Time Value of Money

COMP0164 Lecture 1 (Week 6)

Geoff Goodell (University College London)

3 October 2022

Agenda

Readings

- Brealey Myers Allen chapters **2–3**
- Hull chapter **4**

Topics

- Present and future value
- Annuities
- Types of rates
- The risk free rate
- Interest rates
- Zero rates
- Bond pricing
- Forward rates
- Duration
- Convexity
- Term structure of interest rates

Present and future value

$$FV = PV \times (1 + r)^n \tag{1}$$

- FV = future value (after n time periods)
- PV = present value (principal)
- n = number of time periods
- r = interest rate

Value of a series of cash flows

$$PV = \sum_{t=1}^{n} \frac{C_t}{(1+r)^t} \tag{2}$$

$$NPV = \sum_{t=0}^{n} \frac{C_t}{(1+r)^t} \tag{3}$$

$$= C_0 + PV \tag{4}$$

- PV = present value
- NPV = net present value
- C_t = cash flow at time t
- r = interest rate
- n = number of cash flows following initial investment

Note: C_0 is often <u>negative</u>

Perpetuities

A **perpetuity** (or **consol**) is a bond that offers a fixed income each time period, forever.

$$PV = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} \tag{5}$$

$$PV \times \left[1 - \frac{1}{1+r}\right] = \frac{C}{1+r} \tag{6}$$

$$PV = \frac{C}{r} \tag{7}$$

- PV = present value
- C = cash flow during each time period
- r = interest rate

Annuities

An **annuity** is a bond that offers a fixed income each time period, for a specified number of time periods.

$$PV = C\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right] \tag{8}$$

$$FV = PV \times (1+r)^n = C\left[\frac{(1+r)^n - 1}{r}\right] \tag{9}$$

- $PV =$ present value
- $FV =$ future value (after n time periods)
- $C =$ cash flow during each time period
- $n =$ number of time periods
- $r =$ interest rate

Growing perpetuities

Suppose that $C_i = (1+g)C_{i-1}$ and $g < r$. Then:

$$PV = \sum_{t=1}^{\infty} \frac{C_1(1+g)^{t-1}}{(1+r)^t} = \frac{C_1}{1+r} \sum_{t=0}^{\infty} \left[\frac{1+g}{1+r}\right]^t \tag{10}$$

$$PV \times \left[1 - \frac{1+g}{1+r}\right] = \frac{C_1}{1+r} \tag{11}$$

$$PV = \frac{C_1}{r-g} \tag{12}$$

- PV = present value
- C = cash flow during each time period
- g = payment growth rate
- r = interest rate

Question: What happens if $g \geq r$?

Growing annuities

$$PV = \frac{C_1}{r-g}\left[1-\left[\frac{1+g}{1+r}\right]^n\right] \tag{13}$$

$$FV = PV \times (1+r)^n = C_1\left[\frac{(1+r)^n-(1+g)^n}{r-g}\right] \tag{14}$$

- PV = present value
- FV = future value (after n time periods)
- C = cash flow during each time period
- g = payment growth rate
- r = interest rate

Periodic compounding (1)

$$FV = PV \times \left[1 + \frac{r}{m}\right]^{mn} \tag{15}$$

- FV = future value (after n time periods)
- PV = present value (principal)
- m = compounding frequency (per time period)
- n = number of time periods
- r = interest rate

Periodic compounding (2)

To convert between different compounding frequencies m_1 and m_2:

$$PV \times \left[1 + \frac{r_1}{m_1}\right]^{m_1 n} = PV \times \left[1 + \frac{r_2}{m_2}\right]^{m_2 n} \tag{16}$$

$$r_2 = m_2 \left[\left[1 + \frac{r_1}{m_1}\right]^{m_1/m_2} - 1\right] \tag{17}$$

"equivalent annual interest rate": If $m_2 = 1$, then:

$$r_2 = \left[1 + \frac{r_1}{m_1}\right]^{m_1} - 1 \tag{18}$$

- PV = present value (principal)
- m_1, m_2 = compounding frequency (per time period)
- n = number of time periods
- r_1 = interest rate for compounding frequency m_1
- r_2 = interest rate for compounding frequency m_2

Continuous compounding (1)

$$FV = PV \times e^{rn} \tag{19}$$

- FV = future value (after n time periods)
- PV = present value (principal)
- n = number of time periods
- r = interest rate

Continuous compounding (2)

To convert between continuous and periodic compounding:

$$PV \times e^{r_c n} = PV \times \left[1 + \frac{r_m}{m}\right]^{mn} \tag{20}$$

$$r_c = m \ln\left[1 + \frac{r_m}{m}\right] \tag{21}$$

$$r_m = m(e^{r_c/m} - 1) \tag{22}$$

- PV = present value (principal)
- m = compounding frequency (per time period)
- n = number of time periods
- r_c = interest rate for continuous compounding
- r_m = interest rate for periodic compounding

Types of rates

treasury rate: the rate an investor earns on instruments (bills, notes, bonds) used by a sovereign government to borrow in its own currency.

overnight rate: the interest rate at which a depository institution (generally banks) lends or borrows funds with another depository institution in the overnight market.
Source: `http://investopedia.com/terms/o/overnightrate.asp`

- **overnight market**: the market for borrowing and lending surplus funds with the central bank overnight.

repo rate: a secured rate in which a financial institution that owns securities agrees to sell the securities for a certain price and buy them back at a later time for a slightly higher price.

- **overnight repo**: funds are lent overnight (most common)
- **term repo**: longer-term arrangements

Unsecured overnight rates

Examples

United Kingdom: sterling overnight index average (SONIA)

Eurozone: euro short-term rate (ESTER)

- **replaces** the euro overnight index average (EONIA)

Switzerland: Swiss average rate overnight (SARON)

Japan: Tokyo overnight average rate (TONAR)

United States: effective federal funds rate

- weighted average of **federal funds rate** in brokered transactions

Central banks may intervene with their own transactions to manage these rates.

Reference rates

Reference rates are often used by counterparties to a transaction to define future payment obligations as a function of prevailing market interest rates.

In most of the world, reference rates are **unsecured overnight rates**, determined by transactions between banks when they manage central bank reserves.

In the **United States**, the reference rate is an **overnight repo** rate, the secured overnight financing rate (**SOFR**):

- a volume-weighted median average of the rates on overnight repo transactions.
- generally very slightly lower than the effective federal funds rate

What happened to the London interbank offered rate (**LIBOR**)?

- a combination of quotes from global banks to estimate unsecured rates in interbank transactions (incorporating a credit spread)
- not enough interbank borrowing for market transactions alone to determine this rate, so it is subject to judgement and thus manipulation
- bank regulators are phasing out its use by financial institutions

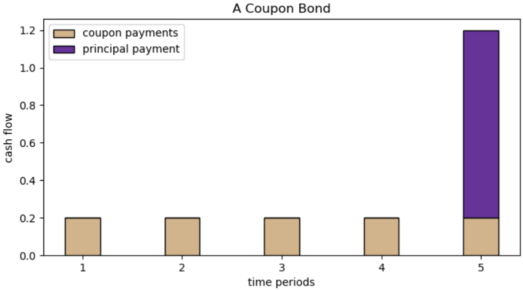
The risk free rate

risk-free rate: the rate of interest that can be earned without assuming any risks.

In practice, the risk-free rate used to price derivatives is created from **overnight rates** and not **treasury rates**.

- Banks are generally not required to maintain capital for investments in treasury securities.
- But for other investments, including low-risk investments, banks are required to maintain capital, so the treasury rate is artificially low.

Coupon bonds



Coupon bonds have a fixed schedule of payments, comprising periodic **coupons** plus a return of **principal** at the end of the final period:

$$PV = c\left[\frac{1}{r} - \frac{1}{r(1+r)^n}\right] + \frac{P}{(1+r)^n} \tag{23}$$

- $P =$ principal
- $c =$ coupon
- $r =$ discount rate
- $n =$ number of time periods

Yield to maturity

The rate of return of a bond is its **yield to maturity**, which can be expressed in terms of periodic or continuous compounding:

$$PV = \frac{P}{(1+y)^n} + \sum_{t=1}^{n} \frac{c}{(1+y)^t} \tag{24}$$

$$PV = Pe^{-yn} + \sum_{t=1}^{n} ce^{-yt} \tag{25}$$

- $P =$ principal
- $c =$ coupon
- $y =$ **yield to maturity**
- $n =$ number of time periods

The **yield to maturity** can be found by solving for y via an iterative process. Going forward, we shall use continuous compounding unless otherwise stated.

Zero rates

The **zero rate** (or **spot rate**, or **zero-coupon interest rate**) is the rate of return that would be earned on a bond that provides no coupons.

$$PV = Pe^{-yn} \tag{26}$$

- $P =$ principal
- $n =$ number of time periods
- $y =$ yield to maturity, which is also the **zero rate**

A **stripped bond** (or **strip**) is an example of a bond with a single cash payment; this can result from an issuer splitting a normal coupon bond into a collection of single-payment bonds.

Par yield

The **par yield** of a bond is the coupon rate c that causes the bond price to equal the value of its principal (its **par value**):

$$P = Pe^{-r(n)n} + \sum_{t=1}^{n} ce^{-r(t)t} \tag{27}$$

- $P =$ principal
- $c =$ **par yield**
- $n =$ number of time periods
- $r(x) =$ zero rate at time x

$$c = \frac{P(1 - e^{-r(n)n})m}{A} \tag{28}$$

- $A =$ value of an annuity that pays one unit on coupon payment dates
- $m =$ coupon payment frequency (per time period)

Bootstrap method to calculate zero rates

Given a table of bond prices and coupon rates for different times to maturity, it is often possible to use the **bootstrap method** to iteratively calculate the zero rates for successive maturities, as follows.

Start with the shortest maturity bond. For each given maturity n, construct an expression for the bond of maturity n with price B, par value P, and coupon c:

$$B = Pe^{-r(n)n} + \sum_{t=1}^{n} ce^{-r(t)t} \tag{29}$$

Solve for $r(n)$, the zero rate at time n, using previously calculated values of $r(t)$ for $t < n$.

The resulting set of zero rates as a function of maturity is called the **zero curve** (or **zero-coupon yield curve**).

Bootstrap method, example (1)

principal	maturity (years)	coupon	price	yield	zero rate (continuous)
100	0.25	0	99.6	1.6064(Q)	**?**
100	0.50	0	99.0	2.0202(SA)	...
100	1.00	0	97.8	2.2495(A)	...
100	1.50	4	102.5	2.2949(SA)	...
100	2.00	5	105.0	2.4238(SA)	...

Source: Hull

Bootstrap method, example (2)

principal	maturity (years)	coupon	price	yield	zero rate (continuous)
100	0.25	0	99.6	1.6064(Q)	1.603
100	0.50	0	99.0	2.0202(SA)	2.010
100	1.00	0	97.8	2.2495(A)	2.225
100	1.50	4	102.5	2.2949(SA)	2.284
100	2.00	5	105.0	2.4238(SA)	2.416

Source: Hull

Forward rates

The **law of one price** states that the same commodity must sell at the same price in a well-functioning market.

A **forward interest rate** is a rate of interest implied by current zero rates for periods of time that start in the future.

$$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1} \quad (30)$$

$$= R_2 + \frac{(R_2 - R_1) T_1}{T_2 - T_1} \quad (31)$$

- R_1 = zero rate for T_1
- R_2 = zero rate for T_2
- R_F = **forward interest rate** for the period between T_1 and T_2

Instantaneous forward rates

An **instantaneous forward interest rate** is a rate of interest applicable to a very short time period starting at some moment in the future.

$$R_F = R + \frac{\partial R}{\partial T} \tag{32}$$

$$= -\frac{\partial}{\partial T} \ln P(0,T) \tag{33}$$

- $R =$ zero rate for T
- $R_F =$ **instantaneous forward interest rate** at time T
- $P(0,T) = e^{-RT} =$ price of a zero-coupon bond maturing at time T

Forward rate agreements

A **forward rate agreement** is an agreement to exchange a predetermined fixed rate for a reference rate that will be observed in the market at a future time.

Suppose there is an agreement between two parties to exchange R_K for some reference rate R_F of maturity τ to be observed at time t, applied to a principal L. Then, at time $t + \tau$, one party (the floating rate seller) would pay $\tau(R_F - R_K)L$ to the other party (the floating rate buyer).

- $t =$ the future time at which the reference rate shall be observed
- $R_K =$ the fixed rate agreed to in the FRA
- $R_F =$ the forward rate for the reference rate to be observed at time t
- $\tau =$ the period of time to which the rates apply
- $L =$ the principal in the contract

Note: The principal itself is not exchanged.

Valuing forward rate agreements

The value of an outstanding forward rate agreement is given by:

$$PV = \tau(R_F - R_K)L \tag{34}$$

- $R_K =$ the fixed rate agreed to in the FRA
- $R_F =$ the current forward rate for the reference rate
- $\tau =$ the period of time to which the rates apply
- $L =$ the principal in the contract

Note: Normally, when a forward rate agreement is established, $R_K = R_F$.

Duration (1)

Suppose that a bond has price B, yield to maturity y, and cash flows c_i at times t_i. Then:

$$B = \sum_{i=1}^{n} c_i e^{-y t_i} \tag{35}$$

The **duration** is the weighted average of the times when payments are made, with the weight assigned to time t_i equal to the share of the present value of the bond represented by the cash flow at time t_i:

$$D = \frac{1}{B} \sum_{i=1}^{n} t_i c_i e^{-y t_i} \tag{36}$$

The duration is thus a measure of the average life of the bond.

Duration (2)

Duration can be used to characterise the relationship between the change in the price of a bond and the corresponding change in its yield.[1] Consider:

$$\Delta B \approx \frac{dB}{dy}\Delta y \tag{37}$$

$$= -\Delta y \sum_{i=1}^{n} c_i t_i e^{-y t_i} \tag{38}$$

$$= -BD\Delta y \tag{39}$$

Thus:

$$D \approx -\frac{\Delta B}{B \Delta y} \tag{40}$$

[1] As shown by Frederick Macaulay in 1938.

Modified duration

If y is expressed with **periodic** rather than continuous compounding, then:

$$\Delta B = -\frac{BD\Delta y}{1 + y/m} \tag{41}$$

- $m =$ compounding frequency (per time period)

Define the **modified duration** as follows:

$$D^* = \frac{D}{1 + y/m} \tag{42}$$

Then the change in the price of the bond can be given by:

$$\Delta B = -BD^*\Delta y \tag{43}$$

The **dollar duration** $D_\$$ is the product of modified duration and bond price, so $D_\$ = BD^*$.

Duration and bond portfolios

The duration D of a **bond portfolio** can be defined as a price-weighted average of the durations of the bonds in the portfolio.

This approach can be used to estimate the change in the value of the portfolio as a function of the change in yields of the underlying bonds.

However:

- The duration approximation is less accurate for larger changes in yield, so it should be considered effective only for **small changes**.
- When the bonds in the portfolio have differing maturities, the duration approximation assumes that the yields of all the bonds will change by about the same amount, so it should be considered effective only for **parallel shifts**.

Convexity

Convexity is a measure of the curvature in the relationship between bond prices and bond yields, given by:

$$C = \frac{1}{B}\frac{d^2B}{dy^2} = \frac{1}{B}\sum_{i=1}^{n} c_i t_i^2 e^{-yti} \tag{44}$$

By applying Taylor series expansions, we can incorporate convexity into our model of the sensitivity of bond prices to changes in bond yields:

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}\Delta y^2 \tag{45}$$

Therefore:

$$\frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2 \tag{46}$$

(Adjusting for convexity can improve the estimate of the change in the price of a bond portfolio for relatively large changes in yield, but not for nonparallel shifts.)

Inflation

Inflation is the increase in the prices of goods and services over time.
Source: `https://www.federalreserve.gov/faqs/economy_14419.htm`

Inflation can be quantified as the difference between real and nominal rates of return.

$$C_r = \frac{C_n}{(1+i)^t} \tag{47}$$

- C_r = real cash flow
- C_n = nominal cash flow
- i = inflation rate

$$1 + r_r = \frac{1 + r_n}{1 + i} \tag{48}$$

- r_r = real interest rate
- r_n = nominal interest rate

Theories of the term structure of interest rates

The **term structure of interest rates** is the relationship between short- and long-term interest rates, evidenced by the shape of the zero curve.

What determines the shape of the zero curve?

- **expectations theory**: long-term interest rates reflect traders' estimates of future short-term interest rates.
- **market segmentation theory**: the curve represents a collection of distinct markets, each with its own set of investors, each with its own equilibrium.
- **liquidity preference theory**: investors prefer to invest funds for short periods of time, and borrowers prefer to borrow at fixed rates for long periods of time.
  - long-term yields are usually greater than short-term yields.
  - forward rates are usually greater than expected future zero rates.

Risk considerations

What happens when an investor (or bank) funds its long-term investments with short-term borrowing?

- During normal times, profit on the upward-sloping yield curve
- Pay a lower rate when **borrowing** over a shorter term, and earn a higher rate by **lending** over a longer term.
- When the term for borrowing is up, borrow again.
- What could go wrong?

When yields rise suddenly: Orange County example

A **yield curve play** is a trading strategy wherein an investor speculates that rates in the future will be different from forward rates today, in a particular direction.

- In 1992, the Treasurer of Orange County (California) entered into a strategy of short-term borrowing and long-term lending.
- Also, he borrowed in the repo market to add leverage to this position.
- Finally, he used **inverse floaters** (more on these later), which paid a fixed rate minus a floating rate.
- And so, he implicitly bet that short-term rates in the future would be the same or lower than today (but that did not happen).

The risk can be addressed with forward rate agreements.

When liquidity evaporates: Northern Rock example

During a **"flight to quality"** financial institutions and investors are less inclined to take credit risks, and they instead look for safe investments.

Northern Rock (Newcastle) financed most of its (long-term) mortgage portfolio with (short-erm) wholesale deposits.

- In 2007, depositors refused to roll over the funding they were providing to Northern Rock.
- As a result, Northern Rock could no longer finance its assets.
- Northern Rock was taken over by the UK government in early 2008.

To be continued in three weeks when we discuss credit risk.

Thank You

Photo Credit: https://www.pinterest.co.uk/pin/736268239051855079/