C161; General Relativity and Black Holes Review

prof. kasen, Spring 2020

The Equivalence Principle

With special relativity, Einstein's reasserted that the laws of physics are valid in all inertial frames and established the transformation rules between these frames, But what about a frame that is accelerating? Do the laws of physics apply is these non-inertial frames? Einstein attempted to generalize his ideas in his theory of general relativity which, interestingly, wound up being a theory of gravity.

An *inertial reference frame* is one in which Newton's first law holds, i.e., where "an undisturbed body stays at rest or moves at a constant velocity". Imagine you are in a windowless spaceship out in deep space, far from any gravitating mass. If your ship is not accelerating and you let go of your phone, it will remain stationary in front of you, floating in exactly the place where you left it.

On the other hand, if your spaceship is accelerating and you let go of your phone, then both you and the ship will accelerate away from it. From your perspective it appears that you are at rest and the phone is accelerating away from you, as if some force was pulling on it. In Newtonian physics, such an apparent force is called "fictitious", since nothing is actually pulling on the phone. Such fictitious forces are generic features of non-inertial frame.

How might you tell that a force was fictitious? An important clue would be that a fictitious force accelerates all objects *at the same rate*, regardless of their mass, shape, composition, etc... This is in contrast to other forces, like the electric force, for which the rate of acceleration depends on the electric charge of an object.

Einstein realized that one force – gravity – appeared to behave like a fictitious force. In Newtonian physics the gravitational force on a mass m by another mass M is

$$F = m_I a = \frac{GMm_G}{r} \implies a = \frac{GM}{r}\frac{m_G}{m_I} \tag{1}$$

Here we have made a conceptual distinction between the *inertial mass*, m_I and the *gravitational charge*, m_G. The inertial mass describes how resistant an object is to acceleration, whereas the gravitational charge[1] describes how strongly an object feels a gravitational force. In Newtonian physics these two quantities turn out to be equal

$$m_I = m_G \tag{2}$$

and so both m_I and m_G are usually just called the "mass". This equality is one form of the *equivalence principle*.

[1] The term gravitational "charge" highlights the similarity to electric charge. The electric charge, q, of a particle describes how strongly it couples to the electric field. In general, of course, the electric charge is not equal to the inertial mass.

According to the equivalence principle m_I and m_G cancel such that all objects experience the same gravitational acceleration, $a = GM/r$. Gravity resembles a fictitious force! Einstein felt that this cancelation was too remarkable to be a coincidence. In his theory of General Relativity, he took a dramatically different approach to gravity.

Einstein suggested that a massive object like the Earth warps the spacetime metric around it. An object moving freely would follow a natural path – or *geodesic* – in this curved spacetime. A spaceship in free fall, for example, is not being pulled towards the Earth, it is simply following a geodesic. An elevator at rest on the surface of earth, on the other hand, deviates from a geodesic, effectively accelerating with respect to one. Such an elevator would be a non-inertial frame, and inside of it we would experience a fictitious force that we call "gravity".

Curved Spacetime

We can provide some quantitative description of curved spacetime. For a 1D space (e.g., ants walking along a string) we could quantify the curvature at any point on the string in the following way: at any point P along the string, draw a circle that follows the bend of the string at P. We define the linear curvature at that point as $1/a$, where a is the radius of the circle[2]

[2] A circle of smaller radius implies a more sharply curving line, that is why we define curvature as an inverse of the radius.

For a 2D space, a single circle does not fully characterize the curvature; we can instead fit two circles at point P: one along the direction of maximum curvature and one along the direction of minimum curvature. We define the Gaussian curvature, K_G, as

$$K_G = \kappa_1 \kappa_2 = \frac{1}{a_1}\frac{1}{a_2} \tag{3}$$

where κ_1 and κ_2 are the minimum and maximum curvatures corresponding to circles of radius a_1 and a_2. The sign of κ_1 and κ_2 are set by the direction in which the circles bend. If the two circles bend in the same direction (as on the surface of a sphere) the curvature is positive. If the two circles bend in opposite directions (as on the surface of a saddle) the curvature is negative. If one of the curvatures is zero (as on a cylinder) the Gaussian curvature is zero.

Can we tell if we are living in a curved spacetime? Yes; the mathematician Gauss showed that the Gaussian curvature could be determined by local measurements made within the space. In flat (uncurved) spacetime, the geometry is *Euclidean*, and the familiar rules from geometry hold, e.g., the circumference of a sphere of radius r is $2\pi r$ and the angles of a triangle add up to 180^o. In a curved, *non-Euclidean* spacetime, the rules of geometry differ.

A useful example is the surface of a sphere of radius a. This is a curved 2D space that we can visualize by *embedding* it in flat 3D space. Ants living on the surface of the sphere would only experience 2 spatial dimensions and may not realize that there was a 3rd. Imagine an ant that started at the North Pole of the sphere and walked south a distance r, and then walked around a line of constant latitude to measure the circumference. The ant would find

$$C = 2\pi a \sin(r/a) \tag{4}$$

and so $C \neq 2\pi r$. If the size of the circle is small, $r \ll a$, we can use the Taylor expansion of $\sin(r/a)$

$$C = 2\pi a \left[\frac{r}{a} - \frac{1}{3!}\frac{r^3}{a^3} + ... \right] \tag{5}$$

where the ... indicate terms of higher order in r/a. This can be written

$$C = 2\pi r - \frac{\pi}{3}\frac{r^3}{a^2} + ... \tag{6}$$

and we see that the circumference of circles drawn on the sphere are *smaller* than $2\pi r$. The Gaussian curvature of a sphere is $K_G = 1/a^2$ which we can relate to the circumference by

$$K_G =_{\lim\ r\to 0} \frac{3}{\pi r^3}(2\pi r - C) \tag{7}$$

where we write limit as $r \to 0$ since we have only kept the leading term in the Taylor expansion. While we derived this relation for a sphere, it turns out that it holds for any space. For a space of negative curvature ($K_G < 0$) we see that the circumference of a circle is *larger* than $2\pi r$. A sphere is an example of a space where the curvature is the same everywhere, but more generally the curvature can vary from place to place in the space.

While the surface of a cylinder looks curved, it's Gaussian curvature is zero. If we roll a flat piece of paper into a cylinder, the geometry of triangles or circles written on it remains Euclidean. The 2D surface of the cylinder has only *extrinsic* curvature (i.e., curvature noticed when seeing it embedded in a higher dimensional 3D space) but no *intrinsic* curvature – i.e., there would be no local measurements an an ant living on the surface could that would distinguish the cylinder from flat space[3]. If we want to bend the paper into a sphere, however, there would be no way to do so without stretching and distorting the shapes drawn on it. This would introduce intrinsic curvature, which would be reflected in the metric.

[3] There are global measurements the ant could make, since it could walk around the cylinder and notice that it wound up back where it started. This feature, however, refers to the *topology* of the surface (i.e., how it is connected) rather than it's curvature.

Geodesics and the Principle of Maximum Proper Time

In Newtonian physics, objects moving freely travel in a straight line at constant velocity. In the curved spacetime of GR, the analogous "natural" motion is along *geodesics*, which are those paths that *maximize* the proper time[4] A complete analysis of geodesics requires the calculus of variations, which we will not get into. We can, however, describe the motion through important constants of motion that arise from the "principle of extremal proper time".

[4] Recall that the proper time is the negative of the spacetime interval. For example, for flat spacetime

$$c^2\Delta\tau^2 = -\Delta s^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

So paths that reduce the spatial distance traveled, $(dx^2 + dy^2 + dz^2)$, will tend to *increase* the proper time. Thus maximizing the proper time is a way of generalizing the idea of the "shortest path" between two spacetime events.

To be precise, the geodesics in GR correspond to *extrema* of proper time. But in most cases we will consider, the extrema are maxima, not minima.

Consider an object that moves through spacetime from an event A at coordinate $(0, 0)$ to an event C at (t_C, x_C) (see Figure 1) For simplicity we imagine the object's path is broken up into two straight line segments that pass through an intermediate event B at coordinate (t_B, x_B). We ask: "Where should event B lie in order to maximize the proper time of the observer?" This will describe the geodesic between A and C.

In the flat spacetime of special relativity, the geodesic path from A to C is clearly a straight line (i.e., motion with constant velocity) with B lying along that line. Let us derive this fact from the principle of extremal proper time, then generalize to curved spacetime. The flat spacetime metric is

$$ds^2 = -c^2 dt^2 + dx^2 \tag{8}$$

The proper time for each of the two segments of the path are

$$\Delta\tau_1 = \sqrt{t_B^2 - (x_B)^2/c^2} \tag{9}$$

$$\Delta\tau_2 = \sqrt{(t_C - t_B)^2 - (x_C - x_B)^2/c^2} \tag{10}$$

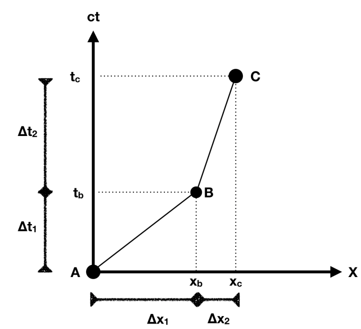

Figure 1: An object moves through spacetime from event A to event C, passing through point B along the way. We ask, "where does event B lie in order to maximize the proper time?" The answer will describe the geodesic from A to C.

The total proper time is $\tau = \Delta\tau_1 + \Delta\tau_2$. Maximizing it's extremal value with respect to x_B (while holding t_B and the endpoints fixed) gives

$$\frac{d\tau}{dx_B} = \frac{d\Delta\tau_1}{dx_B} + \frac{d\Delta\tau_2}{dx_B} = 0 \tag{11}$$

Carrying out the differentiating of each term we find

$$\frac{d\tau}{dx_B} = \frac{\Delta x_1/c^2}{\Delta\tau_1} - \frac{\Delta x_2/c^2}{\Delta\tau_2} = 0 \tag{12}$$

And so

$$\frac{\Delta x_1}{\Delta\tau_1} = \frac{\Delta x_2}{\Delta\tau_2} \tag{13}$$

An arbitrary path through spacetime can be built up by putting together many small straight line segments as in Figure 2. By applying the above reasoning to subsequent pairs line segments we see

$$\frac{\Delta x_1}{\Delta\tau_1} = \frac{\Delta x_2}{\Delta\tau_2} = \frac{\Delta x_3}{\Delta\tau_3} = \frac{\Delta x_4}{\Delta\tau_4} = \ldots \tag{14}$$

Thus $\Delta x/\Delta\tau$ is the same for any path segment. Taking the path segments to be infinitesimally small, we have

$$\frac{dx}{d\tau} = \text{constant} \tag{15}$$

We have found a "constant of the motion".

By maximizing proper time with respect to t_b we can similarly show that $dt/d\tau = $ constant. Analogous arguments hold with respect to the y and z directions. We can write all of these results together as

$$\frac{d}{d\tau}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \gamma\frac{d}{dt}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c\gamma \\ \gamma u_x \\ \gamma u_y \\ \gamma u_z \end{pmatrix} = \text{constant} \tag{16}$$

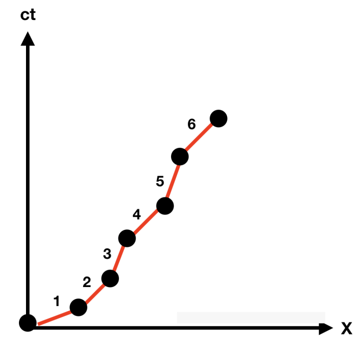

Figure 2: An arbitrary path through spacetime could be broken down into many small straight line segments. Each pair of segments follow the relations you derived

which used the time dilation relation $t = \gamma\tau$ of special relativity. The above is just our definition of the four velocity vector U^μ, and multiplying it by mass gives our energy-momentum four vector $p^\mu = mU^\mu$. This problem shows that the definitions of special relativistic momentum and energy are motivated from the principle of extremal proper time.

The above arguments are easily generalized to the curved spacetime of general relativity. The metric is written

$$ds^2 = -g_{tt}c^2dt^2 + g_{xx}dx^2 \tag{17}$$

where g_{tt} and g_{xx} are components of some general metric[5] that encodes the curvature of the spacetime. The same sort of argument as above shows that *if the metric components are independent of a coordinate* x_i, a constant of motion is

[5] In general, mixed terms like $dxdt$ may appear in the metric, but we won't bother with those here.

$$g_{ii}\frac{dx_i}{d\tau} = \text{constant} \tag{18}$$

For example, if the metric is independent of the coordinate $x_0 = t$, then we have $g_{tt}(dt/d\tau)$ is a constant (which we will generally call energy). We will use these constants of motions to describe the geodesics around black holes.

The Schwarzchild Metric

Soon after Einstein wrote down the equations of general relativity, Schwarzchild derived a solution for the spacetime around a spherical mass M. We simply quote the resulting Schwarzchild metric here

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 + \frac{dr^2}{(1-2GM/c^2r)} + r^2d\theta^2 + r^2\sin\theta^2d\phi^2 \tag{19}$$

The metric has a combination of parameters, $2GM/c^2$ which has dimensions of length. This is called the *Schwarzchild radius*

$$R_s = \frac{2GM}{c^2} \approx 3 \text{ km} \left(\frac{M}{M_\odot}\right) \tag{20}$$

The Schwarzchild radius turns out to be the *event horizon* of a (non-spinning) black hole, in that fall below R_s will never be able to reemerge[6] .

We can rewrite the Schwarzchild metric in terms of R_s

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) c^2 dt^2 + \frac{dr^2}{(1 - R_s/r)} + r^2 d\Omega^2 \tag{21}$$

where we used the shorthand $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The Schwarzchild metric only applies to the region *outside* the mass M. The spacetime outside the Sun, for example, is described by Eq. 21 but inside the sun ($r < R_\odot$) the metric differs. The Schwarzchild radius for the sun $R_s < R_\odot$, so the Sun has no event horizon. Black holes are masses that have been compressed smaller than R_s.

The coordinates $(t, r, \theta\phi)$ used in Equation 21 are called *Schwarzchild coordinates*, or sometimes "far-away" or "bookkeeper" coordinates, because they are a natural coordinate system for an observer situated far away from the black hole. Here θ and ϕ are the ordinary angles we use in spherical coordinate but the radial coordinate r is *not* a measure of the distance from the center[7].

The Schwarzchild metric provides all of the information about the spacetime around a black hole. If we consider a slice of fixed time (i.e., $dt = 0$) then the spacetime interval is a measure of *proper distance*, and we can write $dl = \sqrt{ds^2}$, where l is a proper spatial distance. For example, if we imagine laying out a measuring tape along the radial direction and measuring proper time, then $dt = d\theta = d\phi = 0$ and the metric becomes

$$dl = \sqrt{ds} = \frac{dr}{(1 - R_s/r)^{1/2}} \tag{22}$$

Far away from the mass ($r \gg R_s$) we have $dl \approx dr$ and so the coordinate interval dr is equivalent to the proper distance dl. However, nearer to the mass the proper length is a factor of $(1 - R_s/r)^{-1/2} > 1$ larger than the coordinate interval dr. If we integrate to find the proper length between two coordinates r_1 and r_2 we find

$$l = r_2 - r_1 + \frac{R_s}{2} \log\left(\frac{r_2}{r_1}\right) + ... \tag{23}$$

where the ... represents higher order terms in r/R_s. We see that the proper distance is greater than $r_2 - r_1$ due to the curvature of space around the mass.

[6] The Schwarzchild radius can be motivated by Newtonian physics. If you throw an object of mass m straight up in a gravitational field, it's Newtonian energy is the sum of its kinetic and gravitational potential energies

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

If $E > 0$ the object will escape the gravitational field; solving the above equation for r we find

$$r_{\text{esc}} = \frac{2GM}{v^2}$$

and if the velocity of the object is $v = c$ we get the Schwarzchild radius.

[7] For one thing, we could never lay out a measuring tape from $r = 0$ to some coordinate r because inside the black hole it will be impossible for objects to remain at rest. No matter how strong the measuring tape, it would be ripped apart if we tried to hold it in place. We thus will need another way of defining the coordinate r.

If instead we hold the radial coordinate fixed ($dr = 0$) and lay a measuring tape around the equator ($\theta = \pi/2$) we would find

$$\int dl = \int_0^{2\pi} r d\phi \implies l = 2\pi r \tag{24}$$

And so the circumference of a circle is $C = 2\pi r$. This is actually how r is *defined* in Schwarzchild coordinates – it is not defined as the distance from the center, but rather by measuring the circumference at fixed r and dividing by 2π. The coordinate $r = C/2\pi$ is thus sometimes called a *circumferential* coordinate.

If we imagine a clock held at a fixed position (i.e., $dr = d\theta = d\phi = 0$) the spacetime interval becomes a measure of *proper time*, and we can write $c^2 d\tau = \sqrt{-ds^2}$ where τ is the proper time. From the metric

$$d\tau = \sqrt{-ds^2/c^2} = \left(1 - \frac{R_s}{r}\right)^{1/2} dt \tag{25}$$

If we want we can integrate this over some finite interval

$$\Delta\tau = \left(1 - \frac{R_s}{r}\right)^{1/2} \Delta t \tag{26}$$

We see that an interval of proper time $\Delta\tau$ is smaller than an interval of coordinate (or "faraway") time Δt. Thus clocked held fixed about a mass M will run slowly compared to a clock far away. This effect is known as *gravitational time dilation*.

An effect closely related to gravitational time dilation is *gravitational redshift*. If light is emitted with a wavelength λ_s within a gravitational field, it will be observed far away from the gravitational field to have a longer (i.e., redder) wavelength $\lambda_{\rm obs}$. This is because the period of the light wave $\Delta T = \lambda/c$ is subject to gravitational time dilation, such that Eq. 26 implies

$$\lambda_{\rm s} = \left(1 - \frac{R_s}{r}\right)^{1/2} \lambda_{\rm obs} \tag{27}$$

and so $\lambda_{\rm obs} > \lambda_s$. As an object approaches the event horizon at R_s, the gravitational time dilation and redshift approach infinity.

The Black Hole Event Horizon

The Schwarzchild metric becomes singular at two points: $r = R_s$ and $r = 0$. At $r = 0$ the curvature blows up to infinity and our mathematical description of spacetime intervals breaks down. Presumably some additional physics (e.g., quantum gravity) sets in near $r = 0$ to produce well-defined behavior, but as of yet we do not understand what this physics is.

The singularity at $r = R_s$ is known as a *coordinate singularity*. There is nothing pathological about the spacetime at $r = R_s$; the curvature remains finite and one could pass through R_s without noticing anything special. The singularity at this location is merely an artifact of the specific coordinates chosen, and can be removed by changing coordinates.

There is something special, however, about the Schwarzchild radius that we can understand by examining the paths of light around this location. Light moves on lightlike (or null) geodesics, $ds^2 = 0$, and so for radial motion ($d\Omega^2 = 0$) the Schwarzchild metric becomes

$$0 = -\left(1 - \frac{R_s}{r}\right) c^2 dt^2 + \frac{dr^2}{(1 - R_s/r)} \tag{28}$$

which we can rearrange to find the coordinate speed of light moving radially outward

$$\left(\frac{dr}{dt}\right)_{\text{light}} = c\left(1 - \frac{R_s}{r}\right) \tag{29}$$

This equation makes it look like the speed of light is not always equal to c. Before getting too concerned, we should remind ourselves that the quantity dr/dt is only a "coordinate speed" that tells us how many radial coordinate markers or "flags" a beam of light crosses in some unit of time. Such a coordinate speed depends on how we space the flags and need not always be a constant for light[8].

[8] As discussed in the last section, the r-coordinate flags in the Schwarzchild metric are not spaced equally, rather the proper distance between r flags becomes larger as we get closer to R_s. Thus, as light approaches the black hole it passes fewer and fewer coordinate flags in a given unit of time. This is just a consequence of how we have chosen to place or coordinate flags, and does not violate any principle of relativity; the fundamental property of light is that $ds^2 = 0$, and this remains true regardless of how we choose our coordinates.

Imagine you are at location r_e outside a black hole and emit a radial beam of light to a friend far away at $r = r_o$. The closer you are to R_s the slower your light beam moves in the Schwarzchild coordinates, and the longer it will to take for your light signal to arrive at your friend. We can quantify this by rewriting Eq. 29 as

$$dt = \frac{dr}{c(1 - R_s/r)} \tag{30}$$

and integrating to find the total time (in far-away Schwarzchild coordinates) taken for the light beam to travel from r_e to r_o

$$\Delta t = \frac{r_o - r_e}{c} + \frac{R_s}{c} \ln\left[\frac{r_o - R_s}{r_e - R_s}\right] \tag{31}$$

The first term on the right hand side would be the travel time of light in a flat spacetime, while the second logarithmic term represents the "time delay" due to the curved spacetime. This delay increases for signals emitted closer to R_s and becomes infinite at $r_e = R_s$. If you were to send out periodic light signals to your friend as you fall into the black hole, these signals would take longer and longer to arrive. Your friend would see you just approaching R_s, but would never see you reach it or falling below it, even if he waited infinite time.

At the point $r = R_s$ the metric in Schwarzchild coordinates becomes singular – an alternative coordinate system would be needed to give a continuous description of an object falling through R_s. However, we can pick up the description again inside R_s, where we see from Eq. 29 the sign of $(dr/dt)_{\text{light}}$ becomes *negative*. This means that if you are inside of R_s and emit a light beam radially outward, *the light still moves radially inward*. There is no way for any signal that you send to make it to your friend outside R_s, which is why the Schwarzchild radius R_s is called the *event horizon* of the black hole[9]. Because nothing moves faster than light, you cannot escape the event horizon either, and will inevitably find yourself moving inward towards the singularity at $r = 0$.

[9] We also see from Eq. 31 that the time taken for your signal to arrive at your friend becomes undefined (since natural log of a negative number is undefined). This indicates there is no time at which the light will reach your friend.

Looking back at the Schwarzchild metric, Eq. 21, we see that for $r < R_s$ the sign in front of dt^2 flips to become positive while that in front of dr^2 flips negative. Thus, time becomes spacelike and space becomes limelike. Inside of the event horizon you can no more avoid moving inward in r than you could avoid going forward in time now. There is no way to avoid the fate of the singularity at $r = 0$.

Schwarzchild Geodesics: Falling into a Black Hole

We can quantify the physics of falling into a blackhole using the Schwarzchild metric. In general relativity there is no longer any gravitational "force". Instead, freely moving objects follow the natural paths (i.e., geodesics) which extremize proper time. When spacetime is curved this can give the impression that a force is acting.

A full treatment of geodesics requires studying the calculation of variations. Here we will use analyses that relies simply on the constants of motions discussed in the previous section. We noted that if the metric is independent of time, then

$$\frac{E}{m} = g_{tt}\frac{dt}{d\tau} \tag{32}$$

is a constant of motion that we call the "energy" (per unit mass). For the Schwarzchild metric the energy is then

$$\frac{E}{m} = c^2\left(1 - \frac{R_s}{r}\right)\frac{dt}{d\tau} \tag{33}$$

Consider an object moving only in the radial direction ($d\theta = d\phi = 0$). The Schwarzchild metric is, writing $c^2 d\tau^2 = -ds^2$

$$c^2 d\tau^2 = \left(1 - \frac{R_s}{r}\right)c^2 dt^2 - \frac{dr^2}{(1 - R_s/r)} \tag{34}$$

Dividing both sides by $d\tau^2$ and multiplying both sides by $(1 - R_s/r)$

gives

$$c^2\left(1-\frac{R_s}{r}\right)=\left(1-\frac{R_s}{r}\right)^2c^2\left(\frac{dt}{d\tau}\right)^2-\left(\frac{dr}{d\tau}\right)^2 \tag{35}$$

We recognize the first term on the right hand side as just E^2/m^2c^2. Replacing this and rearranging we find

$$\frac{E^2}{m^2c^2}-c^2=\left(\frac{dr}{d\tau}\right)^2-c^2\frac{R_s}{r} \tag{36}$$

This equation becomes even more suggestive if we multiply through by $m/2$ and put in the expression for R_s

$$\frac{mc^2}{2}\left[\left(\frac{E}{mc^2}\right)^2-1\right]=\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2-\frac{GMm}{r} \tag{37}$$

This looks a lot like the equation for Newtonian energy, where the first term on the right hand side is kinetic energy and the second term gravitational potential energy. The left hand side is not just the energy E but it is still a constant, and we can show [10] that it in the limit it reduces to

$$E_N=\frac{1}{2}m\left(\frac{dr}{dt}\right)^2-\frac{GMm}{r} \tag{38}$$

where E_N is the Newtonian energy. Thus we have derived the Newtonian equation for radial motion using just the Schwarzchild metric and the principle of maximum proper time. In Netwonian physics, the first term on the right hand side is kinetic energy and the second is potential energy. In GR there is no real distinction between kinetic and gravitational potential energy. We instead have a constant of motion E/m that incorporates both.

Imagine now an astronaut that falls into a black hole starting from very far away, $r_0 \gg R_s$. If the person starts out at rest, the energy becomes $E = mc^2$, i.e., the initial energy is just the rest mass energy. Eq. 37 can then be written

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2=\frac{GMm}{r}\implies\left(\frac{dr}{d\tau}\right)^2=\frac{c^2R_s}{r} \tag{39}$$

This differential equation can be solved to find the elapsed proper time to fall from r_0 to a radius r

$$\Delta\tau=\frac{2}{3}\frac{R_s}{c}\left[\left(\frac{r_0}{R_s}\right)^{3/2}-\left(\frac{r}{R_s}\right)^{3/2}\right] \tag{40}$$

Your result shows that an astronaut in free fall will pass through the event horizon of a black hole and reach the center in a finite proper time. In particular, the proper time to fall from the event

[10] The Newtonian energy does not include rest mass energy so $E_N = E - mc^2$, so putting this into the left hand we get

$$\frac{mc^2}{2}\left[\left(\frac{E_N}{mc^2}+1\right)^2-1\right)\right]$$

and expanding out in the limit $E_n/mc^2 \ll 1$ we find the left hand side is just E_N to leading order. In the Newtonian limit where all speeds are $\ll c$ we also have $d\tau = dt$ (i.e., proper time is equal to coordinate time since time dilation is negligible).

horizon ($r_2 = R_s$) to $r_1 = 0$ is simply $(2/3)R_s/c$. For a stellar mass black hole, $R_s \approx 3$ km, this time is only about 6 microseconds. For a supermassive black hole of $M = 10^9 M_\odot$, the time is a bit less than 2 hours.

A friend watching the astronaut falling into the black hole would see something quite different. The friend uses Schwarzchild coordinate time t (i.e., "far away" time) instead of the astronaut's proper time τ. Since our constant of motion (Eq. 32) is $E = mc^2$ we have

$$d\tau = \left(1 - \frac{R_s}{r}\right) dt \tag{41}$$

Then our above expression

$$\frac{dr}{d\tau} = c\sqrt{\frac{R_s}{r}} \tag{42}$$

becomes using coordinate time

$$\frac{dr}{dt} = d\sqrt{\frac{R_s}{r}}\left(1 - \frac{R_s}{r}\right) \tag{43}$$

For $r \gg R_s$, we can ignore the term in parenthesis and the result is just that for free fall in Newtonian gravity. Initially, the friend sees the astronaut falling faster and faster towards the black hole. But as the astronaut approaches $r = R_s$, the term in parenthesis becomes important, and the friend measures the astronaut to slow down and gently approach rest at the event horizon, never to fall below it. Of course, we have shown above that from the astronaut's point of view they do indeed fall through the event and reach $r = 0$ in finite time. The friend does not see this because any light beams sent by the astronaut take longer and longer to get to the friend, with the time taken approaching infinity at R_s.

Circular Schwarzchild Orbits

The last section considered the purely radial motion of somebody plunging directly into a black hole. Let's consider now the orbits of objects around a spherical mass M, which could be a black hole or a star. We will consider orbits that lie in the equatorial plane, so $\theta = \pi/2$ and $d\theta = 0$.

For comparisons sake, first recall that in Newtonian physics the energy equation for an orbiting object is

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2 - \frac{GMm}{r} \tag{44}$$

The first terms on the right hand side are the radial and angular kinetic energy. We have two constants of motion in Newtonian orbits, the energy E and the angular momentum

$$L = mr\left(r\frac{d\phi}{dt}\right) \tag{45}$$

Using this definition of L the energy equation becomes

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{46}$$

which we can rewrite

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V_{\rm eff}(r) \tag{47}$$

where $V_{\rm eff}$ is an *effective potential* that includes both the gravitational potential term $-GMm/r$ (which tends to pull objects to the center) and the centrifugal term $L/2mr^2$ (which tends to push objects out). We have thus isolated the radial motion of the orbit.

A circular orbit is one where the object stays at a fixed radius r_c. This occurs at a minimum of $V_{\rm eff}(r)$, which we can find by setting $dV_{\rm eff}/dr = 0$. One finds

$$r_{c,N} = \frac{L^2}{GMm^2} \tag{48}$$

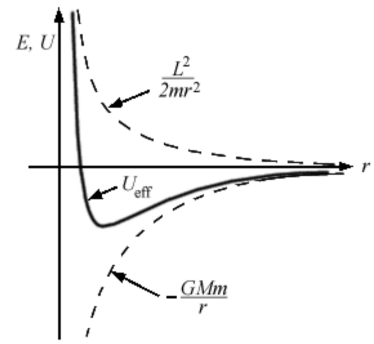

Figure 3: Plot of the effective potential in Newtonian gravity, labeled here as $U_{\rm eff}$. A circular orbit corresponds to the minimum of $U_{\rm eff}$, which is an equilibrium point where an orbiting body can remain at constant r.

Circular orbits of smaller radius have smaller L. Note that in Newtonian physics there are stable orbits all the way down to $r = 0$. This will change in general relativity.

In general relativity, we can study orbits using the Schwarzchild metric, which for the equatorial plane ($\theta = \pi/2$) is

$$c^2 d\tau^2 = -ds^2 = \left(1 - \frac{R_s}{r}\right)c^2 dt^2 - \frac{dr^2}{(1 - R_s/r)} - r^2 d\phi^2 \tag{49}$$

Since the metric is independent of t and ϕ, we can identify two constants of motion

$$\frac{E}{m} = c^2\left(1 - \frac{R_s}{r}\right)\frac{dt}{d\tau} \qquad\qquad \frac{L}{m} = r^2\frac{d\phi}{d\tau} \tag{50}$$

where E/m is the energy (per unit mass) discussed in the last section, and L/m is an angular momentum-like quantity.

Dividing the metric through by $d\tau^2$ and using the constants of motion to replace $dt/d\tau$ and $d\phi/d\tau$ we find after some algebra

$$\epsilon = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + V_{\rm eff} \tag{51}$$

where

$$\epsilon = \frac{mc^2}{2}\left[(E/mc^2)^2 - 1)\right] \tag{52}$$

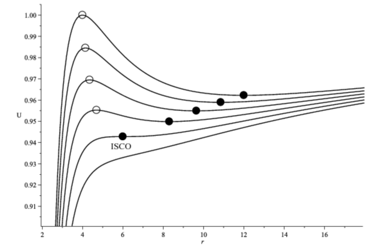

Figure 4: Plots of the effective potential for Schwarzchild orbits of different angular momentum L. The drop in V_{eff} at small radius is a new feature of GR. There are two equilibrium points (marked by dots) but only one is stable (the filled dot). As L is decreased, the stable orbit moves inward, but below some minimum value there is no longer any stable circular orbits.

and the effective potential is

$$V_{\text{eff}}(r) = -\frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2 GM}{r^3 mc^2} \tag{53}$$

The Schwarzchild orbit equation looks very similar to the Newtonian one, with the notable exception that there is an extra terms in the effective potential that goes like $-1/r^3$. This is sometimes called the "pit in the potential", because it causes the effective potential to turn over and become negative at small r (see Figure 4)

Taking $dV_{\text{eff}}/dr = 0$ we can find the equilibrium points corresponding to circular orbits. This results in a quadratic equation which has a solution

$$r_c = \frac{1}{2}\frac{L^2}{GMm^2} \pm \frac{1}{2}\sqrt{\frac{L^4}{G^2M^2m^4} - 4\frac{3L^2}{m^2c^2}} \tag{54}$$

This equation can be made more instructive if we write things in terms of the characteristic length scale of the problem, R_s, and a characteristic angular momentum scale defined as

$$L_c = R_s mc \tag{55}$$

Then the solution for the circular orbits can be written

$$r_c = R_s \left(\frac{L}{L_c}\right)^2 \left[1 \pm \sqrt{1 - \frac{3L_c^2}{L^2}}\right] \tag{56}$$

We see that for the Schwarzchild metric there are *two* positions where circular orbits are possible. This can be seen in Figure 4, where we notice that only the outermost circular orbit is stable. If an object was on the innermost circular orbit, and small perturbation would cause it to fly either into the black hole, or off into space. In the limit $L \gg L_c$ we just get the Newtonian solution of $r_c = r_N$ (and the other circular orbit is at $r = 0$).

From Eq. 56 we see that there is a minimum angular momentum allowed for a circular orbit, because for $L < \sqrt{3}L_c$ the solution is not real. Thus there is a minimum radius of a stable circular orbit in a Schwarzchild metric which (plugging in $L = \sqrt{3}L_c$) is

$$r_{\text{isco}} = 3R_s \quad (\textit{innermost stable circular orbit}) \tag{57}$$

Figure 4 illustrates how as L is decreased the effective potential changes until there is no longer a stable orbit. Any object that tries to orbit at $r < r_{\text{isco}}$ is destined to fall into the black hole. For example, a disk of gas accreting onto a black hole will only extend down to r_{isco}; below that, the gas plunges into the black hole. Similarly, a binary system of two black holes will only remain stable when their orbital separation is greater than r_{isco}; below that, the fall into each other and merge.

We can use the above results to estimate how much energy can be released by gas spiraling into a black hole. Gas in an accretion disk will follow nearly circular orbits, with the gas nearer the center swirling around faster than gas farther out. If there were little friction in the disk, the gas could orbit around indefinitely, like the rings of Saturn. However, as the annuli of gas at different radii are sliding and shearing against each other, the viscosity of the gas releases heat and slows down the inner layers, which fall closer to the black hole. Once gas drops below $r_{\rm isco}$ it can no longer orbit stably and will fall quickly below the event horizon.

Consider a bit of mass m in a disk around a black hole. We previously found the energy equation of Schwarzchild orbits

$$\frac{mc^2}{2}\left[\left(\frac{E}{mc^2}\right)^2 - 1\right] = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2GM}{r^3mc^2} \tag{58}$$

For a circular orbit the radius is constant, so we can set $dr/d\tau = 0$. Rewriting everything in terms of R_s and L_0 the equation becomes

$$\left(\frac{E}{mc^2}\right)^2 - 1 = -\frac{R_s}{r} + 3\frac{L^2}{L_0^2}\left(\frac{R_s}{r}\right)^2\left[1 - \frac{R_s}{r}\right] \tag{59}$$

From this equation we find that the energy of a bit of mass orbiting at the innermost stable orbit (where $L = L_0$ and $r = 3R_s$) is $E = \sqrt{8/9}(mc^2)$. If this bit of mass started off very far from the black hole, $r \gg R_s$, we find that its initial energy is just the rest mass energy $E = mc^2$. The energy difference is

$$\Delta E = \left(1 - \sqrt{\frac{8}{9}}\right)mc^2 \approx 0.052\ mc^2 \tag{60}$$

Thus for a bit of mass to move from far out in the disk to a circular orbit at $r_{\rm isco}$ it must loose about 5% of its energy. It can do this by radiating away the energy as light. The viscosity in the disk turns some of the kinetic energy of the orbiting gas into heat, and the hot gas can then emit light. The remaining 95% or so of the mass/energy winds up being eaten by the black hole[11]

For rotating black holes, the innermost stable orbit turns out to be even closer in. For a maximally spinning black hole, one finds $r_{\rm isco} = R_s$ (assuming the disk rotates in the same direction as the black hole). In this case, even greater energy release is possible, reaching around 12% of mc^2. Not all accretion onto black holes leads to such a large release of energy. If gas lacks angular momentum it may plunge directly into the black hole rather then remaining on circular orbits. In this case it may only radiate a small amount of energy away, with almost all of the energy going into the black hole (or being blown out in winds or jets). This is the case with the supermassive black hole in the center of our own Galaxy, which radiates quite inefficiently.

[11] We can compare this energy release to other processes. Typical chemical processes (e.g., burning coal) involves changes in the atomic and molecular structure, where the energy levels are of order electron volts (eV). The masses of typical atoms are of order 1 GeV (the mass of a proton), so the chemical processes release around eV/GeV $\sim 10^{-9}$ of the rest mass energy. Nuclear reactions such as those that power the energy of the Sun release about an MeV per nucleus, so the efficiency is around 10^{-3}. These are all well below the possible energy release efficiency of black hole accretion.

Light Orbits and Gravitational Lensing

The motion of photons in the Schwarzchild metric can be treated in a way similar to that in the lest section. The key difference is that photons travel on null geodesics, where $ds^2 = 0$. We thus cannot use proper-time as a parameter. We can use, however, some other parameter (call it λ) that marks off positions along the photon trajectory. It will not be important exactly how we define λ here. Simply note that the constants of motion will be the same as those above, just with $d\tau$ replaced by $d\lambda$ Carrying out a similar approach of dividing the Schwarzchild metric through by by $d\lambda$ and replacing the constants of motions we find

$$\frac{e_p^2}{c^2} = \left(\frac{dr}{d\lambda}\right)^2 + V_{\text{eff}}(r) \tag{61}$$

where

$$V_{\text{eff}}(r) = l_p^2 \left[\frac{1}{r^2} - \frac{R_s}{r^3}\right] \tag{62}$$

where e_p and l_p are the two constants of motion[12]. Looking for a minimum $dV_{\text{eff}}/dr = 0$ we find the location of circular orbits

$$r_c = \frac{3}{2} R_s \qquad (\textit{photon sphere}) \tag{63}$$

[12] Previously we called these constants of motion E/m and L/m. However this definition does not make much sense for photons, which have zero mass, $m = 0$. Recall that our derivation of the constants of motion only used the principle of extremal proper time and make no explicit reference to mass, and so the constants relabeled as e_p and l_p are still well defined and non-zero for photons of mass zero.

There is thus only one radius at which photons can have circular orbits, which is called the *photon sphere*. It is the location where light can orbit in a circle around the black hole. If you stood at r_p and looked ahead, you would see the back of your head! This is not a stable orbit, though, so light could not orbit around indefinitely at this radius.

The famous image of taken by the Event Horizon Telescope – often described as "a picture of a black hole" – is of course instead showing light emitted by gas orbiting around a supermassive black hole at the center of a nearby galaxy. . The edge of the dark circle we see from the Event Horizon Telescope image most closely tracks the photon sphere. Light from the disk that approaches this radius will swing around in a circle, though eventually will either fall into the black hole or fly off to be observed. Light that goes below r_p will rapidly fall into the black hole. This produces the so-called "shadow" of the black hole around r_p.

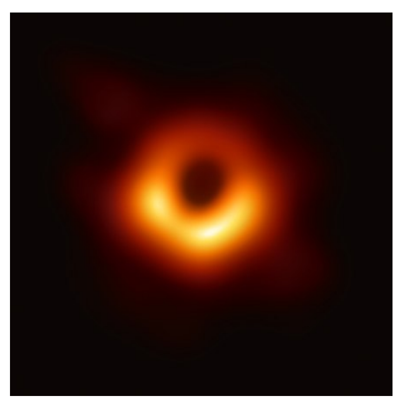

Figure 5: Image of gas around a supermassive black hole, as taken by the Event Horizon Telescope, a set of radio observatories.