Portfolio Theory and Management

COMP0164 Lecture 7 (Week 13)

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Agenda

Readings

- Bodie Kane Marcus chapters **5–11, 24**

Topics

- historical characteristics of asset returns
- tail risk
- short-run versus long-run investments
- risk tolerance and asset allocation
- the capital allocation line
- portfolio optimisation
- index models
- the capital asset pricing model
- arbitrage pricing theory
- multi-factor models of risk and return
- efficient market hypothesis

Portfolio theory and management

Investors are paid to take **risk**.

The challenge is to **manage** this risk.

Measuring **performance** is about measuring the management of risk.

Some useful definitions

The **variance** σ^2 of a variable is defined as follows:

$$\sigma^2(r) = \sum_{i=1}^{n} p(i)(r(i) - E[r])^2 \tag{1}$$

The **covariance** between two variables is defined as follows:

$$\begin{aligned} \mathrm{Cov}(r_1, r_2) &= E[(r_1 - E[r_1])(r_2 - E[r_2])] \quad (2) \\ &= E[r_1 r_2] - E[r_1]E[r_2] \quad (3) \end{aligned}$$

The **correlation** ρ between two variables is defined as follows:

$$\rho(r_1, r_2) = \mathrm{Corr}(r_1, r_2) = \frac{\mathrm{Cov}(r_1, r_2)}{\sigma_1 \sigma_2} \tag{4}$$

Recall: Estimating volatility from historical data

If a stock price is observed at fixed intervals, then the volatility can be estimated as follows:

$$u_i = \ln\left[\frac{S_i}{S_{i-1}}\right] \quad \text{for } i \in (1, n) \tag{5}$$

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2} \tag{6}$$

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}} \tag{7}$$

- $n + 1 =$ number of observations
- $S_i =$ stock price at end of interval i
- $\tau =$ length of time interval in years
- $\hat{\sigma} =$ **estimate** of annualised volatility σ
- standard error is approximately $\hat{\sigma}/\sqrt{2n}$
- $\bar{u}$ is often assumed to be zero for historical estimates of σ

Reward versus volatility: the Sharpe ratio

A simple way to measure the performance of a portfolio is to measure the ratio of return versus risk, using volatility as a proxy for risk. This is the **Sharpe ratio**:

$$\frac{\text{risk premium}}{\text{SD of excess return}} = \frac{E(r) - r_f}{\sigma} \tag{8}$$

- $E(r)$ = expected return[1]
- r_f = risk-free rate
- σ = standard deviation of return in excess of risk-free rate[1]

But: In practice, log returns are seldom normally distributed

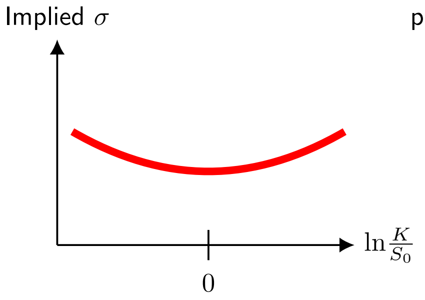
[1] for some time period τ

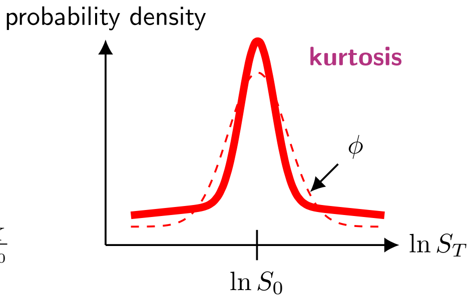
Recall: Implied volatility and volatility smile

implied volatility: volatility implied from option prices observed in the market

volatility smile: implied volatility as a function of moneyness K/S_0

implied distribution: risk-neutral probability distribution for an asset price at a future time T implied by the volatility smile for options maturing at that time

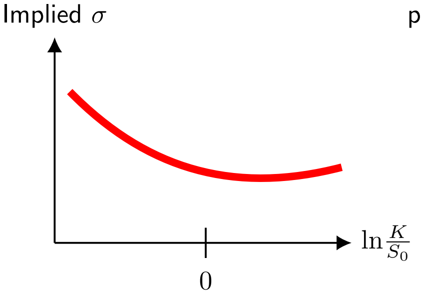




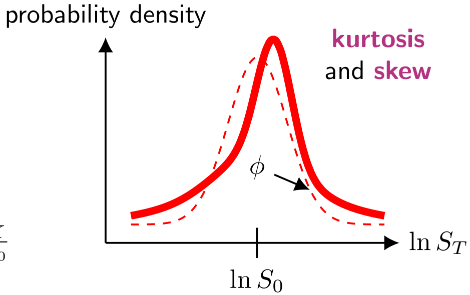
Recall: Volatility skew

volatility skew: asymmetry in volatility smile for options on some assets

volatility smile
for equity options



implied distribution
from equity options



Possible explanations for volatility skew include:

- **leverage**, which increases as prices decrease
- **volatility feedback**, as investors require more return for more risk
- **risk aversion** with respect to market-level crashes

Skew and kurtosis

Using volatility as a measure of risk can underestimate the risk of assets whose log returns exhibit characteristics of **skew** and **kurtosis**, which can be estimated as follows:

$$u_i = \ln\left[\frac{S_i}{S_{i-1}}\right] \quad \text{for } i \in (1, n) \tag{9}$$

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2} \tag{10}$$

$$\textbf{skew} = \tilde{\mu}_3 = \frac{(u - \bar{u})^3}{ns^3} \tag{11}$$

$$\textbf{kurtosis} = \tilde{\mu}_4 = \frac{(u - \bar{u})^4}{ns^4} - 3 \tag{12}$$

- $n + 1 =$ number of observations
- $S_i =$ stock price at end of interval i

Other popular ways to measure tail risk

Value at Risk (**VaR**): loss incurred at a given quantile of the sample distribution

- for example, the degree of loss at the 1st (lowest) or 5th percentile

expected shortfall (**ES**): expectation of loss conditioned upon being in the portion of the sample distribution to the left of a given quantile

- the average of all observations less than the specified quantile

Sortino ratio: like Sharpe ratio, but using the **lower partial standard deviation** (standard deviation of negative values only) instead of the standard deviation

- ignores both positive excess returns and frequency of negative excess returns

relative frequency of -3σ returns: ratio of the fraction of observations with returns more than three standard deviations below the mean to the relative frequency of such returns in the normal distribution

Risk aversion

utility function: function relating specific goods and services in an economy to individual preferences
Collins English Dictionary, 2014

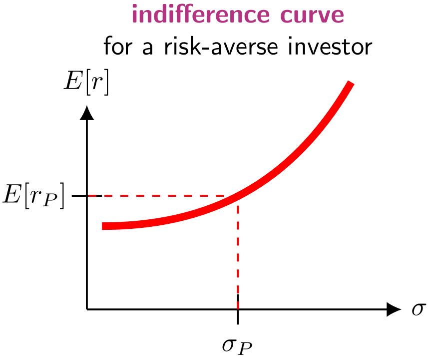
$$U = E[r] - \frac{1}{2}A\sigma^2 \tag{13}$$

- U = utility of an investor
- $E[r]$ = expected return of an asset (or portfolio)
- σ = volatility of the asset (or portfolio)
- A = **risk aversion**:
  - $A > 0 \Rightarrow$ risk-averse investor
  - $A = 0 \Rightarrow$ risk-neutral investor
  - $A < 0 \Rightarrow$ risk-loving investor

Trade-off between risk and return

mean-variance (M-V) criterion: portfolio A dominates portfolio B if $E[r_A] \geq E[r_B]$ and $\sigma_A \leq \sigma_B$

indifference curve: function defining the set of possible portfolios with the same utility value

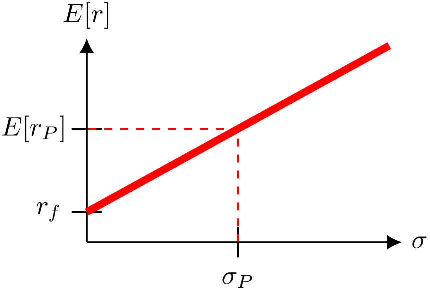


- $E[r_P]$ = expected portfolio return
- σ_P = portfolio volatility

The capital allocation line (1/2)

capital allocation line (**CAL**): available risk-return combinations of a set of possible **complete portfolios** (weighted combinations of a risky portfolio and the risk-free asset such that their respective weights sum to 1)

capital allocation line

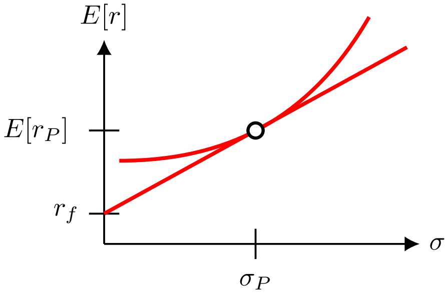


- $E[r_P]$ = expected return of risky portfolio
- σ_P = volatility of risky portfolio
- r_f = risk-free rate

Optimal complete portfolio

The position y^* in an **optimal complete portfolio** P containing a risky asset and the risk-free asset with return r_f is given by:

$$y^* = \frac{E[r_P] - r_f}{A\sigma_P^2} \tag{14}$$

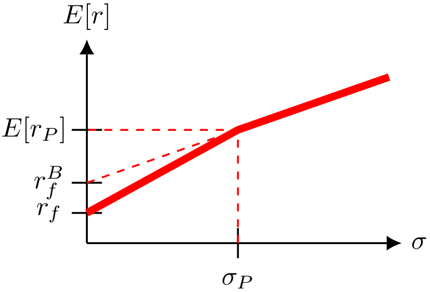


- r_f = risk-free rate

The capital allocation line (2/2)

If the risk-free **borrowing** rate differs from the risk-free **lending** rate, then the CAL is not differentiable at the point wherein the weight of the risky portfolio is 1.

capital allocation line

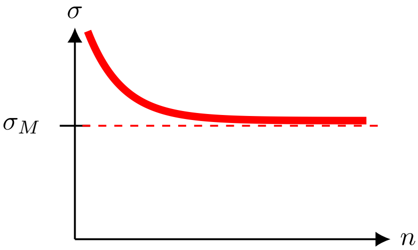


- $E[r_P]$ = expected return of risky portfolio
- σ_P = volatility of risky portfolio
- r_f = risk-free rate for lending
- r_f^B = risk-free rate for borrowing

Diversification and portfolio risk

diversification: strategy of reducing total risk by spreading exposure across multiple assets with different (**unique**, **firm-specific**, or **nonsystematic**) risk factors

market risk (**systemic risk**, **nondiversifiable risk**): the risk that remains after extensive diversification



- σ_M = volatility of market portfolio
- n = number of assets

Portfolio variance

We can express the variance of the return of a two-asset portfolio as follows:

$$\sigma_P^2 = \sum_{i=1}^{n} p(i)(w_1(r_1(i) - E[r_1]) + w_2(r_2(i) - E[r_2])) \tag{15}$$
$$= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\mathrm{Cov}(r_1, r_2) \tag{16}$$
$$= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\mathrm{Corr}(r_1, r_2) \tag{17}$$

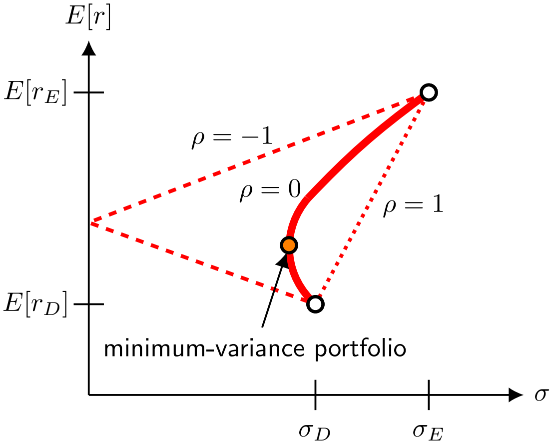
Then the **minimum-variance portfolio** can be found via differentiation:

$$w_{1,\min} = \frac{\sigma_2^2 - \mathrm{Cov}(r_1, r_2)}{\sigma_1^2 + \sigma_2^2 - 2\mathrm{Cov}(r_1, r_2)} \tag{18}$$

- r_1, r_2 = asset returns
- w_1, w_2 = asset weights
- σ_1, σ_2 = asset volatilities

Portfolio variance, illustrated

Consider hypothetical debt and equity assets D and E:



- r_D, r_E = asset returns
- σ_D, σ_E = asset volatilities
- ρ = correlation between D and E

Optimal risky portfolio

The **optimal risky portfolio** can be computed by determining the set of weights w_i that maximises the Sharpe ratio S_P:

$$\max_{w_i} S_P = \frac{E[r_P] - r_f}{\sigma_P} \tag{19}$$

The portfolio weights w_1 and w_2 that maximise the Sharpe ratio of a portfolio of two risky assets is given by:

$$E[R_1] = E[r_1] - r_f \tag{20}$$

$$E[R_2] = E[r_2] - r_f \tag{21}$$

$$w_1 = \frac{E[R_1]\sigma_2^2 - E[R_2]\sigma_1\sigma_2\rho_{1,2}}{E[R_1]\sigma_2^2 + E[R_2]\sigma_1^2 - (E[R_1] + E[R_2])\sigma_1\sigma_2\rho_{1,2}} \tag{22}$$

$$w_2 = 1 - w_1 \tag{23}$$

- $r_1, r_2 =$ asset returns
- $\sigma_1, \sigma_2 =$ asset volatilities
- $\rho_{1,2} =$ correlation between assets

Markowitz portfolio optimisation model (1/2)

We can calculate the expected return $E[r_P]$ and variance σ_P^2 of a portfolio of n assets with weights w_i as follows:

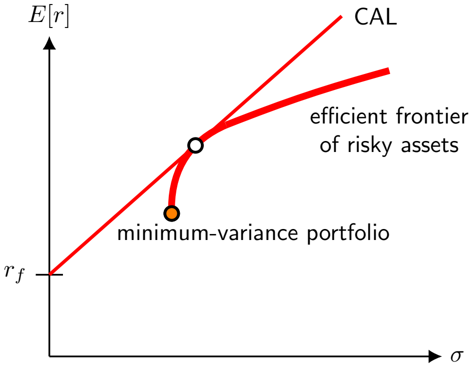
$$E[r_P] = \sum_{i=1}^{n} w_i E[r_i] \tag{24}$$

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \mathrm{Cov}(r_i, r_j) \tag{25}$$

Note: This calculation requires the full **covariance matrix** for the assets in the portfolio.

Markowitz portfolio optimisation model (2/2)

Given input data for expected returns, variances, and covariances, we can calculate the minimum-variance portfolio consistent with any targeted expected return, forming a **minimum-variance frontier**. The **efficient frontier** is the portion of the minimum-variance frontier that lies above the global minimum-variance portfolio. The **optimal risky portfolio** is the point on the efficient frontier of risky assets tangent to the capital allocation line (CAL) defined by the risk-free asset r_f.



A single-factor model for systemic risk

Consider the following **single-factor model** to describe the exposure of the return of an asset r_i to an unexpected market surprise:

$$r_i = E[r_i] + \beta_i m + \epsilon_i \tag{26}$$

- $\beta_i =$ sensitivity coefficient
- $m =$ market factor
- $\epsilon =$ firm-specific random variable

The variance of r_i, therefore, is the sum of the variance due to the market factor and the firm-specific variance:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma^2(\epsilon_i) \tag{27}$$

Since firm-specific variances among firms are mutually independent:

$$\mathrm{Cov}(r_i, r_j) = \mathrm{Cov}(\beta_i m + \epsilon_i, \beta_j m + \epsilon_j) = \beta_i \beta_j \sigma_m^2 \tag{28}$$

The single-index model

single-index model: single-factor model for the rate of return of an asset that uses the market index to stand in for the common factor

security characteristic line (**SCL**): If $R_i(t)$ and $R_M(t)$ represent the excess returns over the risk-free rate in month t for stock i and the market, respectively, then:

$$R_i(t) = \alpha_i + \beta_i R_M(t) + \epsilon_i(t) \tag{29}$$

- α_i = excess return of the stock when the market excess return is zero (y-intercept of the regression line)
- β_i = sensitivity of the stock return to the market return (slope of the regression line)
- $\epsilon_i(t)$ = zero-mean, firm-specific surprise in month t (residual)

The single-index model and diversification

We can describe the excess return on a portfolio of n stocks as:

$$R_P = \alpha_P + \beta_P R_M + \epsilon_P \tag{30}$$

The variance of the portfolio is given by:

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma^2(\epsilon_P) \tag{31}$$

Because the individual values of ϵ_i are independent:

$$\sigma^2(\epsilon_P) = \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma^2(\epsilon_i)}{n} = \frac{1}{n} \bar{\sigma}^2(\epsilon) \tag{32}$$

As the number of stocks increases, the variance decreases, although the **systemic** (**nondiversifiable**) **variance** due to the market factor remains.

Explanatory power of the SCL

The equation for **R-square** is the ratio of the systematic variance to the total variance:

$$R^2 = \frac{\beta^2 \sigma_M^2}{\beta^2 \sigma_M^2 + \sigma^2(\epsilon)} \tag{33}$$

The **adjusted R-square** corrects for an upward bias in R-square that arises because we use estimated values of some parameters (here, $k = 1$):

$$R_A^2 = 1 - (1 - R^2)\frac{n-1}{n-k-1} \tag{34}$$

The Treynor-Black model (1/3)

First, recall the **capital asset pricing model** (**CAPM**) for asset i:

$$E[r_i] = r_f + \beta_i(E[r_M] - r_f) \tag{35}$$

The **Treynor-Black model** depends upon the following inputs:

- Macroeconomic analysis is used to estimate the market return $E[r_M]$ and risk of the market index.
- Statistical analysis is used to estimate β_i and the residual variances $\sigma^2(\epsilon_i)$.
- The expected return $E[r_i]$ is calculated from $E[r_M]$ and β_i, based on information common to all securities, <u>not</u> security analysis of firm i.
- Finally, security-specific **alphas** α_i are derived from one or more **security-valuation models**, distilling the **incremental** risk premium attributable to private information.

The Treynor-Black model (2/3)

Next, normalise the portfolio weights so that they sum to 1 and compute the alpha, beta, and residual variance of the portfolio:

$$\alpha_A = \sum_{i=1}^{n} w_i \alpha_i \tag{36}$$

$$\beta_A = \sum_{i=1}^{n} w_i \beta_i \tag{37}$$

$$\sigma^2(\epsilon_A) = \sum_{i=1}^{n} w_i \sigma^2(\epsilon_i) \tag{38}$$

Next, we must select portfolio weights to maximise the Sharpe ratio of the portfolio:

$$S_P = \frac{E[r_P] - r_f}{\sigma_P} \tag{39}$$

The Treynor-Black model (3/3)

The optimal risky portfolio comprises two portfolios: an **active portfolio** A and a **passive portfolio** M. Suppose $\beta_A = \beta_M = 1$. Then the optimal weight for the active portfolio would balance its contribution to the return to its contribution to variance in the combined portfolio:

$$w_A^0 = \frac{\alpha_A / \sigma_A^2}{(E[r_M] - r_f)/\sigma_M^2} \tag{40}$$

Finally, scale the portfolio to account for the actual beta of the active portfolio, β_A:

$$w_A^* = \frac{w_A^0}{1 + (1 - \beta_A) w_A^0} \tag{41}$$

The information ratio

Assuming positive α_A, the Sharpe ratio of an optimally constructed risky portfolio will exceed that of the index portfolio:

$$S_P^2 = S_M^2 + \left[\frac{\alpha_A}{\sigma(\epsilon_A)}\right]^2 \tag{42}$$

The difference between S_P^2 and S_M^2 is the **square** of the **information ratio**:

$$\left[\frac{\alpha_A}{\sigma(\epsilon_A)}\right]^2 = \sum_{i=1}^{n} \left[\frac{\alpha_i}{\sigma(\epsilon_i)}\right]^2 \tag{43}$$

The weight of each security i is given by:

$$w_i^* = \frac{w_A^* \alpha_i}{\sigma^2(\epsilon_i)} \left[\sum_{i=1}^{n} \frac{\alpha_i}{\sigma^2(\epsilon_i)}\right]^{-1} \tag{44}$$

The market portfolio

market portfolio: value-weighted portfolio of all assets in the investable universe

capital market line (**CML**): capital allocation line constructed from a (presumed) risk-free asset and the market portfolio

Recall that each individual investor chooses a proportion y, allocated to the optimal portfolio M, such that:

$$y = \frac{E[r_M] - r_f}{A\sigma_M^2} \tag{45}$$

Since net borrowing and lending across all investors must be zero, the average position $\bar{y}$ the market portfolio M must be 1. Setting $y = 1$, we have:

$$E[r_M] - r_f = \bar{A}\sigma_M^2 \tag{46}$$

- $\bar{A} =$ degree of risk aversion of the average investor

The security market line and the market cost of risk

beta: covariance of the return of an asset i with that of the market portfolio, as a fraction of the variance of the return of the market portfolio

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_M)}{\sigma_M^2} \tag{47}$$

security market line (**SML**): relationship between beta and expected return under the CAPM

market price of risk: the reward-to-risk ratio for investment in the market portfolio:

$$\frac{\text{market risk premium}}{\text{market variance}} = \frac{E[r_M] - r_f}{\sigma_M^2} \tag{48}$$

Note: The market cost of risk measures contribution to portfolio variance and is <u>not</u> the Sharpe ratio.

Characteristics of efficient frontier portfolios

Any **combination** of efficient frontier portfolios
is also an efficient frontier portfolio.

Because investors choose their optimal risky portfolios from the efficient frontier, **the market portfolio is efficient**.

Every efficient portfolio M has a unique **zero-beta companion portfolio** Z on the lower half of the minimum-variance frontier with the same variance. For any asset i, then:

$$E[r_i] - E[r_Z] = (E[r_M] - E[r_Z]) \frac{\mathrm{Cov}(r_i, r_M)}{\sigma_M^2} \tag{49}$$

$$= \beta_i (E[r_M] - E[r_Z]) \tag{50}$$

Investors who wish to borrow but find it costly or impossible to do so tend to tilt their portfolios toward high-beta (high expected return) stocks and away from low-beta ones.

- As a result, the risk premiums of high-beta stocks tend to be less than what is predicted by the basic CAPM (and the SML is flatter).

Liquidity and the CAPM

Diverse beliefs among investors give ride to **trading** as investors rearrange their portfolios in accordance with heterogeneous demands.

liquidity: the ease and speed with which an asset can be sold at fair market value

immediacy: ability to sell an asset quickly without reverting to fire-sale prices

illiquidity discount: discount from fair market value a seller must accept if an asset is to be sold quickly

Liquidity is an important determinant of prices and expected returns!

Multifactor models

A **multifactor model** with K factors, for which R_i represents the excess return of asset i, can be expressed as:

$$R_i = E[R_i] + \epsilon_i + \sum_{k=1}^{K} \beta_{i,k} F_k \tag{51}$$

The coefficients $\beta_{i,k}$ are the **factor loadings** or **factor betas**.

Example: The **Fama-French** (**FF**) **three-factor model** is given by:

$$R_{it} = \alpha_i + \beta_{iM} R_{Mt} + \beta_{i,\mathrm{SMB}} \mathrm{SMB}_t + \beta_{i,\mathrm{HML}} \mathrm{HML}_t + \epsilon_{it} \tag{52}$$

- SMB = return of a portfolio of small stocks in excess of the return on a portfolio of large stocks (**"small minus big"**)
- HML = return of a portfolio of stocks with a high book-to-market ratio in excess of the return on a portfolio of stocks with a low book-to-market ratio (**"high minus low"**)

Arbitrage pricing theory

arbitrage: opportunity that arises wherein an investor can earn riskless profits without making an investment

law of one price: proposition stating that if two assets are equivalent in all economically relevant respects, then they should have the same market price

arbitrage pricing theory (**APT**): argument linking expected returns to risk, based upon the following propositions:

- (**1**) Security returns can be described by a factor model.
- (**2**) Sufficient securities exist to diversify away all idiosyncratic risk.
- (**3**) Well-functioning securities markets do not allow arbitrage opportunities to persist.

The efficient market hypothesis

efficient market hypothesis (**EMH**): proposition that stock prices reflect all available information

- **weak-form**: Stock prices reflect all information that can be derived from trading data, such as history of past prices, trading volume, short interest, and so on.
- **semistrong-form**: Stock prices reflect all publicly available information regarding the prospects of a firm.
- **strong form**: Stock prices reflect all relevant information, including information available only to insiders or other privileged parties.

Tests of the efficient market hypothesis

Weak-form tests:

- returns over short horizons
- returns over long horizons
- predictors of broad market returns

Semistrong-form tests:

- small-firm effect
- neglected-firm and liquidity effects
- book-to-market ratios
- post-earnings-announcement price drift

Strong-form tests:

- inside information

How to measure your portfolio manager

$$\textbf{Sharpe ratio}, \ S_P = \frac{r_P - r_f}{\sigma_P} \tag{53}$$

$$\textbf{M}^2, \ M^2 = \frac{r_P \sigma_M}{\sigma_P} - r_M \tag{54}$$

- for a portfolio that represents the entire investment fund

$$\textbf{information ratio}: \ \frac{\alpha_P}{\sigma(\epsilon_P)} \tag{55}$$

- for an active portfolio that is to be mixed with a passive portfolio

$$\textbf{Treynor measure}, \ T_P = \frac{r_P - r_f}{\beta_P} \tag{56}$$

- for a portfolio that is to be chosen as one subportfolio of many

$$\textbf{Jensen's alpha}, \ \alpha_P = \bar{r}_P - (\bar{r}_f + \beta_P(\bar{r}_M - \bar{r}_f)) \tag{57}$$

- when the investor wants to avoid paying for systemic risk

Parting thoughts

Consider an appropriate **bogey**, the portfolio that would measure the returns a portfolio manager would make if he or she were to follow a completely passive strategy.

Beware of **survivorship bias** in choosing managers on the basis of their past success.

Market timing is a potential source of variation in portfolio risk and can sometimes be more important in explaining returns than the portfolio itself.

The **Morningstar risk-adjusted rating** (**MRAR**) is the only performance measure that is theoretically impossible to manipulate[2]:

$$\mathrm{MRAR}(\gamma) = \left[\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1+r_t}{1+r_{ft}}\right]^{-\gamma}\right]^{\frac{12}{\gamma}} - 1 \tag{58}$$

- $\gamma =$ measure of investor risk aversion
- $T =$ set of monthly observations

[2] as shown by Ibbotson et al.

Thank You

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