Math 5720 HW 7

Question 1

Let $f(x_1, x_2) = x_1^2 + x_2^2 - x_1x_2 - 2x_1$. Perform the exact line search at $x = (1, 1)$ in direction $d = (2, -1)$.

If $(2, -1)$ is a decent direction then $\nabla f(x_1, x_2)^T d < 0$, $\nabla f(x_1, x_2) = \begin{bmatrix} 2x_1 - x_2 - 2 \\ 2x_2 - x_1 \end{bmatrix}$,

$\nabla f(1, 1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -3 < 0$ and therefore d is a descent direction

$f(x + td) = f(1 + 2t, 1 - t) = (1 + 2t)^2 + (1 - t)^2 - (1 + 2t)(1 - t) - 2(1 + 2t) = (1 + 4t +$
$+ (4 + 1 + 2)t^2 = -1 - 3t + 7t^2$

min $f(x + td)$ occurs when $f'(x + td) = 0$

$f'(x + td) = 14t - 3$ which means that $f'(x + td) = 0$ when $t = \frac{3}{14}$

Then $f(x + (\frac{3}{14})d) = -1 - 3(\frac{3}{14}) + 7(\frac{9}{196}) = 1 - \frac{9}{14} + \frac{9}{28} = \frac{28}{28} - \frac{18}{28} + \frac{9}{28} = \frac{9}{28}$

Question 2

Let $f(x_1, x_2) = x_1^2 + x_2^2 - x_1x_2 - 2x_1$. Perform the backtracking procedure at $x = (1, 1)$ in a direction $d = (2, -1)$ with $s = 1, \alpha = 0.7, \beta = 0.5$.

d is a descent direction for the function, proved in (q1)

$\alpha \nabla f(x)^T d$= $-0.7 \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -.7 * 3 = -2.1$, so while $\frac{f(x+td)-f(x)}{t} < -2.1$ we continue dividing t by β until we find a value of t that satisfies the stopping condition

t	$x + td$	$f(x + td)$	$f(x)$	$\frac{f(x+td)-f(x)}{t}$
1	$(3, 0)$	3	-1	4
$\frac{1}{2}$	$(2, \frac{1}{2})$	$-\frac{3}{4}$	-1	$\frac{1}{2}$
$\frac{1}{4}$	$(\frac{3}{2}, \frac{3}{4})$	$-\frac{21}{16}$	-1	$-\frac{5}{4}$
$\frac{1}{8}$	$(\frac{5}{4}, \frac{7}{8})$	$-\frac{81}{64}$	-1	$-\frac{17}{8}$

As $-\frac{17}{8} = -2.125 < -2.1$ we can stop the backtracking procedure and our step size is .125

Question 3

$f(x) = \sum_{i=1}^{m} (||x - a_i|| - d_i)^2$

Part A

$\nabla f(x) = \sum_{i=1}^{m} 2(||x - a_i|| - d_i)\frac{x-a_i}{||x-a_i||}$

Part B

$\nabla f(x) = 0$, when $0 = \sum_{i=1}^{m} 2(||x - a_i|| - d_i)\frac{x-a_i}{||x-a_i||}$

$\sum_{i=1}^{m} 2(||x - a_i|| - d_i)\frac{x-a_i}{||x-a_i||} = 0 \Leftrightarrow \sum_{i=1}^{m}(x - a_i - d_i\frac{x-a_i}{||x-a_i||}) = 0$

$\Leftrightarrow 0 = \sum_{i=1}^{m} x - \sum_{i=1}^{m} a_i - \sum_{i=1}^{m}(d_i\frac{x-a_i}{||x-a_i||})$

$\Leftrightarrow mx = \sum_{i=1}^{m} a_i + \sum_{i=1}^{m}(d_i\frac{x-a_i}{||x-a_i||}) \Leftrightarrow x = \frac{1}{m}(\sum_{i=1}^{m} a_i + \sum_{i=1}^{m}(d_i\frac{x-a_i}{||x-a_i||})$

Part C

$0 = 2x\sum_{i=1}^{m} \frac{||x-a_i||-d_i}{||x-a_i||} - 2\sum_{i=1}^{m} \frac{a_i(||x-a_i||-d_i)}{||x-a_i||}$

$2\sum_{i=1}^{m} \frac{a_i(||x-a_i||-d_i)}{||x-a_i||} = 2x\sum_{i=1}^{m} \frac{||x-a_i||-d_i}{||x-a_i||}$ we can divide as we assume that all x_ks are different from a_i

$x = \frac{1}{\sum_{i=1}^{m} \frac{||x-a_i||-d_i}{||x-a_i||}} \sum_{i=1}^{m} \frac{a_i(||x-a_i||-d_i)}{||x-a_i||}$

$T(x) = \frac{1}{\sum_{i=1}^{m} \frac{||x-a_i||-d_i}{||x-a_i||}} \sum_{i=1}^{m} \frac{a_i(||x-a_i||-d_i)}{||x-a_i||}$

$x_{k+1} = T(x_k)$

$x_{k+1} = \frac{1}{\sum_{i=1}^{m} \frac{||x_k-a_i||-d_i}{||x_k-a_i||}} \sum_{i=1}^{m} \frac{a_i(||x_k-a_i||-d_i)}{||x_k-a_i||}$

$x_{k+1} = x_k - x_k + \frac{1}{\sum_{i=1}^{m} \frac{||x_k-a_i||-d_i}{||x_k-a_i||}} \sum_{i=1}^{m} \frac{a_i(||x_k-a_i||-d_i)}{||x_k-a_i||}$

$x_{k+1} = x_k - \frac{1}{\sum_{i=1}^{m} \frac{||x_k-a_i||-d_i}{||x_k-a_i||}} \sum_{i=1}^{m} \frac{(x_k-a_i)(||x_k-a_i||-d_i)}{||x_k-a_i||}$

$= x_k - \frac{1}{\sum_{i=1}^{m} \frac{||x_k-a_i||-d_i}{||x_k-a_i||}} \nabla f(x_k)$

Therefore we have a gradient method with a step size of $\frac{1}{\sum_{i=1}^{m} \frac{||x_k-a_i||-d_i}{||x_k-a_i||}}$ as it is set up in the form of a gradient method

Question 4

$min_{x\in\mathbb{R}^3} f(x) = \sum_{i=1}^{9} \omega_i||x - a_i||$

$\nabla f(x) = \sum_{i=1}^{9} \omega_i\frac{x-a_i}{||x-a_i||}$

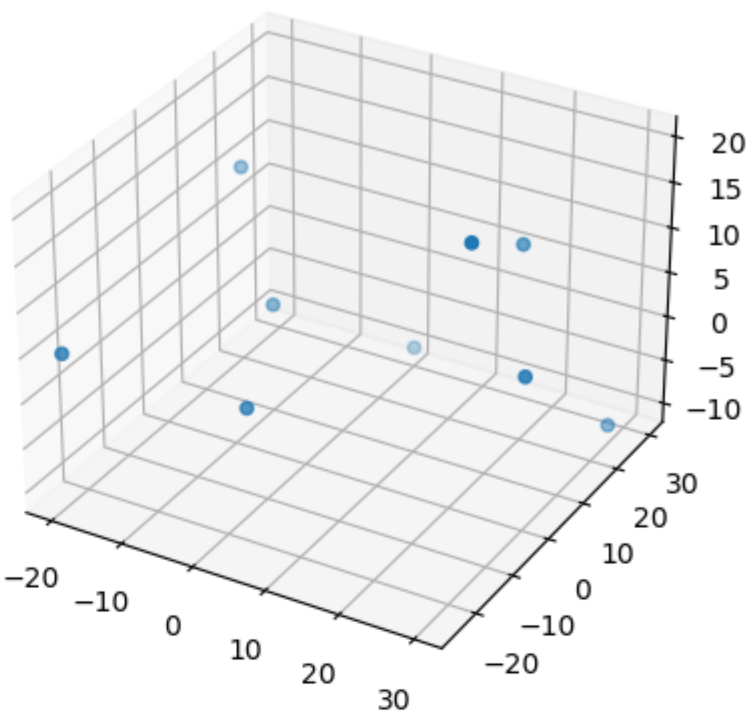
$x_{k+1} = \frac{1}{\sum_{i=1}^{9} \frac{\omega_i}{||x-a_i||}} \sum_{i=1}^{9} \frac{\omega_i a_i}{||x-a_i||}$

In [37]:
```
using LinearAlgebra
using PyPlot
```

In [2]:
```
a = [-10 10 0; 0 30 -10; 20 20 10; 30 0 5; 25 -5 20; -20 -25 5; 30 25 -10; -2(
```

Out[2]:
```
9×3 Matrix{Int64}:
 -10   10    0
   0   30  -10
  20   20   10
  30    0    5
  25   -5   20
 -20  -25    5
  30   25  -10
 -20   20   10
   0  -15    0
```

In [62]:
```
scatter3D(a[:,1],a[:,2],a[:,3]);
```



In [3]:
```
omega = [1;2;.5;.5;1/3;1;1.5;1;1]
```

Out[3]:
```
9-element Vector{Float64}:
 1.0
 2.0
 0.5
 0.5
 0.3333333333333333
 1.0
 1.5
 1.0
```

In [5]:
```
x=[0 0 0]
```
Out[5]:
```
1×3 Matrix{Int64}:
 0  0  0
```
In [68]:
```
function nextX(x,omega,alpha)
    s1 = [0 0 0]
    s2 = 0
    for i=1;9
        s1 = s1 .+ omega[i]/norm(x.-alpha[i])*alpha[i]/norm(x.-alpha[i]);
        s2 = s2 + omega[i]/norm(x.-alpha[i,:]);
    end
    return 1/s2 * s1;
end
```
Out[68]:
```
nextX (generic function with 1 method)
```
In [69]:
```
b = nextX(x,omega,a)
```
Out[69]:
```
1×3 Matrix{Float64}:
 -0.816497  -0.816497  -0.816497
```
This proccess can be continued until $\nabla f(x)$ is close to zero, here it is (-.8,-.8,-.8), so we continue

In [70]:
```
c = nextX(b,omega,a)
```
Out[70]:
```
1×3 Matrix{Float64}:
 -0.972967  -0.972967  -0.972967
```
We continue as $\nabla f(x)$ is (-0.16,-0.16,-0.16)

In [71]:
```
d = nextX(c,omega,a)
```
Out[71]:
```
1×3 Matrix{Float64}:
 -1.00908  -1.00908  -1.00908
```
We continue as $\nabla f(x)$ is -.003 (1,1,1), however that is rather small so continuing more will only generate smaller and smaller returns

In [72]:
```
e = nextX(d,omega,a)
```
Out[72]:
```
1×3 Matrix{Float64}:
 -1.01774  -1.01774  -1.01774
```
In [73]:
```
f = nextX(e,omega,a)
```
Out[73]:
```
1×3 Matrix{Float64}:
 -1.01984  -1.01984  -1.01984
```

In [81]:

```
g = (0,0,0)
for i=1:1000
    g = nextX(f,omega,a);
end
```

In [82]:

```
g
```

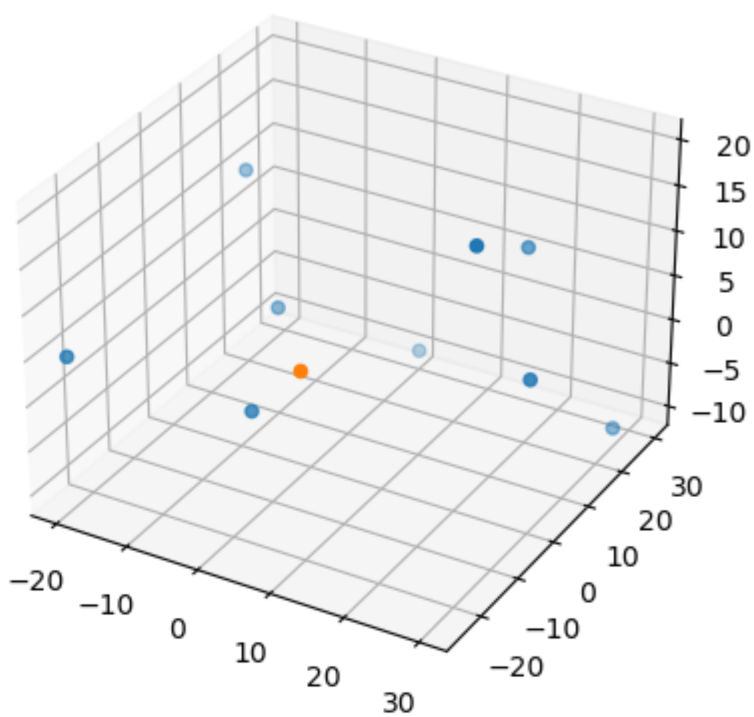
Out[82]:

```
1×3 Matrix{Float64}:
 -1.02035  -1.02035  -1.02035
```

During the next 1000 increments, our solution only improves by roughly -0.00051. As such we could've been fine with f, or even e, or d depending on accuracy wanted

In [87]:

```
scatter3D(a[:,1],a[:,2],a[:,3]);
scatter3D(g[1],g[2],g[3]);
```



In [ ]: