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Light Near a Black Hole

Taken from Faber 231: For $u = 1/r$ and θ, so that (r, θ) is in the plane of the light, its velocity vector and the mass (of mass M) we have

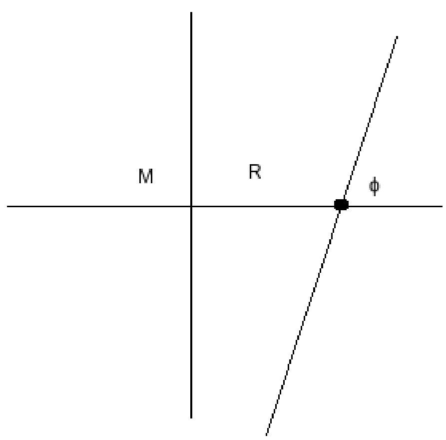
$$\frac{d^2u}{d\theta^2} + u = 3Mu^2.$$

Note that the units don't match up! This is because there are implicit $c = 1$ and $G = 1$. I believe the units work out to

$$\frac{d^2u}{d\theta^2} + u = 3\frac{GM}{c^2}u^2.$$

We can solve this using the energy method.

We are given initial conditions as follows: The camera is a distance R from the origin (the mass) and looking in a direction ϕ from the r vector. Since the equation is θ invariant, we may assume that the camera is at $\theta = 0$. This is given by the following picture:



Clearly $r(0) = R$ so that $u(0) = 1/r(0) = 1/R$. Further, we see that the slope of the line through the camera is $\tan(\phi)$ so that in Cartesian coordinates we have the line

$$\cos(\phi)(y - 0) - \sin(\phi)(x - R) = 0.$$

Now converting this by taking $x = r\cos(\theta)$ and $y = r\sin(\theta)$ gives

$$r\cos(\phi)\sin(\theta) - r\sin(\phi)\cos(\theta) = R\sin(\phi).$$

Recalling the double angle for sin we see

$$u(\theta) = \frac{1}{r} = \frac{1}{R}\frac{\sin(\phi - \theta)}{\sin(\phi)}.$$

Thus

$$u'(0) = -\frac{1}{R}\frac{\cos(\phi)}{\sin(\phi)} = -\frac{1}{R}\cot(\phi).$$

We might be concerned as to what happens when $\sin(\phi) = 0 \iff \phi = 2\pi k,\ k \in \mathbb{Z}$. However, this is the case where the camera is either pointed directly at or away from the mass (and so render normally). Thus, we always assume $u'(0) \in \mathbb{R}$.

Now, we can apply the Energy method in ernest.

We see that by the chain rule

$$\frac{d}{du}\left[\frac{1}{2}\left(\frac{du}{d\theta}\right)^2\right] = \frac{du}{d\theta}\frac{d^2u}{d\theta^2}\frac{d\theta}{du} = \frac{d^2u}{d\theta^2} = \frac{3GM}{c^2}u^2 - u$$

Thus,

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2GM}{c^2}u^3 - u^2 + K.$$

for $K \in \mathbb{R}$. We can find K by applying our initial conditions:

$$K = (u'(0))^2 - \frac{2GM}{c^2}u(0)^3 + u(0)^2 = \frac{1}{R^2}\cot(\phi)^2 - \frac{2GM}{c^2R^3} + \frac{1}{R^2}.$$

In general, we have no cancellation, and so let's leave it as K. Thus, flipping things around we have

$$\theta = \int_{1/R}^{u} \frac{ds}{\sqrt{2GMs^3/c^2 - s^2 + K}}.$$

We can solve this explicitly in special cases: Suppose $R = 2GM/c^2$ and $\phi = \pi/2$. Then $\cot(\phi) = 0$ and

$$K = -\frac{2GM}{c^2}\left(\frac{c^2}{2GM}\right)^3 + \left(\frac{c^2}{2GM}\right)^2 = 0.$$

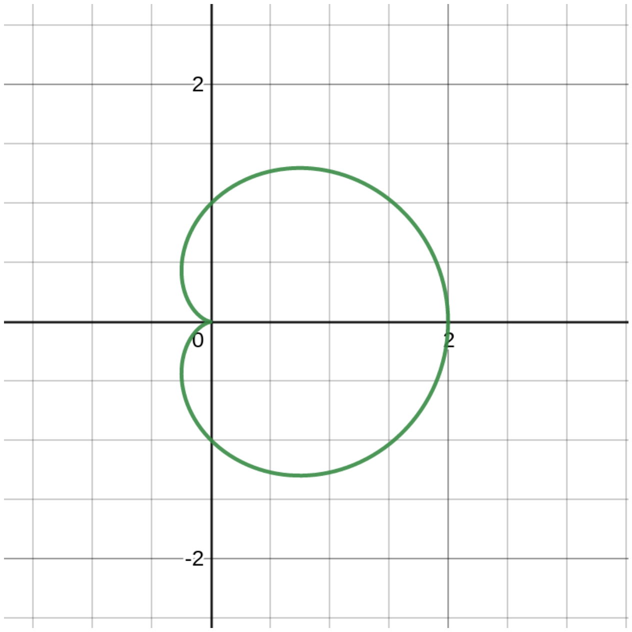
MAPLE spits out for the integral

$$\theta = 2\tan^{-1}\left(\sqrt{\frac{2GM}{c^2}u - 1}\right).$$

Solving for r and remembering a few trig identities gives

$$r = \frac{GM}{c^2}(1 + \cos(\theta)).$$

Which looks like for $GM/c^2 = 1$



Clearly, at this radius, the geodesic gets sucked into 0.

We could also try $R = 3GM/c^2$ and $\phi = \pi/2$. We could do it by hand, or simply notice that $u = c^2/(3GM)$ solves the original DE, with these initial conditions! This is a circle of radius $3GM/c^2$.

Let's use Runge-Kutta to approximate this DE We need to write this as

$$\frac{dy}{dt} = f(t, y)$$

Set

$$y = \begin{bmatrix} u \\ u' \end{bmatrix}$$

So that

$$y' = f(y) = \begin{bmatrix} u' \\ u'' \end{bmatrix} = \begin{bmatrix} u' \\ 3GM/c^2u^2 - u \end{bmatrix}$$

Then, for

$$y_0 = \begin{bmatrix} 1/R \\ 1/R\cot(\phi) \end{bmatrix},$$

$\theta_0 = 0$, and step size h (then the approximate distance traveled will be h/u.)

$$k_1 = f(y_n)$$

$$k_2 = f(y_n + hk_1/2)$$

$$k_3 = f(y_n + hk_2/2)$$

$$k_4 = f(y_n + hk_3)$$

$$y_{n+1} = y_n + 1/6\ h(k_1 + 2k_2 + 2k_3 + k_4).$$