C161; Problem Set #5

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Problem 1: Einstein Ring

In class we showed that a light beam passing by a spherical mass M at an impact parameter b undergoes a deflection by an angle

$$\alpha = \frac{2R_s}{b} = \frac{4GM}{c^2 b} \tag{1}$$

If a source lies directly behind a lens along the observer line of sight, the light rays will be bent symmetrically around the lens and form a beautiful Einstein ring.

The figure shows the geometry for lensing that produces an Einstein ring, from which we can calculate the size of this ring.

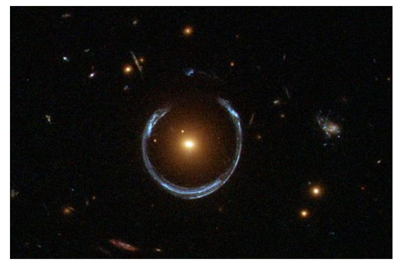
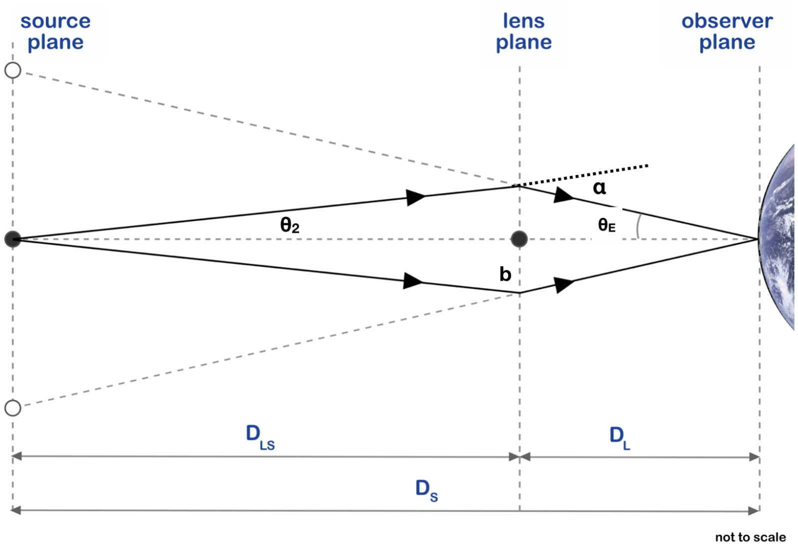


Figure 1: Image of an Einstein ring produced by lensing.



1a) From the geometry of the figure, derive that the angular size of the Einstein ring

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}} \tag{2}$$

where D_L is the distance to the lens, D_S the distance to the source, and D_{LS} the distance between the lens and source[1]

[1] These distances are all *angular diameter distances*. Later when we talk about the expanding Universe, we will see there are various ways to define a distance.

Solution: From the triangle that includes θ_E and θ_2 we have the angles add to 180 degrees

$$\theta_E + \theta_2 + (\pi - \alpha) = \pi \tag{3}$$

which becomes

$$\theta_E + \theta_2 = \alpha = \frac{4GM}{c^2 b} \tag{4}$$

Writing

$$\tan\theta_E \approx \theta_E = \frac{b}{D_L} \implies b = \theta_E D_L \tag{5}$$

and

$$\tan\theta_2 \approx \theta_2 = \frac{b}{D_{LS}} = \theta_E \frac{D_L}{D_{LS}} \tag{6}$$

we find

$$\theta_E + \theta_2 = \theta_E + \frac{b}{D_{LS}} = b\frac{D_{LS} + D_L}{D_L D_{LS}} = \theta_E \frac{D_{LS} + D_L}{D_{LS}} \tag{7}$$

and since $D_{LS} + D_L = D_S$

$$\theta_E + \theta_2 = \theta_E \frac{D_S}{D_{LS}} \tag{8}$$

Putting it all together

$$\theta_E \frac{D_S}{D_{LS}} = \frac{4GM}{c^2} \frac{1}{\theta_E D_L} \tag{9}$$

or

$$\theta_E^2 = \frac{4GM}{c^2} \frac{D_{LS}}{D_S D_L} \tag{10}$$

and so

$$\boxed{\theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_S D_L}}} \tag{11}$$

comment: For a galaxy of mass $M \approx 10^{12}\ M_\odot$, and where the source and lens are at cosmological distances ($D \approx 10^9$ light years) the Einstein ring is about few arcseconds, which is small but resolvable with telescopes. For a stellar mass black hole ($M \approx M_\odot$) lensing a star in our Galaxy ($D \approx 10,000$ light years) the Einstein ring is more like 5×10^{-4} arcseconds. This would be very difficult to resolve, and so for stellar objects in the Galaxy we typically only detected the magnification due to lensing. The unresolved case is called *microlensing*.

Problem 2: Falling into a Black Hole

THE NATURAL MOTION OF OBJECTS in general relativity is along *geodesics*, which are the curved spacetime generalization of a straight line. The timelike geodesics are determined by finding paths that are extrema (usually maxima) of the proper time. In the last homework, you showed that the principle of maximum proper time led to *constants of motion*. For example

$$g_{tt}\frac{dt}{d\tau} = \text{constant} \tag{12}$$

where g_{tt} is the term in front of dt^2 in the metric. The Schwarzchild metric is

$$c^2 d\tau^2 = -ds^2 = \left(1 - \frac{R_s}{r}\right) c^2 dt^2 - \frac{dr^2}{(1 - R_s/r)} - r^2 d\Omega \tag{13}$$

Identifying the term g_{tt} in front of the dt^2 we see that a constant of motion is

$$\left(1 - \frac{R_s}{r}\right) c^2 \frac{dt}{d\tau} = \frac{E}{m} \tag{14}$$

where we called the constant E/m, since we will see it is related to the energy per unit mass.

2a) Consider purely radial motion, so that $d\Omega = 0$. Divide the metric through by $d\tau^2$ and rearrange to show

$$\epsilon = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} \tag{15}$$

where ϵ is a constant that depends on E, m and c.

Solution: Setting $d\Omega = 0$ and diving the metric through be $d\tau$ gives

$$c^2 = \left(1 - \frac{R_s}{r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 \frac{1}{(1 - R_s/r)} \tag{16}$$

Multiplying through by $(1 - R_s/r)$ this becomes

$$\left(1 - \frac{R_s}{r}\right) c^2 = \left(1 - \frac{R_s}{r}\right)^2 c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 \tag{17}$$

We recognize the first term on the right hand side as our constant of motion and replace it

$$\left(1 - \frac{R_s}{r}\right) c^2 = (E/m)^2 - \left(\frac{dr}{d\tau}\right)^2 \tag{18}$$

Rearranging we get

$$(E/m)^2 - c^2 = \left(\frac{dr}{d\tau}\right)^2 - \frac{R_s}{r} c^2 \tag{19}$$

Using the definition of R_s

$$(E/m)^2 - c^2 = \left(\frac{dr}{d\tau}\right)^2 - \frac{2GM}{rc^2}c^2 \tag{20}$$

Now multiplying through by $m/2$ we find

$$\frac{mc^2}{2}\left[(E/mc^2)^2 - 1)\right] = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} \tag{21}$$

Or

$$\epsilon = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} \tag{22}$$

where

$$\epsilon = \frac{mc^2}{2}\left[(E/mc^2)^2 - 1)\right] \tag{23}$$

2b) The above result looks very similar to the energy equation for something falling in Newtonian gravity. The Newtonian energy is $E_N = E - mc^2$ (since it does not include the rest mass energy) and if objects move much slower than c we have $E_N \ll mc^2$. Use an expansion to show that $\epsilon = E_N$ in this limit.

Solution: To see how ϵ is related to energy, write $E_N = E - mc^2$. Then $E = E_N + mc^2$ and the expression for epsilon is

$$\epsilon = \frac{mc^2}{2}\left[\left(\frac{E_n + mc^2}{mc^2}\right)^2 - 1)\right] \tag{24}$$

or

$$\epsilon = \frac{mc^2}{2}\left[(1 + E_N/mc^2)^2 - 1)\right] \tag{25}$$

and expanding out in the limit $E_n/mc^2 \ll 1$

$$\epsilon = \frac{mc^2}{2}\left[(1 + 2E_N/mc^2 + ... - 1)\right] = \frac{mc^2}{2}\left[2E_N/mc^2\right] = E_N \tag{26}$$

comment: In the Newtonian limit where all speeds are $\ll c$ we also have $d\tau = dt$ (i.e., proper time is equal to coordinate time since time dilation is neglible). Therefore our equation becomes

$$E_n = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - \frac{GMm}{r} \tag{27}$$

We have derived the Newtonian equation for radial motion using just the Schwarzchild metric and the principle of maximum proper time. Newtonian physics arises as a limit of GR! In Netwonian physics, the first term on the right hand side is kinetic energy and the second is potential energy. In GR there is no real distinction between kinetic and gravitational potential energy. We instead have a constant of motion E/m that incorporates both.

Imagine throwing a ball straight up – it rises up, slows, then eventually turns around and falls back down to your hand some time later. In Newtonian physics, we say that a gravitational force pulls down on the ball and causes it to turn around and fall back. In GR, there is no mention of gravitational force, rather than ball is following its natural path – a geodesic – which maximizes its proper time through the curved spacetime[2].

[2] Imagine a clock pasted to the ball and consider how the ball can maximize the proper time measured by this clock. Due to gravitational time dilation, clocks run more slowly near the surface of the earth, so to maximize τ the ball will want to rise higher in the sky where its clock runs faster. However, if the ball rises up too quickly, it's gamma factor $\gamma = (1 - u^2/c^2)^{1/2}$ will increase and it's clock will slow down due to kinematic time dilation. Thus there will be optimal path for the ball where these two effects combine to give a maximal τ. This path is just the trajectory familiar from Newtonian physics.

2c) Imagine an astronaut falling into a black hole. She starts from rest very far away, so that her energy is $E \approx mc^2$ (i.e., the only energy is rest mass energy). Solve Eq. 15 to show that the proper time she experiences when falling from a radius r_2 to a radius r_1 is

$$\tau = \frac{2}{3}\frac{R_s}{c}\left[\left(\frac{r_2}{R_s}\right)^{3/2} - \left(\frac{r_1}{R_s}\right)^{3/2}\right] \tag{28}$$

Solution: For $E = mc^2$ we have $\epsilon = 0$. So we write the equation as

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{GMm}{r} \implies \left(\frac{dr}{d\tau}\right)^2 = \frac{2GM}{r} \tag{29}$$

which can be written

$$\left(\frac{dr}{d\tau}\right)^2 = c^2 R_s/r \tag{30}$$

To solve the differential equation we separate variables

$$r dr^2 = R_s c^2 d\tau^2 \implies r^{1/2} dr = R_s^{1/2} c\, d\tau \tag{31}$$

which integrates to

$$\frac{2}{3} r^{3/2} = R_s^{1/2} c\tau \tag{32}$$

and so

$$\tau = \frac{2}{3}\frac{R_s}{c}\left(\frac{r}{R_s}\right)^{3/2} \tag{33}$$

If the limits are r_2 to r_1, the proper time elapsed is

$$\Delta\tau = \frac{2}{3}\frac{R_s}{c}\left[\left(\frac{r_2}{R_s}\right)^{3/2} - \left(\frac{r_1}{R_s}\right)^{3/2}\right] \tag{34}$$

comment: Your result shows that a person in freefall will pass through the event horizon of a black hole and reach the center in a finite proper time. In particular, the proper time to fall from the event horizon ($r_2 = R_s$) to $r_1 = 0$ is simply $(2/3)R_s/c$. For a stellar mass black hole, $R_s \approx 3$ km, this time is only about 6 microseconds. For a supermassive black hole of $M = 10^9 M_\odot$, the time is a bit less than 2 hours. Thus you might have time to watch a Netflix movie before you hit the singularity[3].

[3] Assuming you already downloaded it, since you won't be able to connect to the internet from inside the event horizon.

2d) Consider a friend watching the astronaut falling into the black hole. The friend uses Schwarzchild coordinate time t (i.e., "far away" time) instead of the astronaut's proper time τ. Use your constant of motion to replace τ and show that the friend observes the astronaut (with $E = mc^2$) to move with speed

$$\frac{dr}{dt} = \sqrt{\frac{2GM}{r}}\left(1 - \frac{R_s}{r}\right) \tag{35}$$

Solution: We have $E = mc^2$ so our constant of motion is

$$\left(1 - \frac{R_s}{r}\right) c^2 \frac{dt}{d\tau} = c^2 \tag{36}$$

or

$$d\tau = \left(1 - \frac{R_s}{r}\right) dt \tag{37}$$

Then our above expression

$$\frac{dr}{d\tau} = \sqrt{\frac{2GM}{r}} \tag{38}$$

becomes

$$\frac{dr}{dt} = \sqrt{\frac{2GM}{r}}\left(1 - \frac{R_s}{r}\right) \tag{39}$$

comment: For $r \gg R_s$, we can ignore the term in parenthesis and the result is just that for free fall in Newtonian gravity. Initially, the friend sees the astronaut falling faster and faster towards the black hole. But as the astronaut approaches $r = R_s$, the term in parenthesis becomes important, and the friend measures the astronaut to slow down and gently approach rest at the event horizon, never to fall below it. Of course, we have shown above that from the astronaut's point of view they do indeed fall through the event and reach $r = 0$ in finite time. The friend does not see this because any light beams sent by the astronaut take longer and longer to get to the friend, with the time taken approaching infinity at R_s.

Problem 3: Schwarzchild Orbits

The last problem considered the purely radial motion of somebody plunging directly into a black hole. Let's consider now the orbits of objects around a spherical mass M, which could be a black hole or a star. We will consider orbits that lie in the equatorial plane, so $\theta = \pi/2$ and $d\theta = 0$.

For comparisons sake, first recall that in Newtonian physics the energy equation for an orbiting object is

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2 - \frac{GMm}{r} \tag{40}$$

The first terms on the right hand side are the radial and angular kinetic energy. We have two constants of motion in Newtonian orbits, the energy E and the angular momentum

$$L = mr\left(r\frac{d\phi}{dt}\right) \tag{41}$$

Using this defintition of L the energy equation becomes

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{42}$$

which we can rewrite

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V_{\text{eff}}(r) \tag{43}$$

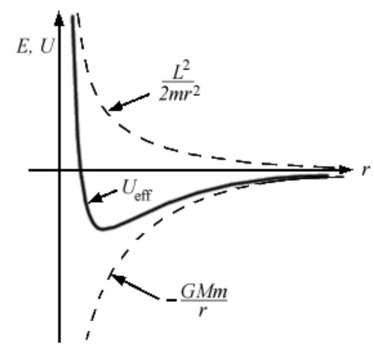


Figure 2: Plot of the effective potential in Newtonian gravity, labeled here as U_{eff}.

where V_{eff} is an *effective potential* that includes both the gravitational potential term $-GMm/r$ (which tends to pull objects to the center) and the centrifugal term $L/2mr^2$ (which tends to push objects out). With this equation we have isolated the radial motion of the orbit.

3a) A circular orbit is one where the object stays at a fixed radius r_c. This occurs at a minimum of $V_{\text{eff}}(r)$. Find the radii r_c of circular Newtonian orbits in terms of the angular momentum L.

Solution: The effective potential in Newtonian gravity is

$$V_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{44}$$

We want to find the minimum of $V_{\text{eff}}(r)$, so we differentiate

$$\frac{dV_{\text{eff}}}{dr} = -2\frac{L^2}{2mr^3} + \frac{GMm}{r^2} = 0 \tag{45}$$

multiplying through by r^3 this becomes

$$-\frac{L^2}{m} + rGMm = 0 \tag{46}$$

And solving for r gives the radius of a circular orbit

$$r_c = \frac{L^2}{GMm^2} \tag{47}$$

comment: Circular orbits of smaller radius have smaller L. Note that there is no minimum value of r_c – in Newtonian physics there are stable orbits as close to the mass as you want. This will change in general relativity.

IN GENERAL RELATIVITY we can study orbits using the Schwarzchild metric, which for the equatorial plane ($\theta = pi/2$) is

$$c^2 d\tau^2 = -ds^2 = \left(1 - \frac{R_s}{r}\right) c^2 dt^2 - \frac{dr^2}{(1 - R_s/r)} - r^2 d\phi^2 \qquad (48)$$

The constants of motion determined from this metric, as discussed in the last problem, are E/m along with an angular momentum-like quantity

$$r^2 \frac{d\phi}{d\tau} = \frac{L}{m} = \text{constant} \qquad (49)$$

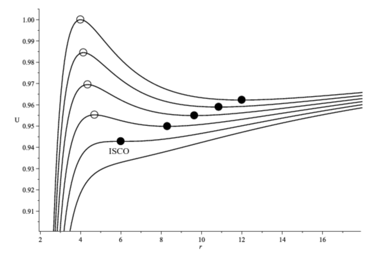


Figure 3: Plots of the effective potential for Schwarzchild orbits of different angular momentum L. The drop in V_{eff} at small radius is a new feature of GR. There are two equilibrium points, but only one is stable. As L is decreased, the stable orbit moves inward, but below some minimum value there is no longer any stable circular orbits.

3b) Rewrite the metric in terms of the constants of motion to find

$$\epsilon = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2 GM}{mc^2 r^3} \qquad (50)$$

Solution: Dividing the metric through by $d\tau$

$$c^2 = \left(1 - \frac{R_s}{r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{(1 - R_s/r)} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \qquad (51)$$

and multiplying through by $(1 - R_s/r)$

$$c^2 \left(1 - \frac{R_s}{r}\right) = \left(1 - \frac{R_s}{r}\right)^2 c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \left(1 - \frac{R_s}{r}\right) \qquad (52)$$

Using the definition of E/M and L/m

$$c^2 \left(1 - \frac{R_s}{r}\right) = c^2 \left(\frac{E}{mc^2}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{r^2 m^2} \left(1 - \frac{R_s}{r}\right) \qquad (53)$$

rearranging

$$c^2 \left(\frac{E}{mc^2}\right)^2 - c^2 = \left(\frac{dr}{d\tau}\right)^2 - \frac{c^2 R_s}{r} + \frac{L^2}{r^2 m^2} - \frac{L^2 R_s}{r^3 m^2} \qquad (54)$$

Multiply through by $m/2$ and putting in R_s

$$\epsilon = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2 GM}{r^3 mc^2} \qquad (55)$$

where

$$\epsilon = \frac{mc^2}{2} \left[(E/mc^2)^2 - 1)\right] \qquad (56)$$

comment: The Schwarzchild orbit equation looks very similar to the Newtonian one, with the notable exception that there is an extra terms in the effective potential that goes like $-1/r^3$. This is sometimes called the "pit in the potential", because it causes the effective potential to turn over and become negative at small r.

3c) Use the condition $dV_{\text{eff}}/dr = 0$ to determine the radii of circular orbits r_c for Schwarzchild orbits in terms of L.

Solution: The effective potential is now

$$V_{\text{eff}}(r) = -\frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2GM}{r^3mc^2} \tag{57}$$

Finding the minimum of this

$$\frac{dV_{\text{eff}}}{dr} = \frac{GMm}{r^2} - 2\frac{L^2}{2mr^3} + 3\frac{L^2GM}{r^4mc^2} = 0 \tag{58}$$

multiply through by r^4

$$GMmr^2 - r\frac{L^2}{m} + \frac{3L^2GM}{mc^2} = 0 \tag{59}$$

and dividing through by GMm

$$r^2 - r\frac{L^2}{GMm^2} + \frac{3L^2}{m^2c^2} = 0 \tag{60}$$

We solve this using the quadratic formula

$$r_c = \frac{1}{2}\frac{L^2}{GMm^2} \pm \frac{1}{2}\sqrt{\frac{L^4}{G^2M^2m^4} - 4\frac{3L^2}{m^2c^2}} \tag{61}$$

This is a fine solution, but we could clean things up a bit by writing this as

$$r_c = \frac{1}{2}\frac{L^2}{GMm^2} \pm \frac{1}{2}\frac{L^2}{GMm^2}\sqrt{1 - \frac{12G^2M^2m^2}{L^2c^2}} \tag{62}$$

or

$$r_c = \frac{1}{2}\frac{L^2}{GMm^2}\left[1 \pm \sqrt{1 - \frac{12G^2M^2m^2}{L^2c^2}}\right] \tag{63}$$

The term L^2/GMm^2 is just the radius of a Newtonian circular orbit that we found in part a). If we notate this as r_N we could if we want rewrite our solution as

$$r_c = \frac{r_N}{2}\left[1 \pm \sqrt{1 - \frac{12G^2M^2m^2}{L^2c^2}}\right] \tag{64}$$

By dimensional analysis we note that the quantity $12G^2M^2m^2/c^2$ must have units of angular momentum squared. So to make things even nicer we could define a characteristic angular momentum $L_0 = sqrt12GMm/c$ so our solution is

$$r_c = \frac{r_N}{2}\left[1 \pm \sqrt{1 - \frac{L_0^2}{L^2}}\right] \tag{65}$$

We now see clearly that for $L \gg L_0$ we just get the Newtonian solution of $r_c = r_N$. There will also be a minimum angular momentum possible $L = L_0$ because for $L < L_0$ we see the solution is not real.

comment: Unlike Newtonian orbits, you will find in general *two* circular orbit positions in Schwarzchild orbits. As can be seen in the figure, the outer position corresponds to a stable orbit, while the inner orbit is unstable[4] (any small pertubation and a object would fly off the circular orbit and either into or away from the central mass).

[4] You can determine for yourself (if you want) which oribts are stable or unstable by calculating d^2V_{eff}/dr^2 and evaluating it at r_c. The stable orbits have $d^2V_{\text{eff}}/dr^2 > 0$ and the unstable ones $d^2V_{\text{eff}}/dr^2 < 0$.

3d) Note that if L becomes too small, there is no real solution for r_c. Find this critical value of L and use it to show that the minimum radius of a stable circular orbit is

$$r_{\text{isco}} = 3R_s \tag{66}$$

Solution: From above, we see that the solution is not real if $L < L_0$ where the critical value of angular momentum is

$$L_0 = \sqrt{12}\frac{GMm}{c} = \sqrt{3}\frac{2GMm}{c} \tag{67}$$

Multiplying top and bottom by c we can write this in terms of $R_s = 2GM/c^2$

$$L_0 = \sqrt{3}\frac{2GMm}{c^2}c = \sqrt{3}R_c mc \tag{68}$$

we can see clearly now that L_0 has units of angular momentum.

At the minimum value of $L = L_0$ the radius of the orbit is

$$r_c = \frac{r_N}{2} = \frac{1}{2}\frac{L_0^2}{GMm^2} \tag{69}$$

writing the last term in terms of R_s we get

$$r_c = \frac{r_N}{2} = \frac{L_0^2}{2GMm^2}\frac{c^2}{c^2} = \frac{L_0^2}{m^2R_sc^2} \tag{70}$$

Plugging in $L_0^2 = 3R_c^2m^2c^2$

$$r_c = \frac{3R_c^2m^2c^2}{m^2R_sc^2} = 3R_s \tag{71}$$

which is the radius of the innermost stable orbit

comment: Black holes have an "innermost stable circular orbit" (ISCO) that is a few times the Schwarzchild radius. No object can stably orbit interior to this radius, and will instead plunge into the black hole. Thus we expect the accretion disks of non-spinning black holes to only extend down to r_{isco} and not all the way to R_s.

The famous image of taken by the Event Horizon Telescope – often described as "a picture of a black hole" – is of course instead showing light emitted by gas orbiting around a supermassive black hole at the center of a nearby galaxy. The edge of the dark circle in the image is not actually the event horizon R_s. It is not exactly $R_{\rm isco}$ either. While the emitting gas only extends down to $r_{\rm isco}$, some of the light it emits can circle around due to lensing, which produces an inner edge somewhere between R_s and $r_{\rm isco}$.

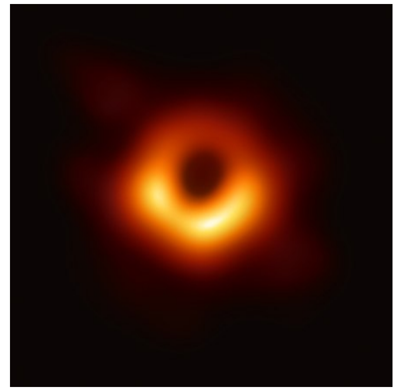


Figure 4: Image of gas around a supermassive black hole, as taken by the Event Horizon Telescope, a set of radio observatories.

FOR LIGHT, THE METRIC INTERVAL $ds^2 = 0$. We thus cannot use proper-term time as a parameter to describe a lightlike geodesic. We can use, however, some other parameter (call it λ) that marks off positions along the photon trajectory. It will not be important exactly how we define λ here. Simply note that the constants of motion will be the same as those above, just with $d\tau$ replaced by $d\lambda$

3e) Consider photons, for which $ds^2 = 0$. Divide the metric by $d\lambda$ and use the constants of motions to find

$$c^2 \left(\frac{E}{mc^2}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 + V_{\rm eff}(r) \tag{72}$$

where

$$V_{\rm eff}(r) = \frac{L^2}{r^2 m^2} - \frac{L^2 R_s}{r^3 m^2} \tag{73}$$

Solution: Setting $ds^2 = 0$ and dividing through by $d\lambda$

$$0 = \left(1 - \frac{R_s}{r}\right) c^2 \left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{(1 - R_s/r)} \left(\frac{dr}{d\lambda}\right)^2 - r^2 \left(\frac{d\phi}{d\lambda}\right)^2 \tag{74}$$

and multiplying through by $(1 - R_s/r)$

$$0 = \left(1 - \frac{R_s}{r}\right)^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\lambda}\right)^2 \left(1 - \frac{R_s}{r}\right) \tag{75}$$

Using the definition of E/M and L/m

$$0 = c^2 \left(\frac{E}{mc^2}\right)^2 - \left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^2 m^2} \left(1 - \frac{R_s}{r}\right) \tag{76}$$

rearranging

$$c^2 \left(\frac{E}{mc^2}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2 m^2} - \frac{L^2 R_s}{r^3 m^2} \tag{77}$$

so

$$V_{\rm eff} = \frac{L^2}{r^2 m^2} - \frac{L^2 R_s}{r^3 m^2} \tag{78}$$

3f) Show that the minimum value of a circular orbit for light is at $r_p = (3/2) R_s$

Solution: The minimum is at

$$\frac{dV_{\rm eff}}{dr} = -\frac{2L^2}{r^3 m^2} + \frac{3L^2 R_s}{r^4 m^2} = 0 \quad (79)$$

so

$$2r_c = 3R_s \implies r_c = \frac{3}{2}R_s \quad (80)$$

comment: The value $r_p = (3/2)R_s$ is known as the *photon sphere*. It is the location where light can orbit in a circle around the black hole. If you stood at r_p and looked ahead, you would see the back of your head! This is not a stable orbit, though, so light could not orbit around indefinitely at this radius.

The edge of the dark circle we see from the Event Horizon Telescope image most closely tracks the photon sphere. Light from the disk that approaches this radius will swing around in a circle, though eventually will either fall into the black hole or fly off to be observed. Light that goes below r_p will rapidly fall into the black hole. This produces the so-called "shadow" of the black hole around r_p.

By measuring the size of the imaged dark circle, we can estimate the mass of the black hole. There are a few complexities though: if the black hole is spinning (as it most likely is) the metric is different than the Schwarzchild one, and the location of the photon sphere and $r_{\rm isco}$ are modified. In addition the inclination of the disk from our viewing angle also affects the image. Astrophysicists use computer models to try to simulate the image and derive the mass; such models are basically just generalizations of what you did in this problem.

Problem 4: Black Hole Accretion

Black holes are perhaps the most efficient energy generation sources around. When matter accretes onto a black hole, i some fraction of its rest mass energy can be tapped an radiated as light. This is the process we think powers some of the most luminous sources in the Universe, such as quasars.

If gas falling into a a black hole has sufficient angular momentum, it will form an accretion disk. The gas in the disk follows nearly circular orbits, with the gas nearer the center swirling around faster than gas farther out. If there were little friction in the disk, the gas could orbit around indefinitely, like the rings of Saturn. However, as the gas at different radii are sliding and shearing against each other, the viscosity of the gas releases heat and slows down the inner layers, which fall closer to the black hole. Once gas drops below $r_{\rm isco}$ it can no longer orbit stably and will fall quickly below the event horizon.

4a) Consider a bit of mass m in a disk around a Schwarzchild black hole. Determine the energy $E_{\rm isco}$ of this mass[5] if it is a circular orbit

[5] Here E is defined as in the previous problem. For circular orbits $dr/d\tau = 0$, so look at the energy for the L and r at the $r_{\rm isco}$.

at the innermost stable orbit, r_{isco}

Solution: From problem 3 we have

$$\epsilon = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2GM}{r^3mc^2} \quad (81)$$

where

$$\epsilon = \frac{mc^2}{2}\left[(E/mc^2)^2 - 1)\right] \quad (82)$$

For a circular orbit $dr/d\tau = 0$ and for the innermost stable orbit the angular momentum $L = L_0 = \sqrt{3}R_c mc$

$$\epsilon = -\frac{GMm}{r} + \frac{3R_c^2m^2c^2}{2mr^2} - \frac{3R_c^2m^2c^2GM}{r^3mc^2} \quad (83)$$

or

$$\epsilon = -\frac{GMm}{r} + \frac{3R_c^2m^2c^2}{2mr^2} - \frac{3R_c^2m^2c^2GM}{r^3mc^2} \quad (84)$$

And the innermost stable circular orbit is $r_c = 3R_s$

$$\epsilon = -\frac{GMm}{3R_s} + \frac{3R_s^2m^2c^2}{2m9R_s^2} - \frac{3R_s^2m^2c^2GM}{27R_s^3mc^2} \quad (85)$$

Cleaning this up

$$\epsilon = -\frac{2GMc^2m}{6c^2R_s} + \frac{3mc^2}{18} - \frac{mc^2GM}{9R_sc^2} \quad (86)$$

$$\epsilon = -\frac{mc^2}{6} + \frac{mc^2}{6} - \frac{mc^2}{9} = -\frac{mc^2}{9} \quad (87)$$

Using the definition of ϵ

$$\epsilon = \frac{mc^2}{2}\left[(E/mc^2)^2 - 1)\right] = -\frac{1}{9}mc^2 \quad (88)$$

$$(E/mc^2)^2 - 1 = -\frac{1}{9} \quad (89)$$

$$(E/mc^2)^2 = \frac{8}{9} \quad (90)$$

or

$$E = mc^2\sqrt{\frac{8}{9}} \quad (91)$$

4b) Assume the initial energy of this piece of mass was $E = mc^2$ (i.e., it's initial orbit was at large r where it was moving slowly, so that its energy was essentially just its rest mass energy). Show that the drop in energy in going from this initial radius to its final circular orbit at r_{isco} is

$$\Delta E = \left(1 - \sqrt{\frac{8}{9}}\right)mc^2 \approx 0.052\; mc^2 \quad (92)$$

Solution: The initial energy is mc^2 and the final energy we found above is $mc^2\sqrt{8/9}$ so the difference is

$$\Delta E = \left(1 - \sqrt{\frac{8}{9}}\right) mc^2 \approx 0.052\, mc^2 \tag{93}$$

comment: You have shown that for a bit of mass to move from far out in the disk to a circular orbit at $r_{\rm isco}$ it must loose about 5% of its energy. It can do this by radiating away the energy as light. The viscosity in the disk turns some of the kinetic energy of the orbiting gas into heat, and the hot gas can then emit light. The remaining 95% or so of the mass/energy winds up being eaten by the black hole.

We can compare this energy release to other processes. Typical chemical processes (e.g., burning coal) involves changes in the atomic and molecular structure, where the energy levels are of order electron volts (eV). The masses of typical atoms are of order 1 GeV (the mass of a proton), so the chemical processes release around eV/GeV $\sim 10^{-9}$ of the rest mass energy. Nuclear reactions such as those that power the energy of the Sun release about an MeV per nucleus, so the efficiency is around 10^{-3}. These are all well below the possible energy release efficiency of black hole accretion.

For rotating black holes, the innermost stable orbit turns out to be even closer in. For a maximally spinning black hole, one finds $r_{\rm isco} = R_s$ (assuming the disk rotates in the same direction as the black hole). In this case, even greater energy release is possible, reaching around 12% of mc^2.

Not all accretion onto black holes leads to such a large release of energy. If gas lacks angular momentum it may plunge directly into the black hole rather then remaining on circular orbits. In this case it may only radiate a small amount of energy away, with almost all of the energy going into the black hole (or being blown out in winds or jets). This is the case with the supermassive black hole in the center of our own Galaxy, and the one in the M87 galaxy (shown in the Event Horizon telescope picture), both of which radiate quite inefficiently.