AN INTRODUCTION TO CHAOS THEORY

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Abstract. An introduction to Chaos Theory. Explores the basic principles as well as provides some numerical methods for graphical represention of the logisitic equation.

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1. Introduction

A common model for population growth/decay is known as the Verhulst Differential Equation, which can be seen below.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1 - \frac{P}{K})$$

Where $X(t)$ is the function describing the population at time t, with growth rate k and carrying capacity K. This continuous form has the solution of:

$$P(t) = \frac{K}{1 - Ae^{-kt}}$$
$$A := \frac{K - X_0}{X_0}$$

This however contains many flaws, one of which is that populations are not continuous. Especially in the case of seasonal breeding populations. A discrete form of this equation is called the logisitic map and is found below.

$$X(n+1) = \mu X(n)(1 - X(n))$$

Where n is the number of itterations or breeding season, $X(n)$ is the percentage of the population to the carying capacity at itteration n, and μ is a constant that is unique to the system.

A natural question to ask, is what is the eventual population of the species being modeled by either equastion. For the Verhaulst differential equation it is quite easy to show that as $t \to \infty$ that $P(t) \to K$ or $P(t) \to 0$, however this question has a not as simple answer for the discrete logisitc. [2]

2. Bifurcation

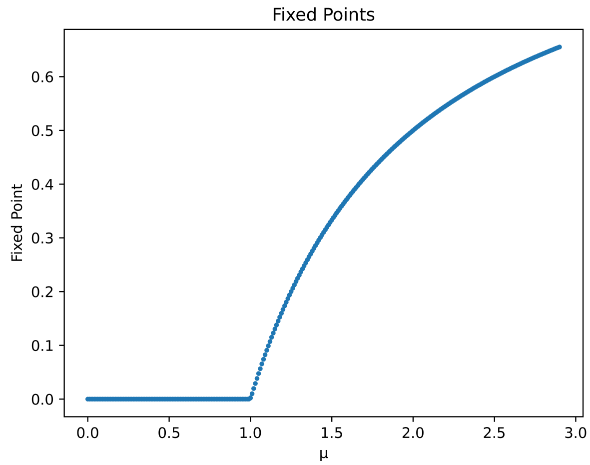
As the logisitc equation is a discrete map, it is important to realize we are not taking abount limits, but rather fixed points.

Definition 1. *A point x_* is a called fixed point of the map f if $f(x_*) = x_*$ [2]*

In particular we are talking about stable fixed points.

Definition 2. *Let $f : I \to I$ be a map and x_* be a fixed point of f, where I is an interval in the set of $\mathbb{R}$. Then x_* is said to be stable if $\forall \epsilon > 0 \exists \delta > 0$ s.t. $\forall x_0 \in I$, $|x_0 - x_*| < \delta \Rightarrow |F^n(x_0) - x^*| < \epsilon \forall n \in \mathbb{Z}^+$ [2]*

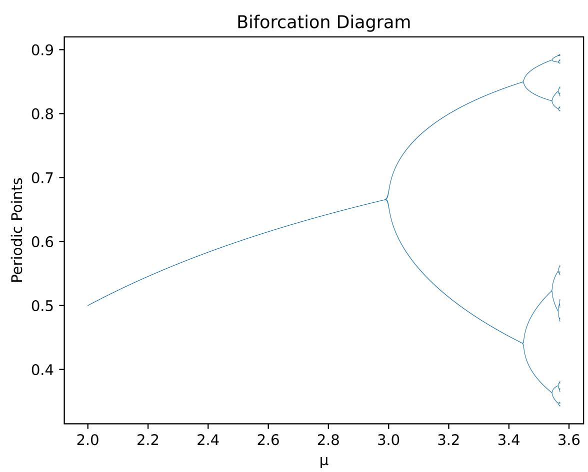
For the logisitic equation it can be shown that there is an eventually fixed point for $\mu \in (0, 3)$ and a graph of the fixed point vs μ can be seen below.



After $\mu = 3$ it is important to introduce the idea of periodic fixed points.

Definition 3. *Let x_* be in the domain of a map f. x_* is said to be a periodic point of f with period k if $f^k(x_*) = x_*$ for $k \in \mathbb{Z}^+$. In this case x_* may be called $k-$periodic. If in addition $f^r(x_*) \neq x_* \forall 0 < r < k$, then k is called the minimal period of x_*. [2]*

The logisitic map for $3 < \mu < 3.57$ can be seen below.



The logisitc map goes from having 1 fixed point before $\mu = 3$ to 2 periodic fixed points after $\mu = 3$. After $1 + \sqrt{6}$ there are 4 periodic solutions. This process of splitting into multiple periodic solutions is refered to as bifurcation, and before $\mu = 3.57$ one can find periodic points of 2^n for $n \in \mathbb{Z}^+$. We will define μ_n as the supremum of μ where 2^n is the minimal period of x_*. A table of values can be found below.

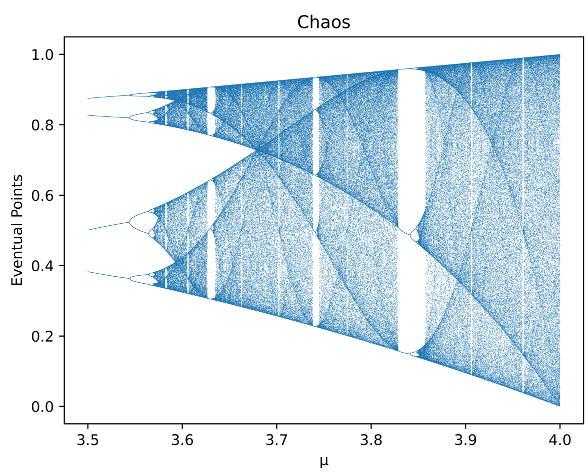
n	μ_n	$\frac{\mu_n-\mu_{n-1}}{\mu_{n+1}-\mu_n}$
1	3.449489	-
2	3.544090	-
3	3.564407	4.751419
4	3.568759	4.656248
5	3.569692	4.668321
6	3.569692	4.668683
7	3.569891	4.669354
$\vdots$	$\vdots$	$\vdots$
∞	3.57	4.669202

[2]

One strange number which arises from bifurcation is Feigenbaum's Constant which is definied as $\lim_{n\to\infty} \frac{\mu_n-\mu_{n-1}}{\mu_{n+1}-\mu_n}$ and is approximately 4.669. While this number is treated as irrational, it has never been proven to be so. It can also be shown that Feigenbaum's constant holds for all unimodal maps.[2]

3. Chaos

After $\mu = 3.57$ a mathematical phenomenon, aptly named chaos, occurs and can be seen below.



In order to define chaos, one must define transitivity, density, and

Definition 4. *Let $f : I \to I$, Then f is said to be topologically tansitive if for any pair of nonempty open intervals J_1 and J_2 in I $\exists k \in \mathbb{Z}^+$ s.t. $f^k(J_1) \cup J_2 \neq \emptyset$ [2]*

Definition 5. *Let $I \subseteq \mathbb{R}$, Then a set A is said to be dense in I if $\forall x \in I, \forall \delta > 0, \exists a \in A$ s.t. $a \in (x - \delta, x + \delta)$*

Definition 6. *A map of an interval I is said to possess senstivie dependence on initial conditions if there exsists $\nu > 0$ such that $\forall x_0 \in I, \forall \delta > 0, \exists y_0 \in (x_0 - \delta, x_0 + \delta), \exists k \in \mathbb{Z}^+$ s.t.*

$$|f^k(x_0) - f^k(y_0) \geq \nu|$$

[2]

After those 3, definitions we can then define chaos.

Definition 7. *A map $f : I \to I$, where I is an interval, is said to be chaotic if: f is transitive, the set of periodic points P is dense in I, and f has a senstivie dependence on initial conditiions. [1]*

We have defined chaos, and clearly the logisitic map has intervals of chaos, but even within the chaos, it can be orderly. To find out when the logisitc map is chaotic we look at the Liapunov Exponent.

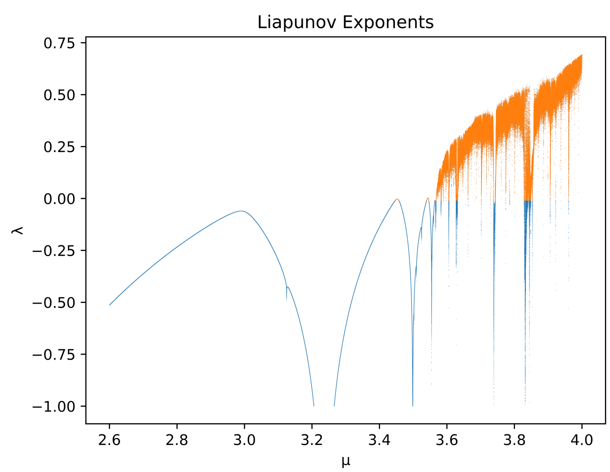
Definition 8. *Liapunov Exponent is given by*

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |f'(x(k))|$$

if $\lambda(x_0) > 0$ then senstitive dependence on initial conditions exsists. [2]

The sensitive dependence on initial conditions is due to the liapunov exponent meauses how quickly two nearby points move away from eachother as you repeatedly apply the map. If f' is greater than one, then the log is positive and sensitive dependence is shown.

A graph of the Liapunov Exponent for the logisitic equation can be seen below.



The numerical technique to calculate the exponents is:

$$\lambda(x_0) = \frac{1}{100} \sum_{k=401}^{500} \log |\mu - 2\mu x|$$

The textbook divided by 500, without explanation. [2] The average of infinitely many points in the orbit of x_0 would be more similar to 400 through 500, than 100 to 400, so I averaged 401 to 500 in order to estimate more closely. There are certainly better estimates, but this one sufficies. [2]

4. Period 3 Implies Period k

Clearly in the chaos graph there is a region of 3 periodic solutions within the chaos. This region is of significant note due to Li-Yorke's Theorem

Theorem 1. *Let $f : I \to I$ be a continuous map on an interval I. If there is a periodic point in I of period 3, then $\forall k \in \mathbb{Z}^+$ there is a periodic point in I having period k*

Which means not only does the logisitic graph have periodic points of period 2^n, but also it has periodic points of every integer. This is remarkable.

Sharkovsky strengthed this idea with his ordering and theorem.

Definition 9. *Sharkosky's ordering of the positive integers is defined as followed:*

$$\begin{aligned} 3 \rhd 5 \rhd 7 \rhd \dots \\ 2 \times 3 \rhd 2 \times 5 \rhd 2 \times 7 \rhd \dots \\ 2^n \times 3 \rhd 2^n \times 5 \rhd 2^n \times 7 \rhd \dots \\ 2^n \rhd \dots \rhd 2 \rhd 1 \end{aligned}$$

Theorem 2. *Let $f : I \to I$ be a continuous map on the interval I, where $I \subseteq \mathbb{R}$. If f has a periodic point of period k, then it has a periodic point of period $r \forall r$ s.t. $k \rhd r$*

Which is remarkable as observing certain periods can tell you information that really is not intuitive at first glance (or prehaps at any glance).

5. Appendix

This short introduction to mathematical chaos cannot do this topic justice. This paper was meant to be a survey of what I learned throughout a end of term project. What this paper lacks in both rigor and reach, I hope it makes up for in approachability. Also LaTeX did something weird on page 6, sorry.

5.1. Code.

The code below is the one I used to explore this topic numerically as well as produce the images above.

```
##Packages
import numpy as np
import matplotlib.pyplot as plt
import math
from collections import Counter

### Given a linear Map X(x), n, x0, returns numpy array
def orbit(X,n,x0):
        arr = np.zeros(n+1)
        arr[0] = x0
        for i in range(n):
                arr[i+1] = X(arr[i])
        return arr

####
def logistic(mu,x):
        return mu*x*(1-x)

def logistic_orbit(mu,n,x0):
        arr = np.zeros(n+1)
        arr[0] = x0
        for i in range(n):
                arr[i+1] = logistic(mu,arr[i])
        return arr
def Arr_const(v,n):
        arr = np.zeros(n)
        for i in range(n):
                arr[i] = v
        return arr
def drange(start,stop,step):
        r=start
        while r<stop:
                yield r
                r+=step
###
def biforcation_portion(start,end,x_0,n):
        X = Arr_const(start,100)
```

```
        Y = logistic_orbit(start,500,x_0)[401:]
        for i in drange(start,end,1/n):
                X = np.append(X, Arr_const(i,100))
                new = logistic_orbit(i,500,x_0)[401:]
                Y = np.append(Y,new)
        return X,Y

def Liapunov_exp(mu,x_0):
        sum = 0
        x=x_0
        for i in range(401,501):
                x = logistic(mu,x)
                sum += np.log(abs(mu-2*mu*x))
        return sum/100

def Liaponv_exp_auto_stop(start,stop,x_0,n,epsilon):
        arr = np.zeros(0)
        I=0
        if(start==0):
                start += n
        arr=np.append(arr,[Liapunov_exp(start,x_0)])
        for i in drange(start+n,stop,n):
                arr=np.append(arr,[Liapunov_exp(i,x_0)])
                if(arr[-1]>=0 ):
                        while(arr[-1]>=0):
                                i+=n
                                arr=np.append(arr,[Liapunov_exp(i,x_0)])
                        return arr,i
                if(arr[-1]-arr[-2]>0 and arr[-1]>-epsilon):
                        while(arr[-1]-arr[-2]>0 and arr[-1]>-epsilon):
                                i+=n
                                arr=np.append(arr,[Liapunov_exp(i,x_0)])
                        return arr,i
                I=i
        return arr,I

#X, Y = biforcation_portion(2,3.57,.1,10000)
#plt.scatter(X,Y,s=.1,linewidth=0)
#plt.title("Biforcation Diagram")
#plt.xlabel("\u03bc")
#plt.ylabel("Periodic Points")
#plt.savefig("Biforcation",dpi=1000)

Y = np.zeros(4000000-2600000)
X = np.zeros(4000000-2600000)
```

```
for i in range(2600000,4000000):
        X[i-2600000] = i/1000000
        Y[i-2600000] = Liapunov_exp(i/1000000,.2)

plt.scatter(X[Y>-1],Y[Y>-1],s=.1,linewidth=0)
plt.scatter(X[Y>=-0.01],Y[Y>=-0.01],s=.1,linewidth=0)
plt.title("Liapunov Exponents")
plt.xlabel("\u03bc")
plt.ylabel("\u03BB")
plt.savefig("liapunov",dpi=1000)


#R_list = np.linspace(2.0,3.6,10000)
#x0 = 0.2
#N = 1200


#non_repetitive=[]
#mu = []
#cool_down=False
#for r in R_list:
#               new = len(Counter(logistic_orbit(r,N,.2)[201:]))
#               non_repetitive.append(new)
#               if(new == 1000 and cool_down==False):
#                       mu.append(r)
#                       cool_down = True
#               if(new <900 ):
#                       cool_down=False

#print(mu)


#plt.scatter(R_list, non_repetitive, s=1, linewidth =0)
#plt.title("Repeated Points")
#plt.xlabel("\u03bc")
#plt.ylabel("Number of Repeated Points")
#plt.savefig("unique",dpi=1000)

#X,Y = biforcation_portion(3.5,4,.2,10000)
#plt.scatter(X,Y,s=.1,linewidth=0)
#plt.title("Chaos")
#plt.xlabel("\u03bc")
#plt.ylabel("Eventual Points")
#plt.savefig("Chaos",dpi=1000)
```

5.2. **Bibliography.**

References

[1] R. Devaney and L. Devaney. *An Introduction To Chaotic Dynamical Systems, Second Edition*. Addison-Wesley advanced book program. Avalon Publishing, 1989. ISBN: 9780201130461. URL: https://books.google.com/books?id=z1PvAAAAMAAJ.

[2] S.N. Elaydi. *Discrete Chaos, Second Edition: With Applications in Science and Engineering*. Taylor & Francis, 2007. ISBN: 9781584885924. URL: https://books.google.mw/books?id=HUmhngEACAAJ.

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