Ryan Napolitano (rn2473)
Professor Dear
COMS 4733
Homework 3 - Written

1 Problem 1: 1-Dimensional Kalman Filter

We have the linear system as follows.

$$x_{k+1} = f_k x_k + g_k u_k + v_k \tag{1}$$
$$y_k = h_k x_k + w_k, \text{ where } v_k \sim \mathcal{N}(0, q_k), \; w_k \sim \mathcal{N}(0, r_k) \tag{2}$$

a) We have the update equations as follows.

- **Prediction step:**

$$\hat{x}_{k+1|k} = f_k \hat{x}_{k|k} + g_k u_k \tag{3}$$
$$p_{k+1|k} = f_k^2 p_{k|k} + q_k \tag{4}$$

- **Update step:**

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K\tilde{y} \tag{5}$$
$$p_{k+1|k+1} = (1 - K h_{k+1}) p_{k+1|k}, \tag{6}$$
$$\text{where} \tag{7}$$
$$\tilde{y} = y_{k+1} - h_{k+1}\hat{x}_{k+1|k} \tag{8}$$
$$K = \frac{p_{k+1|k} h_{k+1}}{s}, \tag{9}$$
$$s = h_{k+1}^2 p_{k+1|k} + r_{k+1} \tag{10}$$

b) When we receive noise free measurement at time T ie $r_T = 0$, then we see from eq(10) that

$$s_T = h_{k+1}^2 p_{k+1|k} \Rightarrow K_T = \frac{1}{h_{k+1}} \Rightarrow p_{T|T} = 0 \tag{11}$$

So, convariance goes to zero in this step as you know measurement exactly.

c) As you saw, in the limit of noise free measurement, K is maximum and in this case $Kh_{k+1} \to 0 \Rightarrow p_{k+1|k+1} \to 0$. In the other limit of worse measurement, when covariance $r_{k+1} \to \infty$(uniform distribution), we see that $s \to \infty \Rightarrow K \to 0 \Rightarrow p_{k+1|k+1} \to p_{k+1|k}$. These are the two limits and clearly, always prooves that $p_{k+1|k+1} \leq p_{k+1|k}$(by also noting that $p_{i|j} \geq 0$ always).

d) When we have an asymptotically stable system ($|f_k| < 1$) and zero process noise ($q_k = 0$), then we see from eq(4) that

$$p_{k+1|k} = f_k^2 p_{k|k} < p_{k|k} \Rightarrow 0 < p_{k+1|k+1} \leq p_{k+1|k} < p_{k|k} \Rightarrow \lim_{k\to\infty} p_k = 0 \tag{12}$$

So, without process noise, covariance of estimate goes to zero asymptotically.

2 Problem 2: Process and Input Noise

We have the transition model as follows.

$$x_{k+1} = F_k x_k + G_k u_k + v_k \tag{13}$$
$$\tag{14}$$

Hence, we have

$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k \tag{15}$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + Q_k \tag{16}$$

a) Now, noting that the process noise can be changed to zero mean noise and other part added to input term, we get

$$x_{k+1} = F_k x_k + G_k u_k + \bar{v}_k + v_k^z, \text{ where } v_k^z \sim \mathcal{N}(0, Q_k) \tag{17}$$

So, we have

$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k + \bar{v}_k \tag{18}$$

Now,

$$\begin{aligned} P_{k+1|k} &= \mathbb{E}\left[(x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|k})^T\right] = \mathbb{E}\left[(F_k x_k + v_k - F_k \hat{x}_{k|k})(F_k x_k + v_k - F_k \hat{x}_{k|k})^T\right] && (19) \\ &= \mathbb{E}\left[F_k (x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T F_k^T + 2F_k (x_k - \hat{x}_{k|k})^T + v_k v_k^T\right] = F_k P_{k|k} F_k^T + \bar{v}_k^2 + Q_k && (20) \end{aligned}$$

b) We have

$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k \bar{u}_k \tag{21}$$

On same lines as before,

$$\begin{aligned} P_{k+1|k} &= \mathbb{E}\left[(x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|k})^T\right] && (22) \\ &= \mathbb{E}\left[(F_k(x_k - \hat{x}_{k|k}) + G_k(u_k - \bar{u}_k) + v_k)(F_k(x_k - \hat{x}_{k|k}) + G_k(u_k - \bar{u}_k) + v_k)^T\right] && (23) \\ &= F_k P_{k|k} F_k^T + G_k U_k G_k^T + Q_k && (24) \end{aligned}$$

3 Problem 3: Ackermann Car EKF

The non-linear Ackermann car steering model is as follows.

$$x_{k+1} = \begin{bmatrix} x_k + u_1(k)\cos\phi_k \\ y_k + u_1(k)\sin\phi_k \\ \phi_k + \frac{1}{l}u_1(k)\tan\psi_k \\ \psi_k + u_2(k) \end{bmatrix} + v_k \tag{25}$$

a) The linearized model about state $x_k = (x_k, y_k, \phi_k, \psi_k)$ for small perturbations (x, y, ϕ, ψ) is as follows.

$$x_{k+1} = \underbrace{\begin{bmatrix} 1 & 0 & -u_1(k)\sin(\phi_k) & 0 \\ 0 & 1 & u_1(k)\cos(\phi_k) & 0 \\ 0 & 0 & 1 & \frac{u_1(k)\sec^2\psi_k}{l} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{F_k} \begin{bmatrix} x \\ y \\ \phi \\ \psi \end{bmatrix} + v_k \tag{26}$$

b) When we compute $P = F_k \cdot F_k^T$, we get

$$P = \begin{bmatrix} -(u_1(k))^2(\cos(\phi_k))^2 + (u_1(k))^2 + 1 & -(u_1(k))^2\sin(\phi_k)\cos(\phi_k) & -u_1(k)\sin(\phi_k) & 0 \\ -(u_1(k))^2\sin(\phi_k)\cos(\phi_k) & 1 + (u_1(k))^2(\cos(\phi_k))^2 & u_1(k)\cos(\phi_k) & 0 \\ -u_1(k)\sin(\phi_k) & u_1(k)\cos(\phi_k) & 1 + \frac{(u_1(k))^2}{l^2(\cos(\psi_k))^4} & \frac{u_1(k)}{l(\cos(\psi_k))^2} \\ 0 & 0 & \frac{u_1(k)}{l(\cos(\psi_k))^2} & 1 \end{bmatrix} \tag{27}$$

c) Clearly, the covariance components of x,y which are P_{21} and P_{12} will get minimized only when $\phi_k = 0$ or $\phi_k = 90°$. This is obvious as in these headings, it is pure x or y motion and there is no variance coming to the orthogonal component here. Hence, covariance goes to zero in these headings due to lack of cross-sensitivity.

d) From P in eq(27), $P_{33}, P_{3,4}, P_{4,3}$ all become unbounded as $\psi_k \to \frac{\pi}{2}$ as $\cos\psi$ in denominator$\to 0$. This means covariance of ϕ to ϕ and ϕ to ψ go unbounded. Because in this limit, the vehicle is trying to take a sharp left 90° turn which makes the change in heading highest and in limit unbounded. The value $\psi \to \frac{\pi}{2}$ is literally in direction normal to current heading and hence, any change will make its effect on ϕ go unbounded as sensitivity is $\to \infty$.

4 Problem 4: Non-Euclidean PRM

1. Done in file in function `sample_points`
2. Done in file in function `construct_prm`
3. The incremental collision check hasnt been done as I am not able to get a full grip on the toroidal space structure. The screenshot of solution otherwise is as shown. However, due to lack of incremental check for collision, two states can have intermediate colliding points. Response to the questions are as follows.

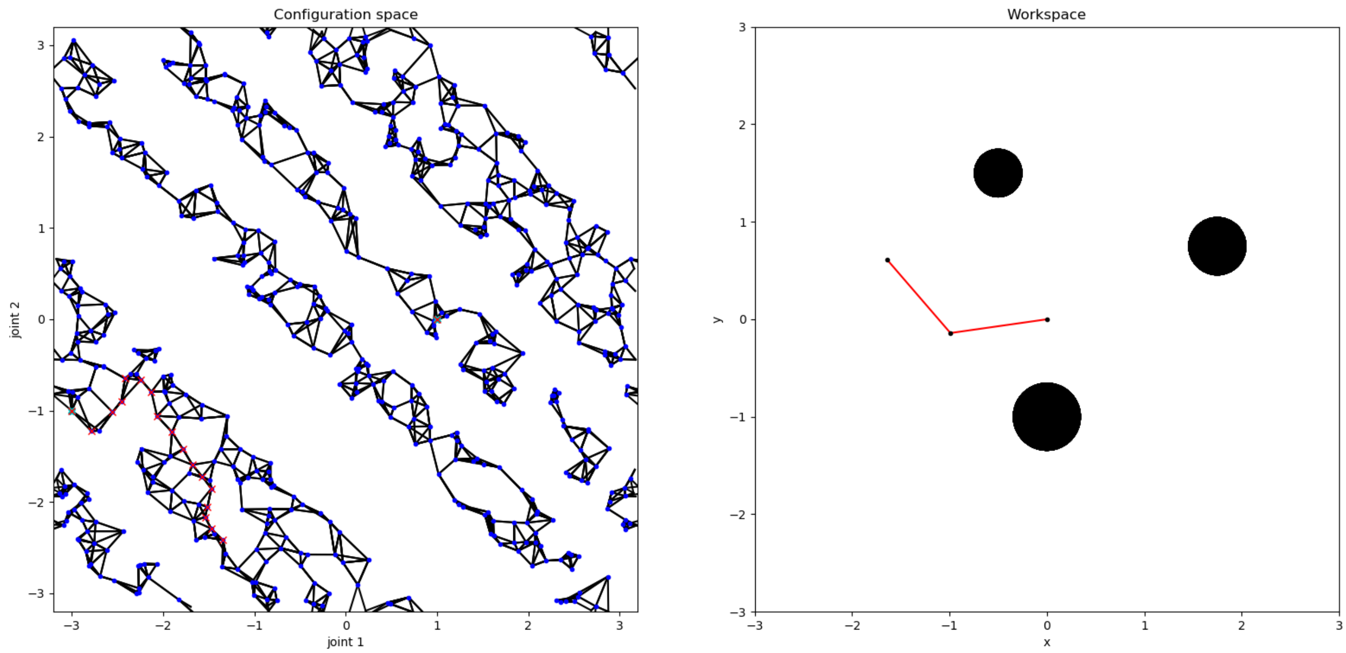

Figure 1: Screen shot of solution for Point set 1

(a) More the samples, higher is the computational overhead of sampling them, but better is a change to build a more spanning sampled population of configurations. In this case, we found around 300 points to be giving good results consistently.

(b) More the neighbors, the chance of configurations far away from current configurations getting connected increases resulting in higher probability for intermiediate states colliding and also for motion being too jerky. On the other hand, too few edges can cause some of the nodes to become disjoint with not connectivity resulting in unreachable sampled points and hence, no path and wastage of sampling compute resources.

(c) Smaller the edge length, lesser the probability of intermediate configurations colliding obstacles and better the chance for a smoother plan. However, this also risks getting disjoint or isolated samples causing no solution cases.