# Continuous-Time VAR(1) models: What, Why and How?

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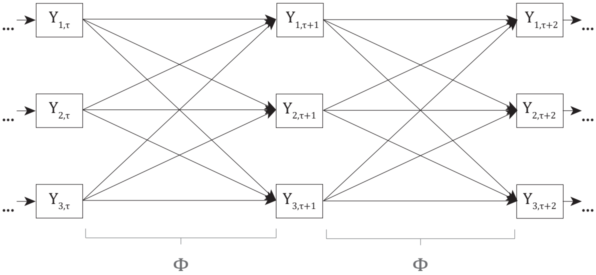
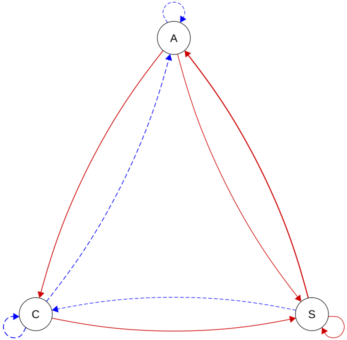
# Overview

1. Review of VAR(1) model
2. The Time-Interval problem
3. The Continuous-Time VAR(1) model
4. The “How” of the CT-VAR(1)

# The VAR(1) Model

$$\boldsymbol{Y}_\tau = \boldsymbol{c} + \boldsymbol{\Phi}\boldsymbol{Y}_{\tau-1} + \boldsymbol{\epsilon}_\tau$$



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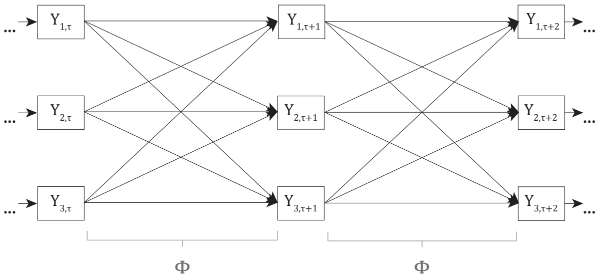


- **Discrete-Time** model
- **Time-interval** between occassions not accounted for

# What's the problem?

The **time-interval** $\Delta t$ is important for our understanding of the psychological process!

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The effect of Stress now on Rumination later

- Little effect: 1 minute later
- Large effect: 1 hour later
- Little effect: 5 hours later

# What's the problem?

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The effect of Stress now on Rumination later

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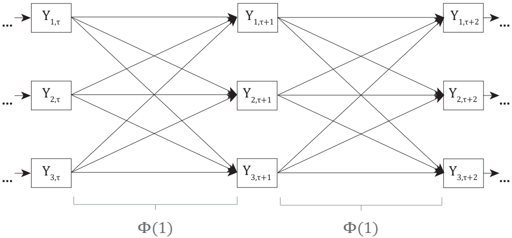
We can say that the regression parameters are a function of the time-interval

- $\mathbf{\Phi} = \mathbf{\Phi}(\Delta t)$

# Consequences of the time-interval problem

1. Equal time-intervals: not generalizable
   - $\mathbf{\Phi}(\Delta t = 1) \neq \mathbf{\Phi}(\Delta t = .5)$



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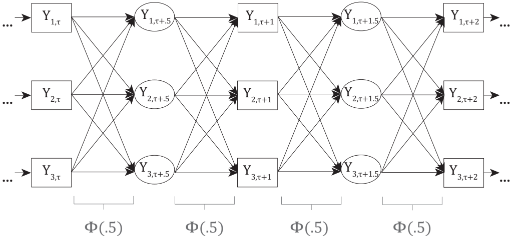

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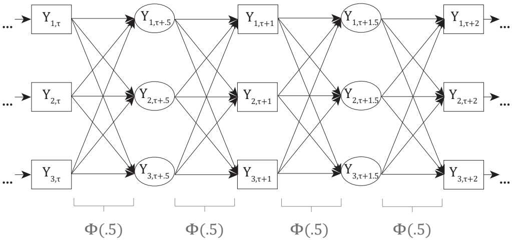

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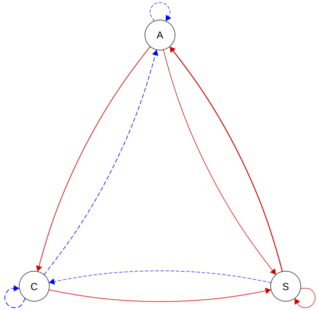

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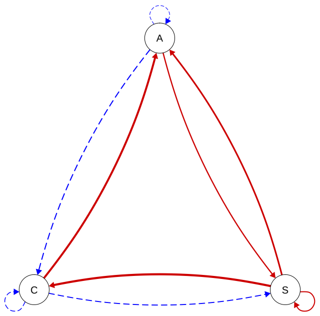

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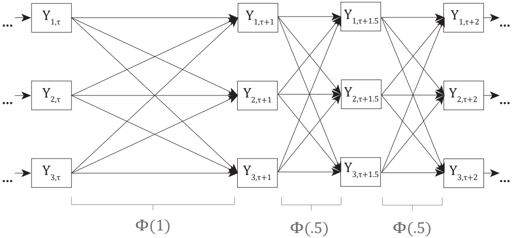
2. Unequal time-intervals: mix of effects
   - $\hat{\mathbf{\Phi}} = ?$
   - If not accounted for, may not reflect effects at *any* time-interval



# Solution: Continuous-Time Modeling

Continuous-Time VAR(1) models [1]

- Assume that processes take on some value **at every moment in time**
- The causal "action" takes place at a very short time-interval
  - Based on differential equation models
  - $\frac{dY(t)}{dt} = \boldsymbol{c} + \boldsymbol{AY}(t) + \gamma$
- Explicitly model effects as a function of time-interval
- Boker 2002, Oravecz, Teurlinckx & Vandekerckhove 2011, Voelkle et al 2012

[1] Also known as the Ornstein-Uhlenbeck model

# The CT-VAR(1) model

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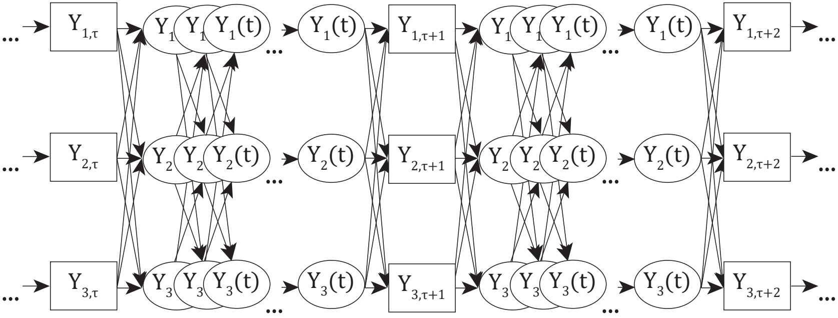

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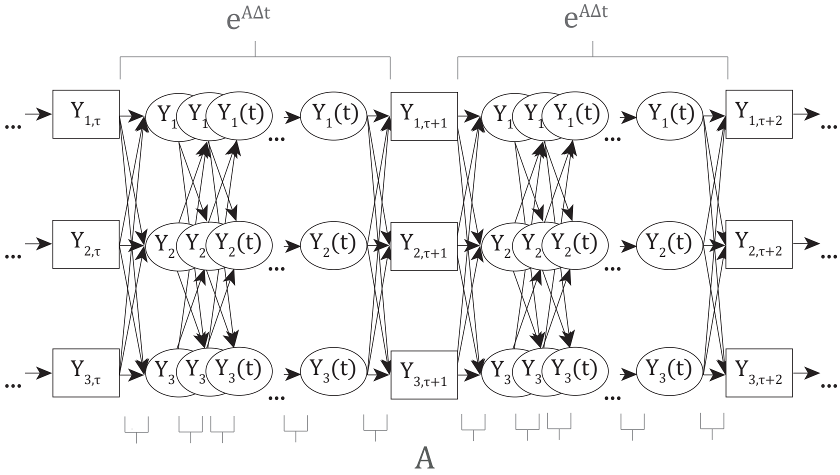
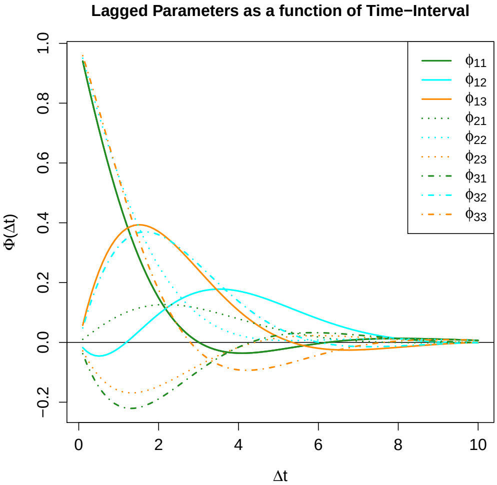


# Time-interval dependency of VAR estimates

$$e^{\boldsymbol{A}\Delta t} = \boldsymbol{\Phi}(\Delta t)$$



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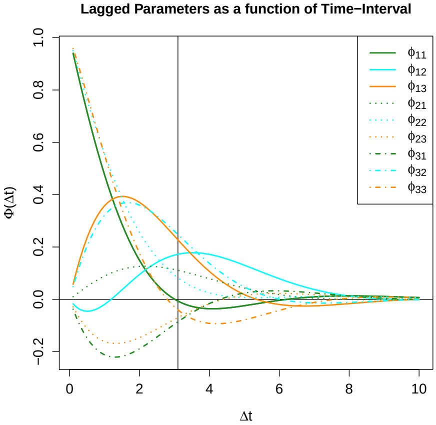


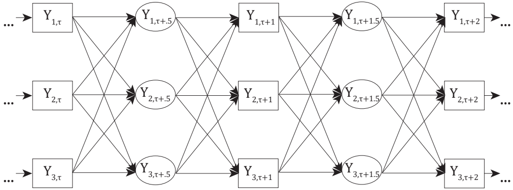

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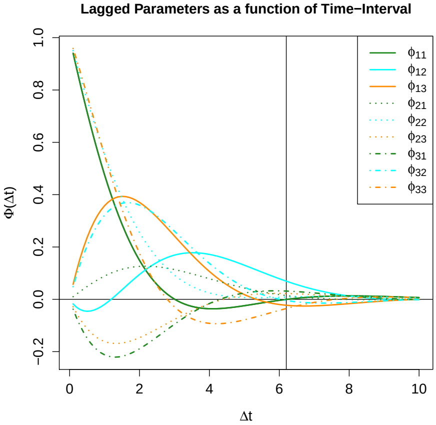
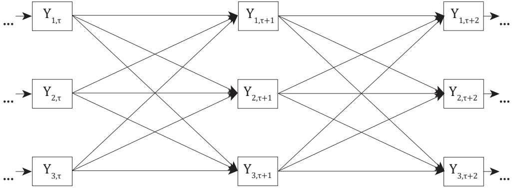
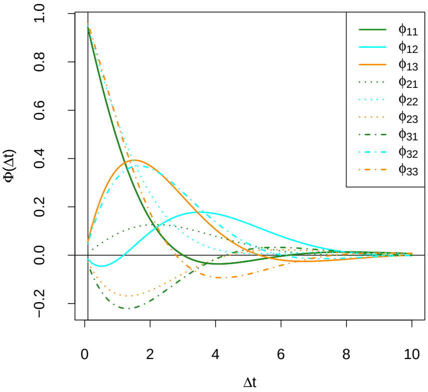
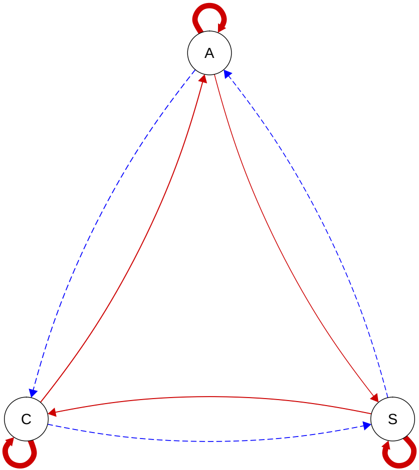


# Network structure as a function of time-interval



# Why adopt a Continuous-Time approach?

Practical benefits:

- Deals well with measurements taken at unequal intervals
- Helps comparability between studies using different time-intervals

Conceptual benefits:

- Matches closer to our substantive ideas about psychological processes (Boker 2002)
- Capture/explore how "the effect" of X on Y evolves and varies over the interval

# What, Why... How?

# Estimation Possibilities

Need only repeated-measures data and time info on observations

- The Indirect Method
  - First fit a VAR(1) model with equal intervals (e.g. using Mplus)
  - Solve for the CT model (Voelkle et al 2012)
- Fit the CT-VAR(1) model
  - ctsem (R package: Driver et al. 2018)
  - BHOUM (Oravecz et al. 2009)
  - Allows for single-subject or multi-level
- Fit the differential equation
  - GLLA (R; Boker et al 2010) and dynr (R; Ou, Hunter and Chow, 2018)

# Making sense of CT models

- Interpret the estimated parameters directly
  - $\boldsymbol{A}$ describes how position $Y(t)$ is related to the *rate of change* $\frac{dY(t)}{dt}$
- Visualise the predicted behaviour of the system
  - Just like the VAR(1), the CT-VAR(1) describes a system that varies around a stable equilibrium
  - The estimated model parameters describe how changes in one variable result in changes in the others

# Example Analysis (Ryan, Kuiper & Hamaker, 2018) [1]

Subset of data from Wichers & Groot (2016)

- Single-subject
- 286 measurements over 42 days
- Modal interval: 1.77 hours (IQR: 1.25 - 3.23 hrs)

Two items selected:

- *Down*: "I feel down"
- *Tired*: "I am tired"
- Both were centered and standardized

How do Down and Tired influence one another over time?

[1] Complete analysis on github.com/ryanoisin/continuous_time-ILD-what-why-how

# Example Analysis (Ryan, Kuiper & Hamaker, 2018)

The drift matrix relating the processes Down ($Do(t)$) and Tired ($Ti(t)$) is given by

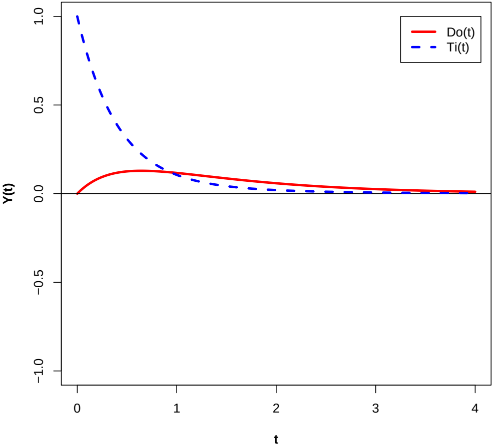
$$\boldsymbol{A} = \begin{bmatrix} -0.995 & 0.573 \\ 0.375 & -2.416 \end{bmatrix}.$$

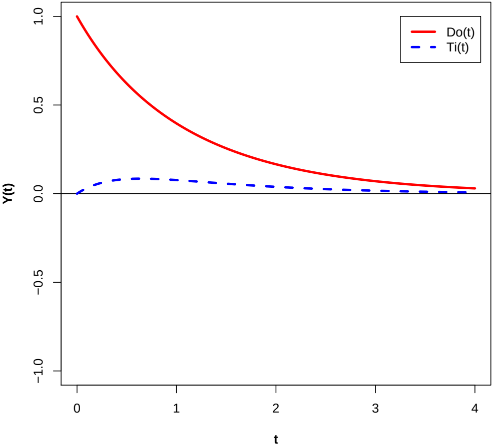
Which means

$$\frac{d\hat{Do}(t)}{dt} = -0.995 Do(t) + 0.573 Ti(t)$$
$$\frac{d\hat{Ti}(t)}{dt} = 0.375 Do(t) - 2.416 Ti(t)$$

# Visualisation I: Impulse Response Functions

IRFs: How does the system react to a given impulse?



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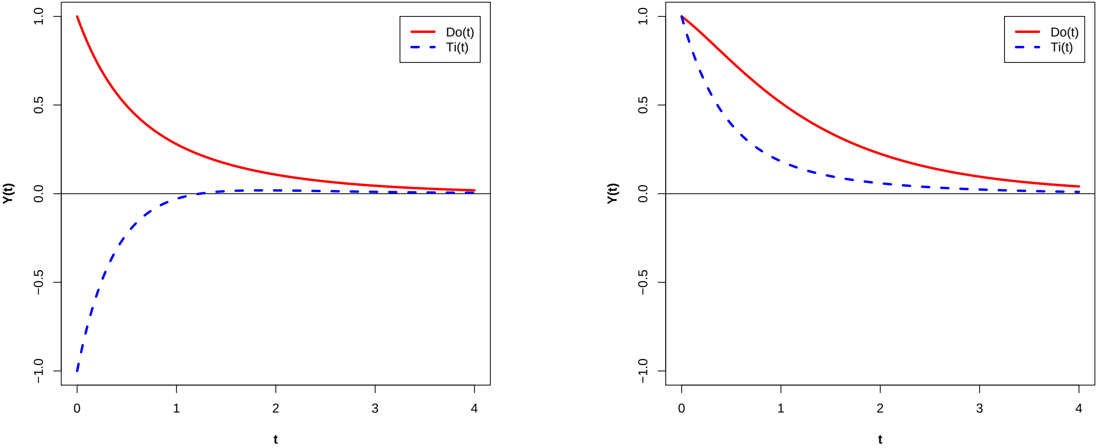
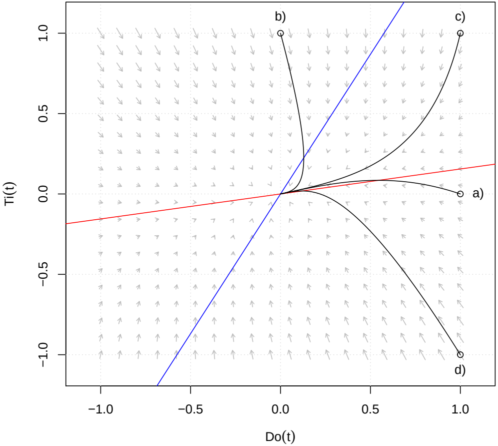


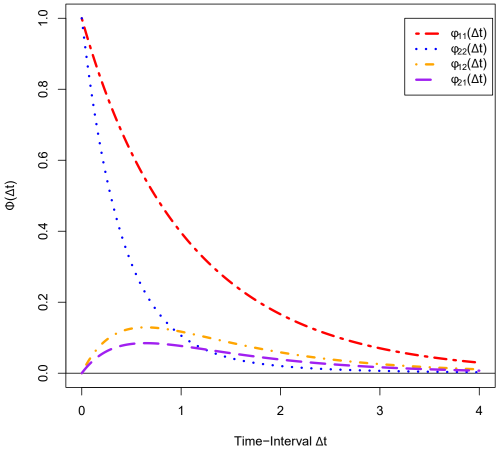
# Visualisation II: Vector Fields

## Vector Fields: What trajectories are possible?



# Visualisation III: $\mathbf{\Phi}(\Delta t)$

How do the lagged parameters depend on time-interval?



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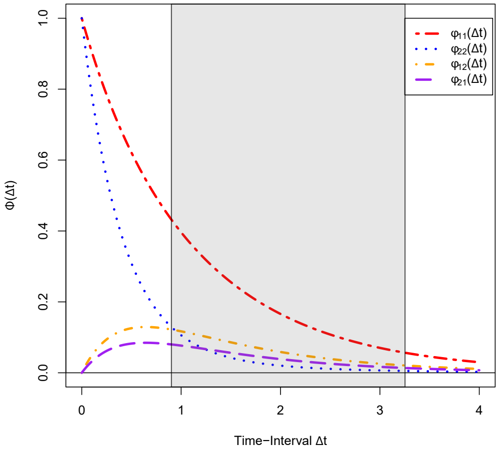

# Summary

The CT-VAR(1) model is a promising alternative to current VAR(1) models

- Better matches our substantive ideas about psychological processes
- Deals well with unequal spacing, gaps for night time etc.
- Gives us a single effects matrix which is independent of $\Delta t$
- Explains the time-interval dependency problem
- Explore how the lagged parameters potentially change as a function of $\Delta t$
- Numerous software packages now available

# CT-VAR(1): It's about time!

Substantive Applications in psychology becoming more popular

- Affect Regulation (Wood et al 2017), Happiness and Religiosity (Meulemann & Oud, 2018), Work Engagement (Sosowska et al 2019)

More reading: van Montfort, Oud & Voelkle (2018)

- This presentation is based on Chapter 2!

Upcoming papers:

- How can CT models be used to inform interventions? (previous version available on PsyArXiv now)
- Meta-analysis of VAR(1) using CT (with Rebecca Kuiper)

# Get in Touch


- ryanoisin.github.io
- o.ryan@uu.nl

# Key References


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# Continuous Time Model

First-Order Stochastic Differential Equation

$$\frac{d\boldsymbol{Y}(t)}{dt} = \boldsymbol{A}(\boldsymbol{Y}(t) - \boldsymbol{\mu}) + \gamma \frac{d\boldsymbol{W}(t)}{dt}$$

CT VAR(1) Model

$$\boldsymbol{Y}(t) = \boldsymbol{e}^{\boldsymbol{A}\Delta t}\boldsymbol{Y}(t - \Delta t) + \boldsymbol{w}(\Delta t)$$

# Example analysis estimates

| Parameter | Value | Std. Error |
|---|---|---|
| $a_{11}$ | -0.995 | 0.250 |
| $a_{21}$ | 0.375 | 0.441 |
| $a_{12}$ | 0.573 | 0.595 |
| $a_{22}$ | -2.416 | 1.132 |
| $\gamma_{11}$ | 1.734 | 0.612 |
| $\gamma_{21}$ | -0.016 | 0.650 |
| $\gamma_{22}$ | 4.606 | 1.374 |

# Numerical Example Network

$$\boldsymbol{A} = \begin{bmatrix} -6 & -.2 & .6 \\ .1 & -.5 & -.3 \\ -.4 & .5 & -.4 \end{bmatrix}$$