# COMS W4733: Computational Aspects of Robotics

Homework 3

## Problem 1: 1-Dimensional Kalman Filter (16 points)

(a) Rewriting from the standard Kalman filter equations [1]:

$$\begin{aligned}
\hat{x}_{k+1|k} &= f_k x_{k|k} + g_k u_k \\
p_{k+1|k} &= f_k^2 p_{k|k} + q_k \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K\tilde{y} = \hat{x}_{k+1|k} + \frac{h_{k+1}p_{k+1|k}}{h_{k+1}^2 p_{k+1|k} + r_{k+1}}(y_{k+1} - h_{k+1}\hat{x}_{k+1|k}) \\
p_{k+1|k+1} &= (1 - Kh_{k+1})p_{k+1|k} = \frac{r_{k+1}}{h_{k+1}^2 p_{k+1|k} + r_{k+1}} p_{k+1|k}
\end{aligned}$$

(b) $p_{T|T} = 0$

(c) The fraction $\frac{r}{h^2 p + r}$ is guaranteed to be between 0 and 1, since its magnitude is smaller than 1 and all terms are positive ($r$ and $p$ are both positive terms as they are both variances).

(d) We've already shown that $p_{k+1|k+1}$ must be smaller than $p_{k+1|k}$. For the variance prediction, since $p_{k+1|k} = f_k^2 p_{k|k}$ and $f_k$ has a magnitude smaller than 1, it is also the case that $p_{k+1|k}$ is smaller than $p_{k|k}$. Thus the uncertainty will tend to 0 over time.

## Problem 2: Process and Input Noise (16 points)

(a) Mean prediction simply incorporates the process noise mean: $\hat{\mathbf{x}}_{k+1|k} = F_k\hat{\mathbf{x}}_{k|k} + G_k\mathbf{u}_k + \overline{\mathbf{v}}_k$

Covariance prediction follows the derivation from lecture notes ($G_k\mathbf{u}_k$ terms cancel out):

$$\begin{aligned}
P_{k+1|k} &= E[(\mathbf{x}_{k+1} - \hat{x}_{k+1|k})(\mathbf{x}_{k+1} - \hat{x}_{k+1|k})^T] \\
&= E[(F_k\mathbf{x}_k + \mathbf{v}_k - F_k\hat{\mathbf{x}}_{k|k} - \overline{\mathbf{v}}_k)(F_k\mathbf{x}_k + \mathbf{v}_k - F_k\hat{\mathbf{x}}_{k|k} - \overline{\mathbf{v}}_k)^T] \\
&= E[F_k(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})^T F_k^T] + E[(\mathbf{v}_k - \overline{\mathbf{v}}_k)(\mathbf{v}_k - \overline{\mathbf{v}}_k)^T] \\
&= F_k P_{k|k} F_k^T + Q_k
\end{aligned}$$

(b) Mean prediction utilizes the input control mean: $\hat{\mathbf{x}}_{k+1|k} = F_k\hat{\mathbf{x}}_{k|k} + G_k\overline{\mathbf{u}}_k$

Covariance prediction follows the derivation from lecture notes ($G_k\mathbf{u}_k$ terms do not cancel out):

$$\begin{aligned}
P_{k+1|k} &= E[(\mathbf{x}_{k+1} - \hat{x}_{k+1|k})(\mathbf{x}_{k+1} - \hat{x}_{k+1|k})^T] \\
&= E[(F_k\mathbf{x}_k + G_k\mathbf{u}_k + \mathbf{v}_k - F_k\hat{\mathbf{x}}_{k|k} - G_k\overline{\mathbf{u}}_k)(F_k\mathbf{x}_k + G_k\mathbf{u}_k + \mathbf{v}_k - F_k\hat{\mathbf{x}}_{k|k} - G_k\overline{\mathbf{u}}_k)^T] \\
&= E[F_k(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})^T F_k^T] + E[G_k(\mathbf{u}_k - \overline{\mathbf{u}})(\mathbf{u}_k - \overline{\mathbf{u}})^T G_k^T] + E[\mathbf{v}_k\mathbf{v}_k^T] \\
&= F_k P_{k|k} F_k^T + G_k U_k G_k^T + Q_k
\end{aligned}$$

[1] All equations for this probelm can be found on page 25 of lecture 6

Note that we are omitting covariance cross-terms in both cases as they simplify to be 0.

## Problem 3: Ackermann Car EKF (18 points)

(a) $F_k = \frac{\partial f}{\partial \mathbf{x}}|_{\mathbf{x}=\hat{\mathbf{x}_k}|k} \begin{bmatrix} 1 & 0 & -u_1(k)\sin\hat{\phi}_{k|k} & 0 \\ 0 & 1 & u_1(k)\cos\hat{\phi}_{k|k} & 0 \\ 0 & 0 & 1 & \frac{1}{l}u_1(k)\sec^2\hat{\psi}_{k|k} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $F_k F_k^T = \begin{bmatrix} 1+u_1(k)^2\sin^2\hat{\phi}_{k|k} & -u_1(k)^2\sin\hat{\phi}_{k|k}\cos\hat{\phi}_{k|k} & -u_1(k)\sin\hat{\phi}_{k|k} & 0 \\ -u_1(k)^2\sin\hat{\phi}_{k|k}\cos\hat{\phi}_{k|k} & 1+u_1(k)^2\cos^2\hat{\phi}_{k|k} & u_1(k)\cos\hat{\phi}_{k|k} & 0 \\ -u_1(k)\sin\hat{\phi}_{k|k} & u_1(k)\cos\hat{\phi}_{k|k} & 1+\frac{1}{l^2}u_1(k)^2\sec^4\hat{\psi}_{k|k} & \frac{1}{l}u_1(k)\sec^2\hat{\psi}_{k|k} \\ 0 & 0 & \frac{1}{l}u_1(k)\sec^2\hat{\psi}_{k|k} & 1 \end{bmatrix}$

(c) The covariance components of $x$ are those in the first row and first column of the above matrix; they are minimized when $\phi = 0$. Conversely, the covariance components of $y$ are those in the second row and second column; they are minimized when $\phi = \frac{\pi}{2}$. In the former case the car is heading only in the $x$ direction so there is minimal uncertainty in this component. In the latter case the car is heading only in the $y$ direction, minimizing uncertainty in this component.

(d) The variance of the $\phi$ component and the covariance between $\phi$ and $\psi$. As $\psi \to \frac{\pi}{2}$, uncertainties in these components increase without bound. This is because the front wheels become more and more perpendicular to the back wheels. This produces high uncertainty in the car's heading, as it becomes more sensitive to heading changes and perturbations, and also increases the correlations between the car's heading and wheel angles.

## Problem 4: Non-Euclidean PRM

### 4.3: Outputs and Discussion (20 points)

(a) More samples: Higher likelihood of connecting PRM and covering the space, but slower sampling and PRM construction processes. Fewer samples: Higher likelihood of disconnected graph components and insufficient coverage.

(b) More neighbors: Greater likelihood of connecting vertices to PRM and forming connected components, but slower graph construction process and expensive nearest neighbor computations. Fewer samples: Higher likelihood of disconnected graph components and isolated useless vertices.

(c) Larger edge length: Greater likelihood of connecting vertices to more or further neighbors, but requires more computations for local planner to determine collisions. Smaller edge length: Fewer neighbors and distances to check, but higher likelihood of disconnected graph components and isolated vertices.