0 = world frame

1 = base frame

2 = 1st link frame

3 = 2nd (final) link frame

- Going to use notation + understanding of rotation matricies from dynamics for our problem

rotation matrix FROM 0 frame TO 1 frame

$$R_{01} = R_Z(q_1) = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{01}^T = R_{10}$$

$$I_0 = R_{01} I_1 R_{01}^T$$

I in 0 frame

- with this notation, I know the answer resolves in the frame furthest to the left

- For our problem:

$$R_{01} = R_Z(q_1) = \begin{bmatrix} cq_1 & -sq_1 & 0 \\ sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{10} = R_{01}^T$$

$$R_{12} = R_Y(q_2) = \begin{bmatrix} cq_2 & 0 & sq_2 \\ 0 & 1 & 0 \\ -sq_2 & 0 & cq_2 \end{bmatrix}$$

$$R_{23} = R_Y(q_3) = \begin{bmatrix} cq_3 & 0 & sq_3 \\ 0 & 1 & 0 \\ -sq_3 & 0 & cq_3 \end{bmatrix}$$

$R_{01} = R_{01}$	$R_{10} = R_{01}^T$
$R_{02} = R_{01} R_{12}$	$R_{20} = R_{12}^T R_{01}^T = R_{21} R_{10}$
$R_{03} = R_{01} R_{12} R_{23}$	$R_{30} = R_{23}^T R_{12}^T R_{01}^T = R_{32} R_{21} R_{10}$

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ME699 - RMC
Assignment 2

$$\boxed{R_{01} = \begin{bmatrix} cq_1 & -sq_1 & 0 \\ sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \boxed{R_{10} = R_{01}^T = \begin{bmatrix} cq_1 & sq_1 & 0 \\ -sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$R_{02} = R_{01}R_{12} = \begin{bmatrix} cq_1 & -sq_1 & 0 \\ sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} cq_2 & 0 & sq_2 \\ 0 & 1 & 0 \\ -sq_2 & 0 & cq_2 \end{bmatrix}$$

$$\boxed{R_{02} = \begin{bmatrix} cq_1cq_2 & -sq_1 & cq_1sq_2 \\ sq_1cq_2 & cq_1 & sq_1sq_2 \\ -sq_2 & 0 & cq_2 \end{bmatrix}}$$

$$R_{20} = R_{12}^T R_{01}^T = R_{21}R_{10} = \begin{bmatrix} cq_2 & 0 & -sq_2 \\ 0 & 1 & 0 \\ sq_2 & 0 & cq_2 \end{bmatrix}\begin{bmatrix} cq_1 & sq_1 & 0 \\ -sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{R_{20} = \begin{bmatrix} cq_1cq_2 & sq_1cq_2 & -sq_2 \\ -sq_1 & cq_1 & 0 \\ cq_1sq_2 & sq_1sq_2 & cq_2 \end{bmatrix} = R_{02}^T}$$

$$R_{03} = R_{01}R_{12}R_{23} = \begin{bmatrix} cq_1 & -sq_1 & 0 \\ sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_2 & 0 & sq_2 \\ 0 & 1 & 0 \\ -sq_2 & 0 & cq_2 \end{bmatrix} \begin{bmatrix} cq_3 & 0 & sq_3 \\ 0 & 1 & 0 \\ -sq_3 & 0 & cq_3 \end{bmatrix}$$

$$R_{03} = \begin{bmatrix} cq_1 & -sq_1 & 0 \\ sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_2cq_3 - sq_2sq_3 & 0 & cq_2sq_3 + sq_2cq_3 \\ 0 & 1 & 0 \\ -sq_2cq_3 - sq_3cq_2 & 0 & -sq_2sq_2 + cq_2cq_3 \end{bmatrix}$$

$$\boxed{R_{03} = \begin{bmatrix} cq_1(cq_2cq_3 - sq_2sq_3) & -sq_1 & cq_1(cq_2sq_3 + sq_2cq_3) \\ sq_1(cq_2cq_3 - sq_2sq_3) & cq_1 & sq_1(cq_2sq_3 + sq_2cq_3) \\ -sq_2cq_3 - sq_3cq_2 & 0 & -sq_2sq_3 + cq_2cq_3 \end{bmatrix}}$$

$$\boxed{R_{30} = \begin{bmatrix} cq_1(cq_2cq_3 - sq_2sq_3) & sq_1(cq_2cq_3 - sq_2sq_3) & -sq_2cq_3 - sq_3cq_2 \\ -sq_1 & cq_1 & 0 \\ cq_1(cq_2sq_3 + sq_2cq_3) & sq_1(cq_2sq_3 + sq_2cq_3) & -sq_2sq_3 + cq_2cq_3 \end{bmatrix}}$$

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ q_1 \end{bmatrix} \quad \dot{q}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \quad \ddot{q}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{q}_1 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0 \\ q_2 \\ 0 \end{bmatrix} \quad \dot{q}_2 = \begin{bmatrix} 0 \\ \dot{q}_2 \\ 0 \end{bmatrix} \quad \ddot{q}_2 = \begin{bmatrix} 0 \\ \ddot{q}_2 \\ 0 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 0 \\ q_3 \\ 0 \end{bmatrix} \quad \dot{q}_3 = \begin{bmatrix} 0 \\ \dot{q}_3 \\ 0 \end{bmatrix} \quad \ddot{q}_3 = \begin{bmatrix} 0 \\ \ddot{q}_3 \\ 0 \end{bmatrix}$$

- from online notes

$KE = \frac{1}{2}\dot{q}^T D(q)\,\dot{q}$, where $D(q)$ is the inertia matrix of the particular link expressed in the world frame (not its own body frame)

$PE = mg\,[0\,0\,1]\,c_i^0(q)$, c_i^0 - location of the center of mass of link i in world frame

$$c_1^0(q_1) = R_{01}\begin{bmatrix} 0 \\ 0 \\ L_1/2 \end{bmatrix}$$

$$c_3^0(q_3) = \overset{(q_1)}{R_{01}}\,\overset{(q_2)}{R_{12}}\,\overset{(q_3)}{R_{23}}\begin{bmatrix} 0 \\ 0 \\ \frac{L_3}{2} \end{bmatrix}$$

E-L Equation for joint 1

$$L = KE - PE$$

$$L = \frac{1}{2}[0\,0\,\dot{q}_1]\; R_{01}\; I_1\; R_{01}^T \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} - mg[0\,0\,1]\, R_{01} \begin{bmatrix} 0 \\ 0 \\ L_1/2 \end{bmatrix}$$

$$L = \frac{1}{2}[0\,0\,\dot{q}_1] \begin{bmatrix} cq_1 & -sq_1 & 0 \\ sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} cq_1 & sq_1 & 0 \\ -sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

$$- mg[0\,0\,1] \begin{bmatrix} cq_1 & -sq_1 & 0 \\ sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ L_1/2 \end{bmatrix}$$

$$\boxed{L = \frac{1}{2} I_{zz,1}\, \dot{q}_1^{\,2} - mg\, L_1/2}$$ Joint 1 Lagrangian

$$\cancel{\frac{\partial L}{\partial \dot{q}_1} = I_{zz}\dot{q}_1} \quad , \quad \cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = I_{zz}\ddot{q}_1}$$

$$\cancel{\frac{\partial L}{\partial q_1} = 0}$$

- Realized I started constructing these too early

$$\cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \tau_1}$$

$$\cancel{\boxed{I_{zz,1}\ddot{q}_1 - 0 = \tau_1}}$$

E-L Equation for joint 2

$$L = KE - PE$$

$$L = \frac{1}{2}\begin{bmatrix}0 & \dot{q}_2 & 0\end{bmatrix}\begin{bmatrix} cq_1 cq_2 & -sq_1 & cq_1 sq_2 \\ sq_1 cq_2 & cq_1 & sq_1 sq_2 \\ -sq_2 & 0 & cq_2 \end{bmatrix}\begin{bmatrix} I_{xx,2} & 0 & 0 \\ 0 & I_{yy,2} & 0 \\ 0 & 0 & I_{zz,2}\end{bmatrix}\begin{bmatrix} cq_1 cq_2 & sq_1 cq_2 & -sq_2 \\ -sq_1 & cq_1 & 0 \\ cq_1 sq_2 & sq_1 sq_2 & cq_2 \end{bmatrix}\begin{bmatrix}0 \\ \dot{q}_2 \\ 0\end{bmatrix}$$

$$- mg\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix} cq_1 cq_2 & -sq_1 & cq_1 sq_2 \\ sq_1 cq_2 & cq_1 & sq_1 sq_2 \\ -sq_2 & 0 & cq_2 \end{bmatrix}\begin{bmatrix}0 \\ 0 \\ L_2/2\end{bmatrix}$$

$$L = \frac{1}{2}\begin{bmatrix}0 & \dot{q}_2 & 0\end{bmatrix}\begin{bmatrix} cq_1 cq_2 & -sq_1 & cq_1 sq_2 \\ sq_1 cq_2 & cq_1 & sq_1 sq_2 \\ -sq_2 & 0 & cq_2 \end{bmatrix}\begin{bmatrix} I_{xx,2} & 0 & 0 \\ 0 & I_{yy,2} & 0 \\ 0 & 0 & I_{zz,2}\end{bmatrix}\begin{bmatrix} sq_1 cq_2 \\ cq_1 \\ sq_1 sq_2 \end{bmatrix}\dot{q}_2$$

$$- mg\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix} cq_1 sq_2 \\ sq_1 sq_2 \\ cq_2 \end{bmatrix} L_2/2$$

~~only~~ these two rows gone due to $\begin{bmatrix}0 & \dot{q}_2 & 0\end{bmatrix}$

$$L = \frac{1}{2}\begin{bmatrix}0 & \dot{q}_2 & 0\end{bmatrix}\begin{bmatrix} cq_1 cq_2 & -sq_1 & cq_1 sq_2 \\ sq_1 cq_2 & cq_1 & sq_1 sq_2 \\ -sq_2 & 0 & cq_2 \end{bmatrix}\begin{bmatrix} I_{xx,2}\, sq_1 cq_2 \\ I_{yy,2}\, cq_1 \\ I_{zz,2}\, sq_1 sq_2 \end{bmatrix}\dot{q}_2$$

$$- \frac{mg L_2 cq_2}{2}$$

$$\boxed{L = \frac{1}{2}\dot{q}_2^{\,2}\left(I_{xx,2}\, s^2q_1 c^2q_2 + I_{yy,2}\, c^2q_1 + I_{zz,2}\, s^2q_1 s^2q_2\right) - \frac{mg}{2} L_2 cq_2}$$

Joint 2 Lagrangian

- going to attempt to shorten the first four steps using work from past joint

E-L Equation for joint 3

$$L = KE - PE$$

$$L = \frac{1}{2}\begin{bmatrix} 0 & \dot{q}_3 & 0 \end{bmatrix} \begin{bmatrix} cq_1(cq_2cq_3 - sq_2sq_3) & -sq_1 & cq_1(cq_2sq_3 + sq_2cq_3) \\ sq_1(cq_2cq_3 - sq_2sq_3) & cq_1 & sq_1(cq_2sq_3 + sq_2cq_3) \\ -sq_2cq_3 - sq_3cq_2 & 0 & -sq_2sq_3 + cq_2cq_3 \end{bmatrix} \begin{bmatrix} I_{xx,3}\, sq_1(cq_2cq_3 - sq_2sq_3) \\ I_{yy,3}\, cq_1 \\ I_{zz,3}\, sq_1(cq_2sq_3 + sq_2cq_3) \end{bmatrix} \dot{q}_3$$

$$- mg \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_1(cq_2sq_3 + sq_2cq_3) \\ sq_1(cq_2sq_3 + sq_2cq_3) \\ -sq_2sq_3 + cq_2cq_3 \end{bmatrix} L_3/2$$

$$\boxed{\begin{aligned} L = \frac{1}{2}\dot{q}_3^2 \left[I_{xx,3} s^2q_1(cq_2cq_3 - sq_2sq_3)^2 + I_{yy,3} c^2q_1 + I_{zz,3} s^2q_1(cq_2sq_3 + sq_2cq_3)^2 \right] \\ - mg\frac{L_3}{2}(cq_2cq_3 - sq_2sq_3) \end{aligned}}$$

Joint 3 Lagrangian

- Expression for the full Lagrangian of the system.

$$\begin{aligned} L = {} & \frac{1}{2} I_{zz,1}\dot{q}_1^2 - mg^{L_1}/_2 \\ & + \frac{1}{2}\dot{q}_2^2 (I_{xx,2} s^2q_1 c^2q_2 + I_{yy,2} c^2q_1 + I_{zz,2} s^2q_1 s^2q_2) - mg^{L_2}/_2\, cq_2 \\ & + \frac{1}{2}\dot{q}_3^2 \left[I_{xx,3} s^2q_1(cq_2cq_3 - sq_2sq_3)^2 + I_{yy,3} c^2q_1 + I_{zz,3} s^2q_1(cq_2sq_3 + sq_2cq_3)^2 \right] \\ & - mg^{L_3}/_2 (cq_2cq_3 - sq_2sq_3) \end{aligned}$$

- going to do some simplification of this expression before next step.

- ~~step~~ since q_2 & q_3 are ~~both~~ subsequent by rotations, $cq_2cq_3 = cq_{2+3}$, $sq_2sq_3 = sq_{2+3}$

$$
\begin{aligned}
L = {} & \tfrac{1}{2}\dot{q}_1^2 I_{zz,1} - mg\,L_1/2 \\
& + \tfrac{1}{2}\dot{q}_2^2\left(I_{xx,2}s^2q_1c^2q_2 + I_{yy,2}c^2q_1 + I_{zz,2}s^2q_1s^2q_2\right) - mg\tfrac{L_2}{2}cq_2 \\
& + \tfrac{1}{2}\dot{q}_3^2\Big[I_{xx,3}s^2q_1\left(c^2(q_2+q_3) - 2c(q_2+q_3)s(q_2+q_3) + s^2(q_2+q_3)\right) + I_{yy,3}c^2q_1 \\
& \qquad + I_{zz,3}s^2q_1\left(c^2q_2s^2q_3 + 2cq_2cq_3sq_2sq_3 + s^2q_2c^2q_3\right)\Big] \\
& \qquad - mg\,L_3/2\left(cq_2cq_3 - sq_2sq_3\right)
\end{aligned}
$$

$$\frac{\partial L}{\partial \dot{q}_1} = I_{zz,1}\dot{q}_1 \qquad\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = I_{zz,1}\ddot{q}_1$$

$$
\begin{aligned}
\frac{\partial L}{\partial q_1} = {} & \tfrac{1}{2}\dot{q}_2^2\left(I_{xx,2}s^2q_1c^2q_2 - I_{yy,2}s^2q_1 + I_{zz,2}s^2q_1s^2q_2\right) \\
& + \tfrac{1}{2}\dot{q}_3^2\Big[I_{xx,3}s^2q_1\left(c^2(q_2+q_3) - 2c(q_2+q_3)s(q_2+q_3) + s^2(q_2+q_3)\right) - I_{yy,3}s^2q_1 \\
& \qquad + I_{zz,3}s^2q_1\left(c^2q_2s^2q_3 + 2cq_2cq_3sq_2sq_3 + s^2q_2c^2q_3\right)\Big]
\end{aligned}
$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \tau_1}$$

$$\cancel{D = \begin{bmatrix} I_{zz,1} & 0 & 0 \\ \xi & \xi & \xi \\ \xi & \xi & \xi \end{bmatrix}} \qquad \cancel{C = \begin{bmatrix} 0 & & \\ \xi & \xi & \xi \\ \xi & \xi & \xi \end{bmatrix}}$$

$$\frac{\partial L}{\partial \dot{q}_2} = \left(I_{xx,2}\, s^2 q_1 c^2 q_2 + I_{yy,2}\, c^2 q_1 + I_{zz,2}\, s^2 q_1 s^2 q_2\right) \dot{q}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = \left(I_{xx,2}\, s^2 q_1 c^2 q_2 + I_{yy,2}\, c^2 q_1 + \cancel{z} I_{zz,2}\, s^2 q_1 s^2 q_2\right) \ddot{q}_2$$

$$\begin{aligned}\frac{\partial L}{\partial q_2} = {} & \tfrac{1}{2} \dot{q}_2^2 \left(-I_{xx,2}\, s^2 q_1 s^2 q_2 + I_{zz,2}\, s^2 q_1 s^2 q_2\right) + mg \tfrac{L_2}{2} s q_2 + \\ & \tfrac{1}{2} \dot{q}_3^2 \Big[I_{xx,3}\, s^2 q_1 \left(-s^2 q_2 + 2 s q_2 c q_2 + s^2 q_2\right) + I_{zz,3}\, s^2 q_1 \big(-s^2 q_2 s^2 q_3 - 2 s \overset{s^2 q_2 c^2 q_3}{\cancel{q_2 c q_3 c q_2 s q_3 +}} \\ & + s^2 q_2 c^2 q_3\big)\Big] + mg\, {}^{L_3}\!/_2 \left(s q_2 c q_3 + c q_2 s q_3\right)\end{aligned}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = \tau_2}$$

$$\begin{aligned}\frac{\partial L}{\partial \dot{q}_3} = {} & \Big[I_{xx,3}\, s^2 q_1 \left(c^2 (q_2+q_3) - 2c(q_2+q_3) s(q_2+q_3) + s^2 (q_2+q_3)\right) + I_{yy,3}\, c^2 q_1 \\ & + I_{zz,3}\, s^2 q_1 \left(c^2 q_2 s^2 q_3 + 2 c q_2 s q_2 c q_3 s q_3 + s^2 q_2 c^2 q_3\right)\Big] \dot{q}_3\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_3}\right) = {} & \Big[I_{xx,3}\, s^2 q_1 \left(c^2 (q_2+q_3) - 2c(q_2+q_3) s(q_2+q_3) + s^2 (q_2+q_3)\right) + I_{yy,3}\, c^2 q_1 \\ & + I_{zz,3}\, s^2 q_1 \left(c^2 q_2 s^2 q_3 + 2 c q_2 s q_2 c q_3 s q_3 + s^2 q_2 c^2 q_3\right)\Big] \ddot{q}_3\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial q_3} = {} & \tfrac{1}{2} \dot{q}_3^2 \Big[I_{xx,3}\, s^2 q_1 \left(-s^2 q_3 + 2 s q_3 c q_3 + s^2 q_3\right) \\ & + I_{zz,3}\, s^2 q_1 \left(c^2 q_2 c^2 q_3 - 2 c q_2 s q_3 s q_2 c q_3 - s^2 q_2 s^2 q_3\right)\Big] \\ & + mg\, {}^{L_3}\!/_2 \left(-c q_2 s q_3 - s q_2 c q_3\right)\end{aligned}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_3}\right) - \frac{\partial L}{\partial q_3} = \tau_3}$$

- We have now derived the equations of motion for this system. These equations, once provided $q, \dot{q}, + \ddot{q}$ will calculate the joint torques required to make those initial conditions.

- We now take these closed-form expressions to Julia and use them ~~=~~ in the inverse dynamics function. The function will request as input from the user three 3x1 vectors for q, $\dot{q}$ + $\ddot{q}$. The function will return one 3x1 vector which will correspond to joint torques $\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$.