options.
  - (FX options and equity index options are notable exceptions.)
- The extra flexibility means that American options are generally more valuable than European options.

Payoff function for option contracts held long



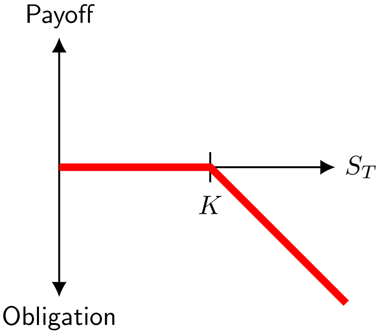
payout at maturity from **long call**
$$= \max(0, S_T - K)$$



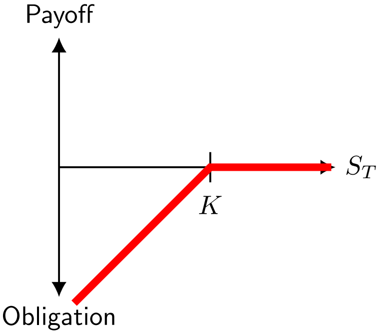
payout at maturity from **long put**
$$= \max(0, K - S_T)$$

- K = exercise price
- S_T = asset price at maturity

Payoff function for option contracts held short



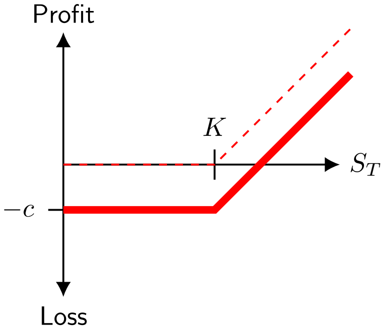
payout at maturity from **short call**
$= \min(0, K - S_T)$



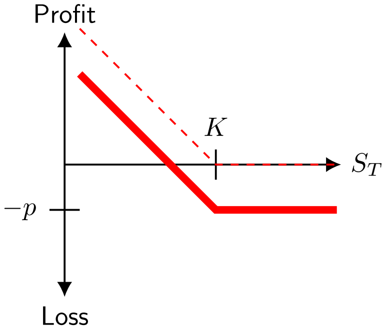
payout at maturity from **short put**
$= \min(0, S_T - K)$

- K = exercise price
- S_T = asset price at maturity

Profit function for buying options



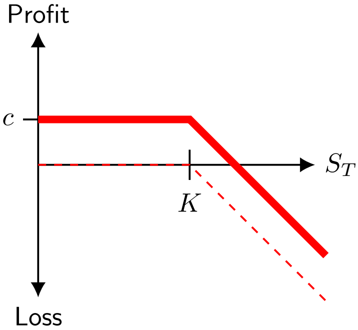
profit from **long call**
$= \max(0, S_T - K) - c$



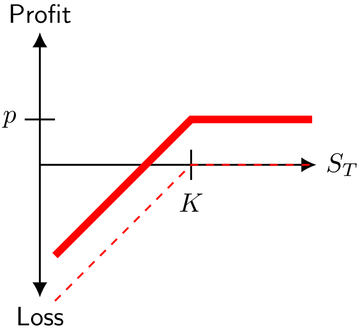
profit from **long put**
$= \max(0, K - S_T) - p$

- c, p = initial option price
- K = exercise price
- S_T = asset price at maturity

Profit function for writing options



profit from **short call**
$= c + \min(0, K - S_T)$

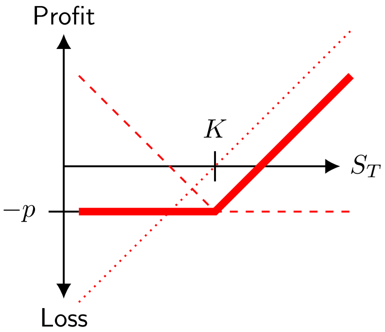


profit from **short put**
$= p + \min(0, S_T - K)$

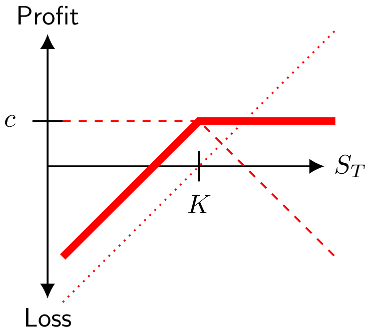
- $c, p =$ initial option price
- $K =$ exercise price
- $S_T =$ asset price at maturity

Basic option strategies

protective put $=$ asset $+$ long put



covered call $=$ asset $+$ short call



- $c, p =$ initial option price
- $K =$ exercise price
- $S_T =$ asset price at maturity

Put-call parity

Consider options on stock that does not pay dividends.

For **European** call and put options c and p, respectively:

$$c + Ke^{-rT} = p + S_0 \tag{1}$$

For **American** call and put options C and P, respectively:

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT} \tag{2}$$

- K = exercise price
- S_0 = asset price at maturity
- T = time to maturity
- r = risk-free rate

Effect of dividends on put-call parity

Consider options on dividend-paying stock.

For **European** call and put options c and p, respectively:

$$c + D + Ke^{-rT} = p + S_0 \tag{3}$$

For **American** call and put options C and P, respectively:

$$S_0 - D - K \leq C - P \leq S_0 - Ke^{-rT} \tag{4}$$

- D = present value of dividends[1]
- K = exercise price
- S_0 = asset price at maturity
- T = time to maturity
- r = risk-free rate

[1] assuming dividends are paid on the ex-dividend date

Exercising American options early

For **American put options** that are deep in the money:

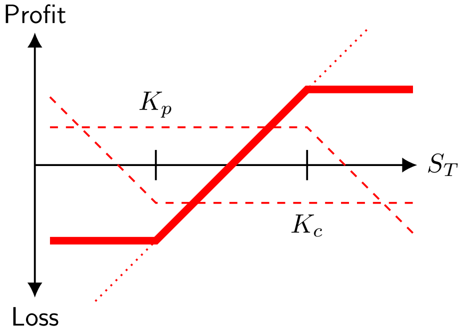
- If $r > 0$, then there are some values of S_0 for which the value of the put is its intrinsic value, $K - S_0$.
- If the put is worth its intrinsic value, then early exercise is optimal.

For **American call options** on stocks that pay dividends:

- It is possible that $D_n > K(1 - e^{-r(T-t_n)})$ for some dividend D_n to be paid at time t_n.
- In this circumstance, it is optimal to exercise the American call option just before t_n.

Collars

option collar: asset + long put + short call

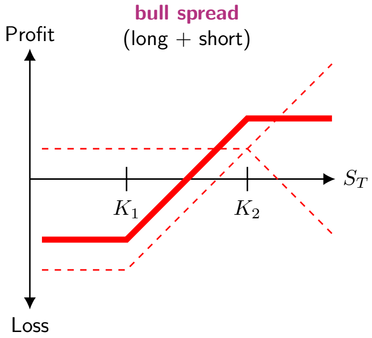


- K_p, K_c = exercise prices of put and call options
- S_T = asset price at maturity

Spreads

spread: trading strategy involving a position in two or more options of the same type (call or put)

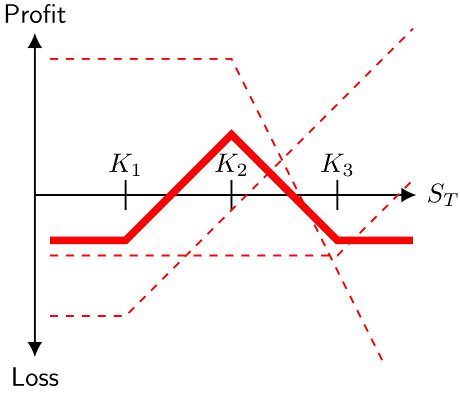
- **money spread**: spread with different strike prices
- **calendar spread** (not shown): spread with different maturity dates



- K_1, K_2 = exercise prices of options
- S_T = asset price at maturity

Butterfly spreads

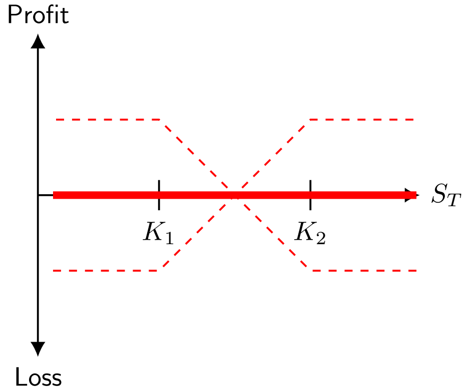
butterfly spread: two short options at a single strike bracketed by two long options of the opposite type at different strikes



- $K_1, K_2 =$ exercise prices of options
- $S_T =$ asset price at maturity

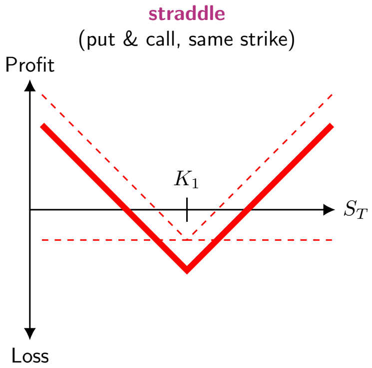
Box spreads

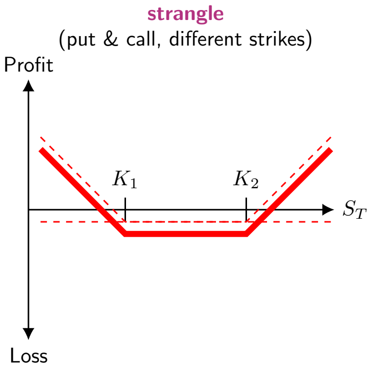
box spread: bull call spread plus bear put spread with the same strikes



- $K_1, K_2 =$ exercise prices of options
- $S_T =$ asset price at maturity

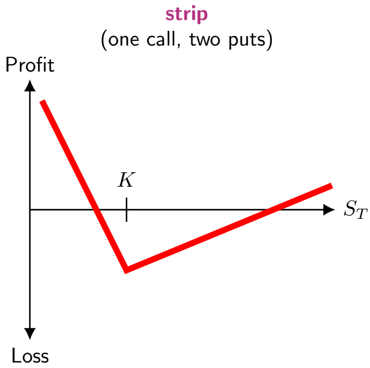
Straddles and strangles





- $K_1, K_2 =$ exercise prices of options
- $S_T =$ asset price at maturity

Strips and straps



- K = exercise price of options
- S_T = asset price at maturity

Factors affecting option prices

Factors affecting the price of a **stock option** include:

- (**1**) the current stock price, S_0
- (**2**) the strike price, K
- (**3**) the time to expiration, T
- (**4**) the volatility of the stock price, σ
- (**5**) the risk-free interest rate, r
- (**6**) the dividends that are expected to be paid

Discrete-time model for stock prices

The discrete-time version of the model for **geometric Brownian motion** is given by:

$$\frac{\Delta S}{S} = \mu \, \Delta t + \sigma \epsilon \sqrt{\Delta t} \tag{5}$$

$$\sim \phi(\mu \, \Delta t, \sigma^2 \, \Delta t) \tag{6}$$

- $S =$ stock price
- $\Delta t =$ small time interval
- $\phi(m, v) =$ normal distribution with mean m and variance v
- $\epsilon =$ normally distributed random variable $\phi(0, 1)$
- $\mu =$ expected rate of return
- $\sigma =$ stock price volatility

Lognormal property of stock prices

Assuming that $\ln S$ follows a general Wiener process with constrant drift rate $\mu - \sigma^2/2$ and constant variance σ^2, then the change in $\ln S$ between 0 and T is normally distributed:

$$\ln S_T = \phi \left[\ln S_0 + \left[\mu - \frac{\sigma^2}{2} \right] T,\ \sigma^2 T \right] \tag{7}$$

- S_0 = initial stock price
- S_T = stock price at time T
- $\phi(m, v)$ = normal distribution with mean m and variance v

Note: The standard deviation of $\ln S_T$ is $\sigma\sqrt{T}$.

Estimating volatility from historical data

If a stock price is observed at fixed intervals, then the volatility can be estimated as follows:

$$u_i = \ln\left[\frac{S_i}{S_{i-1}}\right] \quad \text{for } i \in (1, n) \tag{8}$$

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2} \tag{9}$$

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}} \tag{10}$$

- $n + 1 =$ number of observations
- $S_i =$ stock price at end of interval i
- $\tau =$ length of time interval in years
- $\hat{\sigma} =$ **estimate** of <u>annualised</u> volatility σ
- standard error is approximately $\hat{\sigma}/\sqrt{2n}$
- $\bar{u}$ is often assumed to be zero for historical estimates of σ

The Black-Scholes-Merton pricing formulas

To estimate the price of a European call option c or put option p:

$$c = S_0 N(d_1) - K e^{-rT} N(d_2) \tag{11}$$

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1) \tag{12}$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{13}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{14}$$

- $N(x) =$ cumulative probability distribution for a variable $\phi(0,1)$
- $S_0 =$ initial stock price
- $K =$ option strike price
- $r =$ risk-free rate (continuously compounded)
- $\sigma =$ stock price volatility
- $T =$ time to maturity

Interpreting the Black-Scholes-Merton formulas

The expected risk-neutral payoff is given by:

$$S_0 N(d_1) e^{rT} - K N(d_2) \tag{15}$$

The value of a European call option at time zero is:

$$\begin{align} c &= S_0 N(d_1) - K e^{-rT} N(d_2) \tag{16} \\ &= e^{-rT} N(d_2) [S_0 e^{rT} N(d_1)/N(d_2) - K] \tag{17} \end{align}$$

- e^{-rT} = present value factor
- $N(d_2)$ = probability of exercise
- $S_0 e^{rT} N(d_1)/N(d_2)$ = expected stock price in a risk-neutral world if option is exercised
- K = strike price paid if option is exercised

Binomial trees (1/2)

Suppose that for every interval Δt, a stock price moves up to u times its value or down to d times its value, in accordance with its **volatility** σ. Then the price of an option f can be calculated from its price at the end of the interval, f_u or f_d, according to the following equations:

$$f = e^{-r\Delta t}(pf_u + (1-p)f_d) \tag{18}$$

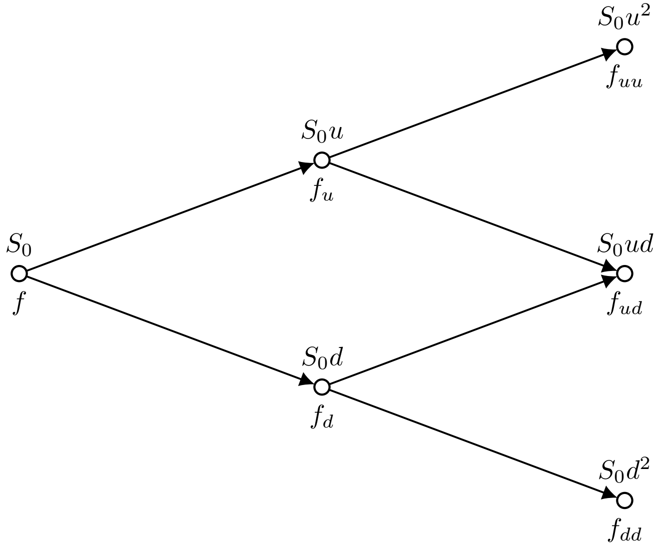
$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{19}$$

$$u = e^{\sigma\sqrt{\Delta t}} \tag{20}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{21}$$

- r = risk-free rate
- σ = volatility of underlying asset

Binomial trees (2/2)



Binomial trees and options on dividend-paying stocks

Suppose that a stock pays a known dividend yield q and the **risk-neutral** total return of dividends and capital gains r. Then capital gains must provide a return of $r - q$. Then the return after one time interval Δt is $S_0 e^{(r-q)\Delta t}$, and:

$$S_0 e^{(r-q)\Delta t} = p S_0 u + (1-p) S_0 d \tag{22}$$

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d} \tag{23}$$

$$u = e^{\sigma\sqrt{\Delta t}} \tag{24}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{25}$$

- r = risk-free rate
- S_0 = initial stock price
- σ = volatility of underlying asset

Binomial trees and options on foreign currencies

We can consider a foreign currency to be an asset with a yield at the foreign risk-free rate r_f, so:

$$p = \frac{e^{(r-r_f)\Delta t} - d}{u - d} \tag{26}$$

$$u = e^{\sigma\sqrt{\Delta t}} \tag{27}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{28}$$

- r = domestic risk-free rate
- r_f = foreign risk-free rate
- σ = volatility of underlying asset

Binomial trees and options on futures

The **risk-neutral** growth rate of a futures contract price should be zero, so for options on futures contracts:

$$p = \frac{1 - d}{u - d} \tag{29}$$

$$u = e^{\sigma\sqrt{\Delta t}} \tag{30}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{31}$$

- F_0 = initial futures contract price
- σ = volatility of underlying asset

Delta

Delta (Δ) represents the sensitivity of the value of an option (c or p, or portfolio thereof) to changes in the price of the underlying asset, S.

$$\Delta(\textbf{call}) = \frac{\partial c}{\partial S} = N(d_1) \tag{32}$$

$$\Delta(\textbf{put}) = \frac{\partial p}{\partial S} = N(d_1) - 1 \tag{33}$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{34}$$

- $N(x) =$ cumulative probability distribution for a variable $\phi(0, 1)$
- $S_0 =$ initial asset price
- $K =$ option strike price
- $r =$ risk-free rate (continuously compounded)
- $\sigma =$ volatility of underlying asset
- $T =$ time to maturity

Theta

Theta (Θ) represents the sensitivity of the value of an option (c or p, or portfolio thereof) to the passage of time, t.

$$\Theta(\textbf{call}) = \frac{\partial c}{\partial t} = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) \tag{35}$$

$$\Theta(\textbf{put}) = \frac{\partial p}{\partial t} = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(d_2) \tag{36}$$

$$N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \tag{37}$$

- $N(x)$ = cumulative probability distribution for a variable $\phi(0,1)$
- S_0 = initial asset price
- K = option strike price
- r = risk-free rate (continuously compounded)
- σ = volatility of underlying asset
- T = time to maturity
- d_1, d_2 as defined earlier

Gamma

Gamma (Γ) represents the sensitivity of the **delta** of a portfolio Π to changes in the price of the underlying asset, S.

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2} = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} \tag{38}$$

$$\Theta + rS\Delta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = r\Pi \tag{39}$$

- S_0 = initial asset price
- K = option strike price
- r = risk-free rate (continuously compounded)
- σ = volatility of underlying asset
- T = time to maturity
- $d_1, N'(x)$ as defined earlier

Vega

Vega ($\mathcal{V}$) represents the sensitivity of the price f of an option to changes in the **volatility** of the underlying asset, S.

$$\mathcal{V} = \frac{\partial f}{\partial \sigma} = S_0 N'(d_1)\sqrt{T} \tag{40}$$

- S_0 = initial asset price
- f = option price
- σ = volatility of underlying asset
- T = time to maturity
- $d_1, N'(x)$ as defined earlier

Note: The vega of a European or American option is always positive.

Rho

Rho (**rho**) represents the sensitivity of the price f of an option to changes in the (risk-free) interest rate r.

$$\textbf{rho}(\textbf{call}) = KTe^{-rT}N(d_2) \tag{41}$$

$$\textbf{rho}(\textbf{put}) = -KTe^{-rT}N(d_2) \tag{42}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{43}$$

- $N(x) =$ cumulative probability distribution for a variable $\phi(0, 1)$
- $K =$ option strike price
- $T =$ time to maturity
- $\sigma =$ volatility of underlying asset
- $r =$ risk-free rate (continuously compounded)
- d_1 as defined earlier

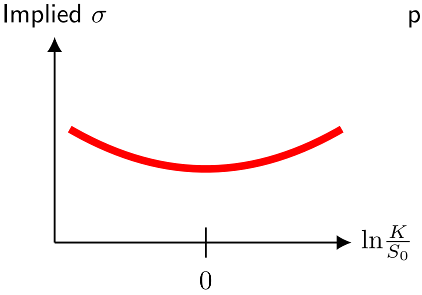
Implied volatility and volatility smile

implied volatility: volatility implied from option prices observed in the market

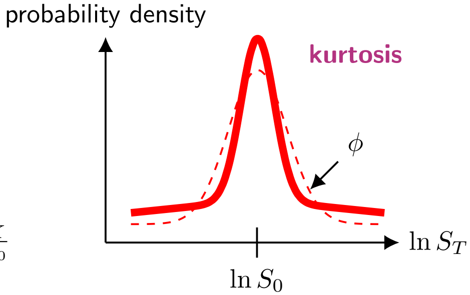
volatility smile: implied volatility as a function of moneyness K/S_0

implied distribution: risk-neutral probability distribution for an asset price at a future time T implied by the volatility smile for options maturing at that time

volatility smile
for foreign currency options

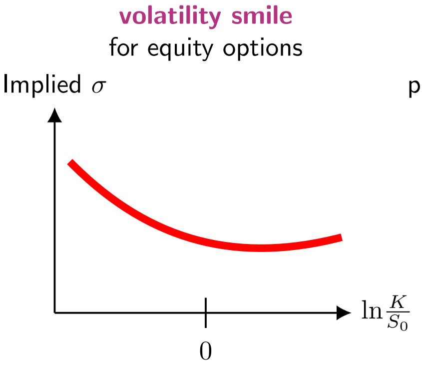


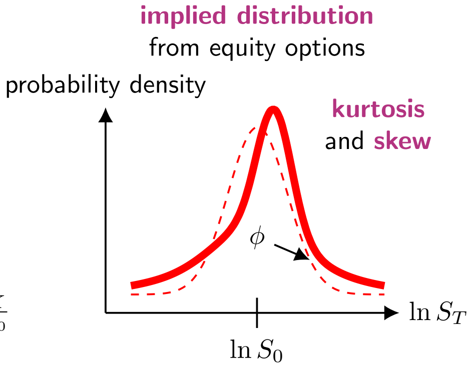
implied distribution
from foreign currency options



Volatility skew

volatility skew: asymmetry in volatility smile for options on some assets





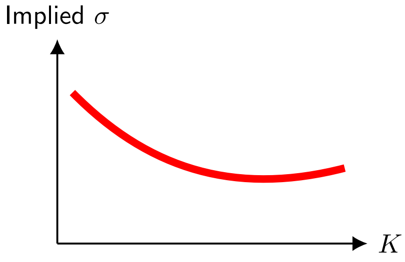
Possible explanations for volatility skew include:

- **leverage**, which increases as prices decrease
- **volatility feedback**, as investors require more return for more risk
- **risk aversion** with respect to market-level crashes

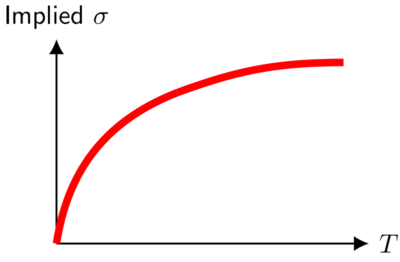
Volatility surface

A **volatility surface** can be created by calculating implied volatility as a two-dimensional function of **strike** K and **time to maturity** T.

volatility smile

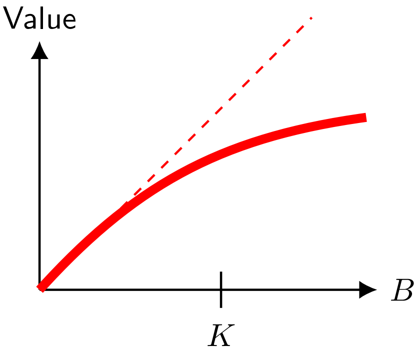


implied volatility versus **time to maturity**



Recall: Callable bonds

callable bond: a straight bond bundled with the issuance of a call option by the investor to the bond-issuing firm



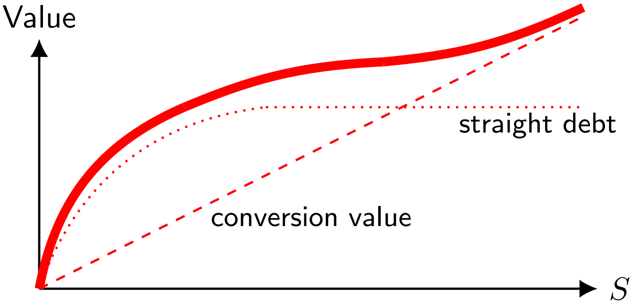
- K = call price
- B = value of straight debt

Recall: Convertible securities

Convertible securities convey options to their holders, typically to convert securities to common stock at a specified **conversion ratio**.

- **convertible bonds**
- **convertible preferred stock**

The **conversion value** is the value a security would have if converted immediately.



- S = stock price

Other option-like securities

A **warrant**, like a call option, grants its holder the right to purchase shares of stock.

- However, a warrant requires the firm to issue new shares, whereas call options allow the writer of the call to deliver an already-issued share.
- **detachable warrant**: warrant issued in conjunction with another security, for example a bond (and might be sold as a "sweetener")
- **fully diluted earnings per share**: financial reporting convention for reporting earnings under the assumption that all convertible securities and warrants are exercised

A **collateralised nonrecourse loan** gives the lender no recourse beyond the right to the collateral S.

- This amounts to an option not to repay: Suppose the value of the collateral at maturity S_T is less than the loan principal L.

Holders of **levered equity** in a firm have an implicit put option to relinquish ownership of the firm in exchange for the face value of the firm's debt.

Exotic options

Asian option: option for which its payoff depends upon the **average** price of the underlying asset during some portion of the life of the option

barrier option: option for which its payoff depends upon not only the asset price but also whether the asset price has crossed through some **barrier**

- specified as either **up** or **down** and either **in** or **out**

lookback option: option for which its payoff depends (in part) on the minimum or maximum price of the underlying asset during the life of the option

quanto option: option allowing an investor to fix in advance the exchange rate at which an investment in a foreign currency can be converted back into domestic currency

digital option: option with a fixed payoff that depends upon whether a condition is satisfied by the price of the underlying asset at maturity S_T

- for example, whether $S_T > K$ for some strike K

Thank You

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